

Translation-invariant generalized topologies induced by probabilistic norms

Bernardo Lafuerza-Guillén

*Departamento de Estadística y Matemática Aplicada,
Universidad de Almería, E-04120 Almería, Spain
blafuerz@ual.es*

José L. Rodríguez

*Área de Geometría y Topología,
Universidad de Almería, E-04120 Almería, Spain
jlrodri@ual.es*

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Abstract. One considers probabilistic normed spaces as defined by Alsina, Sklar, and Schweizer, but with non necessarily continuous triangle functions. Such spaces are endowed with a generalized topology that is Fréchet-separated, translation-invariant and countably generated by radial and circled 0-neighborhoods. Conversely, we show that such generalized topologies are probabilistically normable.

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1 Introduction

Probabilistic normed spaces (briefly, PN spaces) were first defined by Šerstnev in the early sixties [see [13]], thus originating a fruitful theory that extended the theory of ordinary normed spaces. Thirty years later, Alsina, Schweizer, and Sklar gave in [1] a quite general definition of PN space, based on the definition of Menger's betweenness in probabilistic metric spaces; [see [14], p. 232].

We here consider PN spaces in which the involved triangle functions are non necessarily continuous. With regards to a generalized topology in the sense of Fréchet and for probabilistic metric spaces, the problem was treated by Höhle in [6] where he showed that all generalized topologies which are Fréchet-separated and first-numerable are induced by certain probabilistic metrics. The main result of this paper is a similar result for probabilistic norms, where the t -norm has certain restriction:

1 Theorem. *Let T be a t -norm such that $\sup_{0 \leq x < 1} T(x, x) < 1$. Suppose that $T(x, y) \leq xy$, whenever $x, y < \delta$, for some $\delta > 0$. A Fréchet-separated, translation-invariant, generalized topology $(\mathcal{U}_p)_{p \in S}$ on a real vector space S is*

derivable from a Menger PN space $(S, \nu, \tau_T, \tau_{T^*})$, if and only \mathcal{U}_θ admits a countable base of radial and circled subsets, where θ is the origin of S .

In fact, this result also holds if one assumes T to be Archimedean near the origin, i.e. there is a $\delta > 0$ such that $0 < T(x, x) < x$, for all $0 < x < \delta$ [see Remark [8], after Theorem [1]].

We think that a similar result could be interesting for fuzzy normed spaces in the sense of Felbin [3], but allowing non-continuity of the t -norms, and t -conorms involved in the fuzzy structure.

In [10] the authors use this generalized topology to define bounded subsets in PN spaces (with non necessarily continuous triangle functions) and study its relationship with \mathcal{D} -bounded subsets (a concept which is defined in probabilistic terms).

2 PM and PN spaces

Recall from [1] and [14] some definitions on probabilistic metric and probabilistic normed spaces.

As usual, Δ^+ denotes the set of distance distribution functions, i.e. distribution functions with $F(0) = 0$, endowed with the metric topology given by the modified Lévy-Sybley metric d_L [see 4.2 in [14]]. Given a real number a , ε_a denotes the distribution function defined as $\varepsilon_a(x) = 0$ if $x \leq a$ and $\varepsilon_a(x) = 1$ if $x > a$. Hence, the set of non-negative real numbers \mathbb{R}^+ can be viewed as a subspace of Δ^+ . A triangle function τ is a map from $\Delta^+ \times \Delta^+ \rightarrow \Delta^+$ which is commutative, associative, nondecreasing in each variable and has ε_0 as the identity. Such functions give rise to all possible extensions of the sum of real numbers, so that (M3) below corresponds to the triangle inequality.

A *probabilistic metric space* (briefly, a PM space) is a triple (S, F, τ) where S is a non-empty set, F is a map from $S \times S \rightarrow \Delta^+$, called the probabilistic metric, and τ is a triangle function, such that:

$$(M1) \quad F_{p,q} = \varepsilon_0 \text{ if and only if } p = q.$$

$$(M2) \quad F_{p,q} = F_{q,p}.$$

$$(M3) \quad F_{p,q} \geq \tau(F_{p,r}, F_{r,q}).$$

When only (M1) and (M2) are required, the pair (S, F) is said to be a *probabilistic semi-metric space* (briefly, PSM space).

A *PN space* is a quadruple (S, ν, τ, τ^*) in which S is a vector space over \mathbb{R} , the *probabilistic norm* ν is a map $S \rightarrow \Delta^+$, τ and τ^* are triangle functions¹

¹In the definition of PN space given in [1] the triangle functions are assumed to be continuous

such that the following conditions are satisfied for all p, q in S :

(N1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$, where θ is the origin of S .

(N2) $\nu_{-p} = \nu_p$.

(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$.

(N4) $\nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0, 1]$.

Observe that every PN space (S, ν, τ, τ^*) is a PM space, where $F_{p,q} := \nu_{p-q}$.

Recall that a t -norm is a binary operation on $[0, 1]$ that is commutative, associative, nondecreasing in each variable, and has 1 as identity. Dually, a t -conorm is a binary operation on $[0, 1]$ that is commutative, associative, nondecreasing in each variable, and has 0 as identity. If T is a t -norm, its associated t -conorm T^* is defined by $T^*(x, y) := 1 - T(1 - x, 1 - y)$. Given a t -norm T one defines the functions τ_T and τ_{T^*} by

$$\tau_T(F, G)(x) := \sup\{T(F(s), G(t)) : s + t = x\},$$

and

$$\tau_{T^*}(F, G)(x) := \inf\{T^*(F(s), G(t)) : s + t = x\}.$$

Recall that if T is left-continuous then τ_T is a triangle function [14, p. 100], although this is not necessary; For example, if Z denotes the weakest t -norm, defined as $Z(x, 1) = Z(1, x) = x$ and $Z(x, y) = 0$ elsewhere, then τ_Z is a triangle function which is not continuous.

A Šerstnev PN space is a PN space (V, ν, τ, τ^*) where ν satisfies the following Šerstnev condition:

$$(\check{S}) \quad \nu_{\lambda p}(x) = \nu_p\left(\frac{x}{|\lambda|}\right), \text{ for all } x \in \mathbb{R}^+, p \in V \text{ and } \lambda \in \mathbb{R} \setminus \{0\}.$$

which clearly implies (N2) and also [see [1]](N4) in the strengthened form

$$\nu_p = \tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p}), \tag{1}$$

for all $p \in V$ and $\lambda \in [0, 1]$ [see [1, Theorem 1]], where M is the t -norm defined as $M(x, y) = \min\{x, y\}$.

Let T be a t -norm. A Menger PM space under T is a PM space of the form (S, F, τ_T) . Analogously, a Menger PN space under T is a PN space of the form $(S, \nu, \tau_T, \tau_{T^*})$. Note that every metric space (S, d) is a Menger space (S, F, τ_M) where $F_{p,q} = \varepsilon_{d(p,q)}$. Analogously, every normed space $(S, \|\cdot\|)$ is a Menger and Šerstnev PN space (S, ν, τ_M, τ_M) where $\nu_p = \varepsilon_{\|p\|}$.

3 Probabilistic metrization of generalized topologies

In [6] Höhle solved a problem posed by Thorp about the probabilistic metrization of generalized topologies. We recall some definitions and results that we shall use in the next section.

Let S be a (non-empty) set. A *generalized topology (of type V_D)* on S is a family of subsets $(\mathcal{U}_p)_{p \in S}$, where \mathcal{U}_p is a filter on S such that $p \in U$ for all $U \in \mathcal{U}_p$ [see e.g. [14, p. 38], [2, p. 22]]. Elements of \mathcal{U}_p are called *neighborhoods* at p . Such a generalized topology is called *Fréchet-separated* if $\bigcap_{U \in \mathcal{U}_p} U = \{p\}$.

A *generalized uniformity* \mathcal{U} on S is a filter on $S \times S$ such that every $V \in \mathcal{U}$ contains the diagonal $\{(p, p) : p \in S\}$, and for all $V \in \mathcal{U}$, we have that $V^{-1} := \{(q, p) : (p, q) \in V\}$ also belongs to \mathcal{U} . Elements of \mathcal{U} are called *vicinities* (or “entourages”). Every generalized uniformity \mathcal{U} induces a generalized topology as follows: for $p \in S$,

$$\mathcal{U}_p := \{U \subseteq S \mid \exists V \in \mathcal{U} : U \supseteq \{q \in S \mid (p, q) \in V\}\}. \quad (2)$$

A uniformity \mathcal{U} is called *Hausdorff-separated* if the intersection of all vicinities is the diagonal on S . Theorem 1 in [6] claims:

2 Theorem. [Höhle] *Every Fréchet-separated generalized topology $(\mathcal{U}_p)_{p \in S}$ on a given set S is derivable from a Hausdorff-separated generalized uniformity \mathcal{U} in the sense of (2).* □

Let (S, F) be a PSM space. Consider the system $(\mathcal{N}_p)_{p \in S}$, where $\mathcal{N}_p = \{N_p(t) : t > 0\}$ and

$$N_p(t) := \{q \in S : F_{p,q}(t) > 1 - t\}.$$

This is called the *strong neighborhood system*. If we define $\delta(p, q) := d_L(F_{p,q}, \varepsilon_0)$, then δ is a semi-metric on S (i.e. it may not satisfy the triangle inequality of the standard metric axioms), and $N_p(t) = \{q : d_L(F_{p,q}, \varepsilon_0) < t\}$. Clearly $p \in N$ for every $N \in \mathcal{N}_p$, and the intersection of two strong neighborhoods at p is a strong neighborhood at p . Furthermore, \mathcal{N}_p admits a countable filter base given by $\{N_p(1/n) : n \in \mathbb{N}\}$, hence the strong neighborhood system is first-countable. The above explanation yields the following fact [see more details in [14], p. 191]:

3 Theorem. *Let (S, F) be a PSM space, then the strong neighborhood system defines a generalized topology of type V_D which is Fréchet-separated and first-countable.* □

This generalized topology is called the *strong generalized topology* of the PSM space (S, F) .

The main result in [6] is the following.

4 Theorem. [Höhle] Let T be a t -norm such that $\sup_{0 \leq x < 1} T(x, x) < 1$. A Fréchet-separated generalized topology $(\mathcal{U}_p)_{p \in S}$ on a set S is derivable from a Menger PM space (S, F, T) if and only if there exists a Hausdorff-separated, generalized uniform structure \mathcal{U} having a countable filter base, such that \mathcal{U} is compatible with $(\mathcal{U}_p)_{p \in S}$. □ QED

5 Remark. If (S, F, τ) is a PM space with τ continuous, then the associated generalized topology is in fact a topology. This topology is called *the strong topology*. Because of (M1) this topology is Hausdorff. Since it is first-countable and uniformable, it is metrizable [see [14, Theorem 12.1.6]].

Conversely, if $\sup_{0 \leq x < 1} T(x, x) = 1$, then a Fréchet-separated, uniformable topology is derivable from a Menger space (S, F, T) if and only if there exists a Hausdorff uniformity U on S having a countable filter base [6].

4 Translation-invariant generalized topologies

Assume now that S is a vector space over \mathbb{R} . A generalized topology $(\mathcal{U}_p)_{p \in S}$ on S is said to be *translation-invariant* if for all $U \in \mathcal{U}_p$ and $q \in S$, we have $q + U \in \mathcal{U}_{p+q}$. Consequently, a translation-invariant generalized topology is uniquely determined by the neighborhood system \mathcal{U}_θ at the origin θ of S . In this case, the generalized uniformity from which one can derive the generalized topology is:

$$\mathcal{U} := \{V \subseteq S \times S \mid \exists U \in \mathcal{U}_\theta : V \supseteq \{(p, q) \mid p - q \in U\}\}.$$

Recall that a subset U of a vector space is called *radial* if $-U = U$; it is called *circled (or balanced)* if $\lambda U \subset U$ for all $|\lambda| \leq 1$.

6 Theorem. Every PN space (S, ν, τ, τ^*) admits a generalized topology $(\mathcal{U}_p)_{p \in S}$ of type V_D which is Fréchet-separated, translation-invariant, and countably-generated by radial and circled θ -neighborhoods.

PROOF. Let (S, ν, τ, τ^*) be a PN space with τ non-necessarily continuous. Let (S, F) be its associated PSM space, where $F_{p,q} = \nu_{p-q}$. The strong neighborhoods at p are given by $N_p(t) = \{q \in S : \nu_{p-q}(t) > 1 - t\} = p + N_\theta(t)$. In particular, the generalized topology is translation-invariant. By (N1) we have that this generalized topology is Fréchet-separated (as in the case of PSM spaces). The countable base of θ -neighborhoods is $\{N_\theta(\frac{1}{n}) : n \in \mathbb{N}\}$, whose elements are clearly radial and circled, by axioms (N2) and (N4), respectively. □ QED

Note that the generalized topology induced by a PN space (S, ν, τ, τ^*) is derivable from the following generalized uniformity:

$$\mathcal{U} := \left\{ V \subset S \times S \mid \exists n \in \mathbb{N} : V \supseteq \left\{ (p, q) \mid \nu_{p-q} \left(\frac{1}{n} \right) \geq 1 - \frac{1}{n} \right\} \right\},$$

which is translation-invariant and has a countable filter base of radial and circled vicinities.

Adapting the methods in [6], we next show that a converse result holds for such generalized topologies (or generalized uniformities).

Let S be a vector space and $(\mathcal{U}_p)_{p \in S}$ be a Fréchet-separated, translation-invariant, generalized topology of type V_D on S . Then, there is a unique translation-invariant, Hausdorff-separated generalized uniformity, which is defined as follows

$$\mathcal{U} := \{V \subseteq S \times S \mid \exists U \in \mathcal{U}_\theta : V \supseteq \{(p, q) : p - q \in U\}\}.$$

The analogous result of Theorem 4 for PN spaces is the following. (Note that there is an extra assumption on T):

7 Theorem. *Let T be a t -norm such that $\sup_{0 \leq x < 1} T(x, x) < 1$. Suppose that $T(x, y) \leq xy$, whenever $x, y < \delta$, for some $\delta > 0$. A Fréchet-separated, translation-invariant, generalized topology $(\mathcal{U}_p)_{p \in S}$ on a real vector space S is derivable from a Menger PN space $(S, \nu, \tau_T, \tau_{T^*})$, if and only \mathcal{U}_θ admits a countable base of radial and circled subsets.*

PROOF. The direct implication has been shown above. For the converse, let $\mathcal{B} = \{V_n \mid n \in \mathbb{N}\}$ be a countable filter base for \mathcal{U}_θ consisting on radial and circled θ -neighborhoods.

Let $N_0 \in \mathbb{N}$ such that $1 - \frac{1}{N_0} \geq \sup_{0 \leq x < 1} T(x, x)$. We can assume that $\frac{1}{N_0} < \delta$, so that $T(x, y) \leq xy$, for all $x, y \leq \frac{1}{N_0}$, where δ is given by hypothesis.

Before defining ν , recall from [6, Theorem 2] the distribution functions F_n (used to define the probabilistic metric F):

$$F_n(x) := \begin{cases} 0 & : x \leq 0 \\ 1 - 1/(N_0(n + 1)), & 0 < x \leq \frac{1}{n+1}, \\ 1 - 1/(2N_0(n + 1)), & \frac{1}{n+1} < x \leq 1, \\ 1 - 1/(2^{m+1}N_0(n + 1)), & m < x \leq m + 1 \quad \text{for } m \in \mathbb{N}. \end{cases}$$

By putting " $\nu_p = F_{p,\theta}$ " in [6, Theorem 2]) we define:

$$\nu_p(x) := \begin{cases} F_0, & p \notin V_1 \\ F_n, & p \in V_n \setminus V_{n+1}, \text{ for } n \in \mathbb{N} \\ \varepsilon_0, & p \in \bigcap_n V_n. \end{cases}$$

We next check that $(S, \nu, \tau_T, \tau_{T^*})$ is a PN space. Axiom (N1) holds because the generalized topology is Fréchet-separable. (N2) holds because all V_n 's are radial. (N3) holds as in [6]:

$$\tau_T(\nu_p, \nu_q)(x) = \sup_{r+s=x} T(\nu_p(r), \nu_q(s)) \leq 1 - 1/N_0 \leq \nu_{p+q}(r + s) = \nu_{p+q}(x).$$

Finally, for (N4): Let $p \in V_n$ and $\lambda \in [0, 1]$. Then, λp and $(1 - \lambda)p$ are also in V_n , because V_n is circled. For $x = r + s$, we have to show that

$$\nu_p(x) \leq T^*(\nu_{\lambda p}(r), \nu_{(1-\lambda)p}(s)).$$

Suppose first that r and s are strictly greater than 1, $r, s > 1$. Let $a, b, c \in \mathbb{N}$ such that $a < r \leq a + 1$, $b < s \leq b + 1$, and $c < r + s \leq c + 1$. Then,

$$\begin{aligned} \nu_{\lambda p}(r) &= 1 - 1/(2^{a+1}N_0(n + 1)), \\ \nu_{(1-\lambda)p}(s) &= 1 - 1/(2^{b+1}N_0(n + 1)), \\ \nu_p(r + s) &= 1 - 1/(2^{c+1}N_0(n + 1)). \end{aligned}$$

By the properties of T it follows that

$$\begin{aligned} T^*(\nu_{\lambda p}(r), \nu_{(1-\lambda)p}(s)) &= 1 - T(1 - \nu_{\lambda p}(r), 1 - \nu_{(1-\lambda)p}(s)) \\ &= 1 - T(1/(2^{a+1}N_0(n + 1)), 1/(2^{b+1}N_0(n + 1))) \\ &\geq 1 - (1/(2^{a+1}N_0(n + 1))) \cdot (1/(2^{b+1}N_0(n + 1))) \\ &\geq 1 - 1/(2^{c+1}N_0(n + 1)) \\ &= \nu_p(r + s) = \nu_p(x). \end{aligned}$$

In the third line we have used that the arguments of T are smaller than $1/N_0$, thus we can apply $T(x, y) \leq xy$. Then, we obtain $\nu_p \leq \tau_{T^*}(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ as desired. The inequality for the other possible values of r and s , is checked in a similar way. We conclude that $(S, \nu, \tau_T, \tau_{T^*})$ is a Menger PN space under T .

It only remains to show that the generalized topology induced by ν is the same as the one given at the beginning. As in [6], we have by construction that

$$V_n = \left\{ p \in S \mid \nu_p \left(\frac{1}{n + 1} \right) \geq 1 - \frac{1}{N_0(n + 1)} \right\}.$$

Thus, the filter base $\{p \in S \mid \nu_p(\frac{1}{n+1}) \geq 1 - \frac{1}{n+1}\}$ induced by ν is equivalent to \mathcal{B} , hence the proof is finished. \square

8 Remark. Theorem 6 also holds if instead of assuming $T(x, y) \leq xy$ near the origin, one assumes that T is Archimedean near the origin (i.e. there is a $\delta > 0$ such that $0 < T(x, x) < x$, for all $0 < x < \delta$). In that case, the distribution function F_n can be chosen as:

$$F_n(x) := \begin{cases} 0 & : x \leq 0 \\ 1 - z & : 0 < x \leq \frac{1}{n+1} \\ 1 - T(z, z) & : \frac{1}{n+1} < x \leq 1 \\ 1 - T^{m+1}(z, z) & : m < x \leq m + 1 \text{ for } m \in \mathbb{N}, \end{cases}$$

where $z = 1/(N_0(n + 1))$, $T^1(x, y) = T(x, y)$ and recursively

$$T^r(x, y) = T(T^{r-1}(x, y), T^{r-1}(x, y)).$$

References

- [1] C. ALSINA, B. SCHWEIZER, A. SKLAR: *On the definition of a probabilistic normed space*, Aequationes Math. **46** (1993), 91–98.
- [2] A. APPERT, KY-FAN: *Espaces topologiques intermédiaires*, Act- Sci. et Ind., 1121 Herman, Paris (1951).
- [3] C. FELBIN: *Finite-dimensional fuzzy normed linear space*, Fuzzy Sets and Systems, **48** (1992), 239–248.
- [4] R. FRITSCHKE: *Topologies for probabilistic metric spaces*, Fund. Math., **72** (1971), 7–16.
- [5] T. L. HICKS, P.L. SHARMA: *Probabilistic metric structures. Topological classification*, Review of Research Faculty of Science, Univ. of Novi Sad, **14** (1984), 43–50.
- [6] U. HÖHLE: *Probabilistic metrization of generalized topologies*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **25** (1977), 493–498.
- [7] I. M. JAMES: *Introduction to Uniform Spaces*, Cambridge Univ. Press, Cambridge, U. K., (1990).
- [8] B. LAFUERZA-GUILLÉN, J.A. RODRÍGUEZ-LALLENA, C. SEMPI: *A study of boundedness in probabilistic normed spaces*, J. Math. Anal. Appl., **232** (1999), 183–196.
- [9] B. LAFUERZA GUILLÉN, J.A. RODRÍGUEZ LALLENA, C. SEMPI: *Normability of Probabilistic Normed Spaces*, Note di Matematica, Vol. **29**(1), in press, to appear in 2009.
- [10] B. LAFUERZA-GUILLÉN, J. L. RODRÍGUEZ: *Boundedness in generalized Šerstnev PN spaces*, math.PR/0408207.
- [11] B. MORREL, J. NAGATA: *Statistical metric spaces as related to topological spaces*, General Top. Appl. **9** (1978), 233–237.
- [12] D. H. MUSHTARI, A. N. ŠERSTNEV: *On methods of introducing a topology in random metric spaces*, Izv. vyssh. Uch. Zav. Math., **6** **55** (1966), 99–106.
- [13] A. N. ŠERSTNEV: *On the notion of a random normed space*, Dokl. Akad. Nauk SSSR, **149** (2)(1963), 280–283 (English translation in Soviet Math. Dokl., **4** (1963), 338–390).
- [14] B. SCHWEIZER, A. SKLAR: *Probabilistic Metric Spaces*, Dover Publication, Inc. Mineola, New York, (2005).
- [15] R. M. TARDIFF: *Topologies for probabilistic metric spaces*, Pacific J. Math., **65** (1976), 233–251.
- [16] E. THORP: *Generalized topologies for statistical metric spaces*, Fund. Math., **51** (1962), 9–21.
- [17] JIANZHONG XIAO, XINGHUA ZHU: *On linearly topological structure and property of fuzzy normed linear spaces*, Fuzzy Sets and Systems, **125** (2002), 153–161.