UNDERSTANDING INTRODUCTORY STUDENTS' APPLICATION OF INTEGRALS IN PHYSICS FROM MULTIPLE PERSPECTIVES

by

DEHUI HU

B.S., University of Science and Technology of China, 2009

AN ABSTRACT OF A DISSERTATION

Submitted in partial fulfillment of the requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Physics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2013

Abstract

Calculus is used across many physics topics from introductory to upper-division level college courses. The concepts of differentiation and integration are important tools for solving real world problems. Using calculus or any mathematical tool in physics is much more complex than the straightforward application of the equations and algorithms that students often encounter in math classes. Research in physics education has reported students' lack of ability to transfer their calculus knowledge to physics problem solving. In the past, studies often focused on what students fail to do with less focus on their underlying cognition. However, when solving physics problems requiring the use of integration, their reasoning about mathematics and physics concepts has not yet been carefully and systematically studied. Hence the main purpose of this qualitative study is to investigate student thinking in-depth and provide deeper insights into student reasoning in physics problem solving from multiple perspectives.

I propose a conceptual framework by integrating aspects of several theoretical constructs from the literature to help us understand our observations of student work as they solve physics problems that require the use of integration. I combined elements of three important theoretical constructs: mathematical resources or symbolic forms, which are the small pieces of knowledge elements associated with students' use of mathematical ideas; conceptual metaphors, which describe the systematic mapping of knowledge across multiple conceptual domains – typically from concrete source domain to abstract target domain; and conceptual blending, which describes the construction of new learning by integrating knowledge in different mental spaces.

I collected data from group teaching/learning interviews as students solved physics problems requiring setting up integrals. Participants were recruited from a second-semester calculus-based physics course. I conducted qualitative analysis of the videotaped student conversations and their written work. The main contributions of this research include (1) providing evidence for the existence of symbolic forms in students' reasoning about differentials and integrals, (2) identifying conceptual metaphors involved in student reasoning about differentials and integrals, (3) categorizing the different ways in which students integrate their mathematics and physics knowledge in the context of solving physics integration problems, (4)exploring the use of hypothetical debate problems in shifting students' framing of physics problem solving requiring mathematics.

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Acknowledgements

First and foremost, I would like to thank my major advisor Dr. Sanjay Rebello, for his guidance and tremendous support in the past four years. This work would not be possible without his insights and advice. He provided me lots of inspiring conversations to keep me motivated and excited about my research work. He helped me grow and prepared me for my future professional life.

I also want to give a special thanks to Dr. Eleanor Sayre who gave me lots of helpful advice and made me think more thoroughly about my research. I admire her a lot for her unending energy and passion for physics education research.

My other dissertation defense committee members, Dr. Andrew Bennett, Dr. Kay Ann Taylor, Dr. Steve Warren, have given me invaluable feedback on this dissertation work. I am so grateful for having them in my dissertation committee. I want to express my gratitude to Dr. Larry Weaver and Dr. Tim Bolton for mentoring me on my teaching as well as giving me the opportunity to collect data in their classes.

I have been very fortunate to be a member of the KSU PER group. I want to thank everyone in this big family for being so nice to me and giving me confidence all the time. In particular, Dr. Bashirah Ibrahim, Dr. Josh Von Korff, and Dr. Dong-Hai Nguyen provided me so much help on my research. Dr. Adrian Madsen, Amy Rouinfar, and Ying Chen have been so supportive for my life. I cannot imagine what my life would be without you. I also want to thank our administrative assistant Kim Coy for her help. She has done an amazing job!

Finally, I must thank my fiancée Yan Cheng for standing with me and keeping me sane during my graduate life. This dissertation would not exist without your unceasing love and support. My parents, Shunqin Liu and Quanming Hu, have given me consistent support and encouragement. It was their unconditional love that kept me working hard and moving forward.

This research project was supported in part by the National Science Foundation grant 0816207.

Dedication

To my parents.

Chapter 1 - Introduction

1.1 Motivation

This study is concerned with the application of mathematics in physics problem solving. Mathematics is often regarded as the backbone of physics and science in general. Physics classes often involve the intensive use of mathematics and they often require much more than the straightforward application of mathematical operations and algorithms. Mathematics is often used in physics to describe the fundamental relationships between physical quantities. Physics problem solving requiring mathematics could be modeled as the following sequence of steps: mapping a physical situation onto a mathematical model, processing mathematical representations, interpreting mathematical results in physics, and evaluating whether the results adequately describe the physical system (Redish, 2005). A significant portion of students in physics courses often struggle to apply mathematical skills in physics problem solving (Yeatts & R. Hundhausen, 1992; Nguyen & Meltzer, 2003; Shaffer & McDermott, 2005).

This research project is specifically focused on students' use of mathematical integrals in physics problem solving. Mathematical integration is widely used across many physics topics from introductory to upper level physics courses. In introductory mechanics, students may use integration as a tool to find the mass or momentum of inertia of physical objects with non-uniform density. In electrostatics, students need to set up integrals to find the electric field due to a charged bar or disk with non-uniform charge distribution. Developing expertise to use integration in physics is both necessary and important for students to fully understand mathematics and physics concepts as well as transfer their learning to new situations. Setting up integrals in a physics problem can be divided into several steps: setting up the expression for infinitesimal quantity (e.g., dE, dB), accumulating infinitesimal quantity, determining the variable of integration, and turning the integral into a form that can be evaluated mathematically (Nguyen & Rebello, 2011). Hence, using integration in physics requires students to not only know how to evaluate an integral, but also to correctly use differentials and integrals to express relations between physical quantities. For example, to construct the expression from a more fundamental equation from Coulomb's law $E = k\frac{q}{r^2}$, a student needs to develop the understanding of the differential terms (i.e., dq, dE) in the context of an electric field.

Studies have reported that students are typically able to evaluate a pre-determined mathematical integral, but often struggling with setting up integrals in physics problems (Orton, 1983a; Mahir, 2009). Previous studies in physics education research have documented students' deficiencies in the application of mathematical integrals in physics (Cui et. al., 2006; Manogue, Browne, Dray, & Edwards, 2006; McDermott, Rosenquist, & van Zee, 1986; Meredith & Marrongelle, 2008; Pollock, Thompson, & Mountcastle, 2007; Wallace & Chasteen, 2010). These studies investigated students' deficiencies with understanding the mathematical aspect of integration (Orton, 1983a; Orton, 1983b; Artigue, Menigaux, & Viennot, 1990; Mahir, 2009; Sealey, 2006); understanding when and why integration is needed in a physics situation (Meredith & Marrongelle, 2008; Cui, Rebello, Fletcher, & Bennett, 2006; Nguyen & Rebello, 2011); using mathematical symbols and notations in a physical situation (Yeatts & Hundhausen, 1992; Wallace & Chasteen, 2010; Nguyen & Rebello, 2011); and procedural knowledge of setting up integrals (Orton, 1983a; Nguyen & Rebello, 2011).

Studies have reported students' difficulties when applying mathematical integrals in physics; however, student reasoning underlying these difficulties has not yet been carefully studied. We cannot simply attribute students' difficulties to a lack of mathematics or physics knowledge. There is a gap between what mathematics education researchers have found about students' understanding of calculus concepts and what physics education researchers have found about students' poor performance in physics problems. The main purpose of this study is to take one small step toward bridging this gap and to explore students' thinking about mathematical concepts in physics contexts. I use the resources model to discuss the mathematical resources associated with major calculus concepts that students use when they set up integrals. Further, I utilize the analytical tools from cognitive science, such as conceptual metaphors, to analyze the resources that students use in physics contexts. While the resources model and conceptual metaphor theory provide a fine-grain analysis of students' reasoning, the conceptual blending framework offers us a language to discuss the overall picture of how students combine their knowledge in mathematics and physics to set up integrals. In additional to investigating student reasoning, I also explore the use of a non-traditional type of problem -- hypothetical debate problem -- as instructional tool to facilitate students' use of integrals in physics problem solving.

1.2 Brief Introduction to the Theoretical Framework

In the studies described in this dissertation I explored both students' deep reasoning about specific mathematical concepts in physics contexts as well as students' strategies for setting up integrals. I amalgamated elements of three theoretical frameworks -- mathematical resources, conceptual metaphors, and conceptual blending – to construct a more complete picture of student reasoning process when applying integrals in physics problem solving.

Mathematical Resources

Resources are context-sensitive, small-scale knowledge elements that can be applied in many different situations (diSessa & Sherin, 1998; Hammer, 1996a). A resource by itself can neither be deemed correct nor incorrect, but it can be activated appropriately or inappropriately depending upon the context. Physics learning involves the activation of various kinds of resources, such as phenomenological primitives or p-prims that are based on intuitive knowledge about physical world (diSessa, 1993), mathematical resources (Sherin, 2001), as well as epistemological resources (Hammer & Elby, 2002). In some contexts, it may also involve the activation of mathematical resources (Sherin, 2001). When applying mathematics in physics, students bring to bear their intuitive sense about mathematical symbols and notations. Symbolic forms are the kinds of mathematical resources that may be activated by students in conjunction with cognitive processes associated with using and interpreting equations in physics contexts. A symbolic form consists of a symbol template (such as []=[], or []+[]+[]…) and a conceptual schema associated with the template. The conceptual schema is a kind of mathematical primitives, a simple structure corresponding to the understanding of the relations among mathematical entities. As an example, the conceptual schema of *balancing* is associated with the mathematical template $[] = []$. In the physical situation "a book is at rest on a desk", the normal force acting on the book due to the desk is balanced by the gravitational force on the book due to the earth.

Metaphors in Mathematical Thinking

A conceptual metaphor describes a cognitive mechanism which allows the use of sensorimotor experience in the source domain to conceptualize a subjective experience in the target domain (Lakoff & Johnson, 1980). In the phrase "*I am feeling up*," indicates that the speaker is experiencing the emotion of happiness. This emotion is associated with the physical experience of having an upright posture (Lakoff & Johnson, 1999). One important characteristic of conceptual metaphor theory is that we typically conceptualize nonphysical notions such as time, mind, emotion, and ideas in terms of concrete physical terms such as objects, substances, entities, and motions with which we have more experience. According to cognitive linguistics Lakoff and Johnson (1980), our human intellectual system is metaphorically structured and metaphors exist pervasively in our everyday thought and language. Metaphors were first studied in linguistics, but later on evidence for using metaphors was also found in mathematical and scientific language.

Lakoff and Núñez (2000) extended the idea of conceptual metaphors to understand mathematical concepts. They introduced the grounding metaphor as one of the main types of metaphors involved in the conceptualization of basic mathematical ideas, such as arithmetic. The term "grounding" implies that the fundamental mathematical ideas emerge directly from our everyday activities. For example, adding numbers is construed as adding objects to a collection, such as "adding an apple to the basket." This grounding metaphor described the mapping from the domain of physical objects to the domain of numbers. Núñez described conceptual metaphors as "fundamental cognitive mechanisms which project the inferential structure of a source domain onto a target domain, allowing the use of effortless species-specific body-based inference to structure abstract inference" (Núñez, 2000).

Conceptual Blending

Conceptual blending or conceptual integration theory describes how the mind combines two or more mental spaces to make sense of linguistic inputs (Fauconnier & Turner, 1998a). Mental spaces are used as general terms, referring to small interconnected conceptual packets constructed in a local context. Mental spaces are partial assemblies containing elements that are structured by frames and cognitive models. Blending is the process of selectively taking some elements from each mental space to compose a new mental space -- the blended mental space. For example, the phrase "computer virus" is a blend constructed from two input mental spaces–a biological mental space and computer mental space. A biological mental space contains elements like a biological system, cell, virus, and immunity mechanism. A computer mental space includes elements like memory, software, and programs. In the blended space, a computer program and virus are fused into a new, single entity ‘computer virus” - a computer program that can replicate itself and spread from one computer to another, usually making unauthorized and undesirable changes to the computer.

The conceptual blending framework has also been extended and elaborated to understand students’ reasoning and actions in mathematics and physics (Zandieh, Knapp, & Roh, 2008; Wittmann, 2010). Bing and Redish (2006) used the language of conceptual blending to model how students combine physical and mathematical knowledge to construct solutions to physics problems in the context of algebra-based physics. They pointed out that the difficulties students experienced were often not from their lack of prerequisite knowledge but from inappropriate blending of mental spaces.

1.3 Research Questions

In summary, this study has two main purposes. First, I focus on investigating student reasoning and conducting both small-grain and large-grain analysis about how students set up integrals in various physics contexts. Second, I study the extent to which a non-traditional type of problem – the hypothetical debate problem could be potentially used to shift students’ framing of physics problem solving requiring mathematics. Specifically, I address the following research questions:

1. How do students reason about calculus concepts as they set up integrals in physics contexts?
    - What are the mathematical resources that students activate?
    - What are the conceptual metaphors that students use in association with these mathematical resources?
2. How do students blend their knowledge in calculus and physics to set up integrals in physics?
3. How does students’ framing of solving physics problems requiring mathematics differ when dealing with hypothetical debate problems?

1.4 Main Contributions of This Research

This dissertation contributes both to theoretical development and instructional applications in physics education. To answer the first research question “How do students reason about calculus concepts as they set up integrals in physics contexts", I extended Sherin's symbolic forms framework to categorize the mathematical resources that students activated in the context of solving physics integration problems. I also amalgamated two frameworks – symbolic forms from physics education research and conceptual metaphors from cognitive linguistics, to help us make interpretations of student reasoning about mathematical concepts in physics contexts. The first question aims to provide a fine grained analysis of student problem solving and develop an in-depth understanding of student reasoning. The second research question aims at providing us a more complete picture of the overall strategies that students used. To answer this research question, I described the ways in which students combine their knowledge in calculus and physics to set up integrals from the perspective of conceptual blending. Investigating students' problem solving behavior from this perspective also allows us to see the connection between students' small-grained knowledge elements and large-grained problem solving strategies when solving physics integration problems. In addition to exploring and understanding student thinking, I also studied if a nontraditional type of problem (i.e., hypothetical debate problem) could be potentially used to change students' framing of using mathematics in physics problem solving to answer the third research question.

1.5 Overview of Dissertation

Chapter two offers a literature review of previous research on students' understanding of major calculus concepts, specifically integrals and differentials; as well as students' difficulties with applying those concepts to solve physics integration problems. Chapter three is focused on the conceptual frameworks including a review of the relevant frameworks -- resources, conceptual metaphors, and conceptual blending. It also includes a description of how those frameworks are extended and closely integrated in this research. In chapter four I discuss the context of this research study, the student population, and data collection processes used in the study. In chapter five I introduce the resources model and conceptual metaphor framework, and then describe the mathematical resources that students activated about differential concept and the conceptual metaphors associated with those resources implicitly in this study. In chapter six, I provide a review of the conceptual blending framework and then discuss how this framework is used to provide a more complete analysis of student problem solving approaches when dealing with physics integration problems. In chapter seven, I discuss the development of hypothetical debate problems and the pedagogy underlying those problems. Then I compare students' framing of physics problem solving requiring mathematics in conventional problem and hypothetical debate problem. Chapter eight summarizes the main results of the study and talks about the implications of this research on mathematics and physics teaching, as well as the direction for future research.

Chapter 2 - Literature Review on Students' Use of Calculus

2.1 Introduction

Students' performance on calculus and physics problems requiring integrals are two topics that are particularly relevant to my study. In this section, I will present a review of previous studies in both of these two areas. Mathematics education research has well documented students' understanding of calculus concepts, such as differentiation, integration, and Riemann sums. Since all of the students in our calculus-based physics classes have previously taken calculus, previous research in mathematics education provides us the necessary background to understand their prior mathematics knowledge and what I could possibly expect to observe in their mathematical thinking in physics contexts. It also allows us to see what mathematics educators value in students' understanding of these concepts and reflect on which of these aspects physicists might perceive as valuable in students' thinking. The following section reviews the previous studies conducted by mathematics education researchers with regard to students' conceptual and procedural knowledge when solving calculus problems. Then I will also discuss several studies in physics education research on students' ability to apply integrals in physics scenarios from both introductory and upper-division physics courses. Previous findings from both mathematics and physics education research provide insights and motivation for this study.

2.2 Student Understanding of Calculus Concepts

Studies in mathematics education have shown that students are capable of following routine procedures to calculate a derivative, evaluate an integral, or find the area under a curve; however, they may lack conceptual knowledge of those concepts (Orton, 1983a; Mahir, 2009) Orton (1983b) conducted clinical interviews with 110 students to investigate their understanding of elementary calculus. He categorized students' mistakes as structural (conceptual or fundamental), executive (procedural or operational), or arbitrary. Overall, the errors made by students were structural and students found that the execution of a procedure was relatively easy. In an interview study on the topic of rate of change and differentiation, Orton found that the most difficult items were those concerned with understanding differentiation and graphical approaches to rate of change. Students revealed many misunderstandings in explaining the symbols δx, δy, *δy/δx*, *dx*, *dy*, and *dy/dx*. Taking the example of *dx*, three main types of incorrect responses were apparent. Twenty-nine students explained dx as "the differential of *x*," or "the rate of change of *x*." A further 25 students explained *dx* as "the limit of *δx* as *δx*→0." Another 20 students thought that *dx* was an "amount of *x*" or "*x*-increment," in other words it was more or less the same as *δx*. It is clear that students did not understand the symbols of differentiation or the approach to differentiation. Orton pointed out that "some students are introduced to differentiation as a rule to be applied without much attempt to reveal the reasons for and justifications of the procedure."

In another interview study about the integration concept, Orton (1983a) found that very few students understood integration as the limit of a Riemann sum. Most students had little idea of the procedure of applying integration, which involves dissecting an area or volume into narrow sections, summing the areas or volumes of the sections, and obtaining an exact answer for the area or volume by narrowing the sections. Without exploring the structure of Riemann sums, students encountered a lot of trouble in setting up integrals in a real world situation.

Artigue (1990) administered questionnaires to both physics and mathematics students to investigate their conceptions about differentials (e.g., *dx, dl*). In particular, she discussed two extreme tendencies in students' responses. First, the differential element lost all meaning except that it indicated the variable of integration which may have excluded other meanings. Second, the differential element had material content. For instance, "*dl* is a small length" or "a little bit of wire." Artigue argued that the terms such as "small" or "little bit" were associated with "looseness in reasoning" indicating students' lack of rigorous understanding of the mathematical concept, even though this reasoning is convenient and effective in physics.

Mahir (2009) reported on a study that students' conceptual and procedural performance on five calculus problems after a one-year calculus course. Sixty-two students completed interview tasks that covered the major calculus concepts, including integrals, the integral – area relation, integral as a sum of areas, and the fundamental theorem of calculus. Mahir found that almost all the students did very well on the first two problems which only involved the use of integral algorithm and techniques, or the procedural knowledge. When a problem could be solved by either a procedural or a conceptual approach, the majority of students tended to apply a procedural approach. Students who followed a conceptual approach were much more likely to solve the problems correctly (correctness is more than 70%) than those who used a procedural approach (correctness is less than 20%). Mahir concluded that the conceptual performance of the participants on calculus problems was not satisfactory and he suggested that conceptually based instruction might help improve students' conceptual understanding of calculus concepts.

Ferrini-Mundy and Graham (Ferrini-Mundy & Graham, 1994) conducted interviews with six calculus students to probe their understanding of basic concepts of calculus - function, limit, continuity, derivative, and integral. Their paper provided a detailed discussion about the interview with one student and common reasoning patterns revealed by most students. They concluded that most students perceived integral as a signal to "do something" or perceived definite integral as "the area between the graph of the function and the x-axis" and "the sum of the areas of the small rectangles under the curve."

More recently, Bennett et al. (2011) investigated how engineering students develop their understanding of function and integration concepts as they progress through their calculus courses. They used the APOS framework introduced by Dubinsky (2001) as a scale to measure students' mathematical conceptual understanding. They found that about 25-30% of the students in Calculus I or II were at the Action level which means this group of students could only follow algorithms to arrive at an answer but did not know what the procedure meant; about 35% of students were at the Action/Process level which means students could follow prescribed steps and show some understanding. For instance, they might recognize some mistakes and contradictions, but they might not be able to explain what went wrong.

Mathematical researchers also probed students' performance when applying the integral concept in real world situations. Sealey (2006) conducted a teaching experiment in a calculus class to investigate student understanding of definite integrals and Riemann sums. The participants were from a traditional first semester college calculus class. She reported on the work of two groups when they solved two problems requiring the use of integrals in real world contexts. One problem asked for the total force exerted by the water on a dam and the other one involved the compression of a spring and the calculation of spring potential energy. In the first problem, students recognized the use of integral but had difficulty constructing the appropriate integral. Then they used approximation method based on the structure of Riemann sums and were eventually able to set up the correct integral. In the second problem, students attempted to use area under the curve concept, however, they could not explain why the area under the curve was equal to energy and they were not confident about the correctness of the force graph. She concluded that understanding the structure of Riemann sums was necessary for students to construct the correct integral in a real world context. However, students might not be able to relate area under a curve to the structure of a Riemann sum when they used the area under a curve as a tool to evaluate a definite integral.

Thompson & Silverman (2008) specifically addressed the structure of a Riemann sum. A Riemann sum can be viewed as a process of accumulating bits of a quantity and each bit is formed multiplicatively. For example, to understand the concept of work done as accruing incrementally means that one must think of each momentary total amount of work as the sum of past increments, and of every additional incremental bit of work as being composed of a product of force applied through a displacement. They illustrated students' difficulties in understanding accumulation mathematically and they emphasized the importance of giving explicit attention in accumulation function when teaching definite integrals to students.

In summary, previous studies from mathematics education research reported that many students did not acquire a complete conceptual understanding of the integral concept or the structure of a Riemann sum after they took calculus class. They tend to focus on algebraic procedures when solving calculus problems and their understanding of integrals was bereft of contextual meanings. These findings of student understanding of calculus concepts can have important implications for their application of those concepts in a physics context.

2.2 Student Application of Calculus Concepts in Physics

In mathematics, students often learn algorithms to evaluate integrals without necessarily referring to any concrete physical situations. Some of students’ conceptual difficulties with physics integration problems could be attributed to their insufficient understanding of mathematics concepts, such as their understanding of integral as a sum and the area-integral relation (Pollock, Thompson, & Mountcastle, 2007; Wallace & Chasteen, 2010; Bajracharya, Wemyss, & Thompson, 2010; Nguyen & Rebello, 2011). A significant number of students have difficulties with setting up a physics problem rather than with the calculus aspect of the problem. In other words, their difficulty is related to the physics context, and not the procedural aspect of doing the calculus in the problem. (Yeatts & Hundhausen, 1992; Cui et al., 2006; Nguyen & Rebello, 2011;).

Based on their teaching experience on an integrated calculus and physics course, Yeatts and Hundhausen (1992) described students' difficulties in three categories when transferring from calculus to physics. The first category is "notation and symbolism", which means students have strong reliance on the symbols used in each context. The second category -"distraction factor", happens when the surface feature of the problem hinders the underlying mathematical process. The third category is "compartmentalization of knowledge", occurs when students stored knowledge of different disciplines in different "cabinets" and students activate knowledge in each "cabinet" only in the corresponding discipline.

Cui et al. (2006) conducted semi-structured think aloud individual interviews with introductory level calculus-based physics students to investigate their retention and transfer from calculus to physics. In the first interview, students were given a physics problem and an isomorphic calculus problem requiring the same calculus concept. All eight participants were confident about their calculus knowledge, however, none of them were confident about their physics solution and unclear about when calculus is applicable in a given physics problem. Most participants did not see the connection between the physics problem and isomorphic calculus problem. In another interview with a different cohort of students, students were given three different physical situations each consisted of a pair of problems. They found that seven out of the eight interviewees appropriately applied integration but four of them recalled similar in-class examples and the other three only had a rough idea of an integral as a sum of infinitesimal elements. Most students responded that they had difficulties with determining the variable of integration and deciding the limits of integration while applying integrations in physics. Six out of the eight interviewees perceived the use of calculus in physics as 'plug and chug" without the need of understanding.

Then Meredith and Marrongelle (2008) specifically investigated when students recognized the use of integrals in electrostatics problems from the resources perspective. Sherin's (2001) symbolic forms were used as a framework to describe the resources activated by students. A symbolic form is defined as a cognitive mathematical primitive which allows students to "associate a simple conceptual schema with an arrangement of symbols in an equation." The 'dependence' symbolic form is the most common cue used by students and it is identified when students decide to integrate because a quantity changes depends on another quantity, for instance, non-constant density. The 'parts-of-a-whole' cue is identified when students recognize they need to sum up the little pieces of a physical quantity to get the total quantity. The 'parts-of-a-whole' symbolic form is more powerful and flexible in many physics contexts than the 'dependence' symbolic form. Dependence symbolic forms can cue integration in novel situations and allow students to invent equations, however, Meredith and Marrongelle (2008) pointed out that "the use of dependence symbolic form led to inaccuracies if the quantity being integrated was not a rate or a density". Also, students typically do not pay attention to units when they choose variable of integration; instead they pay attention to what is changing.

Recognizing that a problem needs integration is the first important step for setting up integrals in physics contexts. Extending the previous work, Nguyen and Rebello (2011) conducted individual teaching/learning interviews to investigate students' common difficulties when setting up integrals in electricity problems more systematically. They divided the procedure of applying integral concept in physics into four steps: recognizing the need for an integral; setting up the expression for the infinitesimal quantity; accumulating the infinitesimal quantity; and computing the integral. They found that many students did not understand the meaning of differential terms such as dx, dr in physics equations, which were consistent with the findings from Orton (1983b). When setting up the expression for the infinitesimal physical quantities, many students simply ignored the differential term or to appended it to the integrand, or even prefixed d to whatever quantity was changing. When accumulating the infinitesimal quantity, almost all students did not pay attention to how these quantities should be added up. For example, when integrating the differential term $d\vec{E}$, they did not consider that fact that electric field $d\vec{E}$ is a vector and it should be added up by components. They also encountered a number of difficulties in computing the integrals. Some could be attributed primarily to students' misunderstanding of the physical meaning of symbols in the integrals. A few students still had difficulties in determining the limits of integrals. It also supported the findings about students' difficulties with setting up integrals from the study of Cui et al. (2006).

There are research studies that have focused on middle- and upper-division physics students' performance on the application of integrals in physics. Wallace and Chasteen (2010) reported students' performance on a conceptual test they had developed titled Colorado Upper-Division Electrostatics assessment (CUE). They discussed the results of one problem in the context of Ampere's law which involves finding the magnetic field of an infinite non-magnetizable cylinder with a uniform volume current. They found that students from traditionally taught courses had an average score of 28% on this problem and the reformed classes scored from 38% to 69% on average. In order to further determine student difficulties, they conducted individual interviews with six students working on an Ampere's law problems. They identified several difficulties students have and one of students' difficulties was due to students not viewing the integral in Ampère's law as representing a sum, which aligned with the work of Manogue et al. (2006) on the same topic.

Wilcox et al. (2012) analyzed students' responses to traditional exam questions that involved finding the electric potential due to a continuous charge distribution in a junior level Electricity and Magnetism (E&M) course. Students' common challenges included determining expressions for the differential charge element (*dq*) and finding the limits of integration for a specific charge distribution. When setting up the integral, nearly half of the students had difficulty expressing the differential charge element.

2.4 Summary

In summary, a review of the literature regarding students' understanding and application of integral concept shows that students' deficiencies when applying integral concept in physics can be categorized in the following aspects:

(1) *Understanding of integral concept in mathematics aspect*: Students do not view an integral as the limit of a Riemann sum. They often think the limit is an approximation, not an exact answer. Students' lack understanding of an integral as a Riemann sum (Orton, 1983a; Orton, 1983b; Artigue et al., 1990; Mahir, 2009; Sealey, 2006).

(2) *Understanding why integration is needed in a physics situation*: A majority of students did not experience difficulties with recognizing whether integration is required in a situation. However, they might not understand *why* integration is needed. Students typically draw on familiar examples or look at surface features of the problem, such as whether a quantity is changing (e.g., non-constant density) rather than thinking of whether the problem requires

adding up small pieces of quantity to get the total quantity, when deciding whether integration is need or not in a physics context (Meredith & Marrongelle, 2008; Cui et al., 2006; Nguyen & Rebello, 2011b).

(3) *Use of mathematical symbols and notations in a physical situation*: Students experience tremendous difficulties with correctly use mathematical symbols to express relations of physical quantities or to interpret the meaning of mathematical equations in a physics context. For instance, a common difficulty students have is how to correctly use the differential terms such as *dr*, *dm*, *dq* in physical scenarios (Yeatts & Hundhausen, 1992; Pollock et al., 2007;D. Nguyen & Rebello, 2011a; Wilcox et al., 2012).

(4) *Knowing how integration should be used in physics*: Students seem to have deficiencies related to the procedural knowledge of setting up an integral. When setting up integrals, they need to think of how the physical object can be chopped into pieces, find the quantity/effect due to one small piece, and then add up all the effects/quantities. Students' lack of procedural knowledge of setting up integrals has been investigated in the past (Orton, 1983a; Nguyen & Rebello, 2011a).

This review of literature provides us a good knowledge of what students are capable or incapable of when applying integration in physics. However, students' underlying thought process or mechanism of their thinking is still not clear. To take this step further, we bring in several frameworks to help us understand students' reasoning and explain observed student difficulties. In the next chapter we will review in detail each of those frameworks.

Chapter 3 - Conceptual Framework

3.1 Introduction

The main purpose of this study is to extend existing theoretical frameworks to interpret our observation of student performance in the context of solving physics problems that require the use of integration. In order to gain deep insights into students' reasoning, I amalgamated aspects of two theoretical perspectives: the resources framework from physics education research and conceptual metaphor framework from cognitive linguistics. There is very little research on how these two frameworks can be integrated to make sense of student reasoning. I will review each framework and provide a discussion about how they can be tied together to explain student thinking in our context of research. In addition to a fine grained analysis of student reasoning, I also adapted the conceptual blending framework from cognitive science to make sense of the ways in which students combined their knowledge from calculus and physics to set up integrals. I investigated the effect of hypothetical debate problems from the perspective of students' epistemological framing about using mathematics in physics. Hence, this chapter will review three pieces of framework: resources, conceptual metaphor, and conceptual blending.

3.2 Resources Framework

General Description of Resources

In physics education research, there are two main approaches for modeling and characterizing student thinking: the unitary, misconception-based approach (Chi & Taylor, 2005) and the manifold or knowledge-in-pieces approach (DiSessa, 1983). The central idea of the unitary framework is that students possess robust cognitive structures or misconceptions that need to be dismantled in order to build the correct conception (Clement, 1983; Chi & Taylor, 2005). The manifold or knowledge-in-pieces framework is based on the assumption that one's conception is determined by activation of various resources depending on the context (Hammer, 1996a). The knowledge-in-pieces approach has been proven useful for modeling student reasoning and learning in a growing body of evidence about the flexibility and variability in student reasoning (Hammer, 2000; Hammer, Elby, Scherr, & Redish, 2004; Elby & Hammer, 2010).

Consistent with the knowledge-in-pieces perspective, cognitive resources are fine-grained knowledge elements that one brings into a situation, and the activation of one resource often depends on context (DiSessa & Sherin, 1998; Hammer, 2000). Typically, I think of resources as chunks of knowledge elements that can be applied productively in many situations. Resources might be unproductive for learning physics if used inappropriately in a situation. Resources are often associated to each other as a network and the activation of one resource often leads to the activation of other resources or clusters of resources. When the same locally coherent set of resources becomes activated again and again, it may eventually be strongly connected and established to be activated as a unit. Learning physics largely entails the activation of existing resources, including conceptual (Hammer, 2000; Wittmann, 2006), procedural (Black & Wittmann, 2009), epistemological (Hammer & Elby, 2002), and other types of resources.

When making sense of physical phenomena and processes, students use a form of intuitive knowledge through their interaction with the physical world, such as walking, pushing, pulling, and throwing objects. This intuitive sense of knowledge is often referred to as "sense-of-mechanism" or phenomenological primitives (p-prims) according to diSessa (1993). diSessa identified several clusters of p-prims used by students when explaining the motion of objects in mechanics, such as Ohm's p-prims, force as mover, and dying away. This intuitive sense of knowledge could be applied directly to physics learning; however, if misapplied, it could cause conflicts and barriers for students' construction of formal physics knowledge. Epistemological resources control how a student perceives the nature of learning under a situation and what conceptual resources are activated. For example, when learning physics many students appear to view physics knowledge as coming from authority.

A summary of the importance characteristics of the resources framework is mainly based on previous work by Sayre & Wittmann (2008) and Bing & Redish (2009).

1. *Resources are associative networks*. Just like neurons in our brain are structured as a network, resources are associated and not isolated units (Redish, 2004). The activation of one might lead to the activation of another related resource or cluster of resources. Learning can be considered as a process of constructing connections between neurons. Knowledge in long-term memory is well structured as patterns, which are sometimes referred to as knowledge structures. An associational pattern of resources is also described as a coordination class which involves a

readout strategy and a causal net (DiSessa & Sherin, 1998). Wittmann (2006) also created the representation of resources graph to express the network of linked resources. When the same locally coherent set of resources becomes activated again and again, it will eventually be strongly connected and established to be activated as a unit.

2. *Resources can be active or inactive*. A resource might be activated or inhibited in a situation depending on the interaction of the individual with the local environment. In a physics class, a student might activate the resource of "knowledge as memorized stuff" or "knowledge as fabricated stuff" based on his or her own perceptions about learning physics, the way the teacher teachers, how other students behave, as well as other social and physical factors (Redish, 2004). Whether a resource is activated or not depends on the individual's real-time perception or judgment (i.e., framing) about the situation (Hammer et al., 2004). Framing is also called the control structure, determining which resources are selected for activation and which ones are inhibited (Bing & Redish, 2009).

3. *Resources may or may not have internal structure*. Resources are often used as a compiled knowledge element, whose internal structure often seems not explorable to the user. In physics education research, a significant amount of research (diSessa, 1983; Minstrell, 1992; Hammer, 1996b; Sherin, 2006) has primarily focused on the role of intuitive sense of knowledge played in physics learning, such as primitive resources or p-prims (diSessa & Sherin, 1998), meaning that they are used unitarily without the user being concerned about their internal structure. When a cluster of resources are activated together frequently, they might become more tightly related and more likely to be activated together spontaneously by the user in a particular context. This creates "bigger" or "larger-scale" resources with higher level structures, such as a schema or a concept (e.g., force) (Sayre & Wittmann, 2008).

4. *Resources do not have inherent correctness or incorrectness*. Resources are abstracted from our everyday experience and exposure to the physical world. Hence, resources that are appropriate in one situation may not be applicable in another situation. For example, *closer* is *stronger* is one of the most commonly used resources in students' learning physics (Hammer, 2000). This resource may be extracted from a variety of real world experiences: when you are *closer* to a fire in winter you feel *warmer*; the *closer* you are to a light bulb, the *brighter* you can feel. The resource '*closer* is *stronger*' may be applied in many different contexts and it would be

meaningless to say whether this resource is right or wrong without considering the specific context in which the resource is applied. Take a very familiar example in physics: when asked to explain why it is hotter in the summer than that in winter, some students might say because the earth is closer to the sun in the summer. Consider a different situation: when discussing the electric field due to a point charge, students might say that the electric field is *weaker* as it gets *further* away from the point charge. This resource is appropriate in explaining the electric field, however, it is inappropriate in explaining the change of the season. Hence, resources, by nature, are not correct or incorrect; they can only be productively or unproductively used in a situation.

Mathematical Resources

Some resources are abstracted from our observation of everyday phenomenon or interaction with the physical world, such as p-prims (DiSessa & Sherin, 1998). Other types of resources, such as mathematical resources, also play important roles in students' physics learning. Intuitive mathematics knowledge is a kind of mathematical resources formed in the very early stage of a person's life, even before they received any formal mathematical instruction (Fuson, 1992). Those resources include counting (i.e., enumerating a series of objects), subitizing (i.e., distinguishing between sets of one, two, and three objects), pairing (i.e., grouping two objects for collective consideration), and ordering (i.e., ranking relative magnitudes of mathematical objects) (Lakoff & Núñez, 2000). Intuitive mathematical knowledge is useful and involved in both low-level and high-level mathematical learning and thinking, as well as in physics learning. While intuitive mathematical resources are typically formed in young children, even infants, more complex mathematical resources are often developed through receiving years of formal mathematics practices.

Through exposure to formal mathematical expressions in mathematics and other domains requiring application of mathematics, students not only learn the 'rules' of how to perform calculations, but also develop a sense of mathematical notations and equations (Sherin, 2001). In physics, mathematical equations are used to represent relations in a physical system. I also want students to obtain an understanding of what the equation means beyond the symbolic manipulations. Sherin (1996, 2001) provided evidence that successful physics students learn to express simple ideas in equations and to read these same ideas from those equations.

Symbolic Forms

Symbolic Forms in Algebra-based Physics

Sherin (2001) introduced symbolic forms (i.e., cognitive mathematical primitives) to understand student interpretation of mathematical equations in physics problem solving. Symbolic forms allow students to associate meanings with certain structures of mathematical expressions. A symbolic form has two components: a symbolic template (e.g., []=[], $\frac{[]}{[]}$) and a conceptual schema. The conceptual schema is "a simple structure associated with the symbolic form that offers a conceptualization of the knowledge contained in the mathematical expression" (Sherin, 2001). Sherin observed that students not only applied known equations or given principles, but also invented their own equations from intuition. When observing five pairs of intermediate level engineering students solve physics problems, Sherin discovered the existence of symbolic forms based on that fact that students "learn to associate meanings with certain structures in equations." One problem was based on the physical scenario that a person gives a block a shove so that the block slides across a table and then comes to rest. Students were asked to talk about the forces acting on the block and discuss what would happen if the block was heavier. A pair of students invented an equation $\mu = \mu_1 + C\frac{\mu_2}{m}$ for the coefficient of friction based on their understanding of the situation "the coefficient of friction has two components: one that's a constant and one that varies inversely as the weight." This expression was not from the textbook; instead they constructed this equation from their understanding of the physical scenario. One symbolic form identified by Sherin from this expression was "parts-of-a-whole." The symbol template that two or more terms are separated by plus (+) signs contains a conceptual schema "a whole is composed of two or more parts." Sherin's symbolic forms were primarily based on student reasoning of algebraic equations.

Symbolic Forms in Calculus-based Physics

Meredith & Marrongelle (2008) investigated students' use of mathematical resources regarding when they recognize the use of integrals in electrostatics contexts. The 'dependence' symbolic form is the most common cue used by students and it is identified when students decide to integrate because a quantity changes depends on another quantity, for instance, non-constant density, associated with the symbolic template [..x…]. The ‘parts-of-a-whole’ symbolic form, contained in the template []+[]+[]+…, is identified when students recognize they need to sum up the little pieces of a physical quantity to get the total quantity. Meredith & Marrongelle pointed out that ‘parts-of-a-whole’ symbolic form is more powerful and flexible in many physics contexts than the dependence symbolic form.

Jones (2010, 2013) extended the symbolic forms framework to analyze student understanding of the integral concept. He interviewed nine students who were enrolled in an introductory level physics course which was designed primarily for students in physics and engineering. All students were interviewed twice in their study. In the first interview, students were given open-ended mathematics problems related to integrals; in the second interview, students were given open-ended physics integration problems involving real-world objects. He identified four major symbolic forms associated with the integral symbol template “$\int_{[]}^{[]} []\, d[]$.” The *area* symbolic form basically takes the integral expression and interprets it as an area in the x-y plane. The *adding up pieces* symbolic form refers to the evidence that students sliced the area on a graph into infinitely small pieces and added those pieces to find the total amount. However, Jones argued that under this symbolic form, the limiting process occurred before the addition process took place. Thus, this symbolic form diverged from the Riemann sum process. The *function mapping* symbolic form conceives the integral as a “pairing of objects,” which matches the integrand with an “original function.” According to the problematic *add up then multiply* or *adding up integrand* symbolic form, the first box inside the integral “$\int_{[]}^{[]} []\, d[]$” was added up over the infinitesimally small pieces and the resultant summation was then multiplied by the quantity represented by the differential. Among the four major symbolic forms, Jones concluded that the *adding up pieces* symbolic form is a more productive way to view integral in both mathematics and physics contexts.

Other Mathematical Resources

In addition to symbolic forms, several pieces of work examined students’ use and development of mathematical resources in physics contexts requiring mathematics from different subfields. Sayre & Wittmann (2008) described students’ creation of ‘*coordinate system*’ resources when choosing various coordinate systems in physics mechanics problems. Black and

Wittmann (Black & Wittmann, 2009) identified a list of procedural mathematical resources from student work during the formation of the '*Separate Variables*' procedural resource in the context of solving differential equations in an intermediate mechanics problem. Unlike p-prims, these resources are less primitive, usually with internal structures, developed later in their learning.

3.3 Conceptual Metaphor Framework

Metaphors in Everyday Language

Lakoff and Johnson (1980) introduced the theory of conceptual metaphor when studying the linguistic phenomenon of human beings. The conceptual metaphor is a cognitive mechanism which allows the use of sensorimotor experience (e.g., George Lakoff & Johnson, 1980). Conceptual metaphor is a cognitive mechanism, which allows the use of sensorimotor experience (e.g., grasping an object) to conceptualize a subjective experience (e.g., an emotion). This theory is based on the assumption that our ordinary conceptual system is metaphorically structured and the evidence that metaphors are pervasive in our life, mostly reflected in the language we use (Lakoff & Johnson, 1999). Conceptual metaphors describe the systematic mapping across multiple conceptual domains – concrete source domains and abstract target domains. The source domains are the mental schemata derived directly from sensorimotor experience, whereas target domains are domains formed by subjective or non-sensorimotor experience.

Similar to resources, metaphors also include primary metaphors and more complex metaphor. I provide a brief summary of both types of metaphors:

1. *Primary Metaphors* are formed through conflation in the early stage of our life (Grady, 1997). In young children, their sensorimotor experience and subjective experiences are often automatically conflated together. When these two domains of experiences are activated at the same time, they do not separate out these two domains. After a period of differentiation, they are able to distinguish the domains, but the associations still persist. Primary metaphors often arise naturally and unconsciously in our thought and language without the need of exploring its internal structure. Primarily metaphors are often reflected in our everyday language, such as "*a big problem*," "*a warm welcome*."

2. *Complex Metaphors* are made up of primary metaphors. Two or more primary metaphors can be brought together to form larger complex metaphors. Location event-structure

metaphor is one of the basic complex metaphors identified by Lakoff and Johnson (1999) for the purpose of understanding the internal structure of events and causes. "Event" is a general term referring to anything that exists in space and time. The location event-structure metaphor is complex mapping with a number of sub-mappings, such as "states are locations" and "changes are movements." In metaphorical cases like "*He is in trouble*." or "*He was brought out of depression*.", experiencing certain states "trouble" or "depression" are associated with locations and the change of states are correlated with movements "into" or "out of" a location. "States are locations" and "changes are movements" are within the vast collections of primary metaphor (Grady, 1997; Lakoff & Johnson, 1999).

Metaphors in Mathematical and Physics Thinking

Conceptual metaphors have also been used to characterize ideas in mathematical and scientific reasoning. Núñez (2000) brings the notion of embodiment cognition into mathematical thinking and emphasizes the role our body and brain played in the construction of mathematical notions. He described conceptual metaphors as "fundamental cognitive mechanisms which project the inferential structure of a source domain onto a target domain, allowing the use of effortless species-specific body-based inference to structure abstract inference" (Núñez, 2000).

Lakoff and Núñez (2000) described two main types of metaphorical mathematical ideas: grounding metaphors (i.e., "ground our understanding of mathematical ideas in terms of everyday experience") which generate basic arithmetic notions and linking metaphors which generate more sophisticated ideas linking arithmetic to other branches of mathematics. A simple example of grounding metaphor can be found in arithmetic. The notion of addition and subtraction is conceptualized as adding objects to or taking away objects from a collection. Mathematicians also describe this type of metaphor as a grounding metaphor, meaning. Another example of a grounding metaphor is the mathematical notion of "set" is grounded in the notion of a "container" that comes from outside mathematics (Bazzini, 2001). Regarding the concept of function, one of the grounding metaphors is the "function machine" metaphor. In the "function machine" metaphor, a function is regarded as a machine which performs an algorithm or calculates the input-output relationship (Lakoff & Núñez, 2000; Mi-Kyung & Oh Nam, 2004).

Amin et al. (2012) provided an analysis of the lay and scientific usage of the term "energy" from the conceptual metaphor perspective. In phrases involving constructing human activities, such as "full of energy" or "drained of energy", energy is understood as a possession contained in a person. In scientific use, energy transformation is often revealed in phrases such as "in some form" or energy "going back and forth between" different forms. The conceptualization of the energy concept is structured as a location event-structure metaphor, including sub-mappings of forms of energy as locations and energy transformation as movements. The use of conceptual metaphors in scientific language has been described for a number of other topics including thermodynamics (Jeppsson, Haglund, Amin, & Strömdahl, 2012) and quantum mechanics (Brookes & Etkina, 2007).

3.4 Conceptual Blending Framework

Conceptual blending (also called mental space integration) describes how the mind combines two or more mental spaces to make sense of linguistic inputs (Fauconnier & Turner, 1998a). Mental spaces are small interconnected conceptual packets or knowledge elements that tend to be activated together (Fauconnier & Turner, 1998b). According to this framework, one creates new meanings from the combination of different mental spaces that share content or structure. Blending, as a general cognitive process, brings two or more spaces together through selective projection, taking some information from each input to compose a blend. The new space is called the blended space, which inherits partial structure from input spaces but also has its own emergent structure (Wittmann, 2010). The way a person blends several input mental spaces together depends strongly on the cues and contexts provided. Constructing a blend involves three operations: composition, completion, and elaboration. I use an example "computer virus" to illustrate the three operations.

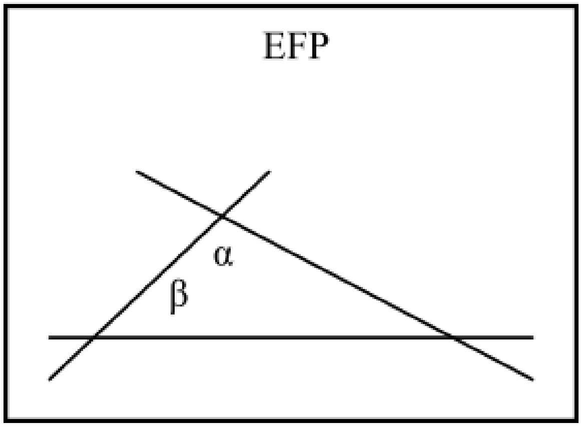
Consider the phrase "computer virus" which we use frequently in our vocabulary nowadays. To understand this phrase, one has to blend knowledge from two different mental spaces – a biological mental space and computer mental space. A biological mental space contains elements like a biological system, cell, virus, and immunity mechanism. A computer mental space includes elements like memory, software, and programs.

Composition: In the blended space, a computer program and virus are fused into a new, single entity 'computer virus' through the process of "composition," meaning that the 'computer virus' is composed of elements from two input spaces. The composed structure often provides relations that are not available in the input spaces.

Completion: When completing the blend, crucial elements from each input (e.g., viruses attack normal body cells, software gives instructions to a computer) are mapped to the third space to form the blend. Meanwhile, an organizing frame (i.e., the property of viruses - reproduction and attacking the normal system) is adopted to organize the knowledge elements in the blended space (Coulson & Oakley, 2000).

Elaboration: When the blend is complete, it can be elaborated to make inferences according to the rules in the blend. The process of elaboration is also called "running the blend." By elaborating on the knowledge in the 'computer virus' blended space, a new meaning emerges - a computer virus as a computer program that can replicate itself and spread from one computer to another, usually making unauthorized and undesirable changes to the computer. Only in the blended space, does the phrase 'computer virus' make sense. In general, the process of composition, completion and elaboration leads to a new emergent structure not pre-existing in the input spaces.

The conceptual blending framework has also been extended to analyze students' mathematical and scientific reasoning. Zandieh et al. (2008) applied the theory of conceptual blending to illustrate how university students construct mathematical proofs. Students were asked to prove the equivalence of two forms of parallel postulate of Euclidean geometry (Figure 3.1& Figure 3.2). Euclid's Fifth Postulate (EFP) states that "if a straight line crossing two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on the side on which are the angles less than the two right angles." Playfair's Parallel Postulate (PPP) states that "for every line and every point not on the line there is a unique line through the point that does not intersect the original line." They explored the ways in which students combined ideas from statements of the postulates and given pictures representing two conditionals. They identified two proving frames: the "Simple Proving Frame" (SPF) and "Conditional Implies Conditional Frame" (CICF). In their initial proving, students tended to apply SPF but it often hampered their efforts due to a lack of necessary theorem to complete the proofs. CICF was recruited by students in later discussion and it was a more powerful and promising frame when constructing proofs.



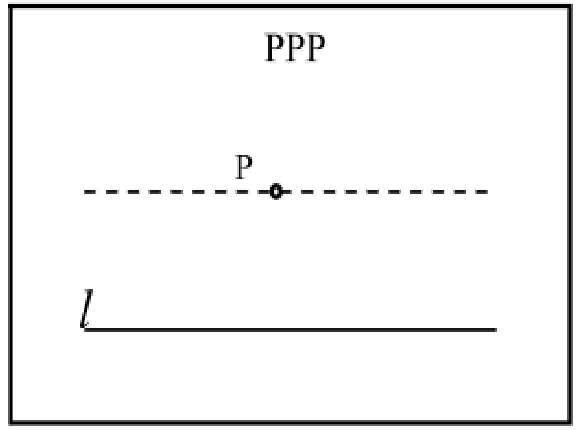


Figure 3.1 Euclid's Fifth Postulate

Figure 3.2 Playfair's Parallel Postulate

Bing and Redish (2006) used the conceptual blending theory to model how students combine physical and mathematical knowledge to construct solutions to physics problems. They described two representative ways in which students blend their knowledge from physics and mathematics mental spaces. A single-scope blend only involves unidirectional mapping of the elements from one input space into the organizing frame of the other while a double-scope blend involves an integration of the organizing frames from input spaces. In the context of using mathematics in physics problem solving, a single-scope blend often refers to a one-way mapping of physical quantities to an existing mathematical equation or template. In a double-scope blend, students not only map an existing mathematical equation to a physics context, but also translate a physical scenario to a mathematical expression. They pointed out that the difficulties students experienced were often not from their lack of prerequisite knowledge but from inappropriate blending of mental spaces.

More recently, Wittmann (2010) used the conceptual blending framework to describe emergent meanings in students' understanding of wave propagation. He analyzed gestural, perceptual, and verbal information to describe how different elements were combined to create new, emergent meaning. The "wave-ball" blend (Figure 3.3) is formed by selective projection from two different mental spaces: observed wave pulse and imagined ball thrown in the air. Then he compared this blend with a "beaded-strong" blend (Figure 3.4) containing the input spaces of the observed spring and falling dominoes. The blend that students make is determined by their selective attention to parts of the observed physical system as they determine which part of the system to use in the process of projection from the input spaces to the blended space

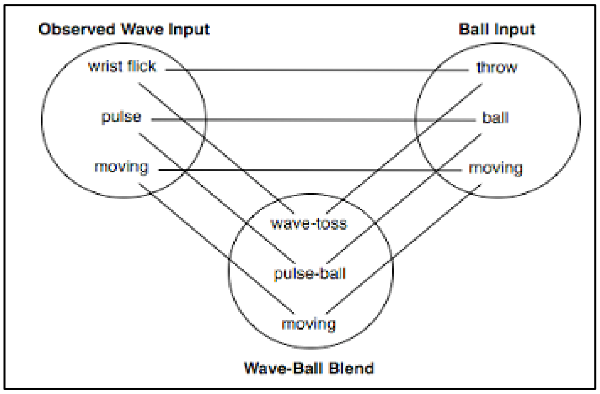


Figure 3.3 Wave-Ball Blend from Wittmann (2010)

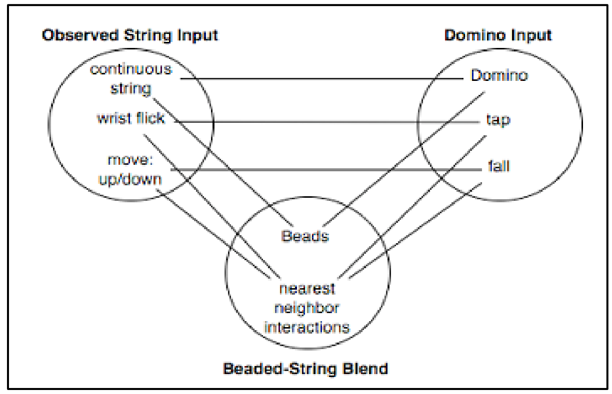


Figure 3.4 Beaded-String Blend from Wittmann (2010)

3.5 Summary

In this chapter I reviewed three pieces of frameworks – resources, conceptual metaphors, and conceptual blending. I discussed how those frameworks have been used in previous research, in particular in mathematics and physics education research. Each framework offers a unique perspective for analyzing student reasoning. I combine three frameworks to analyze the observed student work: exploring mathematical resources to get a sense of the kinds of intuitive knowledge that students bring in their formal mathematical thinking in physics; investigating conceptual metaphors the involved in their reasoning to gain more information about how those resources are grounded from their interaction with the physical world; describing how students blend their knowledge in mathematics and physics to understand the overall strategies that students employ in setting up integrals. In the following chapter I will discuss the settings of this study and our data collection process.

Chapter 4 - Research Methodology

4.1 Introduction

The goal of this research is to examine in detail student reasoning about mathematics and physics concepts as well as to understand how students set up integrals in various physics contexts requiring the use of mathematical integrals. In order to deeply investigate student reasoning, I conducted group teaching/learning interviews (Engelhardt, Corpuz, Ozimek, & Rebello, 2003) and completed a qualitative analysis of the student work. I interviewed 13 students who were organized into five groups. All of the students were enrolled in an introductory calculus-based physics course which primarily focused on the topics of electromagnetism. Each group met with the interviewer eight times during the semester. By interviewing a small number of students over several sessions, I was able to examine in detail students' reasoning with setting up integrals in depth. This chapter explains the research methodology employed in this study, including the context of this study, data collection process, and data sources.

4.2 Data Collection

Settings of This Study

This study was conducted at a Midwestern land grant university. The participants were concurrently enrolled in 'Engineering Physics II'. This course was the second semester of a yearlong sequence of introductory physics courses with a large enrollment of about 250 students. The first semester course -- 'Engineering Physics I', covers the topics of mechanics, waves, fluids, and thermodynamics. Engineering Physics II covers the topics of electricity, magnetism, circuits and optics. About 90% of the students in Engineering Physics II were engineering majors, and the rest of them majored in mathematics, physics and other sciences. The pre-requisite math course is calculus of single variables. This class was taught by a faculty member who has rich experience in teaching Engineering Physics courses as well as other introductory level physics courses. The instructional format is two regular lectures with 50 minutes each and two integrated laboratory problem solving sessions with 110 minutes each. The class uses the modified Studio format that has been used here for over a decade. It was first developed by an award winning faculty member (Sorensen, Churukian, Maleki, & Zollman, 2006) but since then it has been taught by several other faculty members. The lecture follows a traditional format with clicker questions and was taught by the professor. The Studio sessions were taught by teaching assistants, typically one graduate student with prior teaching experience and one undergraduate or graduate student with little or no teaching experience. Each studio had an enrollment of approximately 40 students.

Student population

A sample of 13 student participants were selected from a pool of 40 volunteers based on the convenience of their schedules. Each student was paid $100 after they completed all eight interviews. The demographic information of students is given in Table 4.1. To protect student identity, I use pseudonyms for all 13 students in this dissertation. All participants had taken Engineering Physics I and the pre-requisite math course – 'Calculus of Single Variables' (Calculus I and II). Eight of these participants had previously taken calculus of multi-variables (Calculus III) and two of the participants were concurrently taking that course. The 13 participants were divided into five groups.

Table 4.1 Demographics of 13 Subjects

Name	Gender	Year	Major	Group #
Alan	Male	2	Mechanical Engineering	1
Zad	Male	2	Mechanical Engineering	1
Phil	Male	2	Mechanical Engineering	1
Lee	Male	2	Electrical Engineering	2
Jared	Male	2	Civil Engineering	2
David	Male	2	Mechanical Engineering	3
Alice	Female	2	Chemical Engineering	3
Bob	Male	2	Chemical Engineering	4
Adam	Male	2	Industrial Engineering	4
Kelly	Female	2	Architectural Engineering	4
Mark	Male	2	Computer Science	5
Ron	Male	2	Architectural Engineering	5

Alex	Male	1	Computer Science	5

Group Teaching/Learning Interviews

I organized the 13 participants into five groups of two or three students each. In total, I completed eight 75-minute long interviews over the semester. For each interview, the interviewer met separately with each group in a room which was only used for interview purposes. Students were seated around a table with a white board set up on the top of the table. Each student was provided with a marker. The purpose of using whiteboards was to provide a shared workspace for students to use and to communicate with each other. It also allowed the researcher to easily video-record students' written work and to understand students' conversation. The microphone was mounted in the center of the table to ensure the audio quality. The video camera was mounted about six feet above the floor and fixed above one end of the table to capture students' gestures and written work on the whiteboard. The layout of the interview set up is shown in Figure 4.1.

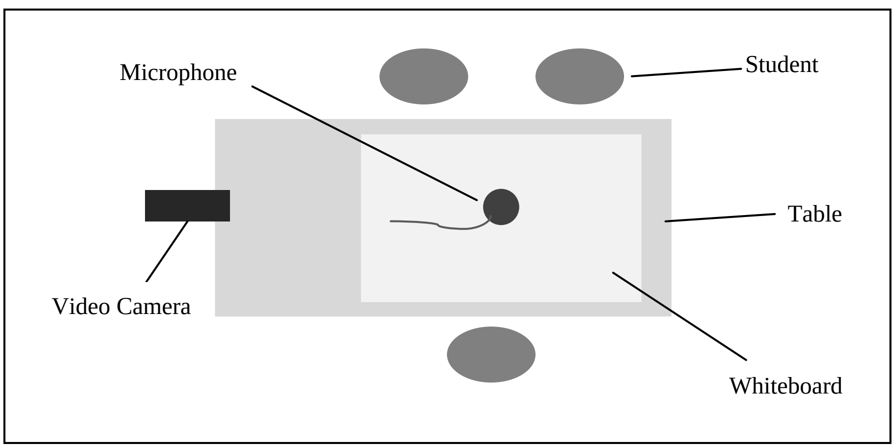


Figure 4.1 Top View of the Interview Layout

Each interview occurred within one week after students covered the relevant concepts in class. I gave students about three physics problems in each session and asked them to discuss the problems as a group using a whiteboard. During the interview sessions, the interviewer was a silent observer for a vast majority of the time, and only interjected to ask students to explain their thinking wherever necessary. When the students were unable to proceed, the interviewer would engage in Socratic dialog in order to provide hints and cues to facilitate the participants in figuring out the next step in the solution process. The interviewer refrained from providing any instruction or feedback during the session.

Interview tasks

Problems presented to students were physics problems that required the use of integration in the contexts of electricity and magnetism. Problems in the context of electromagnetism provide an ideal environment for investigating student's use of integrals. It contains a wide selection of topics (e.g., charge, electric field, resistance, magnetic field) and geometric shapes (e.g., line, circle, disk). Our interview sessions covered most of the typical topics in electricity and magnetism.

The interview tasks were presented with two different formats. The eight interview sessions were designed as four paired sessions: each pair consisted of an odd number session and the following even number session (e.g., interview session 1 and 2). The odd number interview session was presented with conventional physics problems and the following even number interview session was presented with hypothetical debate problems. The topics and formats of the interview tasks in each interview session are given in Table 4.2.

Table 4.2 Topics of Interview Tasks

Interview #	Topics Covered	Problem Format
1	Charge, electric field (application of coulomb's law)	Conventional
2		Debate
3	Electric field (Application of Gauss law)	Conventional
4		Debate
5	Resistance and Capacitance	Conventional
6		Debate
7	Magnetic field (Application of Ampere's law and Biot-Savart's law)	Conventional
8		Debate

In the first interview of each interview pair, students worked together on three conventional physics problems requiring mathematical integration. An example is shown in Figure 4.2. In the second interview, students worked on a problem similar to the first interview problem, but the problem was presented in the hypothetical debate format. An example of hypothetical debate problems is given in Figure 4.3. The hypothetical debate problem shows conflicting strategies employed by four hypothetical students (A through D). The hypothetical debate problem shows conflicting strategies employed by four hypothetical students (A through

D). These strategies were created based on strategies documented in previous studies (Nguyen & Rebello, 2011a) as well as strategies proposed by participants in the previous interview as they worked on conventional physics problems. Students were asked to compare those conflicting strategies and talk about why they agree or disagree with those ideas.

A thin disk with radius R has a non-uniform surface charge density $\sigma(r) = \alpha r$ (α is a constant). Find the total charge on this thin disk.

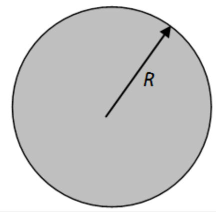
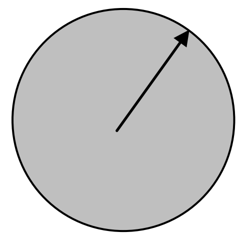


Figure 4.2 Example of a conventional problem

A thin disk with radius R has a non-uniform surface charge density $\sigma(r) = \alpha r$ (α is a constant). Find the total charge on this thin disk. Four students were discussing how to solve this problem and they provided four different strategies. Now discuss in a group why you think each strategy (or part of the strategy) is correct/incorrect.



Student A: The surface charge density is charge per unit area, thus, the total charge is the surface charge density times area, that is $Q = \sigma(r) \cdot A$. σ changes with r, so an integral is needed here. The final equation should be $\int_0^R \sigma(r)(\pi R^2)dr = \pi R^2 \int_0^R \sigma(r)dr$.

Student B: Because we need to add the areas of the rings, $A = \pi r^2$ and it should be inside the integral. Thus, the total charge is $\int_0^R \sigma(r)(\pi r^2)dr$.

Student C: As the surface charge density changes with *r*, we need an integral. First, we need to chop the whole disk into infinite numbers of extremely thin rings with thickness *dr*. For an arbitrary thin ring located at a distance *r* from the center, it has an area of $dA = 2\pi r dr$, and $\sigma(r)dA$ gives the small amount of charge carried by this ring. The total charge is the summation of all the charges carried by each ring, that is $\int_0^R \sigma(r)dA = \int_0^R \sigma(r)2\pi r dr$.

Student D: The surface charge density is changing, the integral $\int_0^R \sigma(r)dr$ means averaging up the charge density along the radius from 0 to *R*. Then we just apply it into the equation $Q = \sigma A$ for a constant charge density. Thus, the total charge is: $\left(\int_0^R \sigma(r)dr\right) * (\pi R^2)$.

Figure 4.3 Example of a hypothetical debate problem

4.3 Data sources

I video-taped all the interview sessions and in total collected about 50 hours of interview data. As I was primarily interested in student thinking, I wanted to gather all the information such as their language and gestures that might reflect how students are thinking. I listened to the students' conversation for all of the interview sessions and transcribed their conversation selectively for detailed analysis. Throughout the interview sessions, students frequently used their hands to convey meanings in their problem solving activity. Hence I also analyzed students' gestures as supplemental information when interpreting their thinking. Finally their written work (diagrams and equations) on whiteboards was also an important source.

Data analysis was conducted from several different perspectives for different purposes. In the next several chapters, I will talk about the data analysis from each perspective and present the findings.

Chapter 5 - Understanding Student Reasoning about Differentials and Integrals – Symbolic Forms and Metaphors

5.1 Introduction

In this chapter I discuss the data analysis and then present the main findings of students' deep reasoning about the mathematical concept of differentials and integrals when solving physics integration problems. When analyzing students' mathematical thinking in physics, I pieced together two frameworks – the resources framework and conceptual metaphors framework. I illustrate how each framework serves different purposes and how they could be fused together to help us better understand student reasoning.

First, I focus on the mathematical resources that students activate with their application of the differential and integral concepts in a physics context. The analysis of mathematical resources provides us deeper insights into how students apply their mathematical learning to physics problem solving. More specifically, I analyzed the conceptual schemas that students activated associated with the mathematical templates $d[]$ and $\int_{[]}^{[]}[]d[]$. Second, I identify the conceptual metaphors involved in students' use of mathematical differentials and prove the existence of metaphors in students' higher-level mathematical thinking (i.e., calculus). Identifying the conceptual metaphors that appear in student language illustrates how concrete experiential notions might affect students' construction of scientific reasoning. It offers an alternative and perhaps fundamental explanation of student thinking. The results of the mathematical resources about the differential template $d[]$ has been published in a conference proceeding (Hu & Rebello, 2013a). A more detailed analysis of students' use of mathematical resources and conceptual metaphors is from Hu & Rebello (2013b).

5.2 Data analysis

Identify symbolic forms about $d[]$ *and* $\int_{[]}^{[]}[]d[]$

To identify students' use of symbolic forms associated with the mathematical templates $d[]$ and $\int_{[]}^{[]}[]d[]$, I focused our data analysis on students' reasoning about differential terms (e.g., *dx, dr, dE*) and integrals in physics equations. I used a phenomenographic approach to analyze the video transcripts, in that I did not decide apriori categories of students' responses, rather the categories were emergent from the data. The video transcripts were analyzed in several stages. First I noted the various kinds of reasoning students provided about differentials and integrals, in particular the ones appeared in various settings. I selected the reasoning that appeared in several different settings. Next, I categorized each kind of reasoning as a conceptual schema contained in the mathematical templates. In other words, I categorized the various kinds of symbolic forms students activated in their reasoning process. In the next section (5.3) I describe the four symbolic forms in association with the differential template $d[]$ and three symbolic forms in association with the integral template $\int_{[]}^{[]}[]d[]$. Our purpose is to demonstrate the existence of types of student reasoning and explain how important they are in students' physics problem solving. Hence it is not our goal to prove how generalizable those symbolic forms are in student reasoning or if one symbolic form is more popular than another.

I selected video clips with students' rich reasoning associated with each symbolic form. I transcribed the video clips and conducted in-depth analysis based on information from student dialogs, gestures, and written work on the whiteboard.

Identifying the Metaphors

For each selected video-clip, I analyzed student conversations to identify the metaphorical expressions involved in their description of mathematical concepts of differentials and integrals. Then I identified the specific conceptual metaphors embedded in their metaphorical language. I also combined the information of student gestures as well as the diagrams or symbols they draw on whiteboards. After generating the conceptual metaphors from the transcripts, I compared our metaphors with the metaphor categories described in the literature. I provide a description of the four metaphors appeared in student thinking in section 5.4.

5.3 Description of Symbolic Forms

I identified four symbolic forms about differentials (Table 5.1) and three symbolic forms about integrals (Table 5.2) emerged from our interviews. I provide an overview of those symbolic forms in this section. Later in the section 5.5, I provide examples to demonstrate how students applied these symbolic forms in physics problems.

Symbolic Forms Associated with Differential Template

Table 5.1 List of symbolic form associated with differential template

Name of symbolic forms	Description
Small Amount	a small amount of a physical quantity, e.g., a very small length, a small amount of charge, often aided by visual representations
Point	a point with no dimensions, sometimes aided by visual representations
Differentiation	taking the derivative of a changing function, often followed by mathematical operations
Variable of Integration	a variable that could be integrated

Small Amount: Differential terms (e.g., *dx*, *dr*, *dE*) often contain concrete physical meanings based on physical systems. When we apply the concept of an integral to physics, we often chop an object or a physical quantity into infinitesimal pieces and add the quantity or its effect due to each infinitesimal piece. Hence, we use differential terms to represent an infinitesimal piece or amount of a physical quantity. Some physicists may prefer to use less rigorous terms such as "extremely small or very small piece or segment" to describe the meaning of differential elements. Thus, I coded this conceptual schema about the differential template *d*[] as 'small amount'.

Point: In mathematics, differential terms such as *dx* carry the meaning of an infinitesimal quantity. When constructing the expression for infinitesimal quantities, students perceive the differential term *dx* as points on a line and they set up their equations accordingly. In Euclid's geometry, a "point" is defined as something that lacks all the dimensions or something with zero dimensions. In geometry, points often represent locations on a line, plane, or space. I discuss how students apply this symbolic form in physics in our case studies.

Differentiation: One of the views about differentials held by students is related to the action of taking the derivative. In mathematics, "*d*" is often used as a symbol for differentiation as in the equation $df(x) = f^{'}(x)dx$ which involves differentiating a function with respect to a quantity. When solving physics integration problems, I find that some students interpret the symbol "*d*" as a cue for a mathematical operation to take the derivative, which I refer to as the 'differentiation' symbolic form.

***Variable of Integration*:** For some students, the symbol "*d*" indicates the variable of integration devoid of any physical meaning. In evaluating a mathematical integral, the notation

$\int f(x)$ is commonly used by many students, simply leaving out the term dx in the expression $\int f(x)dx$. When computing an integral, the only thing that students focus on is finding the anti-derivative of $f(x)$. In physics problems, I often see that students do not include *dx* or *dl* as they set up equations and simply append it to the integrand in the end. Students do not see the differential term as representing any actual meaning other than an abstract, purely mathematical notion.

Symbolic Forms Associated with Integral Template

Table 5.2 Symbolic forms associated with integral template

Name of symbolic forms	Description
Adding Up Small Amounts	Adding up small amounts of physical quantities of an object; '[]d[]' represents the quantity being added up; the two limits represent the physical boundaries.
Adding Up One Quantity Over	Adding up quantities of the integrand over another quantity; the first box '[]'inside the integral is the quantity being added up; 'd[]' represents the variable being integrated with; the limits represent the region that variable varies.
Function Matching	The integrand comes from an "original function," taking the integral would get the "original function." Taking the limits would generate a definite answer without variables.

Adding up small amounts: Under this symbolic form, the integral template $\int_{[]}^{[]}[]d[]$ contains the meaning of adding up small pieces of physical quantities represented by []*d*[]. The two limits often correspond to the physical region or boundaries of the physical object. This symbolic form is often co-activated with "*d*[] as small amount' symbolic form in physical contexts.

Adding up one quantity over: Unlike the 'adding up small amounts' symbolic form, 'adding up one quantity over' symbolic form contains a conceptual schema that adding up a quantity (typically the integrand) over another quantity. The quantity being added up is contained in the first box of "[]*d*[]" while another quantity is the variable of the integral contained in the second box. Take an example of the integral $\int_0^1 f(x)dx$, according to this symbolic form, it is interpreted as adding up the quantity of *f(x)* as *x* varies from 0 to 1. In Jones' work (2013), he defined a symbolic form of 'adding up the integrand', meaning that the "small quantities of the integrand are added up and the result is multiplied by the variable of the differential." However, I found that students did not necessarily perceive the integrand as the quantity being added up. They also thought of the integral as summing up other quantities but not the integrand based on the problem contexts. I will illustrate this symbolic form in detail in our case studies.

Function matching: The conceptual schema of this symbolic form is that the integrand comes from an "original function," and taking the integral is to find the "original function." Taking the limits is to get a definite answer without variables rather than a function. This symbolic form is often applied in companion with the 'd[] as differentiation' symbolic form. Differentiation is the mathematical operation of taking the derivative and taking the integral is the inverse of this operation.

The three conceptual schemas associated with the template $\int_{[]}^{[]} [] d[]$ identified in our work have also been discussed in a very recent work by Jones (2013). However, the mathematics and physics background of the participants as well as the interview tasks in Jones's work were different from ours. Most of our participants had already completed three semesters of university calculus courses and the rest of them were concurrently taking the third semester calculus course. All our participants had finished one semester of calculus-based physics courses. In Jones' work, his participants had completed or nearly completed their second calculus course. The interview tasks in Jones' work include mathematics problems and some physics problems in the context of mechanics while our interview tasks were presented in the context of electromagnetism.

5.4 Description of Conceptual Metaphors

Throughout our interview sessions, there were a number of conceptual metaphors embedded in symbolic forms students activated while they set up integrals. To obtain a deeper understanding of the conceptual schemas I described above, I discuss several metaphors that I identified from student work as they activated symbolic forms in physics contexts (Table 5.3).

Table 5.3 List of conceptual metaphors

Name of Conceptual Metaphors	Description
Objects	'*d[]*' represents a concrete object, such as a little charge or small length; integration is adding up objects.
Locations	'*d[]*' represents locations in space, such as points on a line.
Machines	The template "*d[]*" or "∫[]" is a machine that performs an algorithm, generating an output from a given input.
Motion Along a Path	The variable of integration in the box "*d[]*" represents a traveler; integration is the motion of the traveler along a line.

Objects Metaphor: This metaphor describes the mapping from a concrete object notion in their source domain based on physical experience to the abstract mathematical concept of differentials in the target domain. For example, students viewed the differential term *dx* as a small segment of a bar and *dq* as a small amount of charge. This metaphor has two sub-mapping – 'differentials as objects' and 'integration as adding up objects.' Students associated the differential terms with a small amount of a physical quantity in which the "amount" is an important attribute of an object. Thinking of differentials as objects helped students to quantify them in terms of the amount of length or charge, and to identify relations between different objects such as connecting the total length L with a small length *dx*.

The 'differentials as objects' metaphor is an elaboration of the object event-structure metaphor cluster originally defined by Lakoff and Johnson (1999). The cluster of object event-structure metaphor involving the conceptualization of an abstract notion as a concrete object has been identified in everyday language, mathematical, and scientific reasoning. In mathematics, Lakoff and Núñez (2000) elaborated on the cluster of object event-structure metaphor to describe metaphorical mathematical ideas in basic arithmetic notions, such as "addition as adding objects to a collection". In thermodynamics, an application of the object event-structure metaphor is the entropy is considered as the possession of a system or a physical object in a container (Amin et al., 2012).

Locations Metaphor: This metaphor involves the use of our everyday knowledge of motion in space, coming from our physical experience of movements, to understand the structure of events (Lakoff & Johnson, 1999). When interpreting the mathematical concept of differentials, in particular spatial terms such as *dx*, *dA* and *dV*, students conceptualize the differentials as locations in space. The 'Differentials as Locations' metaphor aligns well with the basic location event-structure metaphor defined by Lakoff and Johnson (1999). In the example of location event-structure metaphor involved in everyday language "He is brought out of a depression" the state of "depression" is construed as location and the change of state is correlated with movement "out of" a location. In thermodynamics the states of a system are understood as locations in cases like "the system is in equilibrium or at a temperature."

Machines Metaphor: This metaphor considers the understanding of differential template $d[]$ and integral template $\int[]$ as machines which perform certain functions. Mathematicians also describe this type of metaphor as a grounding metaphor, meaning "ground our understanding of mathematical ideas in terms of everyday experience" (Lakoff & Núñez, 2000). Regarding the concept of function, one of the grounding metaphors is the 'function machine' metaphor. In the 'function machine' metaphor, a function is regarded as a machine which performs an algorithm or calculates the input-output relationship. Similarly, the symbols "$\int$" and "d" can also be treated as operators that execute algorithms. When solving differentiation or integration problems, those symbols often appear in the structures of $d[]$ or $\int[]$ where the boxes contain variables or functions. When the mathematical templates $d[]$ and $\int[]$ are construed as machines, students often focus their attention on mathematical algorithms and actions without making connection to the physical contexts.

Motion along a Path Metaphor: In the 'Motion along a Path' metaphor, the source domain depicts a fictitious motion along a trajectory. The target domain in our context involves the abstract notion of a differential concept. More specifically, when integrating over the differential terms "$d[]$", the variable contained in the box is conceptualized as a traveler moving along a line. In the phrase "Time flies" the passage of time is conceptualized as one moves along a path or observes a motion (Lakoff & Johnson, 1999). Mi-Kyung and Oh Nam (2004) also identified students' use of a 'Fictive Motion' metaphor when describing the solution of a differential equation. The solution is depicted as a trajectory of a moving point and the variables in the solution function are travelers moving along the axes.

5.5 Case Studies

In this section I present four case studies to illustrate students' activation of symbolic forms and use of conceptual metaphors in their problem solving activities. As discussed in our methodology section, our categorizations of symbolic forms and conceptual metaphors were those frequently used by students and they appeared in student reasoning in multiple contexts. These four case studies allow us to provide an in-depth discussion of student thinking. The four case studies involved two contexts – finding the total electric field due to a line of charge in Episodes 1, 2, and 4, as well as finding the total resistance of the cylindrical resistor with non-constant resistivity in Episode 3.

Setting up an integral to find total electric field is a common electrostatics problems encountered by students. Using this physics scenario gives us the opportunity to see the variability of student reasoning and make comparisons between different groups of students. I do not have clear evidence to show that students explicitly use the 'differentiation' symbolic form in the context of electric field although students frequently use the words "the derivative of". Hence I decided to choose an episode of student work in the context of resistance in which I have more clear evidence to show that students activated the 'differentiation' and 'function mapping' symbolic forms. I claim that students' activation of resources had strong contextual dependence; however, none of the resources were unique to a particular context.

Episode 1: Student Activation of 'Small Amount' and 'Adding Up Small Amounts' Symbolic Forms

This episode involves two students working on an electrostatic problem (Figure 5.1), which I refer to as the line of charge problem. It was selected from the first interview session, which occurred in the second week of their course. This is a classic problem requiring integration in electrostatics. Before the interview students had attended the lecture on this topic in class and completed the related homework assignments prior to their participation in this interview. I select this episode from part of the total 20 minutes of conversation on this problem.

An insulated thin rod with length L has charge +Q uniformly distributed over the rod. Point P is located at a distance d from the right end of the rod. Find the electric field at point P due to this charged rod.

Figure 5.1 Line of charge problem.

In order to find the total electric field at point P due to this line of charge, the typical approach expert physicists take consists of three major steps. The first step is to chop the whole rod into infinitesimal pieces each of length *dx*, carrying an infinitesimal amount of charge *dq*. The second step is to set up the equation for *dE* (Equation 1), which is the infinitesimal electric field at point P due to *dq* (Equation 2). The last step is to find the total electric field *E* by integrating *dE*.

$$dE = k\frac{dq}{r^2} \tag{1}$$

$$dq = \frac{Q}{L}dx \tag{2}$$

Two students (Dave and Alice) discuss how to set up the equation for *dq*.

Dave: *I guess I can do separating little segments of dq.*

As they wrote down equations for *dE* (Equation 1) and *dq* (Equation 2), the interviewer prompted them to explain their thinking.

Dave: *[looking at equation 1] Well, since this is just the value that a particular line segment is putting on to our P, then that would only be a tiny segment of the charge, which is what we described as dq. And I believe dq is what we have evaluated here [pointing to Equation 2]. So, if we plug in further, we can put that in.*

Interviewer: *Okay, can someone explain this equation? [pointing to equation 2]*

Alice: *Well, we have a charge Q over the entire length of L, so this is just saying when you have a little piece, cause you can write it differently, you can write it as* $\frac{dq}{dx} = \frac{Q}{L}$*. So then it is just a ratio of a whole charge over the whole length to a little bit of charge over a little bit of length.*

Students then continued to complete the integral equation for the total electric field. After they finished their work, the interviewer asked students to explain why they used an integral.

Interviewer: *So what made you guys think you should use an integral? Like, why do you decide to use an integral?*

Dave: Basically because we've got a line which you can't separate into, you know, a certain number of points, you have to <u>*separate into infinite amount of points*</u>. *Since we know how to work with points, we figure we just* <u>*sum up all the points of that field on the line*</u>. *And that would give us the total electric field acting on P.*

Both Dave and Alice consistently used similar phrases such as "a tiny segment" or "a little bit" to describe the differential terms *dx* and *dq*. When explaining equation 1, Dave used "a tiny segment of charge" to explain the meaning of *dq* and "the value that a particular line segment is putting on to our P" to describe *dE*. Apart from verbal explanations, he also used gestures to demonstrate "a small section" of the line by making a tiny interval between his thumb and index finger on the picture (Figure 5.2). In describing her reasoning, Alice made use of contrastive terms such as, "the entire length" versus "a little piece" or "a whole charge" versus "a little bit of charge". Both students considered the differential template *d*[] as representing a "small piece/segment" of a physical quantity in this context. Hence, I categorize this group of students as having applied the symbolic form of '*d*[] as small amount' when they set up the infinitesimal equation for *dE* and *dq*.

Figure 5.2 Gesture used in episode one

Dave initiated the conversation by suggesting, "I can do separating little segments of *dq*". By "separating little segments", he attempted to separate the whole rod into little segments of charge and sum up the electric field due to each segment of charge. When asked to explain why they used an integral, Dave explained "separate into infinite amount of points" and "sum up all the points of that field on the line." When talking about "points", students perceived the line as composed of infinite number of points generating a small electric field and integration as summing up the field due to each point on the line. This group eventually found the total electric field by adding up the electric field due to each charge element *dq* successfully. I recognized that students activated the ‘adding up small amounts’ symbolic form about the integral template.

From the perspective of conceptual metaphors, students constructed the abstract notion of differentials in terms of concrete objects such as the ‘amount of a quantity’. In their conversation, students used a number of terms such as “little segment” and “a little bit of…” to describe the differential terms *dx* and *dq*. Near the end of the episode, in her response Alice said, “When you have a little piece”. In this response “a little piece” refers to a thing or object. This indicates that students viewed the mathematical notions *dx* and *dq* as substances or objects and the integral symbol $\int[]$ as adding up objects, an application of the ‘object’ metaphor.

Students in this episode broke down the line into small line segments and then found the charge carried by each line segment. By viewing the total length *L* as composed of a very small line segment *dx*, students were able to construct ratios of the differential or infinitesimal terms (i.e., *dq*, *dx*) to finite quantities (i.e., *Q* and *L*). Then students went on to build the expression for infinitesimal charge element *dq*. They finally set up the total electric field successfully. The fundamental idea of integration is chopping the object into small pieces and adding the quantity (or effect) of each piece. Hence I recognized the approach students invented to set up an integral as “chopping-adding pieces”. The “pieces” correspond to the geometrical and physical constraints of the object. The geometrical constraint refers to the fact that the chopped pieces have the same dimension as the object, such as a line chopped into small line segments or an area chopped into small sections. The physical constraint determines how the object should be chopped. The use of the ‘small amount’ symbolic form and ‘object’ metaphor helped students construct the abstract notions of differentials in terms of concrete things with which they have direct experience. In other words, it led students to make sense of the mathematics in physical scenarios.

Episode 2: Student Activation of ‘Point’ Symbolic Form

In the example of the line of charge problem (Figure 5.1), two students (Aaron and Kelly) seemed to visualize the line as numbers of points each represented by *dx*. Near the beginning of solving this problem, they started to find the expression for infinitesimal charge element *dq*.

Aaron: *Well, we are gonna find, like, would be like summing up little <u>charges at every point</u>?*

Kelly: Yeah, so it's dq, Q/L?

Then, Kelly wrote down an equation for dq (Equation 3) on the whiteboard.

$$dq = \frac{Q}{L} \tag{3}$$

Later on, before she proceeded to construct the integral equation for total electric field, she attempted to add *dx* to the right side of the equation but she was not sure whether *dx* should be included in the equation. Then the interviewer prompted the students to explain their thinking.

Interviewer: *Can you guys explain this equation? What is the meaning of this equation (Equation 3)?*

Aaron: The charge <u>at every single point</u> is charge divided by the distance.

Kelly: Basically, the <u>point charge</u> is <u>at each point</u> along L, is the total charge over its length, so like, what's it called?

Aaron: Charge density.

Later, the interviewer prompted

Interviewer: So what does dq mean exactly?

Aaron: We knew we have to use integral to sum everything up. We need to know what we are summing up the whole time. So, we have to find…

Kelly: Just find the <u>little charges</u> by taking the total charge over the length it's over, to find… since it's uniform, we can find the <u>charge at every point</u>.

At the beginning, Aaron started to talk about "summing up little charges" which indicates that he realized that the total electric field is the sum of (the effect due to) little charges. After that, they set up an equation for *dq* which they described as "little charges" initially. Of particular interest in this episode is that there are two kinds of phrases emerging in their responses: "charge at each point" and "point charge". Students used both phrases interchangeably in their conversation when describing the meaning of *dq* and they did not appear to be aware that there are two ideas involved in the two phrases. From an expert's point of view, "charge at each point" involves a certain amount of charge located at each position on a line; whereas "point charge" involves viewing a certain amount of charge as a point quantity with no physical size. In other words, as students used the phrase "charge at each point", they implicitly associated *dq* with a certain amount of charge, and *dx* with the location in which the charge *dq* is located.

I also noticed that students used the phrases "little charge" in the beginning and the end of this episode and they used "point charge" elsewhere. It seems that students related both terms to the differential term *dq*, implying that they thought *dq* represented a small amount of charge with negligible physical size. In physics, when the size of an object can be neglected compared to the distance from the point of reference, the object is often viewed as a point for simplicity. For instance, in Newtonian mechanics we often use a point to represent the object and draw forces acting on the "point", which we describe in a free body diagram. The point model is also extensively used in E&M, such as Coulomb's law which defines electric force between two point charges. Thus, I conclude that students used the "point" resource in two different ways: differential term *dx* represents a geometrical location in space and *dq* represents a small amount of charge with negligible physical dimensions.

In another context of finding the total charge of a disk with non-uniform charge distribution, I also found that students used similar reasoning with "charge at each point on a disk" as they described *dA* (i.e., small area element) and *dq*. In both geometrical shapes, students related the spatial differential terms *dx* or *dA* with a location on a line or in a plane. From the perspective of conceptual metaphors, students used the 'location' metaphor in their reasoning about spatial differential terms such as *dx*, *dA*, and the 'object' metaphor in their reasoning about *dq* (i.e., amount of charge at each location).

In this episode, students invented a different approach towards setting up an integral, which I refer to as the "separating-adding points" approach. They considered the physical object as being separated into a number of points located on the object or small parts of the object. This approach differs from the "chopping-adding pieces" approach that I identified in the first episode. For example, a line might be viewed as a number of points located along the line rather than line segments; a surface area is viewed as points located on the surface rather than many small pieces of areas. As discussed above, when viewing the object as composed of points, students often failed to relate the points with physical dimensions. Students started with the idea of "summing up little charges at every point". When talking about "charges at each point" students seemed to relate “charge at a point” to the ratio “total charge over the length”. If there were discrete charges distributed and each point carried the same amount of charge, the charge at each point would be the total charge divided by the number of points. However, there was no discrete charge distribution. When constructing the equation relating the point charge with the ratio (Equation 3), students must have considered the length of the rod as a proxy for the number of points on the rod. Similarly, in another problem involving a charge disk with non-uniform charge distribution, I found that students set up an incorrect equation of $dq = \frac{Q}{A}$ for the differential charge element dq. I suspect that students linked the area A of the disk with the numbers of points on the surface without noticing the difference between the two.

I thus infer that when thinking of the differential terms *dx* or *dA* as locations on a line or plane, students did not see the connection between the differential terms and amount of physical quantity as locations that are often regarded as having zero dimensions. Hence, the expression for infinitesimal quantities such as d*q* often did not include a spatial differential term (e.g., *dx*, *dA*).

Episode 3: Student Activation of ‘Differentiation’ and ‘Function Matching’ Symbolic Forms

In the example of a resistor problem (Figure 5.3), the resistivity is changing along the central axis of the cylindrical resistance; thus, integration is required to find the total resistance.

A material with length L and cross-sectional area A lies along the *x*-axis between *x*=0 and *x*=L. Its resistivity varies along the rod according to $\rho(x) = \rho_0 \cdot e^{-x/L}$. Find the total resistance of this cylinder between end faces.

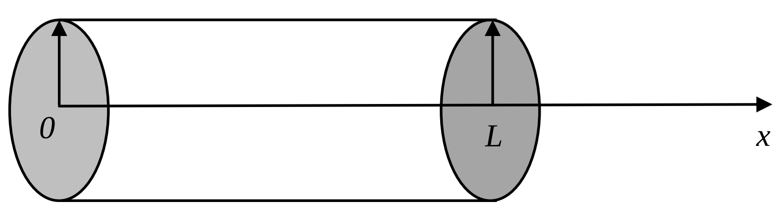


Figure 5.3 Resistor problem

The basic resistance equation for a resistor with constant resistivity is given by Equation 4. To find the infinitesimal expression for *dR*, two students (Zad and Alan) first made up a "resistance function" (Equation 5) by plugging the resistivity function into the basic resistance equation. They then found the expression for *dR* (Equation 6) by taking the derivative of this function with respect to *x*.

$$R = \frac{\rho L}{A} \quad (4)$$

$$R = \frac{\rho(x)L}{A} \quad (5)$$

$$dR = d\left(\frac{\rho(x)L}{A}\right) = \frac{d\rho(x)}{A}L \quad (6)$$

Interviewer: *So why did you take the derivative of this? [Pointing to Equation 4]*

Zad: Um, because we just plugged it into… we plugged what R was… into the integral of R basically. So we need to take the derivative of it. So, we can plug into R. Because we basically pull dx out of nowhere, because the derivative of the only changing function… then we require dx, we need to integrate that.

Zad: We have the function. You have to take the derivative so you can take the integral. I don't know how to explain it other than mathematically.

Alan: You need to take the integral across the whole thing. So, in order to do that, we have to do a derivative, but it's basically just taking the difference from one to the other.

Students continued their calculation and later Alan said he was not sure about whether this method was correct. Zad explained to him about the reason for doing this.

Zad: Oh, we take the derivative, so we can use an integral of two different points. It's not just to get this function, it's just basically the whole reason, we take the derivative, so we can have a way of plugging in two different points…

In Zad's response, "changing function" refers to the resistivity of the cylinder as a function of *x*. In an earlier conversation, Zad said "the only variable was *x*, cause it's from *0* to *L*, so we have to take the integral of *x* from *0* to *L*", which indicates that Zad used the function and variable as a cue for applying integration. Thus, I identified that Zad had activated the 'dependence' symbolic form as described by Sherin (2001) as "a whole depends on a particular symbol appears in the expression". When asked to explain how they set up the expression for *dR* (Equation 6), both students provided similar reasoning – "you have to take the derivative so that you can take the integral." Students perceived *dR* as taking the derivative of the resistance function, and that explains why they made up the resistance function (Equation 5). Once they obtained the resistance function, they took the derivative of the resistance function as they had planned earlier. I recognize that this group of students activated the symbolic form of 'differentiation' when setting up the expression for infinitesimal resistance.

Students then took the integral of the expression from Equation 6 based on the reason that integration would not "get a function", but giving a way of "plugging into two different points". By taking the integral students got an original function and when plugging the limits, they got the expression for the total resistance. I identified the conceptual schema students activated with the integral template $\int_{[]}^{[]}[]d[]$ as 'function mapping', meaning that the integral template serves as a way for getting an original function and taking the limits is to get a definite answer.

In their phrases such as "plug into the integral", "take the derivative", "take the integral", or "do the derivative" I claim that students treated the templates $d[]$ and $\int[]$ as machines, in other words, they used the 'machine' metaphor. In order to find the output (i.e., total resistance) students figured out that the input of the machine $\int[]$ should be *dR*. Next, students needed to find a function as the input of the machine $d[]$. When no such function was available, they made up a "resistance function" (Equation 5) by plugging the resistivity function into the fundamental resistance equation.

Upon setting up an integral, students' solution involved the purely mathematical process of finding a function, taking the derivative of the function, and taking the integral of the differential term. I refer to this chain of mathematical steps as the "differentiating-integrating function" approach. Another interesting phenomenon in students' solution is that they never attempted to talk about the physical meaning of the equations or make use of visual representations (e.g., pictures). While applying this approach, the symbolic forms of 'differentiation' and 'function mapping' as well as 'machine' metaphor are often embedded in student reasoning. The 'differential' symbolic form implies thinking of the symbol "*d*" as taking the derivative of a certain function. It is very likely that the students' abstract notions of "*d*" and "$\int$" as actions come from their experience of performing mathematical calculations. I could also argue that students' notions about mathematical operations are grounded from their physical experience of machines which performs certain functions. Previous studies (Mi-Kyung & Oh

Nam, 2004; Amin et al., 2012) also found that the 'machine' metaphor appeared to be prevalent in students' mathematical reasoning.

Episode 4: Student Activation of 'Variable of Integration' and 'Adding Up One Quantity Over' Symbolic Forms

In this example, two students (Zad and Alan) were discussing how to solve the line of charge problem (Figure 5.1). They first set up the expression for *dq* (Equation 3) which was incorrect. Then they moved on to set up the integral equation (Equation 7). However, they were not sure whether the numerator should be *dq* or *(Q/L)dx*, even though they are in fact equivalent in this situation. In the following conversation, they tried to seek opinions from each other.

$$E = \int_0^L k \frac{dq}{(d+L-x)^2} \tag{7}$$

Zad: *It would be dq or (Q/L)dx?*

Alan: Well, we are just looking at a small part of the bar right now.

Zad: The total Q/L … you don't need to take the derivative or anything, would just give you a basic number since … [looks at the picture again] uniformly distributed over the rod, so you don't need dq, since <u>there is not more charge</u>, so it would be (Q/L)dx.

Alan: Our <u>q is not changing</u> throughout the length, cause it's uniform. So we shouldn't need to integrate q. I didn't think.

When deciding whether *dq* or *(Q/L)dx* should be used as the numerator, both students agreed that *dq* should not be used inside the integral as "there is no more charge" or "*q* is not changing". Students thought any quantity *x* should be a variable in the differential term "*dx*" and can be integrated. In other words, the differential represents the variable of integration. When they were discussing whether *dq* or *(Q/L)dx* should be included inside the integral, Alan talked about the fact that "our *q* is not changing throughout the length". He appeared to be relating *dq* with the notion that the charge is moving along the axis. However, this notion contradicted the fact that the charge is uniformly distributed along the line. Thus, students eventually decided not to use *dq*. In the case of *dx*, because the value of *x* was changing along the axis, students decided to use *(Q/L)dx*.

I also noticed that students activated more than one symbolic forms about differentials. In the conversation above, students also said they should "take the derivative" which could be an indication of using the 'differentiation' symbolic form. Prior to this conversation, students used the 'point' symbolic form to set up the equation for dq (i.e., $dq = \frac{Q}{L}$). When asked to explain their integral equation, they explained "because we need to sum all the charges over the whole bar" and "this whole thing (the bar) does not just have one charge on a point…each one is at a different length away, so you have to integrate over the length." In the integral template $\int_{[]}^{[]} [] d[]$, students did not perceive []*d*[] as representing the quantities (e.g., the electric field) being added up; instead, they interpreted this integral form as adding up one quantity over another quantity. They often did not specify which quantity was added up. In this example here, students talked about "sum all the charges" though the expression they set up in the first box of "[]*d*[]" represent a meaning of small electric field based on their work.

During the interviews, I found that when students wrote down integrals, many often did not include the differential term such as $\int 2\pi r$. And later on, they simply appended a differential term (*dr*) to the integrand as they computed the integral. Consider the example of finding the total charge of a disk with non-uniform charge distribution in which I noticed students setting up an integral $\int_0^R \sigma(r)\pi r^2 dr$. They explained the integral equation as "adding up charges" based on the reasoning that "$\sigma(r)\pi r^2$ is basically charge density times area which gives charge." When asked to explain the differential term *dr*, they said "as you integrate, you would go from *0* to *R*". I suspect that students might have perceived the variable of differential term *dr* as a traveler who moves along the radius from the starting point *0* to ending point *R* when integrating. From the conceptual metaphor perspective, students applied the "motion along a path" metaphor towards the understanding of mathematical differential and integral.

5.6 Summary

In this chapter I have examined the symbolic forms and conceptual metaphors that are involved in students' application of differentials and integrals in physics problems requiring integration. I identified seven symbolic forms and four conceptual metaphors in association with the differential and integral concepts. Analyzing students' reasoning from both perspectives allows us to obtain deeper and broader insights into students' mathematical thinking in physics contexts. I also provided multiple case studies to illustrate how students applied those symbolic forms and used conceptual metaphors in various contexts of solving physics integration problems.

The '*d*[] as small amount' symbolic form refers to a small portion of a physical object. It is used across many physics contexts, such as finding electric field due to a charge distribution. The '$\int_{[]}^{[]}[]d[]$ as adding up small amounts' symbolic form represents adding up small amounts of physical quantities. An 'object' metaphor is often involved with the use of the two symbolic forms. This metaphor allows students to associate the complex mathematical notions with their experiential knowledge of physical objects. The '*d*[] as a point' symbolic form involves two distinct meanings: one represents locations in space and the other represents a point quantity with negligible physical size. However, students seemed to be unaware of this distinction. Our analysis of student work suggested that students' notion of "points as locations in space" is embedded in their reasoning of "spatial" differential terms such as dx, dA. Students' notion of "points as point quantities" is associated with differential terms such as dq. The "point" resource used in the spatial differential terms also aligns with the 'location' metaphor, in which students' metaphorical construct of differential terms as spatial locations might be grounded in their interactions with the physical world.

The '*d*[] as differentiation' symbolic form involves associating differentials with the mathematical action of taking the derivative of a function and the '$\int_{[]}^{[]}[]d[]$ as function matching' symbolic form involve taking the integral to get the original function. I identified the 'machine' metaphor from students' linguistic use while solving integration problems. A 'machine' usually yields an output from a given input. In fact, mathematical operators are often treated as "machines" and students are able to get the correct output when the input is explicitly given to them. However, our analysis of students solving physics problems finds that students tend to encounter difficulties if they need to find the input, such as a function in a physical scenario when using this metaphor. The '*d*[] as variable of integration' resource implies that the symbol "*d*" is purely an abstract entity, which is often followed by a variable. I consider it likely that students activated this resource when they simply dropped or added a differential term in an integral equation. When activating this symbolic form, students often activate '$\int_{[]}^{[]} []d[]$ as adding up one quantity over' symbolic form when setting up integral equation. In the 'motion along a path' metaphor, students view the variable of differential as a traveler "going from one point to another" when doing integration. Mathematically, the concept of a differential has dual aspects: it is treated as an object representing an infinitesimal quantity and it is used as a mathematical operator for differentiating a function with respect to a variable. This dual aspect of a differential concept is also reflected in student use of conceptual metaphors.

I do not claim that applying certain symbolic forms or conceptual metaphors will certainly lead students to set up an integral correctly or incorrectly. For instance, in order to correctly set up the infinitesimal expression, students need to break down the object in a correct way that goes beyond merely the activation of the 'd[] as small amount' symbolic form and 'object' metaphor. The differentiating-integrating functions approach is not necessarily always an unproductive approach. When activating the 'd[] as differentiation' and $\int_{[]}^{[]} []d[]$ as function matching' symbolic form, if students could generate a function in a correct manner, they would eventually set up the integral correctly.

Our findings are based on interviews with 13 engineering students and it is very likely that there are other symbolic forms or conceptual metaphors that were not identified in our study. On the other hand, our eight interview sessions covered many of the physics topics in E&M encountered in second-semester calculus-based physics that involve the use of integration, such as electric field due to a line or disk of charge, Gauss's law, resistance, Ampere's law, and Biot-Savart's law. Though the reasoning and approaches that I identified often appeared in more than one context, students' use of symbolic forms and conceptual metaphors does have contextual dependence. For instance, I have observed students were more likely to use the 'd[] as a point' symbolic form and 'location' metaphor when setting up infinitesimal expressions in charge distribution on a line or disk.

Our study contributes to both the mathematics and physics education research on student use of integration in physics problem solving. Our study investigated the various mathematical resources and conceptual metaphors that students bring to bear while applying the differential and integral concepts in solving physics problems. While Jones investigated nine students' use of symbolic forms associated with various integral templates such as $\int[]d[]$ and $[]\int[]$ in his problem contexts – mathematics problems and physics mechanics problems. I explored the symbolic forms used by a different student population and demonstrated the existence of some previous symbolic forms in the contexts of electromagnetism. More importantly, I identified several symbolic forms associated with the differential template *d*[] which have not been studied carefully in previous work. Our study also adds to the literature by providing evidence about the existence of conceptual metaphors involved in students' higher-level mathematical thinking and quantitative problem solving in physics.

5.7 Discussion about the frameworks

In this chapter, we analyzed students' use of mathematics in physics from resources and conceptual metaphors perspectives. In this section, we would like to discuss the affordances and constraints of using both frameworks in understanding students' mathematical physics problem solving. As the main stream of physics education research framework, resource framework has been significantly affected how researchers looked at and interpret students' learning.

First, using the resources framework allows us to value the diversity and usefulness of students' thinking. From exploring the various mathematical resources that students activated, we can better understand students' reasoning and make inferences about where their difficulties might come from. For example, one mathematical resource in association with the differential structure *d*[] students activated is the variable of integration (i.e., a variable over which to integrate). One way of thinking based on the resource perspective is that this variable of integration is from students' past experience possibly from their mathematical learning; when computing a mathematical integral, the differential did play the role of variable of integration.

Second, this perspective also has important implications for instruction. When analyzing students' reasoning from the resources perspective, we found that students' activation of resources often depends on the context. Even for the same mathematical structure (e.g., differential *d*[]), some students activated different mathematical resources in different physical situations. Hence, we can create teaching lessons which are focused on how to facilitate students to distinguish the various roles that differential terms played in various situations as well as which resources are more powerful in most physical situations.

However, there are also two major limitations of this framework. A hidden assumption of the resources framework is that resources are pre-existing ideas in a student's mind that are activated in various situations. Hence, there are two parts which are not very well developed in this framework: pre-existence of student reasoning and the mechanism of activating certain resources. It is hard to distinguish if a resource pre-existed and was simply active in a context, or if a resource was organically created in a situation. Sayre & Wittmann (2008) described the plasticity nature of resources as an extension to the resources framework. They identified the plasticity continuum and the two directions are more solid and more plastic. Rather than thinking of resources as something ready to be applied in a situation, the plasticity continuum could perhaps provide a way to look at the development of resources.

When analyzing student work from the conceptual metaphors framework, it gives us important information about what concrete experiences students bring to bear in association with their development of resources. The metaphorical language involved in student reasoning conveys a message about how students' mathematical or physics learning interacted with their experiential knowledge from everyday life. Hence, the significance of using this framework is that it describes which domain(s) of knowledge students bring to bear as they conceptualize more abstract ideas in mathematics and physics. In this study, we identified several metaphors involved in students' mathematical reasoning. It provided us a deeper understanding of how students constructed their learning in mathematics and physics.

Chapter 6 - Using Conceptual Blending to Describe How Students Set Up Mathematical Integrals in Physics

6.1 Introduction

In the last chapter I analyzed students' ideas about the mathematical concepts of differential and integral from the perspectives of resources and conceptual metaphors. I categorized students' use of mathematical resources (i.e., symbolic forms) and metaphors in their mathematical discourse in physics contexts. In this chapter I address the second research question: How do students combine their knowledge in calculus and physics to set up integrals in physics? The purpose of this question is to explore students' overall strategies of applying mathematical integrals in physics. I use the language of conceptual blending to describe how students combine their knowledge in mathematics and physics.

Previous studies in physics education research have provided evidence that students' difficulties with using mathematics in physics is often caused by physics contexts rather than mathematical aspects alone. Analyzing students' blending of knowledge of mathematics and physics would help us generate deeper insights into understanding how students transfer their learning from mathematics to physics, or how the physics context affects their problem solving strategies. I conducted a qualitative analysis of student work and in this chapter we will talk about our data analysis and the several different strategies that students invented towards setting up integrals. The main findings are from a recently submitted article (Hu & Rebello, 2013c).

6.2 Data Analysis: Identifying the Blends

I present a detailed analysis that focuses on one physics problem in the context of resistance (Figure 5.3). The problem involves finding the total resistance of a cylindrical resistor with non-constant resistivity. When the resistivity is not a constant, one must first chop the cylinder into infinitesimally thin disks and find the resistance dR of each disk (Figure 6.1). We find the total resistance by summing up i.e., integrating over the resistance, dR of each disk. Finally, we substitute the resistivity function $\rho(x)$ provided in this equation and evaluate the integral.

The basic physics equation for resistance with constant resistivity is given by $R = \frac{\rho L}{A}$. The resistance of one infinite thin disk dR can be expressed as $dR = \frac{\rho dx}{A}$ with dx the thickness of the disk. The total resistance R is the integral of dR, that is $R = \int dR = \int \frac{\rho(x)dx}{A}$

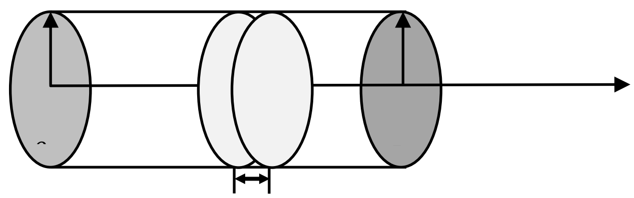


Figure 6.1 Partial solution to the interview task in Figure 5.3.

The problem was presented to students during the fifth interview session. There were two main reasons for choosing this problem for a detailed analysis. First, all of the students were familiar with the resistance concept but they had not seen any specific example similar to the problem nor had they received any instruction on how to find the resistance of a resistor with non-uniform resistivity. Students had to invent their own integral equation as there was no specific integral equation on resistance that they could draw upon from their memory. Second, students used diverse strategies and reasoning when solving this problem. This problem seemed neither too easy nor too difficult for most students. Most students were able to invent strategies to solve this problem and explain their thinking.

Our unit of analysis was a student discussion group. We transcribed the video files for all five groups working on the task. We conducted a qualitative analysis and our unit of analysis was a student discussion group. I transcribed the video files for all five groups working on the task. Next, I conducted a qualitative analysis using a phenomenographic approach. An important aspect of our approach was that I did not categorize students' descriptions based on pre-decided categories. Rather, the categories emerged from the data. Then, I identified the important themes that emerged from students' conversations or written work on the board. Next, I examined these emergent themes through the lens of the conceptual blending framework. For each theme, I determined the input spaces and blended spaces by analyzing the ways in which students connected their ideas from different domains. Finally, I generated a holistic description of how the students blended their ideas. This holistic picture describes how the knowledge elements from each input space are projected to the blended space and how a new structure emerges from the blend.

6.3 Description of Blends

In this section, I present four distinct conceptual blends identified from five groups of students working on the resistor problem (Figure 5.3). I noticed that students within the same group did not necessarily follow the same approach, thus, there was often more than one blend created in a group. I claim that blending is a dynamic process. That is, students do not have explicit ideas in their mind and construct the solutions spontaneously. I found that students constructed one blend at the beginning and later changed the blend as they proceeded through the problem. Some blends eventually led students to solve the problem correctly while others hampered students' efforts to setting up the correct integral. When constructing a blend, students often recruited an organizing frame, which determines how students put together the knowledge elements projected from input mental spaces. Hence, the organizing frame describes the structure of student reasoning. I will mainly focus on the organizing frame adopted by the students. This analysis allows us to classify student work into four frame clusters: Integral-Sum Across frame, Differential Algorithm frame, Equation Mapping frame, and Chopping &Adding frame. I will illustrate the four different frames using examples of student work.

Frame 1: Integral-Sum Across

Two groups of students constructed a blend under the 'Integral-Sum Across' frame. I discuss one group of three students as an example. Prior to this dialog, all three students set up their own equations and they all seemed to be in agreement with each other. I show Phil's solution is shown in Figure 6.2. The interviewer then prompted the students to explain their thinking which resulted in two of the students (Zad and Alan) participating in the conversation below.

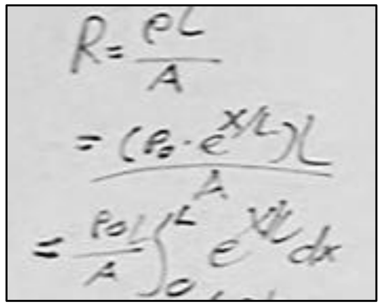


Figure 6.2 Equation set up by Phil

Interviewer: *Can you guys sort of explain what your plan is for solving this problem?*

Alan: *Well, We set up our basic equation. And then I are using our* $\rho(x)$ *that they gave us. We are going to integrate that to give the whole area, right? That's where I are at right now.*

Interviewer: *What do you mean by "integrate to get the whole area"? Can you talk more about that?*

Alan: *We are looking for total resis* (pause) *total* ρ*, yeah, and this* (points to the resistivity function in the problem statement) *gives us* <u>*ρ at any one point*</u>*, so we are* <u>*summing up all of its points*</u>*.*

Then students then discussed the cross sectional area A in the integral and calculating the integral. After they were all done, the interviewer continued

Interviewer: *So can you guys explain your solution from the beginning?*

Zad: *Resistivity times length over area. We have* <u>*function of resistivity*</u>*,* <u>*pull out all the constants*</u>*. Length is a constant, x=L,* ρ_0 *is a constant, over A, we need to* <u>*take the integral of x*</u>*, which, a function of x, which is* $e^{-x/L}$*, from 0 to L, because you are taking from 0 to the length L…(Continues to talk about his calculation.)*

Interviewer: *Can you guys explain what the meaning is for the integral part? Is there a meaning for that?*

Zad: *Um, it gets more resistive as it approaches L. That's a function of* ρ but just the *resistance isn't constant, I guess? So you* have to take, since <u>*it's not constant, you have to take the total integral of… sum all the way across*</u>.

When working on this problem, this group of students mainly focused on mathematical manipulations and looked for ways to relate the symbols to a mathematical expression. First the students related the resistivity function $\rho(x)$ to the basic resistance equation $R = \frac{\rho L}{A}$ by substitution. There was no evidence that students associated the equation or mathematical operation of substitution with the physical situation. In mathematics, substitution or "plugging in" is one of the most basic operations, often made subconsciously by students even though the resulting equation does not have any concrete meaning in this physical situation. Students also described how one quantity changes as another quantity varies, which can be interpreted as demonstrating a basic knowledge of a function. Alan explained that the resistivity function "*gives us ρ at any one point*" so "*we are summing up all of its points.*" In other words, Alan seemed to have activated the "parts-of-a-whole" symbolic form defined by Sherin as "amounts of generic substance, associated with terms that contribute to a whole." In their explanation, Alan talked about "*summing up all of its points*" and Zad said "*take the total integral of, sum all the way across*." Hence, students had the notion that "integration represents summation," but they did not seem to be clear about which physical quantity must be summed up. When asked to explain his integral, Alan talked about "*we are looking for the total ρ, and this (resistivity function) gives us ρ at any point, so we are summing up all of its points*." One possible explanation is that students perceived the integral as a way to sum up the varying quantity regardless of what quantity it was. Another possible explanation is that in the integral template $\int [\,] d[\,]$, students interpreted the first box as the physical quantity at any one point which could be added up through an integral and the second box as the indicator of dimension that they "sum across." The latter explanation is also similar to the problematic *'add up then multiply'* symbolic form identified by Jones (2010).

In this example, students spent the entire time discussing the manipulation of symbols or talking about mathematical ideas such as functions, constants, and integrals. Thus, I concluded that students' reasoning involved two input spaces: symbolic space and mathematical notion space. The symbolic space includes ideas that students tend to express in symbols. In this case, the symbolic space included ideas such as the resistivity function $\rho(x)$ and resistance equation $R = \frac{\rho L}{A}$, as well as the integral form of $\int [\,] d[\,]$. The mathematical notion space refers to the mathematical ideas or concepts that students tended to express in words. In this case, the mathematical notion space included students' ideas of the function as being non-constant and the idea of integral as "summing across." Then the elements in the input spaces were projected to a new space - the blended space.

The main organizing frame involved in the blended space is the notion of "integral as sum across," meaning that the integral is the summation of a quantity at any one point across all the points. However, this group of students did not specify which quantity they were adding up even after the interviewer prompted them several times. Hence, the "Integral-Sum Across" frame refers to the mathematical idea students had about the integral form $\int[]d[]$ where the first box often contained a quantity which could be summed up across the independent variable in the second box. Then the resistance equation and resistivity function were combined using substitution and the resulting expression was then plugged into the first box of the integral form, $\int[]d[]$. The integral and differential symbols were organized to construct the integral expression.

In terms of the conceptual blending framework this process can be described as shown in Figure 6.3. Students blended the elements in the symbolic space and the mathematical notion space and employed the organization frame of "Integral-Sum Across." From students' conversations and written work, it appears that they might have had several connections across the two input spaces. First, they seemed to relate the symbol of resistivity function $\rho(x)$ with the idea of the function representing a non-constant quantity. They also seemed to associate the integral form $\int[]d[]$ with the notion that an integral involved summing a quantity across another independent quantity. When projecting the elements from two input spaces into the blended space, students seemed to map the basic resistance equation ($R = \frac{\rho L}{A}$) into the new space to create a new expression ($R = \frac{\rho(x)L}{A}$). They also seemed to map the integral form $\int[]d[]$ and the notion of integral as summing across into the blended space, where the first box contained the new expression $R = \frac{\rho(x)L}{A}$. In the end, the interviewer asked the students how confident they were about their solution and all three students stated that they were confident.

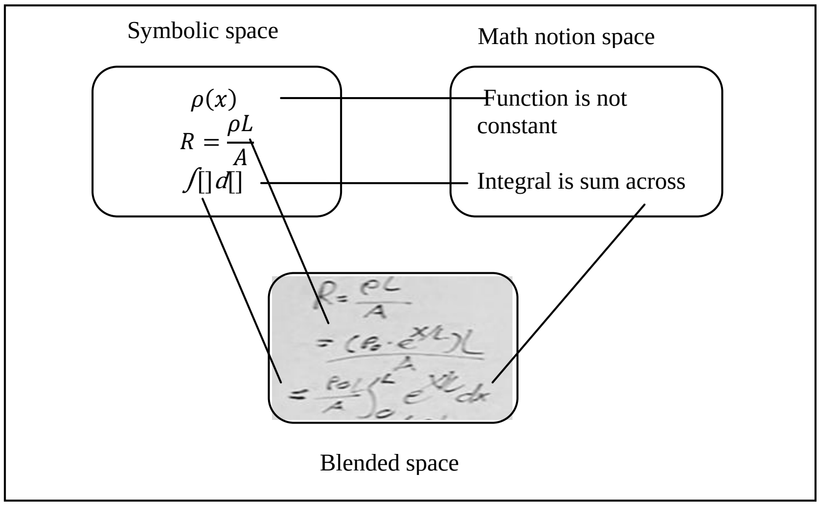


Figure 6.3 Blend under 'Integral-Sum Across' frame

Frame 2: Differential Algorithm

This group of students first solved the problem following the blend under the 'Integral-Sum Across' frame. At the very beginning of their problem solving episode, one student (Zad) wrote "*dR"* but he did not specify what *dR* meant nor did he write the expression for *dR*. In the following episode, the interviewer asked students to explain what *dR* meant and they started to find the expression for *dR*. Students eventually constructed another blend which was different from the blend they had originally created. In the last episode under the 'Integral-Sum Across' frame, students focused on the mathematical notion of the integral as a sum but did not pay attention to the differential terms. However, when prompted to explain the differential term *dR*, students' notion about the differential template *d*[] was activated. Hence, students shifted from one organizing frame to another.

Interviewer: *So here I saw you wrote dR. Can you sort of explain what do you mean by this dR?*

Zad: *Like electric field, anything else uneven, we have to define the integral of dR. We have to find dR, um… with… over an area.*

Alan: *It's a small R. It's just <u>a small part of our resistance</u>* (points to the cylinder of this problem). *Like, we just look at a section of it. We look at… just on our cylinder like we are looking at our <u>resistance at this point</u> right here* (points to the dot on the central axis of the cylinder in Figure 6.4. *It would be just our dR there. So I do that and then I take the integral… adds up the little parts together.*

Interviewer: *How would you set up the equation for dR?*

Then students started to set up the equation for *dR* and talked about their thinking at the same time.

Alan: *So dR would be equal to …* (Figure 6.5)*, but I need dx, I guess.*

Zad: *We would have to take the derivative of this one* (points to the equation in Figure 6.6)*. Um, this is the derivative. Take the derivative of that* (Figure 6.6)*. Would that* (Figure 6.7) *be the derivative of it?*

Alan: *It should equal dR.*

Zad: *dR… cause that will times dx, not equals dR. So, instead, you have to take the derivative of something, we just plug in to dR* (sets up equation in Figure 6.8).

Interviewer: *So why did you take the derivative of this one* (points to Figure 6.6)*?*

Zad: *Um, because we just plugged it in to… we plugged what R was into the integral of R basically. So, we need to take the derivative of it. So, we can plug in to R because we basically pull dx out of nowhere, because the derivative of the only changing function… then we require dx. We need to integrate that.*

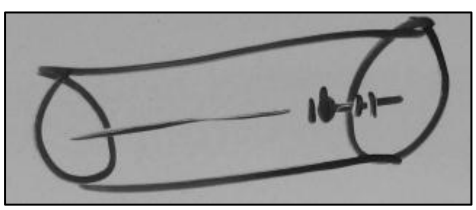


Figure 6.4 Picture drawn by Alan.

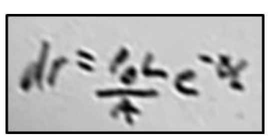


Figure 6.5 Equation set up by Alan.

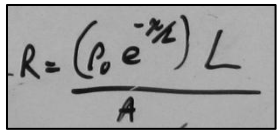


Figure 6.6 Equation set up by Zad.

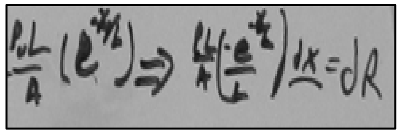


Figure 6.7 Expression for dR.

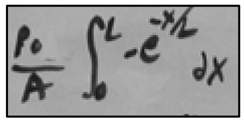


Figure 6.8 Expression for total resistance.

At the beginning of this conversation, when asked to explain what *dR* meant, Zad said "*like electric field…, you have to define the integral of dR*." I noticed that Zad did not actually talk about the meaning of *dR*. Instead, he recalled the electric field problems in which the total electric field $\vec{E}$ is the integral of d$\vec{E}$. Then, he argued that the total resistance *R* should be the integral of *dR*. Hence, his argument seemed to be based on pattern matching with similar examples he had seen before rather than understanding what *dR* was and how it was related to the total resistance *R*. Alan seemed to have a more concrete understanding that *dR* represented "*a small R*" or "*a small part of our resistance*." He drew a picture (Figure 6.4) to explain "*resistance at this point*." When finding the expression for *dR*, Alan just plugged the resistivity function into the basic resistance equation (Figure 6.5) and then he realized there must be something wrong with this equation because he "*need*[ed] *dx*." Zad proposed a different approach. He first substituted the resistivity function into the basic resistance equation to get a new expression for "*R*" (Figure 6.6); then he took the derivative of this new expression with respect to *x* to get the differential of *R*, which was *dR* (Figure 6.7). Finally they set up a definite integral of *dR* to find the total resistance (Figure 6.8). When finding the expression for *dR*, Zad explained "*that will times dx…you have to take the derivative of something*." I suspect that Zad realized that *dx* was needed in the differential equation (Figure 6.5) set up by Alan and this prompted him to "*take the derivative of something*" in order to get *dx*. In this problem, Zad activated the mathematical operation of "taking the derivative" to find the differential expression *dR*. Upon seeing Zad's approach, Alan erased his work and seemed to accept Zad's approach.

Students' construction of their solution can be represented by a blending diagram (Figure 6.9) which organizes the ideas reflected from their conversation and written work. This diagram was mainly based on Zad's work. In this episode, Zad mainly focused on mathematical operations and symbolic manipulations. The two mental spaces involved in this process are the symbolic space and the mathematical concept space. Similar to the blend under the 'Integral-Sum Across' frame (Figure 6.3), the symbolic space includes the symbol of resistivity function $\rho(x)$, the basic resistance equation $R = \frac{\rho L}{A}$, differential template d[], and integral template $\int d$[]. The math notion space includes the mathematical concept of function, differential as the operation of taking the derivative or differentiation, and the integral-differential relation (i.e., total resistance R is the integral of differential form dR).

Across the two input mental spaces, the students seemed to connect the symbol of the resistivity function with the notion of function as a changing quantity. They also seemed to relate the template d[] to the idea of differentiation and the integral template $\int$d[] with the 'integral-differential' relation. When elements in the two input mental spaces were projected into the blended space, the basic resistance equation was mapped into the blended space to create a new expression; the template d[] and the differentiation idea were mapped into the blended space to take the derivative of the new expression; the integral template $\int$d[] and the integral-differential relation were also mapped into the blended space to construct an integral. In the blended space, the solution was organized under the notion that the differential was representing the operation of differentiation. I refer to this organizing frame as the Differential Algorithm frame. Under the Differential Algorithm frame, students found the expression of *dR* by differentiating the equation (Figure 6.6) obtained from substitution without considering the meaning of this equation.

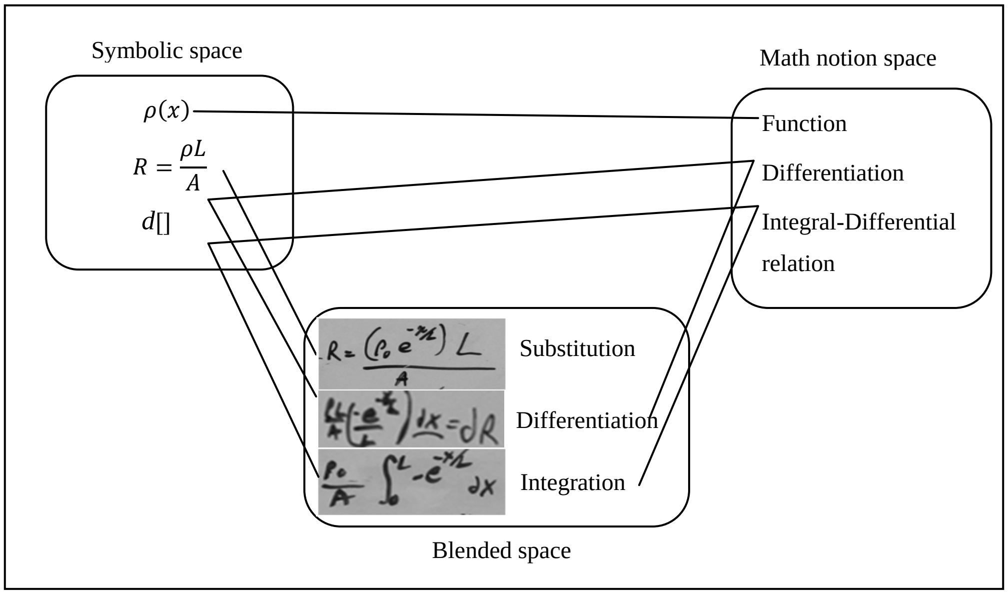


Figure 6.9 Blend under 'Differential Algorithm' frame.

Frame 3: Equation Mapping

In the episode below, two students (Jared and Lee) used different approaches to solve this problem. I will discuss Jared's approach in this example. Jared started with the basic resistance equation and then converted it into an integral form as shown in Figure 6.10. When he completed this equation, the interviewer prompted him to explain his thinking. Below is a transcript from the conversation between Jared and the interviewer.

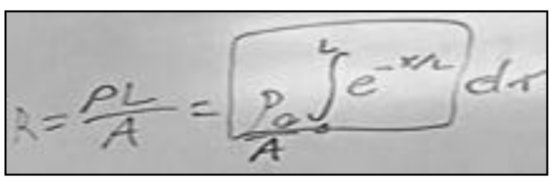


Figure 6.10 Equation set up by Jared.

Interviewer: *Can you guys talk about what you are trying to do before you continue?*

Jared: *Uh, I <u>took this ρL and converted to</u> this right here which is* $\rho(x)$. *And then multiply by d, uh, small distance x, each of the little resistivity? It's <u>all those little pieces going up together</u>. And so it's the integral from 0 to L, of e to the negative x over L, dx.*

Interviewer: *Can you explain more about this part, the integral part?*

Jared: *Basically this part right here* (drew a rectangular box on the equation as shown in Figure 6.10) *is the ρ, resistivity, and the dx is the length, and uh, since it's not just multiplied by one small length, it's from 0 to L, so you get the whole length.*

Interviewer: *Uh huh.*

Jared: *And you are also adding up all of the resistivities together.*

Jared explained that each small piece was represented by "*dx*" and by using integration, he was adding "*all those little pieces going up together*". Jared seemed to view the use of integral in this physical situation as adding up small quantities. By saying "*I took this ρL and converted to this right here*", Jared mapped the basic resistance equation and the summation idea into the structure of the basic resistance equation. He interpreted the variable of integration *dx* as "*little pieces*" and the integral part in the rectangular box (Figure 6.10) as "*adding up all of the little resistivities.*" The expression in the rectangular box is neither a complete nor correct expression without *dx*. By saying "*adding up all of the resistivities together*" he seemed to have understood that an integral represents a Riemann sum. However, he did not seem to realize that the resistivity represents the property of this material and the resistivity at different points could not be added up in this physical scenario.

The blending diagram (Figure 6.11) represents how Jared constructed his solution. There are three mental spaces involved in this blending process: symbolic space, math notion space, and physical world space. The symbolic space contains basic resistance equation, the integral template $\int[]$, and the differential notation $d[]$. The math notion space contains the idea of differential as a small amount of a quantity, and the integral as a sum. The physical world space contains the cylindrical resistor.

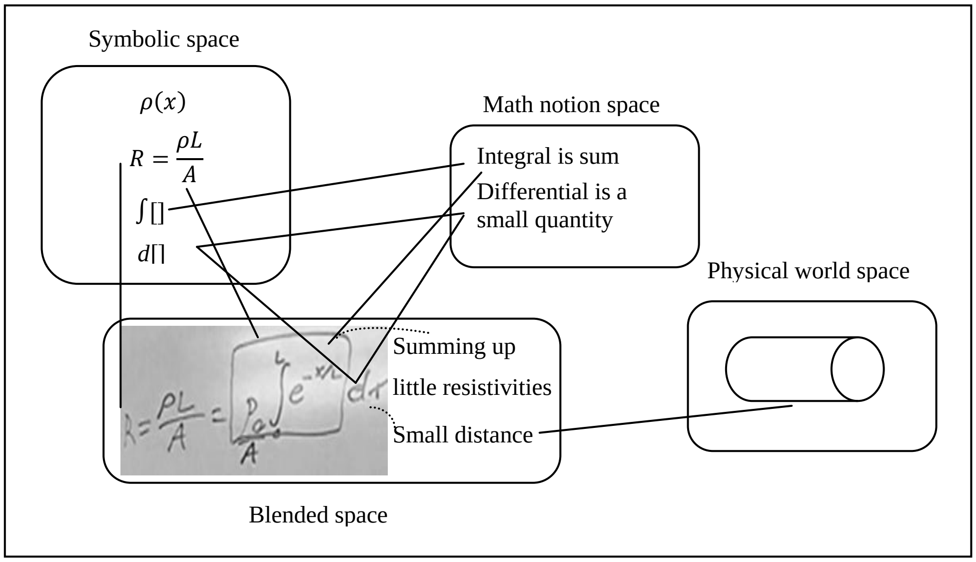


Figure 6.11 Blend under 'Equation Mapping' frame.

Students seemed to make the following connections across the input spaces. First, they seemed to associate the idea of an integral as a sum in the math notion space with the integral template $\int[]$ in the symbolic space. They also seemed to relate the idea of the differential as a small amount of quantity from the math notion space to the differential notation *d*[] in the symbolic space. When composing the blend, I speculate that the students mapped the structure of the basic resistance equation ($R = \frac{\rho L}{A}$) into the blended space. They also mapped the idea of the integral as a sum and the integral template into the blended space to construct an integral of the resistivity function. Further, they seemed to map the idea of the differential as a small amount of quantity from the mathematical notion space, the differential notation *d*[] from the symbolic space, and the cylinder from the physical world space into the new space to construct the differential *dx*. Jared seemed to recruit an organizing frame -- the structure of the resistance equation -- primarily from the symbolic space.

I call this frame the 'Equation Mapping' frame, because it refers to the fact that students tried to use the structure of this basic equation and map it to the physical situation. The structure of their solution strongly relied on this equation rather than the physical situation or the physics concepts. In our problem, students were given the basic resistance equation which could be used when resistivity was constant. The 'Equation Mapping' organizing frame differs from the two organizing frames I described above. Unlike the first two blends, Jared recruited the structure of the equation from the symbolic mental space but also combined the notion of “integral as a summation” and the physical picture of a cylinder to construct the solution.

Frame 4: Chopping and Adding

In the following episode, two students, David and Alice, used an approach which is similar to the typical approach described in Figure 6.1. They first chopped the cylindrical resistor into infinitely thin disks and then added the resistance due to each thin disk to get the total resistance. The following conversation occurred as soon as they started to solve this problem.

David: *Okay, I think I need to <u>separate this into like, little pieces</u>* (Figure 6.12).

Alice: *Yeah.*

David: *Little disks. Disks are fun.*

Interviewer: *So what made you guys think you need to separate it into disks?*

David: *In this case, because the <u>resistivity changes as the length increases</u>* (see gesture in Figure 6.13).

Alice: *So that's the <u>sum</u>, when we do that, it is just an <u>integral</u>.*

David: *It is different here than it is here* (see gesture in Figure 6.14). *So, the easiest way is to <u>picture it as infinitesimal thing at different values</u> and then <u>add up all the different values.</u>*

Figure 6.12 David's gesture about “separating into pieces.”

Figure 6.13 David's gesture about “resistivity.”

Figure 6.14 David's gesture about “different values.”

After this short conversation, they set up the differential form *dR*, which is the infinitesimal resistance due to a thin section of the cylinder of thickness *dx*. They then set up the expression for total resistance *R* by taking the integral of *dR*. Their solution is shown in Figure 6.15 below. Then the interviewer prompted them to explain their thinking. During his explanation, David also drew a picture as shown in Figure 6.16 below.

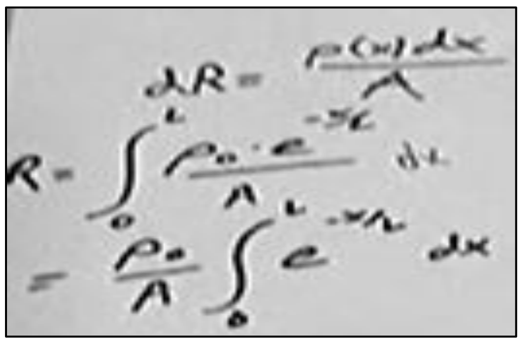


Figure 6.15 Solution from David and Alice.

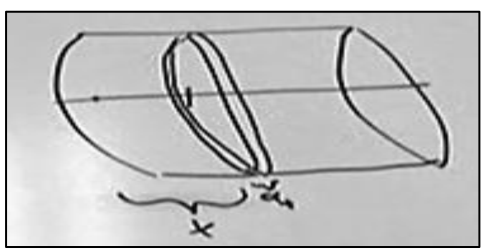


Figure 6.16 Picture drawn by David.

Interviewer: *Okay. Can you guys explain how you got this equation* (referring to the integral equation in Figure 6.15)?

Alice: *Okay. Well, we are taking that one (the basic resistance equation) and so… but since we have to sum up a bunch of little pieces, you have to get little pieces, little piece of R, and then our function for ρ, as ρ changes with respect to x, and where* *x is little pieces of length. And so we cannot use whole of it, we have to use dx, and this is just the area* (points to the equations in Figure 6.15)*. So, it's our resistivity, (laughs) our resistance, is dependent on our length…*

Interviewer: *Okay. If you are going to explain your solution to your classmates, or friends, how would you explain it?*

David: *Well, I guess that would be… Since we have a <u>material that has a varying density</u>, we can <u>separate it into small cylinders</u>, each one having <u>a different density depending on their location</u> x. So, starting out with the basic equation, we would find the <u>resistance of each individual cylinder, which is dR</u>, and that would be our little piece of resistance. And when you <u>sum up those resistances</u> from the left side to the right side, 0 to L, um, you would get the total resistance.*

David started with "*separate this into like, little pieces*," and used his hands (Figure 6.12) to imitate the action of "chopping the whole cylinder into little pieces." He explained that "*resistivity changes as length changes*" and indicated the resistivity was different at different positions by sweeping his hand across using the gestures shown in Figure 6.13 and Figure 6.14. The students realized that they needed to do a "*sum*" or "*integral*" to "*add up all the different values*." In the above episode, the students first analyzed the physical situation, then they developed a story about it using words and gestures, and finally they began to apply formal mathematics to set up an integral. When setting up the integral expression, they first found *dR*, which was the *"little piece of R"* or "*resistance of each individual cylinder*"; then they "*summed up those resistances*" using an integral to get the total resistance. He realized each resistor had a certain amount of resistance and could be added up. When he said "*a material that has varying density*," David viewed the resistivity as a property of the material by making an analogy with the density. I construct the diagram (Figure 6.17) describes how students applied their mathematics knowledge in this physical situation to set up an integral.

The blending diagram (Figure 6.17) includes four input mental spaces: symbolic space, math notion space, physical world space, and physics concept space. In the upper left of the blending diagram, the symbolic space contains the symbol of resistivity function, the basic resistance equation which relates several physical quantities, and the integral template $\int d[]$; the math notion space involves students' knowledge about function, integral as a sum, and differential as representing a small quantity. In the upper right, the physical world space is the physical scenario containing a cylindrical resistor; the physics concept space contains the concept of resistivity as a property of a material and the notion that resistance can be added up. The four input spaces were integrated to create the blend. The cylindrical geometry, the resistivity concept, as well as the symbolic and mathematics concept of function were all associated together when David talked about "*resistivity changes as the length increases*" and "*it is different here than it is here*." When David visualized the change of resistivity as the length x changing in the physical world space, it also cued him to "*separate it into small cylinders*" and apply integration as integral representation a "sum."

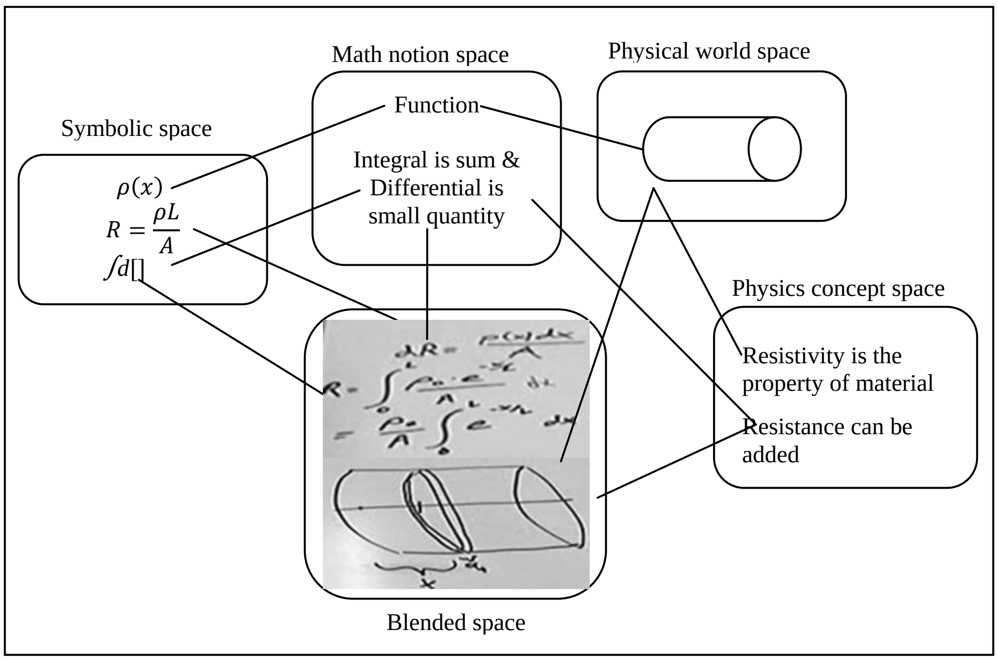


Figure 6.17 Blend under 'Chopping and Adding' frame.

From the students' discussion and written work, I speculate that they might have made the following connections across spaces. First, they seemed to associate the integral template $\int d[]$ in the symbolic math space with the mathematical notion of integral as a sum of small amounts of quantities contained in the notation *d*[]. They also seemed to relate to the idea that resistance could be added up in the physics concept space. In the blended space, the students first constructed the expression of a small amount of resistance from the basic resistance equation; the notion of integral as a sum of small quantities was mapped into the blended space to construct an integral. Hence, the organizing frame involved chopping the cylinder into pieces and adding the resistance due to each piece. This frame comes from the combination of all four input spaces. Using this organizing frame, this group of students then found the resistance of each small resistor *dR* and integrated *dR* to find the total resistance.

Summary of the Four Blends

Our four blends are identified based on our observations of student work. As shown in Table 6.1, those four blends involve different input mental spaces as well as different organizing frames. Organizing frames are the emergent structures that describe the way in which students organize the knowledge elements in the blended space.

Table 6.1 Description of the four blends.

Organizing frame	Integral-Sum Across	Differential Algorithm	Equation Mapping	Chopping & Adding
Input spaces	symbolic and math notion	symbolic and math notion	symbolic, math notion, and physical world	symbolic, math notion, physical world, and physics concept
Use of representations	primarily based on algebraic	primarily based on algebraic	algebraic and narrative (develop a story about the physical situation)	algebraic, narrative, pictorial, and heavy use of gesturing

The first two types of blends are similar: both the ‘Integral-Sum Across’ and the ‘Differential Algorithm’ frames only involve the symbolic space and the math notion space. They are primarily based on mathematical manipulations, leading students to set up integrals incorrectly. As students were constructing those two kinds of blends, they relied heavily on algebraic manipulations. The third blend contains an ‘Equation Mapping’ frame, indicating a mapping of the physical quantities onto the structure of the resistance equation. This blend involves the physical world space of a cylinder but does not include the physical concept of resistivity and resistance. Following this blend, the students set up an integral expression which was mathematically correct but was based on an incorrect physical principle i.e. the resistivity at different points could not be added up. In the fourth blend, students recruited the chopping-adding frame which comes from the integration of several input mental spaces, meaning chopping the cylinder into small pieces and adding the resistance due to each small piece. This is also the type of blend that hypothetical experts would construct.

6.4 Summary

In this chapter, I have illustrated how the conceptual blending framework can be used to analyze students' application of the integration concept in physics problem solving. From a large set of interview data, I analyzed data from five groups of students working on a problem situation and identified four different kinds of blends that emerged from the data as students constructed solutions to the problem. The four blends differ by not only the input spaces but also the organizing frames. I described the four major organizing frames: Integral-Sum Across, Differential Algorithm, Equation Mapping, and Chopping & Adding frames.

The first two types of blends only contain two mental spaces: the symbolic space and mathematical concept space. Students did not relate the mathematical concepts to this specific physical scenario and mapped their mathematics knowledge to physics concepts. They share some similarities with the Recursive Plug-and-Chug epistemic game identified by Tuminaro and Redish (2007) as students worked on algebra-based physics problems. However, this framework explains more about students' thinking about math and physics concepts than epistemic games do. For example, recruiting the integral sum across the frame demonstrates students' lack of understanding or ignorance of the structure of integral concept. In the integral form $\int f(x)dx$, the multiplication of $f(x)dx$ represents the infinitesimal quantity and an integral represents the summation of this infinitesimal quantity. In other words, it seems that the structure of the Riemann sum (Thompson & Silverman, 2008) was not very well understood by students who recruited the 'Integral-Sum Across' frame in this problem context. When recruiting the 'Differential Algorithm' frame, students' solution was primarily based on the mathematical operation of taking a derivative. However, instead of deriving a correct expression from which to take the derivative, they substituted the resistivity function into the basic resistance equation. One could argue that students might not know how to relate the conceptual aspect (i.e. an infinitesimal amount of quantity) and the operational aspect of the differential concept. It is also possible that students perceived the application of mathematics in physics as performing a mathematical operation. The latter is related to students' epistemological framing about using mathematics in physics (Tuminaro, 2004; Bing & Redish, 2009).

The third type of blend mainly contains three mental spaces and an organizing frame of 'Equation Mapping.' Although this organizing frame was predominantly from the symbolic space, students also took into account the physical world. In the blended space, they constructed the notion of "adding up the resistivities" but they did not seem to be making any connection to the physics concepts in order to be able to distinguish between adding resistivities and adding resistances. Hence, students eventually set up an expression which made sense for them but the physics underlying the expression was incorrect.

The last type of blend contains four input mental spaces and the organizing frame of 'Chopping & Adding.' The organizing frame was based on the integration of knowledge elements from the four input spaces: physical world, physics concept, symbol, and math notion space. More specifically, it involves integrating the visualization of the resistivity function and the notion of "separating into small disks" in the physical world space. It also involves associating the integral template $\int d[]$ and the mathematical notion of integral as a sum of small amounts of quantities along with the idea that resistance can be added up in the physics concept space.

I also observed that the representations that students chose were different when constructing different types of blends. During the first two blends, students tended to focus on algebraic expressions and mathematical substitutions. When constructing the last type of blend, students first developed a story about the physical situation and frequently used gestures to facilitate their sense-making. They also drew pictures and related these pictures to the mathematical symbols or equations they set up.

There is a contrastive comparison between the first two types of blends and the last two types of blends. The way that students blended their knowledge in mathematics and physics might be related to how they anticipated or perceived the application of mathematics in physics should be, for example using mathematics in physics as performing a mathematical operation or sense-making in a physics context. In other words, students' framing about their problem solving activity could affect their problem solving strategies. In the following chapter, I will introduce the perspective of framing and how the use of hypothetical debate problem can change students' framing of applying mathematics in physics.

Modeling student responses with the conceptual blending framework allows us to conduct a fine grained analysis of student reasoning in our problem scenario. Here I consider four mental spaces: symbol, mathematical notion, physical world, and physical concept spaces. This classification of the four mental spaces is based on characteristics of our problems as well as students' responses. I do not claim that this is the only way to classify students' mental spaces, however, it does help us to create a representation to describe student reasoning.

6.5 Discussion about Conceptual Blending Framework

According to Fauconnier and Turner's cognitive blending theory, learners must find a way to bridge or blend their knowledge from different mental spaces to make sense of new information. Using mathematics in physics requires the blending of mathematics knowledge and physical situations. In this study, we use the tool of conceptual blending to analyze how novice students bridge their knowledge in two distinct worlds - calculus and physics. The unproductive ways in which novice students blended mathematics and physics knowledge hampered their transfer of learning from mathematics to physics. The cognitive blending framework offers a powerful perspective for understanding students' mathematizing in physics from the following two aspects: what information that students perceive as relevant (i.e., what knowledge elements students select in a mental space and what they ignore) when constructing a solution and how they interpret their solution (i.e., what new meanings are created).

However, there are also constraints of using this framework in our data analysis, which pertain mainly to the blending diagram. As a representation, the blending diagram is difficult to create and interpret. A blending diagram consists of mental spaces often represented by circles and linking (i.e., mappings) across mental spaces often represented by lines. In linguistics phenomenon, it is often not difficult to identify the mental spaces and the linking across spaces. In the example of "This surgeon is a butcher", it is clear that the two input mental spaces are "surgeon" and "butcher" spaces (Fauconnier & Turner, 1998b). The elements (i.e., characteristics) of each space are also easy to identify. However, in the context of solving physics problems which involve symbols, mathematical concepts, physics concepts, and physical objects, how to decide the input mental spaces is often less clear. It is quite challenging to construct a blending diagram which clearly explains students' reasoning and also easy enough to comprehend. Furthermore, different researchers may construct different diagrams depending upon their interpretation of student work.

Another difficulty of this representation is how to create the links across mental spaces. The connections between spaces rely on clear evidence as well as inferences of student reasoning. When there are more than two input mental spaces, the blending diagram contains too many lines indicating the connections across different spaces. The whole diagram is too complex and difficult to create and comprehend. One possible way is to create a dynamic diagram which is composed of many small blends following the sequence of student problem solving. In spite of these limitations, given the complexity of student reasoning and problem solving in physics, the blending diagram is still a powerful representation.

In conclusion, this conceptual blending framework is powerful in explaining how students apply mathematics in physics situations, in particular how they make connections between different domains. Describing student reasoning using the conceptual blending framework provides us an alternative and perhaps more realistic way to interpret students' difficulty with physics problem solving in terms of the inappropriate ways in which students blended their knowledge in mathematics and physics which hampered their use of mathematics in physical situations. However, more work and discussion about the blending diagram are necessary in the future in order to make it as a more powerful representation.

Chapter 7 - Shifting Students' Epistemological Framing Using Hypothetical Debate Problems

7.1 Introduction

In Chapter 5 we discussed the productive and unproductive resources that students activated associated with the differential and integral templates. In Chapter 6 we used the language of conceptual blending to analyze four different types of blends that students create for setting up integrals in physics problems using the language of conceptual blending. From our discussion the blends that students created, the types of blends are also strongly related to their activation of resources. Understanding students' thinking has important implications for instruction. From the resources perspective, as instructors we hope to help students to activate more productive resources in a context. In addition to investigating students' reasoning resources, this study also explores strategies that might improve students' activation of appropriate resources as they solve of problems that require integration in physics. Based on previous research about resources activation, whether or not a resource is activated depends on the individual's real-time perception or judgment (i.e., framing) about the situation (Hammer et al., 2004). Framing is also called the control structure, determining which resources are selected for activation and which ones are inhibited (Bing & Redish, 2009).

In this chapter we discuss how the use of hypothetical debate problems affects students' framing about using mathematics in physics problem solving. We investigate students' physics problem solving from the perspective of epistemological framing. Epistemological framing describes an individual's perceptions about what knowledge tools or skills are appropriate to use in a learning situation (Bing & Redish, 2009). Novice students often perceive the use of mathematics in physics very differently from expert physicists (Redish, 2005) and their expectations or perceptions can play a powerful role in their problem solving in physics (Hammer, 1996a; Redish, 2004). Students' expectations about what they are supposed to do in physics problem solving could strongly affect or even control their selection of knowledge resources in a situation.

Hypothetical debate problems require students to compare discrepant problem solving strategies presented by hypothetical problem solvers and discuss why they agree or disagree with each problem solving strategy. The hypothetical solution strategies provided in the debate problem are developed from common students' approaches to solving the problem that are uncovered through our research. They are designed to help students develop their understanding by honing student awareness of their own as well as other students' ideas as they attempt to explain discrepant strategies. We expect that when working on debate problems students would more likely frame problem solving in physics as making sense of concepts rather than manipulating symbols and equations.

In this chapter we aim to answer the third research question: How does a hypothetical debate problem affect students' framing about using mathematics in physics? We start with an overview of previous research on framing and then we specifically discuss students' epistemological framing related to physics problem solving requiring the use of mathematics. Then we discuss the rationale of our instructional strategy – the development of hypothetical debate problems. Next we present our analysis and findings of students' framing for both conventional physics problems and hypothetical debate problems. Finally we discuss the significance of this research on physics learning and instruction. The main findings are from a recently submitted article (Hu & Rebello, 2013d).

7.2 Epistemological Framing

According to linguist Tannen (1993), a frame describes an individual's expectations (i.e., What is it that's going on here?) about a situation and determines how one thinks and acts in a certain situation. Framing emphasizes the individual's interpretation of the situation as an ongoing process (MacLachlan & Reid, 1994). As per the resources perspective, framing is a cognitive process, which involves the activation of a locally coherent set of cognitive resources. Consider a situation in a lecture hall where a speech is going on: When you enter a lecture hall, you might recall your past experiences of listening to a lecture and think of "sitting quietly," "listening carefully," and "taking notes" as appropriate actions to make. In this example, your framing about this situation is based on the activation of a set of resources from your past experiences while in the meantime your framing determines how you react to the current situation. However, if the speaker stops lecturing and encourages the audience to discuss as a group instead, your original framing is challenged and not appropriate for this situation. You probably feel nervous and uncomfortable when the situation challenges your expectations. Alternatively, you might activate a different set of resources related to your past experience of attending seminars or discussion groups. You might reframe this situation as “making new friends” and “exchanging ideas with others.” Hence, framing is a dynamic process, a part of an individual’s cognition in response to an external environment, such as the speaker’s tone, a question from the audience, or even a spontaneous idea coming to one’s mind. An individual will reframe his or her own activity upon receiving new information.

Framing has many components including social (“Who am I interacting with?”), affective (“How do I expect to feel about it?”), epistemological (“What do I expect to use to answer this question?”), and others (Redish, 2004; Hammer, Elby, Scherr, & Redish, 2005). Epistemological framing describes one’s perceptions or judgments about what knowledge tools could be used in a learning situation. Students’ expectations about what they are supposed to do play an important role in deciding what they should pay attention to and what knowledge resources they will use for creating new knowledge. In a physics class, a student might frame learning physics as “remembering the knowledge from an authority.” The instructor might expect students to construct new knowledge by connecting physics concepts to their everyday experiences. Hence, students activate a different set of epistemological resources than the teacher does. Communication problems can arise when students and the teacher frame a situation differently during an interaction.

When solving physics problems, a student may frame a physics problem as an opportunity for making sense of the concepts or an occasion for manipulating formulas and numbers (Bendixen & Feucht, 2010). Students’ epistemological framing affect the way in which they apply mathematics and interpret mathematics in physics. Tuminaro (2004) observed student groups solving problems in an algebra-based physics course and identified three common frames in which students solving physics problems required the use of mathematics known as: rote equation chasing, qualitative sense-making, and quantitative sense-making. The rote equation chasing frame is when students expect problem solving in physics to involve plugging quantities into a memorized physics equation. The qualitative sense-making frame is when students expect physics problem solving to include the application of physics principles or common sense, but they are devoid of formal mathematical procedures or equations. Finally, in the quantitative sense-making frame, students expect that the solution requires formal mathematics integrated with physical sense making. The three common frames identified by Tuminaro (2004) provide a basis for comparing student behavior in conventional and debate problems.

Bing and Redish (2009) analyzed and classified students' framing about using math in physics by classifying the reasons or warrants students provide to support their points of view. The warrant analysis could be a useful tool for finding evidence of students' epistemological framing, which is typically implicit in their work. For instance, in "calculation" framing, the student relies on mathematical computations and the unspoken warrant might be that the student would accept a result by carefully following a set of computational steps. In "physical mapping" framing, students would make a claim or accept a result based on the evidence that mathematical calculations are consistent with a physical mechanism or their common sense about the physical scenario. The warrants from student work are associated with the epistemological resource activated at that moment, such as "a mathematical symbolic representation faithfully characterizes some feature of the physical or geometric system it is intended to represent (Bing & Redish, 2009)." The researchers can get information about the tacit epistemological resources that students activate or the epistemological framing processes by identifying the warrants involved in their work.

This study is concerned with students' epistemological framing about physics problem solving. The classification and warrant analysis about students' framing provides us a thorough understanding of students' judgments regarding physics problem solving beyond the scope of conceptual learning. As instructors, we are interested in helping students take more productive approaches to learning and frame physics problem solving as sense-making rather than rote use of formulas. Previous research has suggested instructional strategies to facilitate the shifts of student framing, such as manipulating the wording and representation of the problems, changing the classroom setting, and modifying the structure of the lecture (Hammer et al., 2005). In this study, we introduce a non-traditional type of problem – a hypothetical debate problem – and examine students' framing in both a conventional problem and a hypothetical debate problem.

7.3 Rationale for Hypothetical Debate Problems

The hypothetical debate problem (HDP) is a non-traditional type of problem which differs from the conventional problems (CP) we encounter in physics. A conventional physics problem typically involves a physical scenario and a question at the end. To solve a conventional problem, students need to find an answer to the question based on given information in the problem scenario. A major difference between a conventional problem and a hypothetical debate problem (HDP) is that a HDP typically requires students to compare several problem solving strategies or types of reasoning which are often contradictory instead of asking students to create a solution by themselves. Those problem solving strategies provided are often from common students' answers. The structure of a HDP creates an imaginary scenario of "debating with peers", through which students' own knowledge structures are challenged and reevaluated.

The development of a HDP is based on the pedagogical strategies of cognitive conflict and contrasting cases as well as empirical evidence from our earlier studies. Cognitive conflict is derived from a Piagetian view of learning in which the learners' active part in reorganizing their knowledge is central to learning (Piaget, 1952). Instructional approaches based on cognitive conflict involve promoting situations where the students' existing ideas about some phenomenon are made explicit and are then directly challenged in order to create a state of cognitive conflict (Scott, Asoko, & Driver, 1992).

When solving physics problems, students often tend to focus on the surface features of the problems and develop an overly superficial understanding of the concepts, limiting their ability to transfer to new situations (Larkin, McDermott, Simon, & Simon, 1980). Perceptual learning is the process of learning the skills of perception (i.e., noticing and differentiating) and it emphasizes the role of attention in people's learning process (Gibson, 1969). Based on the theory of perceptual learning contrasting cases are important to guide people to notice and differentiate knowledge and experiences by affecting what one notices about subsequent events and how one interprets them (Garner, 1974; Bransford & Schwartz, 1999). Garner (1974) provided an example to illustrate the role of contrasting cases in affecting noticing. First, he asked readers to look at a figure (i.e., standard figure in Figure 7.1) and then asked them to describe the figure. Most people described it as a grid with letters and some described it as a set of lines. Then the same figure was shown alongside another figure (Figure 7.2). The new features such as size, shape, and symmetry became salient and relevant to the readers. Hence, by comparing and contrasting these two figures they allowed readers to attend to distinguishing features and filter out irrelevant features (Gibson, 1969). Schwartz and Bransford (1999) also pointed out that analyzing contrasting cases could help learners generate the differentiated knowledge structure that enabled them to understand a concept or a phenomenon deeply.

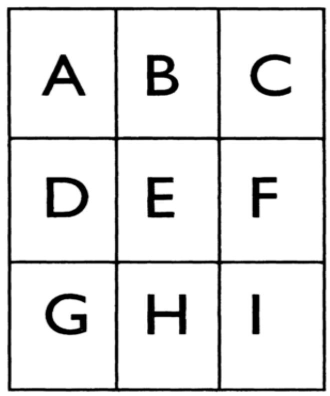


Figure 7.1 Standard figure.

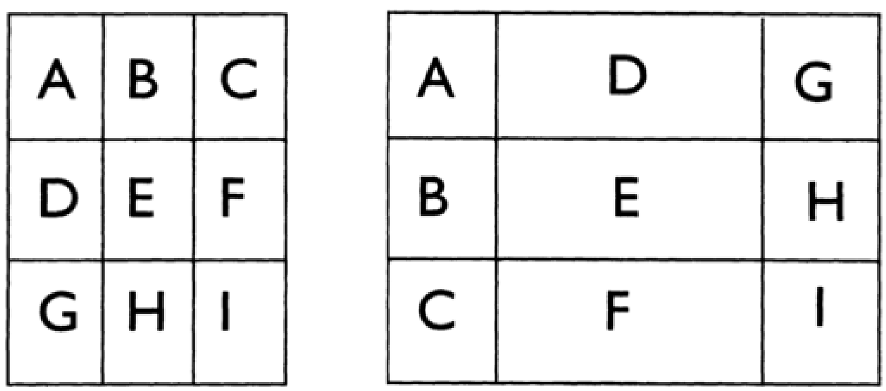


Figure 7.2 Standard figure in the context of another figure.

In summary, the instructional strategies of cognitive conflict and contrasting cases can serve two functions. First, by comparing and differentiating discrepant cases (reasoning or strategies), learners are cued to notice the deep structure that distinguishes the cases rather than the superficial features. Second, when confronting a different strategy, learners' own knowledge structures are challenged and reorganized to resolve the conflict. Compared to conventional problems, hypothetical debate problems can potentially shift students' attention away from pattern matching from existing examples.

Problem tasks similar to HDPs have been used effectively in previous studies for helping students construct conceptual understanding (McDermott & Shaffer, 2002; Nguyen, Gire, & Rebello, 2010), as well as investigating students' deep understanding of math and physics concepts (Von Korff & Rebello, 2012). The purpose of our HDPs in our study is to facilitate students to frame the use of mathematics in physics problems as sense-making instead of the rote use of equations. To facilitate this change of frame, we present students with several problem solving strategies, some of which are based on commonly used albeit unproductive student strategies based on equation manipulations and others based on more sophisticated, productive strategies based on conceptual understanding of the mathematics and physics. We aim to steer students' attention away from strategies that use mathematical manipulations to those that use the deep structure of mathematics and physics concepts.

7.4 Analysis of Students' Framing

Our analysis for this study concentrates on eight interview segments, a total of 150 minutes of video data. On average, students spent around 20 minutes on each problem. In this paper, we show analysis of student work on two problems based on the same problem context which was given in the first and second interview sessions. In the first interview, students solved a conventional problem (Figure 4.2). We analyzed students' solutions for all groups given this problem to create four hypothetical strategies (A through D) for the debate problem in Figure 4.3. There are two main reasons for selecting this problem for analysis. First, it was a relatively easy physical scenario and the problem is not based on rote memorization. Thus, less help was needed from the interviewer regarding the physics concepts so that we could neglect the effect of students-interviewer interaction on their framing. Second, this problem was not similar to problems that students had just encountered in their class so it required them to construct their own solutions rather than directly apply formulas from memory.

We watched the video data to look for evidence of how students frame physics problem solving and identify the moments when there was a shift in students' framing. Students' framing can shift from moment to moment and in this analysis it varies from several seconds to several minutes. Our data analysis was primarily based on the three kinds of frames identified by Tuminaro (2004). The correctness or incorrectness of student solution was not the focus of this study. We also recognize the work of Bing and Redish (2009) and summarize the evidence which indicates students' framing about mathematics use in our problem solving activity.

In the rote equation chasing frame students perceive physics problem solving as plugging quantities into a memorized physics equation. This frame shares some similarity with the "calculation" frame in Bing and Redish's work, in which students rely on mathematical computations. Bing and Redish identified some common indicators of students' "calculation" frame, such as "correctly following algorithmic steps gives trustable result" and "focus on technical correctness." In our data analysis we identified some common student behaviors as an indication of a "rote equation chasing" frame, including "focus on algorithmic steps," "plug physical quantities into a memorized equation," and "map physical quantities to the solution of a memorized example." Bing and Redish also identified the "physical mapping" frame, meaning that students would make a claim or accept a result based on the evidence that mathematical calculations are consistent with a physical mechanism or their common sense about the physical scenario. This is similar to the "quantitative sense-making" frame in Tuminaro's work, in which students expect that the solution requires formal mathematics integrated with physical sense making. According to the main features of a "quantitative sense-making" frame, we identified some common students behaviors, including "map a physical situation to a mathematical equation" and "interpret mathematical symbols or equations in a physical scenario." In the "qualitative sense-making" frame students expect physics problem solving to include the application of physics principles or common sense without the use of formal mathematical equations. Hence students' behaviors include "use qualitative analysis" and "draw diagrams or use words to explain physical mechanism."

Table 7.1 Three frames and their indicators.

Frame	**Evidence of Frame**
Rote-Equation Chasing	Focus on algorithmic steps; Plug physical quantities into a memorized equation; Map physical quantities to the solution of a memorized example
Quantitative Sense-making	Map a physical situation to a mathematical equation; Interpret mathematical symbols or equations in a physical scenario; Sometimes aided by a diagram or gesture
Qualitative Sense-making	Use qualitative analysis without formal mathematics; Often draw diagrams or use words to explain physical mechanism

As students were solving the problem, the interviewer interacted with students on very few occasions and only to clarify the questions for the students. Since the interviewer did not significantly interact with the students, our analysis does not include the time when they were interacting with the interviewer. We also excluded data when it was not possible to identify students' framing, including students reading the problem statement, asking clarification questions, and talking about something irrelevant. We randomly selected two episodes of about 10 minutes each to check the coding reliability and the inter-rater reliability which is about 85%. We aggregated the amount of time that students spent on each frame and then compared the percentage of time for each frame on both the conventional problem in interview one and the hypothetical debate problem in interview two.

7.5 Results

We found that students' framing about solving a physics problem is different on the conventional physics problem; overall students spent significantly less time on rote equation chasing when working on the hypothetical debate problem. Among the five groups there were two groups of students who spent less than 30% of their time on the rote equation chasing frame while solving the conventional problem. During this time, students were mainly focused on doing calculations and reorganizing equations, which are also a necessary part of solving physics problem requiring mathematics. Their entire problem solving activity involved frequent use of diagrams, graphs, gestures, and verbal explanations. In other words, their frames about physics problem solving are very productive and similar to those of experts. In one of the other groups, one student figured out how to solve the problem in about two minutes and explained his strategy to the rest of the group who seemed to agree with him. Thus, the time for student problem solving activity was too short for us to identify the shift in their framing.

In this study we were interested in students who were involved in a less productive (i.e., rote equation chasing) frame when solving a conventional problem. Our hypothesis was that the use of a hypothetical debate problem might help these students, who tended to favor rote equation chasing, to shift their frame to a more productive sense-making frame. To demonstrate how the framing of these students shifted as they moved from a conventional problem to a hypothetical debate problem, we present the results for the two groups of students who were primarily involved in a rote equation chasing frame on the conventional physics problem and provide a detailed discussion about their framing for both types of problems to provide evidence for frame shifting. We aggregated the amount of time that both groups of students were involved in each frame and compared the percentage of time for each type of framing as shown in Figure 7.3.

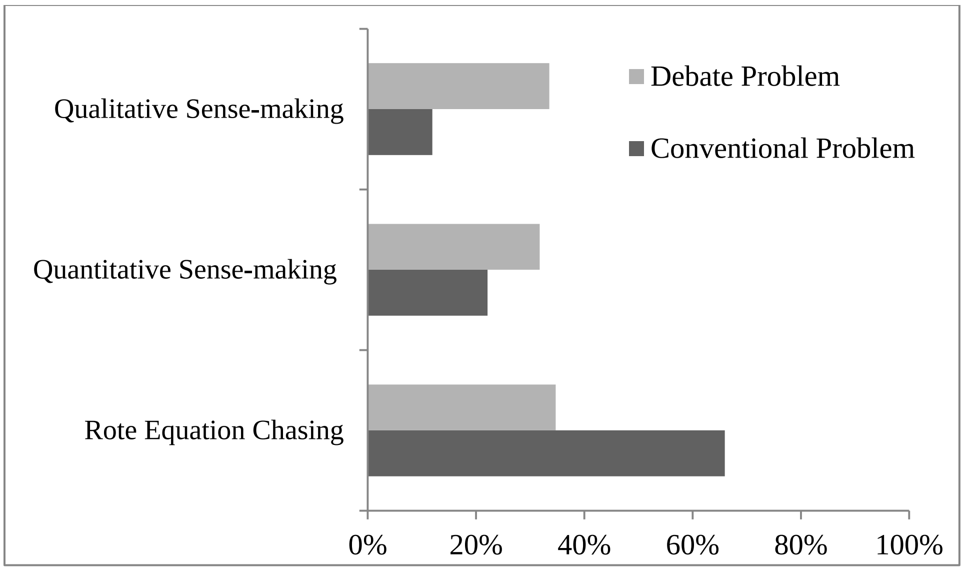


Figure 7.3 Percentage of time students were involved in each frame.

When working on the conventional problem, students perceived the use of mathematics in physics as rote equation chasing about 66% of the time during their problem solving activity, compared with only 20% or less time on qualitative/quantitative sense-making. We found that students often tried different mathematical equations based on their memorization and plugged physical quantities into the equations to get an answer without considering whether the equations were appropriate in this context. We selected an episode from one group of students to illustrate how students typically do as they solve the conventional physics problem. The following conversation occurred at the beginning of this problem solving activity.

Alan: Can't we just say if αr equals our charge density and our charge density equals Q over A? Can we say αr equals Q/A? Just multiply this (charge density αr) by A, and we get Q - our charge.

Zad: Well, write that out.

Then Alan started to write down equations as shown in Figure 7.4.

Alan: Because this (αr) equals our charge density given by that, you just multiply A and then you get a charge.

Zad: Okay. And then take the integral of that, over the area, from 0 to 2π.

Alan: So the area would be, area of our circle is 2π…

Phil: $2\pi r^2$.

Alan then replaced A by $2\pi r^2$ *in the second equation of* Figure 7.4.

Alan: Right?

Phil: Uh huh.

Alan: So we want to integrate that for r? From 0 to R?

Zad: We have several equations where we have charge density, substitute that in, um, charge density, charge over area. We have defined the full circle of the radius, the charge, the derivative of the charge, so we find it out at a certain point, times the area, so we integrate it, uh, over an area, so we integrate it.

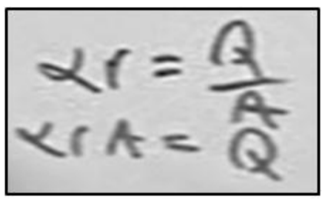


Figure 7.4 Equations set up by students in the conventional problem.

In this episode all three students were primarily focused on how to find the expression for total charge through manipulating the physical quantities and mathematical equations. At the beginning, Alan proposed an equation which related the quantities charge density and total charge in this problem. He seemed to have used the relationship that the charge density is the total charge divided by area. Without considering whether or not this equation was appropriate to use in this context, or what each quantity meant in this context, he plugged the charge density function (σ(r)=αr) into this equation and manipulated this equation to find the charge Q as shown in Figure 7.4.

Alan frequently used the words "equals" and "multiply" to explain his work while he was manipulating mathematical equations. Zad proposed the use of integration and he also started out with mathematical rules such as "equations," "substitute in," "derivative of," and "integrate."

Later, Zad set up another intergral $\int_0^{2\pi} \frac{\sigma}{2\pi r^2}$ although he was not sure whether it was right or wrong. Overall, the group focused primarily on the mathematical operations and did not pay attention to the physical meaning of quantities such as charge density. They seldom talked about how to map the physical situation onto the mathematical equations or to interpret the meaning of mathematical equations. Hence students were mainly involved in a rote equation chasing frame in this episode.

Comparing students' framing on conventional and debate problems in Figure 7.3. clearly shows that while working on a debate problem, the percentage of time that students spent on rote equation chasing decreased from 66% to 35% and their framing shifted to either qualitative or quantitative sense-making. In Figure 7.3, we also observe a tendency that students' framing shifted from rote equation chasing to qualitative sense-making. The percentage of qualitative sense-making is much higher while solving a hypothetical debate problem (36% vs. 12%). When working on a debate problem, they often focused on comparing one strategy with another, discussing how those strategies were different, and which strategy "makes more sense." We also selected an episode from the same group of students as they worked on the hypothetical debate problem.

Zad: The difference between [students] A and B is A is taking area times the charge of each…

Phil: Each ring?

Zad: I think I was wrong.

Alan: So we need to do some dA, not just A.

*Zad: No, you don't have to do the derivative of A, because you can do that as a function. Yeah, cause you can do that as a function of this. Because we... I… drew this circle (*Figure *7.5), this is R, when you find it, when you're finding this little area, (shaded the area of the small circle as shown in* Figure *7.5).*

Alan: A small section would be dA (pointed to the small circle in Figure *7.5).*

Zad: The radius is R, you are not times the radius of the whole (drew the big circle in Figure *7.5), you times the radius of that area (shaded the area of the small circle again). Because that wouldn't make sense to find the charge of this (pointed to the small circle) and times the area of this all (shaded the area of the ring between two circles).*

Alan: So you are looking at some r, some small radius, times some small area dA.

Zad: Yeah.

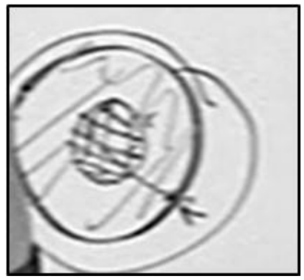


Figure 7.5 Picture drawn by students in the hypothetical debate problem.

In the above episode Zad drew a picture (Figure 7.5) about the physical situation to help them interpret the physical meaning of the equations as he was explaining how strategies used by hypothetical students A and B were different. Instead of focusing on how to find the expression for total charge Q, the students in the group were trying to understand how the strategy of hypothetical student B was different from the strategy of hypothetical student A. We observed that as students were working on the debate problem, they tended to draw diagrams or use verbal explanations instead of drawing from their memorization of mathematical equations or solutions from past experiences. They used pictorial representations to illustrate what the equations represented in each strategy. In their conversation they used “radius of the whole” and “radius of that area” to differentiate symbols R and r; they also used “little area” and “area of this all” to interpret the meaning of symbol A. Their explanations were accompanied by embodied gestures such as drawing a circle or shading an area. We identified that students were mainly involved in a qualitative sense-making frame. The evidence is that they were using a qualitative approach and tools such as a diagram rather than formal mathematical equations in this problem solving activity. In our analysis we do not concern ourselves with the correctness or incorrectness of student reasoning as we only focus on their framing. Occasionally students also talked about mathematical algorithms, such as “do the derivative” and “a function” which is an indication of a rote equation chasing frame.

Overall, we found that both groups spent the majority of time on a rote equation chasing frame while working on the conventional problem. They all seemed to expect solving physics problems as a direct application of formulas or “doing math.” However, when the problem was presented in a different format, instead of finding an "answer" they were asked to "debate with fictitious peers" so their framing shifted from rote equation chasing to sense-making on the hypothetical debate problem.

In discussing students in qualitative or quantitative sense-making, we do not claim that one is better than the other. Problem solving mainly involves the correct use of mathematics (i.e., the mathematical concept of integral) in a physical situation and using mathematical equations is a necessary part of this problem solving activity. Working on the hypothetical debate problem seemed to make students move their attention away from the superficial features of equations (e.g., whether or not a mathematical equation is simplified) and motivated students to explore the underlying meaning of the mathematical equations. We also found that students spent about 36% of the total time on qualitative sense-making. Given that this problem cannot be solved using a purely qualitative approach, with no formal mathematics, we should not expect that percentage to be very high.

7.6 Summary

In this chapter we have analyzed students' physics problem solving from the point of view of framing. We extend Tuminaro's (2004) work on students' framing in algebra-based physics and apply his classification in calculus-based physics. Students' framings were investigated in two different problem types: the conventional free response problem and the hypothetical debate problem. Though hypothetical debate problems have been used extensively in physics and science education in general, very few studies have carefully looked at the effectiveness of this new type of problem as an instructional tool. Our data show that some students in our case study tend to frame their activity as rote equation chasing when solving the conventional problem. However, when working on the debate problem, they were more likely to perceive using math in physics as quantitative or qualitative sense-making. This work provides evidence for using debate problems as an instructional strategy to teach problem solving from the framing perspective.

We do not claim that the way students frame the use of math in physics necessarily relates to the correctness of their claims or responses. However, strategies that help shift students' framing to sense-making could be potentially useful for developing more expert-like problem solving strategies. As students work on the debate problems, they seem to focus more on the deep structure of mathematical equations and trying to understand what they mean in a context rather than executing a mathematical procedure.

This study explores the use of hypothetical debate problems as an instructional tool to shape the way in which students frame mathematics use in physics problem solving. Framing might also play a critical role in students using math in non-mathematical contexts in general. The research reported in this study shows that hypothetical debate problems seem to shift students' frame from rote equation chasing to qualitative or quantitative sense-making. Thus, the use of hypothetical debate problems can potentially be a beneficial strategy in facilitating students in general to engage in sense-making during problem solving.

Chapter 8 - Conclusion and Discussion

8.1 Overview of this Dissertation

The main purpose of this dissertation research was to investigate how students set up mathematical integrals when solving physics problems in the context of electricity and magnetism. It is commonly observed that while students have a reasonably good "technical" mastery of calculus (e.g., take a derivative or perform an integral), they often experience tremendous difficulties when making the connections between mathematics and physics. In physics problems involving non-constant quantities, they struggle to set up the relevant integral. We pieced together three theoretical constructs to understand students' reasoning underneath their problem solving performance. I attempted to answer three research questions: (1) *How do students reason about calculus concepts as they set up integrals in physics contexts?*; (2) *How do students blend their knowledge in calculus and physics to set up integrals in physics?*; and (3) *How does the hypothetical debate problem affect students' framing of physics problem solving requiring mathematics?*

To answer the three research questions, we observed student work as they solve physics integrations problems in group teaching/learning interviews. Then we extended and amalgamated three frameworks to conduct a careful and systematic analysis of student thinking. We analyzed in detail students' deep reasoning of mathematical concepts by categorizing the types of symbolic forms associated with mathematical templates of differentials and integrals. We also conducted an analysis of the conceptual metaphors that involved in students' use of each type of symbolic forms to provide us more insights into how students' conceptual schemas of mathematical symbolism are formed through their everyday interaction with the physical world. In addition to investigating students' deep thinking, we brought the perspective of conceptual blending to understand the overall strategies that students employed when applying mathematical integrals in physics contexts. We described how students combined their knowledge in mathematics and physics to set up integrals. Finally, we explored the role of hypothetical debate problem as an instructional strategy in shifting students' framing about solving physics problems requiring mathematics. In the following three sections, we will discuss how we specifically addressed each research question.

8.2 Research Question One

To understand student thinking of differential and integral concepts as they set up integrals in physics, we conducted a qualitative analysis of student work. We introduced the resources framework and conceptual metaphors theory in Chapter 3. Then in Chapter 5 we identified the mathematical resources (i.e., symbolic forms) and conceptual metaphors as students reason about mathematical concepts in the context of physics problems. Symbolic forms describe students' intuitive thinking of mathematical formalism. Sherin (2001) first introduced this framework and identified a list of symbolic forms (e.g., 'parts-of-a-whole', 'proportionality') that involved in algebra-based physics students' mathematical thinking in physics. We categorized the types of symbolic forms and conceptual metaphors emerged from student thinking, which have not been identified previously.

<u>Symbolic forms associated with the differential template *d*[]:</u> The 'small amount' resource refers to a small portion of a physical object. This resource is used across many physics contexts, such as finding electric field due to a charge distribution. The 'point' resource may contain two distinct meanings based on the local physics context, though students seemed to be unaware of this distinction. We found that students tend to activate 'points as locations in space' when using spatial differential terms such as dx, dA, and dV. In differential terms other than spatial terms, they seemed to activate 'points as point quantities' (i.e., quantities contained by points with negligible physical size). The 'differentiation' resource involves associating differentials with the mathematical action of taking the derivative of a function. The 'variable of integration' resource involves the interpretation of symbol "*d*" as an abstract entity, which is often followed by a variable.

<u>Symbolic forms associated with the integral template $\int_{[]}^{[]}[]d[]$</u>: The 'adding up small amounts' symbolic form represents adding up small amounts of physical quantities. The 'adding up one quantity over' symbolic form contains the meaning of adding up a quantity (typically the integrand) over another quantity. Typically, the quantity in first box of "[]*d*[]" is the quantity being added up and the quantity in the second box is the variable to be integrated over. The 'function matching' symbolic form involves the thinking of integral template as taking the integral to get the original function.

Conceptual metaphors involved in student use of symbolic forms: An 'object' metaphor describes the mapping from complex mathematical notions to the experiential knowledge of concrete physical objects. This metaphor is often involved with the use of the 'd[] as a small amount' and '$\int_{[]}^{[]}[]d[]$ as adding up small amounts' symbolic forms. When activating those two mathematical resources, students conceptualized the abstract mathematical notions of differentials and integrals as physical objects. The 'location' metaphor involves the mapping of mathematical differentials as spatial locations, which are grounded in their interactions with the physical world. This metaphor is used in association with students' activation of the 'spatial differential terms as points' symbolic form. In the 'machine' metaphor, students treated the mathematical symbolic templates differential and integral as a machine which performs a function or algorithm. This metaphor was identified when students activated the 'd[] as differentiation' and '$\int_{[]}^{[]}[]d[]$ as function matching' symbolic forms. In the 'motion along a path' metaphor, students view the use of differentials as a traveler "going from one point to another" when doing integration. In the 'motion along a path' metaphor, students viewed the variable of differential as a traveler "going from one point to another" when doing integration. This metaphor is used when students activated the 'd[] as variable of integration' and '$\int_{[]}^{[]}[]d[]$ as adding up one quantity over' symbolic forms.

8.3 Research Question Two

In Chapter 6, we illustrated how students blended their knowledge from calculus and physics to set up integrals in physics from the perspective of conceptual blending. Using the language of conceptual blending helps us understand how students make connections between their mathematical learning and physics learning, in other words, how they transfer their calculus to physics. We described four different kinds of blends that students created when setting up integrals. The four blends were defined under four organizing frames: Integral-Sum Across, Differential Algorithm, Equation Mapping, and Chopping & Adding frames.

The blends that students created under the 'Integral-Sum Across' and 'Differential Algorithm' contained only two mental spaces - the symbolic space and mathematical concept space. Students' solutions were primarily focused on mathematical operations and algorithms, which were isolated from the specific physical scenario. When recruiting the 'Integral-Sum

Across', students' constructed a solution based on the basic notion of 'integral as a sum' and the matching of physical quantities to an existing integral template. When recruiting the 'Differential Algorithm' frame, students created a solution primarily based on the mathematical operation of taking a derivative. The blend constructed under the organizing frame of 'Equation Mapping' which was adopted predominantly from the symbolic space. This type of blend contains three mental spaces – symbolic space, math notion space, and physical world space. Student attempted to make some connection between the mathematical formalism and the physical world space. However, when the student constructed the notion of "adding up the resistivities" in the blended space, he did not seem to be making any connection to the physics concepts to evaluate the integral. The last type of blend was constructed under the organizing frame of 'Chopping & Adding.' It contains four input mental spaces - physical world, physics concept, symbol, and math notion spaces. When recruiting this organizing frame, students integrated the elements in four mental spaces and often translated back and forth between different spaces.

8.4 Research Question Three

According to the resources (Hammer, 1996a) perspective, there are two possible explanations for students' deficiencies in physics problem solving requiring mathematics: students used resources unproductively in a context or students framed a situation in a problematic way, which in turn activated unproductive knowledge resources or skills. We hypothesized that if students' framing of physics problem solving requiring mathematics can be shifted towards a more appropriate direction, they will be more likely to activate productive mathematical resources in their physics problem solving. Hence we developed hypothetical debate problems and analyzed the role that hypothetical debate problem played in changing students' epistemological framing about their problem solving activity.

In order to build students' mathematical fluency in physics, instructors should be more aware of students' framings in a situation. This will help us determine whether a student's difficulty is due to a lack of particular knowledge or a framing issue. From analyzing students' framing about physics problem solving, we can get information about which epistemological resources students activate in a situation. For example, when a student always frames the process of solving a physics problem as rote equation chasing, he or she might perceive using mathematics in physics as “doing mathematics.” We investigated students’ framings in both the conventional free response problem and the hypothetical debate problem. Overall, we found that when working on the debate problem, students more likely to perceive using math in physics as quantitative or qualitative sense-making rather than rote equation chasing.

8.5 Discussion about Theoretical Frameworks

I combined several theoretical constructs to analyze students’ use of calculus concepts in physics problems in this dissertation research. Each framework has its own explanatory power and uncovers students’ mathematical reasoning from a different perspective. Previously, there were very little research on synthesizing and comparing these theoretical frameworks. In this section, I would like to provide a further discussion about how these theoretical frameworks are connected with each other and what their roles are in understanding students’ mathematical reasoning in physics from the following three aspects.

(1) The conceptual blending framework describes the generation of new resources. One implicit assumption about the resource framework is that resources pre-exist in students’ minds and ready to be activated in a context. Resources are considered as the ontological component for describing students’ physics problem solving and students activate certain resources in a physical situation. This assumption as well as the mechanism of activating certain resources in a specific context is not very well defined and explored in the resources framework. The language of conceptual blending can be used to describe how new resources are created from other resources in a context. Through the language of conceptual blending, we can also explore how a resource was generated from more fundamental resources. Those fundamental resources are considered as the knowledge elements in the input mental spaces. After years of learning and practice, learners may often activate some resources as a unit (or a blend) in which the input mental spaces are no longer distinguishable. Those resources are activated and in a situation and combined to create more new resources. The conceptual blending framework is used to describe how students organize their existing knowledge resources in a situation that they have little or no experience with to create new learning.

We analyzed students’ application of calculus concepts in physics problems from the perspective of conceptual blending. Conceptual blending framework can be used to understand the creation of resources. In other words, a resource can be described as a blend. For example, in blend A, students constructed a mathematical expression $\int_0^L \frac{\rho(x)L}{A} dx$, which contains a meaning that the total resistance is the sum of a varying resistance across the length over which the resistivity varies. It can also be described as a mathematical resource (i.e., symbolic form) in which the mathematical structure $\int_{[]}^{[]} [] d[]$ provides a way for students to associate mathematical symbols with their intuitive mathematical knowledge (i.e., conceptual schema). Jones [9] descried the conceptual schema as "add up then multiply" – adding up one and the resultant summation was then multiplied by the quantity represented by the differential. From the resources perspective, students activated the "add up then multiply" resource in this physics context. Instead of assuming that this resource pre-exists in students' minds and becomes activated in this context, we describe how this resource is constructed based on more generic resources (e.g., integral as adding up quantities) using the language of conceptual blending.

(2) Conceptual metaphors are considered as a unique type of blend (i.e., metaphorical blends) and can be described using the language of conceptual blending (Grady, Oakley, & Coulson, 1999). However, these two theories play different roles and they are complimentary in several aspects (Grady et al., 1999). Many conventionalized metaphors are stored in long-term memory and they provide "ready-made" conceptual associations for the real-time construction of blends. Blending is conceived as an opportunistic process, often generating a richer and more complex image based on the elaboration of other simple conceptual associations.

(3) Framing is the control structure, which determines the activation of resources in a context or the kinds of blends to create in a situation. Students' framing of a situation i.e. their expectations about what they are supposed to do in that situation, play an important role in deciding what they should pay attention to and what they should ignore in a situation. A student who frames physics problem solving as "rote equation chasing" might activate mathematical resources of "taking the derivative" or "variable of integration with differential structure of *d*[]; A student who frames physics problem solving as sense-making might activate resource of "small amount" with differential structure *d*[].

In general, these theoretical constructs serve different purposes in my work: the resources are considered as the ontological component of student reasoning, and the use of conceptual blending framework is to model the conceptual component of student use of mathematics in physics problem solving; framing is the control structure of students' problem solving.

8.6 Future Directions

This study provides a springboard for a possible future studies as well as the development of pedagogical strategies for students' application of integrals in physics problem solving.

Extending The Study

Based on our needs of investigating students' deep thinking, we conducted this qualitative study in the context of introductory electromagnetism with 13 students. It is possible that our categorization of students' use of symbolic forms and conceptual metaphors is not a comprehensive list. We could investigate students' reasoning in other physics topics, such as kinematics, center of mass, and moment of inertia at the introductory level. This study could potentially lead to a further study to investigate students' reasoning with a larger student population. These investigations would contribute to the robustness and generalizability of the findings of this study. Previous research (Wilcox et al., 2012) also reported students' lack of ability to set up integrals in upper-division physics course. The quantitative study will potentially explore more resources or conceptual metaphors that were not identified in this study and provide evidence. It could also provide data to support the robustness and generalizability of each type of reasoning. One possible way is to design a 'two-level' multiple choice survey, in which students, in addition to selecting one of the choices, provide reasoning to support their choice or suggest alternatives that were not presented as part of the multiple choices. Such a survey would enable us to probe the resources and conceptual metaphors used by a larger student population, including those that were not previously uncovered in our study. We can also extend this study to an upper-level physics course, such as mechanics, E&M or thermodynamics. We might expect to observe a transition in students' reasoning from introductory level to upper-level physics course as students learn "more" math and physics.

Development of Pedagogical Strategies

This study explored the use of hypothetical debate problems as an instructional tool to shift students' framing of using mathematics in physics problem solving. The development of hypothetical debate problem is based on the assumption that shifting students' framing from a rote equation chasing frame to a qualitative/quantitative sense-making frame will help students to activate more productive mathematical resources. For future studies, we are interested in exploring whether the use of debate problem can actually facilitate students to activate more productive conceptual resources and metaphors or create different kinds of conceptual blends as they set up mathematical integrals in physics problems compared with conventional free response problems.

8.7 Implications of Research

This study has several important implications for instruction. First, our categorization of symbolic forms, conceptual metaphors, and blends provides different lenses for describing and understanding students' underlying reasoning process. Students' approaches to physics problems are often more visible to instructors but the reasoning process underlying their approaches are often less visible. For instance, when students write down an equation such as $dA = 2\pi r$ for an infinitesimal ring, some instructors might simply make judgments that students just forgot to add *dr* on the right side of the equation due to their carelessness. However, it is very likely that students did not attribute any physical meaning to the term *dr*, which is consistent with the variable of integration resource view of the differential element. When thinking of the differentials simply as an indicator of doing integration, students encounter several conceptual difficulties in an unfamiliar situation even though they might be able to solve familiar problems by simply remembering the equations.

Analysis of students' work using conceptual blending framework shows that blending is a complex process and students do it in many different ways. The difficulties students experience seem to be not necessarily from a lack of prerequisite knowledge of mathematics, but rather from inappropriate blending of the knowledge of mathematics with their knowledge of physics concepts and the physical scenario at hand. For example, almost all of the students mentioned the idea that the integral was representing a sum, however, only some of them were able to project it correctly onto the blended space. The classification under this framework might help instructors better understand students' difficulties and the important features in student solutions.

Second, a comparison of different types of reasoning may suggest which reasoning seems to be productive for students in the context of physics integration problems. For example, when activating the 'small amount' resource and 'object' metaphor, students' solution involved chopping an object into pieces and adding the effect due to each piece, which seems to involve more mathematical sense-making. Students appeared to be able to translate back and forth between the math and physics concepts. Conceptualizing the symbol "*d*[]" in terms of a concrete object made the abstract mathematical notion more transferrable to physical scenarios.

Finally, our results can also suggest several instructional strategies that instructors may use from different perspectives. For instance, if students use the 'point' resource, they may be unable to relate the differential terms to their respective dimensions. To help students relate the "point" quantity with its dimension, it is necessary to help students make the connection between the "point" and the "amount" carried by the "point". Hence it might be useful to lead students through the process of breaking down a rod into pieces and then shrink each piece into infinitesimal length. Another possible strategy is to suggest a different metaphor (e.g., an 'object' metaphor) which can be more productive in this situation.

In terms of the language of conceptual blending, we found that it is possible to steer students away from one blend to another when certain resources are provided. The construction of a blend is a dynamic process in which we observed that students changed their thinking and created a new blend when they were reflecting on their work or were asked questions that prompted them to activate different resources. For example, one student first constructed a blend under the 'Integral-Sum Across' frame and later switched to the 'Chopping & Adding' frame as he saw a cylinder being sliced up drawn by one of the other students in his group.

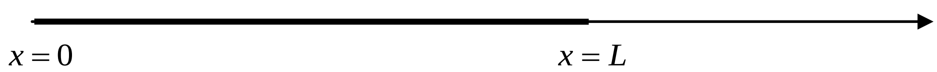
From the framing perspective, a student's difficulty with using mathematics in physics could be due to a framing issue. From analyzing students' framing about physics problem solving, we can get information about their epistemological resources activated. For example, when a student always frames solving physics problem as rote equation chasing, he or she might perceive using mathematics in physics as "doing mathematics." Hence strategies (e.g., hypothetical debate problems) that help shift students' framing to sense-making could be potentially useful for developing more expert-like problem solving strategies.

Appendix: Interview Tasks

Interview One

Problem 1

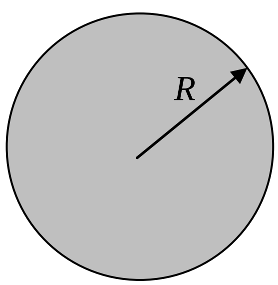
An insulated thin rod with length L has charge +Q uniformly distributed over the rod. Point P is located at a distance d from the right end of the rod.



Find the electric field at point P due to this charged rod. Note: You only need to set up the mathematical expression for the electric field and turn the equation into a form that you (or computer) can compute mathematically. [Given: The electric field due to a point charge q at a distance r is $\vec{E} = k\frac{q}{r^2}\hat{r}$]

Problem 2

A thin disk with radius R has a non-uniform surface charge density $\sigma(r) = \alpha r$ (α is constant).

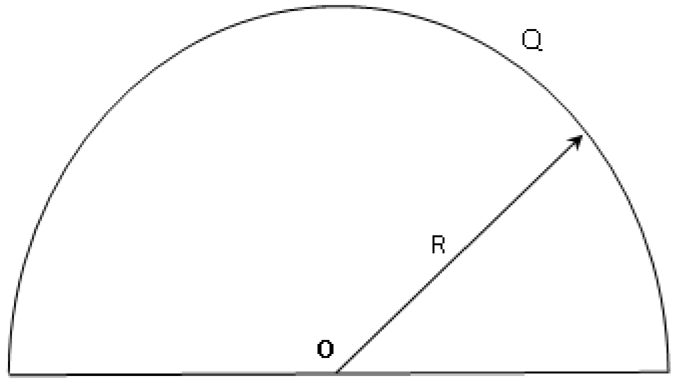


(a) Use your own words to explain what surface charge density means.

(b) Find the total charge on this thin disk.

Problem 3

A circular arch which has radius R is electrically insulated from the ground, as shown in the figure below. The arch is positively charged with Q uniformly distributed along the arch. Find the magnitude and direction of the electric field at the center O of the arch. Find the total electric field at point O due to this arch.



Interview Two

Problem 1

A very thin plastic rod with length L carries charge +Q and the charge is uniformly distributed along the rod. Find the electric field due to the rod at point P, a perpendicular distance d from the right end of the rod. [Given: The electric field due to a point charge q at a distance r is $\vec{E} = k\frac{q}{r^2}\hat{r}$]

We can find the total electric field from this equation: $\vec{E} = \int d\vec{E}$. What does $d\vec{E}$ mean in this problem? Now express $d\vec{E}$ in terms of the given quantities in the problem.

Several students are discussing about how to solve the problem and they proposed different solutions. Now write down your thoughts about why you think **each strategy (or part of the strategy)** is correct/incorrect.

<u>Student A</u>: We are given the equation $\vec{E} = k\frac{q}{r^2}\hat{r}$, $d\vec{E}$ is just the differential of this equation. Because the charges are distributed everywhere along the rod, so the distance from the charge to point P is changing, thus $d\vec{E} = k\frac{qdr}{r^2}\ \hat{r}$.

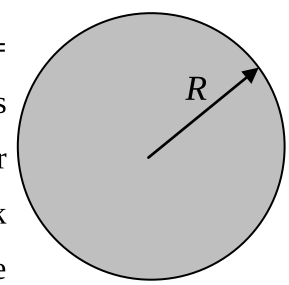
Student B: Equation $\vec{E} = k\frac{q}{r^2}\hat{r}$ is the electric field due to a point charge. In this problem, we do not have a point charge, but we can pick a small tiny bit of charge dq on the rod. This dq can be viewed as a point charge, so $d\vec{E} = k\frac{dq}{r^2}\hat{r}$ and r=L-x.

Student C: As the charge is uniformly distributed over the rod, the charge of any arbitrary point is Q/L which is the total charge divide by the total length. The electric field due to the small point charge on the rod is $d\vec{E} = k\frac{(\frac{Q}{L})}{r^2}\hat{r}$ and $r = \sqrt{d^2 + (L - x)^2}$ is the distance from the small point charge to point P.

Student D: The electric field due to a small arbitrary amount of charge on the rod is $d\vec{E} = k\frac{\lambda dx}{x^2}\hat{r}$. dx is an infinitesimal length on the rod and $\lambda = \frac{Q}{L}$ is linear charge density.

Problem 2

A thin disk with radius R has a non-uniform surface charge density $\sigma(r) = \alpha r$ (α is constant). Find the total charge on this thin disk. Four students were discussing about how to solve this problem and they provided four different strategies. Now write down your thoughts about why you think each strategy (or part of the strategy) is correct/incorrect. If you think none of them is correct, provide your own solution.



Student A: The surface charge density is charge per unit area, thus, the total charge is the surface charge density times area, that is $Q = \sigma(r)A$, but σ changes with r, so an integral is needed here. The final equation should be $\int_0^R \sigma(r)(\pi R^2)dr = \pi R^2 \int_0^R \sigma(r)dr$

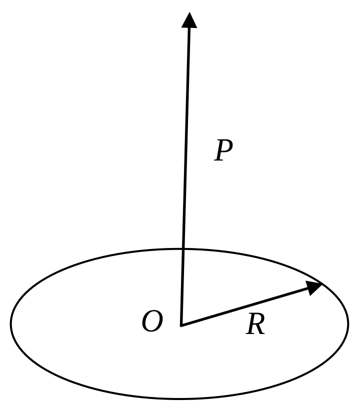
Student B: As the surface charge density changes with r, we should use $A = \pi r^2$ and it should be inside the integral, thus the total charge is $\int_0^R \sigma(r)\,\pi r^2 dr$.

Student C: As the surface charge density changes with r, we need an integral. First, we need to chop the whole disk into infinite numbers of extremely thin rings with thinness dr. For an arbitrary thin ring located at a distance r from the center, it has an area of $dA = 2\pi r dr$, and $\sigma(r)dA$ gives the small amount of charge carried by this ring. The total charge is the summation of all the charges carried by each ring, that is $\int_0^R \sigma(r)dA = \int_0^R \sigma(r)2\pi r dr$.

Student D: The surface charge density is not uniform, so the integral $\int_0^R \sigma(r)dr$ means averaging up the charge density along the radius from 0 to R. Then we just apply it into the equation $Q = \sigma A$ for a constant charge density. Thus, the total charge is: $\left(\int_0^R \sigma(r)dr\right) * (\pi R^2)$

Problem 3

The figure below shows a disk of radius R and uniform surface charge density σ. Find the total electric field due to the charged disk at point P on the central axis of the disk, at distance d from the center of the disk. Find both the direction and the magnitude of the electric field.



Interview Three

Problem 1

Electric charge is distributed uniformly along an infinitely long, thin wire with linear charge density λ. We assume the charge is positive. Find the electric field at a distance r from the wire.

Problem 2

A very thin, infinite, nonconducting sheet has a uniform positive surface charge density σ. Find the electric field at a distance r above the sheet.

Problem 3

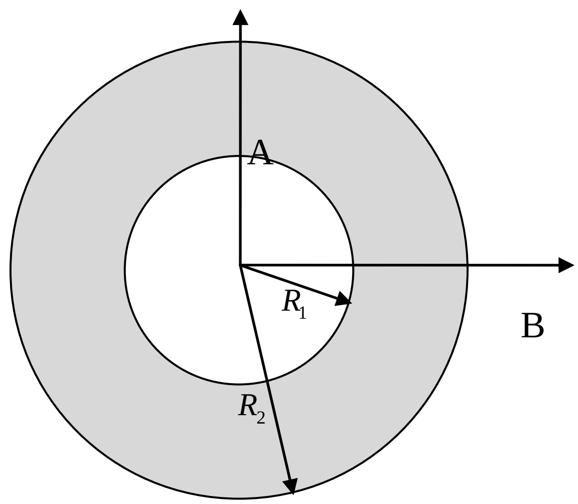
A nonconducting spherical shell with inner radius R_1 and outer radius R_2. Positive charges are distributed nonuniformly in the spherical shell with volume charge density $\rho(r) = \frac{\alpha r}{R_2}$ (α is a positive constant), ($r>R_1$, r is the radial distance from the sphere's center.

Part I:

Find the electric field at point A located inside the spherical shell with radial distance a ($R_1 < a < R_2$).

Part II:

Find the electric field at point B located outside the spherical shell with radius distance b ($b > R_2$).



Interview four

Problem 1

Electric charge is distributed uniformly along an infinitely long, thin wire with linear charge density λ. We assume the charge is positive. Find the electric field at a distance r from the wire. Several students are discussing about how to solve the problem and they proposed different solutions as well as their explanations. Each strategy may not be entirely right or entirely wrong. Discuss with your partners about each strategy. Imagine you are trying to convince those students why their strategy (or part of the strategy) is incorrect.

Student A: We can find the total electric field by integration, $\vec{E} = \int d\,\vec{E}$ where $d\vec{E} = k\frac{dq}{R^2}\hat{r}$ is the electric field due to a tiny bit of charge dq on the rod. dq is the tiny bit of charge carried by dx, that is $\lambda dx. R = \sqrt{r^2 + x^2}$ is the distance from dq to the point located at distance r from the wire.

The following students all start with Gauss law $\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$ as the wire is infinitely long and the electric field is distributed symmetrically around the wire. They all draw a cylindrical surface with radius r around the wire.

Student B: This equation can be simplified as $E \cdot A = \frac{Q}{\varepsilon_0}$,, and the area should be all the surface area of the cylinder, that is $A = \pi r^2 L$. $Q=\lambda L$ is the charge enclosed by the Gaussian surface. Then we can plug in the A and Q, so $E = \frac{Q}{\varepsilon_0 \cdot A} = \frac{\lambda L}{\varepsilon_0 \cdot \pi r^2 L}$.

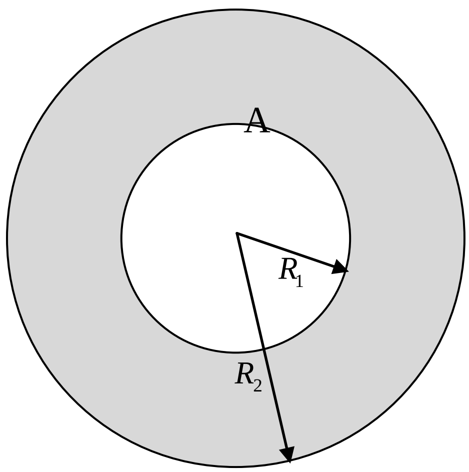
Student C: The area only includes the cylindrical surface area around the wire. $d\vec{A}$ is just a tiny piece of the cylindrical area and this vector is parallel to the wire. So the dot product $\vec{E} \cdot d\vec{A}$ can be turned into $E \cdot dA$. So the integral can be simplified as $\int EdA = E\int dA$. We take this E out of the integral because the magnitude of the electric field is the same on the cylindrical surface. $\int dA$ means summing up all the little pieces of area, so it is equal to the entire cylindrical surface area which is $2\pi rL$.

Student D: $\oint \vec{E} \cdot d\vec{A}$ means we integrate over a closed surface, so we should consider the whole area which includes the two ends and cylindrical part. $d\vec{A}$ of the two ends is parallel to the ends, so $d\vec{A}$ is also parallel to the electric field. Thus, the dot product $\vec{E} \cdot d\vec{A}$ is zero. So we only need to consider the cylindrical area which is $2\pi rL$.

Student E: In the equation $\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$, the area is $2\pi rL$ and the charge enclosed should be λdx. So $E = \frac{Q}{\varepsilon_0 \cdot A} = \frac{\lambda dx}{\varepsilon_0 \cdot 2\pi rL}$.

Problem 2

A nonconducting spherical shell with inner radius R_1 and outer radius R_2. Positive charges are distributed nonuniformly in the spherical shell with volume charge density $\rho(r) = \frac{\alpha r}{R_2}$ (α is a positive constant), ($r>R_1$, r is the radial distance from the sphere's center.



Find the electric field at point A located inside the spherical shell with radial distance a ($R_1 < a < R_2$).

Several students are discussing about how to solve the problem and they proposed different solutions with their explanations. Each strategy may not be entirely right or entirely wrong. Discuss with your partners about each strategy. Imagine you are trying to convince those students why their strategy (or part of the strategy) is incorrect.

They all started with Gauss law $\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$ due to the spherical symmetry of the electric field.

First of all, they draw a spherical Gaussian surface at *r*=*a*, and turn $\oint \vec{E} \cdot d\vec{A}$ into $E \cdot 4\pi r^2$. Then they tried to find the charge enclosed.

Student A: Volume charge density is total charge divide by volume, so the total charge is charge density multiplied by the volume, that is $Q = \rho V = \frac{\alpha r}{R_2} \cdot \frac{4}{3}\pi(r^3 - R_1^3)$.

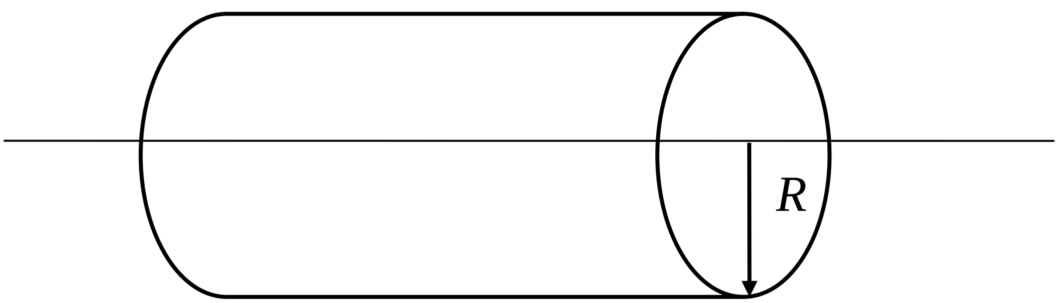
Student B: In the density function, r could be any distance from the center, so we need to integrate over the radius to get the total charge, that is $Q = \int_{R_1}^{a} \frac{\alpha r}{R_2} \cdot \frac{4}{3}\pi(r^3 - R_1^3)\, dr$

Student C: As the volume charge density changes with r, we should chop the whole shell into infinite number of extremely thin shells with thickness dr. Then find the charge carried by the thin shells, that is $dq = \rho(r)dV$. The volume of the very thin shell is $dV = 4\pi r^2 dr$. In the equation $\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$, Q should be the total charge of the shell that is $Q = \int_{R_1}^{R_2} \frac{\alpha r}{R_2} \cdot 4\pi r^2\, dr$

Student D: To find the total charge Q, we should find $dq = \rho(r)dr$ first. Then the total charge is $Q = \int_{R_1}^{a} \frac{\alpha r}{R_2} dr$.

Problem 3

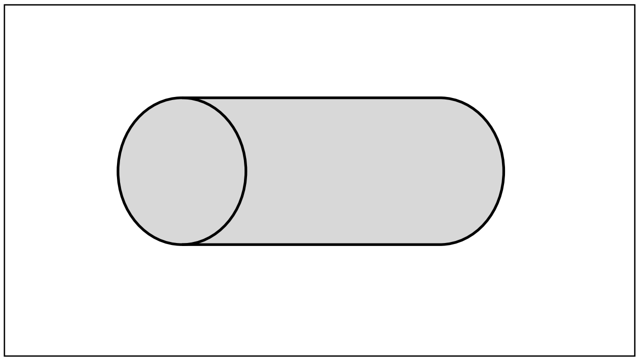
An infinite long, solid insulating cylinder with radius R has a volume charge density $\rho(r) = \rho_0(1 - \frac{r}{R})$, $0 \le r \le R$. Find the magnitude and direction of the electric field at a point inside the cylinder located at a distance *a* from the central axis.



Interview Five

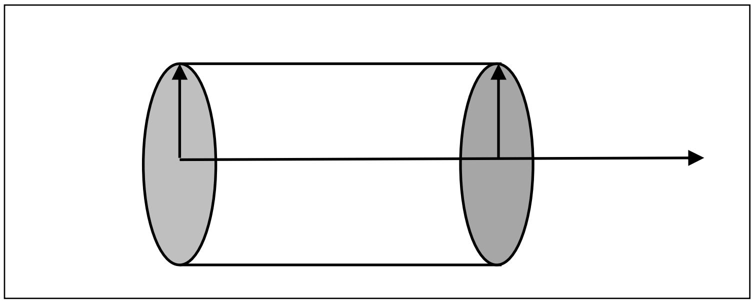
Problem 1

A material of resistivity ρ is formed into a solid cylinder with length L, radius r. Find the total resistance of this cylinder between the two end faces.



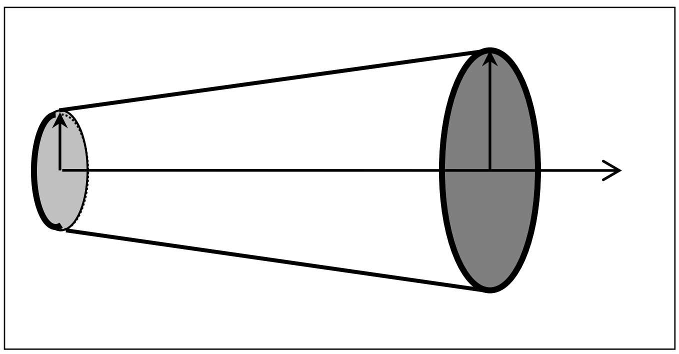
Problem 2

A material with length L and cross-sectional area A lies along the x-axis between x=0 and x=L. Its resistivity varies along the rod according to $\rho(x) = \rho_0 \cdot e^{-x/L}$. Find the total resistance of this cylinder between the two end faces.



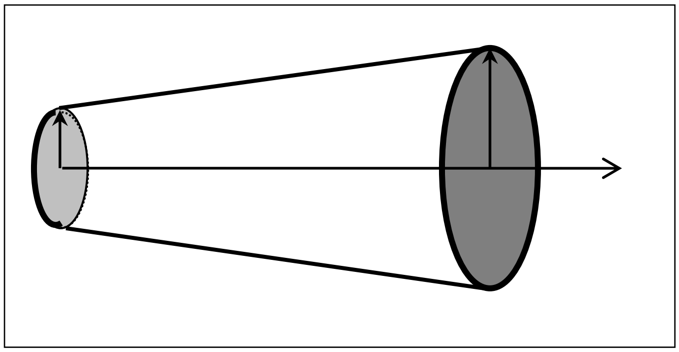
Problem 3

A material of resistivity ρ is formed into a solid, truncated cone of length L and radii a and b at either end. Calculate the total resistance of the cone between the two flat end faces in terms of resistivity ρ, a, b, and L.



Problem 4

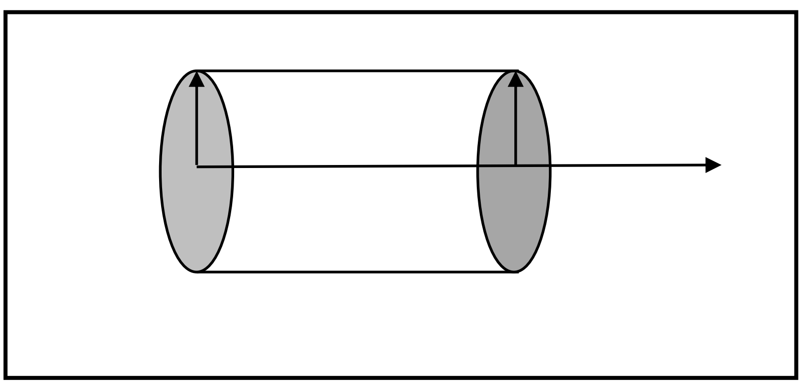
Consider a capacitor of material, permittivity ε. The capacitor consists of two circular plates of radii *a* and *b* placed at a distance *L* apart. Find the capacitance of this capacitor between the two flat ends in terms of its length L, radius a, radius b, and permittivity ε.



Interview Six

Problem 1

A material with length L and cross-sectional area A lies along the x-axis between x=0 and x=L. Its resistivity varies along the rod according to $\rho(x) = \rho_0 e^{-x/L}$. Find the total resistance of this cylinder between the two end faces.



Several students are discussing about how to solve the problem and they proposed different solutions as well as their explanations. Each strategy may not be entirely right or entirely wrong. Discuss with your partners about each strategy. Imagine you are trying to convince those students why their strategy (or part of the strategy) is incorrect.

Student A: As ρ is changing, we have to use an integral. Based on the original equation $R = \frac{\rho L}{A}$, we plug in the $\rho(x)$ function and integrate it, that is $\frac{L}{A}\int_0^L \rho_0 e^{-x/L} dx$. Everything else is outside the integral, except the $\rho(x)$. The two bounds are from *0* to L, as *0* is the origin and L is the length of the rod.

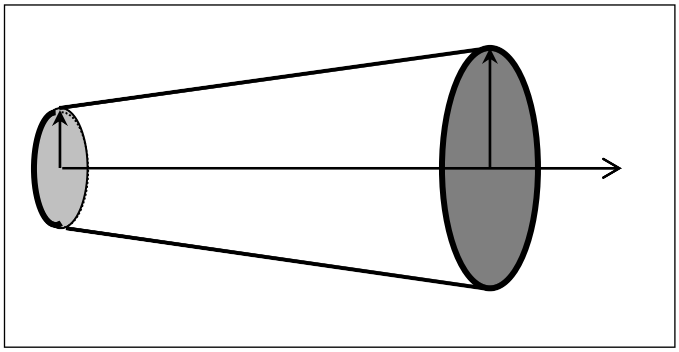
Student B: In order to find the total resistance R, we first need to find the expression for dR. As ρ is changing, $dR = \frac{d\rho * x}{A}$ with $d\rho = d\rho(x) = (-\frac{1}{L})\rho_0 e^{-x/L} dx$. Then we can find the expression for R by taking the integral of dR, that is $R = \int dR = \int \frac{d\rho * x}{A} = \frac{\rho_0}{A}\int_0^L (-\frac{x}{L}) e^{-x/L} dx$

Student C: As resistivity is changing along the x axis, at different points along the x axis, it has different resistivity. Thus, we have to chop the whole cylinder into extremely thin disks with length dx. Then we can find the resistance of an arbitrary thin disk, that is $dR = \frac{\rho(x) * dx}{A}$. The total resistance can be found by adding up all the resistance of the thin disks, that is $R = \int dR = \int \frac{\rho(x) * dx}{A} = \frac{\rho_0}{A}\int_0^L e^{-x/L} dx$.

Student D: We can find the total resistance without the use of integration. We can divide the whole cylinder into several, say 1000 pieces along the x axis. Each piece is a very tiny disk with length *L/1000*. For the n^{th} piece located at $x = \frac{n \cdot L}{1000}$, the resistivity can be found by plugging in $x = \frac{n \cdot L}{1000}$ to the $\rho(x)$ function. Thus, the resistance of this piece is $R_n = \frac{\rho_0 e^{-\frac{n \cdot L}{1000}/L}}{A} * \frac{L}{1000}$. Then the total resistance is the summation of all the pieces: $R = \sum_{n=1}^{1000} R_n$

Problem 2

A material of resistivity ρ is formed into a solid, truncated cone of length *L* and radii *a* and *b* at either end. Calculate the total resistance of the cone between the two flat end faces in terms of resistivity ρ, *a*, *b*, and *L*.



Several students are discussing about how to solve the problem and they proposed different solutions as well as their explanations. Each strategy may not be entirely right or entirely wrong. Discuss with your partners about each strategy. Imagine you are trying to convince those students why their strategy (or part of the strategy) is incorrect.

Student A: Since the cross sectional area A is changing, we need to set up the expression for dR first, that is $\mathrm{dR} = \frac{\rho \mathrm{L}}{\mathrm{dA}}$., dA representing the changing area. From $\mathrm{A} = \pi \mathrm{r}^2$, then $\mathrm{dA} = 2\pi \mathrm{rdr}$ and $\mathrm{dR} = \frac{\rho \mathrm{L}}{2\pi \mathrm{rdr}}$. In order to integrate dR to find the total resistance, we need to put the "dr" in the denominator, that is $\mathrm{R} = \int_a^b \frac{\rho \mathrm{L}}{2\pi \mathrm{r}} \mathrm{dr}$. The two bounds are from a to b as the radius changes from a to b.

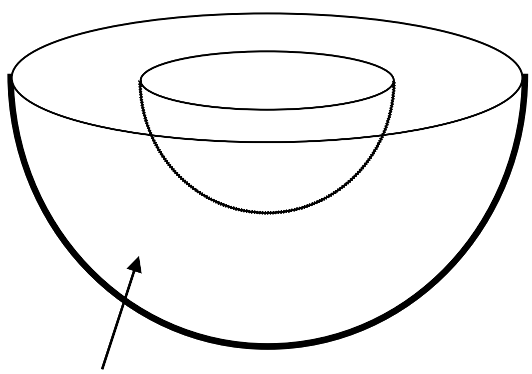
Student B: We need a double integral here because both the radius and the length change. We can set up our equation for dR as $\mathrm{dR} = \frac{\rho \mathrm{dxdr}}{\pi \mathrm{r}^2}$.Then our total resistance is the integral of dR, that is $\mathrm{R} = \int_0^L \int_a^b \frac{\rho \mathrm{dxdr}}{\pi \mathrm{r}^2}$.

Student C: We start with the original equation $\mathrm{R} = \frac{\rho \mathrm{L}}{\mathrm{A}} = \frac{\rho \mathrm{L}}{\pi \mathrm{r}^2}$ and then we can find dR by taking the derivative of this equation: $\mathrm{dR} = -\frac{2\rho \mathrm{L}}{\pi \mathrm{r}^3} \mathrm{dr}$. Then the total resistance is the integral of dR, that is $\mathrm{R} = \int_a^b \left(-\frac{2\rho \mathrm{L}}{\pi \mathrm{r}^3}\right) \mathrm{dr}$.

Student D: As the cross sectional area or the radius of the cross section is changing with x, we can find the area as a function of x. From the geometry relationship, we can the radius r as a function of x: $r(x) = a + \frac{b-a}{L}x$. We can slice the whole cone into little pieces of cones with length dx and radius r. The resistance for an arbitrary slice is given by $dR = \frac{\rho dx}{\pi r^2} = \frac{\rho dx}{\pi(a+\frac{b-a}{L}x)^2}$. The total resistance is the summation of resistances due to each extremely small disk, that is $R = \int_0^L \frac{\rho dx}{\pi(a+\frac{b-a}{L}x)^2}$

Problem 3

The region between two half concentric metal spheres with radii a and b is filled with some material. The resistivity of the material varies according to the function: $\rho(r) = \frac{\rho_0 r}{b}$ $(a<r<b)$, where 'a' is the inner radius and 'b' is the outer radius. Find the total resistance between the inner and outer spherical surfaces.

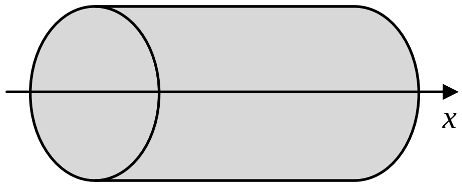


Interview Seven

Problem 1

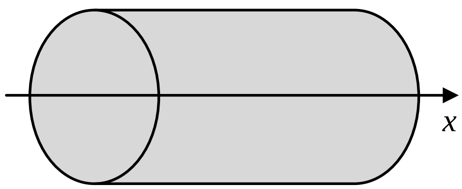
A long, straight, solid cylinder with radius R, oriented with its axis in the x direction, carries a current whose current density J varies according to the relationship: $J(r) = \frac{j_0 r}{R}$, $0<r<R$. The

current is flowing along the x axis. Find the total current passing through the entire cross section of the wire.



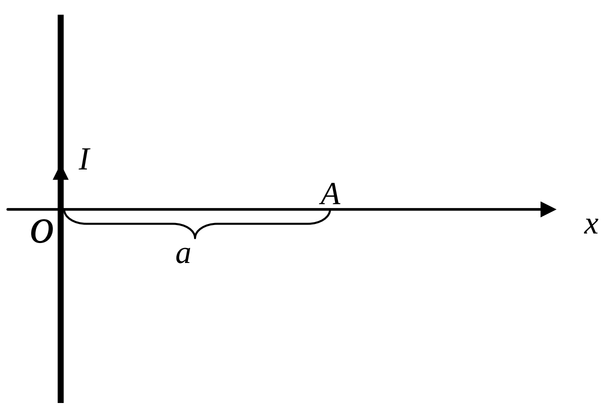
Problem 2

A infinitely long, straight, solid cylinder with radius R, oriented with its axis in the x direction, carries a current whose current density J varies according to the relationship: $J(r) = \frac{j_0 r}{R}$, 0<r<R. The current is flowing along the x axis. Find the magnetic field at a point located a distance a (a<R) from the central axis of the cylinder.



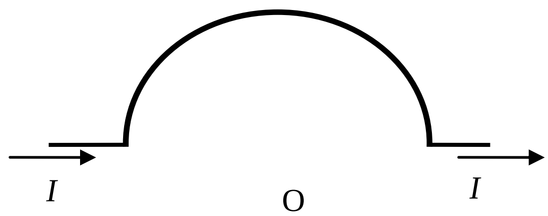
Problem 3

A straight conducting wire with length L carries a current I. Find the magnitude and direction of the magnetic field due to this wire at a point A located a distance a from the center of the wire O.



Problem 4

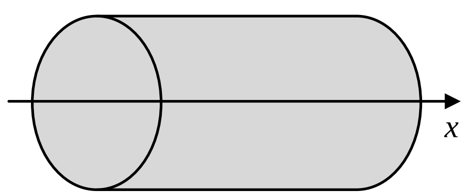
The current I is flowing through a semicircular wire with radius R as shown below. Find the magnitude and direction of the magnetic field at point O due to the current in the semicircular section of the wire shown in the figure below.



Interview Eight

Problem 1

A long, straight, solid cylinder with radius R, oriented with its axis in the x direction, carries a current whose current density J varies according to the relationship: $J(r) = \frac{j_0 r}{R}$, 0<r<R. The current is flowing along the x axis. Find the current passing through the cross section of the wire.



Several students are discussing about how to solve the problem and they proposed different solutions as well as their explanations. Each strategy may not be entirely right or entirely wrong. Discuss with your partners about each strategy. Imagine you are trying to convince those students why their reasoning is incorrect.

Student **A**: To find the total current I, as the current density is changing with radius r, we need to use integration. Then we find the changing current $dI = dJ(r) * A = \frac{j_0 dr}{R} * \pi r^2$ and dJ(r) means the changing current density. So the total current is the integral of dI, that is $I = \int dI = \int_0^R \pi r^2 * \frac{j_0 dr}{R} = \frac{\pi j_0}{R} \int_0^R r^2 dr$.

Student B: As the current density changes with the radius r, at different radius r, the current density is different. We need to think of the cross section in terms of infinitesimal rings with thickness dr and area dA, so the current density can be considered as constant for each infinitesimal ring. Thus, the infinitesimal bit of current flowing through an infinitesimal ring can be found by the equation: $dI = J(r) * dA = J(r) * 2\pi dr$. So the total current flowing through the entire cross section is the summation of all the small bit of current flowing through each infinitesimal ring, that is $I = \int dI = \int_0^R J(r) * 2\pi dr = \int_0^R \frac{j_0 r}{R} * 2\pi dr$.

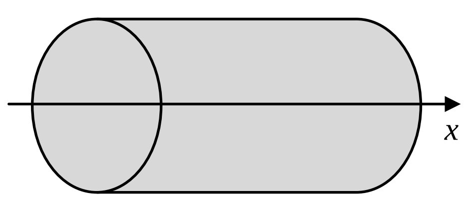
Student C: As the current density changes with radius r, we can consider the whole cross section in terms of rings. As each ring is infinitesimal thin, the area of each ring is equivalent to its circumference $2\pi r$. So the current flowing through each ring is $I = J(r) * 2\pi r$. The total current is the integral, that is $I = \int J(r) 2\pi r$.

Student D: I remember the equation $I = J * A$ for finding the current. As the area is changing with r, we need to integrate over dA to find A and then multiplied by current density. So the total current is $I = J * \int dA = J * \int 2\pi r dr$.

Student E: I also started with the basic equation $I = J * A$ and as the current density is changing, we need to integrate J(r) to find the total current density, that is $\int J(r) dr$. So the total current is $I = \int_0^R J(r) dr * (\pi R^2)$.

Problem 2

An infinitely long, straight, solid cylinder with radius R, oriented with its axis in the x direction, carries a current whose current density J varies according to the relationship: $J(r) = \frac{j_0 r}{R}$, 0<r<R. The current is flowing along the x axis. Find the magnetic field at a point A located a distance a (a<R) from the central axis of the cylinder.



Several students are discussing about how to solve the problem and they proposed different solutions as well as their explanations. Each strategy may not be entirely right or entirely wrong. Discuss with your partners about each strategy. Imagine you are trying to convince those students why their reasoning is incorrect.

They all attempted to use Ampere's law $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$. They first found the current, and then they proposed different strategies for evaluating $\oint \vec{B} \cdot d\vec{l}$.

Student A: dl means an infinitesimal amount of distance from the center of wire to point A, so $\oint \vec{B} \cdot d\vec{l}$ can be simplified as B*L and L is equal to the distance from the central axis to point A, that is r.

The following students drew Amperian loop, then they simplified the dot product $\vec{B} \cdot d\vec{l}$ as $B * dl$ as the magnetic field on the loop is along the tangential direction which is the same direction as dl. Then they take B out of the integral, that is $B * \oint dl$. However, they proposed different strategies as evaluating $\oint dl$.

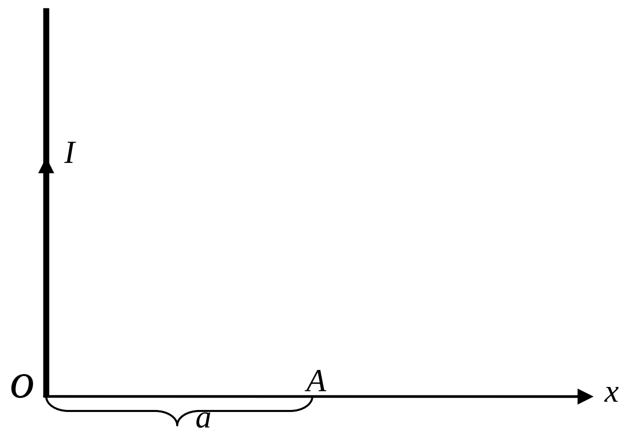
Student B: As we integrate around a circle, the bounds are from 0 to 2π. Thus, the integral is equal to 2π.

Student C: $\oint dl$ means adding up all the infinitesimal length around the Amperian loop and dl is an infinitesimal segment on the loop. Thus, $\oint dl$ is the total length of the loop (or circumference of the Amperian loop), this is $2\pi r$.

Student D: dl is infinitesimal arc length on the loop, thus dl can be written as $r d\theta$. When we integrate around the loop, the angle θ is changing from 0 to 2π. So this integral is equal to $2\pi r$.

Problem 3

A straight conducting wire with length L carries a current I. Find the magnitude and direction of the magnetic field due to this wire at a point A located a distance a from bottom end of the wire.



Several students are discussing about how to solve the problem and they proposed different solutions as well as their explanations. Each strategy may not be entirely right or entirely wrong. Discuss with your partners about each strategy. Imagine you are trying to convince those students why their reasoning is incorrect.

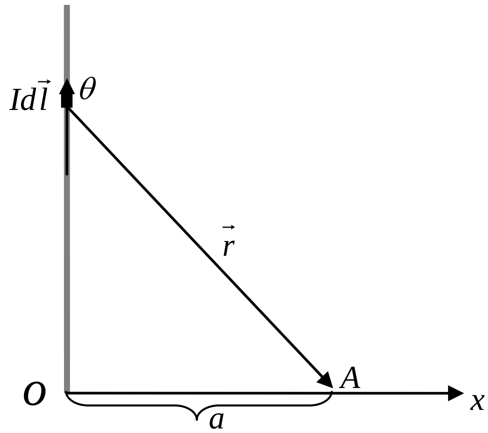
<u>Student A</u>: We can use Ampere's law as the magnetic field is symmetric around the wire (i.e., the magnitude of B is the same at the same distance from the wire). In this equation, $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$, the left side is equal to $2\pi rB$ and current enclosed by the loop is I. So we can get $2\pi rB = \mu_0 I$, thus $B = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I}{2\pi a}$.

The following students disagree with student A, and they started with Biot-Savart law $d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\hat{r}}{r^2}$

<u>Student B</u>: In the Biot-Savart law equation, r is the distance from the wire to point A, the unit vector $\hat{r}$ is pointing from O to A (along positive x direction). $d\vec{l}$ represents an infinitesimal segment of the wire along the direction of the current.. Based on right hand rule, cross product $d\vec{l} \times \hat{r}$ is pointing into the page. It has the magnitude of dl. Thus, dB can be written as $\frac{\mu_0}{4\pi}\frac{Idl}{a^2}$. Then the total magnetic field due to the wire is the integral of dB, that is $B = \int dB = \int_0^L \frac{\mu_0}{4\pi}\frac{Idl}{a^2} = \frac{\mu_0}{4\pi}\frac{IL}{a^2}$

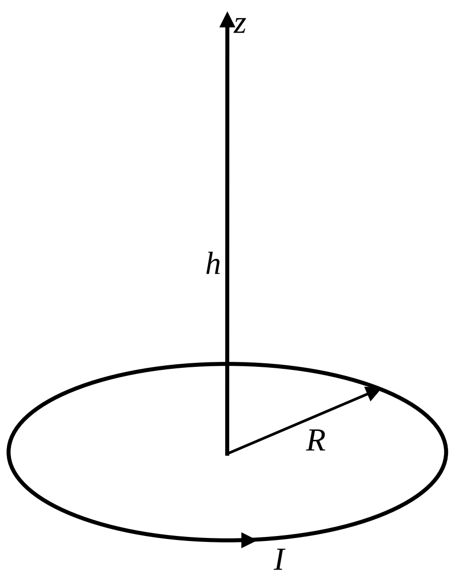
<u>Student C</u>: In the equation, $Id\vec{l}$ represents an extremely small bit of current segment, r is the distance from the small current segment to point A. The cross product $d\vec{l} \times \hat{r}$ has the magnitude of $\sin\theta dl$ with $\sin\theta = \frac{a}{r}$ and this vector is pointing into the page. As l is the variable, we need to

express r in terms of l, that is $r = \sqrt{l^2 + a^2}$. So dB can be expressed as: $dB = \frac{\mu_0}{4\pi}\frac{I\sin\theta dl}{r^2} = \frac{\mu_0}{4\pi}\frac{Iadl}{r^3} = \frac{\mu_0 Ia}{4\pi}\frac{dl}{(l^2+a^2)^{3/2}}$. Then we can find B by integrate dB from 0 to L.



Problem 4

A current I is flowing through a circular loop as shown in the picture below. P is located at a distance h from the circular loop's center. Find the magnetic field at point P due to the circular current loop.



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