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The impact of monetary policy on wage dispersion and economic growth

by

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Biography

Ricardo Fernando Azevedo da Silva was born in the 16^{th} of June of 1994 in Vila Nova de Famalicão, the same city that saw him grow up.

He have completed the high school level on Dom Sancho I School in June of 2012 and in July of 2015, he concluded his bachelor degree in Economics in University of Minho. Currently he is attending the 2^{nd} year of the Master in Economics in the Faculty of Economics of Porto, in which he obtained an average of 14 out of 20 points in the curricular part.

Since he was a young child that sport influenced him, what lead to eight years of regular competition in the Portuguese regional and national championships of Badminton. In May of 2012, he was the singles champion in the regional championships.

During the university, Ricardo was volunteer in the Youth Red Cross - Braga Delegation, promoting the awareness of children to avoid dangerous substances.

Between December of 2016 and August of 2017, he worked in the financial shared service of Adidas Group. In September of the current year, he started to work in the Porto office of PwC.

Acknowledgments

The journey that we propose to when we start to write a dissertation is a very ambitious and challenging time of our life. In order to overtake all the these obstacles that appear in our path, we should have the right people by our side to help us achieve our main goal that is to obtain the master title.

Following this idea, I would like to thank you to Professor António Neto for all the support and guidance that he gave me during this year and that allow me now to conclude this dissertation with the feeling of job complete. I am sure that the professor will have a great research career and I hope that, in the next years, many other students may have the same opportunity to work with him.

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Abstract

This study presents a novel theoretical framework to understand the impact of monetary policy on wage dispersion and labor allocation. We build an endogenous growth model with cash-in-advance (CIA) constraints on R&D and two types of workers, high- and low-skilled. The monetary authorities sets the nominal interest rate to maximize welfare, which allows to study not only the impact on monetary policy on wage dispersion but also to test the optimality of the Friedman rule (i.e., whether optimal nominal interest rate should be zero). The main conclusions are the following. First, under inelastic labor supply, Friedman rule might not be optimal for low economic growth rates. Furthermore, a positive but low interest rate can contribute to a lower wage dispersion between high- and low-skilled workers. Second, under elastic labor supply, Friedman rule seems to be optimal for all the considered scenarios.

Keywords: Wage Dispersion, Labor Skills, Economic Growth, Monetary Policy.

JEL-Codes: E52, J31, O42

Resumo

Esta dissertação apresenta um modelo teórico que pretende explicar o impacto da política monetária na dispersão salarial e no crescimento económico. Construímos um modelo endógeno de crescimento com restrições monetárias na I&D e com dois tipos de trabalhadores, qualificados e não qualificados. A autoridade monetária estabelece a taxa de juro nominal de modo a maximizar o bem-estar económico, o que permite estudar, não só o impacto da política monetária na dispersão salarial, mas também se a regra de Friedman é ótima (isto é, se a taxa de juro nominal ótima deve ser zero). As principais conclusões são as seguintes. Primeiro, no caso da oferta de trabalho ser inelástica, a regra de Friedman pode não ser ótima para baixas taxas de crescimento. Adicionalmente, uma taxa de juro positiva, mas pequena pode contribuir para uma redução da dispersão salarial entre trabalhadores qualificados e não qualificados. Segundo, no caso da oferta de trabalho ser elástica, a regra de Friedman aparenta ser ótima para todos os cenários considerados.

Palavras-Chave: Dispersão Salarial, Qualificações de Trabalho, Crescimento Económico, Política Monetária.

Códigos JEL: E52, J31, O42

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1 Introduction

This dissertation aims to understand the role of monetary policy on wage dispersion and economic growth. To achieve this objective, we develop an endogenous growth model combining employment, wage dispersion and monetary policy.

Since the two industrial revolutions the world economy, as a whole, has been raising at incredible rates of growth due to constant increases in the productivity of workers and machines. Following these improvements in technology, there was also an increase in wage dispersion between workers, i.e., the earnings of the high skilled workers have been rising faster than the ones from the low skilled workers.

Taking this into account, what is the relationship between monetary policy and wages dispersion? In other words, could different monetary policies influence, positively or negatively, this apparent relationship between technology and wage dispersion? Regarding the economic relevance of this research, note that, to the best of our knowledge, we present the first paper studying and combining the three fields of research, i.e., monetary policy, wage dispersion and economic growth.

Hence, we developed an endogenous growth model with cash-in-advance constraints in R&D (as in Chu and Cozzi, 2014) and two types of workers: (a) high-skilled workers who can be employed in the final and R&D sector; and (b) low-skilled workers who can be employed in the final and intermediate good sector (as in Afonso, 2016). A monetary authority is introduced and maximizes welfare by defining the nominal interest rate.

The main conclusion can be summarized as follows: (i) under inelastic labor supply, Friedman rule might not be optimal for low economic growth rates; hence, a positive but low interest rate can contribute to a lower wage dispersion between high and low skill labor; (ii) under elastic labor supply, Friedman rule seems to be optimal for all the considered scenarios; (iii) hence, a common monetary policy might not be optimal for all types of countries (i.e., depending on the economic growth rate) and can contribute to lower the wage dispersion between high- and low-skilled workers.

The rest of the dissertation is organized as follows: section 2 provides an in-dept literature review of this topic; section 3 presents and describes the model; section 4 analysis the impact of monetary policy on welfare; section 5 provides a sensitive analysis of the economy; and section 6 concludes.

2 Literature Review

2.1 Identification of key concepts

This section provides a brief overview of the main concepts under analysis in this dissertation.[1]

2.1.1 Monetary Economics

According to Walsh (2005, p. 1), "monetary economics focuses on the behavior of prices, monetary aggregates, nominal and interest rates and output". In other words, monetary economics coordinates the actions/policies that must be set to generate an increase on the overall welfare of economies. In a more specific topic, Arestis and Mihailov (2011) describe monetary policy as the role of central bank policies in the variations of money supply.

2.1.2 Structural Change

Matsuyama (2008) interprets structural change as a complex phenomenon, because not only economic growth foments complementary changes in various sectors of the economy, just as output or employment, but these changes also influence the economic growth, i.e., the growth process affects and it is affected by structural changes.

2.1.3 Wage Dispersion

Mortensen (2004) defines wage dispersion as the unequal compensation of workers who have similar productive attributes. Salverda and Checchi (2014) clarifies the difference between dispersion and inequality of wages, since that it is very common to have different interpretations regarding these two fields. In a broad sense,

[1]This section closely follows the final report prepared for the unit Plan of Dissertation under the subject: "The impact of monetary policy on wage dispersion and economic growth"

for these authors, dispersion should be interpreted as the mathematical logic of the word, i.e., when we consider a range of different wages, we should apply a numerical approach only. This does not happen when wage inequality is discussed, because wage inequality requires an analytical explanation of the data. Summing up, dispersion refers to quantitative analysis and inequality represents qualitative analysis of wages.

2.1.4 Creative Destruction

Caballero (2008) explains creative destruction as an endless research process to discover new production units that succeed the outdated ones and with this development, economies are modified essentially in terms of long-run economic growth, structural changes and economic fluctuations. This concept was first introduced by Schumpeter and later on introduced in a famous economic growth model by Aghion and Howitt (1992).

2.1.5 Economic Growth

Howitt and Weil (2008) argues that "Economic growth is typically measured as the change in per capita gross domestic product", but this is the general definition of economic growth. According to the same authors, this concept can be defined as "the increase in a country's standard of living over time", which is a more complete version of the definition of it.

2.1.6 Endogenous Growth Theory

Howitt (2008) defines endogenous growth theory as the long-run economic growth, but only the growth that emerge due to actions taken by forces internal to the economy, particularly the economies forces that allow technological change to occur by managing the opportunities of growth created. According to Howitt (2008), in the neoclassical theory, long-run growth is taken as an exogenous variable. Hence, the challenge for endogenous growth theory is to prove and explain how technological improvements and long-run economic growth can be determined by economic factors.

2.2 Historical context and theoretical framework

For the purpose of this dissertation it is extremely important to understand the historical evolution of the monetary theory as well as the different theories regarding wage dispersion and economic growth.

In terms of monetary policy, there were many models related to the subject developed during the twentieth century. Arestis and Mihailov (2011) presents a compilation of the most important research in the monetary field, from the first authors to work exhaustively on the topics until recent times. They divided the branch of monetary economics into three different sections: monetary theory, monetary policy and public finance. According to Arestis and Mihailov (2011), the monetary theory comprehends theories such as the classical models of Fisher (1911), Friedman (1956), and the Keynesian models from Keynes (1936) to Baumol (1952). On the other hand, the monetary policy covers the topics regarding the relation between central banks and money supply, with the systems of the gold standard (1776-1914) or the Bretton Woods system (1944-1971) as the earlier main contributions for the area. More recently we have the contributions of Lucas (1972), Barro and Gordon (1983) and Galí (2008), for example). Finally, the public finance can be characterized by the classic theories of Ramsey (1927), which are being advanced throughout time by different authors like Friedman (1960), Leeper (1991) or Benigno and Woodford (2003).

Following the same pattern, the economic growth theory has been evolving particularly since the middle of the last century, with the contribution for the literature from several authors demonstrating the importance of this field in the economic environment.

Two of the first major contributors for the literature are Kuznets (1947) with his article "Measurement of Economic Growth" (a few years later, this author derived the well know Kuznets Curve, relating inequality and income per capita) and Schumpeter (1947) where he developed the "theoretical problems of economics growth".

A decade later, Solow (1956) develop what is considered by many the first long-run economic growth model, commonly known as the Solow model, which is still used as one of the principal references of growth theory. Rostow (1959) is another interesting approach for the literature with the idea of economic growth through stages.

The end of the last century was a period rich in new material available with the research of Barro (1991), Aghion and Howitt (1992) and Mankiw et al (1992). For example, Mankiw et al (1992) introduced a new contribution in terms of empirics' analysis to this theory.

Recently, papers like Acemoglu (2002), Atkinson et al (2011) and Chu and Cozzi (2014) consolidate the existent literature with some improvements in their researches.

2.3 Integration and critical analysis of the different contributions to the literature

This dissertation aims to explore exhaustively the combined field of endogenous growth and monetary economics.

According to Bordo (2007), monetary policy is the major support for governments, through the action of central banks, to control the economies of the nations. To pursue this goal, policy makers have two main instruments able to influence the macroeconomic behavior of a country: (a) changes in interest rates (mainly short-term variations); and (b) changes in monetary base. Bordo (2007) argues that these instruments are critical to central banks achieve targets as low inflation or sustained increases in output.

Regarding the thematic of monetary policy, it is important to discuss a specific related topic, which is money neutrality. Patinkin (1987) defines money neutrality as a quantity-theory proposition that, in the long-run, only the level of prices in an economy changes with variations of the supply of money and not the level of real output, such as real wages and employment. Which according to this theory, are not affected by the quantity of money in circulation. In the literature there are several different perspectives relatively to money neutrality, some accepting the theory and others refuting it. Some examples of articles supporting money neutrality are Serletis and Krause (1996) and Bae et al (2005), whereas some refusing the argument are Bertocco (2007) and Pasten and Schoenle (2016).

Regarding the relationship between monetary policy and wage dispersion, according to Ahrens and Snower (2014). Under the presence of Calvo nominal wage contracts, a higher level of wage dispersion is caused by a higher level of inflation. This situation of higher inequality between workers will cause envy (for the workers that have lower incomes) and guilt (for the employees that receive higher wages). These different experiences have opposite impacts on aggregate economy, since that if the envy effect is bigger than guilt effect, an increase on inflation is associated to an augment of employment and output, not allowing a vertically long-run Phillips curve.

Another perspective relatively to this subject is the one proposed by Thomas (2008). This author analyses the optimal monetary policy in the context of a New Keynesian model and within a searching and matching framework. The main conclusions of Thomas (2008) are as follows: (a) if the economy is on an efficient steady state equilibrium and all hiring wages are identical, inflation should be zero to have an optimal equilibrium; and (b) if the bargaining of the nominal wages creates differences between workers (which according to Thomas (2008), corresponds to a more realistic scenario), there will be price instability, which should be mitigated with a controlled monetary policy of price inflation to guarantee that wage dispersion is not too excessive.

Chu and Cozzi (2014) studies the effects of monetary policy on economic growth within a Schumpeterian growth model featuring cash-in-advance (CIA) constraints on consumption and R&D. After the development of the model, Chu and Cozzi (2014) reaches to some conclusions regarding CIA constraints, namely that: (a) if there are CIA constraints on consumption and R&D, an increase in the nominal interest rate would decrease R&D investment and economic growth; and (b) if the effect of CIA constraint on R&D dominates the CIA constraint on consumption, the nominal interest rate generates negative impacts on R&D and economic growth. These authors also discussed the optimality and suboptimality of the Friedman rule (hypothetical zero or near-zero nominal interest rate) and how that monetary policy influences the investment of economies on R&D.

According to Burdett and Mortensen (1998), wage dispersion occurs due to the different characteristics of the labor force since workers can be employed or unemployed and that situation changes the value of the necessary compensation that firms need to give to workers. These authors argue that if an individual is employed, the wage required for him to switch from on job to another is higher than the required wage by other individuals who are unemployed. The rationale behind this theory is that unemployed workers are willing to accept a lower wage to enter the market and the ones that are already employed demand a higher compensation to change their professional occupation. If we consider that these two types of workers have similar skills to execute a job, according to Burdett and Mortensen (1998), we can observe the phenomenon of wage dispersion.

Postel-Vinay and Robin (2002) follows the theory developed by Burdett and Mortensen (1998) and also discusses the on-the-job search as one factor for the existence of wage dispersion, but with some improvements to their model. These authors argue that search frictions are a source of inefficiency and that is why there are wage differences between firms and wage dispersion between workers. A key aspect in the research of these authors is the introduction of asymmetric information between employers and employees. On one hand, both know the type of each other (in the case of employees, if they are employed or not and in the case of employers, the type of the firm). One the other hand, when a firm makes a proposal to a potential worker, he may have alternative offers from other firms, so there is more bargaining power between individuals and firms at the time of wage definition, allowing this way, variations in the wage dispersion between identical workers.

According to Aghion and Howitt (1992), growth results exclusively from technological progress, which means that firms have incentives to invest in research to innovate the production system. The concept of creative destruction was introduced in a mathematical model by these authors and, in a simply way, it suggests that if firms are innovative enough, they will benefit from monopolistic rents until the next innovation is introduced in the market.

Regarding the connection between economic growth and wage dispersion, Carré and Drouot (2004) argues that with improvements in the technology, workers must be able to adapt to new productive realities, leading to a situation of "on-the-job learning", in which workers must adapt to the technological progress present in the economy. According to Carré and Drouot (2004), this learning effect can smooth the dispersion in wages as well as to offset the concept of creative destruction, once that less skilled workers can learn on-the-job, so they are not affected by the pace of technological change that could drag them to an unemployment situation.

Acemoglu (2002) supports the idea that there are two forces affecting the equilibrium bias of technology, the price effect and the market size effect. To achieve this results, Acemoglu (2002) studies the influence of the direction of technical change on the equilibrium bias of technology and the result is that price effect and market size effect are substitutes, so it is the elasticity of substitution between the factors the key to understand the power of the effects. Acemoglu (2002) also presents the innovation possibilities frontier as determinant of equilibrium bias of technology.

Contrarily to Acemoglu (2002), Afonso (2006) eliminates the market size and scale effects, and argues that the rise of skill premium can be explained through a combination between the price channel effect and what he called as the technological-knowledge-absorption effect. Afonso (2006) supports the idea that technological-knowledge progress is influenced by the stock of skilled workers and that stock of qualified workers will determine the technological-knowledge bias. Afonso (2006) argues that if the amount of skilled labor available increase, there is an expansion in the technological-knowledge-absorption effect, creating conditions for the R&D to be redirected to improve intermediate goods, reducing this way the final price of goods for the existent technology.

3 Model

In the following section, we introduce, describe, and analyze an endogenous growth model with cash-in-advance (CIA) constraints on R&D and two types of workers: low and high-skilled.

As a baseline, we closely follow Chu and Cozzi (2014) approach but we allow for two types of workers, i.e., high and low-skilled workers (Barro and Sala-i-Martin, 2004).

As in the standard R&D literature, our model have three different sectors: final good sector, intermediate sector and R&D sector. The low-skilled workers can only participate on the final and intermediate good sectors, whereas the high-skilled workers can only work on the final good and R&D sectors.

3.1 The final good sector

Following Afonso (2016), the final good sector produce an homogeneous good, in a perfect competition scenario, which production function at time t is given by:

$$y_t = \frac{1}{1-\alpha-\beta} L_{y,t}^{\alpha} \left(\gamma H_{y,t}\right)^{\beta} \left(\int_0^1 x_t\left(j\right)^{1-\alpha-\beta} dj\right), \tag{1}$$

Where $x_t(j)$ denotes intermediate goods $j \in [0,1]$,and $H_{y,t}$ and $L_{y,t}$ corresponds to the high-skilled and low-skilled labor, respectively, used in the final goods production. Since that there are two types of workers in the final good sector, firms must adjust their labor force by hiring low skilled workers $L_y \leq L$ and high skilled workers $H_y \leq H$. Note that, although both of the skilled types are required, their obligations are different: while low skilled workers only execute straightforward tasks, the high skilled must perform more demanding ones. Furthermore, by allowing $\gamma \geq 1$, we imply that there is a difference in the productivity of workers, this is, the high skill labor is, in absolute terms, more productive that the low skill labor. To produce the final good, firms apply j^{th} types of non-durable intermediate goods X_j. Moreover, α , β and $1-\alpha-\beta$ represent, respectively, the shares of unskilled labor, skilled labor and intermediate goods. In equilibrium, the skill premium, i.e., the relative wage of skilled over unskilled labor, is greater than one.

From profit maximization, the demand function for $x_t\left(j\right)$, $H_{y,t}$ and $L_{y,t}$ are, respectively:

$$x_t\left(j\right)=\left(\frac{1}{p_t\left(j\right)}\right)^{\frac{1}{\alpha+\beta}}\left[L_{y,t}^{\alpha}\left(\gamma H_{y,t}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}}, \tag{2}$$

$$w_{h,t}=\frac{\beta}{1-\alpha-\beta}L_{y,t}^{\alpha}\left(\gamma H_{y,t}\right)^{\beta-1}\left(\gamma\right)\left(\int_0^1 x_t\left(j\right)^{1-\alpha-\beta}dj\right) \tag{3}$$

$$w_{l,t}=\frac{\alpha}{1-\alpha-\beta}L_{y,t}^{\alpha-1}\left(\gamma H_{y,t}\right)^{\beta}\left(\int_0^1 x_t\left(j\right)^{1-\alpha-\beta}dj\right) \tag{4}$$

where $p_t(j)$ is the price of $x_t\left(j\right)$, $w_{h,t}$ is the wage for high-skilled workers, and $w_{l,t}$ for low-skilled workers.

3.2 The intermediate good sector

In the final good sector we have multiple identical firms producing a homogeneous product under perfect competition, using j^{th} types of intermediate goods. In the intermediate sector there are symmetric firms, but contrary to the final good sector, these firms produce differentiated intermediate goods. There is a firm temporarily leading each industry until the arrival of the next innovation.Upon each new innovation, the industry leader is surrogate by the firm owning the new innovation. The production function for the industry leader j at time t is:

$$x_t(j) = Z^{q_t(j)} L_{y,t}(j) \tag{5}$$

The parameter z ($z > 1$) measures the step size of a productivity improvement, $q_t(j)$ is the number of improvements that took place in industry j at time t and $L_{x,t}(j)$ is product0ion low skill labor on industry j. Following Chu and Cozzi (2014) and Peretto (1998), $MC_t(j) = \frac{w_{l,t}}{z^{q_j(t)}}$, i.e., we follow a cost-reducing perspective of vertical innovation, for a given $Z^{q_t(j)}$.

As firms compete on prices, if we follow the Bertrand price competition approach, the price that maximizes the profit of the firm will be a markup $\mu = p_t(j)/MC_t(j)$ over the marginal cost. For the purpose of our model, we will assume that the markup $\mu > 1$ is a policy instrument defined by the patent authority who regulates the sector (Chu and Cozzi, 2014). Therefore the amount of monopolistic profit is defined as:

$$\Pi_t(j) = p_t(j)\, x_t(j) - w_{L,t} L_{x,t} \tag{6}$$

Following Neto et al (2017) and Acemoglu (2002), we normalize the prices to unity, i.e., $p_t j = 1$. Hence:

$$\Pi_t(j) = \left(\frac{\mu - 1}{\mu}\right) p_t(j) X_t(j) = \left(\frac{\mu - 1}{\mu}\right) G, \tag{7}$$

for matters of simplicity, lets consider that $G = \left[L_{y,t}^{\alpha} \left(\gamma H_{y,t}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}}$.

Finally, we can conclude that production-labor income is:

$$w_{L,t} L_{x,t} = \frac{1}{\mu} p_t(j)\, x_t(j) = \frac{1}{\mu} G \tag{8}$$

3.3 R&D Sector

The value of the monopolistic firm in industry j is symbolized by $v_t(j)$. If we assume that there is a symmetric equilibrium between industries, $v_t(j)$ will simply be v_t. For this reason, the familiar no-arbitrage condition for the value of the monopolistic firm is:

$$r_t = \frac{\pi_t + \dot{v}_t - \lambda_t v_t}{v_t} \tag{9}$$

With this equation, we state that the real interest rate r_t is equal to the asset return per unit of asset. That asset return is estimated by the sum of the monopolist profit (π_t), potential capital gain ($\dot{v}_t$) and expected capital loss due to creative destruction ($\lambda_t v_t$). λ_t is defined as the arrival rate of the next innovation.

Following Chu and Cozzi (2014), in our model, there is an unit continuum of R&D firms indexed by $k \in [0, 1]$, and we make two essential assumptions. The first one is that only high skill labor $H_{r,t}(k)$ works on this sector and the second assumption is that R&D firms faces a CIA constraint and need to borrow money from households, $B_t(k)$, subject to the nominal interest rate to pay the entire wage bill. Therefore, the total amount of money borrowed is $B_t(k) = w_{t,H}.H_{r,t}(k)$, and the total cost of R&D per unit of time is $(1 + i_t)\left[w_{t,H} H_{r,t}(k)\right]$.

If the interest rate fluctuates, it will influence the hiring decisions of the entrepreneurs, thus the monetary authority can affect the allocation equilibrium of labor resources. The zero-expected-profit condition of firm k is give by:

$$v_t \lambda_t \left(k\right) = \left(1 + i_t\right) w_{H,t} H_{r,t} \left(k\right) \tag{10}$$

where $\lambda_t(k)$ is the innovation arrival rate per unit of time t of firm k, that is given by $\lambda_t(k) = \bar{\varphi} H_{r,t}(k)$, with $\bar{\varphi} = \frac{\varphi}{N_t}$ capturing the dilution effect that removes the scale effects as described in Lainez and Peretto (2006). If we combine the different arrival rates of innovation from all the k firms in the sector, we will get the aggregate arrival rate of innovation as follows:

$$\lambda_t = \int_0^1 \lambda_t(k)\partial k = \frac{\varphi H_{r,t}}{N_t} = \varphi h_{r,t} \tag{11}$$

with $h_{r,t} \equiv \frac{H_{r,t}}{N}$ as the R&D high skill labor per capita.

3.4 Households

At time t, the population size of each household is N_t, and its law of motion is $\dot{N}_t = nN_t$, where $n \geq 0$ is the exogenous growth rate of the population. The utility function of the identical households of the population is give by:

$$U = \int_0^1 e^{-\rho t}\left[lnc_{u,t} + \theta ln\left(1 - u_t\right)\right] dt, \tag{12}$$

where $u_t = h_t, l_t$, i.e., high-skilled and low-skilled labor, $c_{u,t}$ is the consumption of final goods per capita and u_t is the supply of labor per capita at time t. The parameters $\rho > 0$ and $\theta \geq 0$ measure subjective discounting and leisure preference, respectively.

In order to maximize their utility function, households are subject to the next asset-accumulation equation:

$$\dot{a_{u,t}} + \dot{m}_t = \left(r_t - n\right) a_{u,t} + w_{u,t}u_t + \tau_t - c_{u,t} - \left(\pi_t + n\right) m_{u,t} + i_t b_{u,t} \tag{13}$$

Where $a_{u,t}$ is the real value of assets owned by each member of households and r_t is the real interest rate. To earn a wage $w_{u,t}$ each household supplies labor u_t. The government obtains or gives a lump-sum transfer τ_t to households. The inflation rate is given by π_t and the real money balance that individuals retain is $m_{u,t}$. $b_{u,t}$ is the amount of money lend by each household to the investors of R&D firms and i_t is the interest rate associated to $b_{u,t}$.

Applying standard dynamic optimization, we compute a no-arbitrage condition given by $i_t = r_t + \pi_t$ and, then we can infer that i_t also represents the nominal interest rate.

The optimality conditions for consumption and labor supply are, respectively,

$$\frac{1}{c_{u,t}} = \eta_{u,t} \tag{14}$$

$$w_{u,t}\left(1 - u_t\right) = \theta c_{u,t} \tag{15}$$

For simplicity, we assume that there is an exogenous threshold such that $c_{h,t} = sy_t$ and $c_{l,t} = \left(1 - s\right) y_t$, with $s > 0.5$. Hence:

$$w_{h,t}\left(1 - h_t\right) = \theta s y_t \tag{16}$$

$$w_{l,t}\left(1 - l_t\right) = \theta\left(1 - s\right) y_t \tag{17}$$

The familiar inter-temporal optimality condition is:

$$-\frac{\dot{\eta}_t}{\eta_t} = r_t - \rho - n \tag{18}$$

3.5 The monetary authority

The aim of this dissertation is to analyze the impact of the monetary policy on wage dispersion and economic growth and, therefore, the role of the monetary authority is key to understand this same impact. Despite all the monetary policy instruments that exist, we will only study the impact of exogenously changing the nominal interest rate on wages at the equilibrium, in line with Chu and Cozzi (2014).

Hence, M_t stands for the nominal money supply and $\frac{\dot{M}_t}{M_t}$ is the growth rate. Assuming that the monetary authority set an exogenously i_t, then we can calculate endogenously the inflation rate as $\pi_t = i_t - r_t$. Given π_t, the growth rate of the nominal money supply is endogenously given by $\frac{\dot{M}_t}{M_t} = \frac{\dot{m}_t}{m_t} + \pi_t + n$. Finally, households receive a lump transfer $\tau_t N_t = \frac{\dot{M}_t}{P_t} = (\dot{m}_t + [\pi + n] m_t) N_t$ due to seigniorage revenue from the monetary authority.

3.6 Decentralized equilibrium

The equilibrium is a combination of allocations and prices, namely a time path of allocations $\{c_{l,t}, c_{h,t}, m_{l,t}, m_{h,t}, l_{x,t}, l_{y,t}, h_{y,t}, h_{r,t}, y_t, x_t(j), L_t(j), H_t(k)\}$ and a time path of prices $\{p_t(j), w_{l,t}, w_{h,t}, r_t, i_t, v_t\}$

Additionally, at time t,

- households maximize utility assuming $\{w_{l,t}, w_{h,t}, r_t, i_t\}$;
- competitive final-good firms maximize their profit by producing y_t and taking $\{p_{x,t}(j), w_{l,t}, w_{h,t}\}$ as given;
- in the intermediate good sector, as firms have market power, they choose to produce $\{x_t(j)\}$, to hire $\{L_{x,t}(j)\}$ and sell the goods at $\{p_t(j)\}$ to maximize profit, assuming $\{w_{l,t}\}$ as given;
- R&D firms maximize expect profit by employing $\{H_{r,t}(k)\}$, taking $\{w_{h,t}, i_t, v_t\}$ as given;
- the market clearing condition for low-skilled labor is given by $L_{x,t} + L_{y,t} = L_t$;
- the market clearing condition for high-skilled labor is given by $H_{r,t} + H_{y,t} = H_t$;

- the market clearing condition for final good sector holds that $y_t = (c_{l,t} + c_{h,t})N_t$;
- the value of the assets of households increases with the value of monopolistic firms such that $v_t = a_{u,t}N_t$;
- R&D entrepreneurs borrow money from households and the total amount holds such that $w_{h,t}H_{r,t} = b_{u,t}N_t$

Substituting (5) into (1), we find the aggregate production function:

$$y_t = \frac{1}{1-\alpha-\beta} Z_t^{1-\alpha-\beta} L_{x,t}^{1-\alpha-\beta} L_{y,t}^{\alpha} \left(\gamma H_{y,t}\right)^{\beta}, \tag{19}$$

where aggregate technology Z_t is defined as:

$$Z_t = exp(\int_0^1 q_t(j) dj ln z) \tag{20}$$

Following Chu and Cozzi (2014), after some mathematical manipulations, the growth rate of aggregate technology is given by:

$$g_t = \lambda_t ln(z) = \varphi ln(z) h_{r,t} \tag{21}$$

It is possible to prove that given a constant nominal interest rate i, the economy immediately jumps to a unique and saddle point stable balanced growth path along which each variable grows at a constant rate, in line with Chu and Cozzi (2014).

Following the previous paragraph, we know that labor allocations are stationary under the balanced growth path. Imposing balanced growth on 9, we set $v_t = \frac{\Pi_t}{\rho+\lambda}$, taking into account that $\frac{\dot{\pi}}{\pi} = g + n$ and $r = g + p + n$, from (18). Hence, following Chu and Cozzi (2014), combining the next four equations we get:

$$\begin{cases} v_t = \frac{\Pi_t}{\rho+\lambda} \\ v_t \lambda_t \left(k\right) = \left(1 + i_t\right) w_{h,t} H_{r,t} \left(k\right) \\ \lambda = \varphi h_{r,t} \\ \Pi_t = (\frac{\mu-1}{\mu}) \left[L_{y,t}^{\alpha} \left(\gamma H_{y,t}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}} \end{cases}$$

$$(\frac{\mu-1}{\mu})(1-\alpha-\beta)h_y = (1+i)\beta(\frac{\rho}{\varphi} + h_r) \tag{22}$$

This corresponds to the first of three equations to obtain the allocation if high-skilled workers. To get the second one needs to substitute (3) on (16) to obtain:

$$\beta(1-h) = \theta s h_y \tag{23}$$

Finally, the last equation will be the market clearing for high-skilled labor, as:

$$h = h_r + h_y \tag{24}$$

Solving (22) - (24), we find the equilibrium for high skill labor allocations:

$$h_y = \frac{(1+i)\beta}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta) + (1+i)(\theta s + \beta)]}(\frac{\rho}{\varphi} + 1) \tag{25}$$

$$h_r = \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta) + (1+i)(\theta s + \beta)]}(\frac{\rho}{\varphi} + 1) - \frac{\rho}{\varphi} \tag{26}$$

$$h = \frac{(1+i)\beta + (\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta) + (1+i)(\theta s + \beta)]}(\frac{\rho}{\varphi} + 1) - \frac{\rho}{\varphi} \tag{27}$$

Equation (26) gives insights of how R&D labor is affected by the nominal interest rate since an increase of i leads to a decrease in h_r under both elastic and inelastic labor supply. Moreover, the economic growth g, which is defined by $g = \varphi ln(z)h_r$, is also decreasing in the nominal interest rate. Note that, substituting h_r on g, we get $g = [\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)(\rho+\varphi)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]} - \rho]ln(z)$ and $\frac{\partial g}{\partial i} < 0$. Chu and Lai (2013) already support the idea that i have a negative impact on g through the inflation rate, i.e., the authors defend that there is a negative relationship between R&D and inflation. As in Chu and Cozzi (2014), $\pi = i - r = i - g(i) - \rho - n$, so with an increase on i, π also increases, as opposed to h_r and g that decreases under these circumstances.

PROPOSITION 1 *R&D and economic growth both decrease with an increase of the nominal interest rate.*

Regarding low-skilled labor, it is interesting to note that, due to the definition of the model, its allocation across sectors will not depend in the nominal interest rate.

Hence, following a similar approach, combining (4) with (8):

$$\alpha\mu l_x = (1 - \alpha - \beta) l_y \tag{28}$$

and (4) with (17), we get:

$$\alpha (1 - l) = \theta (1 - s) l_y \tag{29}$$

Finally, the market clearing condition for low-skilled labor is:

$$l = l_x + l_y \tag{30}$$

Solving (28)-(30), we obtain the equilibrium allocations for low-skilled labor:

$$l_x = \frac{(1-\alpha-\beta)}{\theta\mu\,(1-s) + (1-\alpha-\beta) + \alpha\mu} \tag{31}$$

$$l_y = \frac{\alpha\mu}{\theta\mu\,(1-s) + (1-\alpha-\beta) + \alpha\mu} \tag{32}$$

$$l = \frac{(1-\alpha-\beta) + \alpha\mu}{\theta\mu\,(1-s) + (1-\alpha-\beta) + \alpha\mu} \tag{33}$$

3.7 Socially Optimal allocation

Following Chu and Cozzi (2014), it is possible to derive the socially optimal allocations of the model. Imposing balanced growth on (12), yields:

$$U = \frac{1}{\rho}\left[lnc_0 + \frac{g}{\rho} + ln\,(1-l_u)\right], \tag{34}$$

where $C_0 = \frac{1}{1-\alpha-\beta} z_0^{1-\alpha-\beta} l_{y,t}^{\alpha} (\gamma h_{y,t})^{\beta} l_x^{1-\alpha-\beta}$, $g = (\varphi lnz)h_r$ and the exogenous z_0 is normalized to unity.

Maximizing the previous equation subject to $l = l_y + l_x$ and to $h = h_y + h_r$, we get the first best allocations, denoted with a subscript *:

$$h_r^* = 1 - \frac{(\theta+\beta)\rho}{(\varphi lnz)} \tag{35}$$

$$h_y^* = \frac{\rho\beta}{(\varphi lnz)} \tag{36}$$

$$h^* = 1 - \frac{\theta\rho}{(\varphi lnz)} \tag{37}$$

$$l_y^* = \frac{\alpha}{(1+\theta-\beta)} \tag{38}$$

$$l_x^* = \frac{1-\beta-\alpha}{1+\theta-\beta} \tag{39}$$

$$l^* = \frac{1-\beta}{1+\theta-\beta} \tag{40}$$

4 Optimal Monetary Policy and Friedman Rule

In this section we provide a detailed analysis regarding the optimal nominal interest rate. As stated in the previous section, taking into account that only high-skilled labor is directly affected by the nominal interest rate, the identification on the optimal monetary policy will only take into account $h_y,\ h_r,\ and\ h$. Nevertheless, note that the relative allocations of labor and the respective wage dispersion will be affected by the optimal interest rate.

We first analyze the case of inelastic labor supply (section 4.1); and, then, we study the general scenario of elastic labor supply (section 4.2). Note that, as in Chu and Cozzi (2014) and Neto et al (2017), it is possible to study where (a) the optimal interest rate enables the first best socially optimal allocations, and (b) R&D overinvestment or underinvestment is a necessary and sufficient condition for the Friedman rule to be suboptimal. Nevertheless, for simplicity, we will not focus our analysis on these two topics, but rather on the simulations and the respective wage dispersion, once the optimal monetary policy is defined.

4.1 Friedman Rule under Inelastic Labor Supply

Under inelastic labor supply ($\theta = 0$), the equilibrium allocations simplifies to:

$$h_y = \frac{(1+i)\beta}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)\beta}(\frac{\rho}{\varphi}+1) \tag{41}$$

$$h_r = \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)\beta}(\frac{\rho}{\varphi}+1)-\frac{\rho}{\varphi} \tag{42}$$

As $\theta = 0$, $h = 1$. Just by evaluating the equations (41) and (42), it is clear that an increase in the nominal interest rate prompts a decrease (increase) in the R&D

(final good) high-skilled labor h_r (h_y). Thus, i can be seen as a CIA constraint on R&D investment since an increase in i raises R&D costs, which drives a transfer of high-skilled labor from the R&D to the final good sector.

Under inelastic labor supply, by choosing the optimal interest rate i^*, the monetary authority may be able to achieve the first best allocations $\{h_r^*, h_y^*\}$, as follows:

$$i^* = max[\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta) - [(1+\frac{\varphi}{\rho})lnz - \beta]}{(1+\frac{\varphi}{\rho})lnz - \beta}\ ,\ 0] \tag{43}$$

We impose $i^* > 0$ to respect the zero lower bound on the nominal interest rate.

Following Chu and Cozzi (2014), it is possible to infer (or suspect) that if $i^* = 0$, then Friedman rule is found to be optimal, nevertheless the monetary authority is unable to match the first-best allocations. In the case of $i^* > 0$, the Friedman rule is suboptimal, nonetheless the monetary authority is able to reach the first-best allocations assuming that $i = i^*$. This will be analyzed in the next section.[2]

Overinvestment in R&D is a necessary and sufficient condition for the Friedman rule to be suboptimal, well as whether a positive i^* leads to the first-best labor allocations.

Regarding the several variables that influence i^*(when $i^* > 0$), we can infer that an increase in μ, leads to an increase in the optimal interest rate. Intuitively, following Chu and Cozzi (2014), a higher μ implies a larger protection of an innovation, leading to a higher probability of overinvestment in R&D. Additionally, i^* is also increasing in ρ, because if the discount rate is higher, the probability of overinvestment in R&D is larger. On the other hand, i^* is decreasing in φ *and* z, since that when the R&D productivity φ or the step size z of innovation is larger, for the same economic growth rate, we need less R&D workers, hence the probability of underinvestment is higher. One of the differences of our model when compared with Chu and Cozzi (2014) is that we allow the participation of workers (high- and low-skilled) in the production function. Interestingly, $i^* > 0$ is decreasing in α and

[2]Indeed, making a comparison between (42) and (45) under $\theta = 0$, it is possible to check whether mathematically.

β. One explanation might be that the more workers needed in the final good sector, the more likely underinvestment is to occur.

4.2 Friedman Rule under Elastic Labor Supply

Following a similar approach as in the previous section, we can derive the optimal nominal interest rate under elastic labor supply as:

$$i^* = max[\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)-\Omega}{\Omega} \ , \ 0], \tag{44}$$

with Ω as a parameter for the following condition:.

$$\Omega = \frac{(\theta s+\beta)}{(\beta+\theta)}(1+\frac{\varphi}{\rho})lnz - \theta s - \beta \tag{45}$$

Note that, in this case, i has a distortionary effect on the consumption-leisure decision. Hence, it is possible to infer that, following Chu and Cozzi (2014), i^* can no longer achieve the first-best allocations.

From (43), we can prove that $h_r^* > 0$ is sufficient to ensure that $\Omega > 1$.

Following, once again, Chu and Cozzi (2014), we can infer that $(\frac{\mu-1}{\mu})(1-\alpha-\beta) > \Omega$ implies R&D overinvestment. Furthermore, note that the comparative statistics from the previous section apply.

In the next section, we numerically simulate in detail the dynamics and main implications of i^* .

5 Simulations

This section closely follows the methodology proposed by Chu and Cozzi (2014). We provide a numerical simulation on the optimality of the Friedman rule, under both inelastic and elastic labor supply. To provide the results available below, we need to identify the following set of parameters $\{\rho,\ z,\ \mu,\ \varphi,\ \theta,\ i_{LR},\ \beta,\ \alpha,\ s,\ h\}$. i_{LR} defines the nominal interest rate in the long run, which is set to 0.08, in line with Chu and Cozzi (2014). Considering Acemoglu and Akcigit (2012), we set the step size of innovation z to 1.02 and the discount rate ρ to 0.05. For the markup μ, we follow Neto et al (2017) and Reis and Sequeira (2007) to set $\mu = 2$, a slightly higher value thank Chu and Cozzi (2014). For the other set of parameters $\{\beta,\ \alpha,\ s,\ h\}$, we have established $\beta > \alpha$, which means that the share of high-skilled workers is bigger than the share of low-skilled workers in the final good sector, in line with the average values reported by Afonso (2016), based on OECD statistics. $s = 0.5$, i.e., it is assumed that the consumption of goods is equal for both types of workers, and $\varphi = 1.2$.

In line with Chu and Cozzi (2014), we fix $g = 0.02$, corresponding to the long-run economic growth rate of the United States. Nevertheless, it is important to consider the argument that R&D might not be the only source of economic growth (Comin, 2004). Hence, we consider several other possible lower economic growth rates, based on this idea that R&D investments can only explain a fraction, f, of the long-run economic growth rate. Taking into account that f corresponds to a lower growth rate, we can extend our analysis to different levels of growth.

Finally, regarding θ, we set it in a way to match a standard moment of $h = 0.3$[3].

Table 1 summarizes the values for each parameter:

Table 1: Baseline Parameters

ρ	z	μ	g	i_{LR}	α	β	s	h
0.05	1.02	2	0.02	0.08	0.2	0.5	0.5	0.3

[3] $h = 0.3$ can be seen as high-skilled workers working one third of the day.

The simulation results under inelastic and elastic labor supply are reported in table 2 and table 3. It is interesting to note that the Friedman rule is optimal for almost all of the considered scenarios, i.e., it seems that there is no room for the monetary authority to improve social welfare. Nevertheless, there is a particular case where the Friedman rule is not optimal: it corresponds to the lowest growth rate scenario considered ($g = 0.60\%$). In other words, an $i^* > 0$ is actually socially desired if the growth rate is below a particular level.

By analyzing the allocation of the high-skilled labor, it is possible to conclude that $h_r(under\ i^* > 0) < h_r(under\ i^* = 0)$. The mechanism behind can be explained as follows: under a low economic growth rate, setting $i^* = 0$ leads to an overinvestment in R&D (through the allocation of high-skilled labor) in terms of social welfare. Hence, setting $i^* > 0$ (in this case, $i^* = 0.05$) leads to a reallocation of high-skilled labor between the two sectors, which connects to the overinvestment phenomenon. Additionally, by analyzing and comparing $h_r(under\ i^* > 0)$, $h_r(under\ i^* = 0)$, and the optimal, h_r^*, we can see that, as expected in section 4.1, $i^* > 0$ (in this case under study, $i^* = 0.05$), can achieve the first best optimal allocation of labor. Hence, $h_r(under\ i^* > 0) = h_r^* < h_r(under\ i^* = 0)$. [4]

Finally, regarding wage dispersion, by comparing its values under $i^* > 0$ and $i^* = 0$, we can conclude that applying the Friedman rule (and not $i^* = 0.08$) contributes to an increase in the wage dispersion. This result states that, below a specific economic threshold, wage dispersion can be reduced by setting $i^* = 0$. In this case, we go from a wage ratio 1.800 to 1.781.

Table 2: Inelastic Labor Supply

f	1	0,9	0,8	0,7	0,6	0,5	0,4	0,3
g	2,00%	1,80%	1,60%	1,40%	1,20%	1,00%	0,80%	0,60%
φ	4,83	4,36	3,90	3,43	2,97	2,50	2,04	1,57
i*	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,05
$\frac{W_h}{W_l}$	5,461	4,935	4,409	3,884	3,358	2,832	2,307	1,781

[4] In this case, $0.198 < 0.2063$;

Table 3: Elastic Labor Supply

f	1	0,9	0,8	0,7	0,6	0,5	0,4	0,3
g	2,00%	1,80%	1,60%	1,40%	1,20%	1,00%	0,80%	0,60%
φ	16,09	14,54	12,99	11,44	9,89	8,34	6,79	5,25
θ	2,95	2,95	2,94	2,94	2,93	2,92	2,91	2,89
i*	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
$\frac{W_h}{W_l}$	1,161	1,160	1,160	1,160	1,159	1,159	1,158	1,156

6 Conclusions

In this thesis we have developed an endogenous growth model with cash-in-advance (CIA) constraints in R&D to study the impacts of monetary policy on wage dispersion and labor allocation.

Section 2 provided an in-depth literature review on the topic. We concluded that, to the best of our knowledge, there was no theoretical study dealing specifically with the relationship between monetary policy and wage dispersion, under an endogenous growth model with CIA in R&D.

Section 3 presented the theoretical model, following the contributions of Chu and Cozzi (2014). We introduced two types of workers into the model: (a) high-skilled workers who can be employed in the final and R&D sectors; and (b) low-skilled workers who can be employed in the final and intermediate good sectors. A monetary authority was introduced aiming to maximize welfare by setting the optimal nominal interest rate. This allowed us to study not only the impact on monetary policy on wage dispersion but also to test the optimality of the Friedman rule (i.e., whether optimal nominal interest rate should be zero). Hence, from the theoretical model, the main conclusions are the following. First, under inelastic labor supply, Friedman rule might not be optimal for low economic growth rates. Furthermore, a positive but low interest rate can contribute to a lower wage dispersion between high- and low-skilled workers. Second, under elastic labor supply, Friedman rule seems to be optimal for all the considered scenarios. Therefore, it is possible to infer that: (i) a common monetary policy among several countries, as the Eurozone, might not be optimal for all types of countries, based on their economic growth rates; (ii) a correct setting of the nominal interest rate might contribute to lower the wage dispersion between high- and low-skilled workers.

Finally, as future work, it would be interesting to extend this analysis by introducing trade unions and their role of bargaining low-skilled wages to study the interdependency between labor market policies and monetary policy.

7 Appendix

7.1. Final good sector

We consider production output function:

$y_t = \frac{1}{1-\alpha-\beta} L_{y,t}^{\alpha} (\gamma H_{y,t})^{\beta} (\int_0^1 x_t(j)^{1-x-\beta} \partial j)$ (1)

From profit maximization we get:

$\frac{\partial Y_t}{\partial x_t(j)} = 0 \iff$

$\left[\frac{1}{1-\alpha-\beta} . L_{y,t}^{\alpha} (\gamma H_{y,t})^{\beta} \right] \left[\int_0^1 x_t(j)^{1-\alpha-\beta} \partial j \right]' = 0 \iff$

$\left[\frac{1}{1-\alpha-\beta} . L_{y,t}^{\alpha} (\gamma H_{y,t})^{\beta} \right]' . \int_0^1 x_t(j)^{1-\alpha-\beta} \partial j + \left[\frac{1}{1-\alpha-\beta} . L_{y,t}^{\alpha} (\gamma H_{y,t})^{\beta} \right] . \left[\int_0^1 x_t(j)^{1-\alpha-\beta} \partial_y \right]' = 0 \iff$

$\left[\frac{1}{1-\alpha-\beta} . L_{y,t}^{\alpha} (\gamma H_{y,t})^{\beta} \right] . \left[x_t^{1-\alpha-\beta} \right]' = 0 \iff$

$\frac{1}{1-\alpha-\beta} . L_{y,t}^{\alpha} (\gamma H_{y,t})^{\beta} . (1-\alpha-\beta) x_t^{-\alpha-\beta} = 0 \iff$

$L_y^{\alpha} (\gamma H_y)^{\beta} . x_t^{-\alpha-\beta} = 0$

$\pi = P_t . y_t - P_t(j) . X_t(j) \iff$

$\pi = P_t . \left[\frac{1}{1-\alpha-\beta} . L_{y,t}^{\alpha} (\gamma H_{y,t})^{\beta} (\int_0^1 x_t(j)^{1-\alpha-\beta} \partial j) \right] - p_t(j) . x_t(j)$

$\frac{\partial \pi}{\partial x_t} = 0 \iff$

$p_t . L_y^{\alpha} (\gamma H_y)^{\beta} . X_t(j)^{-\alpha-\beta} - p_t(j) = 0 \iff$

$p_t . L_y^{\alpha} (\gamma H_y)^{\beta} . X_t(j)^{-\alpha-\beta} = p_t(j) \iff$

$x_t(j)^{-\alpha-\beta} = \frac{p_t(j)}{p_t . L_y^{\alpha} (\gamma H_y)^{\beta}} \iff$

$x_t(j)^{\alpha+\beta} = \frac{1}{p_t(j)} . p_t L_y^{\alpha} (\gamma H_y)^{\beta}$

Assuming $P_t = 1$,

$x_t(j) = (\frac{1}{p_t(j)})^{\frac{1}{\alpha+\beta}}.\left[L_y^{\alpha}(\gamma H_y)^{\beta}\right]^{\frac{1}{\alpha+\beta}}$,considering $G = \left[L_y^{\alpha}(\gamma H_y)^{\beta}\right]^{\frac{1}{\alpha+\beta}}$

$\Longleftrightarrow x_t = (\frac{1}{p_t(j)})^{\frac{1}{\alpha+\beta}}.G$ (2)

To get the wages of the final good sector, we must derivate $y_t$in order to labor:

$$\frac{\partial y_t}{\partial H_y} = w_h \Longleftrightarrow$$
$$w_{h,t} = \left[\frac{1}{1-\alpha-\beta}.L_{y,t}^{\alpha}(\gamma H_{y,t})^{\beta}(\int_0^1 x_t(j)^{1-\alpha-\beta}\partial j)\right]' \Longleftrightarrow$$
$$w_{h,t} = \left[\frac{\beta}{1-\alpha-\beta}.L_{y,t}^{\alpha}(\gamma H_{y,t})^{\beta-1}.\gamma(\int_0^1 x_t(j)^{1-\alpha-\beta}\partial j)\right] \Longleftrightarrow$$
$$w_{h,t} = \left[\frac{\beta}{1-\alpha-\beta}.L_{y,t}^{\alpha}\frac{(\gamma H_{y,t})^{\beta}}{\gamma.H_{y,t}}.\gamma(\int_0^1 x_t(j)^{1-\alpha-\beta}\partial j)\right]$$

If $G = \left[L_{y,t}^{\alpha}(\gamma H_{y,t})^{\beta}\right]^{\frac{1}{\alpha+\beta}}$,then $L_{y,t}^{\alpha}.(\gamma H_{y,t})^{\beta} = G^{\alpha+\beta}$,so:

$w_{h,t} = \frac{\beta}{1-\alpha-\beta}.\frac{G^{\alpha+\beta}}{H_{y,t}}(\int_0^1 x_t(j)^{1-\alpha-\beta}\partial j)$ (3)

And, therefore:

$w_{h,t} = \frac{\alpha}{1-\alpha-\beta}.\frac{G^{\alpha+\beta}}{L_{y,t}}(\int_0^1 x_t(j)^{1-\alpha-\beta}\partial j)$ (4)

7.2. Intermediate good sector

$x_t(j) = Z^{q_t(j)}.L_{y,t}(j)$ (5)

z: step of productivity improvement;

q : # of improvements that have occurred in industry j at time t;

$L_{x,t}(j)$: production labor/low skill labor on industry j.

Following Perreto (1998), $MC_t(j) = \frac{W_t}{z^{q_j(t)}}$

Markup: $\mu = \frac{p_t(j)}{MC_t(j)}$

Notice that

$\pi_t(j) = \left(\frac{\mu-1}{\mu}\right).p_t(j).X_t(j) \iff$
$\pi_t(j) = (\frac{\mu}{\mu}).p_t(j).x_t(j) - (\frac{1}{\mu}).p_t(j).x_t(j) \iff$
$\pi_t(j) = p_t(j).x_t(j) - (\frac{1}{\mu}).p_t(j).x_t(j) \iff$

$\pi_t(j) = p_t(j).x_t(j) - w_l.L_{x,t}(j)$ (6)

$\Pi_t(j) = \left(\frac{\mu-1}{\mu}\right).p_t(j).X_t(j) = \left(\frac{\mu-1}{\mu}\right)G$ (7)

The production-labor income is:

$w_{L,t}L_{x,t} = \frac{1}{\mu}p_t\left(j\right)x_t\left(j\right) = \frac{1}{\mu}.G$ (8)

7.3. R&D sector

$r_t = \frac{\pi_t + \dot{v}_t - \lambda_t.v_t}{v_t}$ (9)

Two assumptions:

. R&D borrows money to pay wages
. only skilled workers work in the R&D sector

Total amount of money borrowed: $B_t(k) = \alpha w_{t,H}.H_{r,t}(k)$

Total amount of money borrowed plus interest: $B_t(k)(1+i) = [\alpha w_{t,H}.H_{v,t}(k)]\,(1+i)$

Since that R&D firms borrow money, the zero-expected-profit condition:

$$v_t \lambda_t (k) = (1 + i_t)\, w_{H,t} H_{r,t} (k) \quad (10)$$

Considering that the firm-level innovation arrival rate per unit of time is $\lambda_t(k) = \varphi.\bar{H_{r,t}}(k)$,

where $\bar{\varphi} = \frac{\varphi}{w}$ captures the dillution effect that removes the scale effects, we will have that:

$\lambda_t = \int_0^1 \lambda_t(k)\partial k = \frac{\varphi.H_{r,t}}{N_t} = \varphi.h_{r,t} \quad (11)$

7.4. Households

We have the utility function for households:

$$U = \int_0^1 e^{-\rho t} \left[\ln c_{u,t} + \theta \ln (1 - u_t)\right] dt, \quad (12)$$

And we have the restriction to maximize the utility respecting the asset-accumulation equation:

$$\dot{a}_t + \dot{m}_t = (r_t - n)\, a_t + w_{u,t} u_t + \tau_t - c_{u,t} - (\pi_t + n)\, m_t + i_t b_t \quad (13)$$

$$\text{Max Ham}_{C_{u_t},L_{u_t}} = U + \eta_t \left[\dot{a}_t + \dot{m}_t - \left[(r_t - n)\, a_t + w_{u,t} u_t + \tau_t - c_{u,t} - (\pi_t + n)\, m_t + i_t b_t\right]\right]$$

Note:

$\int_0^1 e^{-\rho t} \left[\ln c_{u,t} + \theta \ln (1 - u_t)\right] dt =$

$\left[e^{-\rho t}\left[\ln c_{u,t} + \theta \ln\left(1-u_t\right)\right]\right]_0^1 =$

$e^{-\rho t}\left[\ln c_u + \theta \ln\left(1-u\right)\right] - \left[e^{-\rho .0}\left[\ln c_u + \theta \ln\left(1-u_t\right)\right]\right] =$

$0 - \left[1.\left[\ln c_u + \theta \ln\left(1-u\right)\right]\right] =$

$-\left[\ln c_u + \theta \ln\left(1-u\right)\right]$

$\frac{\partial Ham}{\partial C_u} = 0 \iff$

$-\left[\ln c_u + \theta \ln\left(1-u\right)\right]' + \left[\eta_t\left[\dot{a}_t + \dot{m}_t - \left[\left(r_t - n\right)a_t + w_{u,t}u_t + \tau_t - c_{u,t} - \left(\pi_t + n\right)m_t + i_t b_t\right]\right]\right]' \iff$

$-\frac{1}{C_u} + \eta_{u,t} \iff$

$\frac{1}{C_u} = \eta_{u,t} \quad (14)$

If $U_t = h_z$, then $\frac{1}{C_u} = \eta_{u,t} \iff \frac{1}{C_H} = \eta_{H,t}$

If $U_t = l_l$, then $\frac{1}{C_u} = \eta_{u,t} \iff \frac{1}{C_L} = \eta_{L,t}$

$\frac{\partial Ham}{\partial l_U} = 0 \iff$

$-\left[\ln c_u + \theta \ln\left(1-l_u\right)\right]' + \left[\eta_t\left[\dot{a}_t + \dot{m}_t - \left[\left(r_t - n\right)a_t + w_{u,t}u_t + \tau_t - c_{u,t} - \left(\pi_t + n\right)m_t + i_t b_t\right]\right]\right] = 0 \iff$

$-\left[\theta.\frac{-1}{1-l_u}\right] - \eta_{u,t}.w_{u,t} = 0 \iff$

$\frac{\theta}{1-l_u} = \eta_{u,t}.w_{u,t}$

As $\frac{1}{C_{u,t}} = \eta_{u,t}$, then: $\frac{\theta}{1-l_u} = \frac{1}{C_{u,t}}.w_{u,t}(=)\ w_{u,t}(1-l_u) = \theta.C_{u,t}$ (15)

. If $u_t = h_t$,$w_{h,t}(1-h_t) = \theta.C_{h,t}$

. If $u_t = l$,$w_{l,t}(1-l_t) = \theta.C_{l,t}$

We assume that there is a threshold exogenously to the model such that $c_{h,t} = sy_t$ and $c_{l,t} = \left(1-s\right)y_t$, with $s > 0.5$. Hence:

$$w_{h,t}\left(1-h_t\right)=\theta s y_t \quad (16)$$

$$w_{l,t}\left(1-l_t\right)=\theta\left(1-s\right)y_t \quad (17)$$

The familiar intertemporal optimality condition is:

$$-\frac{\dot{\eta}_t}{\eta_t}=r_t-\rho-n \quad (18)$$

7.5. Decentralized Equilibrium

Substituting (5) into (1), we find the aggregate production function:

$$y_t=\frac{1}{1-\alpha-\beta}Z_t^{1-\alpha-\beta}L_{x,t}^{1-\alpha-\beta}L_{y,t}^{\alpha}\left(\gamma H_{y,t}\right)^{\beta}, \quad (19)$$

where aggregate technology Z_t is defined as:

$$Z_t=exp(\int_0^1 q_t(j)djlnz) \quad (20)$$

Growth rate of aggregate technology is given by:

$g_t=\lambda_t ln(z)=\varphi ln(z)h_{r,t}$ (21)

Imposing balanced growth for high skill labor, yields that:

-If we substitute (3) on (9), we have:

$v_t \lambda_t(k) = (1+i_t)[\frac{\beta}{1-\alpha-\beta}.\frac{G^{\alpha+\beta}}{H_{y,t}}(\int_0^1 x_t(j)^{1-\alpha-\beta}\partial j)]H_{r,t}(k)$

We also know that $v_t = \frac{\Pi_t}{\rho+\lambda}, \Pi_t = (\frac{\mu-1}{\mu})G, \lambda = \varphi.h_{r,t}(k)$, therefore:

$$\frac{\Pi_t}{\rho+\lambda}\lambda_t(k) = (1+i_t)[\frac{\beta}{1-\alpha-\beta}.\frac{G^{\alpha+\beta}}{H_{y,t}}(\int_0^1 x_t(j)^{1-\alpha-\beta}\partial j)]H_{r,t}(k) \Longleftrightarrow$$
$$\frac{(\frac{\mu-1}{\mu})G}{\rho+\varphi.h_{r,t}(k)}\varphi.\underline{h_{r,t}(k)} = (1+i_t)[\frac{\beta}{1-\alpha-\beta}.\frac{G^{\alpha+\beta}}{\underline{N}.h_{y,t}}(\int_0^1 x_t(j)^{1-\alpha-\beta}\partial j)]\underline{N.h_{r,t}(k)} \Longleftrightarrow$$
$$\frac{(\frac{\mu-1}{\mu})G}{\rho+\varphi.h_{r,t}(k)}\varphi = (1+i_t)[\frac{\beta}{1-\alpha-\beta}.\frac{G^{\alpha+\beta}}{h_{y,t}}(G)^{1-\alpha-\beta}] \Longleftrightarrow$$
$$\frac{(\frac{\mu-1}{\mu})\underline{G}}{\rho+\varphi.h_{r,t}(k)}\varphi = (1+i_t)[\frac{\beta}{1-\alpha-\beta}.\frac{\underline{G}}{h_{y,t}}] \Longleftrightarrow$$
$$(\frac{\mu-1}{\mu})\varphi(1-\alpha-\beta)h_y = (1+i_t)\beta(\rho+\varphi h_r) \Longleftrightarrow$$
$$(\frac{\mu-1}{\mu})(1-\alpha-\beta)h_y = (1+i_t)\beta(\frac{\rho}{\varphi}+h_r) \quad (22)$$

To obtain the second equation, we have to substitute (3) on (16):

$$w_{h,t} = \frac{\beta}{1-\alpha-\beta}.\frac{G^{\alpha+\beta}}{H_{y,t}}(\int_0^1 x_t(j)^{1-\alpha-\beta}\partial j) \quad (3)$$
$$\vdots$$
$$w_{h,t}(1-h_t) = \theta s y_t \quad (16)$$

$$[\frac{\beta}{1-\alpha-\beta}.\frac{G^{\alpha+\beta}}{h_{y,t}}(\int_0^1 x_t(j)^{1-\alpha-\beta}\partial j)](1-h) = \theta s y_t \Longleftrightarrow$$
$$[\frac{\beta}{1-\alpha-\beta}.\frac{G^{\alpha+\beta}}{h_{y,t}}(G)^{1-\alpha-\beta}](1-h) = \theta s y_t \Longleftrightarrow$$
$$[\frac{\beta G}{(1-\alpha-\beta)h_{y,t}}](1-h) = \theta s y_t \Longleftrightarrow$$

If $G = \left[L_{y,t}^{\alpha}(\gamma H_{y,t})^{\beta}\right]^{\frac{1}{\alpha+\beta}}$,then $L_{y,t}^{\alpha}.(\gamma H_{y,t})^{\beta} = G^{\alpha+\beta}$, so $y_t = \frac{1}{1-\alpha-\beta}L_{y,t}^{\alpha}(\gamma H_{y,t})^{\beta}\left(\int_0^1 x_t(j)^{1-\alpha-\beta}dj\right) \Longleftrightarrow y_t = \frac{1}{1-\alpha-\beta}G^{\alpha+\beta}G^{1-\alpha-\beta} \Longleftrightarrow$ $y_t = \frac{G}{1-\alpha-\beta}$, therefore:

$$[\frac{\beta \underline{G}}{\underline{(1-\alpha-\beta)}h_{y,t}}](1-h) = \theta s \underline{\frac{G}{1-\alpha-\beta}} \Longleftrightarrow \frac{\beta}{h_y}(1-h) = \theta s \Longleftrightarrow \beta(1-h) = \theta s h_y \quad (23)$$

Finally, the last equation will be the market clearing for high skill labor, so:

$$h = h_r + h_y \quad (24)$$

Solving (22) - (24), we will find the equilibrium for high skill labor allocations:

$$\begin{cases} (\frac{\mu-1}{\mu})(1-\alpha-\beta)h_y = (1+i_t)\beta(\frac{\rho}{\varphi}+h_r) \\ \beta(1-h) = \theta s h_y \\ h = h_r + h_y \end{cases} \iff$$

$$\begin{cases} (\frac{\mu-1}{\mu})(1-\alpha-\beta)h_y = (1+i_t)\beta(\frac{\rho}{\varphi}+1-\frac{\theta s}{\beta}h_y - h_y) \\ h = 1-\frac{\theta s}{\beta}h_y \\ 1-\frac{\theta s}{\beta}h_y - h_y = h \end{cases} \iff$$

$$\begin{cases} (\frac{\mu-1}{\mu})(1-\alpha-\beta)h_y = (1+i_t)\beta(\frac{\rho}{\varphi}+1-\frac{\theta s}{\beta}h_y - h_y) \\ ----- \\ ----- \end{cases} \iff$$

$$\begin{cases} (\frac{\mu-1}{\mu})(1-\alpha-\beta)h_y = (1+i_t)\beta(\frac{\rho}{\varphi}+1) - (1+i_t)\beta(\frac{\theta s}{\beta}+1)h_y \\ ----- \\ ----- \end{cases} \iff$$

$$\begin{cases} [(\frac{\mu-1}{\mu})(1-\alpha-\beta) + (1+i_t)\beta(\frac{\theta s}{\beta}+1)]h_y = (1+i_t)\beta(\frac{\rho}{\varphi}+1) \\ ----- \\ ----- \end{cases} \iff$$

$$\begin{cases} h_y = \frac{(1+i_t)\beta}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1) \quad (25) \\ ----- \\ ----- \end{cases}$$

Notice that, h_r will be:

$$(\frac{\mu-1}{\mu})(1-\alpha-\beta)h_y = (1+i_t)\beta(\frac{\rho}{\varphi}+h_r) \iff$$
$$(\frac{\mu-1}{\mu})(1-\alpha-\beta)\frac{(1+i_t)\beta}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1) = \underline{(1+i_t)\beta}(\frac{\rho}{\varphi}+h_r) \iff$$
$$h_r = \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1) - \frac{\rho}{\varphi} \quad (26)$$

Finally, h will be:

$$h = h_r + h_y \Longleftrightarrow$$
$$h = \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1) - \frac{\rho}{\varphi} + \frac{(1+i_t)\beta}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1) \Longleftrightarrow$$
$$h = \frac{(1+i_t)\beta+(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1) - \frac{\rho}{\varphi} \quad (27)$$

Imposing balanced growth for low skill labor, yields that:

We already know that:

$$w_{l,t} = [\frac{\alpha}{1-\alpha-\beta} \cdot \frac{G^{\alpha+\beta}}{Ly,t}(\int_0^1 x_t(j)^{1-\alpha-\beta}\partial j)] \iff w_{l,t} = [\frac{\alpha}{1-\alpha-\beta}\frac{G^{\alpha+\beta}}{L_{y,t}}(G)^{1-\alpha-\beta}] \iff$$
$$w_{l,t} = \frac{\alpha}{1-\alpha-\beta} \cdot \frac{G}{L_{y,t}}$$

If we consider $P_t(j) = 1$ and $X_t(j) = G$, therefore:

$$w_{l,t} L_x = \frac{1}{\mu} G \Longleftrightarrow$$
$$\frac{\alpha}{1-\alpha-\beta} \cdot \frac{G}{L_{y,t}} L_x = \frac{1}{\mu} \underline{G} \Longleftrightarrow$$
$$\frac{\alpha}{1-\alpha-\beta} \cdot \frac{1}{l_{y,t}} l_x = \frac{1}{\mu} \Longleftrightarrow$$
$$\alpha\mu l_x = (1-\alpha-\beta) l_{y,t} \quad (28)$$

Similar to high skill calculations, we know that:

$w_{h,t} = \frac{\alpha}{1-\alpha-\beta} \cdot \frac{G}{l_{y,t}}$ and $y_t = \frac{G}{1-\alpha-\beta}$

$$\frac{\alpha}{\underline{1-\alpha-\beta}} \cdot \frac{\underline{G}}{l_{y,t}} (1-l_t) = \theta (1-s) \frac{\underline{G}}{\underline{1-\alpha-\beta}} \Longleftrightarrow$$
$$\frac{\alpha}{l_{y,t}} (1-l_t) = \theta (1-s) \Longleftrightarrow$$
$$\alpha (1-l_t) = \theta (1-s) l_{y,t} \quad (29)$$

The market Clearing for low skill labor will be:

$$l = l_x + l_y \quad (30)$$

Solving (28)-(30), we will obtain the equilibrium for low skill labor:

$$\begin{cases} \alpha\mu l_x = (1-\alpha-\beta) l_{y,t} \\ l = l_x + l_y \\ \alpha\left(1 - l_t\right) = \theta\left(1-s\right) l_{y,t} \end{cases} \Longleftrightarrow$$

$$\begin{cases} l_x = \frac{(1-\alpha-\beta)}{\alpha\mu} l_{y,t} \\ l = (\frac{(1-\alpha-\beta)}{\alpha\mu} + 1) l_y \\ \alpha\left(1 - (\frac{(1-\alpha-\beta)}{\alpha\mu} + 1) l_y\right) = \theta\left(1-s\right) l_{y,t} \end{cases} \Longleftrightarrow$$

$$\begin{cases} -\,-\,-\,-\,- \\ -\,-\,-\,-\,- \\ \alpha - \frac{(1-\alpha-\beta)}{\mu} l_y - \alpha l_y = \theta\left(1-s\right) l_y \end{cases} \Longleftrightarrow$$

$$\begin{cases} -\,-\,-\,-\,- \\ -\,-\,-\,-\,- \\ \theta\left(1-s\right) l_y + \frac{(1-\alpha-\beta)}{\mu} l_y + \alpha l_y = \alpha \end{cases} \Longleftrightarrow$$

$$\begin{cases} -\,-\,-\,-\,- \\ -\,-\,-\,-\,- \\ \theta\mu\left(1-s\right) l_y + (1-\alpha-\beta) l_y + \alpha\mu l_y = \alpha\mu \end{cases} \Longleftrightarrow$$

$$\begin{cases} -\,-\,-\,-\,- \\ -\,-\,-\,-\,- \\ [\theta\mu\left(1-s\right) + (1-\alpha-\beta) + \alpha\mu] l_y = \alpha\mu \end{cases} \Longleftrightarrow$$

$$\begin{cases} l_x = \frac{(1-\alpha-\beta)}{\alpha\mu} \frac{\alpha\mu}{\theta\mu(1-s)+(1-\alpha-\beta)+\alpha\mu} \\ -\,-\,-\,-\,- \\ l_y = \frac{\alpha\mu}{\theta\mu(1-s)+(1-\alpha-\beta)+\alpha\mu} \end{cases} \Longleftrightarrow$$

$$\begin{cases} l_x = \frac{(1-\alpha-\beta)}{\theta\mu(1-s)+(1-\alpha-\beta)+\alpha\mu} \\ ----- \\ l_y = \frac{\alpha\mu}{\theta\mu(1-s)+(1-\alpha-\beta)+\alpha\mu} \end{cases} \iff$$

$$\begin{cases} l_x = \frac{(1-\alpha-\beta)}{\theta\mu(1-s)+(1-\alpha-\beta)+\alpha\mu} & (31) \\ l = \frac{(1-\alpha-\beta)+\alpha\mu}{\theta\mu(1-s)+(1-\alpha-\beta)+\alpha\mu} & (32) \\ l_y = \frac{\alpha\mu}{\theta\mu(1-s)+(1-\alpha-\beta)+\alpha\mu} & (33) \end{cases}$$

7.6 Optimal Monetary Policy and Friedman Rule

7.6.1 Socially Optimal Allocation

Imposing balanced growth on (11), yields that:

$$U = \frac{1}{\rho}\left[lnc_0 + \frac{g}{\rho} + ln\,(1 - l_u)\right] \;\; (34)$$

where $u = h, l$, $C_0 = z_0 l = y$ and $g = (\varphi lnz)h_r$

$$U = \tfrac{1}{\rho}\left[ln\tfrac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta} + \tfrac{(\varphi lnz)}{\rho}h_r + \theta ln\,(1 - l_y - l_x) + \theta ln\,(1 - h_y - h_r)\right]$$

To find the socially optimal allocations for high and low skill, we must maximize the welfare to obtain the first best allocations.

$$\tfrac{\partial U}{\partial h_y} = [\tfrac{1}{\rho}\left[ln\tfrac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta} + \tfrac{(\varphi lnz)}{\rho}h_r + \theta ln\,(1 - l_y - l_x) + \theta ln\,(1 - h_y - h_r)\right]' \Longleftrightarrow$$

$$\tfrac{\partial U}{\partial h_y} = 0 \Longleftrightarrow$$

$$\tfrac{1}{\rho}[\tfrac{(\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta})'}{\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta}} + \theta\tfrac{(1-h_y-h_r)'}{(1-h_y-h_r)}] = 0 \Longleftrightarrow$$

$$\tfrac{1}{\rho}[\tfrac{(\underline{\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}}\beta(\gamma h_{y,t})^{\beta-1}\gamma \underline{l_x^{1-\alpha-\beta}})}{\underline{\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}}(\gamma h_{y,t})^{\beta}\underline{l_x^{1-\alpha-\beta}}} + \theta\tfrac{-1}{(1-h_y-h_r)}] = 0 \Longleftrightarrow$$

$$\tfrac{1}{\rho}[\tfrac{(\beta(\gamma h_{y,t})^{\beta-1}\gamma)}{(\gamma h_{y,t})^{\beta}} - \tfrac{\theta}{(1-h_y-h_r)}] = 0 \Longleftrightarrow$$

$$\tfrac{1}{\rho}[\tfrac{(\beta\frac{(\gamma h_{y,t})^{\beta}}{\underline{\gamma} h_{y,t}}\underline{\gamma})}{\underline{(\gamma h_{y,t})^{\beta}}} - \tfrac{\theta}{(1-h_y-h_r)}] = 0 \Longleftrightarrow$$

$$\tfrac{1}{\rho}[(\tfrac{\beta}{h_{y,t}}) - \tfrac{\theta}{(1-h_y-h_r)}] = 0 \Longleftrightarrow$$

$$(\tfrac{\beta}{h_{y,t}}) - \tfrac{\theta}{(1-h_y-h_r)} = 0 \Longleftrightarrow$$

$$(\tfrac{\beta}{h_{y,t}}) = \tfrac{\theta}{(1-h_y-h_r)} \Longleftrightarrow$$

$$\beta(1 - h_y - h_r) = \theta h_{y,t} \Longleftrightarrow$$

$$(1 - h_y - h_r) = \tfrac{\theta}{\beta}h_{y,t} \Longleftrightarrow$$

$$(1 - h_r) = (\tfrac{\theta}{\beta} + 1)h_y$$

$$\frac{\partial U}{\partial h_r} = [\frac{1}{\rho}\left[ln\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta} + \frac{(\varphi lnz)}{\rho}h_r + \theta ln\left(1-l_y-l_x\right) + \theta ln\left(1-h_y-h_r\right)\right]' \Longleftrightarrow$$

$$\frac{\partial U}{\partial h_r} = 0 \Longleftrightarrow$$
$$\frac{1}{\rho}[(\frac{(\varphi lnz)}{\rho}h_r)' + \theta\frac{(1-h_y-h_r)'}{(1-h_y-h_r)}] = 0 \Longleftrightarrow$$
$$\frac{1}{\rho}[(\frac{(\varphi lnz)}{\rho}) + \theta\frac{-1}{(1-h_y-h_r)}] = 0 \Longleftrightarrow$$
$$\frac{1}{\rho}[(\frac{(\varphi lnz)}{\rho}) - \frac{\theta}{(1-h_y-h_r)}] = 0 \Longleftrightarrow$$
$$[(\frac{(\varphi lnz)}{\rho}) - \frac{\theta}{(1-h_y-h_r)}] = 0 \Longleftrightarrow$$
$$\frac{(\varphi lnz)}{\rho} = \frac{\theta}{(1-h_y-h_r)} \Longleftrightarrow$$
$$(\varphi lnz)\left(1-h_y-h_r\right) = \theta\rho \Longleftrightarrow$$
$$(1-h_y-h_r) = \frac{\theta\rho}{(\varphi lnz)}$$

$$\begin{cases} (1-h_r) = (\frac{\theta}{\beta}+1)h_y \\ \qquad\qquad\qquad\qquad \Longleftrightarrow \\ (1-h_y-h_r) = \frac{\theta\rho}{(\varphi lnz)} \end{cases}$$
$$\begin{cases} ----- \\ \qquad\qquad\qquad\qquad \Longleftrightarrow \\ \left((\frac{\theta}{\beta}+1)h_y - h_y\right) = \frac{\theta\rho}{(\varphi lnz)} \end{cases}$$
$$\begin{cases} ----- \\ \qquad\qquad \Longleftrightarrow \\ (\frac{\theta}{\beta})h_y = \frac{\theta\rho}{(\varphi lnz)} \end{cases}$$
$$\begin{cases} ----- \\ \qquad\qquad \Longleftrightarrow \\ h_y = \frac{\rho\beta}{(\varphi lnz)} \end{cases}$$
$$\begin{cases} (1-h_r) = (\frac{\theta}{\beta}+1)\frac{\rho\beta}{(\varphi lnz)} \\ \qquad\qquad\qquad\qquad \Longleftrightarrow \\ h_y = \frac{\rho\beta}{(\varphi lnz)} \end{cases}$$

$$\begin{cases} (1-h_r) = \frac{\theta\rho}{(\varphi lnz)} + \frac{\rho\beta}{(\varphi lnz)} \\ \qquad\qquad\qquad\qquad\qquad \Longleftrightarrow \\ ------ \end{cases}$$

$$\begin{cases} (1-h_r) = \frac{\theta\rho}{(\varphi lnz)} + \frac{\rho\beta}{(\varphi lnz)} \\ \qquad\qquad\qquad\qquad\qquad \Longleftrightarrow \\ ------ \end{cases}$$

$$\begin{cases} -h_r = \frac{\theta\rho}{(\varphi lnz)} + \frac{\rho\beta}{(\varphi lnz)} - 1 \\ \qquad\qquad\qquad\qquad\qquad \Longleftrightarrow \\ ------ \end{cases}$$

$$\begin{cases} h_r^* = 1 - \frac{(\theta+\beta)\rho}{(\varphi lnz)} \quad (35) \\ \qquad\qquad\qquad \Longleftrightarrow \\ h_y^* = \frac{\rho\beta}{(\varphi lnz)} \qquad\quad (36) \end{cases}$$

Finally,

$h^* = h_r^* + h_y^* \Longleftrightarrow$

$h^* = 1 - \frac{(\theta+\beta)\rho}{(\varphi lnz)} + \frac{\rho\beta}{(\varphi lnz)} \Longleftrightarrow$

$h^* = 1 - \frac{\theta\rho}{(\varphi lnz)} \quad (37)$

$$\frac{\partial U}{\partial l_x} = [\frac{1}{\rho}\left[ln\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta} + \frac{(\varphi lnz)}{\rho}h_r + \theta ln\left(1-l_y-l_x\right) + \theta ln\left(1-h_y-h_r\right)\right]' \Longleftrightarrow$$

$\frac{\partial U}{\partial l_x} = 0 \Longleftrightarrow$

$\frac{1}{\rho}[\frac{(\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta})'}{\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta}} + \theta\frac{(1-l_y-l_x)'}{(1-l_y-l_x)}] = 0 \Longleftrightarrow$

$\frac{1}{\rho}[\frac{(\underline{\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}}(1-\alpha-\beta)l_x^{-\alpha-\beta})}{\underline{\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}}l_x^{1-\alpha-\beta}} + \theta\frac{(1-l_y-l_x)'}{(1-l_y-l_x)}] = 0 \Longleftrightarrow$

$\frac{1}{\rho}[\frac{(1-\alpha-\beta)l_x^{-\alpha-\beta})}{l_x^{1-\alpha-\beta}} + \theta\frac{(-1)}{(1-l_y-l_x)}] = 0 \Longleftrightarrow$

$\frac{1}{\rho}[\frac{(1-\alpha-\beta)}{l_x^{1-\alpha-\beta+\alpha+\beta}} - \frac{\theta}{(1-l_y-l_x)}] = 0 \Longleftrightarrow$

$\frac{(1-\alpha-\beta)}{l_x} - \frac{\theta}{(1-l_y-l_x)} = 0 \Longleftrightarrow$

$\frac{(1-\alpha-\beta)}{\underline{\rho}l_x} = \frac{\theta}{\underline{\rho}(1-l_y-l_x)} \Longleftrightarrow$

$\frac{(1-\alpha-\beta)}{l_x} = \frac{\theta}{(1-l_y-l_x)} \Longleftrightarrow$

$$(1-\alpha-\beta)\left(1-l_y-l_x\right)=\theta l_x \Longleftrightarrow$$

$$(1-\alpha-\beta)\left(1-l_y\right)=(\theta+(1-\alpha-\beta))l_x \Longleftrightarrow$$

$$l_x=\tfrac{(1-\alpha-\beta)}{(\theta+(1-\alpha-\beta))}\left(1-l_y\right)$$

$$\frac{\partial U}{\partial l_y}=[\tfrac{1}{\rho}\left[ln\tfrac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta}+\tfrac{(\varphi lnz)}{\rho}h_r+\theta ln\left(1-l_y-l_x\right)+\theta ln\left(1-h_y-h_r\right)\right]' \Longleftrightarrow$$

$$\frac{\partial U}{\partial l_y}=0 \Longleftrightarrow$$

$$\tfrac{1}{\rho}[\tfrac{(\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta})'}{\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta}}+\theta\tfrac{(1-l_y-l_x)'}{(1-l_y-l_x)}]=0 \Longleftrightarrow$$

$$\tfrac{1}{\rho}[\tfrac{(\frac{1}{1-\alpha-\beta}\alpha l_{y,t}^{\alpha-1}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta})}{\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta}}+\theta\tfrac{(-1)}{(1-l_y-l_x)}]=0 \Longleftrightarrow$$

$$\tfrac{1}{\rho}[\tfrac{\underline{(\frac{1}{1-\alpha-\beta}}\alpha l_{y,t}^{\alpha-1}\underline{(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta})}}{\underline{(\frac{1}{1-\alpha-\beta})}l_{y,t}^{\alpha}\underline{(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta}}}-\tfrac{\theta}{(1-l_y-l_x)}]=0 \Longleftrightarrow$$

$$\tfrac{1}{\rho}[\tfrac{\alpha\frac{\underline{\underline{l_y^{\alpha}}}}{l_y}}{\underline{l_y^{\alpha}}}-\tfrac{\theta}{(1-l_y-l_x)}]=0 \Longleftrightarrow$$

$$\tfrac{\alpha}{\underline{\rho}l_y}-\tfrac{\theta}{\underline{\rho}(1-l_y-l_x)}=0 \Longleftrightarrow$$

$$\tfrac{\alpha}{l_y}-\tfrac{\theta}{(1-l_y-l_x)}=0 \Longleftrightarrow$$

$$\tfrac{\alpha}{l_y}=\tfrac{\theta}{(1-l_y-l_x)} \Longleftrightarrow$$

$$\alpha\left(1-l_y-l_x\right)=\theta l_y \Longleftrightarrow$$

$$\alpha\left(1-l_x\right)-\alpha l_y=\theta l_y \Longleftrightarrow$$

$$\theta l_y+\alpha l_y=\alpha\left(1-l_x\right) \Longleftrightarrow$$

$$(\theta+\alpha)l_y=\alpha\left(1-l_x\right) \Longleftrightarrow$$

$$l_y=\tfrac{\alpha}{(\theta+\alpha)}\left(1-l_x\right)$$

$$\begin{cases} l_y=\tfrac{\alpha}{(\theta+\alpha)}\left(1-l_x\right) \\ \\ l_x=\tfrac{(1-\alpha-\beta)}{(\theta+(1-\alpha-\beta))}\left(1-l_y\right) \end{cases} \Longleftrightarrow$$

$$\begin{cases} l_y=\tfrac{\alpha}{(\theta+\alpha)}-\tfrac{\alpha}{(\theta+\alpha)}l_x \\ \\ l_x=\tfrac{(1-\alpha-\beta)}{(\theta+(1-\alpha-\beta))}\left(1-l_y\right) \end{cases} \Longleftrightarrow$$

$$\begin{cases} l_y = \frac{\alpha}{(\theta+\alpha)} - \frac{\alpha}{(\theta+\alpha)}[\frac{(1-\alpha-\beta)}{(\theta+(1-\alpha-\beta))}(1-l_y)] \\ \iff \\ ----- \end{cases}$$

$$\begin{cases} l_y = \frac{\alpha}{(\theta+\alpha)} - \frac{\alpha(1-\alpha-\beta)}{(\theta+\alpha)(\theta+(1-\alpha-\beta))} + \frac{\alpha(1-\alpha-\beta)}{(\theta+\alpha)(\theta+(1-\alpha-\beta))}l_y \\ \iff \\ ----- \end{cases}$$

$$\begin{cases} l_y - \frac{\alpha(1-\alpha-\beta)}{(\theta+\alpha)(\theta+(1-\alpha-\beta))}l_y = \frac{\alpha}{(\theta+\alpha)} - \frac{\alpha(1-\alpha-\beta)}{(\theta+\alpha)(\theta+(1-\alpha-\beta))} \\ \iff \\ ----- \end{cases}$$

$$\begin{cases} \frac{(\theta+\alpha)(\theta+(1-\alpha-\beta))-\alpha(1-\alpha-\beta)}{\underline{(\theta+\alpha)(\theta+(1-\alpha-\beta))}}l_y = \frac{\alpha(\theta+\underline{(1-\alpha-\beta)})}{\underline{(\theta+\alpha)(\theta+(1-\alpha-\beta))}} - \frac{\alpha\underline{(1-\alpha-\beta)}}{\underline{(\theta+\alpha)(\theta+(1-\alpha-\beta))}} \\ \iff \\ ----- \end{cases}$$

$$\begin{cases} [(\theta+\alpha)(\theta+(1-\alpha-\beta))-\alpha(1-\alpha-\beta)]l_y = \alpha\theta \\ \iff \\ ----- \end{cases}$$

$$\begin{cases} [(\theta+\alpha)\theta+\theta(1-\alpha-\beta)+\underline{\alpha(1-\alpha-\beta)-\alpha(1-\alpha-\beta)]}l_y = \alpha\theta \\ \iff \\ ----- \end{cases}$$

$$\begin{cases} [\theta\theta+\underline{\alpha\theta}+\theta-\underline{\theta\alpha}-\theta\beta]l_y = \alpha\theta \\ \iff \\ ----- \end{cases}$$

$$\begin{cases} [\theta(\theta+1-\beta)]l_y = \alpha\theta \\ \iff \\ ----- \end{cases}$$

$$\begin{cases} l_y = \frac{\alpha\underline{\theta}}{\underline{\theta}(\theta+1-\beta)} \\ \qquad\qquad\qquad\Longleftrightarrow \\ ----- \end{cases}$$

$$\begin{cases} l_y = \frac{\alpha}{(1+\theta-\beta)} \\ \qquad\qquad\qquad\Longleftrightarrow \\ ----- \end{cases}$$

Therefore, $l_x$will be:

$$\frac{\alpha}{(1+\theta-\beta)} = \frac{\alpha}{(\theta+\alpha)} - \frac{\alpha}{(\theta+\alpha)}l_x \Longleftrightarrow$$
$$\frac{\underline{\alpha}}{(\theta+\alpha)}l_x = \frac{\underline{\alpha}}{(\theta+\alpha)} - \frac{\underline{\alpha}}{(1+\theta-\beta)} \Longleftrightarrow$$
$$l_x = \frac{(\theta+\alpha)}{(\theta+\alpha)} - \frac{(\theta+\alpha)}{(1+\theta-\beta)} \Longleftrightarrow$$
$$l_x = 1 - \frac{(\theta+\alpha)}{(1+\theta-\beta)} \Longleftrightarrow$$
$$l_x = \frac{(1+\underline{\theta}-\beta)-(\underline{\theta}+\alpha)}{(1+\theta-\beta)} \Longleftrightarrow$$
$$l_x = \frac{1-\beta-\alpha}{1+\theta-\beta}$$

$$\begin{cases} l_y^* = \frac{\alpha}{(1+\theta-\beta)} \quad (38) \\ \\ l_x^* = \frac{1-\beta-\alpha}{1+\theta-\beta} \quad (39) \end{cases}$$

As for high skill labor, $l^* = l_y^* + l_x^*$:

$$l^* = l_y^* + l_x^* \Longleftrightarrow$$
$$l^* = \frac{\underline{\alpha}}{1+\theta-\beta} + \frac{1-\beta-\underline{\alpha}}{1+\theta-\beta} \Longleftrightarrow$$
$$l^* = \frac{1-\beta}{1+\theta-\beta} \quad (40)$$

7.6.2 Friedman Rule Under Inelastic Labor Supply

Under the inelastic labor supply, $\theta = 0$, so the welfare function will be:

$U = \frac{1}{\rho}\left[ln\frac{1}{1-\alpha-\beta}l_{y,t}^{\alpha}(\gamma h_{y,t})^{\beta}l_x^{1-\alpha-\beta} + \frac{(\varphi lnz)}{\rho}h_r\right]$

As the interest rate only affects the high skill labor, the optimal interest rate will be found through the high skill allocations, therefore the welfare function will turn into:

$U = \frac{1}{\rho}\left[\beta lnh_y + \frac{(\varphi lnz)}{\rho}h_r\right]$

Note that, if we simply the welfare function in respect to i, we will get:

$U = \frac{1}{\rho}\left[ln\frac{1}{1-\alpha-\beta} + \alpha lnl_{y,t} + \beta ln(\gamma h_{y,t}) + (1-\alpha-\beta)lnl_x + \frac{(\varphi lnz)}{\rho}h_r\right] \Longleftrightarrow$
$U = \frac{1}{\rho}\left[ln\frac{1}{1-\alpha-\beta} + \alpha lnl_{y,t} + \beta ln(\gamma) + \beta ln(h_{y,t}) + (1-\alpha-\beta)lnl_x + \frac{(\varphi lnz)}{\rho}h_r\right] \Longleftrightarrow$

As $ln\frac{1}{1-\alpha-\beta}$, $\alpha lnl_{y,t}$, $\beta ln(\gamma)$ and $(1-\alpha-\beta)lnl_x$ are not affected by i, when the maximization of the welfare function occur, these variables are simplified, so the function will simply be:

$U = \frac{1}{\rho}\left[\beta lnh_y + \frac{(\varphi lnz)}{\rho}h_r\right]$

With $\theta = 0$:

$h_y = \frac{(1+i)\beta}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)\beta}(\frac{\rho}{\varphi}+1) \quad (41)$

$h_r = \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)\beta}(\frac{\rho}{\varphi}+1) - \frac{\rho}{\varphi} \quad (42)$

$U = \frac{1}{\rho}\left[\beta ln[\frac{(1+i)\beta}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)\beta}(\frac{\rho}{\varphi}+1)] + \frac{(\varphi lnz)}{\rho}\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)\beta}(\frac{\rho}{\varphi}+1) - \frac{\rho}{\varphi}\right] \Longleftrightarrow$

$\frac{\partial U}{\partial i} = 0 \Longleftrightarrow$
$\frac{1}{\rho}[\beta\frac{[\frac{(1+i)\beta}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)\beta}(\frac{\rho}{\varphi}+1)]'}{[\frac{(1+i)\beta}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)\beta}(\frac{\rho}{\varphi}+1)]} + (\frac{(\varphi lnz)}{\rho}\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)\beta}(\frac{\rho}{\varphi}+1) - \frac{\rho}{\varphi})'] = 0 \Longleftrightarrow$

$$\frac{1}{\rho}[\beta\frac{[\frac{\beta[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)\beta]-((1+i)\beta)\beta}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)\beta]^2}(\frac{\rho}{\varphi}+1)]}{[\frac{(1+i)\beta}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)\beta}(\frac{\rho}{\varphi}+1)]}+(\frac{(\varphi lnz)}{\rho}\frac{-\beta(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)\beta]^2}(\frac{\rho}{\varphi}+1)]=0\Longleftrightarrow$$

$$\frac{1}{\rho}[\beta[\frac{\beta[(\frac{\mu-1}{\mu})(1-\alpha-\beta)]}{(1+i)\beta[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)\beta]}]+\frac{-\beta(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)\beta]^2}(1+\frac{\varphi}{\rho})lnz]=0\Longleftrightarrow$$

$$\frac{1}{\rho}[\frac{\beta[(\frac{\mu-1}{\mu})(1-\alpha-\beta)]}{(1+i)}-\frac{\beta(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)\beta]}(1+\frac{\varphi}{\rho})lnz]=0\Longleftrightarrow$$

$$\frac{\beta[(\frac{\mu-1}{\mu})(1-\alpha-\beta)]}{(1+i)}=\frac{\beta(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)\beta]}(1+\frac{\varphi}{\rho})lnz\Longleftrightarrow$$

$$\frac{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)\beta]}{(1+i)}=(1+\frac{\varphi}{\rho})lnz\Longleftrightarrow$$

$$\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(1+i)}+\frac{(1+i_t)\beta}{(1+i)}=(1+\frac{\varphi}{\rho})lnz\Longleftrightarrow$$

$$\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(1+i)}+\beta=(1+\frac{\varphi}{\rho})lnz\Longleftrightarrow$$

$$\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(1+i)}=(1+\frac{\varphi}{\rho})lnz-\beta\Longleftrightarrow$$

$$\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(1+\frac{\varphi}{\rho})lnz-\beta}=(1+i)\Longleftrightarrow$$

$$\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(1+\frac{\varphi}{\rho})lnz-\beta}-1=i\Longleftrightarrow$$

$$i=\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(1+\frac{\varphi}{\rho})lnz-\beta}-1\Longleftrightarrow$$

$$i=\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)-(1+\frac{\varphi}{\rho})lnz-\beta}{(1+\frac{\varphi}{\rho})lnz-\beta}\Longleftrightarrow$$

$$i^*=\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)-[(1+\frac{\varphi}{\rho})lnz-\beta]}{(1+\frac{\varphi}{\rho})lnz-\beta}\qquad(43)$$

7.6.3 Friedman Rule Under Elastic Labor Supply

As under inelastic labor supply, under elastic labor supply, the interest rate will only affect high skill workers. The welfare will become the following:

$$U=\frac{1}{\rho}\left[\beta lnh_y+\frac{(\varphi lnz)}{\rho}h_r+\theta ln(1-h_y-h_r)\right]$$

To simply the welfare function, lets consider the following notation:

$$\Phi=\beta lnh_y$$
$$\mathit{\Gamma}=\frac{(\varphi lnz)}{\rho}h_r$$
$$\Psi=\theta ln(1-h_y-h_r)$$

Therefore:

$U = \frac{1}{\rho}[\Phi + \mathit{\Gamma} + \Psi]$

$\Phi = \beta ln h_y \Longleftrightarrow$

$\Phi = \beta ln[\frac{(1+i_t)\beta}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)}(\frac{\rho}{\varphi}+1)] \Longleftrightarrow$

$$\frac{\partial\Phi}{\partial i} = \beta\frac{[\frac{(1+i)\beta}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)}(\frac{\rho}{\varphi}+1)]'}{[\frac{(1+i)\beta}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)}(\frac{\rho}{\varphi}+1)]} \Longleftrightarrow$$

$$\frac{\partial\Phi}{\partial i} = \beta\frac{[\frac{\underline{\beta}[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+\underline{(1+i)(\theta s+\beta)]-[(1+i)(\theta s+\beta)]\beta}}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]^{\underline{2}}}\underline{(\frac{\rho}{\varphi}+1)}]}{[\frac{(1+i)\underline{\beta}}{\underline{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)}}\underline{(\frac{\rho}{\varphi}+1)}]} \Longleftrightarrow$$

$$\frac{\partial\Phi}{\partial i} = \frac{\beta[(\frac{\mu-1}{\mu})(1-\alpha-\beta)]}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)](1+i)} \Longleftrightarrow$$

$\mathit{\Gamma} = \frac{(\varphi ln z)}{\rho}h_r$

$\mathit{\Gamma} = \frac{(\varphi ln z)}{\rho}[\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1)-\frac{\rho}{\varphi}] \Longleftrightarrow$

$\frac{\partial\mathit{\Gamma}}{\partial i} = \frac{(\varphi ln z)}{\rho}[\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1)-\frac{\rho}{\varphi}]' \Longleftrightarrow$

$\frac{\partial\mathit{\Gamma}}{\partial i} = [\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]}(1+\frac{\varphi}{\rho})ln z) - \frac{(\varphi ln z)}{\rho}\frac{\rho}{\varphi}]' \Longleftrightarrow$

$\frac{\partial\mathit{\Gamma}}{\partial i} = [\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]}(1+\frac{\varphi}{\rho})ln z - ln z]' \Longleftrightarrow$

$\frac{\partial\mathit{\Gamma}}{\partial i} = -\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)(\theta s+\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]^2}(1+\frac{\varphi}{\rho})ln z \Longleftrightarrow$

$\Psi = \theta ln(1-h_y-h_r) \Longleftrightarrow$

$\Psi = \theta ln(1-h)$

$(1-h) = [1-\{\frac{(1+i_t)\beta+(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1)-\frac{\rho}{\varphi}\}] \Longleftrightarrow$

$(1-h) = [1+\frac{\rho}{\varphi}]-[\frac{(1+i_t)\beta+(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1)] \Longleftrightarrow$

$(1-h) = [1-\frac{(1+i_t)\beta+(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}](\frac{\rho}{\varphi}+1) \Longleftrightarrow$

$(1-h) = [\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)-[(1+i_t)\beta+(\frac{\mu-1}{\mu})(1-\alpha-\beta)]}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}](\frac{\rho}{\varphi}+1) \Longleftrightarrow$

$(1-h) = [\frac{\underline{\underline{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}}+(1+i_t)(\theta s+\underline{\beta})-[\underline{(1+i_t)\beta}+\underline{\underline{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}}]}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}](\frac{\rho}{\varphi}+1) \Longleftrightarrow$

$(1-h) = \frac{(1+i_t)\theta s}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1)$

$$\Psi = \theta ln(1-h) \Longleftrightarrow$$
$$\Psi = \theta ln(\frac{(1+i_t)\theta s}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1)) \Longleftrightarrow$$
$$\frac{\partial \Psi}{\partial i} = [\theta ln(\frac{(1+i_t)\theta s}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1))]' \Longleftrightarrow$$
$$\frac{\partial \Psi}{\partial i} = \theta \frac{(\frac{(1+i_t)\theta s}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1))'}{(\frac{(1+i_t)\theta s}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i_t)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1))} \Longleftrightarrow$$
$$\frac{\partial \Psi}{\partial i} = \theta \frac{\frac{\theta s[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)-(\theta s+\beta)(1+i)\theta s]}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]^2}(\frac{\rho}{\varphi}+1)}{\frac{(1+i)\theta s}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]}(\frac{\rho}{\varphi}+1)} \Longleftrightarrow$$
$$\frac{\partial \Psi}{\partial i} = \frac{\theta[(\frac{\mu-1}{\mu})(1-\alpha-\beta)]}{(1+i)[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]}$$

As$U = \frac{1}{\rho}[\Phi + \mathit{\Gamma} + \Psi]$,

$$\frac{\partial U}{\partial i} = 0 \Longleftrightarrow$$
$$\frac{1}{\rho}[\Phi + \mathit{\Gamma} + \Psi]' = 0 \Longleftrightarrow$$
$$\frac{1}{\rho}[\frac{\beta[(\frac{\mu-1}{\mu})(1-\alpha-\beta)]}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)](1+i)} - \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)(\theta s+\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]^2}(1+\frac{\varphi}{\rho})lnz + \frac{\theta[(\frac{\mu-1}{\mu})(1-\alpha-\beta)]}{(1+i)[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]}] = 0 \Longleftrightarrow$$
$$\frac{1}{\rho}[\frac{\beta[(\frac{\mu-1}{\mu})(1-\alpha-\beta)]}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)](1+i)} - \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)(\theta s+\beta)}{[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]^2}(1+\frac{\varphi}{\rho})lnz + \frac{\theta[(\frac{\mu-1}{\mu})(1-\alpha-\beta)]}{(1+i)[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]}] = 0 \Longleftrightarrow$$
$$\frac{(\beta+\theta)[(\frac{\mu-1}{\mu})(1-\alpha-\beta)]}{\rho(1+i)} - \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)(\theta s+\beta)}{\rho[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]}(1+\frac{\varphi}{\rho})lnz = 0 \Longleftrightarrow$$
$$\frac{(\beta+\theta)[(\frac{\mu-1}{\mu})(1-\alpha-\beta)]}{\rho(1+i)} - \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)(\theta s+\beta)}{\rho[(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)]}(1+\frac{\varphi}{\rho})lnz = 0 \Longleftrightarrow$$
$$\frac{(\beta+\theta)}{(1+i)} = \frac{(\theta s+\beta)}{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)}(1+\frac{\varphi}{\rho})lnz \Longleftrightarrow$$
$$\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)+(1+i)(\theta s+\beta)}{(1+i)} = \frac{(\theta s+\beta)}{(\beta+\theta)}(1+\frac{\varphi}{\rho})lnz \Longleftrightarrow$$
$$\frac{(1+i)(\theta s+\beta)}{(1+i)} + \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(1+i)} = \frac{(\theta s+\beta)}{(\beta+\theta)}(1+\frac{\varphi}{\rho})lnz \Longleftrightarrow$$
$$(\theta s+\beta) + \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(1+i)} = \frac{(\theta s+\beta)}{(\beta+\theta)}(1+\frac{\varphi}{\rho})lnz \Longleftrightarrow$$
$$\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{(1+i)} = \frac{(\theta s+\beta)}{(\beta+\theta)}(1+\frac{\varphi}{\rho})lnz - \theta s - \beta \Longleftrightarrow$$
$$\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{\frac{(\theta s+\beta)}{(\beta+\theta)}(1+\frac{\varphi}{\rho})lnz-\theta s-\beta} = (1+i) \Longleftrightarrow$$
$$\frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)}{\frac{(\theta s+\beta)}{(\beta+\theta)}(1+\frac{\varphi}{\rho})lnz-\theta s-\beta} - 1 = i \Longleftrightarrow$$
$$i^* = \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)-[\frac{(\theta s+\beta)}{(\beta+\theta)}(1+\frac{\varphi}{\rho})lnz-\theta s-\beta]}{\frac{(\theta s+\beta)}{(\beta+\theta)}(1+\frac{\varphi}{\rho})lnz-\theta s-\beta} \Longleftrightarrow$$

$$i^* = \frac{(\frac{\mu-1}{\mu})(1-\alpha-\beta)-\Omega}{\Omega} \quad (44)$$

With: $\Omega = \frac{(\theta s+\beta)}{(\beta+\theta)}(1+\frac{\varphi}{\rho})lnz - \theta s - \beta \quad (45)$

8 References

Acemoglu, D. (2002), "Directed Technical Change", *The Review of Economic Studies*, Vol. 69, N$^{\underline{o}}$ 4., pp. 781-809.

Acemoglu, D. and U. Akcigit (2012). "Intellectual property rights policy, competition and innovation", *Journal of the European Economic Association*, Vol. 10, N$^{\underline{o}}$. 1, pp. 1–42.

Afonso, O. (2006), "Skill-biased technological knowledge without scale effects", *Applied Economics*, Vol. 38, N$^{\underline{o}}$. 1, pp. 13-21.

Afonso, O. (2016), "Effects of labour-market institutions on employment, wages, R&D intensity and growth in 27 OECD countries: From theory to practice", *Economic Modelling*, Vol. 53, N$^{\underline{o}}$. 1, pp. 48-62.

Aghion, P. and P. Howitt (1992), "A Model of Growth Through Creative Destruction", *Econometrica*, Vol. 60, N$^{\underline{o}}$. 2, pp. 323-351.

Ahrens, S. and D. J. Snower (2014), "Envy, guilt, and the Phillips curve", *Journal of Economic Behavior & Organization*, Vol. 99, pp. 69-84.

Arestis, P. and A. Mihailov (2011), "Classifying Monetary Economics: Fields and Methods from Past to Future". *Journal of Economic Surveys*, Vol. 25, N$^{\underline{o}}$. 4, pp. 769-800.

Atkinson, A. B., T. Piketty and E. Saez (2011), "Top incomes in the long run of history", *Journal of Economic Literature,* Vol. 49, N$^{\underline{o}}$. 1, pp. 3-71.

Bae, S.-K-, Jensen, M. J. and S. G. Murdock (2005), "Long-run neutrality in a fractionally integrated model", *Journal of Macroeconomics*, Vol. 27, pp. 257-274.

Barro, R. J. (1991), "Economic Growth in a Cross Section of Countries", *The Quarterly Journal of Economics*, Vol. 106, N$^{\underline{o}}$. 2, pp. 407-443.

Barro, R.J. and D. Gordon (1983), "Rules, discretion and reputation in a model of monetary policy", *Journal of Monetary Economics*, Vol. 12, pp. 101–121.

Barro, R. J. and X. Sala-i-Martin (2004), *Economic Growth*, 2^{th} ed. Massachusetts: MIT Press

Baumol, W. (1952), "The transaction demand for cash: an inventory-theoretic approach" *Quarterly Journal of Economics*, Vol. 66, pp. 545–556.

Benigno, P. and M. Woodford (2003), "Optimal monetary and fiscal policy: a linearquadratic approach", *NBER Macroeconomics Annual*, Vol. 18, pp. 271–333

Bertocco, G. (2007), "The characteristics of a monetary economy: a Keynes-Schumpeter approach", *Cambridge Journal of Economics*, Vol. 31, pp. 101-122.

Bordo, M. D. (2008), "Monetary Policy, History of", in S. Durlauf and L. Blume (Eds) (2008), *The New Palgrave Dictionary of Economics* (2^{th} ed.), Palgrave Macmillan.

Burdett, K. and D. T. Mortensen (1998), "Wage Differentials, Employer Size, and Unemployment", *International Economic Review*, Vol. 39, Nº. 2, pp. 257-273.

Caballero, R. (2008), "Creative Destruction", in S. Durlauf and L. Blume (Eds) (2008), *The New Palgrave Dictionary of Economics* (2^{th} ed.), Palgrave Macmillan.

Carré, M. and D. Drouot (2004), "Pace versus type: The efeect of economic growth on unemployment and wage patterns", *Review of Economic Dynamics*, Vol. 7, Nº. 3, pp. 737-757.

Chu, A. (2010), "Effects of patent length on R&D: A quantitative DGE analysis", *Journal of Economics*, Vol. 99, Nº. 2, pp. 117–140.

Chu, A. and G. Cozzi (2014), "R&D and Economic Growth in a cash-in-advance economy", *International Economic Review*, Vol. 55, Nº. 2, pp. 507-524.

Chu, A. and C. Lai (2013), "Money and the Welfare Cost of Inflation in an R&D Growth Model", *Journal of Money, Credit and Banking*, Vol. 45, pp. 233-249.

Comin, D. (2004), "R&D: A small contribution to productivity growth", *Journal of Economic Growth*, Vol. 9, Nº. 4, pp. 391–421.

Fisher, I. (1911), *The Purchasing Power of Money.* New York: Macmillan

Friedman, M. (1956) "The quantity theory of money: a restatement". In M. Friedman (ed.), *Studies in the Quantity Theory of Money*, pp. 3–21. Chicago, University of Chicago Press

Friedman, M. (1960) *A Program for Monetary Stability.* New York: Fordham University Press.

Galí J. (2008), *Monetary Policy, Inflation and the Business Cycle.* Princeton, NJ: Princeton University Press.

Howitt, P. and D. N. Weil (2008), "Economic Growth", in S. Durlauf and L. Blume (Eds) (2008), *The New Palgrave Dictionary of Economics* (2^{th} ed.), Palgrave Macmillan.

Howitt, P. (2008), "Endogenous Growth Theory", in S. Durlauf and L. Blume (Eds) (2008), *The New Palgrave Dictionary of Economics* (2^{th} ed.), Palgrave Macmillan.

Keynes, J.M. (1936), *The General Theory of Employment, Interest and Money.* London: Macmillan.

Kuznets, S. (1947), "Measurement of Economic Growth", *The Journal of Economic History*, Vol. 7, Nº. 1, pp.10-34.

Lainez, C. and P. Peretto (2006), "Scale Effects in Endogenous Growth Theory: An Error of Aggregation Not Specification," *Journal of Economic Growth*, Vol. 11, pp. 263–288.

Leeper, E.M. (1991), "Equilibria under 'active' and 'passive' monetary and fiscal policies", *Journal of Monetary Economics*, Vol. 27, pp. 129–147.

Lucas, R.E., Jr (1972) "Expectations and the neutrality of money", *Journal of Economics*, Vol. 4, pp. 103-124.

Mankiw Gregory, N., D. Romer, and D. N. Weil (1992), "A contribution to the empirics of economic growth", *Quarterly Journal of Economics*, Vol. 107, Nº. 2, pp. 407-437.

Matsuyama, K. (2008), "Structural Change", in S. Durlauf and L. Blume (Eds) (2008), *The New Palgrave Dictionary of Economics* (2^{th} ed.), Palgrave Macmillan.

Mortensen, D. T. (2004), *Wage Dispersion: Why Are Similar Workers Paid Differently?*. Cambridge: MIT Press.

Neto, A., Y. Furukawa and A. P. Ribeiro (2017), "Can Trade Unions Increase Social Welfare? An R&D Model with Cash-in-Advance Constraints", *MPRA Paper*, no. 77312, Munich Personal RePEc Archive.

Pasten, E. and R. Schoenle (2016), "Rational inattention, multi-product firms and the neutrality of money", *Journal of Monetary Economics*, Vol. 80, pp. 1-16.

Patinkin, D. (1987), "Neutrality of Money", in S. Durlauf and L. Blume (Eds) (2008), *The New Palgrave Dictionary of Economics* (2^{th} ed.), Palgrave Macmillan.

Peretto, P. (1998), "Technological Change and Population Growth", *Journal of Economic Growth*, Vol. 3, pp. 283–311.

Postel-Vinay, F. and J.-M. Robin (2002), "Equilibrium Wage Dispersion with Worker and Employer Heterogeneity", *Econometrica*, Vol. 70, Nº. 6, pp. 2295-2350.

Ramsey, F.P. (1927), "A contribution to the theory of taxation", *Economic Journal*, Vol 37, pp 47–61.

Reis, A. B. and T. N. Sequeira (2007), "Human capital and overinvestment in R&D", *The Scandinavian Journal of Economics*, Vol. 109, Nº. 3, pp. 573–591.

Rostow, W. W. (1959), "The stages of economic growth", *The Economic History Review*, Vol. 12, Nº. 1, pp. 1-16.

Salverda, W. and D. Checchi (2014), "Labour-Market Institutions and the Dispersion of Wage Earnings", in A. Atkinson and F. Bourguignon (editors) (2015), *Handbook of Income Distribution* Volume 2. Oxford. Elsevier. p. 1539.

Schumpeter, J. A. (1947), "Theoretical Problems of Economic Growth", *The Journal of Economic History*, Vol. 7, Nº. 1, pp. 1-9.

Serletis, A. and D. Krause (1996), "Empirical evidence on the long-run neutrality hypothesis using low-frequency international data", *Economic Letters,* Vol. 50, pp. 323-327.

Solow, R. M. (1956), "A contribution to the theory of economic growth", *Quarterly Journal of Economics*, Vol. 70, Nº. 1, pp. 65-94.

Thomas, C. (2008), "Search and matching frictions and optimal monetary policy", *Journal of Monetary Policy*, Vol. 55, pp. 936-956.

Walsh, C. E (2010), *Monetary Theory and Policy*, 3^{rd} ed. Massachusetts, Massachusetts Institute of Technology