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S. V. Rakovic (sasa.v.rakovic@fe.up.pt),
F. A. C. C. Fontes (faf@fe.up.pt), and
I. V. Kolmanovsky (ilya@umich.edu)

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SYSTEC Report 2017-SC2, July 2017.

This is a preprint of the article to be published in the Proceedings of the 20th IFAC World Congress, July 9-14, 2017, Toulouse, France.

Reachability and Invariance for Linear Sampled–data Systems

S. V. Raković [*,1], **F. A. C. C. Fontes** [*], and **I. V. Kolmanovsky** [**]

[*] *Universidade do Porto, Porto, Portugal*
[**] *The University of Michigan at Ann Arbor, Ann Arbor, U.S.*

Abstract: We consider linear sampled–data dynamical systems subject to additive and bounded disturbances, and study properties of their forward and backward reach sets as well as robust positively invariant sets. We propose topologically compatible notions for the sampled–data forward and backward reachability as well as robust positive invariance. We also propose adequate notions for maximality and minimality of related robust positively invariant sets.

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1. INTRODUCTION

Theoretical and computational aspects of reachability and set invariance have been subjest to an extensive research over the last several decades. The prime reasons for academic and practical interest in these set–valued notions stem from their intimate relationships with fundamental aspects of control and systems theory as well as theory of dynamical systems. A selection of comprehensive and influential research monographs (Aubin, 1991; Blanchini and Miani, 2008; Rawlings and Mayne, 2009; Kurzhanski and Varaiya, 2014), together with numerous references therein, provides overwhelming evidence that reachability and set invariance have made a huge impact across a wide spectrum of classical fields of dynamics, controls and systems in general, and deterministic/robust/stochastic optimal and model predictive control syntheses as well as safety and stability analyses in particular. The contemporary, increasingly sophisticated autonomous engineered systems frequently require *a prior–to–operation* assurances of, at least, safe, resilient, secure and fault tolerant operability. The related high quality dynamical *modus operandi* can be realized through utilization of classical reachability and set invariance notions. For instance, the maximal safe sets are, in fact, positively invariant sets. Furthermore, resilience and security can be achieved by using robust control/positively invariant sets that offer a high degree of flexibility and robustness to uncertainty. Similarly, fault tolerance is only possible from certain sets of states. Quite naturally, reachability and set invariance are highly potent platforms for developing appropriate analysis and synthesis methodologies for versatile smart autonomous systems.

Classical and modern research have reached, and also exploited, a well–established conclusion that the analysis of uncertain constrained dynamics utilizing reachability and set invariance enables one to guarantee *a–priori* relevant strtural properties. Properties of the backward and forward reach sets as well as robust positively invariant sets, such as monotonicity, compactness, convexity, have been in the spotlight due to their instrumental role in characterization and computation of the associated maximal and minimal robust positively invariant sets. Interestingly enough, the majority of existing research efforts on the subjects have been focused almost exclusively on either discrete or continuous times formulations. A nonexhaustive list of selected works includes (Bertsekas, 1972; Schweppe, 1973; Gilbert and Tan, 1991; Aubin, 1991; Artstein, 1995; Kolmanovsky and Gilbert, 1998; Raković, 2007; Raković and Kouramas, 2007; Artstein and Raković, 2008; Blanchini and Miani, 2008; Kurzhanski and Varaiya, 2014).

Despite elaborate literature on, and plethora of results for, discrete and continuous time reachability and set invariance problems, there is a limited progress on related aspects for sampled–data systems. The sampled–data setting arises naturally in many classical and modern applications, and it plays a key role for control synthesis carried out using discrete time techniques with the goal to ensure constraint satisfaction in continuous time sense. The literature addressing theory and computations of forward and backward reach sets as well as robust positively invariant sets within sampled–data setting is very scarce excluding a sequence of more recent articles (Mitchell et al., 2012, 2013; Mitchell and Kaynama, 2015; Raković et al., 2016). A major peculiarity of sampled–data setting is inapplicability, to a prohibiting extent, of discrete and continuous time reachability and set invariance (Raković et al., 2016).

In this paper, we follow up on our recent research (Raković et al., 2016), and we first establish key properties of the related sampled–data forward and backward reach sets, and then use these properties to introduce topologically compatible notions of robust positive invariance for sampled–data systems. We consider generalized and relaxed notions of robust positively invariant families, that allow for a flexible combination of robust positive invariance in discrete time sense and safety in continuous time sense. The minimality and maximality of robust positively invariant families are addressed in the natural sense of “pointwise–in–time–over–the–sampling–intervals” minimality and maximality (w.r.t. set inclusion) of sets.

[1] Corresponding author. E-mail address: sasa.v.rakovic@fe.up.pt.

Paper Structure: Section 2 describes basic setting and preliminaries. Sections 3 and 4 focus on key topological properties of the forward and backward reach sets. Section 5 discusses generalized notions of robust positive invariance as well as related notions of minimality and maximality of robust positively invariant sets. A summarizing discussion is provided in Section 6.

Nomenclature and Definitions: The sets of nonnegative integers and real numbers are denoted by $\mathbb{Z}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$, respectively. Any given sampling period $T \in \mathbb{R}_{\geq 0},\ T > 0$ induces sequences of sampling instances π and sampling intervals θ both w.r.t. $\mathbb{R}_{\geq 0}$ specified via:

$$\pi := \{t_k\}_{k\in\mathbb{Z}_{\geq 0}} \text{ and } \theta := \{\mathcal{T}_k\}_{k\in\mathbb{Z}_{\geq 0}}, \text{ where } \forall k \in \mathbb{Z}_{\geq 0},$$
$$t_{k+1} := t_k + T \text{ with } t_0 := 0 \text{ and } \mathcal{T}_k := [t_k, t_{k+1}).$$

$\rho(M)$ denotes the spectral radius of a matrix $M \in \mathbb{R}^{n\times n}$. For any two sets $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{R}^n$, the Minkowski set addition is specified by

$$\mathcal{X} \oplus \mathcal{Y} := \{x + y\ :\ x \in \mathcal{X},\ y \in \mathcal{Y}\},$$

while the Minkowski set subtraction (a.k.a. the Pontryagin or geometric set difference) is specfied by

$$\mathcal{X} \ominus \mathcal{Y} := \{z\ :\ z \oplus \mathcal{Y} \subseteq \mathcal{X}\}.$$

Given a set $\mathcal{X}$ and a real matrix M of compatible dimensions the image of $\mathcal{X}$ under M is denoted by

$$M\mathcal{X} := \{Mx\ :\ x \in \mathcal{X}\},$$

while the preimage of $\mathcal{X}$ under M is denoted by

$$M^{-1}\mathcal{X} := \{x\ :\ Mx \in \mathcal{X}\}.$$

A set $\mathcal{X}$ in $\mathbb{R}^n$ is *a C–set* if it is compact, convex, and contains the origin. A set $\mathcal{X}$ in $\mathbb{R}^n$ is *a proper C–set* if it is a C–set and contains the origin in its interior.

Given any two compact sets $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{R}^n$, their Hausdorff distance is defined by

$$\mathrm{H}_{\mathcal{L}}(\mathcal{X}, \mathcal{Y}) := \min_{\alpha\geq 0}\{\alpha\ :\ \mathcal{X} \subseteq \mathcal{Y} \oplus \alpha\mathcal{L} \text{ and } \mathcal{Y} \subseteq \mathcal{X} \oplus \alpha\mathcal{L}\},$$

where $\mathcal{L}$ is a given symmetric proper C–set in $\mathbb{R}^n$ inducing vector norm

$$|x|_{\mathcal{L}} := \min_{\eta}\{\eta\ :\ x \in \eta\mathcal{L},\ \eta \geq 0\}.$$

Henceforth, unless stated otherwise, we work with nonempty sets, fixed sampling period $T \in \mathbb{R}_{\geq 0},\ T > 0$ and fixed sequences of related sampling instances π and intervals θ.

2. SETTING AND OBJECTIVES

Consider a linear system described, for all $t \in \mathbb{R}_{\geq 0}$, by:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \text{ with } w(t) \in \mathcal{W}, \quad (2.1)$$

where, for any time $t \in \mathbb{R}_{\geq 0}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $w(t) \in \mathbb{R}^p$ denote, respectively, state, control and disturbance values, while $\dot{x}(t)$ denotes the value of the state derivative with respect to time, while the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $E \in \mathbb{R}^{n\times p}$ and the set $\mathcal{W} \subset \mathbb{R}^p$ are known exactly.

The linear system (2.1) is controlled via sampled–data linear state feedback so that

$$\forall k \in \mathbb{Z}_{\geq 0},\ \forall t \in \mathcal{T}_k,\ u(t) := Kx(t_k), \quad (2.2)$$

where $K \in \mathbb{R}^{m\times n}$ is a known exactly control gain matrix. Clearly, the linear sampled–data feedback at each time t is not a function of the state at time instant t, rather it is a function of the state at the last sampling instance t_k.

For any $k \in \mathbb{Z}_{\geq 0}$, within the sampled–data setting, the admissible disturbance functions $w(\cdot)$ in (2.1) are, like the controls $u(\cdot)$, piecewise constant right continuous functions from time interval $[0, t_k]$ to the set $\mathcal{W}$ so that

$$\forall k \in \mathbb{Z}_{\geq 0},\ \forall t \in \mathcal{T}_k,\ w(t) := w(t_k) \in \mathcal{W}, \quad (2.3)$$

i.e., maps $w(\cdot)$ are constant in sampling intervals $\mathcal{T}_k$ and right continuous at sampling instants t_k for all $k \in \mathbb{Z}_{\geq 0}$. Such a class of disturbance maps captures adequately the actuation errors, noise related errors in sampled–data measurements in (2.2) and, in fact, it also represents a reasonably rich model for various types of uncertainty.

To define sampled–data solutions, let, for any $t \in [0, T]$,

$$A_d(t) := e^{tA}, \quad B_d(t) := \left(\int_0^t e^{\tau A} d\tau\right) B \text{ and}$$
$$E_d(t) := \left(\int_0^t e^{\tau A} d\tau\right) E, \quad (2.4)$$

where the related integrals are the standard matrix–valued integrals, and let also, for any $t \in [0, T]$,

$$A_S(t) := A_d(t) + B_d(t)K \text{ and } E_S(t) := E_d(t) \text{ and}$$
$$A_D := A_S(T) \text{ and } E_D := E_S(T). \quad (2.5)$$

We work throughout this note under the following mild and natural set of conditions.

Assumption 1. The sampling period T is such that the matrix pair $(A_d(T), B_d(T))$ is strictly stabilizable. The control matrix K is such that the matrix A_D is strictly stable (i.e., $\rho(A_D) < 1$) and the matrix pair (A_D, E_D) is controllable. The set $\mathcal{W}$ is a proper C–set in $\mathbb{R}^p$.

In view of (2.1)–(2.3), the sampled–data solutions satisfy at the sampling instances t_k, for all $k \in \mathbb{Z}_{\geq 0},\ k > 0$,

$$x(t_k) = A_D^k x + \sum_{i=0}^{k-1} A_D^{k-1-i} E_D w(t_i) \text{ with}$$
$$x(t_0) = x. \quad (2.6)$$

During sampling intervals $\mathcal{T}_k,\ k \in \mathbb{Z}_{\geq 0}$, the sampled–data solutions satisfy, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$, the property that

$$x(t_k + t) = A_S(t)x(t_k) + E_S(t)w(t_k). \quad (2.7)$$

Our main objectives are to study the finite– and infinite–time forward and backward reachability as well as robust positive invariance for linear sampled–data systems. More precisely, the note characterizes, and discusses properties of, the forward reach sets $\mathcal{R}(\mathcal{X}, t)$ at time t from initial set $\mathcal{X}$.(These are the sets of all sampled–data solutions/states that can be reached at time t as the initial state x varies within a given set of initial states $\mathcal{X}$ and the disturbance functions $w(\cdot)$ vary within a class of admissible disturbance maps.) Furthermore, the note also characterizes, and discusses properties of, the backward reach sets $\mathcal{B}(\mathcal{X}, t)$ at time t w.r.t. the target set $\mathcal{X}$. (These are the sets of all initial states x for which the related sampled–data solutions/states satisfy given constraints for all times $\tau \in [0, t)$ and reach the given target set $\mathcal{X}$ at time t despite the presence and effects of admissible disturbance functions $w(\cdot)$.) Finally, the note also considers limiting behaviour of forward and backward reach sets with the aim to provide flexible and topologically compatible definitions of ordinary, minimal and maximal robust positively invariant sets within the considered sampled–data setting.

3. FORWARD REACHABILITY

As already mentioned, the forward reach set $\mathcal{R}(\mathcal{X}, t)$ at time t from the set of initial states $\mathcal{X}$ is the set of all sampled–data solutions/states at time t satisfying (2.6) and (2.7) that can be generated as the initial states x vary within the set $\mathcal{X}$ and the disturbance functions $w(\cdot)$ vary within the class of admissible disturbance maps (namely, piecewise constant right continuous disturbance functions from the time interval $[0, t]$ to the disturbance set $\mathcal{W}$). Thus, the forward reach sets $\mathcal{R}(\mathcal{X}, t)$ are essentially values of the forward reach set map $\mathcal{R}(\cdot, \cdot)$ (that is a function mapping subsets of $\mathbb{R}^n$ and nonnegative time instances to subsets of $\mathbb{R}^n$) evaluated at time t and set of initial states $\mathcal{X}$. Since the sampled–data solutions are entirely characterized by relations (2.6) and (2.7), the forward reach set map $\mathcal{R}(\cdot, \cdot)$ satisfies, by definition, for all subsets $\mathcal{X}$ in $\mathbb{R}^n$ and all sampling instances t_k, $k \in \mathbb{Z}_{\geq 0}$, $k > 0$,

$$\mathcal{R}(\mathcal{X}, t_k) := A_D^k \mathcal{X} \oplus \bigoplus_{i=0}^{k-1} A_D^i E_D \mathcal{W}$$
$$\text{with } \mathcal{R}(\mathcal{X}, t_0) := \mathcal{X}. \tag{3.1}$$

During sampling intervals $\mathcal{T}_k$, $k \in \mathbb{Z}_{\geq 0}$, the forward reach set map $\mathcal{R}(\cdot, \cdot)$ satisfies, by definition, for all subsets $\mathcal{X}$ in $\mathbb{R}^n$, all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$,

$$\mathcal{R}(\mathcal{X}, t_k + t) = A_S(t)\mathcal{R}(\mathcal{X}, t_k) \oplus E_S(t)\mathcal{W}. \tag{3.2}$$

A direct inspection of (3.1) and (3.2) reveals that the forward reach set map $\mathcal{R}(\cdot, \cdot)$ is form–wise identical to the discrete time forward reach set maps at sampling instances t_k, but it is essentially different from the continuous time forward reach set map during the sampling intervals $\mathcal{T}_k$. In turn, its topological behaviour is substantially more complicated, as we summarize next. (A more detailed discussion can be found in (Raković et al., 2016).)

In light of relations (3.1) and (3.2) and since involved linear transformations and Minkowski additions preserve both compactness and convexity, the forward reach set map $\mathcal{R}(\cdot, \cdot)$ also preserves both compactness and convexity.

Proposition 1. Suppose Assumption 1 holds. The forward reach sets $\mathcal{R}(\mathcal{X}, t)$ are compact and convex for all $t \in \mathbb{R}_{\geq 0}$ and for any compact and convex set $\mathcal{X}$ in $\mathbb{R}^n$.

Since $0 \in \mathcal{X}$ implies directly that $0 \in \mathcal{R}(\mathcal{X}, t)$ for all $t \in \mathbb{Z}_{\geq 0}$, the above proposition can be strengthened for the case of C– and proper C–sets in $\mathbb{R}^n$ as follows.

Corollary 1. Suppose Assumption 1 holds. The forward reach sets $\mathcal{R}(\mathcal{X}, t)$ are C–sets for all $t \in \mathbb{R}_{\geq 0}$ and for any C– or proper C–set $\mathcal{X}$ in $\mathbb{R}^n$. Furthermore, the forward reach sets $\mathcal{R}(\mathcal{X}, t_k)$ at sampling instances t_k are guaranteed to be proper C–sets for all large enough k.

Due to periodicity and thanks to the continuity of the matrices $A_S(\cdot)$ and $E_S(\cdot)$, the forward reach set map $\mathcal{R}(\cdot, \cdot)$ is continuous in time w.r.t. Hausdorff distance.

Proposition 2. Suppose Assumption 1 holds and take any compact set $\mathcal{X}$ in $\mathbb{R}^n$. The forward reach sets $\mathcal{R}(\mathcal{X}, t)$, $t \in \mathbb{Z}_{\geq 0}$ change continuously in time w.r.t. Hausdorff distance.

The first relevant peculiarity of the sampled–data setting is the fact that the related forward reach set map $\mathcal{R}(\cdot, \cdot)$ does not, in general, satisfy semi–group property in a proper sense. Instead, the following assertions can be verified.

Proposition 3. Suppose Assumption 1 holds. The forward reach set map $\mathcal{R}(\cdot, \cdot)$ satisfies semi–group property at the sampling instances (i.e. in discrete time sense). In particular, for all compact subsets $\mathcal{X}$ of $\mathbb{R}^n$, all $i \in \mathbb{Z}_{\geq 0}$ and all $j \in \mathbb{Z}_{\geq 0}$, we have

$$\mathcal{R}(\mathcal{X}, t_i + t_j) = \mathcal{R}(\mathcal{R}(\mathcal{X}, t_i), t_j). \tag{3.3}$$

Remark 1. It is worth observing that the forward reach set map $\mathcal{R}(\cdot, \cdot)$ is not guaranteed to satisfy semi–group property in the sampling intervals (i.e. in continuous time sense). In other words, it is generally not possible to guarantee that

$$\mathcal{R}(\mathcal{X}, \tau_1 + \tau_2) = \mathcal{R}(\mathcal{R}(\mathcal{X}, \tau_1)\tau_2). \tag{3.4}$$

holds true for all compact subsets $\mathcal{X}$ of $\mathbb{R}^n$, all $\tau_1 \in \mathbb{R}_{\geq 0}$ and all $\tau_2 \in \mathbb{R}_{\geq 0}$. This absence of generic semi–group property of the forward reach set map $\mathcal{R}(\cdot, \cdot)$ has been exemplified in (Raković et al., 2016, Example 2).

The forward reach set map $\mathcal{R}(\cdot, \cdot)$ remains monotone in the first argument for all $t \in \mathbb{R}_{\geq 0}$.

Proposition 4. Suppose Assumption 1 holds. For any two compact subsets $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{R}^n$ and all $t \in \mathbb{R}_{\geq 0}$ we have

$$\mathcal{X} \subseteq \mathcal{Y} \Rightarrow \mathcal{R}(\mathcal{X}, t) \subseteq \mathcal{R}(\mathcal{Y}, t). \tag{3.5}$$

Due to the absence of the generic semi–group property, and despite the monotonicity in the first argument, the forward reach set map $\mathcal{R}(\cdot, \cdot)$ is not guaranteed to entirely preserve positive invariance. However, it certainty preserves it at the sampling instances as verified by the following observation.

Proposition 5. Suppose Assumption 1 holds. The forward reach set map $\mathcal{R}(\cdot, \cdot)$ preserves positive invariance property at the sampling instances. In particular,

$$\mathcal{R}(\mathcal{X}, T) \subseteq \mathcal{X} \Rightarrow \forall k \in \mathbb{Z}_{\geq 0},\ \mathcal{R}(\mathcal{X}, t_{k+1}) \subseteq \mathcal{R}(\mathcal{X}, t_k). \tag{3.6}$$

Remark 2. The forward reach set map $\mathcal{R}(\cdot, \cdot)$ is not guaranteed to preserves positive invariance property in the sampling intervals. In particular, the implication

$$\forall \tau \in [0, \delta],\ \mathcal{R}(\mathcal{X}, \tau) \subseteq \mathcal{X} \Rightarrow$$
$$\forall t \geq 0,\ \forall \tau \in [0, \delta],\ \mathcal{R}(\mathcal{X}, t + \tau) \subseteq \mathcal{R}(\mathcal{X}, t) \tag{3.7}$$

is not guaranteed to hold in general case.

Likewise, the positive anti–invariance is not guaranteed to be entirely preserved by the forward reach set map $\mathcal{R}(\cdot, \cdot)$.

Proposition 6. Suppose Assumption 1 holds. The forward reach set map $\mathcal{R}(\cdot, \cdot)$ preserves positive anti–invariance property at the sampling instnces.In particular,

$$\mathcal{X} \subseteq \mathcal{R}(\mathcal{X}, T) \Rightarrow \forall k \in \mathbb{Z}_{\geq 0},\ \mathcal{R}(\mathcal{X}, t_k) \subseteq \mathcal{R}(\mathcal{X}, t_{k+1}). \tag{3.8}$$

Remark 3. The forward reach set map $\mathcal{R}(\cdot, \cdot)$ is not guaranteed to preserves positive anti–invariance property in the sampling intervals. In particular, the implication

$$\forall \tau \in [0, \delta],\ \mathcal{X} \subseteq \mathcal{R}(\mathcal{X}, \tau) \Rightarrow$$
$$\forall t \geq 0,\ \forall \tau \in [0, \delta],\ \mathcal{R}(\mathcal{X}, t) \subseteq \mathcal{R}(\mathcal{X}, t + \tau) \tag{3.9}$$

is not guaranteed to hold in general case. The failure of $\mathcal{R}(\cdot, \cdot)$ to preserve positive invariance and positive anti–invariance properties has been exemplified in (Raković et al., 2016, Example 3).

The above outlined properties of the forward reach set map $\mathcal{R}(\cdot, \cdot)$ provide basis for studying its limiting behaviour as well as for introducing topologically compatible notions of ordinary and minimal robust positively invariant sets for linear sampled–data systems under consideration.

4. BACKWARD REACHABILITY

The backward reach set $\mathcal{B}(\mathcal{X}, t)$ at time t w.r.t. the target set $\mathcal{X}$ is the set of all initial states x for which the related sampled–data solutions/states generated by (2.6) and (2.7) satisfy given constraints for all times $\tau \in [0, t)$ (i.e. $x(\tau) \in \mathbb{X}$ where $\mathbb{X}$ is a given state constraint set) and reach the given target set $\mathcal{X}$ at time t despite the presence and effects of admissible disturbance functions $w(\cdot)$. Henceforth, we invoke an additional and relatively mild assumption on the state constraint set $\mathbb{X}$ of interest.

Assumption 2. The set $\mathbb{X}$ is a proper C–set in $\mathbb{R}^n$.

The backward reach sets $\mathcal{B}(\mathcal{X}, t)$ are essentially values of the backward reach set map $\mathcal{B}(\cdot, \cdot)$ (that is a function mapping subsets of $\mathbb{R}^n$ and nonnegative time instances to subsets of $\mathbb{R}^n$) evaluated at time t and the target set $\mathcal{X}$. The backward reach set map $\mathcal{B}(\cdot, \cdot)$ is specified, for all compact subsets (target sets) $\mathcal{X}$ of $\mathbb{R}^n$ and all nonnegative times t by

$$\mathcal{B}(\mathcal{X}, t) := \{x \in \mathbb{R}^n \ : \ \forall \tau \in [0, t), \ \mathcal{R}(\{x\}, \tau) \subseteq \mathbb{X} \text{ and } \mathcal{R}(\{x\}, t) \subseteq \mathcal{X}\}, \quad (4.1)$$

where $\mathcal{R}(\{x\}, t)$ is the forward reach set at time t from the initial state x as defined in (3.1) and (3.2). In view of explicit form of the forward reach set map $\mathcal{R}(\cdot, \cdot)$, we can derive an explicit form of the backward reach set map $\mathcal{B}(\cdot, \cdot)$. To this end, we observe that, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$, it holds that

$$\mathcal{R}(\{x\}, t_k + t) = A_S(t) A_D^k x \oplus \mathcal{R}(\{0\}, t_k + t), \quad (4.2)$$

where $\mathcal{R}(\{0\}, t_k + t)$ is the forward reach set at time $t_k + t$ from the origin whose explicit form follows from (3.1) and (3.2) by setting $\mathcal{X} = \{0\}$ in these relations. A relatively direct set algebra now yields the desired explicit form of the backward reach set map $\mathcal{B}(\cdot, \cdot)$. In particular, let, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$,

$$M(t_k + t) := A_S(t) A_D^k, \ \mathbb{X}(t_k + t) := \mathbb{X} \ominus \mathcal{R}(\{0\}, t_k + t), \text{ and } \mathcal{X}(t_k + t) := \mathcal{X} \ominus \mathcal{R}(\{0\}, t_k + t), \quad (4.3)$$

so that

$$\mathcal{B}(\mathcal{X}, t_k + t) = \bigcap_{j \in \mathbb{Z}_{\geq 0}, \ j \leq k, \ \tau \in [0, t)} M^{-1}(t_j + \tau) \mathbb{X}(t_j + \tau) \bigcap M^{-1}(t_k + t) \mathcal{X}(t_k + t) \quad (4.4)$$

The structural properties of the forward reach set map $\mathcal{R}(\cdot, \cdot)$, and the explicit form of the backward reach set map $\mathcal{B}(\cdot, \cdot)$ allow for a deeper understanding of the structural properties of the backward reach set map $\mathcal{B}(\cdot, \cdot)$, the most important of which are now summarized.

In analogy to Proposition 1, the following can be asserted.

Proposition 7. Suppose Assumptions 1 and 2 hold. The backward reach sets $\mathcal{B}(\mathcal{X}, t)$ are, possibly empty, compact and convex for all $t \in \mathbb{R}_{\geq 0}$ and for any compact and convex set $\mathcal{X}$ in $\mathbb{R}^n$.

The analogous statement to Corollary 1 requires additional hypotheses, while analogous statement to Proposition 2 is not possible in general since the backward reach set can be empty and, thus, it can exhibit discontinuous behaviour. The lack of generic semi–group property of the forward reach set map $\mathcal{R}(\cdot, \cdot)$ propagates to the backward reach set map $\mathcal{B}(\cdot, \cdot)$ so that, in analogy to Proposition 3, only the partial semi–group properties can be established in general case.

Proposition 8. Suppose Assumptions 1 and 2 hold. The backward reach set map $\mathcal{B}(\cdot, \cdot)$ satisfies semi–group property at the sampling instances (i.e. in discrete time sense). In particular, for all compact subsets $\mathcal{X}$ of $\mathbb{R}^n$, all $i \in \mathbb{Z}_{\geq 0}$ and all $j \in \mathbb{Z}_{\geq 0}$, we have

$$\mathcal{B}(\mathcal{X}, t_i + t_j) = \mathcal{B}(\mathcal{B}(\mathcal{X}, t_i), t_j). \quad (4.5)$$

Remark 4. In analogy to Remark 1, it is worth noting that the backward reach set map $\mathcal{B}(\cdot, \cdot)$ is not guaranteed to satisfy semi–group property in the sampling intervals (i.e. in continuous time sense). Namely, the condition that for all compact subsets $\mathcal{X}$ of $\mathbb{R}^n$, all $\tau_1 \in \mathbb{R}_{\geq 0}$ and all $\tau_2 \in \mathbb{R}_{\geq 0}$, we have

$$\mathcal{B}(\mathcal{X}, \tau_1 + \tau_2) = \mathcal{B}(\mathcal{B}(\mathcal{X}, \tau_1), \tau_2) \quad (4.6)$$

is not guaranteed to hold in general case.

An analogue of Proposition 4 verifies the monotonicity in the first argument of the backward reach set map $\mathcal{B}(\cdot, \cdot)$.

Proposition 9. Suppose Assumptions 1 and 2 hold. For any two compact subsets $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{R}^n$ and all $t \in \mathbb{R}_{\geq 0}$ we have

$$\mathcal{X} \subseteq \mathcal{Y} \Rightarrow \mathcal{B}(\mathcal{X}, t) \subseteq \mathcal{B}(\mathcal{Y}, t). \quad (4.7)$$

As in the case of Propositions 5 and 6 relevant for the the forward reach set map $\mathcal{R}(\cdot, \cdot)$, it is not dificult to show that the backward reach set map $\mathcal{B}(\cdot, \cdot)$ is guaranteed to preserve positive invariance and positive anti–invariance at the sampling instances.

Proposition 10. Suppose Assumptions 1 and 2 hold. The backward reach set map $\mathcal{B}(\cdot, \cdot)$ preserves positive invariance property at the sampling instances. In particular,

$$\mathcal{B}(\mathcal{X}, T) \subseteq \mathcal{X} \Rightarrow \forall k \in \mathbb{Z}_{\geq 0}, \ \mathcal{B}(\mathcal{X}, t_{k+1}) \subseteq \mathcal{B}(\mathcal{X}, t_k). \quad (4.8)$$

Proposition 11. Suppose Assumption 1 holds. The backward reach set map $\mathcal{B}(\cdot, \cdot)$ preserves positive anti–invariance property at the sampling instances. In particular,

$$\mathcal{X} \subseteq \mathcal{B}(\mathcal{X}, T) \Rightarrow \forall k \in \mathbb{Z}_{\geq 0}, \ \mathcal{B}(\mathcal{X}, t_k) \subseteq \mathcal{B}(\mathcal{X}, t_{k+1}). \quad (4.9)$$

Remark 5. The question of the preservation of positive invariance and anti–positive invariance in the sampling intervals (i.e. continuous time sense) by the backward reach set $\mathcal{B}(\cdot, \cdot)$ is technically slightly more involved than in the case of the forward reach set map $\mathcal{R}(\cdot, \cdot)$. The related discussion is omitted as it does not strictly fall within the intended scope of this conference paper.

A particularly important set trajectory induced by the related backward reach set map $\mathcal{B}(\cdot, \cdot)$ is the one initiated at the state constraint set $\mathbb{X}$. Namely, the backward reach sets $\mathcal{B}(\mathbb{X}, t), \ t \geq 0$ form this special set trajectory $\{\mathcal{B}(\mathbb{X}, t) \ : \ t \geq 0\}$, which is, in fact, "pointwise–in–time" maximal (w.r.t. set inclusion) set trajectory. More precisely, for any subset $\mathcal{X}$ of $\mathbb{X}$ and for all nonegative times t, it holds that

$$\mathcal{B}(\mathcal{X}, t) \subseteq \mathcal{B}(\mathbb{X}, t). \quad (4.10)$$

The above outlined properties of the backward reach set map $\mathcal{B}(\cdot, \cdot)$ provide basis for studying its limiting behaviour and introducing topologically compatible notions of maximal robust positively invariant sets for linear sampled–data systems under consideration.

5. ROBUST POSITIVE INVARIANCE

The topological properties of the forward and backward reach set maps $\mathcal{R}(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ dictate the utilization of generalized notions of ordinary, minimal and maximal robust positively invariant sets. In this sense, the preceding analysis implies that, within the sampled–data setting, a demand for a subset $\mathcal{S}$ of $\mathbb{X}$ to be robust positively invariant at the sampling instances

$$\forall x \in \mathcal{S},\ \forall w \in \mathcal{W},\ A_D x + E_D w \in \mathcal{S},$$

equivalently expressed by either of its set–theoretic reformulations

$$\mathcal{R}(\mathcal{S},T) \subseteq \mathcal{S} \subseteq \mathbb{X} \text{ or } \mathcal{S} \subseteq \mathcal{B}(\mathcal{S},T), \tag{5.1}$$

is natural and is, in fact, a minimal requirement to be imposed. However, the implication is also that a condition for a subset $\mathcal{S}$ of $\mathbb{X}$ to be robust positively invariant at the sampling instances and in the sampling interval:

$$\forall x \in \mathcal{S},\ \forall w \in \mathcal{W},\ \forall t \in [0,T],\ A_S(t)x + E_S(t)w \in \mathcal{S},$$

equivalently expressed by either of its set–theoretic reformulations

$$\forall t \in [0,T],\ \mathcal{R}(\mathcal{S},t) \subseteq \mathcal{S} \subseteq \mathbb{X} \text{ or } \mathcal{S} \subseteq \mathcal{B}(\mathcal{S},t) \tag{5.2}$$

is not natural and is, in fact, an overly conservative requirement. Consequently, a natural and non–conservative notion of sampled–data robust positive invariance should guarantee robust positive invariance at the sampling instances and it should relax robust positive invariance in the sampling intervals but also facilitate it if it is attainable. Clearly, it is not possible to guarantee such a flexibility with utilization of a single set $\mathcal{S}$. Instead, similarly as it is done for set invariance under output feedback in Artstein and Raković (2011), we introduce a generalized, and, in fact, relaxed, notion of robust positive invariance based on the utilization of a suitable family of sets.

Definition 1. A family of sets

$$\mathfrak{S} := \{\mathcal{S}(t) \ : \ t \in [0,T]\}, \tag{5.3}$$

where, for every $t \in [0,T]$, $\mathcal{S}(t)$ is a subset of $\mathbb{R}^n$, is a robust positively invariant family of sets for uncertain sampled–data linear dynamics, specified via (2.1)–(2.3), and constraint sets $(\mathbb{X},\mathcal{W})$ if and only if for all $t \in [0,T]$

(*I*) $\mathcal{S}(t) := A_S(t)\mathcal{S}(0) \oplus E_S(t)\mathcal{W}$ and $\mathcal{S}(T) \subseteq \mathcal{S}(0)$; and
(*II*) $\mathcal{S}(t) \subseteq \mathbb{X}$. (5.4)

The above generalized notion of robust positive invariance

$$\forall t \in [0,T],\ \mathcal{S}(t) := \mathcal{R}(\mathcal{S}(0),t) \subseteq \mathbb{X} \text{ and } \mathcal{S}(T) \subseteq \mathcal{S}(0).$$

is, in fact, equivalent to weak positive invariance of the family of sets $\mathfrak{S}$ w.r.t. forward reach set map $\mathcal{R}(\cdot,\cdot)$:

$$\forall t \in [0,T],\ \mathcal{S}(t) := \mathcal{R}(\mathcal{S}(0),t) \in \mathfrak{S} \text{ and } \mathcal{S}(T) \subseteq \mathcal{S}(0) \in \mathfrak{S}.$$

Clearly, a family of sets $\mathfrak{S}$ satisfying dynamic relation (I) of (5.4) can be constructed easily given a subset $\mathcal{S}$ in $\mathbb{R}^n$ that satisfies only robust positive invariance at the sampling instances (i.e. $\mathcal{R}(\mathcal{S},T) \subseteq \mathcal{S}$). To this end, it suffices to put, for all $t \in [0,T]$,

$$\mathcal{S}(t) := \mathcal{R}(\mathcal{S},t) = A_S(t)\mathcal{S} \oplus E_S(t)\mathcal{W}. \tag{5.5}$$

Finally, when the above sets $\mathcal{S}(t),\ t \in [0,T]$ are subsets of $\mathbb{X}$, the corresponding robust positively invariant family of sets $\mathfrak{S} = \{\mathcal{S}(t) \ : \ t \in [0,T]\}$ (more precisely any of its members $\mathcal{S}(t)$) is implicitly characterized by sets $\mathcal{S}$ and $\mathcal{W}$ (and matrices $A_S(t)$ and $E_S(t)$), as specified in (5.5).

For typographical convenience, in what follows we simply use terminology "robust positively invariant family" instead of complete expressions as specified in Definition 1. Since the sampled–data dynamics and constraint sets are fixed no confusion should arise.

5.1 Minimality

The limiting behaviour of the set–dynamics induced by the forward reach set map $\mathcal{R}(\cdot,\cdot)$ plays a key role in understanding the minimality of generalized sampled–data robust positively invariant sets. The classical results on the minimal robust positively invariant set for discrete time problems (Kolmanovsky and Gilbert, 1998) assert that the set

$$\mathcal{X}_\infty := \bigoplus_{k=0}^{\infty} A_D^k E_D \mathcal{W} \tag{5.6}$$

is a proper C–set in $\mathbb{R}^n$ and the unique solution to the fixed point set equation (Raković, 2007; Artstein and Raković, 2008)

$$\mathcal{R}(\mathcal{X},T) = \mathcal{X},\ \text{i.e. } A_D\mathcal{X} \oplus E_D\mathcal{W} = \mathcal{X}. \tag{5.7}$$

In addition, for any compact subset $\mathcal{S}$ in $\mathbb{R}^n$, the related sequence of the forward reach sets $\mathcal{R}(\mathcal{S},t_k)$ at sampling instances $t_k,\ k \geq 0$ converges to $\mathcal{X}_\infty$ exponentially fast w.r.t. Hausdorff distance. In fact, the set $\mathcal{X}_\infty$ is the unique set that satisfies, for all $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}\mathcal{R}(\mathcal{X}_\infty,t_{k+1}) &= A_D\mathcal{R}(\mathcal{X}_\infty,t_k) \oplus E_D\mathcal{W}\\ &= \mathcal{R}(\mathcal{X}_\infty,t_k) = \mathcal{X}_\infty.\end{aligned} \tag{5.8}$$

The compactness of $\mathcal{X}_\infty$ and the continuity of the reach set $\mathcal{R}(\cdot,\cdot)$ in time w.r.t. Hausdorff distance guarantee that the forward reach sets generated by $\mathcal{R}(\cdot,\cdot)$ starting from any compact subset $\mathcal{S}$ in $\mathbb{R}^n$ remain bounded, preserve compactness and exhibit a well–defined limiting behavior. In particular, during sampling intervals $\mathcal{T}_k$, we have, for all $t \in [0,T)$,

$$\begin{aligned}\mathcal{R}(\mathcal{X}_\infty,t_k+t) &= A_S(t)\mathcal{R}(\mathcal{X}_\infty,t_k) \oplus E_S(t)\mathcal{W}\\ &= A_S(t)\mathcal{X}_\infty \oplus E_S(t)\mathcal{W}.\end{aligned} \tag{5.9}$$

Consequently, for any fixed $t \in [0,T)$, and any compact subset $\mathcal{S}$ of $\mathbb{R}^n$, the forward reach set $\mathcal{R}(\mathcal{S},t_k+t)$ converges to $A_S(t)\mathcal{X}_\infty \oplus E_S(t)\mathcal{W}$ exponentially fast w.r.t. the Hausdorff distance (as k and, hence, t_k go to infinity).

In view of above analysis, let, for all $t \in [0,T]$,

$$\mathcal{X}_\infty(t) := A_S(t)\mathcal{X}_\infty \oplus E_S(t)\mathcal{W}, \tag{5.10}$$

and define a collection of C–sets in $\mathbb{R}^n$

$$\mathfrak{X}_\infty := \{\mathcal{X}_\infty(t) \ : \ t \in [0,T]\}. \tag{5.11}$$

For any robust positively invariant family of sets $\mathfrak{S} = \{\mathcal{S}(t) \ : \ t \in [0,T]\}$, the set $\mathcal{S}(0)$ is robust positively invariant at the sampling instances. In particular, it satisfies $\mathcal{S}(0) \subseteq \mathbb{X}$ and $A_D\mathcal{S}(0) \oplus E_D\mathcal{W} \subseteq \mathcal{S}(0)$. These conditions and the robust positive invariance and minimality (w.r.t. set inclusion) of the set $\mathcal{X}_\infty(0) = \mathcal{X}_\infty$ in discrete time sense (i.e. at the sampling instances) imply that $\mathcal{X}_\infty(0) \subseteq \mathcal{S}(0)$, which, in turn, implies directly that, for all $t \in [0,T]$, $A_S(t)\mathcal{X}_\infty(0) \oplus E_S(t)\mathcal{W} \subseteq A_S(t)\mathcal{S}(0) \oplus E_S(t)\mathcal{W}$. Consequently, it holds that:

$$\forall t \in [0,T],\ \mathcal{X}_\infty(t) \subseteq \mathcal{S}(t). \tag{5.12}$$

Since, for all $t \in [0,T]$, $\mathcal{S}(t) \subseteq \mathbb{X}$, the relation (5.12) reveals necessary and sufficient conditions for the existence of robust positively invariant families of sets under Assumptions 1 and 2.

Proposition 12. Suppose Assumptions 1 and 2 hold. There exists a robust positively invariant family $\mathfrak{S} = \{\mathcal{S}(t) \ : \ t \in [0,T]\}$ of sets if and only if

$$\forall t \in [0,T], \ \mathcal{X}_\infty(t) \subseteq \mathbb{X}. \tag{5.13}$$

The equivalent forms of the necessary and sufficient conditions (5.13) are given by either

$$\bigcap_{t\in[0,T]} (\mathbb{X} \ominus \mathcal{X}_\infty(t)) \neq \emptyset \tag{5.14}$$

or

$$0 \in \bigcap_{t\in[0,T]} (\mathbb{X} \ominus \mathcal{X}_\infty(t)). \tag{5.15}$$

The preceding justifies a natural assumption that discriminates, and enables focus on, a nontrivial case of interest.

Assumption 3. The collection $\mathfrak{X}_\infty$ of C–sets in $\mathbb{R}^n$, specified by (5.10) and (5.11), is such that the relation (5.13) holds true.

A natural "pointwise–in–time–over–the–sampling–interval" notion of the minimal robust positively invariant families, which is compatible with Definition 1 and preceding analysis, can be summarized as follows.

Definition 2. A robust positively invariant family

$$\mathfrak{S}_\infty := \{\mathcal{S}_\infty(t) \ : \ t \in [0,T]\}, \tag{5.16}$$

of sets is the minimal robust positively invariant family if and only if for any other robust positively invariant family $\mathfrak{S} = \{\mathcal{S}(t) \ : \ t \in [0,T]\}$ of sets it holds that

$$\forall t \in [0,T], \ \mathcal{S}_\infty(t) \subseteq \mathcal{S}(t). \tag{5.17}$$

In light of this definition and Proposition 12, the following fact can be concluded.

Proposition 13. Suppose Assumptions 1, 2 and 3 hold. The family of sets $\mathfrak{X}_\infty$, specified by (5.10) and (5.11), is the minimal robust positively invariant family of sets.

Example. (Minimality Property). Our illustrative example is borrowed from our recent paper Raković et al. (2016), which can be consulted for concrete numerical values and additional details. This part of the example illustrates the minimal robust family $\mathfrak{X}_\infty$ and its related invariance properties. The forward reach sets $\mathcal{R}(\mathcal{X}_\infty, t), \ t \in [0, 5T]$ are plotted in Figure 1. using different levels of gray–scale shading. As expected, the forward reach sets $\mathcal{R}(\mathcal{X}_\infty, t), \ t \in [0, 5T]$ exhibit periodic behavior and never leave the minimal robust positively invariant family $\mathfrak{X}_{S\infty}$.

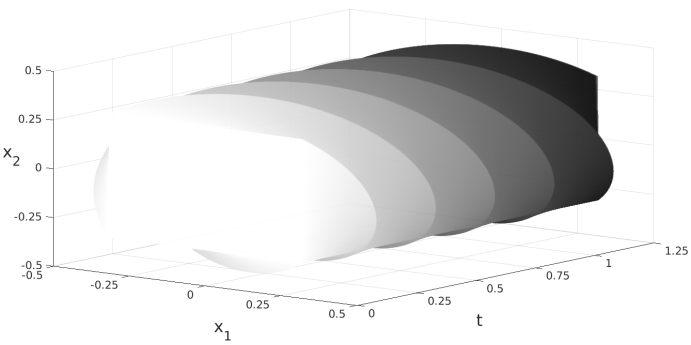

Fig. 1. Minimal Robust Positively Invariant Family $\mathfrak{X}_\infty$.

Under Assumptions 1 and 2, the family of sets $\mathfrak{X}_\infty$, specified by (5.10) and (5.11), is the minimal robust positively invariant family of sets for uncertain sampled–data linear dynamics, specified via (2.1)–(2.3), and constraint sets $(\mathbb{R}^n, \mathcal{W})$. In view of Proposition 12, Assumption 3 ensures existence of a robust positively invariant family of sets for uncertain sampled–data linear dynamics, specified via (2.1)–(2.3), and constraint sets $(\mathbb{X}, \mathcal{W})$. This, in turn, identifies nontrivial case, and it also yields Proposition 13.

5.2 *Maximality*

A central role in identifying maximal robust positively invariant sets is played by the backward reach sets $\mathcal{B}(\mathbb{X}, t)$ and, in fact, their limiting behaviour. The backward reach sets $\mathcal{B}(\mathbb{X}, t)$ enjoy stronger topological properties (relative to behaviour of arbitrary backward reach sets $\mathcal{B}(\mathcal{X}, t)$ discussed in Section 4) as summarized next. The explicit form of the backward reach sets $\mathcal{B}(\mathbb{X}, t)$ is, in view of (4.4) given, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$, by:

$$\mathcal{B}(\mathbb{X}, t_k + t) = \bigcap_{j\in\mathbb{Z}_{\geq 0}, \ j \leq k, \ \tau\in[0,t]} M^{-1}(t_j + \tau)\mathbb{X}(t_j + \tau) \tag{5.18}$$

Proposition 8 yields the following fact.

Lemma 1. Suppose Assumptions 1 and 2 hold. For all $k \in \mathbb{Z}_{\geq 0}$, it holds that

$$\mathcal{B}(\mathbb{X}, t_{k+1}) = \mathcal{B}(\mathcal{B}(\mathbb{X}, t_k), T) = \mathcal{B}(\mathcal{B}(\mathbb{X}, T), t_k). \tag{5.19}$$

By definition, the backward reach sets $\mathcal{B}(\mathbb{X}, t)$ are monotonically nonincreasing w.r.t. time.

Lemma 2. Suppose Assumptions 1 and 2 hold. For all $t_1 \in \mathbb{R}_{\geq 0}$ and $t_2 \in \mathbb{R}_{\geq 0}$, it holds that

$$t_1 \leq t_2 \ \Rightarrow \ \mathcal{B}(\mathbb{X}, t_2) \subseteq \mathcal{B}(\mathbb{X}, t_1). \tag{5.20}$$

Lemma 2 verifies that for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$, it holds that

$$\mathcal{B}(\mathbb{X}, t_{k+1}) \subseteq \mathcal{B}(\mathbb{X}, t_k + t) \subseteq \mathcal{B}(\mathbb{X}, t_k). \tag{5.21}$$

The above nested set inclusions are very helpful for the analysis of the limiting behaviour of the backward reach sets $\mathcal{B}(\mathbb{X}, t)$ as t goes to ∞. In fact, the above two Lemmata imply directly that studying limiting behaviour of the backward reach sets $\mathcal{B}(\mathbb{X}, t)$ as t goes to ∞ can be effectively achieved by examining limiting behaviour of the backward reach sets $\mathcal{B}(\mathbb{X}, t_k)$ at the sampling instances as k (and, hence, t_k) goes to ∞. Proposition 7 asserts that the backward reach sets $\mathcal{B}(\mathbb{X}, t_k), \ k \in \mathbb{Z}_{\geq 0}$ are compact and convex, but possibly empty. Lemma 2 asserts that the backward reach sets $\mathcal{B}(\mathbb{X}, t_k), \ k \in \mathbb{Z}_{\geq 0}$ are monotonically nonincreasing. Thus, the sequence of backward reach sets $\{\mathcal{B}(\mathbb{X}, t_k)\}_{k\in\mathbb{Z}_{\geq 0}}$ converges to a compact and convex set $\mathcal{X}^\infty$ that is possibly empty. Indeed, the limit of the sequence of backward reach sets $\{\mathcal{B}(\mathbb{X}, t_k)\}_{k\in\mathbb{Z}_{\geq 0}}$ is the set

$$\mathcal{X}^\infty = \bigcap_{k\geq 0} \mathcal{B}(\mathbb{X}, t_k). \tag{5.22}$$

As a matter of fact, the set $\mathcal{X}^\infty$ is the maximal set (w.r.t. set inclusion) that is a fixed point of the backward reach set map $\mathcal{B}(\cdot, T)$ at the sampling instances:

$$\mathcal{B}(\mathcal{X}, T) = \mathcal{X}, \text{ i.e. } \bigcap_{t\in[0,T]} M^{-1}(t)\mathcal{X}(t) = \mathcal{X}, \tag{5.23}$$

where, for all $t \in \mathbb{R}_{\geq 0}$, $\mathcal{X}(t) := \mathcal{X} \ominus \mathcal{R}(\{0\}, t)$.

Assumption 3 implies that the backward reach sets $\mathcal{B}(\mathbb{X},t)$ are nonempty for all $t \in \mathbb{R}_{\geq 0}$. In particular, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$, it holds that $\mathcal{X}_\infty(t_k+t) \subseteq \mathbb{X}$ so that, in turn, $\mathcal{X}_\infty(0) \subseteq \mathcal{B}(\mathbb{X}, t_k+t) \neq \emptyset$.

Proposition 14. Suppose Assumptions 1, 2 and 3 hold. Then, for all $t \geq 0$, the backward reach sets $\mathcal{B}(\mathbb{X},t)$, and hence the set $\mathcal{X}^\infty$ given by (5.22), are nonempty, convex and compact subsets of $\mathbb{X}$.

A particularly important case is the one of the finite determination of the set $\mathcal{X}^\infty$. Similarly to a well–understood discrete time case (Kolmanovsky and Gilbert, 1998; Raković and Fiacchini, 2008), the existence of a finite integer k^* such that $\mathcal{X}^\infty = \mathcal{B}(\mathbb{X}, t_{k^*}) = \mathcal{B}(\mathbb{X}, t_{k^*+1})$ can be guaranteed under a slightly stronger version of Assumption 3.

Proposition 15. Suppose Assumptions 1 and 2 hold and that the family $\mathfrak{X}_\infty$ of C–sets in $\mathbb{R}^n$, specified by (5.10) and (5.11), is such that

$$\forall t \in [0,T],\ \mathcal{X}_\infty(t) \subseteq \text{interior}(\mathbb{X}). \tag{5.24}$$

Then, in addition to assertions of Proposition 14, there exists a finite integer k^* such that

$$\mathcal{B}(\mathbb{X}, t_{k^*}) = \mathcal{B}(\mathbb{X}, t_{k^*+1}). \tag{5.25}$$

Moreover, the set $\mathcal{X}^\infty$, specified by (5.22), satisfies

$$\mathcal{X}^\infty = \mathcal{B}(\mathbb{X}, t_{k^*}). \tag{5.26}$$

To summarize the preceding analysis, let, for all $t \in [0,T]$,

$$\mathcal{X}^\infty(t) := A_S(t)\mathcal{X}^\infty \oplus E_S(t)\mathcal{W} \tag{5.27}$$

and define a collection of C–sets in $\mathbb{R}^n$

$$\mathfrak{X}^\infty(t) := \{\mathcal{X}^\infty(t) \ : \ t \in [0,T]\}. \tag{5.28}$$

In light of above discussion and Definition 1, the family $\mathfrak{X}^\infty$ of sets is robust positively invariant. Namely, the sets $\mathcal{X}^\infty(t)$ satisfy $\mathcal{X}^\infty(t) = A_S(t)\mathcal{X}^\infty \oplus E_S(t)\mathcal{W} \subseteq \mathbb{X}$ for all $t \in [0,T]$ and $\mathcal{X}^\infty(T) \subseteq \mathcal{X}^\infty(0) = \mathcal{X}^\infty$. More importantly, given any robust positively invariant family $\mathfrak{S} = \{\mathcal{S}(t) \ : \ t \in [0,T]\}$ of sets, we have, by construction, $\mathcal{S}(0) \subseteq \mathcal{B}(\mathcal{S}(0), T) \subseteq \mathcal{B}(\mathbb{X}, T)$ so that $\mathcal{S}(0) \subseteq \mathcal{X}^\infty(0)$. The latter relation, in turn, implies that, for all $t \in [0,T]$, $A_S(t)\mathcal{S}(0) \oplus E_S(t)\mathcal{W} \subseteq A_S(t)\mathcal{X}^\infty(0) \oplus E_S(t)\mathcal{W}$. Consequently, it holds that:

$$\forall t \in [0,T],\ \mathcal{S}(t) \subseteq \mathcal{X}^\infty(t). \tag{5.29}$$

The preceding facts justify a natural "pointwise–in–time–over–the–sampling–interval" notion of the maximal robust positively invariant families, which is compatible with Definition 1.

Definition 3. A robust positively invariant family

$$\mathfrak{S}^\infty := \{\mathcal{S}^\infty(t) \ : \ t \in [0,T]\}, \tag{5.30}$$

of sets is the maximal robust positively invariant family if and only if for any other robust positively invariant family $\mathfrak{S} = \{\mathcal{S}(t) \ : \ t \in [0,T]\}$ of sets it holds that

$$\forall t \in [0,T],\ \mathcal{S}(t) \subseteq \mathcal{S}^\infty(t). \tag{5.31}$$

In light of this definition and preceding discussion, the following fact can be verified.

Proposition 16. Suppose Assumptions 1, 2 and 3 hold. The family of sets $\mathfrak{X}^\infty$, specified by (5.27) and (5.28), is the maximal robust positively invariant family.

5.3 Attractivity

We close our analysis by summarizing attractivity properties of the minimal robust positively invariant family of sets $\mathfrak{X}_f$. To this end, consider the forward reach sets, specified, for all compact subsets $\mathcal{S}$ of the set $\mathcal{X}^\infty(0)$ and for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$,

$$\mathcal{X}(t_k+t) := \mathcal{R}(\mathcal{S}, t_k+t) \text{ with } \mathcal{X}(0) := \mathcal{S}. \tag{5.32}$$

Proposition 16 implies that, for all compact subsets $\mathcal{S}$ of the set $\mathcal{X}^\infty(0)$ and for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$,

$$\mathcal{X}(t_k+t) \subseteq \mathcal{X}^\infty(t) \subseteq \mathbb{X}, \tag{5.33}$$

so that the forward reach sets , and hence all related samled–data state trajectories, satisfy state constraint for all nonnegative times t.

By (Raković, 2007), Assumption 1, ensures that, for all compact subsets $\mathcal{S}$ of the set $\mathcal{X}^\infty(0)$ and for all $k \in \mathbb{Z}_{\geq 0}$, we have:

$$\mathrm{H}_\mathcal{L}(\mathcal{X}(t_k), \mathcal{X}_\infty(0)) \leq \lambda^k \, \mathrm{H}_\mathcal{L}(\mathcal{X}(t_0), \mathcal{X}_\infty(0)). \tag{5.34}$$

Above, $\mathcal{L}$ is symmetric proper C–set in $\mathbb{R}^n$ and scalar $\lambda \in [\rho(A_D), 1)$ is the minimal scalar such that $A_D\mathcal{L} \subseteq \lambda\mathcal{L}$. As shown in Raković (2007), the existence of such a set $\mathcal{L}$ and scalar λ is guaranteed under Assumption 1.

Let, for all $t \in [0,T]$,

$$\eta(t) := \min_\eta\{\eta \ : \ A_S(t)\mathcal{L} \subseteq \eta\mathcal{L},\ \eta \geq 0\}, \tag{5.35}$$

and note that $\eta(0) = 0$ and $\eta(T) = \lambda$, while for all $t \in (0,T)$, $0 \geq \eta(t) < \infty$. Relations (5.34) and (5.35) imply that, for all compact subsets $\mathcal{S}$ of the set $\mathcal{X}^\infty(0)$ and for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$, we have

$$\mathcal{X}(t_k+t) \subseteq \mathcal{X}_\infty(t) \oplus \lambda^k \, \mathrm{H}_\mathcal{L}(\mathcal{X}(t_0), \mathcal{X}_\infty(0))\eta(t)\mathcal{L}$$

as well as

$$\mathcal{X}_\infty(t) \subseteq \mathcal{X}(t_k+t) \oplus \lambda^k \, \mathrm{H}_\mathcal{L}(\mathcal{X}(t_0), \mathcal{X}_\infty(0))\eta(t)\mathcal{L}$$

so that, in fact,

$$\mathrm{H}_\mathcal{L}(\mathcal{X}(t_k+t), \mathcal{X}_\infty(t)) \leq \lambda^k \, \mathrm{H}_\mathcal{L}(\mathcal{X}(t_0), \mathcal{X}_\infty(0))\eta(t). \tag{5.36}$$

Thus, for all compact subsets $\mathcal{S}$ of the set $\mathcal{X}^\infty(0)$ and for $t \in \mathcal{T}_0$, $\mathrm{H}_\mathcal{L}(\mathcal{X}(t_k+t), \mathcal{X}_\infty(t))$ vanishes as k (and, hence, t_k) goes to ∞. In turn, it follows that the related forward reach sets $\mathcal{X}(t)$ upper–converge, exponentially fast and in a stable manner, w.r.t. Hausdorff distance to the family of sets $\mathfrak{X}_\infty$ as t goes to ∞. In particular, the set–to–family distance function

$$d(\mathcal{X}(t), \mathfrak{X}_f) := \min_\mathcal{Y}\{\mathrm{H}_\mathcal{L}(\mathcal{Y}, \mathcal{X}(t)) \ : \ \mathcal{Y} \in \mathfrak{X}_\infty\} \tag{5.37}$$

vanishes as t goes to ∞ for all compact subsets $\mathcal{S}$ of $\mathcal{X}^\infty(0)$. Indeed, letting $\bar{\eta} := \max_t\{\eta(t) \ : \ t \in [0,T]\}$, we have, for all compact subsets $\mathcal{S}$ of $\mathcal{X}^\infty(0)$ and for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$,

$$\begin{aligned} d(\mathcal{X}(t_k+t), \mathfrak{X}_f) &\leq \lambda^k \, \mathrm{H}_\mathcal{L}(\mathcal{S}, \mathcal{X}_\infty(0))\eta(t) \\ &\leq \lambda^k \, \mathrm{H}_\mathcal{L}(\mathcal{S}, \mathcal{X}_\infty(0))\bar{\eta}. \end{aligned} \tag{5.38}$$

Now, by construction, $\lambda \in [\rho(A_D), 1)$, by compactness, $0 \leq \mathrm{H}_\mathcal{L}(\mathcal{S}, \mathcal{X}_\infty(0)) < \infty$ and, by definition, $0 \leq \bar{\eta} < \infty$. Thus, the set–to–family distance function $d(\mathcal{X}(t), \mathfrak{X}_f)$ vanishes as t goes to ∞ for all compact subsets $\mathcal{S}$ of $\mathcal{X}^\infty(0)$.

Keeping in mind that the minimal robust positively invariant family $\mathfrak{X}_\infty$ is weakly positively invariant w.r.t. the forward reach set map $\mathcal{R}(\cdot,\cdot)$, the above discussion leads to our concluding summary of attractivity properties.

Proposition 17. Suppose Assumptions 1, 2 and 3 hold. The family $\mathfrak{X}_\infty$ of C–sets in $\mathbb{R}^n$, specified by (5.10) and (5.11), is an exponentially stable weak upper–attractor for set–dynamics whose trajectories are induced by the forward reach set map $\mathcal{R}(\cdot,\cdot)$ of (3.1) and (3.2) with the basin of attraction being the family of nonempty compact subsets of the set $\mathcal{X}^\infty(0)$ specified by (5.27).

Example. (Attractivity Property). This part of the example illustrates the above discussed attractivity properties. The forward reach sets $\mathcal{R}(\{0\}, t),\ t \in [0, 5T]$ are plotted

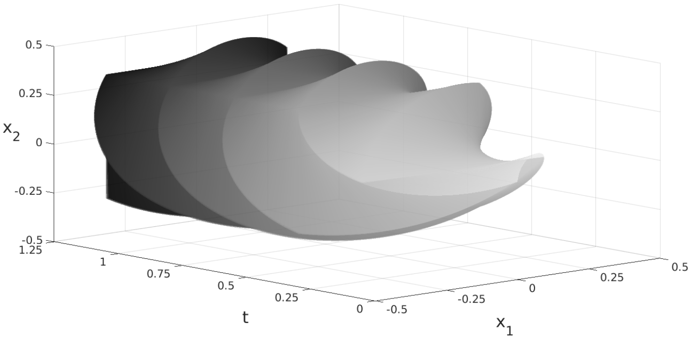

Fig. 2. Forward Reach Sets $\mathcal{R}(\{0\}, t)$.

in Figure 2. using different levels of gray–scale shading (the darker color indicates the larger time t). In this example, the convergence occurs in 2 sampling periods since $A_D^2 = 0$. As expected in view of our analysis and evident by inspection of the figure, the forward reach sets $\mathcal{R}(\{0\}, t),\ t \in [2T, 5T]$ exhibit periodic limiting behavior.

Remark 6. We close by noting that, under monotonicity of the reach set map $\mathcal{R}(\cdot,\cdot)$ in the second argument, i.e., for all subsets $\mathcal{S}$ in $\mathbb{R}^n$, all $\tau_1 \in \mathbb{R}_{\geq 0}$ and all $\tau_2 \in \mathbb{R}_{\geq 0}$,

$$\tau_1 \leq \tau_2 \Rightarrow \mathcal{R}(\mathcal{S}, \tau_1) \subseteq \mathcal{R}(\mathcal{S}, \tau_2), \tag{5.39}$$

the sets $\mathcal{X}_\infty(t),\ t \in [0, T]$ are all equal to the set $\mathcal{X}_\infty$ of (5.6) so that the family of sets $\mathfrak{X}_\infty$ can be reduced to a singleton set $\mathcal{X}_\infty$ that also becomes a strong attractor (instead of a weak upper–attractor) for the related set–dynamics induced by the forward reach set map $\mathcal{R}(\cdot,\cdot)$.

6. CLOSING REMARKS

We studied properties of sampled–data forward and backward reach sets and generalized robust positively invariant sets for constrained sampled–data systems. We also proposed adequate notions for maximality and minimality of related robust positively invariant sets.

This note spans several research directions, one of which is the study of approximate, computationally simpler "uniform–over–sampling–interval" notions of ordinary, minimal and maximal robust positive invariance. Another line of research would be study, and understand, degree of for potential utility of well–understood discrete time notions within the sampled–data setting. Finally, the proposed notions can be, with relative ease, conceptually extended to general sampled–data systems, which, in turns, rises computational applicability of such an extension.

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