EQUILIBRIUM STATES FOR IMPULSIVE SEMIFLOWS

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ABSTRACT. We consider impulsive semiflows defined on compact metric spaces and give sufficient conditions, both on the semiflows and the potential functions, for the existence and uniqueness of equilibrium states. We also generalize the classical notion of topological pressure to our setting of discontinuous semiflows and prove a variational principle.

CONTENTS

1. INTRODUCTION

Impulsive dynamical systems may be interpreted as suitable mathematical models of real world phenomena that display abrupt changes in their behavior, and are described

by three objects: a continuous semiflow on a metric space X; a set $D \subset X$ where the flow experiments sudden perturbations; and an impulsive function $I : D \to X$ which determines the change on a trajectory each time it collides with the impulsive set D. See for instance reference [20], where one may find several examples of evolutive processes which are analyzed through differential equations with impulses.

Dynamical systems with impulse effects seem to be the most adequate mathematical models to describe real world phenomena that exhibit sudden changes in their states. For example, the theoretical characterizations of wormholes [27], also called Einstein-Rosen bridges, seem to fit the description of the traverse effects an impulsive function I acting on a set D induces on a semiflow, thereby possibly creating odd shortcuts in space-time [28]. While at present it appears unlikely that nature allows us to observe a wormhole, these hypothetical entities, with unusual and inherently unstable topological, geometrical and physical properties, show up as valid solutions of the Einstein field equations for the gravity. We also refer the reader to the reference [20], where other examples of nature evolution processes are analyzed within the new branch of differential equations with impulses; in addition, see [3, 8, 13, 14, 16, 18, 21, 22, 26, 29].

For many years the achievements on the theory of impulsive dynamical systems concerned the behavior of trajectories, their limit sets and their stability; see e.g. [5, 6, 10, 11, 19] and references therein. The first results on the ergodic theory of impulsive dynamical systems were established in [1], where sufficient conditions for the existence of invariant probability measures on the Borel sets were given. Afterwards, it was natural to look for some special classes of invariant measures. So far, a useful approach has been to use potential functions and finding equilibrium states. However, as the classical notion of topological entropy requires continuity and impulsive semiflows exhibit discontinuities, it became necessary to introduce a generalized concept of topological entropy, and this has been done in [2]. Moreover, it was proved that the new notion coincides with the classical one for continuous semiflows, and also a partial variational principle for impulsive semiflows: the topological entropy coincides with the supremum of the metric entropies of time-one maps.

Our aim in this paper was to extend the results of [2] in two directions. Firstly we establish a variational principle for a wide class of potential functions; then we present sufficient conditions for the existence and uniqueness of equilibrium states for those potential functions. Once more, due to the discontinuities of the impulsive semiflows, we needed to define a generalized concept of topological pressure; and again we show that this new definition coincides with the classical one for continuous semiflows.

1.1. **Impulsive semiflows.** Consider a compact metric space (X, d), a continuous semiflow $\varphi : \mathbb{R}_0^+ \times X \to X$, a nonempty compact set $D \subset X$ and a continuous map $I : D \to X$ such that $I(D) \cap D = \emptyset$. Under these conditions we say that (X, φ, D, I) is an *impulsive dynamical system*. The first visit of each φ-trajectory to D will be registered by the function $\tau_1 : X \to [0, +\infty]$, defined as

$$\tau_1(x) = \begin{cases} \inf\left\{t > 0 \colon \varphi_t(x) \in D\right\}, & \text{if } \varphi_t(x) \in D \text{ for some } t > 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

The *impulsive trajectory* γ_x and the subsequent *impulsive times* $\tau_2(x), \tau_3(x), \ldots$ (possibly finitely many) of a given point $x \in X$ are defined according to the following rules: for $0 \le t < \tau_1(x)$ we set $\gamma_x(t) = \varphi_t(x)$. Assuming that $\gamma_x(t)$ is defined for $t < \tau_n(x)$ for some $n \ge 1$, we set

$$\gamma_x(\tau_n(x)) = I(\varphi_{\tau_n(x)-\tau_{n-1}(x)}(\gamma_x(\tau_{n-1}(x)))).$$

Defining the $(n+1)^{\text{th}}$ impulsive time of x as

$$\tau_{n+1}(x) = \tau_n(x) + \tau_1(\gamma_x(\tau_n(x))),$$

for $\tau_n(x) < t < \tau_{n+1}(x)$, we set

$$\gamma_x(t) = \varphi_{t-\tau_n(x)}(\gamma_x(\tau_n(x))).$$

We define the *time duration* of the trajectory of x as $\Upsilon(x) = \sup_{n \ge 1} \{\tau_n(x)\}$. Since we are assuming $I(D) \cap (D) = \emptyset$, it follows from [1, Remark 1.1] that we have $\Upsilon(x) = \infty$ for all $x \in X$. Thus we have the impulsive trajectories defined for all positive times. This allows us to introduce the *impulsive semiflow* ψ of an impulsive dynamical system (X, φ, D, I) as

$$\begin{array}{rccc} \psi: & \mathbb{R}_0^+ \times X & \longrightarrow & X \\ & (t, x) & \longmapsto & \gamma_x(t), \end{array}$$

where γ_x stands for the impulsive trajectory of x determined by (X, φ, D, I). It was proved in [4, Proposition 2.1] that ψ is indeed a semiflow, though not necessarily continuous. In what follows, for each $t \ge 0$ we will denote by ψ_t the time-t map of the semiflow ψ.

Remark 1.1. It is known that the function τ_1 is lower semicontinuous on the set $X \setminus D$; see [9, Theorem 2.7]. Since we are assuming that $I(D) \cap (D) = \emptyset$ and $I(D)$ is compact, then there exists some $\eta > 0$ such that for all $x \in X$ and all $n \in \mathbb{N}$ we have $\tau_{n+1}(x) - \tau_n(x) \ge \eta$.

Now we state some conditions about the continuous semiflow φ on the sets D and $I(D)$ which will be useful for the statements of our main results. We define for $t > 0$

$$D_t = \bigcup_{x \in D} \{\varphi_s(x) \colon 0 < s < t\}. \tag{1.1}$$

Given $\xi > 0$, we say that φ is ξ-*regular on* D if
  (1) D_t is an open set for all $0 < t \le \xi$;
  (2) if $x \in X \setminus D_\xi$ and $\varphi_t(x) \in D_\xi$ for some $t > 0$, then $\varphi_s(x) \in D$ for some $0 \le s < t$.
We say that φ satisfies a ξ-*half-tube condition* on a compact set $A \subset X$ if
  (1) $\varphi_t(x) \in A \Rightarrow \varphi_{t+s}(x) \notin A$ for all $0 < s < \xi$;
  (2) $\{\varphi_t(x_1) \colon 0 < t \le \xi\} \cap \{\varphi_t(x_2) \colon 0 < t \le \xi\} = \emptyset$ for all $x_1, x_2 \in A$ with $x_1 \ne x_2$;
  (3) there exists $C > 0$ such that, for all $x_1, x_2 \in A$ with $x_1 \ne x_2$, we have

$$0 \le t < s \le \xi \quad \Rightarrow \quad d(\varphi_t(x_1), \varphi_t(x_2)) \le C\, d(\varphi_s(x_1), \varphi_s(x_2)).$$

In our main results we will assume that φ satisfies a ξ-half-tube condition on the compact sets D and $I(D)$. In particular, the first condition in the definition of ξ-half-tube for $A = D$ implies that $\tau_1(x) \ge \xi > 0$ for all $x \in D$. Given $\xi > 0$ we define

$$X_\xi = X \setminus (D_\xi \cup D).$$

Since D is compact, I is continuous and $I(D) \cap D = \emptyset$, we may choose ξ small enough so that $I(D) \cap D_\xi = \emptyset$. Therefore, the set X_ξ is forward invariant under ψ, that is

$$\psi_t(X_\xi) \subseteq X_\xi, \quad \forall t \geq 0. \tag{1.2}$$

For future use, we introduce the function

$$\tau_\xi^* : X_\xi \cup D \to [0, +\infty]$$

defined as

$$\tau_\xi^*(x) = \begin{cases} \tau_1(x), & \text{if } x \in X_\xi; \\ 0, & \text{if } x \in D. \end{cases}$$

For latter reference we gather as (C1)-(C5) all the properties that we need about impulsive dynamical systems in the list below. We assume that there exists $\xi_0 > 0$ such that for all $0 < \xi < \xi_0$ we have:

(C1) $I : D \to X$ is Lipschitz with $\text{Lip}(I) \leq 1$ and $I(D) \cap D = \emptyset$;
(C2) $I(\Omega_\psi \cap D) \subset \Omega_\psi \setminus D$, where Ω_ψ denotes the set of non-wandering points of ψ;
(C3) φ is ξ-regular on D;
(C4) φ satisfies a ξ-half-tube condition on both D and $I(D)$;
(C5) τ_ξ^* is continuous.

Notice that conditions (C3)-(C4) hold, for instance, when φ is a C^1 semiflow on a manifold for which D and $I(D)$ are submanifolds transversal to the flow direction. Moreover, condition (C2) ensures that $\Omega_\psi \setminus D$ is invariant by ψ (cf. [1, Theorem B]) and conditions (C2) and (C5) are essential to guarantee that $\mathcal{M}_\psi(X) \neq \emptyset$ (cf. [1, Theorem A]).

1.2. **Expansiveness.** Here we recall the classical definition of expansiveness for a continuous semiflow and introduce an adapted version for an impulsive semiflow.

Continuous semiflow. Let φ be a continuous semiflow on a metric space (X, d). We say that φ is *positively expansive* on X if for every $\delta > 0$ there exists $\varepsilon > 0$ such that, if $x, y \in X$ and a continuous map $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $s(0) = 0$ satisfy $d(\varphi_t(x), \varphi_{s(t)}(y)) < \varepsilon$ for all $t \geq 0$, then $y = \varphi_t(x)$ for some $0 < t < \delta$.

Impulsive semiflow. Let ψ be the semiflow of an impulsive dynamical system (X, φ, D, I). Given $\varepsilon > 0$, consider $B_\varepsilon(D)$ the ε-neighborhood of D in X. We say that ψ is *positively expansive* on X if for every $\delta > 0$ there exists $\varepsilon > 0$ such that, if $x, y \in X$ and a continuous map $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $s(0) = 0$ satisfy $d(\psi_t(x), \psi_{s(t)}(y)) < \varepsilon$ for all $t \geq 0$ such that $\psi_t(x), \psi_{s(t)}(y) \notin B_\varepsilon(D)$, then $y = \psi_t(x)$ for some $0 < t < \delta$.

1.3. **Specification.** Let ψ be a semiflow on a metric space (X, d). We say that ψ has the *specification* property on X if for all $\varepsilon > 0$ there exist $L > 0$ such that, for any sequence $x_0, \ldots, x_n$ of points in X and any sequence $0 \leq t_0 < \cdots < t_{n+1}$ such that $t_{i+1} - t_i \geq L$ for all $0 \leq i \leq n$, there are $y \in X$ and $r : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ constant on each interval $[t_i, t_{i+1}[$, whose values depend only on $x_0, \ldots, x_n$, that also satisfy

$$r([t_0, t_1[) = 0 \quad \text{and} \quad |r([t_{i+1}, t_{i+2}[) - r([t_i, t_{i+1}[)| < \varepsilon,$$

for which

$$d\left(\psi_{t+r(t)}(y), \psi_{t-t_i}(x_i)\right) < \varepsilon, \quad \forall t \in [t_i, t_{i+1} - L[\quad \forall\, 0 \leq i \leq n.$$

The specification is said to be *periodic* if we can always choose y periodic with the minimum period in $[t_{n+1} - t_0 - n\varepsilon, t_{n+1} - t_0 + n\varepsilon]$.

1.4. Equilibrium states.

Let ψ be a semiflow on a compact metric space (X, d). In what follows we will denote by $\mathcal{M}_\psi^t(X)$ the set of probability measures defined on the σ-algebra of the Borel subsets of X which are invariant under ψ_t, and set

$$\mathcal{M}_\psi(X) = \bigcap_{t \geq 0} \mathcal{M}_\psi^t(X).$$

An *equilibrium state* for a continuous potential function $f : X \to \mathbb{R}$ is a probability measure $\mu_f \in \mathcal{M}_\psi(X)$ which maximizes the map

$$\mu \in \mathcal{M}_\psi(X) \quad \longmapsto \quad h_\mu(\psi_1) + \int f \, d\mu,$$

where $h_\mu(\psi_1)$ stands for the *metric entropy* of the time-one map of the semiflow ψ with respect to the measure μ. One of the main issues concerning equilibrium states is to determine an appropriate space of potential functions. We consider again the cases of continuous and impulsive semiflows separately.

Continuous semiflow. Given a continuous semiflow φ on X, denote by $V(\varphi)$ the space of continuous maps $f : X \to \mathbb{R}$ for which there are $K > 0$ and $\varepsilon > 0$ such that for every $t > 0$ we have

$$\left| \int_0^t f(\varphi_s(x)) \, ds - \int_0^t f(\varphi_s(y)) \, ds \right| < K, \tag{1.3}$$

whenever

$$d\left(\varphi_s(x), \varphi_s(y)\right) < \varepsilon, \quad \forall s \in [0, t].$$

It was proved in [17] that each $f \in V(\varphi)$ has a unique equilibrium state if φ is continuous and satisfies positive expansiveness and periodic specification. The same conclusion was obtained in [12] under slightly different assumptions.

Remark 1.2. Actually, the results in [17] are proved for flows, but it is straightforward to check that, after modifying the notion of expansiveness as done in Subsection 1.2, the argument still holds for semiflows.

Impulsive semiflow. Consider now ψ as the semiflow of an impulsive dynamical system (X, φ, D, I). In this context we need to restrict the set of potential functions for which we are going to find an equilibrium state, introducing a slightly more demanding version of the space of potential functions. We define $V^*(\psi)$ as the set of continuous maps $f : X \to \mathbb{R}$ for which

(1) $f(x) = f(I(x))$ for all $x \in D$;

(2) there are $K > 0$ and $\varepsilon > 0$ such that for every $t > 0$ we have

$$\left| \int_0^t f(\psi_s(x)) \, ds - \int_0^t f(\psi_s(y)) \, ds \right| < K, \tag{1.4}$$

whenever $d(\psi_s(x), \psi_s(y)) < \varepsilon$ for all $s \in [0, t]$ such that $\psi_s(x), \psi_s(y) \notin B_\varepsilon(D)$.

For instance, constant potential functions belong to $V^*(\psi)$. The aim of our first result is to extend Franco's Theorem [17] on the existence and uniqueness of equilibrium states for potential functions in $V^*(\psi)$. As in [17], to ensure uniqueness we need to assume some finite dimensionality condition on the metric space X; see e.g. [15, Chapter 3]. Here we also need a uniform control on the number of preimages under the impulsive function I.

Theorem A. *Let X be a compact metric space and ψ the semiflow of an impulsive dynamical system (X, φ, D, I) for which (C1)-(C5) hold. If ψ is positively expansive and has the periodic specification property in $\Omega_\psi \setminus D$, then any potential function $f \in V^*(\psi)$ has an equilibrium state. Moreover, if $\dim(X) < \infty$ and there is $k > 0$ such that $\#I^{-1}(\{y\}) \leq k$ for every $y \in I(D)$, then the equilibrium state is unique.*

In particular, taking f the null function, we deduce that the impulsive semiflow ψ has a probability measure of maximum entropy, which in some cases is unique.

1.5. **Topological pressure.** Here we briefly recall the classical definition of topological pressure for continuous semiflows (see [17] for details) and generalize this concept to impulsive semiflows.

Classical definition. Let (X, d) be a compact metric space and $\varphi : X \times \mathbb{R}_0^+ \to X$ be a continuous semiflow. Given $\varepsilon > 0$ and $t \in \mathbb{R}^+$, a subset E of X is said to be $(\varphi, \varepsilon, t)$-*separated* if for any $x, y \in X$ with $x \neq y$ there is some $s \in [0, t]$ such that $d(\varphi_s(x), \varphi_s(y)) > \varepsilon$. Given $f : X \to \mathbb{R}$ a continuous potential function, define

$$Z(\varphi, f, \varepsilon, t) = \sup \left\{ \sum_{x \in E} e^{\int_0^t f(\varphi_s(x)) \, ds} : E \text{ is } (\varphi, \varepsilon, t)\text{-separated} \right\},$$

$$P(\varphi, f, \varepsilon) = \limsup_{t \to +\infty} \frac{1}{t} \log Z(\varphi, f, \varepsilon, t).$$

The *topological pressure* of f with respect to φ is defined as

$$P(\varphi, f) = \lim_{\varepsilon \to 0} P(\varphi, f, \varepsilon).$$

New definition. Let (X, d) be a compact metric space and $\psi : \mathbb{R}_0^+ \times X \to X$ a semiflow (possibly not continuous). Consider a function T assigning to each $x \in X$ a sequence $(T_n(x))_{n \in A(x)}$ of nonnegative numbers, where either $A(x) = \mathbb{N}$ or $A(x) = \{1, \ldots, \ell\}$ for some $\ell \in \mathbb{N}$. We say that T is *admissible* if there exists $\eta > 0$ such that for all $x \in X$ and all $n \in \mathbb{N}$ with $n + 1 \in A(x)$ we have

(1) $T_{n+1}(x) - T_n(x) \geq \eta$;

$$(2) \ T_n(\psi_t(x)) = \begin{cases} T_n(x) - t, & \text{if } T_{n-1}(x) < t < T_n(x); \\ T_{n+1}(x), & \text{if } t = T_n(x). \end{cases}$$

For each admissible function T, $x \in X$, $t > 0$ and $0 < \delta < \eta/2$, we define

$$J_{t,\delta}^T(x) = [0,t] \setminus \bigcup_{j \in A(x)}]T_j(x) - \delta, T_j(x) + \delta[.$$

Observe that $J_{t,\delta}^T(x) = [0,t]$, whenever $T_1(x) > t$. Given $\varepsilon > 0$ and $t \in \mathbb{R}^+$, we say that $E \subset X$ is $(\varphi, \delta, \varepsilon, t)$-*separated* if for any $x, y \in X$ with $x \neq y$ there is some $s \in J_{t,\delta}^T(x)$ such that $d(\varphi_s(x), \varphi_s(y)) > \varepsilon$. Given a continuous potential function $f : X \to \mathbb{R}$, we define

$$Z^T(\varphi, f, \delta, \varepsilon, t) = \sup \left\{ \sum_{x \in E} \exp \left(\int_0^t f(\varphi_s(x)) \, ds \right) : E \text{ is finite and } (\varphi, \delta, \varepsilon, t)\text{-separated} \right\},$$

$$P^T(\varphi, f, \delta, \varepsilon) = \limsup_{t \to +\infty} \frac{1}{t} \log Z_t^T(\varphi, f, \delta, \varepsilon),$$

$$P^T(\varphi, f, \delta) = \lim_{\varepsilon \to 0} P^T(\varphi, f, \delta, \varepsilon).$$

Finally, the T-*topological pressure* of f with respect to φ is defined as

$$P^T(\varphi, f) = \lim_{\delta \to 0} P^T(\varphi, f, \delta).$$

Notice that, as in the classical case, the T-*topological pressure* is well defined, because

(1) if $0 < \varepsilon_1 < \varepsilon_2$, then $Z^T(\varphi, f, \varepsilon_1, \delta, t) \geq Z^T(\psi, f, \varepsilon_2, \delta, t)$;

(2) if $0 < \varepsilon_1 < \varepsilon_2$, then $P^T(\psi, f, \varepsilon_1, \delta) \geq P^T(\psi, f, \varepsilon_2, \delta)$;

(3) if $0 < \delta_1 < \delta_2$, then $P^T(\psi, f, \delta_1) \geq P^T(\psi, f, \delta_2)$.

The next result shows that for continuous semiflows the classical and new notions of topological pressure coincide.

Theorem B. *Let X be a compact metric space, φ a continuous semiflow on X and T an admissible function. If $f : X \to \mathbb{R}$ is a continuous potential function, then $P^T(\varphi, f) = P(\varphi, f)$.*

The previous result motivates our definition of topological pressure for an impulsive semiflow. First of all observe that given ψ the impulsive semiflow of an impulsive dynamical system (X, φ, D, I), the function τ assigning to each point in X its sequence of impulsive times is admissible (recall Remark 1.1). Therefore, we may define the *topological pressure* of a potential function $f : X \to \mathbb{R}$ with respect to an impulsive semiflow ψ as $P^\tau(\psi, f)$. In the sequel we establish a *variational principle* which generalizes [2, Theorem C]. Actually, for the particular choice of $f = 0$ the next result gives [2, Theorem C].

Theorem C. *Let X be a compact metric space and ψ the semiflow of an impulsive dynamical system (X, φ, D, I) for which (C1)-(C5) hold. Then for any potential function $f \in V^*(\psi)$ we have*

$$P^\tau(\psi, f) = \sup_{\mu \in \mathcal{M}_\psi(X)} \left\{ h_\mu(\psi_1) + \int f \, d\mu \right\},$$

where ψ_1 stands for the time one map of the semiflow ψ.

Under both expansivity and specification assumptions, it follows from Theorem A that the supremum in this last result is attained. As $\Omega_{\psi_1} \subseteq \Omega_\psi$ and for any probability measure $\mu \in \mathcal{M}_\psi(X)$ we have $\mu(D) = 0$ (see [1, Lemma 4.7]), then

$$h_\mu(\psi_1) + \int f \, d\mu = h_\mu(\psi_1|_{\Omega_\psi \setminus D}) + \int_{\Omega_\psi \setminus D} f \, d\mu,$$

for any $\mu \in \mathcal{M}_\psi(X)$, and so it follows from Theorem C that

$$P^\tau(\psi, f) = P^\tau(\psi|_{\Omega_\psi \setminus D}, f|_{\Omega_\psi \setminus D}).$$

2. Classical and new pressure

Here we prove that the modified definition of topological pressure coincides with the classical one for continuous semiflows and continuous potential functions defined on compact metric spaces, thus proving Theorem B.

Let (X, d) be a compact metric space, φ a continuous semiflow on X and $f : X \to \mathbb{R}$ a continuous potential function. Given T admissible, consider $\eta > 0$ as in the definition of an admissible function and fix constants $0 < \delta < \eta/2$, $\varepsilon > 0$ and $t > 0$. Notice that for every $x \in X$ we have

$$Z^T(\varphi, f, \delta, \varepsilon, t) \leq Z(\varphi, f, \varepsilon, t),$$

and so

$$P^T(\varphi, f) \leq P(\varphi, f).$$

We will now prove the reverse inequality. We start by stating a useful lemma whose proof can be found in [2, Lemma 2.1].

Lemma 2.1. *Let φ be a continuous semiflow on a compact metric space X. For each $\varepsilon > 0$ there is $\alpha > 0$ such that $d(\varphi_s(x), \varphi_u(x)) < \varepsilon$ for all $x \in X$ and $s, u \geq 0$ with $|s - u| < \alpha$.*

Consider an arbitrary $\varepsilon > 0$. By Lemma 2.1 there exists $\alpha > 0$ such that for all $z \in X$ and all $s, u > 0$ with $|s - u| < \alpha$, we have

$$d(\varphi_s(z), \varphi_u(z)) < \varepsilon/4. \tag{2.1}$$

Hence, if $x, y \in X$ and $s \geq 0$ satisfy

$$d(\varphi_s(x), \varphi_s(y)) > \varepsilon, \tag{2.2}$$

then, for every $u \in (s - \alpha, s + \alpha)$, we get

$$d(\varphi_s(x), \varphi_s(y)) \leq d(\varphi_s(x), \varphi_u(x)) + d(\varphi_u(x), \varphi_u(y)) + d(\varphi_u(y), \varphi_s(y))$$

which, together with (2.1) and (2.2), implies

$$d(\varphi_u(x), \varphi_u(y)) > \varepsilon/2. \tag{2.3}$$

Consider now $E \subseteq X$ being $(\varphi, t, \varepsilon)$-separated. As φ is continuous, the set E is finite. By definition, for every $x, y \in E$, $x \neq y$, there exists $s \in [0, t]$ such that

$$d(\varphi_s(x), \varphi_s(y)) \geq \varepsilon.$$

Choose $0 < \delta < \min\{\eta, \alpha/2, \varepsilon\}$ and $0 < \varepsilon' < \varepsilon/2$. By (2.3), if $u \in (s - 2\delta, s + 2\delta)$, then

$$d(\varphi_u(x), \varphi_u(y)) > \varepsilon/2 > \varepsilon'.$$

If $s \in J_{t,\delta}^T(x)$ for some $t > 0$, then $y \notin B^T(x, \varphi, \delta, \varepsilon', t)$, where

$$B^T(x, \varphi, \delta, \varepsilon', t) = \left\{ z \in X : d(\psi_s(x), \psi_s(z)) < \varepsilon', \ \forall s \in J_{t,\delta}^T(x) \right\}.$$

Otherwise, $J_{t,\delta}^T(x) \cap (s - 2\delta, s + 2\delta) \neq \emptyset$, and then $y \notin B^T(x, \varphi, \delta, \varepsilon', t)$. So, E is $(\varphi, T, \delta, \varepsilon', t)$-separated. Hence

$$Z(\varphi, \varepsilon, t) \leq Z^T(\varphi, \delta, \varepsilon', t),$$

and so

$$\frac{1}{t} \log Z(\varphi, f, \varepsilon, t) \leq \frac{1}{t} \log Z^T(\varphi, f, \delta, \varepsilon', t).$$

Taking the upper limit as $t \to +\infty$, we get

$$P(\varphi, f, \varepsilon) \leq P^T(\varphi, f, \delta, \varepsilon').$$

Now, taking $\varepsilon' \to 0$ we obtain

$$P(\varphi, f, \varepsilon) \leq P^T(\varphi, f, \delta).$$

Noticing that $\delta \to 0$ when $\varepsilon \to 0$, we have

$$P(\varphi, f) \leq P^T(\varphi, f).$$

This finishes the proof of Theorem B.

3. Refinements and semiconjugacies

Given T and T' two admissible functions in X, we say that T' *refines* T, and write $T' \succ T$, if for all $x \in X$ and $n \in \mathbb{N}$ there exists $m = m(n, x) \in \mathbb{N}$ such that $T_n(x) = T'_m(x)$. Our new concept of topological pressure is monotone with respect to the refinement of admissible functions and invariant by semiflow equivalences that respect the fixed admissible functions. The proofs of these properties are given in the two lemmas below and differ only in minor details from analogous results in [2, Section 2.2].

Lemma 3.1. *Let ψ be a semiflow on X. If T and T' are admissible functions such that $T' \succ T$, then $P^T(\psi, f) \geq P^{T'}(\psi, f)$ for any continuous potential function $f : X \to \mathbb{R}$.*

Proof. Given $\varepsilon, t > 0$, $0 < \delta < \eta/2$ and a finite $(\psi, T', \delta, \varepsilon, t)$-separated subset E, as $T' \succ T$, the set E is a $(\psi, T, \delta, \varepsilon, t)$-separated as well. Therefore

$$Z^{T'}(\psi, \delta, \varepsilon, t) \leq Z^T(\psi, \delta, \varepsilon, t),$$

which clearly yields our conclusion. $\qquad \square$

Given two semiflows $\psi : \mathbb{R}_0^+ \times X \to X$ and $\psi' : \mathbb{R}_0^+ \times X' \to X'$, acting on the metric spaces (X, d) and (X', d'), and two admissible functions T and T' defined on X and X', respectively, we say that a uniformly continuous surjective map $h : X \to X'$ is a (T, T')-*semiconjugacy* between ψ and ψ' if

(1) $\psi'_t \circ h = h \circ \psi_t$ for all $t \geq 0$;

(2) $T'(h(x)) = T(x)$ for all $x \in X$.

Lemma 3.2. *Let $h : X \to X'$ be a (T, T')-semiconjugacy between the semiflows ψ and ψ' on X and X', respectively, such that the pre-image of each point under h is finite. Then $P^T(\psi, f \circ h) \geq P^{T'}(\psi', f)$ for any continuous potential function $f : X' \to \mathbb{R}$.*

Proof. Let $\psi : \mathbb{R}_0^+ \times X \to X$ and $\psi' : \mathbb{R}_0^+ \times X' \to X'$ be two semiconjugate semiflows and h be such a semiconjugacy. As h is uniformly continuous, given $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that for $a, b \in X$

$$d(a, b) < \varepsilon' \quad \Rightarrow \quad d'(h(a), h(b)) < \varepsilon.$$

Fix $t > 0$ and $0 < \delta < \eta/2$, and consider a finite $(\psi', T', t, \varepsilon, \delta)$-separated set $B \subseteq X'$. Then $A = h^{-1}(B)$ is finite, although it may have a cardinal bigger or equal than the one of B. Moreover, A is a $(\psi, T, t, \varepsilon', \delta)$-separated set of X. Indeed, for all $a, b \in A$, there are $t_n \in J_{t,\delta}^{T'}(h(a))$ and $s_n \in J_{t,\delta}^{T'}(h(b))$ such that

$$d'(\psi'_{t_n}(h(a)), \psi'_{t_n}(h(b))) \geq \varepsilon \quad \text{and} \quad d'(\psi'_{s_n}(h(a)), \psi'_{s_n}(h(b))) \geq \varepsilon$$

that is,

$$d'(h \circ \psi_{t_n}(a), h \circ \psi_{t_n}(b)) \geq \varepsilon \quad \text{and} \quad d'(h \circ \psi_{s_n}(a), h \circ \psi_{s_n}(b)) \geq \varepsilon.$$

Therefore,

$$d(\psi_{t_n}(a), \psi_{t_n}(b)) \geq \varepsilon' \quad \text{and} \quad d(\psi_{s_n}(a), \psi_{s_n}(b)) \geq \varepsilon'.$$

Taking into account that, by definition of semiconjugacy, $t_n \in J_{t,\delta}^T(a)$ and $s_n \in J_{t,\delta}^T(b)$, we deduce that

$$Z^T(\psi, f \circ h, \delta, \varepsilon', t) \geq Z^{T'}(\psi', f, \delta, \varepsilon, t).$$

Noticing that $\varepsilon' \to 0$ when $\varepsilon \to 0$, we finally conclude our proof. $\qquad \square$

4. Quotient dynamics

Given an impulsive dynamical system (X, φ, D, I), consider the equivalence relation $\sim$ on X given by

$$x \sim y \quad \Leftrightarrow \quad x = y, \quad y = I(x), \quad x = I(y) \quad \text{or} \quad I(x) = I(y). \tag{4.1}$$

Let $\widetilde{X}$ denote the quotient space, $\tilde{x}$ the equivalence class of a point $x \in X$ and $\pi : X \to \widetilde{X}$ the natural projection. It follows from [1, Lemma 4.1] that $\widetilde{X}$ is a metrizable space. Actually, if d denotes the metric on X, a metric $\tilde{d}$ on $\widetilde{X}$ that induces the quotient topology is given by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \inf \{d(p_1, q_1) + d(p_2, q_2) + \cdots + d(p_n, q_n)\},$$

where $p_1, q_1, \ldots, p_n, q_n$ is any chain of points in X such that $p_1 \sim x$, $q_1 \sim p_2$, $q_2 \sim p_3$, ... $q_n \sim y$. In particular, we have for all $x, y \in X$

$$\tilde{d}(\tilde{x}, \tilde{y}) \leq d(x, y). \tag{4.2}$$

In general, an inequality in the opposite direction is much more complicated. In the case that I does not expand distances we have the following result whose proof may be found in [2, Lemma 4.1].

Lemma 4.1. *If* $\operatorname{Lip}(I) \leq 1$*, then for all* $\tilde{x}, \tilde{y} \in \pi(X)$ *there exist* $p, q \in X$ *such that* $p \sim x$, $q \sim y$ *and* $d(p, q) \leq 2\, \tilde{d}\, (\tilde{x}, \tilde{y})$.

Now take $\xi > 0$ such that conditions (C1)-(C5) hold. Since $I(D) \cap D = \emptyset$, then each point in the set $X_\xi = X \setminus (D_\xi \cup D)$ has a representative of its equivalence class in $X \setminus D_\xi$. This implies that

$$\pi(X_\xi) = \pi(X \setminus D_\xi), \tag{4.3}$$

and by the ξ-regular condition (C3) this is a compact set. As we are assuming that φ satisfies a ξ-half-tube condition, then X_ξ is invariant under ψ; recall (1.2). With no risk of confusion, the restriction of ψ to X_ξ will still be denoted by ψ.

Given $x, y \in X_\xi$ we have $x \sim y$ if and only if $x = y$. So $\pi|_{X_\xi}$ induces a continuous bijection from X_ξ onto the set

$$\widetilde{X}_\xi = \pi(X_\xi)$$

that we shall denote by H. The map H allows us to introduce a semiflow $\widetilde{\psi}$ on $\widetilde{X}_\xi$ given by

$$\widetilde{\psi}(t, \tilde{x}) = H \circ \psi(t, x), \tag{4.4}$$

for all $x \in X_\xi$ and $t \geq 0$. Since the impulsive semiflow ψ satisfies conditions (C1) and (C5), it follows from [2, Lemma 4.2] that the semiflow $\widetilde{\psi}$ is continuous. Moreover, H gives a semiconjugacy between the semiflows, i.e.

$$\widetilde{\psi}_t(H(x)) = H(\psi_t(x)) \tag{4.5}$$

for all $x \in X_\xi$ and $t \geq 0$.

We are looking for measures of maximal entropy for the impulsive semiflow ψ and, more generally, equilibrium states for potential functions in $V^*(\psi)$. The strategy is to bring to X, via H, equilibrium states for the continuous quotient dynamics $\widetilde{\psi}$ defined on the compact quotient metric space $\widetilde{X}_\xi$. For this purpose, we will need to carry the expansiveness and periodic specification properties from the impulsive semiflow ψ to the quotient semiflow $\widetilde{\psi}$. However, condition (C3) is incompatible with the periodic specification property for ψ in X_ξ (cf. Section 7). This is why we will restrict the impulsive semiflow ψ to $\Omega_\psi \setminus D$, a set that is contained in X_ξ for every $0 < \xi < \xi_0$. Recall that, as ψ is not continuous, its non-wandering set Ω_ψ, although closed, may be not ψ-invariant. Yet, condition (C2) guarantees that $\Omega_\psi \setminus D$ is ψ-invariant (see [1, Theorem B]) and, moreover, that $\pi(\Omega_\psi \setminus D) = \pi(\Omega_\psi)$. In what follows we will consider $\widetilde{\psi}$ restricted to the compact set

$$\widetilde{Y} = \pi(\Omega_\psi \setminus D).$$

Lemma 4.2. *Let* ψ *be an impulsive semiflow.*

(1) *If* $\psi : \mathbb{R}_0^+ \times (\Omega_\psi \setminus D) \to \Omega_\psi \setminus D$ *is positively expansive, then* $\widetilde{\psi} : \mathbb{R}_0^+ \times \widetilde{Y} \to \widetilde{Y}$ *is positively expansive.*

(2) *If* $\psi : \mathbb{R}_0^+ \times (\Omega_\psi \setminus D) \to \Omega_\psi \setminus D$ *has the (periodic) specification property, then* $\widetilde{\psi} : \mathbb{R}_0^+ \times \widetilde{Y} \to \widetilde{Y}$ *has the (periodic) specification property.*

Proof. (1) Suppose that ψ is expansive on $\Omega_\psi \setminus D$. Given $\delta > 0$, consider $\varepsilon > 0$ associated to δ as in the definition of expansiveness for the impulsive semiflow ψ. Let $\tilde{x}, \tilde{y} \in \widetilde{Y}$ and a continuous map $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $s(0) = 0$ satisfying $\tilde{d}(\widetilde{\psi}_t(\tilde{x}), \widetilde{\psi}_{s(t)}(\tilde{y})) < \varepsilon/2$ for all $t \geq 0$. Letting $x = H^{-1}(\tilde{x})$ and $y = H^{-1}(\tilde{y})$, we have for all $t \geq 0$

$$\psi_t(x) = H^{-1}(\widetilde{\psi}_t(\tilde{x})) \quad \text{and} \quad \psi_{s(t)}(y) = H^{-1}(\widetilde{\psi}_{s(t)}(\tilde{y})).$$

Recall that the equivalence classes of $\psi_s(x)$ and $\psi_s(y)$ are reduced to a single point. Then, if $\psi_t(x), \psi_{s(t)}(y) \notin B_\varepsilon(D)$, it follows from Lemma 4.1 that $d(\psi_t(x), \psi_{s(t)}(y)) < \varepsilon$. Therefore, $y = \psi_t(x)$ for some $0 < t < \delta$. This implies that $\tilde{y} = \widetilde{\psi}_t(\tilde{x})$, and so $\widetilde{\psi}$ is expansive.

(2) Assume now that ψ has the (periodic) specification property on $\Omega_\psi \setminus D$. Given $\varepsilon > 0$, consider $L > 0$ assigned to ε by the specification property. For each sequence $\tilde{x}_0, \dots, \tilde{x}_n$ in $\widetilde{Y}$ and each sequence $0 \leq t_0 < \cdots < t_{n+1}$ such that $t_{i+1} - t_i \geq L$ for all $0 \leq i \leq n$, we have $x_0, \dots, x_n$ the unique representatives in $\Omega_\psi \setminus D$ in the equivalence classes of $x_0, \dots, x_n$, respectively. Then, as ψ has the (periodic) specification property in $\Omega_\psi \setminus D$, there are a (periodic) point $y \in \Omega_\psi \setminus D$ and a function $r : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ constant on each interval $[t_i, t_{i+1}[$ satisfying

$$r([t_0, t_1[) = 0 \quad \text{and} \quad |r([t_{i+1}, t_{i+2}[) - r([t_i, t_{i+1}[)| < \varepsilon,$$

for which

$$d(\psi_{t+r(t)}(y), \psi_t(x_i)) < \varepsilon, \quad \forall t \in [t_i, t_{i+1}[\quad \forall 0 \leq i \leq n.$$

Therefore, the $\widetilde{\psi}$-orbit of $\tilde{y}$ is well defined (and periodic) and, using (4.2), we get

$$\tilde{d}(\widetilde{\psi}_{t+r(t)}(\tilde{y}), \widetilde{\psi}_t(\tilde{x}_i)) \leq d(\psi_{t+r(t)}(y), \psi_t(x_i)) < \varepsilon, \quad \forall t \in [t_i, t_{i+1}[\quad \forall 0 \leq i \leq n,$$

thus proving that $\widetilde{\psi}$ has the (periodic) specification property. $\qquad \square$

Letting $i : \Omega_\psi \setminus D \to X$ be the inclusion map and the subscript $*$ stand for the push-forward of a measure, the next result follows from [1, Theorem A] and [2, Lemma 4.7].

Lemma 4.3. *The map* $i_* : \mathcal{M}_\psi(\Omega_\psi \setminus D) \to \mathcal{M}_\psi(X)$ *is a bijection.*

The following lemma is a straightforward consequence of the previous considerations concerning the map H together with the fact that H^{-1} is measurable by [24].

Lemma 4.4. *The map* $H_* : \mathcal{M}_\psi(\Omega_\psi \setminus D) \to \mathcal{M}_{\widetilde{\psi}}(\widetilde{Y})$ *is a bijection.*

5. Existence and uniqueness of equilibrium states

Here we finish the proof of Theorem A. In the next result we show that H preserves some good properties of the potential functions.

Lemma 5.1. *If* $f \in V^*(\psi)$, *then* $f \circ H^{-1} \in V(\widetilde{\psi})$.

Proof. By definition of $V^*(\psi)$, we have f continuous and $f(x) = f(I(x))$ for all $x \in D$. Recalling the definition of the equivalence relation $\sim$, we can easily see that $f \circ H^{-1}$ is continuous. We are left to check that condition (1.3) holds.

Given $f \in V^*(\psi)$, there are constants $K > 0$ and $\varepsilon > 0$ for which (1.4) holds whenever $d(\psi_s(x), \psi_s(y)) < \varepsilon$ for all $s \in [0, t]$ such that $\psi_s(x), \psi_s(y) \notin B_\varepsilon(D)$. Consider $\tilde{x}, \tilde{y} \in \widetilde{Y}$ and $t > 0$ such that $d(\widetilde{\psi}_s(\tilde{x}), \widetilde{\psi}_s(\tilde{y})) < \varepsilon/2$ for all $s \in [0, t]$. Letting $x = H^{-1}(\tilde{x})$ and $y = H^{-1}(\tilde{y})$, we have for all $s \in [0, t]$

$$\psi_s(x) = H^{-1}(\widetilde{\psi}_s(\tilde{x})) \quad \text{and} \quad \psi_s(y) = H^{-1}(\widetilde{\psi}_s(\tilde{y})). \tag{5.1}$$

Observe that the equivalence classes of $\psi_s(x)$ and $\psi_s(y)$ are reduced to a single point. Then, if $\psi_s(x), \psi_s(y) \notin B_\varepsilon(D)$, it follows from Lemma 4.1 that $d(\psi_s(x), \psi_s(y)) < \varepsilon$. Therefore,

$$\left| \int_0^t f(\psi_s(x)) \, ds - \int_0^t f(\psi_s(y)) \, ds \right| < K. \tag{5.2}$$

Recalling (5.1), we have

$$\left| \int_0^t f(H^{-1}(\widetilde{\psi}_s(\tilde{x}))) \, ds - \int_0^t f(H^{-1}(\widetilde{\psi}_s(\tilde{y}))) \, ds \right| = \left| \int_0^t f(\psi_s(x)) \, ds - \int_0^t f(\psi_s(y)) \, ds \right|,$$

which together with (5.2) gives the desired conclusion. $\qquad\square$

Given a potential function $f \in V^*(\psi)$ and considering $\tilde{f} = f \circ H^{-1}$, it follows from Lemma 5.1 that $\tilde{f} \in V(\widetilde{\psi})$. Therefore, we may use [17] and obtain an equilibrium state $\tilde{\mu}_{\tilde{f}}$ for $\tilde{f}$ (with respect to $\widetilde{\psi}$); recall Remark 1.2. Taking $\mu_f = i_* H_*^{-1} \tilde{\mu}_{\tilde{f}}$, we are going to verify that μ_f is an equilibrium state for f (with respect to ψ). Firstly notice that it follows from Lemmas 4.3 and 4.4 that for every $\nu \in M_\psi(X)$ we have

$$\int \tilde{f} \, dH_* i_*^{-1} \nu = \int f \circ H^{-1} \, dH_* i_*^{-1} \nu = \int f \, d\, i_*^{-1} \nu = \int f \, d\nu, \tag{5.3}$$

and also that, using (4.5) we get

$$h_{\tilde{\nu}}(\widetilde{\psi}_1) = h_\nu\left(\psi_1|_{\Omega_\psi \setminus D}\right) = h_\nu(\psi_1), \tag{5.4}$$

where $\tilde{\nu} = H_* i_*^{-1} \nu$. From (5.3) and (5.4) we deduce that

$$\sup\left\{ h_\nu(\psi_1) + \int f \, d\nu \colon \nu \in M_\psi(X) \right\} = \sup\left\{ h_{\tilde{\nu}}(\widetilde{\psi}_1) + \int \tilde{f} \, d\tilde{\nu} \colon \nu \in M_\psi(X) \right\}$$

$$= \sup\left\{ h_\eta(\widetilde{\psi}_1) + \int \tilde{f} \, d\eta \colon \eta \in M_{\widetilde{\psi}}(\widetilde{Y}) \right\}, \tag{5.5}$$

where the last equality is due to Lemmas 4.3 and 4.4. As $\tilde{\mu}_{\tilde{f}}$ is an equilibrium state for $\tilde{f}$ and (5.4) holds for $\nu = \mu_f$, we conclude that

$$h_{\mu_f}(\psi_1) = \sup\left\{ h_\nu(\psi_1) + \int f \, d\nu \colon \nu \in M_\psi(X) \right\}.$$

Additionally, assuming that $\dim(X) < \infty$ and $\#I^{-1}(\{y\}) \leq k$ for every $y \in I(D)$, it follows from [15, Theorem 3.3.7] that $\dim(\widetilde{Y}) < \infty$, and so $\tilde{\mu}_{\tilde{f}}$ is unique by [17, Theorem 2.15]. Since we have $\mu_f = i_* H_*^{-1} \tilde{\mu}_{\tilde{f}}$, the equilibrium state μ_f is unique as well.

6. Variational principle

According to [17], given a continuous semiflow φ on a metric space X and a continuous potential function $f : X \to \mathbb{R}$, the topological pressure of f may be described thermodynamically as

$$P(\varphi, f) = \sup_{\mu \in \mathcal{M}_\varphi(X)} \left\{ h_\mu(\varphi_1) + \int f \, d\mu \right\}. \tag{6.1}$$

The aim of this section is to prove a generalization of this equality to impulsive semiflows, replacing the classical notion of topological pressure by the new one (cf. Section 1.5).

Consider a compact metric space (X, d), an impulsive dynamical system (X, φ, D, I) satisfying conditions (C1)-(C5) and τ the admissible function of the corresponding impulsive times. The assumptions on $I(D)$ ensure that the function that assigns to each $x \in X$ the sequence of visit times to $I(D)$, say $\theta(x) := (\theta_n(x))_{n \in \mathbb{N}}$, is an admissible function with respect to $I(D)$. Moreover, as $I(D) \cap D = \emptyset$, we may re-index the sequences $\tau(x)$ and $\theta(x)$ in order to assemble them in a unique admissible function τ', where $\tau'_n(x)$ is either $\tau_m(x)$ or $\theta_m(x)$, for some m. This way, we have $\tau' \succ \tau$.

Lemma 6.1. *For every continuous potential function $f : X \to \mathbb{R}$, we have $P^\tau(\psi, f) = P^{\tau'}(\psi, f)$.*

Proof. Let f be a continuous potential function in X. As $\tau' \succ \tau$, by Lemma 3.1 we have

$$P^{\tau'}(\psi, f) \leq P^\tau(\psi, f).$$

Concerning the other inequality, it follows from the proof of [2, Lemma 3.1] that, given $t > 0$ and small enough $\varepsilon > 0$ and $\delta > 0$, then a set E which is finite and $(\psi, \tau', \delta, \varepsilon, t)$-separated is $(\psi, \tau, \delta, \varepsilon/2, t)$-separated as well. Therefore

$$Z^\tau(\psi, \delta, \varepsilon, t) \leq Z^{\tau'}(\psi, \delta, \varepsilon/2, t)$$

and so $P^\tau(\psi, f) \leq P^{\tau'}(\psi, f)$. $\qquad\qquad\square$

As the distance between the compact sets D and $I(D)$ is strictly positive, fixing $\eta > 0$ (recall Remark 1.1) and $\xi_0 > 0$ associated to the conditions (C3)-(C5), we may choose

$$0 < \xi < \min \{\eta/4, \xi_0/2\} \tag{6.2}$$

small enough so that $I(D) \cap D_\xi = \emptyset$. The next result shows that, with this suitable choice of ξ, the τ and τ'-topological pressures of the semiflows ψ and $\psi_{|X_\xi}$ coincide for potential functions in $V^*(\psi)$.

Lemma 6.2. *If $f \in V^*(\psi)$, then there is $0 < \xi < \xi_0$ such that $P^{\tau'}(\psi, f) = P^{\tau'}(\psi|_{X_\xi}, f)$.*

Proof. Consider a potential function $f \in V^*(\psi)$. As $X_\xi \subset X$, then $P^{\tau'}(\psi|_{X_\xi}, f) \leq P^{\tau'}(\psi, f)$. We are left to prove the other inequality. It follows from the assumption that D satisfies condition (C3) and the proof of [2, Lemma 3.2] that, given $t > 0$ and small enough $\varepsilon > 0$ and $\delta > 0$, if a set $E \subset X$ is finite and $(\psi, \tau', \delta, \varepsilon, t)$-separated and we take the subsets

$$A = E \cap (D \cup D_\xi) \quad \text{and} \quad B = E \cap X_\xi,, \tag{6.3}$$

then B is $(\psi_{|X_\xi}, \tau', \delta, \varepsilon, t)$-separated and that there exists $\varepsilon' < \varepsilon$ such that $\psi_\xi(A)$ is $(\psi_{|X_\xi}, \tau', \delta, \varepsilon', t)$-separated. Moreover, $\#\psi_\xi(A) = \#A$. Hence, $\psi_\xi(A) \cup B$ is finite and $(\psi_{|X_\xi}, \tau', \delta, \varepsilon', t)$-separated.

Since $f \in V^*(\psi)$, there are $\rho > 0$ and $K > 0$ such that for every $t > 0$ we have

$$\left| \int_0^t f(\psi_s(x))\, ds - \int_0^t f(\psi_s(y))\, ds \right| < K,$$

whenever $d(\psi_s(x), \psi_s(y)) < \rho$ for all $s \in [0, t]$ such that $\psi_s(x), \psi_s(y) \notin B_\rho(D)$. Now, using Lemma 2.1, we may find $\alpha > 0$ such that $d(\varphi_s(x), \varphi_u(x)) < \rho$ for all $x \in X$ and $s, u \geq 0$ with $|s - u| < \alpha$. Therefore, choosing $0 < \xi < \alpha$, we have

$$d(\psi_s(x), \psi_s(\psi_\xi(x))) = d(\varphi_s(x), \varphi_s(\varphi_\xi(x))) < \rho$$

for all $s \in [0, t]$ such that $\psi_s(x), \psi_s(\psi_\xi(x)) \notin B_\rho(D)$. Hence,

$$\sum_{x \in A} \exp \left(\int_0^t f(\psi_s(x)) ds \right) =$$

$$= \sum_{x \in A} \exp \left(\int_0^t f(\psi_s(x)) ds - \int_0^t f(\psi_s(\psi_\xi(x))) ds + \int_0^t f(\psi_s(\psi_\xi(x))) ds \right)$$

$$\leq \sum_{x \in A} \exp \left(\left| \int_0^t f(\psi_s(x)) ds - \int_0^t f(\psi_s(\psi_\xi(x))) ds \right| \right) \exp \left(\int_0^t f(\psi_s(\psi_\xi(x))) ds \right)$$

$$\leq e^K \sum_{x \in A} \exp \left(\int_0^t f(\psi_s(\psi_\xi(x))) ds \right) = e^K \sum_{y \in \psi_\xi(A)} \exp \left(\int_0^t f(\psi_s(y)) ds \right).$$

On the other hand, recalling (6.3) we may write

$$\sum_{x \in E} \exp \left(\int_0^t f(\psi_s(x)) ds \right) = \sum_{x \in A} \exp \left(\int_0^t f(\psi_s(x)) ds \right) + \sum_{x \in B} \exp \left(\int_0^t f(\psi_s(x)) ds \right).$$

Therefore, as $e^K > 1$ and $\varepsilon' < \varepsilon$ we get

$$\sum_{x \in E} \exp \left(\int_0^t f(\psi_s(x)) ds \right) \leq e^K \sum_{y \in \psi(A) \cup B} \exp \left(\int_0^t f(\psi_s(y)) ds \right),$$

and so

$$Z^{\tau'}(\psi, f, \delta, \varepsilon, t) \leq e^K Z^{\tau'}(\psi|_{X_\xi}, f, \delta, \varepsilon', t).$$

This implies that $P^{\tau'}(\psi, f) \leq P^{\tau'}(\psi|_{X_\xi}, f)$. As $I(D)$ also satisfies condition (C3), a similar argument shows that $P^{\tau'}(\psi, f) = P^{\tau'}(\psi|_{X_\xi \setminus I(D)}, f)$. $\qquad\square$

Using the semiconjugacy H between the semiflows ψ and $\tilde{\psi}$, we now project on $\widetilde{X}_\xi$ the admissible functions τ and τ', as done in [2, Lemma 4.5], thus getting admissible functions on $\widetilde{X}_\xi$. This way, we may compare the corresponding pressures.

Lemma 6.3. If $f \in V^*(\psi)$, then $P^{\tau'}(\psi, f) = P^{\tilde{\tau}'}(\tilde{\psi}, \tilde{f})$.

Proof. Given $f \in V^*(\psi)$, let $0 < \xi < \xi_0$ be given by Lemma 6.2. It follows from Lemma 3.2 applied to the semiconjugacy $H : X_\xi \to \widetilde{X}_\xi$ that $P^{\tilde{\tau}'}(\tilde{\psi}, \tilde{f}) \leq P^{\tau'}(\psi|_{X_\xi}, f)$. Additionally, Lemma 6.2 ensures that $P^{\tau'}(\psi|_{X_\xi}, f) = P^{\tau'}(\psi, f)$. Thus,

$$P^{\tilde{\tau}'}(\tilde{\psi}, \tilde{f}) \leq P^{\tau'}(\psi, f). \tag{6.4}$$

Conversely, as $H^{-1} : \widetilde{X}_\xi \setminus \pi(D) \to X_\xi \setminus I(D)$ is a $(\tilde{\tau}', \tau')$-semiconjugacy between $\tilde{\psi}|_{\widetilde{X}_\xi \setminus \pi(D)}$ and $\psi|_{X_\xi \setminus I(D)}$, it follows from Lemma 3.2 that $P^{\tilde{\tau}'}(\tilde{\psi}|_{\widetilde{X}_\xi \setminus \pi(D)}, \tilde{f}) \geq P^{\tau'}(\psi|_{X_\xi \setminus I(D)}, f)$. More-over, as $f \in V^*(\psi)$, from Lemma 6.2 we have $P^{\tau'}(\psi|_{X_\xi \setminus I(D)}, f) = P^{\tau'}(\psi, f)$. Besides, as $\widetilde{X}_\xi \setminus \pi(D) \subset \widetilde{X}_\xi$, we also know that $P^{\tilde{\tau}'}(\tilde{\psi}, \tilde{f}) \geq P^{\tilde{\tau}'}(\tilde{\psi}|_{\widetilde{X}_\xi \setminus \pi(D)}, \tilde{f})$. Hence,

$$P^{\tilde{\tau}'}(\tilde{\psi}, \tilde{f}) \geq P^{\tilde{\tau}'}(\tilde{\psi}|_{\widetilde{X}_\xi \setminus \pi(D)}) \geq P^{\tau'}(\psi|_{X_\xi \setminus I(D)}, f) = P^{\tau'}(\psi, f),$$

which together with (6.4) gives the result.   □

Let us resume the proof of Theorem C. Given $f \in V^*(\psi)$, it follows from Theorem B that

$$P^{\tilde{\tau}'}(\tilde{\psi}, \tilde{f}) = P(\tilde{\psi}, \tilde{f}),$$

which together with Lemma 6.1, Lemma 6.2 and Lemma 6.3 yields

$$P^\tau(\psi, f) = P(\tilde{\psi}, \tilde{f}).$$

Now, it follows from (5.5) that

$$\sup_{\eta \in \mathcal{M}_{\tilde{\psi}}(X)} \left\{ h_\eta(\tilde{\psi}_1) + \int \tilde{f} \, d\eta \right\} = \sup_{\nu \in \mathcal{M}_\psi(X)} \left\{ h_\mu(\psi_1) + \int f \, d\nu \right\}.$$

Moreover, equation (6.1) gives

$$P(\tilde{\psi}, \tilde{f}) = \sup_{\eta \in \mathcal{M}_{\tilde{\psi}}(X)} \left\{ h_\eta(\tilde{\psi}_1) + \int \tilde{f} \, d\eta \right\}.$$

Therefore,

$$P^\tau(\psi, f) = \sup_{\nu \in \mathcal{M}_\psi(X)} \left\{ h_\mu(\psi_1) + \int f \, d\nu \right\},$$

which ends the proof of Theorem C.

7. EXAMPLES

Here we give two examples of impulsive dynamical systems for which the variational principle and the existence and uniqueness of equilibrium states hold. This follows from the simple fact that the impulsive semiflow is uniquely ergodic in the first example. In the second one we show that the impulsive semiflow is positively expansive and has the specification property and then use Theorem A and Theorem C.

7.1. **Suspension of a rotation.** Here we define an impulsive semiflow on a suspension of an irrational rotation on S^1 which is uniquely ergodic. Consider the unit circle $S^1 = \{e^{2\pi i x} \in \mathbb{C} \colon 0 \leq x < 1\}$ and an irrational number $\theta_1 \in \,]0, 1[$. Let $R_{\theta_1} \colon S^1 \to S^1$ be the irrational rotation $R_{\theta_1}(e^{2\pi i x}) = e^{2\pi i (x + \theta_1)}$ on S^1. Consider the cylinder

$$Y = \{(z, u) \colon z \in S^1, \ 0 \leq u \leq 1\}$$

and the 2-torus

$$X = Y/\approx,$$

where $\approx$ is the equivalence relation in X given by $(z, 1) \approx (R_{\theta_1}(z), 0)$. We define the suspension flow $\varphi \colon \mathbb{R}_0^+ \times X \to X$ over the rotation R_{θ_1} as

$$\varphi(t, (z, u)) = \begin{cases} (z, t + u), & 0 \leq t + u < 1; \\ (R_{\theta_1}(z), 0), & t + u = 1. \end{cases}$$

As R_{θ_1} is uniquely ergodic, the unique invariant probability measure invariant by the suspension flow φ is Lebesgue measure on $X = S^1 \times S^1$; see [23, Chapter 6].

Now, take another irrational number $\theta_2 \in \,]0, 1[$ and consider the set

$$D = S^1 \times \left\{ \frac{1}{2} \right\} \subset X.$$

Let the impulsive function $I \colon D \to X$ be the map defined as

$$I\left(z, \frac{1}{2} \right) = \left(R_{\theta_2}(z), \frac{3}{4} \right),$$

where R_{θ_2} is the irrational rotation of angle θ_2. Then define $\psi \colon \mathbb{R}_0^+ \times X \to X$ as the impulsive semiflow of the impulsive dynamical system (X, φ, D, I). It is straightforward to check that the impulsive semiflow ψ satisfies conditions (C1)-(C5).

Consider the suspension flow $\widetilde{\varphi}$ on X over the rotation $R_{\theta_1 + \theta_2}$; as in the case of φ, the flow $\widetilde{\varphi}$ is uniquely ergodic. We note that the map $\mathcal{F} \colon \Omega_\psi \setminus D \to \widetilde{X}$ defined as

$$\mathcal{F}(x, u) = \begin{cases} (x, u), & 0 \leq u \leq \frac{1}{2}; \\ (R_{\theta_2}^{-1}(x), 2u - 1), & \frac{3}{4} \leq u \leq 1; \end{cases}$$

is a continuous bijection. Moreover, $\mathcal{F}$ conjugates the flows $\widetilde{\varphi}$ and $\psi_{|\Omega_\psi \setminus D}$, that is

$$\widetilde{\varphi}_t \circ \mathcal{F} = \mathcal{F} \circ \psi_t$$

for every $t \geq 0$. Consequently, the semiflow ψ is uniquely ergodic and so it has a unique equilibrium state for any continuous potential function.

7.2. **Suspension of a shift.** Here we define an impulsive semiflow on a suspension of a shift which is positively expansive and has the specification property. Let (Σ_2, σ) be the two-sided full shift on 2 symbols. Given two irrational numbers $a, b > 3$ linearly independent over $\mathbb{Q}$, let $c \colon \Sigma_2 \to \mathbb{R}^+$ be the ceiling function defined as $c(x) = a$ if $x_0 = 0$ and $c(x) = b$ if $x_0 = 1$. Let

$$Y_c = \{(x, u) \colon x \in \Sigma_2, \ 0 \leq u \leq c(x)\} \tag{7.1}$$

and

$$X_c = Y_c/\sim_c \tag{7.2}$$

where $\sim_c$ is the equivalence relation in X_c given by $(x, c(x)) \sim_c (\sigma x, 0)$. We define the suspension flow $\varphi_c : \mathbb{R}_0^+ \times X_c \to X_c$ as

$$\varphi_c(t, (x, u)) = \begin{cases} (x, t + u), & 0 \leq t + u \leq c(x); \\ (\sigma x, 0), & t + u = c(x). \end{cases}$$

As σ is an expansive map, it follows from [7, Theorem 6] that φ_c is an expansive semiflow. It is straightforward to check that c is not cohomologous to a function taking values in $\beta\mathbb{Z}$ for some $\beta > 0$. Therefore, by [25, Proposition 5] the flow φ_c has the periodic specification property. Consider

$$D = \Sigma_2 \times \{1\} \subset X_c$$

and the impulsive function $I : D \to X_c$ defined as

$$I((x_n)_n, 1) = ((1 - x_n)_n, 3).$$

Notice that, on the first coordinate, the map I acts as the isometry $\mathcal{R} : \Sigma_2 \to \Sigma_2$ that reverses each digit, so condition (C1) holds. Moreover, $Y_c \subset \Sigma_2 \times [0, \max\{a, b\}]$ is a finite dimensional metric space; see [15, Theorems 4.1.7, 4.1.21 & 4.1.25]. Since each class of the equivalence relation $\sim_c$ has at most two elements, by [15, Theorem 3.3.7] the dimension of X_c is finite as well.

Let $\psi_c : \mathbb{R}_0^+ \times X_c \to X_c$ be the impulsive semiflow of the impulsive dynamical system (X_c, φ_c, D, I). As

$$\Omega_{\psi_c} \setminus D = \Big(\{(x, u) \colon x \in \Sigma_2, \ 0 \leq u < 1\} \cup \{(x, u) \colon x \in \Sigma_2, \ 3 \leq u \leq c(x)\} \Big)/\sim_c,$$

condition (C2) is also valid. Besides, the semiflow ψ_c satisfies conditions (C3)-(C5) for any $0 < \xi < \xi_0 = 2$. We are left to verify that ψ_c restricted to $\Omega_{\psi_c} \setminus D$ is expansive and has the periodic specification property.

Consider the ceiling function $\tilde{c} = c \circ \mathcal{R} - 2$ and the corresponding spaces $Y_{\tilde{c}}$ and $X_{\tilde{c}}$ defined as in (7.1) and (7.2), respectively. The suspension flow $\varphi_{\tilde{c}}$ on $X_{\tilde{c}}$ is expansive, as in the case of φ_c; moreover, as $\mathcal{R}$ commutes with σ and is an involution, it is not difficult to show that $\tilde{c}$ is not cohomologous to a function taking values in $\beta\mathbb{Z}$ for some $\beta > 0$, and so the flow $\varphi_{\tilde{c}}$ has the specification property.

Let $\mathcal{F} : \Omega_{\psi_c} \setminus D \to X_{\tilde{c}}$ be defined as

$$\mathcal{F}(x, u) = \begin{cases} (x, u), & 0 \leq u < 1; \\ (\mathcal{R}^{-1}(x), u - 2), & 3 \leq u \leq c(x). \end{cases}$$

It is easy to verify that the map $\mathcal{F}$ is a continuous bijection. Besides, as $I(x, 1) = (\mathcal{R}(x), 3)$ for all $x \in \Sigma_2$, the map $\mathcal{F}$ conjugates the semiflows ψ_c on $\Omega_{\psi_c} \setminus D$ and $\varphi_{\tilde{c}}$. Hence, ψ_c is expansive and has the periodic specification property in $\Omega_{\psi_c} \setminus D$. Therefore, by Theorem A the semiflow ψ has a unique equilibrium state for any potential function in $V^*(\psi)$.

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