A Stochastic Diffusion Process Based on the Two-Parameters Weibull Density Function

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Abstract—Stochastic modeling concerns the use of probability to model real-world situations in which uncertainty is present. Therefore, the purpose of stochastic modeling is to estimate the probability of outcomes within a forecast, i.e. to be able to predict what conditions or decisions might happen under different situations. In the present study, we present a model of a stochastic diffusion process based on the bi-Weibull distribution function (its trend is proportional to the bi-Weibull probability density function). In general, the Weibull distribution has the ability to assume the characteristics of many different types of distributions. This has made it very popular among engineers and quality practitioners, who have considered it the most commonly used distribution for studying problems such as modeling reliability data, accelerated life testing, and maintainability modeling and analysis. In this work, we start by obtaining the probabilistic characteristics of this model, as the explicit expression of the process, its trends, and its distribution by transforming the diffusion process in a Wiener process as shown in the Ricciardi theorem. Then, we develop the statistical inference of this model using the maximum likelihood methodology. Finally, we analyse with simulated data the computational problems associated with the parameters, an issue of great importance in its application to real data with the use of the convergence analysis methods. Overall, the use of a stochastic model reflects only a pragmatic decision on the part of the modeler. According to the data that is available and the universe of models known to the modeler, this model represents the best currently available description of the phenomenon under consideration.

Keywords—Diffusion process, discrete sampling, likelihood estimation method, simulation, stochastic diffusion equation, trends functions, bi-parameters Weibull density function.

I. INTRODUCTION

THE statistical analysis of what are referred to as lifetime, survival time, or failure time data has become a topic of considerable interest to statisticians and workers in many areas, including the biomedical, engineering, and social sciences (e.g. Woolson and Clarke [1], Mason et al. [2]). Applications of lifetime distribution methodology range from investigations of the durability of manufactured items to studies of human diseases and their treatment (e.g. Blischke and Murthy [3], Klugman and Parsa [4]). Some methods of dealing with lifetime data are quite old, but starting about 1970

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the field expanded rapidly with respect to methodology, theory, and fields of application. For instance, Davis [5] described applications of the exponential distribution to reliability, Feigl and Zelen [6] provided an early application of an exponential model with covariates to medical survival data, and Cox [7] discussed the gamma distribution in connection with failure times. These include various parametric models and their associated statistical methods, nonparametric and distribution free methods, and graphical procedures.

Various parametric families of models are used in the analysis of lifetime data and the modeling of aging or failure processes. Among univariate models, a few distributions occupy a central position because of their demonstrated usefulness in a wide range of situations. Foremost in this category are the exponential, Weibull, log-normal, log-logistic, and gamma distributions (e.g. Gumbel [8], Lieblein and Zelen [9], Pike [10] and Boag [11]). Actually, the Weibull distribution is perhaps the most widely used lifetime distribution model. Application to the lifetimes or durability of manufactured items is common, and it is used as a model with diverse types of items, such as ball bearings, automobile components, and electrical insulation.

In recent years important advances have been made in modeling based on stochastic diffusion processes, which are defined and studied by several approaches. A number of authors have treated these diffusion processes from the viewpoint of the corresponding Itô stochastic diffusion process (SDE), for instance Giovanis and Skiadas [12] with the Bass distribution model, Katsamaki and Skiadas [13] with the exponential model, and Giovanis and Skiadas [14] with the logistic model. In spite of this, another way of defining and studying stochastic diffusion processes is based on backwards and forwards Kolmogorov equations, which are also known as Fokker-Plank equations, associated with the corresponding infinitesimal moments. For example, [15] for the the gamma model, [16] for the diffusion model with cubic drift, and [17] for the case of the use of the lognormal and Gompertz diffusion process. This method of approaching and studying the topic is particularly interesting when we wish to construct nonhomogeneous versions of a diffusion by introducing just time functions (exogenous factors) into the infinitesimal moments. Furthermore, the question of statistical inference and the problem of parameters estimation in these processes have received recently considerable attention, in situations in which the process is observed continuously or discretely. In most cases, the parameter estimation is based on approximating the maximum likelihood methodology. A large body of literature has this question, both in general and in particular cases,

see [18]-[20]. The problem becomes yet more complex when the distribution parameters are unknown, which we need to estimate from the samples and the uncertainty increases even more.

This paper introduces a stochastic diffusion process based on the bi-parameter Weibull distribution. Section I contains the definition of the model and its characteristics where, in order to obtain the distribution of the process, we employed the theorem of Ricciardi. Section II deals with the inference study of the unknown parameters by the mean of maximum likelihood method, although there are many ways to estimate the parameters, the maximum likelihood is generally the most popular method. This is extended in Section III, where we use simulated data to sort out the computational problems associated with the parameters and to obtain the estimators of the model.

II. FORMULATION OF THE MODEL AND ITS BASIC PROBABILISTIC CHARACTERISTICS

A. The Model

One-dimensional stochastic differential equations of the Itô type have the following general form:

$$dx(t) = a\left(x(t),t\right)dt + b(x(t),t)dW(t), \qquad (1)$$

where the functions $a\left(x(t),t\right)$ and $b(x(t),t)$ are so-called drift and diffusion terms, respectively, and $W(t)$ is a standard Wiener process. The drift and diffusion terms in (1) determines the statistical properties of the variable $x(t)$.

The proposed model in this work is the stochastic Weibull diffusion process defined as the time-nonhomogeneous one-dimensional diffusion process $\{x(t), t \in [t_1, T], 0 < t_1 \le T\}$, with values in $(0, +\infty)$ by the following Itô's stochastic differential equation (SDE)

$$dx(t) = \left(\frac{\alpha}{t} - \beta t^\alpha\right)x(t)dt + \sigma x(t)dw(t), \qquad (2)$$

with the initial condition $P(x(t_1) = x_1) = 1$ and where $w(t)$ is a standard Wiener process.

Alternatively, the above-defined process can be considered by the Kolmogoroff approach with infinitesimal moments (drift and diffusion coefficients) given by

$$\begin{cases} a(x,t) = \left(\dfrac{\alpha}{t} - \beta t^\alpha\right)x, \\ b(x,t) = \sigma^2 x^2. \end{cases} \qquad (3)$$

It can be shown that the functions $a(x,t)$ and $b(x,t)$, $0 < x < +\infty$, are Borel measurables and satisfy the uniform Lipschitz and the growth conditions (see Kloeden and Platen [21]). As a result, there exist a separable, measurable and almost surely sample continuous process $\{x(t), t \in [t_1, T], t_1 > 0\}$ which is the unique (a.s.) solution of the SDE (2). Thus, we denote the probability density function (p.d.f.) of the process by $f(y,t \mid x,s)$, which is the unique solution to the following equations, known as the Fokker-Planck and the backward Kolmogorov equations:

$$\begin{cases} \dfrac{\partial f(y,t \mid x,s)}{\partial t} = -\dfrac{\partial}{\partial x}[a(y,t)f(y,t \mid x,s)] \\ \qquad\qquad + \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}[b(y,t)f(y,t \mid x,s)], \\ \dfrac{\partial f(y,t \mid x,s)}{\partial s} = -a(x,t)\dfrac{\partial f(y,t \mid x,s)}{\partial x} \\ \qquad\qquad - \dfrac{1}{2}b(x,t)\dfrac{\partial^2 f(y,t \mid x,s)}{\partial x^2}, \end{cases} \qquad (4)$$

with the delta-type initial condition

$$\lim_{t \to s} f(y,t \mid x,s) = \delta(y-x),$$

where $\delta(.)$ is the Dirac delta function on $\mathbb{R}$.

B. Distribution of the Model

The common solution to the Kolmogorov equations (4) is obtained by the use of Ricciardi's theorem [22] which basically transforms a diffusion process to a Wiener process. In fact, the infinitesimal moments (3) verify the conditions of the Ricciardi's theorem; therefore, such a transformation exists and has the following form:

$$\begin{cases} \phi(t) = t, \\ \psi(x,t) = \dfrac{1}{\sigma}[\log(x) - \alpha\log(t) + \dfrac{\beta}{\alpha+1}t^{\alpha+1} + \dfrac{\sigma^2}{2}t]. \end{cases} \qquad (5)$$

From the above, the p.d.f. of the considered process have the following expression:

$$f(y,t \mid x_s, s) = \frac{1}{\sqrt{2\pi\sigma^2(t-s)y^{-1}}} \\ \exp\left(-\frac{[\log(y) - \mu(s,t,x_s)]^2}{2\sigma^2(t-s)}\right), \qquad (6)$$

where $\mu(s,t,x_s)$ is the mean of the desired probability distribution.

$$\mu(s,t,x_s) = \log(x_s) + \alpha\log\left(\frac{t}{s}\right) - \frac{\beta}{\alpha+1}(t^{\alpha+1} - s^{\alpha+1}) \\ - \frac{\sigma^2}{2}(t-s). \qquad (7)$$

Consequently, according to (6), the p.d.f. of the process is the density function of the one-dimensional lognormal distribution.

$$x(t) \mid x(s) = x_s \sim \Lambda_1[\mu(s,t,x_s), \sigma^2(t-s)].$$

C. Moments of the Process

We will make strong use of the fact that the random variable $x(t) \mid x(s) = x_s$ is distributed as $\Lambda_1\left[\mu(s,t,x_s), \sigma^2(t-s)\right]$ and bearing in mind that the r-th conditional moment of the process is expressed by:

$$\mathbb{E}\left[x^r(t)|x(s) = x_s\right] = \exp\left(r\mu(s,t,x_s) + \frac{r\sigma^2}{2}(t-s)\right). \qquad (8)$$

In short, by considering the case $r = 1$ in the (8), the conditional trend function of the process is:

$$\mathbb{E}\left[x(t) \mid x(s) = x_s\right] = x_s \left(\frac{t}{s}\right)^{\alpha} e^{-\frac{\beta}{\alpha+1}(t^{\alpha+1} - s^{\alpha+1})}. \quad (9)$$

Thereby, assuming the initial condition $P(x(t_1) = x_1) = 1$, the trend function of the process has the following form:

$$\mathbb{E}\left[x(t)\right] = \frac{x_{t_1} e^{\frac{\beta}{\alpha+1} t_1^{\alpha+1}}}{t_1^{\alpha}} t^{\alpha} e^{-\frac{\beta}{\alpha+1} t^{\alpha+1}}. \quad (10)$$

Remark 1: Note that in the absence of white noise (i.e. $\sigma = 0$), by a simple integration, the solution of the ordinary differential equation associated with the SDE (2) is $x(t) = kt^{\alpha} e^{-\frac{\beta}{\alpha+1} t^{\alpha+1}}$, which is proportional to the bi-parameter Weibull density function.

We can also see that the trend function given in (10) is proportional to the density function of the bi-parameter Weibull distribution. For those reasons, the process received the name of Weibull diffusion process.

III. STATISTICAL INFERENCE ON THE MODEL

A. Maximum Likelihood Function

As long as we obtain the explicit expression of the p.d.f. of the process (lognormal distribution), we can estimate the parameters involved in the process, making use of discrete sampling, based on the conditioned likelihood function which is the product of the corresponding process transitions (given by (6)).

Let us consider a discrete sampling of the process $\{X(t_i) = x_{t_i} = x_i, 1 \leq i \leq n\}$ for the instants $t_1, \ldots, t_n$, with $t_i - t_{i-1} = h$ ($h > 0$), for $i = 2, \ldots, n$. Assuming the initial distribution $P[x(t_1) = x_1] = 1$, the conditioned likelihood estimate of the parameters $\theta = (\theta_1, \theta_2, \theta_3) \equiv (\alpha, \beta, \sigma^2)$ is that value $\theta = \hat{\theta} = (\hat{\alpha}, \hat{\beta}, \widehat{\sigma^2})$ which maximizes the likelihood function associated.

$$\mathbb{L}(x_1, \ldots, x_n; \theta) = \prod_{i=2}^{n} f_{\theta}(x_i, t_i \mid x_{i-1}, t_{i-1}), \quad (11)$$

over θ, i.e., which gives highest local probability to the observed sample $(X(t_1), \ldots, X(t_n)) = (x_1, \ldots, x_n)$. In other words,

$$\mathbb{L}(x_1, \ldots, x_n; \hat{\theta}) = \sup_{\theta} \left\{ \prod_{i=2}^{n} f_{\theta}(x_i, t_i \mid x_{i-1}, t_{i-1}) \right\}. \quad (12)$$

Often such maximizing value $\hat{\theta}$ is unique and it can be obtained by solving

$$\frac{\partial}{\partial \theta_j} \prod_{i=2}^{n} f_{\theta}(x_i, t_i \mid x_{i-1}, t_{i-1}) = 0, \quad j = 1, \ldots, 3. \quad (13)$$

Equation (13) reflects the fact that a smooth function has a horizontal tangent plane at its maximum. Thus solving such

equations is necessary but not sufficient, since it still needs to be shown that it is the location of a maximum.

Since taking derivatives of a product is tedious, one usually resorts to maximizing the log of the likelihood (11), i.e.,

$$l_v(\alpha, \beta, \sigma^2) = -\frac{n-1}{2} \log(2\pi h) - \frac{n-1}{2} \log(\sigma^2)$$
$$- \sum_{j=2}^{n} \log(x_j) - \frac{1}{2\sigma^2 h} \sum_{j=2}^{n} \left[B_j + \frac{\sigma^2}{2} h \right]^2,$$

with $v = (x_1, \ldots, x_n)$, and

$$B_j = \log(x_j/x_{j-1}) - \alpha \log(t_j/t_{j-1}) + \frac{\beta}{\alpha+1} \left(t_j^{\alpha+1} - t_{j-1}^{\alpha+1} \right),$$

for all $j = 2, \ldots, n$, being simpler to deal with the likelihood equations

$$\begin{cases} \dfrac{\partial l_{x_1, \ldots, x_n}(\alpha, \beta, \sigma^2)}{\partial \alpha} = 0, \\[2mm] \dfrac{\partial l_{x_1, \ldots, x_n}(\alpha, \beta, \sigma^2)}{\partial \beta} = 0, \\[2mm] \dfrac{\partial l_{x_1, \ldots, x_n}(\alpha, \beta, \sigma^2)}{\partial \sigma^2} = 0. \end{cases} \quad (14)$$

In a word, by solving the (14), we obtain $\hat{\alpha}$, $\hat{\beta}$, and $\widehat{\sigma^2}$, the maximum likelihood estimators of, respectively, α, β, and σ^2.

B. Maximum Likelihood Estimators

In our case, the derivatives of the log-likelihood function, with respect to the parameters α, β, and σ^2 are

$$\begin{cases} \dfrac{\partial l_v(\alpha, \beta, \sigma^2)}{\partial \sigma^2} = -\dfrac{n-1}{2\sigma^2} + \dfrac{1}{2\sigma^4 h} \sum_{j=2}^{n} A_j^2 - \dfrac{1}{2\sigma^2} \sum_{j=2}^{n} A_j, \\[3mm] \dfrac{\partial l_v(\alpha, \beta, \sigma^2)}{\partial \beta} = -\dfrac{1}{\sigma^2 h} \sum_{j=2}^{n} A_j \left(\dfrac{t_j^{\alpha+1} - t_{j-1}^{\alpha+1}}{\alpha+1} \right), \\[3mm] \dfrac{\partial l_v(\alpha, \beta, \sigma^2)}{\partial \alpha} = -\dfrac{1}{\sigma^2 h} \sum_{j=2}^{n} \dfrac{\partial B_j}{\partial \alpha} A_j, \end{cases} \quad (15)$$

where $A_j = B_j + \frac{\sigma^2}{2} h$, $j = 2, \ldots, n$. Then

$$\frac{\partial B_j}{\partial \alpha} = -\log\left(\frac{t_j}{t_{j-1}}\right) - \frac{\beta}{(\alpha+1)^2}(t_j^{\alpha+1} - t_{j-1}^{\alpha+1})$$
$$+ \frac{\beta}{\alpha+1} \left[\ln(t_j) t_j^{\alpha+1} - \ln(t_{j-1}) t_{j-1}^{\alpha+1} \right].$$

We obtain after some calculation, the following likelihood equations:

$$-(n-1)\sigma^2 h + \sum_{j=2}^{n} A_j^2 + \sigma^2 h \sum_{j=2}^{n} A_j = 0, \quad (16a)$$

$$\sum_{j=2}^{n} A_j \left[t_j^{\alpha+1} - t_{j-1}^{\alpha+1} \right] = 0, \quad (16b)$$

$$\sum_{j=2}^{n} A_j \frac{\partial B_j}{\partial \alpha} = 0. \quad (16c)$$

From (16a), we can get (as a positive solution) the expression of the estimator $\widehat{\sigma}^2$

$$\frac{\widehat{\sigma}^2}{2} = \frac{1}{(n-1)h} \sum_{j=2}^{n} \frac{\hat{B}_j^2}{\left(1 + \frac{1}{n-1} \sum_{j=2}^{n} \hat{B}_j^2\right)^{1/2} + 1} \quad (17)$$

Then, by replacing the expression of $\widehat{\sigma}^2$ in (16b) and (16c), we obtain the estimators $\hat{\alpha}$ and $\hat{\beta}$, from the following nonlinear equations

$$\begin{cases} \sum_{j=2}^{n} \hat{A}_j \left[t_j^{\hat{\alpha}+1} - t_{j-1}^{\hat{\alpha}+1} \right] = 0, \\ \sum_{j=2}^{n} \hat{A}_j \hat{C} = 0. \end{cases} \quad (18)$$

Here, for all $j = 2, \ldots, n$, we have

$$\hat{B}_j = \log(x_j/x_{j-1}) - \hat{\alpha} \log(\frac{t_j}{t_{j-1}}) + \frac{\hat{\beta}}{\hat{\alpha}+1}(t_j^{\hat{\alpha}+1} - t_{j-1}^{\hat{\alpha}+1}),$$

$$\hat{A}_j = \hat{B}_j + \frac{\hat{\sigma}^2}{2}h,$$

and

$$\hat{C}_j = -\log\left(\frac{t_j}{t_{j-1}}\right) - \frac{\beta}{(\hat{\alpha}+1)^2}\left(t_j^{\hat{\alpha}+1} - t_{j-1}^{\hat{\alpha}+1}\right) + \frac{\beta}{\hat{\alpha}+1}\left[\ln(t_j)t_j^{\hat{\alpha}+1} - \ln(t_{j-1})t_{j-1}^{\hat{\alpha}+1}\right].$$

Remark 2: From Zenha's theorem [23], we can obtain the maximum likelihood estimated trend function and the conditional estimated trend function of the process, by substituting the parameters by their estimators in (9) and (10):

$$\hat{\mathbb{E}}\left[x(t) \mid x(s) = x_s\right] = x_s \left(\frac{t}{s}\right)^{\hat{\alpha}} e^{-\frac{\hat{\beta}}{\hat{\alpha}+1}(t^{\hat{\alpha}+1} - s^{\hat{\alpha}+1})}, \quad (19)$$

$$\hat{\mathbb{E}}\left[x(t)\right] = \frac{x_{t_1} e^{\frac{\hat{\beta}}{\hat{\alpha}+1} t_1^{\hat{\alpha}+1}}}{t_1^{\hat{\alpha}}} \; t^{\hat{\alpha}} \; e^{-\frac{\hat{\beta}}{\hat{\alpha}+1} t^{\hat{\alpha}+1}}. \quad (20)$$

IV. APPLICATION TO SIMULATED DATA

This section will complete the inference study of the parameters of the model, with the specifications and improvement previously mentioned, to obtain the maximum likelihood estimations for parameters α, β, and σ^2 in a stochastic Weibull diffusion process with infinitesimal moments given earlier by (3).

A. Simulated Trajectory of the Process

The trajectory of the model can be obtained by simulating the exact solution of the SDE (2), which is found by the mean of Itô's formula. That's why we consider the transformation $y(t) = \log(x(t))$, thereby after applying the Itô's formula, we have the following SDE:

TABLE I
MEAN AND STANDARD ERROR OF $\hat{\alpha}$

h	num.obs.	$mean(\alpha)$	$SE(\alpha)$
0.007	100	0.5035	0.0093
0.007	500	0.5012	0.0096
0.007	1000	0.4915	0.0185
0.0035	100	0.4958	0.0147
0.0035	500	0.5024	0.0077
0.0035	1000	0.5009	0.0070
0.0007	100	0.5137	0.0210
0.0007	500	0.5008	0.0101
0.0007	1000	0.5142	0.0220

$$\begin{cases} dy(t) = \left(\frac{\alpha}{t} - \beta t^\alpha - \frac{\sigma^2}{2}\right) dt + \sigma dw(t), \\ y(t_1) = \log(x_1). \end{cases} \quad (21)$$

By integrating (21), we obtain:

$$\begin{aligned} y(t) =& y(t_1) + \alpha \log(\frac{t}{t_1}) - \frac{\beta}{\alpha+1}(t^{\alpha+1} - t_1^{\alpha+1}) \\ & - \frac{\sigma^2}{2}(t - t_1) + \sigma\left(w(t) - w(t_1)\right), \end{aligned} \quad (22)$$

In short, we deduce the analytical expression of the solution of SDE(2) from the (22)

$$\begin{aligned} x(t) =& x_1 \exp\left(\alpha \log\left(\frac{t}{t_1}\right) - \frac{\beta}{\alpha+1}\left(t^{\alpha+1} - t_1^{\alpha+1}\right) \right. \\ & \left. - \frac{\sigma^2}{2}(t - t_1) + \sigma\left(w(t) - w(t_1)\right) \right). \end{aligned} \quad (23)$$

Therefore, the simulated trajectories of the process are obtained from the following discretizing time interval $[t_1, T]$: $t_i = t_1 + (i-1)h$, for $i = 1, \ldots, N$ (N is an integer and $h > 0$ is the discretization step), taking into account that the random variable in the expression $\sigma(w(t) - w(t_1))$, in (23), is distributed as a one-dimensional normal distribution $\mathcal{N}_1\left(0, \sigma^2(t - t_1)\right)$.

B. Simulated Data

In this simulation, we consider M process trajectories, each of which has N observations, estimating the parameters by means of the system (18). In total M estimators are obtained for each parameter (i.e. one vector of M components), from which we compute the sample mean and the standard error (SE) of each estimator.

Let us study the evolution of the mean and the standard error of the estimators with respect to the variation in the number N and h. The results of this study are shown in Tables I-III.

A Matlab program was implemented to carry out the calculation required for this study. The true parameter values considered in this simulation are $\alpha = 0.5$, $\beta = 0.8$, $\sigma = 0.04$ and the start point is $x_{t_1} = 0.001$, and $t_1 = 0.05$.

Fig. 1 shows some simulated trajectories of the process and the estimated trend function of the process obtained using te Zenha theorem, replacing the parameters by their estimators
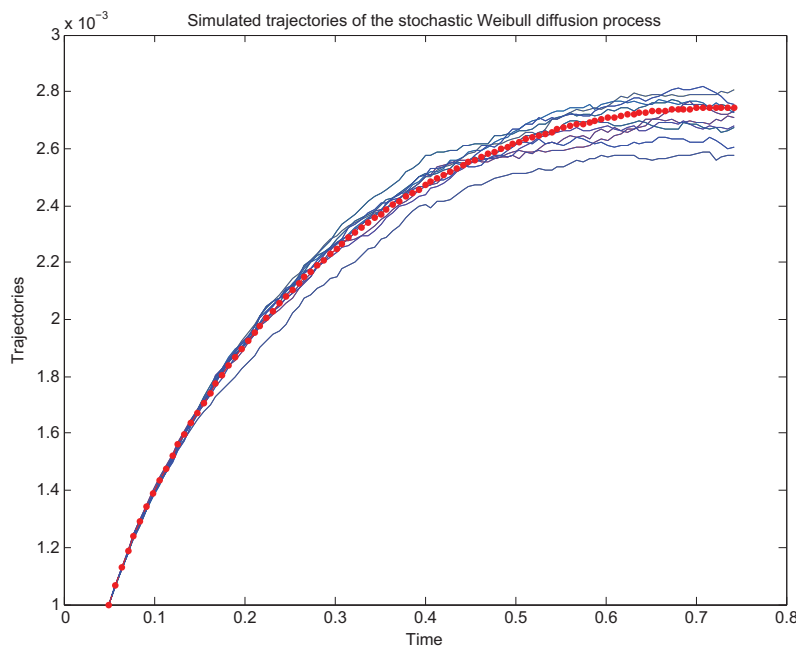
Fig. 1 Simulated trajectories of the stochastic Weibull diffusion process and its estimated trend function

TABLE II
MEAN AND STANDARD ERROR OF $\hat{\beta}$

h	num.obs.	$mean(\beta)$	$SE(\beta)$
0.007	100	0.8183	0.0921
0.007	500	0.8046	0.0137
0.007	1000	0.8129	0.0218
0.0035	100	0.7894	0.2394
0.0035	500	0.7879	0.0354
0.0035	1000	0.7637	0.0964
0.0007	100	1.3072	0.8539
0.0007	500	0.8358	0.1891
0.0007	1000	1.0253	0.3835

TABLE III
MEAN AND STANDARD ERROR OF $\hat{\sigma}$

h	num.obs.	$mean(\sigma)$	$SE(\sigma)$
0.007	100	0.0394	0.0025
0.007	500	0.0401	0.0008
0.007	1000	0.0403	0.0009
0.0035	100	0.0380	0.0021
0.0035	500	0.0404	0.0008
0.0035	1000	0.0399	0.0009
0.0007	100	0.0380	0.0030
0.0007	500	0.0398	0.0015
0.0007	1000	0.0405	0.0011

in (10). In this simulation we assume $h = 0.007$, $N = 100$, $M = 10$.

V. CONCLUSION

The methodology introduced, based on stochastic nonhomogeneous Weibull diffusion process, which perform the possibility to incorporate exogenous factors. From a theoretical point of view, we conclude that the bi-parameter Weibull process presented, which is of a nonhomogeneous nature, is such that we can explicitly establish its probability transition density function in terms of a log-normal distribution (6) together with its moment functions, and in particular its trend functions (10). We can also establish parameter estimation results using the maximum likelihood method and construct approximated confidence intervals, on the basis of discrete sampling. Therefore, the bi-parameter Weibull process we described is followed by a set of statistical results that enable it to be applied to real data.

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