Necessary Conditions of Optimality for Abnormal Problems with Equality and Inequality Constraints[1]

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The aim of this paper is to derive second-order necessary conditions for an abnormal local minimizer of problem $(\mathcal{P})$ which improve the ones presented earlier, [1, 2].

We consider the following mathematical programming problem:

$$f(x) \to \min, \quad F_1(x) = 0, \quad F_2(x) \le 0. \qquad (\mathcal{P}).$$

Here, X is a linear space, and $f : X \to R^1$, $F_1 : X \to R^{k_1}$, and $F_2 : X \to R^{k_2}$ are given mappings (where k_1 and k_2 are fixed) assumed to be smooth in a sense to be specified.

If x_0 is abnormal, i.e. $\operatorname{im} \frac{\partial F_1}{\partial x}(x_0) \ne R^{k_1}$, then it is not difficult to find a simple example revealing that the usual second-order necessary conditions do not hold in general.

Moreover, in [2], meaningful second order necessary conditions for problem $(\mathcal{P})$ were obtained without a priori normality assumptions of the point x_0. In order to formulate these results from [2], let us introduce the Lagrange function $\mathcal{L} : X \times R^1 \times R^{k_1} \times R^{k_2} \to R^1$ of problem $(\mathcal{P})$

$$\mathcal{L}(x, \lambda) = \lambda^0 f(x) + \langle \lambda^1, F_1(x)\rangle + \langle \lambda^2, F_2(x)\rangle,$$

$$\lambda = (\lambda^0, \lambda^1, \lambda^2),\ \lambda^0 \in R^1,\ \lambda^1 \in R^{k_1},\ \lambda^2 \in R^{k_2}.$$

Let x_0 be a local minimizer for problem $(\mathcal{P})$, and the mappings F_i and f be twice continuously differentiable. Denote by $\Lambda(x_0)$ the set of all Lagrange multipliers $\lambda = (\lambda^0, \lambda^1, \lambda^2)$ satisfying the Lagrange multipliers rule at the point x_0:

$$\frac{\partial \mathcal{L}}{\partial x}(x_0, \lambda) = 0, \quad \lambda^0 \ge 0, \quad \langle \lambda^2, F_2(x_0)\rangle = 0, \quad \lambda^2 \ge 0, \quad |\lambda| = 1.$$

For the sake of simplicity assume that $F_2(x_0) = 0$, and denote by Λ_a the set of all Lagrange multipliers $\lambda \in \Lambda(x_0)$ for which there exists a linear subspace $\Pi = \Pi(\lambda) \subseteq X$ satisfying

$$\Pi \subseteq \ker \frac{\partial F_1}{\partial x}(x_0) \cap \ker \frac{\partial F_2}{\partial x}(x_0), \quad \operatorname{codim} \Pi \le k_1 + k_2, \quad \frac{\partial^2 \mathcal{L}}{\partial x^2}(x_0, \lambda)[x, x] \ge 0,\ \forall\, x \in \Pi,$$

where codim means codimension of a linear subspace.

In [2], it was proved that, for any vector $h \in X$ satisfying $\frac{\partial F_1}{\partial x}(x_0)h = 0$, $\frac{\partial F_2}{\partial x}(x_0)h \le 0$, and $\langle \frac{\partial f}{\partial x}(x_0), h\rangle \le 0$, there exists a Lagrange multiplier $\lambda \in \Lambda_a(x_0)$ (depending on h), such that

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(x_0, \lambda)[h, h] \ge 0.$$

These necessary conditions are a natural generalization of the classical ones, [3], in the abnormal case. Note that the non-emptiness of the set $\Lambda_a(x_0)$ is by itself a significant necessary optimality condition.

The above mentioned result in [2] was afterwards generalized in [4] (with the help of the technique developed in [5]) to a problem featuring the more general set-inclusion constraints $F(x) \in C$, where the set C is assumed to be merely closed. On the other hand, the necessary optimality conditions for the problem with only equality type constraints were, under the additional assumption of abnormality of the point x_0, strengthened in [6]. More specifically, in this reference, it is shown that iff́ the local minimizer x_0 is abnormal, then, the set $\Lambda_r(x_0)$ in the necessary optimality conditions presented above can be replaced

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by the smaller set that contains all $\lambda \in \Lambda(x_0)$ such that $|\lambda| = 1$ and for which there exists a linear subspace $\Pi = \Pi(\lambda) \subseteq X$ satisfying:

$$\Pi \subseteq \ker \frac{\partial F_1}{\partial x}(x_0),\ \operatorname{codim}\Pi \leq k_1 - 1,\ \frac{\partial^2 \mathcal{L}}{\partial x^2}(x_0, \lambda)[x, x] \geq 0,\ \forall x \in \Pi.$$

The main goal of this article is to extend this result to abnormal minimizers of the more general mathematical programming problem $(\mathcal{P})$. The approach we use is based on a perturbation method developed in [2] and on methods of real algebraic geometry, [7]

In order to state our main result, we consider $x_0 \in X$ to be the local minimizer in problem $(\mathcal{P})$, and assume that the mappings f, F_1 and F_2, are twice continuously differentiable in a neighborhood of x_0 with respect to the finite topology. By this, we mean that, for any finite dimensional subspace $M \in \mathcal{M}$ containing the point x_0, the restrictions of f, F_1 and F_2 to M are twice continuously differentiable in some (M-dependent) neighborhood of vector x_0.

We also consider the Lagrange multipliers λ, the Lagrange function $\mathcal{L} : X \times R^1 \times R^{k_1} \times R^{k_2} \to R^1$ and the set $\Lambda(x_0)$ defined as above. Now, denote by $I = I(x_0)$ the set of all indexes $i \in \{1, .., k_2\}$ such that $F_2^i(x_0) = 0$. ($F_s^i(x)$ are the coordinates of the vector $F_s(x)$, $s = 1, 2$). For an integer nonnegative number r, we denote by $\Lambda_r(x_0)$ the set of vectors $\lambda \in \Lambda(x_0)$ such that there exists a linear subspace $\Pi = \Pi(\lambda)$ satisfying

$$\Pi(\lambda) \subseteq \ker \frac{\partial F_1}{\partial x}(x_0) \bigcap \Big(\bigcap_{i \in I} \ker \frac{\partial F_2^i}{\partial x}(x_0) \Big) \text{ such that } \operatorname{codim}\Pi \leq r, \text{ and } \frac{\partial^2 \mathcal{L}}{\partial x^2}(x_0, \lambda)[x, x] \geq 0 \ \ \forall x \in \Pi.$$

Consider the cone of critical directions at point x_0:

$$\mathcal{K}(x_0) = \Big\{ x \in X : \ \Big\langle \frac{\partial f}{\partial x}(x_0), x \Big\rangle \leq 0,\ \frac{\partial F_1}{\partial x}(x_0)x = 0,\ \Big\langle \frac{\partial F_2^i}{\partial x}(x_0), x \Big\rangle \leq 0,\ i \in I \Big\}.$$

Put $k = k_1 + |I(x_0)|$, where $|I|$ denotes the number of elements in the set I. We shall say that a point x_0 is *abnormal*, if $k > 0$ and the vectors $\frac{\partial F_1^j}{\partial x}(x_0)$, $j = 1, .., k_1$, $\frac{\partial F_2^i}{\partial x}(x_0)$, $i \in I$ are linearly dependent.

Theorem 2.1 Let the point x_0 be a local minimizer for problem $(\mathcal{P})$. Assume that x_0 is an abnormal point. Then, $\Lambda_{k-1}(x_0) \neq \emptyset$, and the following inequality holds for any vector $h \in \mathcal{K}(x_0)$

$$\max_{\lambda \in \Lambda_{k-1}(x_0)} \frac{\partial^2 \mathcal{L}}{\partial x^2}(x_0, \lambda)[h, h] \geq 0.$$

This theorem only deals with abnormal minimizers. However, if a minimizer is normal then second-order necessary conditions are well known. See for example [2, 3]. These classical second-order necessary conditions do not hold for abnormal minimizers, and so, the issues of when do classical second-order necessary conditions still follow from our theorem and when it is possible to use an universal Lagrange multiplier in our main result by omitting the maximum operation arise.

References

[1] Arutyunov, A.V.: Optimality conditions in abnormal extremal problems. Systems & Control Letters, **27** (5), 279–284 (1996).

[2] Arutyunov, A.V.: *Optimality Conditions: Abnormal and Degenerate Problems.* Kluwer Academic Publishers, Dordrecht/Boston/London (2000).

[3] Alekseev, V.M., Tikhomirov, V.M., Fomin, S.V.: Optimal Control. Consultants Bureau, New York (1987).

[4] Arutyunov, A.V., Pereira, F.L.: Second-order necessary optimality conditions for problems without a priori normality assumptions. Mathematics of Operations Research, INFORMS, **31** (1), 1–12 (2006)

[5] Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation I and II, Springer (2006).

[6] Arutyunov, A.V., Karamzin, D.Yu.: Necessary Optimality Conditions in an Abnormal Optimization Problem with Equality Constraints. Computational Mathematics and Mathematical Physics, **46** (8), 1293–1298 (2006).

[7] Bochnak, J., Coste, M., Roy, M.F.: Real Algebraic Geometry. Springer, A Series of Modern Surveys in Mathematics, (1998).