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Keywords:. Positive Invariance, Constrained Systems, Linear Sampled-data Dynamical Systems. Positive Invariance, Constrained Systems, Linear Sampled-data Dynamical Systems.

SYSTEC Report 2017-SC4, July 2017.

This is a preprint of an article that was mentioned as an output of research produced within the FCT project PTDC-EEI-AUT/2933-2014|16858—TOCCATA, namely in the activities report of Dr. Sasa Rakovic when he was a PostDoc scholarship holder at Univ. Porto, funded by the FCT project. This preprint is placed in the U.Porto Repository with open access to comply with the requirements of making the project research available.

Positive Invariance for Linear Sampled–data Systems

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Abstract

The sampled–data framework captures conventional sampled–data systems, and it also provides an adequate representational tool for contemporary cyber–physical and smart autonomous systems. One of the major concerns for analysis and synthesis of such systems is safe operation under constraints. This paper contributes to resolving this cornerstone aspect by focusing on related positive invariance notions within sampled–data setting. More specifically, we introduce generalized positive invariance notions that are topologically compatible with sampled–data framework and that overcome inevitable conservatism of the classical positive invariance notions. We propose exact generalized positive invariance and complement it with the guaranteed generalized positive invariance. The former notion is topologically nonconservative, while the latter notion is approximate and guaranteed but it leads to finitely parameterizable and practicaly computable generalized positively invariant sets. The limiting behaviour and computational aspects are also discussed, and an example illustrating the proposed notions is provided.

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1 Introduction

The contemporary engineered systems make extensive use of the interface between cyber and physical realms, and frequently operate in autonomous mode. Indeed, the cyber–phyisical and smart autonomous systems build the bridge between digital and actual worlds in order to push performance envelopes to previously daring limits, and to facilitate previously unattainable operational capabilities. Naturally, the digital realm is almost always utilized for a variety of computational tasks encapsulating, *inter alia*, discovery of the actual reality, gathering information about operating environments and processing observed data as well as related decision making leading to enhanced and optimally tunned performance of the overall system. A variety of systems typically perceive continuous appearance of the physical world, and translate the observed data to its digital representation in order to process it effectively with the aid of available computing units [1]. The interface between continuous reality and its digital representation is naturally captured by the important framework of sampled–data systems [2]. Not surprisingly, the very nature of cyber–physical systems and the ever increasing levels of autonomy and sophistication as well as nonconventional contemporary demands and performance specifications call for suitable extensions of classical approaches to the analysis and synthesis of sampled–data systems. Interestingly enough, even the nonconservative and systematic analysis and synthesis of sampled–data systems subject to classical constraints and conventional uncertainty offers nontrivial challenges. These existing research issues are further amplified by contemporary and nonconventional demands for a prior–to–operation assurances of, at least, safe, resilient, secure and fault tolerant operability.

The systematic treatment of classical and contemporary performance constraints and objectives requires utilization of adequately sophisticated analysis and synthesis tools. An almost ideal platform for developing appropriate analysis and synthesis methodologies for versatile modern systems is provided by reachability and set invariance or, more generaly, by set–valued analysis. The corresponding set–valued methods [3–7] have proved their beneficial value over a wide spectrum of fundamental problems within fields of dynamics, controls and systems in general, and deterministic/robust/stochastic optimal and model predictive control syntheses as well as safety and stability analyses in particular. Not surprisingly, influential studies [4, 5, 7–14] have reached a clear conclusion that the analysis of dynamics, and the synthesis of control systems, under constraints and uncertainty by utilizing reachability and set invariance techniques enables one to provide formal and *a–priori* guarantees of the desired structural properties.

Within the context of constrained dynamics one of the

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most important concerns is safe operation under constraints. Likewise, within the setting of constrained control systems the related key issue is the design of control laws ensuring safe operation under constraints. The ability to operate safely under constraints is mathematically captured by the notions of safe sets and positively invariant sets in the case of constrained dynamics as well as control safe sets and control invariant sets in the case of constrained control systems. Naturally, the prime consideration is given to the maximal such sets for obvious reasons (one of which is the deployment of the largest domains of autonomous/controlled safe operability). Even the early literature has provided answers to the important questions concerned with the characterization and computation of maximal positively invariant sets [8] also known as viability kernels [4]. As a matter of fact, the topics of characterization and computation of safe sets, positively invariant sets, control safe sets and control invariant sets have formed an improtant research area; See above mentioned articles [4, 5, 7–14], and numerous references therein, for a more detailed overview of these basics notions and their robust variants applicable to the systems under uncertainty. The rather rich literature offers a plethora of results for characterizing and computing control safe, safe, control invariant and positively invariant sets. However, the major part of the existing and ellaborate research in the field focuses almost exclusively on either discrete or continuous times formulations. Rather surprisingly, there is a limited literature addressing, to a sufficient extent, the related research questions within the setting of sampled–data systems. Indeed, the literature on reachability and set invariance within sampled–data setting is very scarce with the exception of a sequence of more recent articles [15–19].

As already pointed out, the sampled–data setting arises naturally in many traditional and modern applications, and it is applicable to the conventional sampled–data systems as well as to the contemporary cyber–physical and smart autonomous systems. More importantly, sampled–data setting plays a key role for enabling analysis of dynamics and synthesis of control systems to be carried out using discrete time techniques while ensuring constraint satisfaction in continuous time sense. However, a major peculiarity of sampled–data setting is inapplicability, to a prohibiting extent, of discrete and continuous time reachability and set invariance [19]. Indeed, the formulations of the standard reachability and set invariance problems in either continuous or discrete time preserve the underlying semi–group property and, thus, allow for the utilization of the conventional notions. This is in a stark contrast to sampled–data setting in which the control actions are evaluated and implemented at prescribed sampling time instances and kept constant throughout sampling intervals, while the satisfaction of underlying constraints is imposed onto pieces of state and control trajectories throughout whole sampling intervals. Unfortunately, such a setting naturally leads to a lack of semi–group property and renders classical notions conservative and, in fact, inapplicable.

In this paper, we introduce a topologically compatible and flexible notion of positive invariance for constrained linear sampled–data systems controlled by linear sampled–data feedbacks. The characterization and computation of the ordinary and maximal positively invariant sets for constrained linear dynamics have been addressed within both discrete and continuous time settings [4, 5, 7, 8, 10]. However, as already hinted, these standard approaches lead to a conservative and/or noncomputable notions of ordinary and positively invariant sets within sampled–data setting. In particular, in discrete time setting the controls are implemented, and the constraints are imposed, at the discrete times. Likewise, in continuous time setting, the control is applied, and the constraints are invoked, at all times. In sampled–data setting, the controls are implemented in discrete time sense while the constraints are imposed in the continuous time sense. Thus, we introduce notions of generalized positively invariant sets. These sets are positively invariant in discrete time sense (the related state trajectories belong/return to such sets at the sampling instances) and safe in continuous time sense (the related state trajectories might leave such sets during sampling intervals but never leave original state constraint set). The rational behind such generalized and relaxed notion is topological compatability with the underlying sampled–data control process: controls are updated at sampling instances and kept constant until successor sampling instance, so related set should be positively invariant at these sampling times, while no violation of original state constraints should be allowed throughout sampling intervals. Strictly speaking any such set is not positively invariant in the conventional sense. However, as detailed in what follows, any such set induces a positively invariant family of sets which is the generalized notion originally introduced for output feedback set invariance in a relatively recent article [20].

We consider exact and guaranteed notions of generalized positive invariance, and we show that the exact notion (as the wording suggests) does not introduce any conservatism and, thus, it is theoretically “optimal”. Unfortunately, the exact generalized positively invariant sets are obtained by considering discrete time dynamics subject to semi–infinite constraints and, thus, such sets might fail to be finitely parameterizable and practically computable. This computationally crucial issue is handled by introducing the notion of a guaranteed generalized positive invariance. This notion leads to generalized positively invariant sets that preserve finite parameterizability and enhance practical computability. In fact, we derive natural and mild conditions under which such sets are computable by using purely techniques for discrete time positive invariance; More precisely, by utilizing discrete time dynamics subject to finitely parameterizable constraints. Furthermore, we establish formally that the maximal guaranteed generalized positively invariant set can be constructed to be finitely parameterizable, practically computable and arbitrarily close approximation, w.r.t. the Hausdorff distance, of the related maximal exact generalized positively invariant set. The main philosophy of our approach is to convert the sampled–data positive invariance problem into a suitably formulated discrete time positive invariance problem. To realize our approach, we consider a standard sampling of the dynamics and controls, and we also allow for an artificial subsampling of the state constraints. The former sampling is implemented in a manner customary to sampled–data systems, while the latter subsampling is used only offline for analysis; More precisely, for the definition of modified constraint sets that are specified by invoking the state constraints at a finite number of subsampling instances (thus in discrete time sense), and that ensure the satisfaction of state constraints throughout sampling intervals (thus in continuous time sense).

Paper Structure: Section 2 describes a constrained linear system controlled by linear state feedback in a sampled–data setting. It also comments on plausible notions of positively invariant sets, and it outlines the goals of this paper. Section 3 focuses on characterization and computability of the ordinary and maximal exact generalized positively invariant sets. Section 4 introduces approximate but finitely parametrizable and guaranteed generalized positively invariant sets. Section 5 discusses computational aspects of the proposed notions and delivers an illustrative example. Section 6 draws conclusions and comments on extensions and future research.

Basic Nomenclature and Definitions: The sets of nonnegative integers and real numbers are denoted by $\mathbb{Z}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$, respectively. Any given sampling period $T \in \mathbb{R}_{\geq 0}$, $T > 0$ induces sequences of sampling instances π and sampling intervals θ both w.r.t. $\mathbb{R}_{\geq 0}$ specified via:

$$\pi := \{t_k\}_{k\in\mathbb{Z}_{\geq 0}} \text{ and } \theta := \{\mathcal{T}_k\}_{k\in\mathbb{Z}_{\geq 0}}, \text{ where } \forall k \in \mathbb{Z}_{\geq 0}, t_{k+1} := t_k + T \text{ with } t_0 := 0 \text{ and } \mathcal{T}_k := [t_k, t_{k+1}).$$

Given a set $\mathcal{X}$ and a real matrix M of compatible dimensions the image of $\mathcal{X}$ under M is denoted by

$$M\mathcal{X} := \{Mx \ : \ x \in \mathcal{X}\},$$

while the preimage of $\mathcal{X}$ under M is denoted by

$$M^{-1}\mathcal{X} := \{x \ : \ Mx \in \mathcal{X}\}.$$

For any two sets $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{R}^n$, the Minkowski set addition is specified by

$$\mathcal{X} \oplus \mathcal{Y} := \{x + y \ : \ x \in \mathcal{X},\ y \in \mathcal{Y}\}.$$

A set $\mathcal{X}$ in $\mathbb{R}^n$ is *a C–set* if it is compact, convex, and contains the origin. A set $\mathcal{X}$ in $\mathbb{R}^n$ is *a proper C–set* if it is a C–set and contains the origin in its interior. A set in $\mathbb{R}^n$ described by a finitely many constraints is referred to as the finitely parameterizable set. Likewise, a set in $\mathbb{R}^n$ described by a uncountably infinite number of constraints is referred to as the semi–infinite set.

Given any two compact sets $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{R}^n$, Hausdorff distance is defined by

$$\mathrm{H}_{\mathcal{L}}(\mathcal{X}, \mathcal{Y}) := \min_{\alpha \geq 0}\{\alpha \ : \ \mathcal{X} \subseteq \mathcal{Y} \oplus \alpha\mathcal{L} \text{ and } \mathcal{Y} \subseteq \mathcal{X} \oplus \alpha\mathcal{L}\},$$

where $\mathcal{L}$ is a given symmetric proper C–set in $\mathbb{R}^n$ inducing vector norm

$$|x|_{\mathcal{L}} := \min_{\eta}\{\eta \ : \ x \in \eta\mathcal{L},\ \eta \geq 0\}.$$

$\rho(M)$ denotes the spectral radius of a matrix $M \in \mathbb{R}^{n\times n}$. $\varrho_{\mathcal{L}}(\mathcal{X})$ denotes the radius of a compact set $\mathcal{X}$ in $\mathbb{R}^n$, and it is specified by

$$\varrho_{\mathcal{L}}(\mathcal{X}) := \mathrm{H}_{\mathcal{L}}(\mathcal{X}, \{0\}).$$

We work with nonempty sets unless stated otherwise. We distinguish row and column vectors only when needed and we use the same symbol for a variable x and its vectorized form. For clarity of presentation, we provide proofs of less obvious statements in the appendices.

2 Background

2.1 Constrained Sampled–data Linear Dynamics

Consider a linear system described, for all $t \in \mathbb{R}_{\geq 0}$, by:

$$\dot{x}_t = Ax_t + Bu_t, \tag{2.1}$$

where, for any time $t \in \mathbb{R}_{\geq 0}$, $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ denote, respectively, state and control values, while $\dot{x}_t$ denotes the value of the state derivative w.r.t. time, and the matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known exactly. The linear system (2.1) is controlled via sampled–data linear state feedback so that

$$\forall k \in \mathbb{Z}_{\geq 0},\ \forall t \in \mathcal{T}_k,\ u_t(x_t) := Kx_{t_k}, \tag{2.2}$$

where $K \in \mathbb{R}^{m\times n}$ is a known exactly control gain matrix. We stress that the control u_t at time instance t in the sampling intervals $(t_k, t_k + T)$ is not a function of the state x_t at that time instance, rather it is a function of the state at the last sampling instance so that $u_t = u_t(x_{t_k})$. To define sampled–data solutions, let, for any $t \in [0, T]$,

$$A_t := e^{tA} \text{ and } B_t := \left(\int_0^t e^{\sigma A} d\sigma\right) B, \tag{2.3}$$

where the related integral is the standard matrix–valued integral, and let also, for any $t \in [0,T]$,

$$M_t := A_t + B_t K. \tag{2.4}$$

In view of (2.1)–(2.4), given any initial state $x \in \mathbb{R}^n$, the sampled–data solutions satisfy at the sampling instances t_k, for all $k \in \mathbb{Z}_{\geq 0}$,

$$x_{t_k} = M_T^k x, \tag{2.5}$$

while during sampling intervals $\mathcal{T}_k$, the sampled–data solutions satisfy, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$,

$$x_{t_k+t} = M_t x_{t_k}. \tag{2.6}$$

The hard constraints imposed on the values of state and control trajectories, x_t and u_t, are, for all $t \in \mathbb{R}_{\geq 0}$,

$$x_t \in \mathbb{X} \text{ and } u_t \in \mathbb{U}. \tag{2.7}$$

Herein, we work under the following natural conditions.

Assumption 1 *The sampling period T is such that the matrix pair (A_T, B_T) is strictly stabilizable. The control matrix K is such that the matrix $M_T = A_T + B_T K$ is strictly stable. The state and control constraint sets, $\mathbb{X}$ and $\mathbb{U}$, are proper C–sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.*

2.2 Positive Invariance in Discrete Time Sense

Within the above setting, a positively invariant set in discrete time sense is any set $\mathcal{S}$ in $\mathbb{R}^n$ satisfying that

$$\forall x \in \mathcal{S},\ M_T x \in \mathcal{S},\ x \in \mathbb{X} \text{ and } Kx \in \mathbb{U}$$

or equivalently and more compactly written

$$M_T \mathcal{S} \subseteq \mathcal{S},\ \mathcal{S} \subseteq \mathbb{X} \text{ and } K\mathcal{S} \subseteq \mathbb{U}. \tag{2.8}$$

Positive invariance in discrete time sense for sampled–data setting is, in fact, identical to positive invariance for constrained discrete time linear dynamics. The latter topic is a thoroughly studied and deeply understood research theme in set invariance [5, 10], with a plethora of conceptual and concrete algorithmic methods for the design of ordinary as well as maximal positively invariant sets for the dynamics $x^+ = M_T x$ and the constraint set $\{x \in \mathbb{X} \ : \ Kx \in \mathbb{U}\}$. However, despite potential theoretical and computational convenience, positive invariance in discrete time sense is not really applicable to sampled–data setting, as it does not allow one to assert that, in general, the related sampled–data solutions actually satisfy constraints as specified in (2.7).

2.3 Positive Invariance in Sampled–data Sense

A direct, topologically inflexible and almost surely conservative, attempt to obtain a sampled–data analogue of positive invariance in discrete time sense would be to demand that a set $\mathcal{S}$ in $\mathbb{R}^n$ satisfies that

$$\forall x \in \mathcal{S},\ \forall t \in [0,T],\ M_t x \in \mathcal{S},\ x \in \mathbb{X} \text{ and } Kx \in \mathbb{U}$$

or equivalently and more compactly written

$$\forall t \in [0,T],\ M_t \mathcal{S} \subseteq \mathcal{S},\ \mathcal{S} \subseteq \mathbb{X} \text{ and } K\mathcal{S} \subseteq \mathbb{U}. \tag{2.9}$$

The main problem with such an approach stems from the fact that an uncountably infinite number of dynamic, positive invariance related, conditions is imposed on a single set, i.e. $\forall t \in [0,T],\ M_t \mathcal{S} \subseteq \mathcal{S}$. Clearly, the fact that the controls are updated at the sampling instances suggests that it is overly optimistic (or even impossible) to expect the related sampled–data state trajectories to remain with a nontrivial set $\mathcal{S}$ throughout the whole sampling interval. Indeed, such a requirement is overly conservative and particularly so when the matrix A is unstable and the sampling period T is of practical size.

A generalized, and topologically compatible, notion of positive invariance in sampled–data sense would be the one that guarantees positive invariance in discrete time sense as well as safety, or positive invariance if it is attainable, in continuous time sense. Such a notion would require a set $\mathcal{S}$ in $\mathbb{R}^n$ to satisfy that

$$\forall x \in \mathcal{S},\ M_T x \in \mathcal{S},\ \forall t \in [0,T),\ M_t x \in \mathbb{X} \text{ and } Kx \in \mathbb{U}$$

or equivalently and more compactly written

$$M_T \mathcal{S} \subseteq \mathcal{S},\ \forall t \in [0,T),\ M_t \mathcal{S} \subseteq \mathbb{X} \text{ and } K\mathcal{S} \subseteq \mathbb{U}. \tag{2.10}$$

Strictly speaking, the above conditions postulate positive invariance in discrete time sense and safety in continuous time sense. In plain words, the trajectories commencing in $\mathcal{S}$ are allowed to leave the set $\mathcal{S}$ during the sampling intervals as long as they do not leave the state constraint set $\mathbb{X}$, and are, in addition, required to return to the set $\mathcal{S}$ at the sampling instances. Consequently, such a set should not really be refereed to as a positively invariant set in the conventional sense. Nevertheless, letting, for any such set $\mathcal{S}$,

$$\forall t \in [0,T],\ \mathcal{S}_t := M_t \mathcal{S} \tag{2.11}$$

it follows that a family $\mathfrak{S}$ of sets $\mathcal{S}_t$, induced by the set $\mathcal{S}$ and specified by

$$\mathfrak{S} := \{\mathcal{S}_t \ : \ t \in [0,T]\} \tag{2.12}$$

is a positively invariant family of sets in the sense that

$$\forall t \in [0,T],\ M_t\mathcal{S} = \mathcal{S}_t \in \mathfrak{S} \text{ and } \mathcal{S}_T \subseteq \mathcal{S}_0 = \mathcal{S} \in \mathfrak{S},$$
$$\forall t \in [0,T],\ \mathcal{S}_t \subseteq \mathbb{X} \text{ and } K\mathcal{S}_0 \subseteq \mathbb{U}. \tag{2.13}$$

Thus, the latter notion is, indeed, a generalized and relaxed notion of positive invariance within a sampled–data setting. For this reason, typographical convenience as well as for technical simplicity, reflected by our preference towards the utilization of a single set instead of a family of sets induced by an adequate set, we deploy a term "generalized positively invariant set" for any set $\mathcal{S}$ satisfying conditions (2.10). No confusion should arise.

2.4 Objectives

Our main objectives are to analyze the characterization and computation of ordinary and maximal generalized positively invariant sets. In particular, we focus on:

- The exact, semi–infinite, characterization, and computation, of the maximal generalized positively invariant set.
- The guaranteed, finitely paramaterizable, characterization, and computation, of the positively invariant approximations of the maximal generalized positively invariant set.

3 Exact Generalized Positive Invariance

The characterization and computation of an ordinary or the maximal generalized positively invariant set for sampled–data setting can be transformed to a problem of characterizing and computing an ordinary or the maximal positively invariant set for discrete time setting. Indeed, such a transformation is possible by an adequate, and, in fact, equivalent reformulation of state and control constraints imposed on sampled–data system at all times to adequately modified state and control constraints imposed on sampled–data system at the sampling instances instead of all times.

3.1 Exact Semi–infinite Constraint Set

The state constraints on sampled–data solutions emanating in a generalized positively invariant set $\mathcal{S}$ induce state constraint admissibility conditions taking form

$$\forall t \in [0,T),\ M_t\mathcal{S} \subseteq \mathbb{X}, \tag{3.1}$$

which can be equivalently rewritten as

$$\forall t \in [0,T),\ \mathcal{S} \subseteq M_t^{-1}\mathbb{X}. \tag{3.2}$$

Thus, letting,

$$\mathbb{X}_S := \bigcap_{t\in[0,T)} M_t^{-1}\mathbb{X} \tag{3.3}$$

the state constraint admissibility conditions can be compactly and equivalently written as the set inclusion

$$\mathcal{S} \subseteq \mathbb{X}_S. \tag{3.4}$$

Likewise, letting,

$$\mathbb{X}_K := \{x \in \mathbb{R}^n \ :\ Kx \in \mathbb{U}\}, \tag{3.5}$$

the control constraint admissibility conditions read as

$$\mathcal{S} \subseteq \mathbb{X}_K. \tag{3.6}$$

Thus, letting,

$$\mathbb{X}_E := \mathbb{X}_S \bigcap \mathbb{X}_K, \tag{3.7}$$

the exact overall constraint admissibility conditions are simply stated as

$$\mathcal{S} \subseteq \mathbb{X}_E. \tag{3.8}$$

3.2 Exact Generalized Positively Invariant Sets

The condition for generalized positive invariance, specified in (2.10), takes the following simplified form

$$M_T\mathcal{S} \subseteq \mathcal{S} \text{ and } \mathcal{S} \subseteq \mathbb{X}_E. \tag{3.9}$$

Now, the definition of $\mathbb{X}_E$ in (3.7) is exact in the sense that no conservatism is induced by converting conditions (2.10) to aggregated form (3.9). Thus, we simply use the term "exact generalized positively invariant set" for any set $\mathcal{S}$ satisfying conditions (3.9). This convention applies to both ordinary and maximal such sets.

The latter conditions (3.9) are, in fact, analogue to the positive invariance conditions for discrete time dynamics $x^+ = M_Tx$ subject to constraints $x \in \mathbb{X}_E$. However, a key point here is that, even when the sets $\mathbb{X}$ and $\mathbb{U}$ are finitely parameterizable, the set $\mathbb{X}_S$ is an uncountably infinite intersection of finitely parameterizable sets. Thus, the overall state constraint set $\mathbb{X}_E$ is itself a semi–infinite set described by the uncountably infinite number of conditions taking the form $M_t\mathcal{S} \subseteq \mathbb{X}$. This fact does not represent major obstacle for the characterization of the exact generalized positively invariant sets, as indeed any such set satisfies set inclusions of (3.9) or, equivalently, the set inclusion

$$\mathcal{S} \subseteq \left(M_T^{-1}\mathcal{S}\right) \bigcap \mathbb{X}_E. \tag{3.10}$$

Furthermore, the characterization of the maximal exact generalized positively invariant set, say $\mathcal{S}_E^\infty$, can also be derived with relative ease. Namely, the maximal exact generalized positively invariant set can be characterized, and potentially computed, by employing the standard set recursion [4, 5, 8, 10] specified within the sampled–data setting, for all $k \in \mathbb{Z}_{\geq 0}$, by

$$\mathcal{S}_{k+1} := \left(M_T^{-1}\mathcal{S}_k\right) \bigcap \mathbb{X}_E \text{ with } \mathcal{S}_0 := \mathbb{X}_E. \quad (3.11)$$

In fact, the maximal exact generalized positively invariant set is the maximal set, w.r.t. set inclusion, such that

$$\mathcal{S} = \left(M_T^{-1}\mathcal{S}\right) \bigcap \mathbb{X}_E. \quad (3.12)$$

By a direct application of well–known results concerning the maximal positively invariant set for discrete time setting [4, 5, 8, 10] , the maximal exact generalized positively invariant set is, in fact, the Hausdorff limit of the set sequence $\{\mathcal{S}_k\}_{k\geq 0}$ generated by (3.11) and thus, since the sets $\mathcal{S}_k$ are monotonically nonincreasing (i.e. $\forall k \geq 0,\ \mathcal{S}_{k+1} \subseteq \mathcal{S}_k$) by construction, it is also given by

$$\mathcal{S}_E^\infty = \bigcap_{k\geq 0} \mathcal{S}_k. \quad (3.13)$$

In our setting, the iterates $\mathcal{S}_k$ of the set recursion (3.11) are guaranteed to be proper C–sets in $\mathbb{R}^n$ and, furthermore, the existence of a finite integer k^* such that $\mathcal{S}_{k^*} = \mathcal{S}_{k^*+1}$ and, thus, $\mathcal{S}_E^\infty = \mathcal{S}_{k^*}$ is also guaranteed. These facts are collected together by the following result.

Theorem 1 *Suppose Assumption 1 holds, and consider the set sequence $\{\mathcal{S}_k\}_{k\geq 0}$ generated by (3.11). The sets $\mathcal{S}_k,\ k \geq 0$ are proper C–sets in $\mathbb{R}^n$ such that for all $k \geq 0$ it holds that $\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$. Furthermore, there exists finite integer k^* such that*

$$\mathcal{S}_{k^*} = \mathcal{S}_{k^*+1} \quad (3.14)$$

and, thus, the maximal exact generalized positively invariant set $\mathcal{S}_E^\infty$ is given by

$$\mathcal{S}_E^\infty = \mathcal{S}_{k^*}. \quad (3.15)$$

4 Guaranteed Generalized Positive Invariance

Despite the conceptually elegant and finitely determined characterization of the maximal exact generalized positively invariant set $\mathcal{S}_E^\infty$, the semi–infinite nature of the overall state constraint set $\mathbb{X}_E$ (and, thus, the semi–infinite nature of the iterates $\mathcal{S}_k,\ k \geq 0$ and their limit $\mathcal{S}_E^\infty$) renders the associated algorithmic procedures for the construction of explicit or implicit forms of the iterates of set recursion (3.11) computationally highly demanding and almost surely intractable in the general case. This potential computational impracticability motivates development of numerically more applicable algorithmic schemes. A natural way forward is to characterize the positively invariant approximations of the maximal exact generalized positively invariant set $\mathcal{S}_E^\infty$ whose explicit or implicit forms can be computed effectively by means of finitely parameterizable sets. Our take on this issue aligns with this approach, and we employ an inner, finitely parameterizable, approximation of the semi–infinite set $\mathbb{X}_S$ specified in (3.3). To this end, sampling interval $[0, T)$ is further subpartitioned into subintervals $[k\delta, (k+1)\delta),\ k \in \mathbb{Z}_{q-1} := \{k \in \mathbb{Z}_{\geq 0} : k \leq q-1\}$. Then, the stringent state constraints $x_t \in (1-\alpha)\mathbb{X}$ are invoked at corresponding subsampling instances $k\delta,\ k \in \mathbb{Z}_{q-1}$,

$$\forall k \in \mathbb{Z}_{q-1},\ x_{k\delta} \in (1-\alpha)\mathbb{X}.$$

The scalar $\alpha \in [0, 1)$ is utilized, via scaling $(1-\alpha)$ of state constraint set $\mathbb{X}$, to enforce constraints $x_{k\delta+\tau} \in \mathbb{X}$ during subsampling intervals $[k\delta, (k+1)\delta),\ k \in \mathbb{Z}_{q-1}$,

$$\forall k \in \mathbb{Z}_{q-1},\ x_{k\delta} \in (1-\alpha)\mathbb{X} \ \Rightarrow$$
$$\forall k \in \mathbb{Z}_{q-1},\ \forall \tau \in (0, \delta),\ x_{k\delta+\tau} \in \mathbb{X}.$$

This scheme is applied to each sampling interval. Thus, from a conceptual point of view, our approach employs usual sampling of control constraints and controlled dynamics in conjunction with faster subsampling of state constraints. A more detailed discussion follows next.

4.1 Auxiliary Facts

The matrices A_t, B_t and M_t specified by (2.3) and (2.4) are continuous and are, in fact, uniformly continuous over the compact sampling interval $[0, T]$. This fact yields the following, and very helpful, observation linking the sampling and subsamplng periods T and δ.

Lemma 1 *Fix any $T \in (0, \infty)$, any $K \in \mathbb{R}^{m\times n}$, and consider the matrices A_t, B_t and M_t specified by (2.3) and (2.4). Then, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $T = q\delta$ for some $q \in \mathbb{Z}_{\geq 0}$ and*

$$\forall k \in \mathbb{Z}_{q-1},\ \forall \tau \in [0, \delta],\ \|M_{k\delta+\tau} - M_{k\delta}\| \leq \varepsilon. \quad (4.1)$$

For simplicity, the sampling and subsampling instances are, respectively, equispaced, and the sampling period T is an integer multiple of subsampling period δ, i.e. $T = q\delta$ for some finite integer $q \in \mathbb{Z}_{\geq 0}$.

Assumption 2 *The sampling period T is an integer multiple of subsampling period δ, i.e. $T = q\delta,\ q \in \mathbb{Z}_{\geq 0}$.*

The above selection of the sampling and subsampling periods, T and δ, reveals the following, and very helpful, relation between the matrices $M_{k\delta+\tau}$ and $M_{k\delta}$.

Proposition 1 *Suppose Assumptions 1 and 2 hold, and consider matrices A_t, B_t and M_t specified by (2.3) and (2.4). Then,*

$$\forall k \in \mathbb{Z}_{q-1}, \forall \tau \in [0, \delta],$$
$$M_{k\delta+\tau} = M_{k\delta} + A_{k\delta}(M_\tau - I). \quad (4.2)$$

4.2 Inner Finitely Parameterizable Constraint Set

To account for state constraint admissibility without invoking an uncountably infinite number of conditions of the form $\mathcal{S} \subseteq M_t^{-1}\mathbb{X}$ for all $t \in [0, T)$ as specified in (3.2), we postulate only finitely many, but more constricted, constraints

$$\forall k \in \mathbb{Z}_{q-1},\ M_{k\delta}\mathcal{S} \subseteq (1-\alpha)\mathbb{X} \text{ or, equivalently,}$$
$$\forall k \in \mathbb{Z}_{q-1},\ \mathcal{S} \subseteq M_{k\delta}^{-1}(1-\alpha)\mathbb{X} \tag{4.3}$$

where the scalar α is selected in $\alpha \in [0, 1)$. By defining

$$\mathbb{X}_F := \bigcap_{k\in\mathbb{Z}_{q-1}} M_{k\delta}^{-1}(1-\alpha)\mathbb{X}, \tag{4.4}$$

an equivalent form of (4.3) reads as

$$\mathcal{S} \subseteq \mathbb{X}_F. \tag{4.5}$$

Our goal is to select T, δ and α so that the set inclusion (4.5) guarantees state constraint admissibility of $\mathcal{S}$,

$$\forall k \in \mathbb{Z}_{q-1},\ M_{k\delta}\mathcal{S} \subseteq (1-\alpha)\mathbb{X} \ \Rightarrow$$
$$\forall k \in \mathbb{Z}_{q-1},\ \forall \tau \in (0, \delta),\ M_{k\delta+\tau}\mathcal{S} \subseteq \mathbb{X}, \tag{4.6}$$

without having to either invoke directly the latter conditions or to utilize explicitly the set $\mathcal{S}$. More precisely, we aim to provide conditions on T, δ and α that guarantee that the set $\mathbb{X}_F$ is an inner approximation of the semi–infinite set $\mathbb{X}_S$ of (3.7), i.e. that the set inclusion

$$\mathbb{X}_F \subseteq \mathbb{X}_S \tag{4.7}$$

is guaranteed without having to compute explicitly the semi–infinite set $\mathbb{X}_S$. To this end, we observe that by Proposition 1, for all $k \in \mathbb{Z}_{q-1}$ and all $\tau \in (0, \delta)$,

$$x_{k\delta+\tau} = M_{k\delta+\tau}x = M_{k\delta}x + A_{k\delta}(M_\tau - I)x. \tag{4.8}$$

Since relations $M_{k\delta}x \in (1-\alpha)\mathbb{X}$ for all $x \in \mathcal{S}$ are already stipulated, the sufficient conditions for ensuring $x_{k\delta+\tau} = M_{k\delta+\tau}x \in \mathbb{X}$ for all $x \in \mathcal{S}$ and, thus, for guaranteeing relation (4.7) are obtained by recognizing the fact that $\mathcal{S} \subseteq (1-\alpha)\mathbb{X}$ and requiring that $A_{k\delta}(M_\tau - I)x \in \alpha\mathbb{X}$ for all $x \in (1-\alpha)\mathbb{X}$.

Assumption 3 *The sampling period T, the subsampling period δ and the scalar $\alpha \in [0, 1)$ are such that the set inclusions*

$$\forall k \in \mathbb{Z}_{q-1}, \forall \tau \in (0, \delta),$$
$$A_{k\delta}(M_\tau - I)(1-\alpha)\mathbb{X} \subseteq \alpha\mathbb{X}. \tag{4.9}$$

hold true.

Remark 1 *In view of Lemma 1 and Proposition 1, the conditions* (4.9) *are natural and mild, as pointed out next. Let,*

$$\underline{\eta} := \max_\eta\{\eta \ : \ \eta\mathcal{B} \subseteq \mathbb{X},\ \eta \geq 0\},$$

and

$$\overline{\eta} := \min_\eta\{\eta \ : \ \mathbb{X} \subseteq \eta\mathcal{B},\ \eta \geq 0\},$$

where $\mathcal{B} := \{x \in \mathbb{R}^n \ : \ \|x\| \leq 1\}$ and $\|\cdot\|$ is compatible with the norm used in Lemma 1. Then, the conditions (4.9)*, are guaranteed to hold if $\alpha \in [0, 1)$ and $\delta > 0$ are chosen in order to ensure that*

$$\varepsilon\overline{\eta} \leq \alpha(1-\alpha)^{-1}\underline{\eta},$$

and that ε verifies relations (4.1)*. Clearly, a suitable choice of $\delta > 0$ is possible for all $\alpha \in (0, 1)$. Namely, in this case, the existence of the corresponding $\delta > 0$ is guaranteed by Lemma 1 for all $\varepsilon \in (0, \alpha(1-\alpha)^{-1}\underline{\eta}\overline{\eta}^{-1})$. Naturally, the above simplified guaranteed relations can be relaxed, possibly considerable, by employing the exact analogues based on the relations* (4.1)*,* (4.2) *and* (4.9)*.*

The relations (4.3) together with the conditions (4.9) lead to the desired set inclusion (4.7). In fact, the set inclusion of (4.7) can be extended and refined since, by construction, it holds that $\forall k \in \mathbb{Z}_{q-1},\ k\delta \in [0, T)$ and

$$\mathbb{X}_S = \bigcap_{t\in[0,T)} M_t^{-1}\mathbb{X} \subseteq \bigcap_{k\in\mathbb{Z}_{q-1}} M_{k\delta}^{-1}\mathbb{X}, \tag{4.10}$$

and since $\bigcap_{k\in\mathbb{Z}_{q-1}} M_{k\delta}^{-1}\mathbb{X} = (1-\alpha)^{-1}\mathbb{X}_F$, in turn that

$$\mathbb{X}_S \subseteq (1-\alpha)^{-1}\mathbb{X}_F. \tag{4.11}$$

Thus, under our reasonable and mild assumptions, the conditions $M_t\mathcal{S} \subseteq \mathbb{X}$ for all $t \in [0, T)$ can be ensured by requiring that $M_{k\delta}\mathcal{S} \subseteq (1-\alpha)\mathbb{X}$ for all $k \in \mathbb{Z}_{q-1}$. This desired relation is captured equivalently by the set inclusion (4.7) that, under invoked assumptions, can be shown to be true. Our next result formally verifies that the claimed set inclusions (4.7) and (4.11) hold true.

Theorem 2 *Suppose Assumptions 1, 2 and 3 hold, and consider sets $\mathbb{X}_S$ and $\mathbb{X}_F$ specified by* (3.3) *and* (4.4)*. Then, the two–sided set inclusion*

$$\mathbb{X}_F \subseteq \mathbb{X}_S \subseteq (1-\alpha)^{-1}\mathbb{X}_F \tag{4.12}$$

holds true.

Indeed, the two-sided set inclusion established in the above Theorem provides both finitely parameterizable inner and outer approximations, $\mathbb{X}_F$ and $(1-\alpha)^{-1}\mathbb{X}_F$, of the semi–infinite set $\mathbb{X}_S$, and it also provides an estimate of the quality of these inner and outer approximations as a function of the parameter α. As demonstrated in what follows, these facts provide a ground for deriving meaningful estimates of the closeness of the related maximal exact and guaranteed generalized positively invariant sets. Furthermore, with above construction in mind, the state constraint admissibility is, indeed, guaranteed by requiring set inclusion (4.5). It should be clear that the control constraint admissibility remains unchanged and, thus, it reduces to requiring set inclusion (3.6). With above construction in mind, we utilize a constricted constraint set

$$\mathbb{X}_G := \mathbb{X}_F \bigcap \mathbb{X}_K, \tag{4.13}$$

and employ stringent overall constraint admissibility conditions expressed by the following set inclusion

$$\mathcal{S} \subseteq \mathbb{X}_G. \tag{4.14}$$

The tighter overall constraint set $\mathbb{X}_G$ is, indeed, an inner, finitely parameterizable, approximation of the exact overall constraint set $\mathbb{X}_E$.

Corollary 1 *Suppose Assumptions 1, 2 and 3 hold, and consider sets $\mathbb{X}_E$ and $\mathbb{X}_G$ specified by (3.7) and (4.13). Then, the two–sided set inclusion*

$$\mathbb{X}_G \subseteq \mathbb{X}_E \subseteq (1-\alpha)^{-1}\mathbb{X}_G \tag{4.15}$$

holds true.

We close this subsection by pointing out that, in view of the preceding construction, the set $\mathbb{X}_G$ inherits the finite parameterizability from the state and control constraint sets $\mathbb{X}$ and $\mathbb{U}$. In particular, if the sets $\mathbb{X}$ and $\mathbb{U}$ are proper C polytopic sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, the set $\mathbb{X}_G$ is also guranteed to be a proper C polytopic set in $\mathbb{R}^n$.

4.3 Guaranteed Generalized Positively Invariant Sets

We now focus on guaranteed generalized positive invariance, in which the relations of (3.9) are tightened to

$$M_T\mathcal{S} \subseteq \mathcal{S} \text{ and } \mathcal{S} \subseteq \mathbb{X}_G. \tag{4.16}$$

In this case, the definition of $\mathbb{X}_G$ in (4.13) induces a degree of conservatism when applied to conversion of conditions (2.10) to tighter, but guaranteed, requirements (4.16). Thus, we simply use the term "guaranteed generalized positively invariant set" for any set $\mathcal{S}$ satisfying conditions (4.16). This convention applies to both ordinary and maximal such sets. In this setting, it is meaningful to explore finitely parameterizable generalized positively invariant sets since the set $\mathbb{X}_G$ inherits the finite parameterizability from the state and control constraint sets $\mathbb{X}$ and $\mathbb{U}$. Indeed, the finite parameterizability of the tighter overall state constraints $\mathbb{X}_G$ provides a gateway for the construction of finitely parameterizable ordinary and maximal guaranteed generalized positively invariant sets. In particular, the maximal guaranteed generalized positively invariant set, say $\mathcal{S}_G^\infty$, can be computed by using an analogous iteration of the set recursion (3.11) specified, for all $k \in \mathbb{Z}_{\geq 0}$, by

$$S_{k+1} := \left(M_T^{-1}\mathcal{S}_k\right) \bigcap \mathbb{X}_G \text{ with } \mathcal{S}_0 := \mathbb{X}_G. \tag{4.17}$$

Remark 2 *As in the case of the maximal exact generalized positively invariant set $\mathcal{S}_E^\infty$, the maximal guaranteed positively invariant set $\mathcal{S}_G^\infty$ is the Hausdorff limit of the set sequences $\{\mathcal{S}_k\}_{k\geq 0}$ generated by (4.17). The related sets S_k are also monotonically nonincreasing ($\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$) and the corresponding limit is given as in (3.13), i.e.*

$$\mathcal{S}_G^\infty = \bigcap_{k\geq 0} \mathcal{S}_k, \tag{4.18}$$

and it is the maximal set, w.r.t. set inclusion, satisfying

$$\mathcal{S} = \left(M_T^{-1}\mathcal{S}\right) \bigcap \mathbb{X}_G, \tag{4.19}$$

which is the relevant analogue of (3.12).

Theorem 1 is applicable relatively directly to the maximal guaranteed generalized positively invariant set $\mathcal{S}_G^\infty$, as summarized by its relevant corollary.

Corollary 2 *Suppose Assumptions 1, 2 and 3 hold, and consider the set sequence $\{\mathcal{S}_k\}_{k\geq 0}$ generated by (4.17). The sets $\mathcal{S}_k$, $k \geq 0$ are proper C–sets in $\mathbb{R}^n$ such that for all $k \geq 0$ it holds that $\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$. Furthermore, there exists finite integer k^* such that*

$$\mathcal{S}_{k^*} = \mathcal{S}_{k^*+1} \tag{4.20}$$

and, thus, the maximal guaranteed generalized positively invariant set $\mathcal{S}_G^\infty$ is given by

$$\mathcal{S}_G^\infty = \mathcal{S}_{k^*}. \tag{4.21}$$

More importantly, estimates of the Hausdorff distance between the maximal exact and guaranteed generalized positively invariant sets, $\mathcal{S}_E^\infty$ and $\mathcal{S}_G^\infty$, are possible.

Theorem 3 *Suppose Assumptions 1, 2 and 3 hold, and consider the maximal exact and guaranteed generalized positively invariant sets, $\mathcal{S}_E^\infty$ and $\mathcal{S}_G^\infty$, respectively. Then, the two–sided set inclusion*

$$\mathcal{S}_G^\infty \subseteq \mathcal{S}_E^\infty \subseteq (1-\alpha)^{-1}\mathcal{S}_G^\infty, \tag{4.22}$$

and the estimate of the Hausdorff distance

$$\mathrm{H}_{\mathcal{B}}(\mathcal{S}_G^\infty, \mathcal{S}_E^\infty) \leq \alpha(1-\alpha)^{-1}\varrho_{\mathcal{B}}(\mathcal{S}_G^\infty), \tag{4.23}$$

hold true.

We close this subsection by noting that, within our setting, the sets $\mathcal{S}_k,\ k \in \mathbb{Z}_{\geq 0}$ generated by (4.17) preserve the finite parameterizability of the set $\mathbb{X}_G$. Furthermore, the maximal guaranteed generalized positively invariant set $\mathcal{S}_G^\infty$ is finitely determined and, hence, itself inherits the finite parameterizability from the set $\mathbb{X}_G$. The finite parameterizability of the sets $\mathcal{S}_k,\ k \in \mathbb{Z}_{\geq 0}$ and their finitely determined limit $\mathcal{S}_G^\infty$ is highly beneficial for the related set computations.

4.4 Convergence Aspects

Since $\mathcal{S}_G^\infty \subseteq \mathbb{X}_G \subseteq \mathbb{X}$, the guaranteed estimates of the Hausdorff distance in (4.23) can be obtained by utilizing either $\varrho_\mathcal{B}(\mathbb{X}_G)$ or $\overline{\eta} = \varrho_\mathcal{B}(\mathbb{X})$ instead of $\varrho_\mathcal{B}(\mathcal{S}_G^\infty)$. In particular,

$$\mathrm{H}_\mathcal{B}(\mathcal{S}_G^\infty, \mathcal{S}_E^\infty) \leq \alpha(1-\alpha)^{-1}\overline{\eta}, \tag{4.24}$$

and, thus, for values of α satisfying

$$0 < \alpha \leq \alpha^* \text{ with } \alpha^* := \epsilon(\epsilon + \overline{\eta})^{-1} < 1, \tag{4.25}$$

we have

$$\mathrm{H}_\mathcal{B}(\mathcal{S}_G^\infty, \mathcal{S}_E^\infty) \leq \epsilon. \tag{4.26}$$

In view of Remark 1, the lower bound α^* of the maximal admissible value α restricts the range of values of $\varepsilon > 0$ and, hence, $\delta > 0$ appearing in Lemma 1 and conditions of Assumption 3. More precisely, for all values of ε satisfying

$$0 < \varepsilon \leq \varepsilon^* \text{ with } \varepsilon^* = \alpha(1-\alpha)^{-1}\underline{\eta}\overline{\eta}^{-1}, \tag{4.27}$$

Lemma 1 guarantees the existence of $\delta > 0$ verifying (4.1) and conditions of Assumption 3 with $T = q\delta$ for some $q \in \mathbb{Z}_{\geq 0}$. Consequently, the above discussion points out that, for all $\epsilon > 0$, it is possible to select scalars $\delta > 0$ and $\alpha \in (0, \alpha^*)$ so that the relations (4.1) (with $0 < \varepsilon < \varepsilon^*$) and conditions of Assumption 3 hold true simultaneously while guaranteeing *a–priori* that the Hausdorff distance between the maximal exact and guaranteed generalized positively invariant sets, $\mathcal{S}_E^\infty$ and $\mathcal{S}_G^\infty$ is as small as desirable (and without having to compute explicitly any of these sets).

Theorem 4 *Suppose Assumption 1 holds. Then, for all* $\epsilon > 0$*, there exist* $\alpha \in [0,1)$ *and* $\delta > 0$ *satisfying Assumptions 2 and 3 and ensuring that the Hausdorff distance between the maximal exact and guaranteed generalized positively invariant sets,* $\mathcal{S}_E^\infty$ *and* $\mathcal{S}_G^\infty$*, is smaller than* ϵ*, i.e. relation* (4.26) *holds true.*

5 Discussion

5.1 Role of parameters

The sampling period T is, in this paper, considered to be fixed. Clearly, the rate at which the controls are implemented and controlled dynamics are sampled is very important. Naturally, the faster sampling the higher impact of the control actions and the higher the quality of the controlled dynamics (particularly so for naturally unstable systems). The sampling rate affects both the maximal exact and guaranteed generalized positively invariant sets $\mathcal{S}_E^\infty$ and $\mathcal{S}_G^\infty$. However, the related dependence, being a research topic in its own right, does not lie within the intended scope of this note.

Naturally, the larger value of the subsampling period δ the less complex the representation of the constraint set $\mathbb{X}_F$. In fact, the set $\mathbb{X}_F$ admits the simplest representation for $\delta = T$. Intuitively, this might lead to the simplest representation of the maximal guaranteed generalized positively invariant set $\mathcal{S}_G^\infty$. On the other hand, the smaller values of δ lead to smaller values of α and, thus, the better the quality of the approximation in the sense of the Hausdorff distance. Thus, the value of δ can be selected according to the preference for the quality or the simplicity of the approximation of the maximal exact generalized positively invariant set $\mathcal{S}_E^\infty$. When the values of T and δ are selected and fixed, the corresponding value of α can be evaluated with relative ease. In particular, it suffice to set $\beta = \alpha(1-\alpha)^{-1}$ and evaluate the smallest value of a non negative scalar β,

$$\beta^0 := \min_\beta\{\beta \in \mathbb{R}_{\geq 0} : \forall k \in \mathbb{Z}_{q-1},\ \forall \tau \in (0, \delta), \quad A_{k\delta}(M_\tau - I)\mathbb{X} \subseteq \beta\mathbb{X}\}, \tag{5.1}$$

by using the standard numerical computations for optimal control. Indeed, the smallest value of α ensuring the satisfaction of conditions (4.9) takes then the following simple form

$$\alpha^0 := \beta^0(1+\beta^0)^{-1}. \tag{5.2}$$

Clearly, α^0 is a function of T and δ and, for all finite values of T and δ, $\alpha^0 \in [0,1)$ since $0 \leq \beta^0(1+\beta^0)^{-1} < 1$.

Finally, the value of α affects the size of the constraint set $\mathbb{X}_F$ and, thus, the size of the maximal guaranteed generalized positively invariant set $\mathcal{S}_G^\infty$. The smaller values of α yield the higher quality of the approximations in the sense of the Hausdorff distance. Naturally, smaller values of α might require smaller values of δ or T to be utilized. On the other hand, the bigger values of α allow potentially for the utilization of bigger values of δ and, thus, can lead to simpler maximal guaranteed generalized positively invariant sets $\mathcal{S}_G^\infty$, which are, however, poorer approximations of the maximal exact generalized positively invariant set $\mathcal{S}_E^\infty$. When the values of T and $\alpha \in (0,1)$ are selected and fixed, the corresponding value of δ can be also evaluated with relative ease. In particular, it suffices to determine the smallest integer q via

$$q^0 := \min_r\{r \in \mathbb{N} : \forall k \in \mathbb{Z}_{r-1},\ \forall \tau \in (0, Tr^{-1}), \quad A_{k\delta}(M_\tau - I)(1-\alpha)\mathbb{X} \subseteq \alpha\mathbb{X}\}, \tag{5.3}$$

by using the standard numerical computations for optimal control and a direct integer–valued bisection over r. Indeed, the largest value of δ satisfying Assumption 2 and ensuring the satisfaction of conditions (4.9) takes then the following simple form

$$\delta^0 := Tq^{0^{-1}}. \tag{5.4}$$

Clearly, δ^0 is a function of T and α and, as already commented on, for all finite values of T and $\alpha \in (0,1)$, the sufficiently small value of δ is guaranteed to exist under our mild assumptions.

5.2 Computational Aspects

A fundamental aspect for underlying exact and guaranteed set computations is the ability to effectively implement the set iterations specified in (3.11) and (4.17), respectively. In this sense, the issues of paramount importance are to understand the meaning of "computing a set" as well as to utilize in a smart manner explicit or implicit representations of the related set iterates. The standard approaches [4,5,8] to set computations demand that the set iterates $\mathcal{S}_k$ are computed explicitly at every step of the corresponding set recursions. These standard approaches to set computations can be rendered more or less efficient but are nevertheless limited to problems of relatively small dimensions.

A computationally more convenient approach is to employ implicit representations of the set iterates $\mathcal{S}_k$, and to utilize an incremental form of the set recursions (3.11) and (4.17), which we now propose. To this end, we observe that, for all $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}\mathcal{S}_{k+1} = \{x \in \mathbb{R}^n \ &: M_T^{k+1}x \in \mathcal{S}_0, \\ &M_T^k x \in \mathcal{S}_0,\ \ldots,\ M_T x \in \mathcal{S}_0,\ x \in \mathcal{S}_0\} \\ = \{x \in \mathbb{R}^n \ &: M_T^{k+1}x \in \mathcal{S}_0\} \bigcap \mathcal{S}_k,\end{aligned} \tag{5.5}$$

where $\mathcal{S}_0 = \mathbb{X}_E$ in the case of the set iteration (3.11) and $\mathcal{S}_0 = \mathbb{X}_G$ in the case of the set iteration (4.17). With above relations (5.5) in mind, the incremental form of the related set iterations (3.11) and (4.17) is obtained by letting, for all $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}&\mathcal{S}_{k+1} = \partial\mathcal{S}_{k+1} \bigcap \mathcal{S}_k \text{ with} \\ &\partial\mathcal{S}_{k+1} := \{x \in \mathbb{R}^n \ :\ M_T^{k+1}x \in \mathcal{S}_0\},\end{aligned} \tag{5.6}$$

and where, as above, $\mathcal{S}_0 = \mathbb{X}_E$ in the case of the set iteration (3.11) and $\mathcal{S}_0 = \mathbb{X}_G$ in the case of the set iteration (4.17), and $\partial\mathcal{S}_0 = \{x \in \mathbb{R}^n\ :\ x \in \mathcal{S}_0\} = \mathcal{S}_0$.

The incremental form of the related set iterations allows for an alternative and equivalent form of the test for the finite determination of the maximal guaranteed and exact generalized positively invariant sets $\mathcal{S}_G^\infty$ and $\mathcal{S}_E^\infty$, denoted in the remainder of this subsection by $\mathcal{S}_\infty$ for typographical convenience. Namely, by inspection of (5.6), we have

$$\mathcal{S}_k = \mathcal{S}_{k+1} \ \Leftrightarrow\ \mathcal{S}_k \subseteq \partial\mathcal{S}_{k+1}, \tag{5.7}$$

and thus testing whether $\mathcal{S}_k = \mathcal{S}_{k+1}$, which is equivalent to testing $\mathcal{S}_k \subseteq \mathcal{S}_{k+1}$ since $\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$ is true by construction, can be also equivalently performed by testing $\mathcal{S}_k \subseteq \partial\mathcal{S}_{k+1}$. Furthermore, it is possible to obtain sufficient conditions for guaranteeing $\mathcal{S}_k = \mathcal{S}_{k+1}$ without having to compute explicitly or implicitly any of sets $\mathcal{S}_k$ excluding $\mathcal{S}_0$. To this end, recall that $\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$ for all $k \in \mathbb{Z}_{\geq 0}$ so that

$$\begin{aligned}&\mathcal{S}_0 \subseteq \partial\mathcal{S}_{k+1} \ \Rightarrow\ \mathcal{S}_k \subseteq \partial\mathcal{S}_{k+1} \text{ and, in turn,} \\ &\mathcal{S}_0 \subseteq \partial\mathcal{S}_{k+1} \ \Rightarrow\ \mathcal{S}_k = \mathcal{S}_{k+1} = \mathcal{S}_\infty.\end{aligned} \tag{5.8}$$

Since in view of (5.6), $\mathcal{S}_k = \bigcap_{j \in \mathbb{Z}_k} \partial\mathcal{S}_j$, the two most important consequences from the theoretical and computational points of view are summarized by

$$\begin{aligned}&\mathcal{S}_k \subseteq \partial\mathcal{S}_{k+1} \ \Leftrightarrow\ \mathcal{S}_\infty = \mathcal{S}_k = \bigcap_{j \in \mathbb{Z}_k} \partial\mathcal{S}_j, \text{ and} \\ &\mathcal{S}_0 \subseteq \partial\mathcal{S}_{k+1} \ \Rightarrow\ \mathcal{S}_\infty = \mathcal{S}_k = \bigcap_{j \in \mathbb{Z}_k} \partial\mathcal{S}_j.\end{aligned} \tag{5.9}$$

The representation of the set $\mathcal{S}_\infty = \bigcap_{j \in \mathbb{Z}_k} \partial\mathcal{S}_j$ can take implicit or explicit form. In the former case, the implicit representation of each of the sets $\partial\mathcal{S}_j$ and the intersection defining set $\mathcal{S}_\infty$ is obtained. Likewise, in the latter case, the explicit representation of each of the sets $\partial\mathcal{S}_j$ and the intersection defining set $\mathcal{S}_\infty$ is constructed. All above considerations, in conjunction with already mentioned finite parameterizability aspects, can be directly utilized for the related set computations, as demonstrated next by our illustrative example.

5.3 Illustrative Example

Our illustrative example is a simple two dimensional sampled–data system of (2.1) for which

$$A = 2\pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{5.10}$$

while the state and control constraints of (2.7) are proper C–polytopic sets

$$\begin{aligned}&\mathbb{X} = \{x = (\xi_1, \xi_2) \in \mathbb{R}^2\ :\ -1 \leq \xi_1 \leq 1,\ -1 \leq \xi_2 \leq 1\} \text{ and} \\ &\mathbb{U} = \{u \in \mathbb{R}\ :\ -1 \leq u \leq 1\}.\end{aligned} \tag{5.11}$$

Two values of the sampling period T are considered, i.e. $T = 4^{-1}s$ and $T = 8^{-1}s$, and in each case the corresponding linear state feedback control gain K is computed as the solution to a discrete time infinite horizon

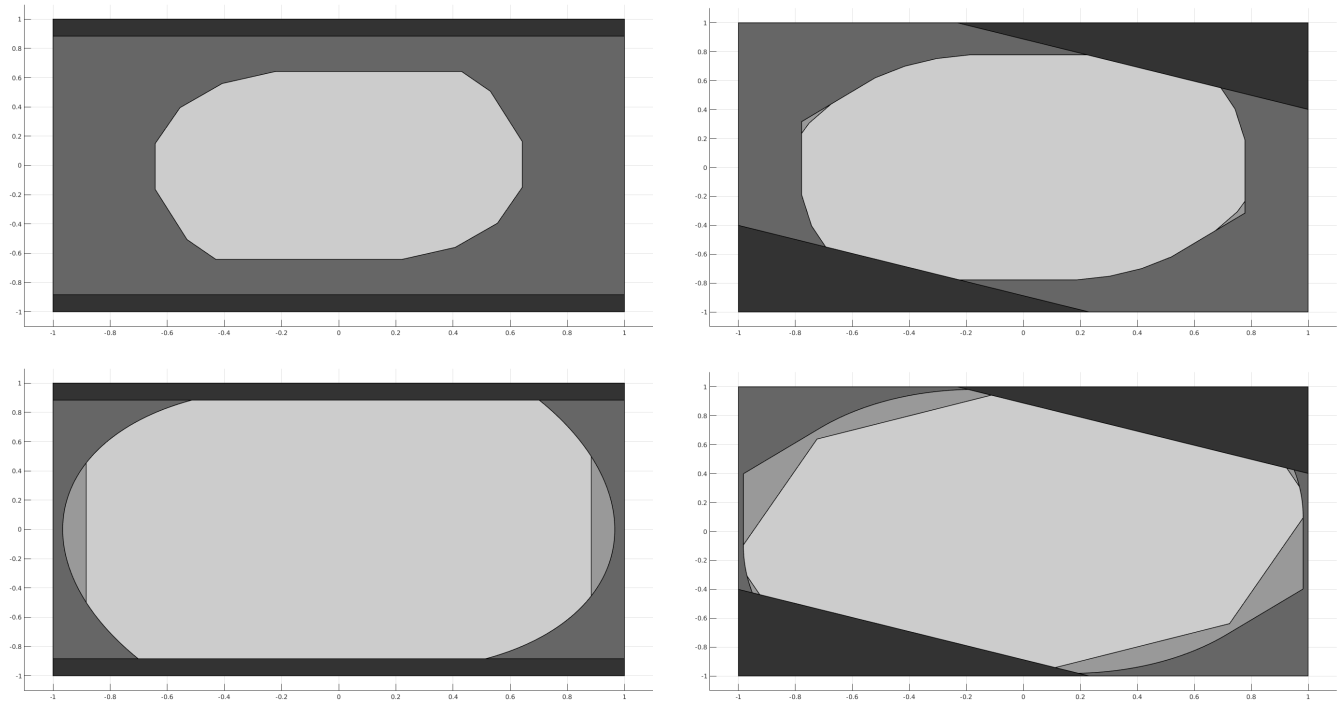

Fig. 1. The variety of illustrative constraint sets and corresponding maximal guaranteed generalized positively invariant sets. Color key: ■ $\mathbb{X}$, ■ $\mathbb{X}_D := \mathbb{X} \cap \mathbb{X}_K$, ■ $\mathbb{X}_G$ and ■ $\mathcal{S}_G^{\infty}$. (Note: $\mathcal{S}_G^{\infty} \subseteq \mathbb{X}_G \subseteq \mathbb{X}_D \subseteq \mathbb{X}$.)

linear quadratic regulator problem for the system matrices (A_T, B_T) and cost weights $Q = I$ and $R = 1$. Four values of the subsampling period δ, obtained by setting $\delta = Tq^{-1}$ with $q \in \{4, 16, 64, 256\}$, are considered for each of two sampling periods $T = 4^{-1}s$ and $T = 8^{-1}s$. The related values $\alpha^0 = \alpha^0(T, \delta)$ computed via (5.1) and (5.2) are reported in Table 1 for these eight cases. As it can be seen, $\alpha^0 = \alpha^0(T, \delta)$ decreases as δ decreases

Case	T	q	δ	α^0
1	0.25	4	0.0625	0.35743
2	0.25	16	0.015625	0.12269
3	0.25	64	0.00390625	0.033808
4	0.25	256	0.0009765625	0.0086722
5	0.125	4	0.03125	0.22149
6	0.125	16	0.0078125	0.06653
7	0.125	64	0.001953125	0.017508
8	0.125	256	0.00048828125	0.0044353

Table 1
The optimal values of $\alpha^0(T, \delta)$.

(i.e., as number of time points at which state constraints are invoked is increased). In fact, for both sampling periods $T = 4^{-1}s$ and $T = 8^{-1}s$ invoking state constraints at 256 uniformly spaced intersample time points leads to values of $\alpha^0 < 0.01$. This, in turn, ensures that the maximal guaranteed generalized positively invariant sets $\mathcal{S}_G^{\infty}$ are close approximations of the maximal exact generalized positively invariant sets $\mathcal{S}_E^{\infty}$ in the sense of the Hausdorff distance, the values of which are guaranteed to be less than 0.01 in absolute terms and less than 1% in relative terms (both w.r.t. usual ∞ vector norm). Figure 1 provides additional illustration of the interplay between the values of δ and α and the related maximal guaranteed generalized positively invariant sets $\mathcal{S}_G^{\infty}$. The sub–figures in the left hand column illustrate cases 1 and 3, while the sub–figures in the right hand column illustrate cases 5 and 7 from Table 1. In all sub–figures, the state constraints $\mathbb{X}$, the discrete time admissibility constraints $\mathbb{X}_D = \mathbb{X} \cap \mathbb{X}_K$, the sampled–data guaranteed admissibility constraints $\mathbb{X}_G$ and the maximal guaranteed generalized positively invariant sets $\mathcal{S}_G^{\infty}$ are shown using different levels of gray–scale shading as detailed by the provided color key. As already asserted, the smaller values of δ the smaller values of α and the better approximation of the maximal exact generalized positively invariant set. In all cases, finite paramaterizability and, in fact, proper C–polytopic structure of all the involved sets is preserved. As expected and as it can be observed in the Figure, the quality of the related maximal guaranteed generalized positively invariant sets is attained at the cost of the complexity of their representations.

Figure 2 depicts the involved sets for case 4 specified in Table 1. For this particular setting, the values of the sampling and subsampling periods are $T = 4^{-1}$ and $\delta = T256^{-1} = 1024^{-1}$. The corresponding value of $\alpha^0 = 0.0088$ guarantees that the Hausdorff distance between the maximal guaranteed and exact generalized positively invariant sets $\mathcal{S}_G^{\infty}$ and $\mathcal{S}_E^{\infty}$ is less than 0.0089 in

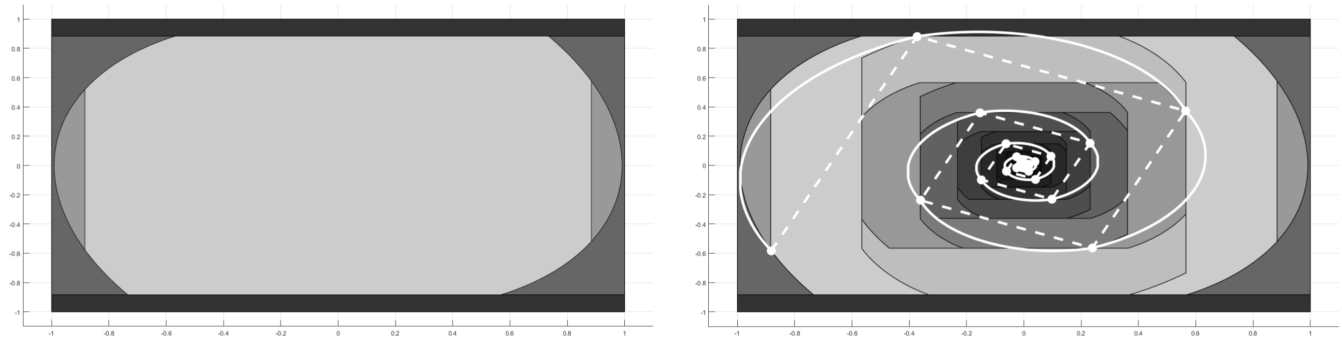

Fig. 2. The constraint sets and corresponding maximal guaranteed generalized positively invariant set for Case 4 of Table 1. Color key: ■ $\mathbb{X}$, ■ $\mathbb{X}_D := \mathbb{X} \cap \mathbb{X}_K$, ■ $\mathbb{X}_G$ and ■ $\mathcal{S}_G^{\infty}$. (Note: $\mathcal{S}_G^{\infty} \subseteq \mathbb{X}_G \subseteq \mathbb{X}_D \subseteq \mathbb{X}$.)

absolute terms and less than 0.9% in relative terms. The figure also shows the sequences of sets $\{M_T^k \mathcal{S}_G^{\infty}\}_{k \in \mathbb{Z}_{\geq 0}}$. The terms of this set sequence spiral in towards the origin while converging to it, and are depicted in different layers of gray–scale shading (the darker shade the larger k). A sample of continuous time trajectory of the considered sampled–data linear dynamics is shown in white using solid line. The related discrete time trajectory is shown in white by filled circles, while the related linear interpolation of this trajectory is shown in white using dashed line. All these trajectories spiral in towards the origin, while converging to it as time goes to ∞. As expected in view of our construction, the discrete time trajectory lies entirely within the maximal guaranteed generalized positively invariant set $\mathcal{S}_G^{\infty}$, and it converges to 0. In fact and as a direct inspection of the figure confirms, it holds that $x_{kT} \in M_T^k \mathcal{S}_G^{\infty}$. On the other hand and as expected, the continuous time trajectory leaves the maximal guaranteed generalized positively invariant set $\mathcal{S}_G^{\infty}$ during sampling intervals. It should be observed that the actual continuous time trajectory also leaves the set $\mathbb{X}_G$ for periods of time (i.e., during some subsampling intervals $(k\delta, (k+1)\delta)$) as well as the control constraint admissibility set $\mathbb{X}_K$. However, it never leaves the state constraint set $\mathbb{X}$ and it also belongs to the set $\mathbb{X}_K$ at the sampling instances when the related control actions are implemented. Finally, a direct inspection of the figure does reveal that the corresponding continuous and discrete time trajectories (more precisely, the linear interpolation of the latter one) differ substantially highlighting once again inadequacy of the utilization of simple discrete time positive invariant sets.

6 Closing Remarks

We have introduced topologically flexible and generalized positive invariance notions within the sampled–data setting. Both the exact and guaranteed generalized positive invariance notions have been proposed. The former notion is theoretically "optimal", while the latter notion is approximate but it enhances significantly finite parameterizability and practical computability. The limiting behaviour and computational aspects have also been discussed, and the notions were illustrated by means of an academic example.

The work reported in this paper opens up a number of questions and lines for future research. First and foremost, the extension of the developed notions to general setting of constrained sampled–data nonlinear systems is relatively direct from the theoretical point of view, but possibly challenging from the computational point of view. Secondly, it would be of much interest to deploy our generalized notions to study ordinary and maximal control and robust control invariant sets in conjunction with efficient and sampling–based computational techniques introduced in [17]. Another relevant aspect would be a more detailed study of computational aspects in conjunction with computational approaches considered in [15, 16, 18]. The derivation of notions allowing for classical robustness within our framework, as initiated in [19], is also of much interest as well as for robustness towards numerical approximations and sampling related imprecisions, as initiated in [17, 18].

Within the context of the present manuscript, there are also several possible extensions. First point is utilization of an alternative version of conditions in (4.9). We have identified a relaxed, but somewhat computationally more complex, form of this main set of conditions; it reads as

$$\forall k \in \mathbb{Z}_{q-1},\ \forall \tau \in (0, \delta),$$
$$M_{k\delta+\tau}\left(\bigcap_{j=0,1,\ldots,k} M_{j\delta}^{-1}\mathbb{X}\right) \subseteq (1-\alpha)^{-1}\mathbb{X}, \tag{6.1}$$

and the benefits of its use are subject of an ongoing investigation. Naturally, a more sophisticated modifications of the state constraints (instead of simple scaling employed herein for convenience) can be developed and are also being considered. Finally, in this paper we have employed separate but uniform sampling and subsampling strategies. It might be beneficial for a number of reasons to allow for adaptive sampling and subsampling, i.e. use of adaptive time partitions for dynamics and controls as well as adaptive subpartitions for state constraints. This aspect is also under current investigation.

APPENDIX I: Proof of Theorem 1

First, we establish that the set $\mathbb{X}_E$ is a proper C–set in $\mathbb{R}^n$. By its definition in (3.3), the set $\mathbb{X}_S$ is closed and convex. But, $0 \in \mathbb{X}_S \subseteq \mathbb{X}$ so $\mathbb{X}_S$ is guaranteed to be a C–set in $\mathbb{R}^n$. However, since $\mathbb{X}$ is a proper C–set in $\mathbb{R}^n$, for each $t \in [0, T]$ there is ball $\mathcal{B}$ of radius $r_t > 0$ centered at the origin such that $M_t r_t \mathcal{B} \subseteq \mathbb{X}$. Thus, letting $r := min_t\{r_t \ : \ t \in [0, T]\}$ it follows that $r > 0$ and, in turn, $r\mathcal{B} \subseteq r_t\mathcal{B} \subseteq M_t^{-1}\mathbb{X}$ for all $t \in [0, T) \subseteq [0, T]$. Thus, $r\mathcal{B} \subseteq \mathbb{X}_S = \bigcap_{t\in[0,T)} M_t^{-1}\mathbb{X}$ and the set $\mathbb{X}_S$ is, in fact, a proper C–set in $\mathbb{R}^n$. Now, the set $\mathbb{U}$ is a proper C–set in $\mathbb{R}^n$ so that $\mathbb{X}_K$ is guaranteed to be a closed, convex set in $\mathbb{R}^n$ that contains the origin in its interior. Thus, the set $\mathbb{X}_E$ is a proper C–set in $\mathbb{R}^n$.

Next, we establish that the sets $\mathcal{S}_k, \ k \in \mathbb{Z}_{\geq 0}$ are proper C–sets in $\mathbb{R}^n$. Now, suppose that $\mathcal{S}_k$ is a proper C–set for some $k \in \mathbb{Z}_{\geq 0}$. Then, the set $M_T^{-1}\mathcal{S}_k$ is guaranteed to be a closed, convex set in $\mathbb{R}^n$ that contains the origin in its interior. Thus, since $\mathbb{X}_E$ is a proper C–set in $\mathbb{R}^n$, the set $\mathcal{S}_{k+1} = (M_T^{-1}\mathcal{S}_k) \bigcap \mathbb{X}_E$ is a proper C–set in $\mathbb{R}^n$. Since $\mathcal{S}_0 = \mathbb{X}_E$ is a proper C–set in $\mathbb{R}^n$, the claimed fact that the sets $\mathcal{S}_k, \ k \in \mathbb{Z}_{\geq 0}$ are proper C–sets in $\mathbb{R}^n$ follows by induction.

Next, $\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$ yields that $M_T^{-1}\mathcal{S}_{k+1} \subseteq M_T^{-1}\mathcal{S}_k$, and, in turn, $(M_T^{-1}\mathcal{S}_{k+1}) \bigcap \mathbb{X}_E \subseteq (M_T^{-1}\mathcal{S}_k) \bigcap \mathbb{X}_E$. Thus, $\mathcal{S}_{k+2} \subseteq \mathcal{S}_{k+1}$. Since $\mathcal{S}_1 = (M_T^{-1}\mathbb{X}_E) \bigcap \mathbb{X}_E \subseteq \mathbb{X}_E = \mathcal{S}_0$, the claimed facts that $\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$ for all $k \in \mathbb{Z}_{\geq 0}$ follow by induction.

A generalized variant of [10, Theorem 4.1.] yields the existence of a finite integer such that $\mathcal{S}_{k^*} = \mathcal{S}_{k^*+1}$. So, only an outline of the proof is provided. As shown in [13], since M_T is strictly stable there is a symmetric proper C–set in $\mathbb{R}^n$, say $\mathcal{L}$, such that $M_T\mathcal{L} \subseteq \lambda\mathcal{L}$ for some $\lambda \in [0, 1)$. Furthermore, since $\mathbb{X}_E$ is a proper C–set in $\mathbb{R}^n$, there are two finite and strictly positive scalars, say $\underline{\eta}$ and $\overline{\eta}$, such that

$$\underline{\eta}\mathcal{L} \subseteq \mathbb{X}_E \subseteq \overline{\eta}\mathcal{L}.$$

By strict stability of M_T, there is a finite $k^* \in \mathbb{Z}_{\geq 0}$ such that $M_T^k\overline{\eta}\mathcal{L} \subseteq \underline{\eta}\mathcal{L}$. We note that k^* is guaranteed to be less or equal to the smallest integer k such that $\lambda^k \leq \underline{\eta}\overline{\eta}^{-1}$. In turn,

$$\overline{\eta}\mathcal{L} \subseteq M_T^{-k^*}\underline{\eta}\mathcal{L}.$$

In view of set iteration specified in (3.11), this implies that $\mathcal{S}_{k^*} = (M_T^{-1}\mathcal{S}_{k^*}) \bigcap \mathbb{X}_E = \mathcal{S}_{k^*+1}$.

Finally, $\mathcal{S}_{k^*+1} = \mathcal{S}_{k^*}$ implies that $\mathcal{S}_{k^*+j} = \mathcal{S}_{k^*}$ for all $j \in \mathbb{Z}_{\geq 0}$ and, in turn, $\mathcal{S}_E^\infty = \mathcal{S}_{k^*}$.

APPENDIX II: Proof of Proposition 1

First, by definition of A_t, we have

$$A_{k\delta+\tau} = e^{(k\delta+\tau)A} = e^{k\delta A}e^{\tau A} = A_{k\delta}A_\tau.$$

Second, by definition of B_t, we have

$$\begin{aligned}
B_{k\delta+\tau} &= \left(\int_0^{k\delta+\tau} e^{\sigma A}d\sigma\right)B \\
&= \left(\int_0^{k\delta} e^{\sigma A}d\sigma + \int_{k\delta}^{k\delta+\tau} e^{\sigma A}d\sigma\right)B \\
&= \left(\int_0^{k\delta} e^{\sigma A}d\sigma + e^{k\delta A}\int_0^{\tau} e^{\sigma A}d\sigma\right)B \\
&= \left(\int_0^{k\delta} e^{\sigma A}d\sigma\right)B + e^{k\delta A}\left(\int_0^{\tau} e^{\sigma A}d\sigma\right)B \\
&= B_{k\delta} + A_{k\delta}B_\tau.
\end{aligned}$$

Third, by definition of M_t, we have

$$\begin{aligned}
M_{k\delta+\tau} &= A_{k\delta+\tau} + B_{k\delta+\tau}K = A_{k\delta}A_\tau + B_{k\delta}K + A_{k\delta}B_\tau K \\
&= A_{k\delta} - A_{k\delta} + B_{k\delta}K + A_{k\delta}(A_\tau + B_\tau K) \\
&= (A_{k\delta} + B_{k\delta}K) + A_{k\delta}(A_\tau + B_\tau K - I) \\
&= M_{k\delta} + A_{k\delta}(M_\tau - I),
\end{aligned}$$

which proves that the claimed relations are affirmative.

APPENDIX III: Proof of Theorem 2

First, relation (4.10) verifies that

$$\mathbb{X}_S \subseteq (1-\alpha)^{-1}\mathbb{X}_F,$$

as asserted in (4.11). So, it suffices to verify relation (4.7). To this end, we observe that the conditions

$$\forall t \in [0, T), \ M_t\mathcal{S} \subseteq \mathbb{X}$$

equivalently read as

$$\forall k \in \mathbb{Z}_{q-1}, \ \forall \tau \in [0, \delta), \ M_{k\delta+\tau}\mathcal{S} \subseteq \mathbb{X}.$$

By construction, since $\alpha \in [0, 1)$ and $\mathbb{X}$ is a proper C–set, we have that the relations

$$\forall k \in \mathbb{Z}_{q-1}, \ M_{k\delta}\mathcal{S} \subseteq (1-\alpha)\mathbb{X} \subseteq \mathbb{X}$$

are guaranteed. Now, by Proposition 1, for all $k \in \mathbb{Z}_{q-1}$ and all $\tau \in (0, \delta)$,

$$M_{k\delta+\tau}\mathcal{S} = [M_{k\delta} + A_{k\delta}(M_\tau - I)]\mathcal{S}.$$

Since $(L_1 + L_2)\mathcal{X} \subseteq L_1\mathcal{X} \oplus L_2\mathcal{X}$ is true for arbitrary matrices L_1 and L_2 of compatible dimensions and a set $\mathcal{X}$, it further follows that

$$[M_{k\delta} + A_{k\delta}(M_\tau - I)]\mathcal{S} \subseteq M_{k\delta}\mathcal{S} \oplus A_{k\delta}(M_\tau - I)\mathcal{S}.$$

Since $\mathcal{S} \subseteq (1 - \alpha)\mathbb{X}$, it follows that

$$M_{k\delta}\mathcal{S} \oplus A_{k\delta}(M_\tau - I)\mathcal{S} \subseteq M_{k\delta}\mathcal{S} \oplus A_{k\delta}(M_\tau - I)(1-\alpha)\mathbb{X}.$$

But, by construction, $M_{k\delta}\mathcal{S} \subseteq (1 - \alpha)\mathbb{X}$ and, by (4.9), we have $A_{k\delta}(M_\tau - I)(1 - \alpha)\mathbb{X} \subseteq \alpha\mathbb{X}$. Thus,

$$M_{k\delta}\mathcal{S} \oplus A_{k\delta}(M_\tau - I)(1 - \alpha)\mathbb{X} \subseteq (1 - \alpha)\mathbb{X} \oplus \alpha\mathbb{X} = \mathbb{X}.$$

Hence,

$$\forall k \in \mathbb{Z}_{q-1},\ \forall \tau \in (0, \delta),\ M_{k\delta+\tau}\mathcal{S} \subseteq \mathbb{X}.$$

The claimed facts are verified and the proof is concluded.

APPENDIX IV: Proof of Corollary 1

By Proposition 1, $\mathbb{X}_F \subseteq \mathbb{X}_S \subseteq (1 - \alpha)^{-1}\mathbb{X}_F$, and, thus,

$$\mathbb{X}_F \bigcap \mathbb{X}_K \subseteq \mathbb{X}_S \bigcap \mathbb{X}_K \subseteq (1 - \alpha)^{-1}\mathbb{X}_F \bigcap \mathbb{X}_K.$$

Since $\alpha \in [0, 1)$ and $\mathbb{X}_K$ is guranteed to be a convex set in $\mathbb{R}^n$ that contains the origin, it follows that $\mathbb{X}_K \subseteq (1 - \alpha)^{-1}\mathbb{X}_K$ and, in turn, $(1 - \alpha)^{-1}\mathbb{X}_F \bigcap \mathbb{X}_K \subseteq (1 - \alpha)^{-1}\mathbb{X}_F \bigcap (1 - \alpha)^{-1}\mathbb{X}_K = (1 - \alpha)^{-1} (\mathbb{X}_F \bigcap \mathbb{X}_K)$. Hence,

$$\mathbb{X}_F \bigcap \mathbb{X}_K \subseteq \mathbb{X}_S \bigcap \mathbb{X}_K \subseteq (1 - \alpha)^{-1} \left(\mathbb{X}_F \bigcap \mathbb{X}_K\right),$$

or, equivalently,

$$\mathbb{X}_G \subseteq \mathbb{X}_E \subseteq (1 - \alpha)^{-1}\mathbb{X}_G,$$

as claimed.

APPENDIX V: Proof of Theorem 4

When $\alpha = 0$ the claim is obvious, so we focus on the case $\alpha \in (0, 1)$. By Corollary 1, $\mathbb{X}_G \subseteq \mathbb{X}_E \subseteq (1 - \alpha)^{-1}\mathbb{X}_G$ so that, by construction,

$$\mathcal{S}_G^\infty \subseteq \mathcal{S}_E^\infty \subseteq \mathcal{X}^\infty,$$

where $\mathcal{X}^\infty$ is the maximal positively invariant set for the dynamics $x^+ = M_T x$ and constraint set $(1 - \alpha)^{-1}\mathbb{X}_G$. Thus, it suffices to prove that $\mathcal{X}^\infty = (1 - \alpha)^{-1}\mathcal{S}_G^\infty$. We will assume that this is not the case and reach contradiction. Before proceeding, observe that, by construction, all three sets $\mathcal{S}_G^\infty$, $\mathcal{S}_E^\infty$ and $\mathcal{X}^\infty$ are proper C–sets. Let us assume that $\mathcal{X}^\infty$ is not equal to $(1 - \alpha)^{-1}\mathcal{S}_G^\infty$. The set $(1 - \alpha)\mathcal{X}^\infty$ satisfies $(1 - \alpha)\mathcal{X}^\infty \subseteq \mathbb{X}_G$ and $M_T(1 - \alpha)\mathcal{X}^\infty \subseteq (1 - \alpha)\mathcal{X}^\infty$. Thus, $(1 - \alpha)\mathcal{X}^\infty$ is positively invariant set for the dynamics $x^+ = M_T x$ and constraint set $\mathbb{X}_G$. But $\mathcal{S}_G^\infty$ is the maximal positively invariant set for the dynamics $x^+ = M_T x$ and constraint set $\mathbb{X}_G$. Thus, $(1-\alpha)\mathcal{X}^\infty \subseteq \mathcal{S}_G^\infty$ and, as both sets are proper C–sets, $\mathcal{X}^\infty \subseteq (1 - \alpha)^{-1}\mathcal{S}_G^\infty$. But, $(1 - \alpha)^{-1}\mathcal{S}_G^\infty$ satisfies $(1 - \alpha)^{-1}\mathcal{S}_G^\infty \subseteq (1 - \alpha)^{-1}\mathbb{X}_G$ and $M_T(1 - \alpha)^{-1}\mathcal{S}_G^\infty \subseteq (1 - \alpha)^{-1}\mathcal{S}_G^\infty$ and, thus, $(1 - \alpha)^{-1}\mathcal{S}_G^\infty$ is positively invariant set for the dynamics $x^+ = M_T x$ and constraint set $(1 - \alpha)^{-1}\mathbb{X}_G$. The desired contradiction is revealed, as the set $(1 - \alpha)^{-1}\mathcal{S}_G^\infty$ refutes the maximal positive invariance of $\mathcal{X}^\infty$. Consequently, it follows that these sets are equal, i.e. $\mathcal{X}^\infty = (1 - \alpha)^{-1}\mathcal{S}_G^\infty$, so that, as asserted,

$$\mathcal{S}_G^\infty \subseteq \mathcal{S}_E^\infty \subseteq (1 - \alpha)^{-1}\mathcal{S}_G^\infty.$$

Since $\mathcal{S}_G^\infty$ is a proper C–set, it follows that

$$(1 - \alpha)^{-1}\mathcal{S}_G^\infty = \mathcal{S}_G^\infty \oplus \alpha(1 - \alpha)^{-1}\mathcal{S}_G^\infty,$$

so that

$$\mathcal{S}_G^\infty \subseteq \mathcal{S}_E^\infty \subseteq \mathcal{S}_G^\infty \oplus \alpha(1 - \alpha)^{-1}\mathcal{S}_G^\infty.$$

Thus, as claimed,

$$\mathrm{H}_\mathcal{B}(\mathcal{S}_G^\infty, \mathcal{S}_E^\infty) \leq \alpha(1 - \alpha)^{-1}\varrho_\mathcal{B}(\mathcal{S}_G^\infty).$$

APPENDIX VI: Proof of Theorem 4

The claim follows from the discussion preceding it.

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