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R&D DYNAMICS

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Abstract. We study a Cournot duopoly model using Ferreira-Oliveira-Pinto's R&D investment function. We find the multiple perfect Nash equilibria and we analyse the economical relevant quantities like output levels, prices, consumer surplus, profits and welfare.

1. **Introduction.** In this paper we consider a Cournot duopoly competition model where each of the firms invest in R&D projects to reduce its initial production costs ([4, 8]). This competition is modeled, as usual, by a two stages game (see [1]). In the first subgame, two firms choose, simultaneously, the R&D investment strategies and in the second subgame, the two firms are involved in a Cournot competition with production costs equal to the reduced costs obtained in the previous stage. The R&D investment function considered is the one introduced in [5]. We find the Nash investment equilibria for the two stages game and study the economical impacts resulting from having distinct equilibria ((see [2, 9])). As it is well known from the literature, the second subgame consists of a Cournot competition and has a unique Nash equilibrium. For the first subgame, consisting of an R&D investment program there are at most four distinct Nash investment equilibria: (i) a Nash equilibrium where both firms invest (see [1]); (ii) a Nash equilibrium where firm F_1 invests and firm F_2 does not; (iii) a Nash equilibrium where firm F_2 invests and firm F_1 does not; (iv) a Nash equilibrium where neither of the firms invest. We consider a competitive investment region C where both firms invest, a single investment region S_1 for firm F_1 where just firm F_1 invests, and a single investment region S_2 for firm F_2 where just firm F_2 invests. We observe that these regions can have non-empty intersections, i.e. the strategic optimal investment equilibrium might not be unique.

2. **The Cournot competition model.** The Cournot competition with R&D investment programs consists of two subgames in one period of time (see [1]). We fully characterize the perfect Nash equilibria of the game that is determined by the Nash investment equilibria for the first subgame (non-unique) and the Nash equilibrium output level for the second subgame (see [5]).

The first subgame is an R&D investment program, where both firms have initial production costs and choose, simultaneously, their R&D investment strategies to obtain new production costs. The second subgame is a standard Cournot duopoly

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competition with production costs equal to the reduced cost obtained in the previous stage. We consider an economy with a monopolistic sector with two firms, F_1 and F_2, each one producing a differentiated good, where q_i denotes the output quantity of the firm F_i. In the region of quantity space where prices are positive, we assume that the inverse demands are linear and the price p_i of the good produced by the firm F_i is given by

$$p_i = \alpha - \beta q_i - \gamma q_j,$$

where $\alpha, \beta > 0$. Furthermore, we assume that the goods are substitutes, i.e. $\gamma > 0$ (see [11]).

The firm F_i invests an amount v_i in an R&D program

$$a_i : [0, +\infty] \times [c_L, \alpha] \to [c_i - \epsilon(c_i - c_L), c_i]$$

that reduces its production cost c_i to a new production cost $a_i = a_i(v_i, c_i)$ given by

$$a_i = c_i - \frac{\epsilon(c_i - c_L)v_i}{\lambda + v_i}. \tag{1}$$

All the results presented hold in an open region of parameters $(\lambda, \alpha, \beta, \gamma)$ containing the point $(0.2, 10, 0.013, 0.013)$.

The profit $\pi_i(q_i, q_j)$ of firm F_i is given by:

$$\pi_i(q_i, q_j) = \pi_i(q_i, q_j; v_1, v_2, c_1, c_2) = q_i\left(\alpha - \beta q_i - \gamma q_j - a_i\right) - v_i, \tag{2}$$

for $i, j \in \{1, 2\}$ and $i \neq j$. Let

$$R_i = \frac{2\beta\alpha - \gamma\alpha - 2\beta a_i + \gamma a_j}{4\beta^2 - \gamma^2},$$

with $i, j \in \{1, 2\}$ and $i \neq j$. As it is well-known (see [5, 7]), the *Nash equilibrium output level* (q_1, q_2) is given by:

$$q_i = q_i(v_1, v_2; c_1, c_2) = \begin{cases} 0, \text{ if } R_i \leq 0 \\ R_i, \text{ if } 0 < R_i < \frac{\alpha - a_j}{\gamma} \\ \frac{\alpha - a_i}{2\beta}, \text{ if } R_i \geq \frac{\alpha - a_j}{\gamma} \end{cases}. \tag{3}$$

At the Nash equilibrium output level, the price p_i of firm F_i is given by

$$p_i = p_i(v_1, v_2; c_1, c_2) = \alpha - \beta q_i(v_1, v_2; c_1, c_2) - \gamma q_j(v_1, v_2; c_1, c_2).$$

Furthermore, the profit $\pi_i(v_1, v_2; c_1, c_2)$ of firm F_i is given by:

$$\pi_i(v_1, v_2; c_1, c_2) = \begin{cases} -v_i, \text{ if } R_i \leq 0 \\ \beta R_i^2 - v_i, \text{ if } 0 < R_i < \frac{\alpha - a_j}{\gamma} \\ \frac{(\alpha - a_i)^2}{4\beta} - v_i, \text{ if } R_i \geq \frac{\alpha - a_j}{\gamma} \end{cases}. \tag{4}$$

Given initial production costs c_1 and c_2, the sets A_i of new production costs for firms F_1 and F_2 are given by:

$$A_i = A_i(c_1, c_2) = [c_i - \epsilon(c_i - c_L), c_i],$$

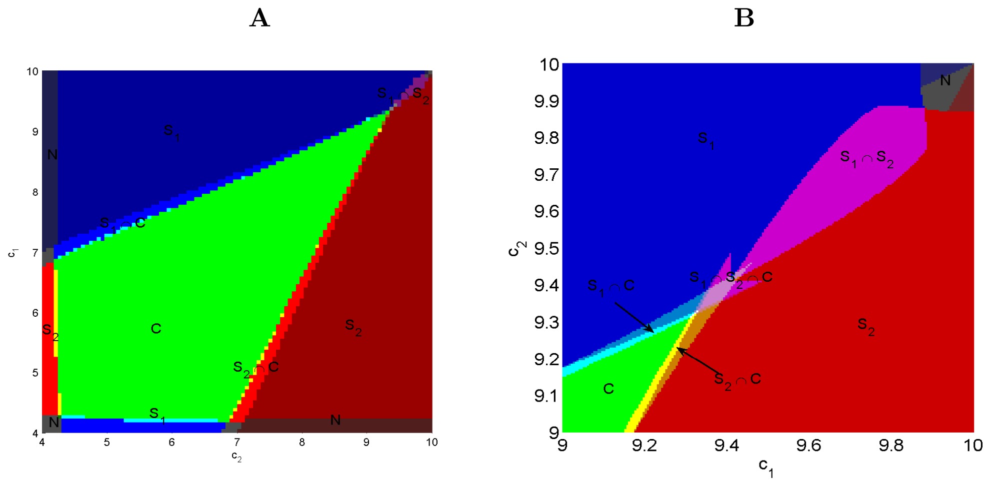

FIGURE 1. (**A**) Full characterization of the Nash investment regions in terms of the firms' initial production costs $(c_1, c_2) \in [4, 10]^2$. The single investment regions S_1 and S_2 are shown in blue and red, respectively; the competitive investment region C is shown in green; and the Nil Nash investment region N is shown in grey, dark blue and dark red. (**B**) Zoom of Figure (**A**) in the region $(c_1, c_2) \in [9, 10]^2$. The intersection $S_1 \cap S_2$ between the region S_1 and the region S_2 is shown in pink. The intersection $S_1 \cap C$ between the region S_1 (respectively S_2) and the region C is shown in light blue (respectively light red); The intersection $S_1 \cap C \cap S_2$ between the region S_1, the region S_2 and the region C is shown in light grey.

for $i \in \{1, 2\}$. The R&D cost reduction investment programs a_1 and a_2 of the firms determine a bijection between the *investment region* $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ of both firms and the *new production costs region* $A_1 \times A_2$ given by the map

$$\begin{array}{rcccc} \mathbf{a} = (a_1, a_2) & : & (\mathbb{R}_0^+)^2 \times [c_L, \alpha]^2 & \longrightarrow & A_1 \times A_2 \\ & & (v_1, v_2; c_1, c_2) & \longmapsto & (a_1(v_1), a_2(v_2)), \end{array}$$

where, due to the non-existence of spillovers

$$a_i(v_i) = a_i(v_i; c_i, c_j) = c_i - \frac{\epsilon(c_i - c_L)v_i}{\lambda + v_i}.$$

The new production costs region can be decomposed, at most, in three disconnected economical regions characterized by the optimal output level of the firms: the *monopoly region* M_i of firm F_i; the *duopoly region* D characterized by the optimal output levels of both firms being non-zero and consequently below their monopoly output levels (see [5]).

The *best investment response (multivalued) function* $V_1 : \mathbb{R}_0^+ \times [c_L, \alpha]^2 \to \mathbb{R}_0^+$ of firm F_1 is given by:

$$V_1(v_2; c_1, c_2) = \arg\max_{v_1} \pi_1(v_1, v_2; c_1, c_2).$$

In [5], an explicit computational algorithm to find the best investment response function $V_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is presented.

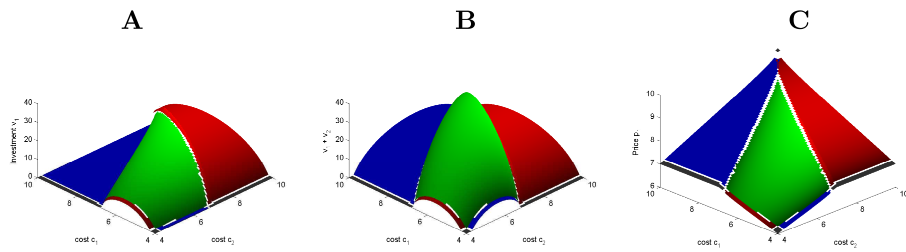

FIGURE 2. (**A**) Plot of the Nash investment $v_1 = v_1(c_1, c_2)$ of Firm F_1 in terms of the initial production costs (c_1, c_2); (**B**) Plot of the aggregated investments $v = v_1 + v_2$ of Firms F_1 and F_2 in terms of the initial production costs (c_1, c_2); (**C**) Plot of the price $p_1 = p_1(v_1(c_1, c_2), v_2(c_1, c_2); c_1, c_2)$ of Firm F_1 in terms of the initial production costs (c_1, c_2) using the Nash investment equilibria $(v_1(c_1, c_2), v_2(c_1, c_2))$.

The *Nash investment equilibria* are given by the (multivalued) function

$$v : [c_L, \alpha]^2 \to (\mathbb{R}_0^+)^2,$$

where $v(c_1, c_2) = (v_1, v_2)$ are the solutions of the system:

$$\begin{cases} v_1 = V_1(v_2; c_1, c_2) \\ v_2 = V_2(v_1; c_1, c_2) \end{cases}.$$

We find, at most, four distinct types of Nash investment equilibria (v_1, v_2): (i) a *competitive Nash equilibrium* where both firms invest, i.e. $v_1 > 0$ and $v_2 > 0$; (ii) a *single Nash equilibrium of firm* F_1 where firm F_1 invests and firm F_2 does not, i.e. $v_1 > 0$ and $v_2 = 0$; (iii) a *single Nash equilibrium of firm* F_2 where firm F_2 invests and firm F_1 does not, i.e. $v_2 > 0$ and $v_1 = 0$; (iv) a *Nil Nash equilibrium* where neither firm F_1 neither firm F_2 invest, i.e. $v_1 = 0$ and $v_2 = 0$ (see [5]). We define a *competitive investment region* C consisting of Nash investment equilibria where both firms invest, a *single investment region* S_1 for firm F_1, consisting of Nash investment equilibria where just firm F_1 invests, a *single investment region* S_2 for firm F_2, consisting of Nash investment equilibria where just firm F_2 invests and a *nil investment region* N where neither of the firms invest (see [5]).

We note that in every figure of this paper we use the same colors to identify the different regions. The single investment regions S_1 and S_2 are shown in blue and red, respectively. The competitive investment region C is shown in green. The Nil Nash investment region N is shown in grey, dark blue and dark red. The intersection $S_1 \cap S_2$ between the region S_1 and the region S_2 is shown in pink. The intersection $S_1 \cap C$ between the region S_1 (respectively S_2) and the region C is shown in light blue (respectively light red); The intersection $S_1 \cap C \cap S_2$ between the region S_1, the region S_2 and the region C is shown in light grey (see the right hand figure).

3. **Investment analysis.** We are going to study the Nash investment equilibria $v : [c_L, \alpha]^2 \to (\mathbb{R}_0^+)^2$. Let us denote $v(c_1, c_2)$ by $(v_1(c_1, c_2), v_2(c_1, c_2))$ (see Figure 2).

If the new production costs $(a_1(v_1), a_2(v_2)) \in M_2$, then the Nash investment equilibrium v_1 of firm F_1 is $v_1 = 0$ (see Figure 2).

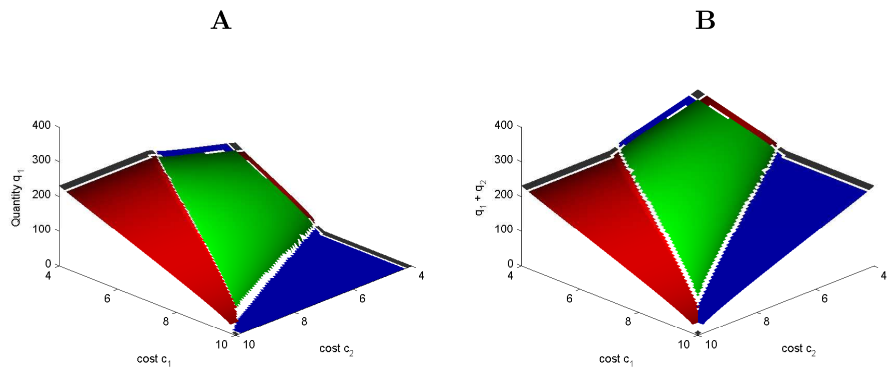

FIGURE 3. (**A**) Plot of the output level $q_1 = q_1(v_1(c_1, c_2), v_2(c_1, c_2); c_1, c_2)$ of Firm F_1 in terms of the initial production costs (c_1, c_2) using the Nash investment equilibria $(v_1(c_1, c_2), v_2(c_1, c_2))$; (**B**) Plot of the aggregated output levels $Q = q_1 + q_2$ of Firms F_1 and F_2 in terms of the initial production costs (c_1, c_2) using the Nash investment equilibria $(v_1(c_1, c_2), v_2(c_1, c_2))$.

Let $\eta_i = \epsilon(c_i - c_L)$, $L_1 = 6\beta\lambda^2 - \lambda\eta_1^2 - \eta_1\lambda(\alpha - c_1)$ and $N_1 = 2\beta\lambda^3 - \eta_1\lambda^2(\alpha - c_1)$. If the new production costs $(a_1(v_1), a_2(v_2)) \in M_1$, then the Nash investment equilibrium v_1 of firm F_1 is a solution of the following polynomial equation (see [5]):

$$2\beta v_1^3 + 6\beta\lambda v_1^2 + L_1 v_1 + N_1 = 0. \quad (5)$$

Let $A_i = 4\beta^2\eta_i\lambda$, $G_i = -2\beta\eta_i\lambda$, $H_i = \gamma\eta_i\lambda$, $C = (4\beta^2 - \gamma^2)^2$, $E_i = \alpha - c_i + \eta_i$ and $F_i = 2\beta E_i - \gamma E_j$. Let $I_i = -A_i F_i C^{-1}$, $J_i = -A_i H_i C^{-1}$ and $K_i = -A_i G_i C^{-1}$. If the new production costs $(a_1(v_1), a_2(v_2)) \in D$, then the Nash investment equilibrium v_1 of firm F_1 is a solution of the following polynomial equation (see [5]):

$$J_1^3 v_1^3 + (J_1(I_2 v_1 + J_2) - K_2)(v_1^3 + I_1 v_1 + K_1)^2 = 0. \quad (6)$$

4. **Profit and welfare analysis.** We study the output levels q_1, the prices p_1 and the profits π_1 of Firm F_1 depending upon the initial production costs (c_1, c_2), when both firms choose to invest accordingly with the Nash investment equilibria $(v_1(c_1, c_2), v_2(c_1, c_2))$. We find the consumer surplus cs_1 of the consumers of firm F_1 and the consumer surplus $cs = cs_1 + cs_2$ of the consumers of both firms depending upon the initial production costs (c_1, c_2), when both firms choose to invest accordingly with the Nash investment equilibria $(v_1(c_1, c_2), v_2(c_1, c_2))$. We analyse the welfare $w_1 = cs_1 + \pi_1$ of the market of firm F_1 and the welfare $w = w_1 + w_2$ of the consumers of both firms depending upon the initial production costs (c_1, c_2), when both firms choose to invest accordingly with the Nash investment equilibria $(v_1(c_1, c_2), v_2(c_1, c_2))$.

Let $q_i = q_i(v_1(c_1, c_2), v_2(c_1, c_2); c_1, c_2)$ be the output level of Firm F_i in terms of the production costs (c_1, c_2) using the Nash investment equilibria $(v_1(c_1, c_2), v_2(c_1, c_2))$ (see Figure 3**A**). The marginal rate of the output level q_i with respect to the

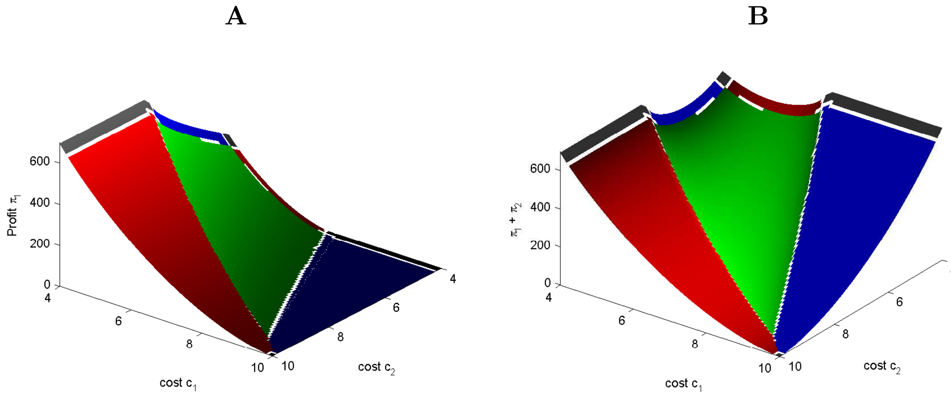

FIGURE 4. (**A**) Plot of the profit $\pi_1 = \pi_1(v_1(c_1,c_2), v_2(c_1,c_2); c_1,c_2)$ of Firm F_1 in terms of the initial production costs (c_1,c_2) using the Nash investment equilibria $(v_1(c_1,c_2), v_2(c_1,c_2))$; (**B**) Plot of the aggregate profit $\pi = \pi_1 + \pi_2$ in terms of the initial production costs (c_1,c_2) using the Nash investment equilibria $(v_1(c_1,c_2), v_2(c_1,c_2))$.

production costs c_i is given by:

$$\frac{dq_i}{dc_i} = \begin{cases} \frac{\eta_i\lambda}{2\beta(\lambda+v_i)^2}\frac{\partial v_i}{\partial c_i}, \text{ if } (a_i(v_i), a_j(v_j)) \in M_i \\ \\ \frac{2\beta\eta_i\lambda}{(4\beta^2-\gamma^2)(\lambda+v_i)^2}\frac{\partial v_i}{\partial c_i} - \frac{\gamma\epsilon(c_i-c_L)\lambda}{(4\beta^2-\gamma^2)(\lambda+v_j)^2}\frac{\partial v_j}{\partial c_i}, \text{ if } (a_i(v_i), a_j(v_j)) \in D \\ \\ 0, \text{ if } (a_i(v_i), a_j(v_j)) \in M_j \end{cases} \quad (7)$$

Let $p_i = p_i(v_1(c_1,c_2), v_2(c_1,c_2); c_1,c_2)$ be the price of Firm F_i in terms of the production costs (c_1,c_2) using the Nash investment equilibria $(v_1(c_1,c_2), v_2(c_1,c_2))$ (see Figure 3**C**). The marginal rate of the price p_i with respect to the production costs c_i is given by:

$$\frac{dp_i}{dc_i} = \begin{cases} -\frac{\eta_i\lambda}{2(\lambda+v_i)^2}\frac{\partial v_i}{\partial c_i}, \text{ if } (a_i(v_i), a_j(v_j)) \in M_i \\ \\ -\frac{2\beta^2\eta_i\lambda}{(4\beta^2-\gamma^2)(\lambda+v_i)^2}\frac{\partial v_i}{\partial c_i} + \frac{\beta\gamma\epsilon(c_i-c_L)\lambda}{(4\beta^2-\gamma^2)(\lambda+v_j)^2}\frac{\partial v_j}{\partial c_i}, \text{ if } (a_i(v_i), a_j(v_j)) \in D \\ \\ 0, \text{ if } (a_i(v_i), a_j(v_j)) \in M_j \end{cases} . \quad (8)$$

Let $\pi_i = \pi_i(v_1(c_1,c_2), v_2(c_1,c_2); c_1,c_2)$ be the profit of Firm F_i in terms of the production costs (c_1,c_2) using the Nash investment equilibria $(v_1(c_1,c_2), v_2(c_1,c_2))$ (see Figure 4**A**). The marginal rate of the profit π_i with respect to the production costs c_i is given by:

$$\frac{\partial \pi_i}{\partial c_i} = \frac{q_i \partial p_i}{\partial c_i} + \frac{p_i \partial q_i}{\partial c_i}.$$

The consumer surplus $cs_i = cs_i(v_1(c_1,c_2), v_2(c_1,c_2), c_1,c_2)$ of the consumers of firm F_i (see Figure 5**A**) is given by:

$$cs_i = (\beta_i q_i^2)/2. \quad (9)$$

In Figure 5**C** we present the welfare $w_i = w_i(v_1(c_1,c_2), v_2(c_1,c_2), c_1,c_2)$ of the market of firm F_i .

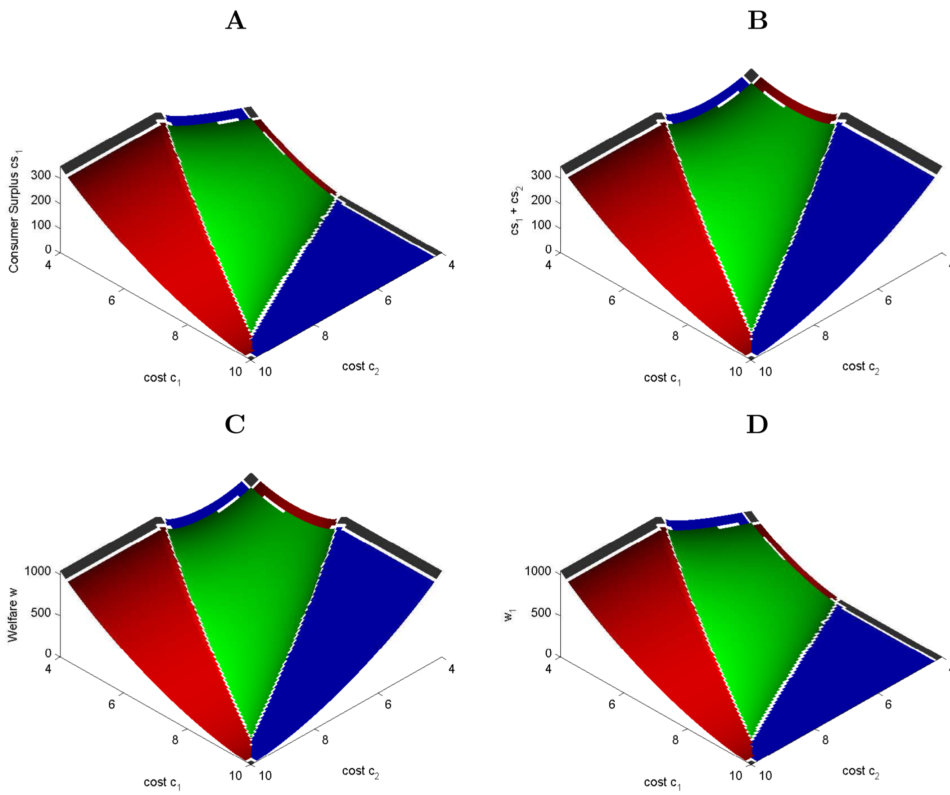

FIGURE 5. (**A**) Plot of the consumer surplus cs_1 of Firm F_1 in terms of the initial production costs (c_1, c_2) using the Nash investment equilibria $(v_1(c_1, c_2), v_2(c_1, c_2))$; (**B**) Plot of the aggregate consumer surplus $cs = cs_1 + cs_2$ of Firm F_1 in terms of the initial production costs (c_1, c_2) using the Nash investment equilibria $(v_1(c_1, c_2), v_2(c_1, c_2))$; (**C**) Plot of the welfare w_1 of firm F_1 in terms of the initial production costs (c_1, c_2) using the Nash investment equilibria $(v_1(c_1, c_2), v_2(c_1, c_2))$; (**D**) Plot of the aggregate welfare $w = w_1 + w_2$ in terms of the initial production costs (c_1, c_2) using the Nash investment equilibria $(v_1(c_1, c_2), v_2(c_1, c_2))$.

5. **Conclusions.** In this paper we studied the mathematical and economical properties of the multiple perfect Nash equilibria of a Cournot duopoly competition model where each of the firms invest in R&D projects to reduce its initial production costs. We analysed the output levels q_1, the prices p_1 and the profits π_1 of Firm F_1; the consumer surplus cs_1 of the consumers of firm F_1; and the welfare $w_1 = cs_1 + \pi_1$ of the market of firm F_1. Furthermore, we analysed the aggregated output levels $Q = q_1 + q_2$ and the aggregated profits $\pi = \pi_1 + \pi_2$ of both firms; the aggregated consumer surplus $cs = cs_1 + cs_2$ of the consumers of both firms and the aggregated welfare $w = w_1 + w_2$ of the full market.

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REFERENCES

[1] C. d'Aspremont and A. Jacquemin, *Cooperative and noncooperative R&D in duopoly with spillovers,* In "American Economic Review," **78** (1988), 1133–1137, (Erratum. In " American Economic Review," **80** 641–642.

[2] J. A. Brander and B. J. Spencer, *Strategic commitment with R&D: The symmetric case,* In "The Bell Journal of Economics," **14** (1983), 225–235.

[3] A. Cournot, *Recherches sur les Principes Mathématiques de la Théorie des Richesses*, Paris, 1838. English edition: Researches into the Mathematical Principles of the Theory of Wealth. Edited by N. Bacon. New York: Macmillan, 1897.

[4] F. A. Ferreira, F. Ferreira, M. Ferreira and A. A. Pinto, "Quantity Competition in a Differentiated Duopoly. Intelligent Engineering Systems and Computational Cybernetics," Springer Netherlands, Chapter 30, 2008.

[5] M. Ferreira, B. M. P. M. Oliveira and A. A. Pinto, *Patents in new technologies,* In "Journal of Difference equations and Applications," **15** (2009), 1135–1149.

[6] M. Ferreira, I. F. Figueiredo, B. M. P. M. Oliveira and A. A. Pinto, *Strategic optimization in R&D Investment*, In "Optimization," (Accepted) 2011 .

[7] M. Ferreira, B. M. P. M. Oliveira and A. A. Pinto, *Piecewise R&D Dynamics on costs,* In " Fasciculi Mathematici," **44** (2010), 29–42.

[8] A. A. Pinto, B. Oliveira, F. A. Ferreira and M. Ferreira, "Investing to Survive in a Duopoly Model. Intelligent Engineering Systems and Computational Cybernetics," Springer Netherlands, Chapter 23, 2008.

[9] L. Ruff,. *Research and technological progress in a Cournot economy,* In "Journal of Economic Theory," **1** (1969), 397–415.

[10] J. Tirole, "The Theory of Industrial Organization," MIT Press, MA, 1988.

[11] N. Singh and X. Vives, *Price and quantity competition in a differentiated duopoly,* In "RAND Journal of Economics," **15** (1984), 546–554.

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