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Bifurcation of periodic solutions of differential equations with finite symmetry groups

Adrian C. Murza

Director: Prof. Isabel S. Labouriau

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Abstract

In this PhD thesis we are concerned with differential equations which are equivariant under the action of certain finite groups. The thesis consists of three articles.

In the first place we investigate the Hopf bifurcation and synchronization properties of a $N \times N$ system of FitzHugh-Nagumo cells, the whole system of coupled cells being invariant under the $\mathbb{Z}_N \times \mathbb{Z}_N$ group.

We are interested in the periodic solutions arising at a first Hopf bifurcation from the fully synchronized equilibrium. Our approach involves the coupling types between the cells in the array: associative couplings reduce the difference between consecutive cells, while for dissociative couplings differences are increased. In the associative type of coupling we find bifurcation into a stable periodic solution where all the cells are synchronized with identical behavior. When the coupling is dissociative in either one or both directions, the first Hopf bifurcation gives rise to rings of N fully synchronized cells. All the rings oscillate with the same period, with a $\frac{1}{N}$-period phase shift between rings. When there is one direction of associative coupling, the synchrony rings are organized along it. Dissociative coupling in both directions yields rings organized along the diagonal. The stability of these periodic solutions was studied numerically and they were found to be unstable for small numbers of cells, stability starts to appear at $N \geqslant 11$.

In the second place, we investigate which periodic solutions predicted by the $H \bmod K$ theorem are obtainable by the Hopf bifurcation when the group Γ is tetrahedral and/or octahedral. We find that not all periodic solutions predicted by the $H \bmod K$ theorem occur as primary Hopf bifurcations from the trivial equilibrium. We also analyze submaximal bifurcations taking place in $4-$dimensional invariant subspaces and giving rise to periodic solutions with very small symmetry groups.

In the third place, we investigate the number of limit cycles that a family of $\mathbb{Z}_6-$equivariant polynomial planar system can give rise to. This question is related to the Hilbert XVI problem. The goal is studying the global

phase portrait of the quintic $\mathbb{Z}_6$-equivariant systems without infinite critical points, paying special attention to the existence, location and uniqueness of limit cycles surrounding 1, 7 or 13 critical points. We show that there is at most one limit cycle and that it surrounds all the equilibria.

Resumo

Nesta tese de doutoramento, estamos preocupados com as equações diferenciais que são equivariantes sob a ação de certos grupos finitos. A tese consiste de três artigos.

Em primeiro lugar investigamos a bifurcação de Hopf e sincronização dum sistema de $N \times N$ de células FitzHugh-Nagumo, todo o sistema de células acopladas sendo invariante sob o grupo $\mathbb{Z}_N \times \mathbb{Z}_N$.

Estamos interessados nas soluções periódicas decorrentes de uma primeira bifurcação Hopf do equilíbrio totalmente sincronizado. A nossa abordagem envolve os tipos de acoplamento entre as células no sistema: acoplamentos associativos reduzem a diferença entre células consecutivas, enquanto que para os acoplamentos dissociativos diferenças são aumentadas. No tipo associativo de acoplamento encontramos bifurcação para uma solução periódica estável onde as células são sincronizadas com um comportamento idêntico. Quando o acoplamento é dissociativo em qualquer um dos sentidos ou em ambos, a primeira bifurcação Hopf dá origem a anéis de N células totalmente sincronizadas. Todos os anéis oscilam com o mesmo período, com uma diferança de fase de $\frac{1}{N}$ entre anéis. Quando há uma direção de acoplamento associativo, os anéis de sincronia são organizados ao longo dela. Acoplamento dissociativo em ambos os sentidos produz anéis organizados ao longo da diagonal. A estabilidade destas soluções periódicas foi estudada numericamente. As soluções são inestáveis quando o número de células é pequeno; começa a aparecer a estabilidade para $N \geqslant 11$.

Em segundo lugar, investigamos quais as soluções periódicas previstas pelo teorema $H \bmod K$ são obtidas pela bifurcação Hopf quando a simetria do grupo Γ é tetraédrica e / ou octaédrica. Descobrimos que nem todas as soluções periódicas previstas pelo teorema $H \bmod K$ ocorrem em bifurcações Hopf primárias do equilíbrio trivial. Também analisamos bifurcações submaximais ocorrendo em subespaços invariantes de dimensão 4, e dando origem a soluções periódicas com grupos muito pequenos de simetria.

Em terceiro lugar, investigamos o número de ciclos limites que pode resultar duma família de sistemas planares

polinomiais $\mathbb{Z}_6-$ equivariantes. Esta questão está relacionada com o XVI problema Hilbert. O objetivo é estudar a diagrama de fase global de um sistema quíntico $\mathbb{Z}_6$-equivariante sem pontos críticos infinitos, dando especial atenção para a existência, localização e unicidade de ciclos limite que cercam 1, 7 ou 13 pontos críticos. Mostra-se que há no máximo um ciclo limite e que rodeia todos os equilíbrios.

A Dolores

"By a small sample we may judge of the whole piece"

– Miguel de Cervantes, *Don Quixote*–

Contents

Chapter 1

Introduction

"There is nothing mysterious, as some have tried to maintain, about the applicability of mathematics. What we get by abstraction from something can be returned."
– Raymond Louis Wilder, *Introduction to the Foundations of Mathematics*–

Bifurcation from periodic solutions in systems with symmetry is different than in systems with no symmetry. When studying specific model equations it may be possible to ignore symmetries and to analyze the bifurcation directly. However, as it is shown in the book "Singularities and Groups in Bifurcation Theory II" [8], it in this way it may be impossible to highlight those aspects of the analysis due to symmetry alone. Besides, taking symmetry into account often simplifies the bifurcation analysis. In [8] it is shown that there are three basic techniques that exploit symmetries of a system to make easier the bifurcation analysis:

(a) Restriction to fixed-point subspaces,

(b) Invariant theory,

(c) Equivariant singularity theory.

Roughly speaking the presence of symmetry may complicate the bifurcation analysis because it often forces eigenvalues of high multiplicity. When symmetry is part of the analysis, reduction to isotypic components may simplify the analysis by reducing it to a lower dimension.

More specifically, in the study of periodic solutions a theory of Hopf bifurcation with symmetry has been developed (see [8]), which gives model-independent information about bifurcation from periodic solutions, depending only on the symmetry of the system. The main result of this theory, The Equivariant Hopf Theorem, guarantees the existence of branches of periodic solutions to equations with symmetry group Γ, corresponding to isotropy subgroups of $\Gamma \times \mathbb{S}^1$ with two-dimensional fixed-point subspace.
Why periodic solutions? Because it is of interest analyzing, for example, the α and ω limit sets or the long-term asymptotic behavior of solutions and periodic solutions are a common limit behavior. The well-known condition for asymptotic stability, known as linear stability is that all the eigenvalues of the linearization have negative real part. The standard theorem states that if a periodic solution is linearly stable, then it is asymptotically stable. Periodic solutions bifurcate when the real part of some eigenvalue crosses the imaginary axis of the complex plane, away from zero.

This thesis consists of three articles. In the paper "Periodic solutions in an array of coupled FitzHugh-Nagumo cells", published in the Journal of Mathematical Analysis and Applications $412\ (2014),\ 29-40$, we analyze the dynamics of an array of N^2 identical cells coupled on the surface of an empty a torus. Each cell is a 2-dimensional ordinary differential equation of FitzHugh-Nagumo type and the global system is invariant under the action of $\mathbb{Z}_N \times \mathbb{Z}_N$ group. We are concerned with the Hopf bifurcation and the possible patterns of oscillation, compatible with the mentioned symmetry. The dynamics of a similar system, but with 1-dimensional ordinary differential equations coupled in a square array, of arbitrary size $(2N)^2$, with the symmetry $\mathbb{Z}_N \times \mathbb{Z}_N$, was studied by Gillis and Golubitsky [6]. In addition their study concerns the possible ways patterned solution to the original $N \times N$ cell system are expected to appear by static bifurcation from a trivial group invariant equilibrium.

In contrast, our analysis concerns the types of patterns that effectively arise through Hopf bifurcation; they are shown to depend on the signs of the coupling constants, under conditions ensuring that the equations have only one equilibrium state.

The patterns of oscillation arising in systems with abelian symmetry was developed by Filipsky and Golubitsky [5]. They identify which solutions from the ones predicted the $H \bmod K$ Theorem are produced from the Equivariant Hopf Theorem, when the group acting on the differential system is finite abelian.

We are interested in the periodic solutions arising at a first Hopf bifurcation from the fully synchronized equilibrium. Our approach involves the coupling types between the cells in the array. For associative couplings we reduce the difference between consecutive cells, while for dissociative couplings differences are increased. In the associative type of coupling we find bifurcation into a stable periodic solution where all the cells are synchronized with identical behavior.

Not surprisingly, when the coupling is dissociative in either one or both directions, the first Hopf bifurcation gives rise to rings of N fully synchronized cells. All the rings oscillate with the same period, with a $\frac{1}{N}$-period phase shift between rings. When there is one direction of associative coupling, the synchrony rings are organized along it. Dissociative coupling in both directions yields rings organized along the diagonal. The stability of these periodic solutions was studied numerically and were found to be unstable for small numbers of cells, stability starts to appear at $N \geqslant 11$.

Within the formalism developed in [7], [8] and [9], two methods for obtaining periodic solutions have been described: the $H \bmod K$ theorem [9, Ch.3] and the equivariant Hopf theorem [9, Ch.4]. The equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for all $\mathbf{C}-$axial subgroups of $\Gamma \times \mathbb{S}^1$, under some generic conditions. It also guarantees the existence of a model with this symmetry having these periodic solutions, but it is not an existence result for any specific equation. The $H \bmod K$ theorem offers the complete set of possible periodic solutions based exclusively on the structure of the group Γ acting on the differential equation.
Possible solutions predicted by the $H \bmod K$ theorem cannot always be obtained by a generic Hopf bifurcation [9] from the trivial equilibrium. Steady-state bifurcation problems with octahedral symmetry are analyzed in [10] using results from singularity theory. For non-degenerate bifurcation problems equivariant with respect to the standard action of the octahedral group they find three branches of symmetry-breaking steady-state bifurcations corresponding to the three maximal isotropy subgroups with one-dimensional fixed-point subspaces. Generic Hopf bifurcation in a system that is symmetric under the action of the rotational symmetries of the cube is studied in [1]. Hopf bifurcation with the rotational symmetry of the tetrahedron is studied in [11] where evidence of chaotic dynamics is presented, deriving from secondary bifurcations from periodic branches created at Hopf bifurcation.

In the article "Hopf bifurcation with tetrahedral and octahedral symmetry", we pose a more specific question: which periodic solutions predicted by the $H \bmod K$ theorem are obtainable by the Hopf bifurcation when the group Γ is tetrahedral or octahedral? The two groups are isomorphic but their representations are not and this creates differences in their action. We answer this question by finding that not all periodic solutions predicted by the $H \bmod K$ theorem occur as primary Hopf bifurcations from the trivial equilibrium. We also analyze secondary bifurcations taking place in $4-$dimensional invariant subspaces and giving rise to periodic solutions with very small symmetry groups.

Hilbert XVI problem represents one of the open questions in mathematics and mathematicians have investigated it throughout the last century. The study of this problem in the context of equivariant dynamical systems is a relatively new branch of analysis and is based on the development within the last twenty years of the theory of Golubitsky, Stewart and Schaeffer in [7, 8]. Other authors [4] have specifically considered this theory when studying the limit cycles and related phenomena in systems with symmetry. Roughly speaking the presence of symmetry may complicate the bifurcation analysis because it often forces eigenvalues of high multiplicity. This is not the case of planar systems; on the contrary, it simplifies the analysis because of the reduction to isotypic components. More precisely it allows us to reduce the bifurcation analysis to a region of the complex plane.
In the paper "Limit cycles for a class of quintic $\mathbb{Z}_6$-equivariant systems without infinite critical points", which is accepted for publication by the Bulletin of the Belgian Mathematical Society, we analyze a $\mathbb{Z}_6$-equivariant system. The study of this class of equations is developed in several books, see [3, 4], when the resonances are strong. The special case of $\mathbb{Z}_4$-equivariant systems without infinite critical points is also treated in several other articles, see [2, 4, 12]. In these mentioned works it is said that the weak resonances are easier to study than the other cases. This is true if the interest lies in obtaining a bifurcation diagram near the origin, but it is no longer true if the analysis is global and involves the study of limit cycles. The goal of the present work is studying the global phase portrait of the quintic $\mathbb{Z}_6$-equivariant systems without infinite critical points, paying special attention to the existence, location and uniqueness of limit cycles surrounding 1, 7 or 13 critical points. As far as we know this is the first work in which the existence of limit cycles is studied for this kind of systems.

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Chapter 2

Background to Equivariant Dynamical Systems

This section is made of copies from the background on Equivariant Bifurcation Theory in the books "The Symmetry Perspective", by Martin Golubitsky and Ian Stewart [2], "Singularities and Groups in Bifurcation Theory" by Martin Golubitsky David Schaeffer and Ian Stewart [1] and the article "The Abelian Hopf $H \bmod K$ Theorem" [3] by Filipsky and Golubitsky. The equivariant bifurcation theory for differential equation has been developed since 1983, as a result of the pioneering work of Martin Golubitsky and Ian Stewart. In this chapter we provide, without proofs, background results required for this thesis. Equivariant bifurcation theory characterizes the systems of ordinary differential equations with symmetry.
Let V be a finite dimensional vector space and let

$$\frac{dx}{dt} = f(x, \lambda) \tag{2.0.1}$$

be a system of ODEs where $x \in V$, $\lambda \in \mathbb{R}$ is a bifurcation parameter and $f : V \times \mathbb{R} \to V$ is a smooth, nonlinear map. We say that $f(x, \lambda)$ in (2.0.1) is $\Gamma-$equivariant -where Γ is a finite compact Lie group, if

$$f(\gamma \cdot x, \lambda) = \gamma \cdot f(x, \lambda),\ \forall x \in V,\ \forall \gamma \in \Gamma.$$

To formalize the notion of symmetry of a solution to an equivariant ODE, we define the isotropy subgroups of Γ. Equilibria of (2.0.1) are solutions of

$$f(x, \lambda) = 0.$$

Assume that x is such an equilibrium. The a symmetry σ of x is an element of Γ that leaves x invariant. The set of all such σ form a subgroup of Γ, called the isotropy subgroup of x. Formally we have

Definition 2.0.1 *Let $v \in \mathbb{R}^n$. The isotropy subgroup of v is*

$$\Sigma_v = \{\gamma \in \Gamma : \gamma v = v\}.$$

2.1 The isotropy lattice

If $x \in \mathbb{R}^n$ is an equilibrium of (2.0.1) and $\gamma \in \Gamma$, then so is γx, because $f(\gamma x, \lambda) = \gamma f(x, \lambda) = \gamma \cdot 0 = 0$. The group orbit of x is

$$\Gamma x = \{\gamma x : \gamma \in \Gamma\}.$$

Both x and γx have conjugate isotropy subgroups, since

$$\Sigma_{\gamma x} = \gamma \Sigma_x \gamma^{-1}.$$

Solutions of (2.0.1) can be classified in terms of conjugacy classes of isotropy subgroups, it is to say, the set of all conjugates of a given subgroup. Subgroups of a given group Γ are related to each other by the notion of containment, which means whether or not a subgroup is contained in another. If we carry this relation over the conjugacy classes of subgroups, we obtain the following structure:

Definition 2.1.1 *Let $H = \{H_i\}$ and $K = \{K_i\}$ be two conjugacy classes of isotropy subgroups of Γ. Define a partial ordering $\preceq$ on the set of conjugacy classes by*

$$H \preceq K \Leftrightarrow H_i \subseteq K_j$$

for some representatives H_i, K_j. The isotropy lattice of Γ in its action on $\mathbb{R}^n$ is the set of all conjugacy classes of isotropy subgroups, partially ordered by $\preceq$.

2.2 Fixed-Point subspaces

The structure of the isotropy lattice can be used to carry out research for solutions with a given symmetry. In addition, for certain type of isotropy subgroups, it can be guaranteed that generically solutions branches with such symmetries exist.

Definition 2.2.1 *Let $\Sigma \subseteq \Gamma$ be a subgroup. Then the fixed-point subspace of Σ is*

$$\mathrm{Fix}(\Sigma) = \{v \in \mathbb{R}^n : \sigma v = v, \ \forall \sigma \in \Sigma\}.$$

This definition leads to the following theorem, whose proof can be found in [2].

Theorem 2.2.2 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $\Gamma-$equivariant and let $\Sigma \subseteq \Gamma$ be a subgroup. Then*

$$f(\mathrm{Fix}(\Sigma)) \subseteq \mathrm{Fix}(\Sigma).$$

This theorem says that one can find an equilibrium solution with isotropy subgroup Σ by restricting the original vector field to the subspace $\mathrm{Fix}(\Sigma)$. In addition we have the following proposition, whose proof can also be found in [2].

Proposition 2.2.1 *Let $x(t)$ be a solution trajectory of an equivariant ODE. Then*

$$\Sigma_{x(t)} = \Sigma_{x(0)}, \ \forall t \in \mathbb{R}.$$

That is, isotropy subgroups remain constant along trajectories.

Lemma 2.2.1 *Let Γ act on $\mathbb{R}^n$ and let $\Sigma \subset \Gamma$ be a subgroup. Suppose that γ is in the normalizer N_Σ, that is $\gamma\Sigma\gamma^{-1} = \Sigma$. Then $\gamma\mathrm{Fix}(\Sigma) = \mathrm{Fix}(\Sigma)$.*

2.3 Invariant and irreducible subspaces

In [2] it is shown that if $f(0, \lambda) = 0$ then the kernel of the linearization of (2.0.1) at the origin is invariant under the action of Γ, and this property is valid for any equivariant system of ODEs.

Definition 2.3.1 *$V \subseteq \mathbb{R}^n$ is a $\Gamma-$invariant subspace if $\gamma V = V$, $\forall \gamma \in \Gamma$.*

It is shown in [2] that when a compact group acts on a space via a system of transformations, we can often decompose the space into subspaces, so that the group acts separately on each subspace, and the ”building-blocks” for the decomposition are said to be ”irreducible”:

Definition 2.3.2 *The subspace $W \subseteq \mathbb{R}^n$ is $\Gamma-$irreducible if the only $\Gamma-$invariant subspaces of W are W and* $\{0\}$.

If Γ acts irreducibly on $\mathbb{R}^n$, then $\text{Fix}(\Gamma) = 0$ or $\text{Fix}(\Gamma) = \mathbb{R}^n$. If Γ acts trivially, then $\text{Fix}(\Gamma) = \mathbb{R}^n$. If Γ acts nontrivially, then $\text{Fix}(\Gamma) = \{0\}$. If $\text{Fix}(\Gamma) = \{0\}$, then $f(0) = 0$ by Theorem 2.2.2. $\Gamma-$irreducible subspaces are important for the following reason.

Theorem 2.3.3 *Let $\Gamma \subseteq \mathbb{O}(n)$ be a compact Lie group. Then there exist $\Gamma-$irreducible subspaces $V_1, \ldots, V_s$ such that*

$$R^n = V_1 \oplus \cdots \oplus V_s$$

2.4 Commuting linear maps

When trying to understand symmetric bifurcations problems, the first step is to linearize the system, and understanding the linear maps that arise. These maps commute with the set of transformations by which the group acts.

Definition 2.4.1 *Let Γ act on V and let $A : V \to V$ be a linear map. A commutes with Γ if $A\gamma = \gamma A$, $\forall \gamma \in \Gamma$. We say that A is $\Gamma-$equivariant.*

Remark 2.4.1 *(1)* ker A *is a $\Gamma-$invariant subspace, since*

$$Av = 0 \Rightarrow A(\gamma v) = \gamma(Av) = 0, \text{ so } \gamma v \in \ker A.$$

If Γ acts irreducibly, then $\ker A = \{0\}$ *or* $\ker A = \mathbb{R}^n$. *Note that* $\ker A = \{0\}$ *implies that A is invertible, while* $\ker A = \mathbb{R}^n$ *implies that $A = 0$.*

(2) If A commutes with Γ and A^{-1} exists, then A^{-1} also commutes with Γ :

$$A^{-1}\gamma = (\gamma^{-1}A)^{-1} = (A\gamma^{-1})^{-1} = \gamma A^{-1}.$$

As shown in [2], there exists a fundamental classification of irreducible subspaces, according to the structure of the set of commuting linear maps. There are three basic types and the type has a major effect on bifurcation behavior. Let **D** be the set of commuting linear maps when the group acts on an irreducible subspace. It can be shown that $\mathbf{D} \subseteq GL(\mathbb{R}^n)$ is a linear subspace and is also a skew field. This statement is known as Schur's Lemma and it implies that **D** is either $\mathbb{R}, \mathbb{C}$ or the group of quaternions.

Definition 2.4.2 Γ *acts absolutely irreducibly on $\mathbb{R}^n$ if* $\mathbf{D} = \mathbb{R}$; *that is, the only commuting linear maps are scalar multiples of identity matrix:* $\{cI : c \in \mathbb{R}\}$.

Again, as shown in [2], absolute irreducibility has an important implication for equivariant bifurcation problems. Consider (2.0.1) where f is $\Gamma-$equivariant, and take into account the parameter λ. By applying the chain rule, we have

$$(df)_{\gamma x, \lambda\gamma} = \gamma (df)_{x,\lambda}$$

and

$$(df)_{0,\lambda\gamma} = \gamma (df)_{0,\lambda}.$$

Thus, $(df)_{0,\lambda}$ commutes with Γ. Therefore, if Γ acts absolutely irreducibly on $\mathbb{R}^n$, then

$$(df)_{0,\lambda} = c(\lambda) I.$$

2.5 Symmetry of the Jacobian and isotypic components

Suppose that we have a $\Gamma-$equivariant system $f(x, \lambda)$ with equilibrium (x_0, λ_0), such that $\Sigma_{x_0} \subseteq \Gamma$. To analyze stability we have to compute the eigenvalues of $(df)_{x_0,\lambda_0}$ at the solution (x_0, λ_0) of the equation $f(x, \lambda) = 0$.

Proposition 2.5.1 *With the above notation, $(df)_{x_0,\lambda_0}$ commutes with Σ_{x_0}.*

In order to deduce the restrictions imposed on $(df)_{x_0,\lambda_0}$ by commutativity, we have

Definition 2.5.1 *Suppose that Γ acts on V and also on W. We say that V is $\Gamma-$isomorphic to W, denoted $V \cong W$ if there exists a linear isomorphism $A : V \to W$ such that $A(\gamma x) = \gamma A(x)$, $\forall x \in V$, $\forall \gamma \in \Gamma$.*

Let Γ act on $\mathbb{R}^n$. We can decompose $\mathbb{R}^n$ into a direct sum of $\Gamma-$irreducible subspaces; in general, this decomposition is not unique. But, if we use components that combine together all of the $\Gamma-$irreducible subspaces that lie in a fixed isomorphism class, we obtain a decomposition that is unique.

Definition 2.5.2 *Choose a $\Gamma-$irreducible representation $V \subseteq \mathbb{R}^n$. Let $\hat{V}$ be the sum of all $\Gamma-$irreducible subspaces of $\mathbb{R}^n$ that are isomorphic to V. Then $\hat{V}$ is the isotypic component of $\mathbb{R}^n$ corresponding to V.*

Isotypic components are unique. They have a decomposition as a direct sum of isomorphic irreducibles:

Lemma 2.5.1 *Let V be an irreducible subspace. Then there exist irreducible subspaces $V_1, \dots, V_t$, all isomorphic to V, such that $\hat{V} = V_1 \oplus \cdots V_t$.*

Within a given isotypic component, all irreducible subspaces are isomorphic:

Lemma 2.5.2 *Let U, V be irreducible subspaces of $\mathbb{R}^n$ and suppose that $U \subseteq \hat{V}$. Then $U \cong V$.*

Lemma 2.5.3 *Choose irreducible subspaces $V_j \subseteq \mathbb{R}^n$, $j = 1, \dots, s$, such that every irreducible subspace is isomorphic to exactly one of the V_j. Then*

$$\mathbb{R}^n = \hat{V}_1 \oplus \cdots \oplus \hat{V}_s.$$

Lemma 2.5.4 *If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map that commutes with Γ, then $A(\hat{V}) \subseteq \hat{V}$.*

Proof. See [2]. ■

Definition 2.5.3 *The decomposition $\mathbb{R}^n = \hat{V}_1 \oplus \cdots \oplus \hat{V}_s$ is the isotypic decomposition of $\mathbb{R}^n$ with respect to the $\Gamma-$action. The subspaces $\hat{V}_j$ are isotypic components.*

2.6 Symmetries of periodic solutions

Suppose that $x(t)$ is a $T-$periodic solution of (2.0.1) and that $\gamma \in \Gamma$. As in [2], we discuss the ways in which γ can be a symmetry of $x(t)$. As a main tool, we use the uniqueness theorem for solutions to the initial value problem (2.0.1). We know that if $x(t)$ is a $T-$periodic solution of (2.0.1), then $\gamma x(t)$ is another $T-$periodic solution of (2.0.1). If the two solutions intersect, then the common point would be the initial point for the two solutions. Based on the uniqueness of solutions, we can affirm that trajectories $\gamma x(t)$ and $x(t)$ must be identical. In consequence, the two trajectories are identical or they do not intersect.
Suppose that the two trajectories are identical. The uniqueness of solutions implies that there exists $\theta \in \mathbb{S}^1 = [0, T]$ such that $\gamma x(t) = x(t - \theta)$, or

$$\gamma x(t + \theta) = x(t).$$

We call $(\gamma, \theta) \in \Gamma \times \mathbb{S}^1$ a spatio-temporal symmetry of the solution $x(t)$. A spatio-temporal symmetry of the solution $x(t)$ for which $\theta = 0$ is called a spatial symmetry, since it fixes the point $x(t)$ at every moment of time. The group of all spatio-temporal symmetries of $x(t)$ is denoted

$$\Sigma_{x(t)} \subseteq \Gamma \times \mathbb{S}^1.$$

The symmetry group $\Sigma_{x(t)}$ can be identified with a pair of subgroups H and K of Γ and a homomorphism $\Theta : H \to \mathbb{S}^1$ with kernel K. Define

$$\begin{aligned} K &= \{\gamma \in \Gamma : \gamma x(t) = x(t)\ \forall t\} \\ H &= \{\gamma \in \Gamma : \gamma\{x(t)\} = \{x(t)\}\}. \end{aligned} \tag{2.6.1}$$

The subgroup $K \subseteq \Sigma_{x(t)}$ is the group of spatial symmetries of $x(t)$ and the subgroup H consists of those symmetries that preserve the trajectory of $x(t)$; in short, the spatial part of the spatio-temporal symmetries of $x(t)$. The groups $H \subseteq \Gamma$ and $\Sigma_{x(t)} \subseteq \Gamma \times \mathbb{S}^1$ are isomorphic; the isomorphism is the restriction to $\Sigma_{x(t)}$ of the projection of $\Gamma \times \mathbb{S}^1$ onto Γ. The group $\Sigma_{x(t)}$ can be written as

$$\Sigma^\Theta = \{(h, \Theta(h)) : h \in H\}.$$

We call Σ^Θ a twisted subgroup of $\Gamma \times \mathbb{S}^1$. There are algebraic restrictions on the pair H and K defined in (2.6.1) in order for them to correspond to symmetries of a periodic solution.

Lemma 2.6.1 *Let $x(t)$ be a periodic solution of* (2.0.1) *and let H and K be the subgroups of* Γ *defined in* (2.6.1). *Then*

(a) *K is an isotropy subgroup for the $\Gamma-$action.*

(b) *K is a normal subgroup of H and H/K is either cyclic or $\mathbb{S}^1$.*

(c) $\dim \mathrm{Fix}(K) \geqslant 2$. *If* $\dim \mathrm{Fix}(K) = 2$, *then either $H = K$ or $H = N(K)$.*

Definition 2.6.1 *When $H/K \cong \mathbb{Z}_m$ the periodic solution $x(t)$ is called either a standing wave or (usually for $m \geqslant 3$) a discrete rotating wave; and when $H/K \cong \mathbb{S}^1$ it is called a rotating wave.*

2.7 A characterization of possible spatio-temporal symmetries

Let Γ be a finite group acting on $\mathbb{R}^n$ and let $x(t)$ be a periodic solution of a $\Gamma-$equivariant system of ODEs. Let K be the subgroup of spatial symmetries of $x(t)$ and let H be the subgroup of spatio-temporal symmetries defined in (2.6.1). Lemma 2.6.1 proves that H and K satisfy three algebraic conditions. As in [2], we show that there is one additional geometric restriction on H and K. Define

$$L_K = \bigcup_{\gamma \notin K} Fix(\gamma) \cap Fix(K).$$

Since K is an isotropy subgroup (Lemma 2.6.1), L_K is the union of proper subspaces of $\mathrm{Fix}(K)$. Suppose that $\mathrm{Fix}(\gamma) \supseteq \mathrm{Fix}(K)$. Then the isotropy subgroup of every point in $\mathrm{Fix}(K)$ contains both K and $\gamma \notin K$. Therefore, the isotropy subgroup of any point in $\mathrm{Fix}(K)$ is larger than K, and K is not an isotropy subgroup. Following [2], we claim that

$$H \text{ fixes a connected component of } \mathrm{Fix}\backslash L_K. \tag{2.7.1}$$

To verify (2.7.1), following [2], we observe that any $\delta \in N(K)$ permutes connected components of $\mathrm{Fix}(K)\backslash L_K$. We have

$$\delta(\mathrm{Fix}(\gamma) \cap \mathrm{Fix}(K)) = \mathrm{Fix}(\delta\gamma\delta^{-1}) \cap \mathrm{Fix}(\delta K \delta^{-1}) = \mathrm{Fix}(\delta\gamma\delta^{-1}) \cap \mathrm{Fix}(K).$$

Moreover, $\delta\gamma\delta^{-1} \notin K$. Since H/K is cyclic, we can choose an element $h \in H$ that projects onto a generator of H/K. We show that h (and hence H) must fix one of the connected components of $\mathrm{Fix}(K)\backslash L_K$. Suppose that the trajectory of $x(t)$ intersects the flow invariant space $\mathrm{Fix}(\gamma) \cap \mathrm{Fix}(K)$. Flow-invariance of $\mathrm{Fix}(\gamma)$ implies that γ is a spatial symmetry of the solution $x(t)$ and by definition $\gamma \in K$. Therefore, the trajectory of $x(t)$ does not intersect L_K. Since h is the spatial part of a spatio-temporal symmetry of $x(t)$, it preserves the trajectory of $x(t)$. Therefore, h must map the connected component of $\mathrm{Fix}(K)\backslash L_K$ that contains the trajectory of $x(t)$ into itself, thus verifying (2.7.1). We have

Theorem 2.7.1 *[The H mod K Theorem] Let Γ be a finite group acting on $\mathbb{R}^n$. There is a periodic solution to some $\Gamma-$equevariant system of ODEs on $\mathbb{R}^n$ with spatial symmetries K and spatio-temporal symmetries H if and only if*

(a) H/K *is cyclic,*

(b) K *is an isotropy subgroup,*

(c) $\dim \mathrm{Fix}(K) \geqslant 2$. *If* $\dim \mathrm{Fix}(K) = 2$, *then either* $H = K$ *or* $H = N(K)$.

(d) H *fixes a connected component of* $\mathrm{Fix}(K)\backslash L_K$.

Moreover, when these conditions hold, hyperbolic asymptotically stable limit cycles with the desired symmetry exist.

2.8 Hopf bifurcation with symmetry

Suppose we have a $\Gamma-$equivariant system system (2.0.1), where $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $f(x_0, \lambda_0) = 0$, which undergoes a local bifurcation at (x_0, λ_0).

Definition 2.8.1 *A local bifurcation occurs when $(df)_{(x_0,\lambda_0)}$ has an eigenvalue on the imaginary axis.*

The two principal types of local bifurcation can be distinguished by the nature of the critical eigenvalues:

(1) Steady-state bifurcation: a zero eigenvalue

(2) Hopf bifurcation: a pair of nonzero purely imaginary eigenvalues.

In this thesis we are interested on the Hopf bifurcation. In the first place is to analyze the cases when the linearization of equivariant equations have purely imaginary eigenvalues.

Lemma 2.8.1 *Suppose $(df)_{(x_0,\lambda_0)}$ has a non-real eigenvalue. Then there exists an irreducible representation $V \subseteq \mathbb{R}^n$ such that either*

(a) $V \oplus V \subseteq \mathbb{R}^n$ *and* Γ *acts absolutely irreducibly on* V, *or*

(b) Γ *acts non-absolutely irreducibly on* V.

In [2] it is shown that if Lemma 2.8.1(a) or (b) holds, then not only can *df* have non-real eigenvalues-it can have nonzero purely imaginary eigenvalues. This leads to

Definition 2.8.2 *The vector space W is $\Gamma-$simple if either:*

(a) $W = V \oplus V$ *where* V *is an absolutely irreducible representation of* Γ, *or*

(b) Γ *acts irreducibly but not absolutely irreducibly on* W.

Theorem 2.8.3 *Generically, at a Hopf bifurcation, the action of Γ on the center subspace is $\Gamma-$simple*

Proof. See [1], Chapter XVI, Proposition 1.4. ■

Theorem 2.8.3 means that when studying Hopf bifurcation in a one-parameter family of ODEs, we may assume that Γ acts $\Gamma-$simply on $\mathbb{R}^{2m}$.

Remark 2.8.1 $\mathrm{Fix}_{\mathbb{R}^{2m}}(\Gamma) = \{0\}$ *when* Γ *acts* $\Gamma-$*simply on* $\mathbb{R}^{2m}$. *It follows that* $f(0) = 0$ *for any* $\Gamma-$*equivariant* f; *thus* f *always has a trivial solution.*

For the remainder of the chapter, we assume that $(x_0, \lambda_0) = (0, 0)$.

Lemma 2.8.2 *Assume that* $\mathbb{R}^n$ *is* $\Gamma-$*simple and* $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *is a* $\Gamma-$*equivariant mapping, such that* $(df)_{(0,0)}$ *has eigenvalues* $\pm\omega i$ *with* $\omega > 0$. *Then the following conditions hold:*

(a) The eigenvalues of $(df)_{(0,\lambda)}$ *are* $\sigma(\lambda) \pm i\rho(\lambda)$, *each of multiplicity* $m = n/2$, *and* σ *and* ρ *depend smoothly on* λ *with* $\sigma(0) = 0$ *and* $\rho(0) = \omega$.

(b) There exists a linear invertible mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ *that commutes with* Γ *such that*

$$(df)_{(0,0)} = \omega S J S^{-1}$$

where

$$J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix} \tag{2.8.1}$$

As in [2], we can rescale time to ensure that $\omega = 1$, and therefore we may assume from now on that

$$(df)_{(0,0)} = J. \tag{2.8.2}$$

Linearization (2.8.2) provides us with an action of $\mathbb{S}^1$ on $\mathbb{R}^n$, namely

$$\theta x = e^{-i\theta J} x. \tag{2.8.3}$$

The action (2.8.3) proves the existence of periodic solutions in the Equivariant Hopf Theorem. The $\mathbb{S}^1$ (2.8.3) commutes with the action of Γ because J commutes with Γ. The Equivariant Hopf Theorem asserts the existence of branches of periodic solutions corresponding to $\mathbb{C}-$axial subgroups, in the following sense:

Definition 2.8.4 *Let* W *be a* $\Gamma-$*simple representation and let* $\mathbb{S}^1$ *act on* W *as in* (2.8.3). *A subgroup* $\Sigma \subseteq \Gamma \times \mathbb{S}^1$ *is* $\mathbf{C}-$*axial if* Σ *is an isotropy subgroup and*

$$\dim \mathrm{Fix}(\Sigma) = 2.$$

Theorem 2.8.5 (Equivariant Hopf Theorem) *Let a compact Lie group* Γ *act* $\Gamma-$*symply, orthogonally, and non-trivially on* $\mathbb{R}^{2m}$. *Assume that*

(a) $f : \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}^{2m}$ *is* $\Gamma-$*equivariant. Then* $f(0, \lambda) = 0$ *and* $(df)_{(0,\lambda)}$ *has eigenvalues* $\sigma(\lambda) \pm i\rho(\lambda)$ *each of multiplicity* m.

(b) $\sigma(0) = 0$ *and* $\rho(0) = 1$.

(c) $\sigma'(0) \neq 0-$ *the eigenvalue crossing condition.*

(d) $\Sigma \subseteq \Gamma \times \mathbb{S}^1$ *is a* $\mathbf{C}-$*axial subgroup.*

Then there exists a unique branch of periodic solutions with period $\approx 2\pi$ *emanating from the origin, with spatio-temporal symmetries* Σ.

2.9 The Abelian Hopf H mod K Theorem

The $H \bmod K$ Theorem 2.7.1 gives necessary and sufficient conditions for the existence of periodic solution to some $\Gamma-$equivariant system of ODEs with specified spatio-temporal symmetries $K \subset H \subset \Gamma$.
Let V be the center subspace of the bifurcation at x_0, and note that Σ_{x_0} acts on V.

Theorem 2.9.1 (Abelian Hopf Theorem) *In systems with abelian symmetry, generically, Hopf bifurcation at a point x_0 occurs with simple eigenvalues, and there exists a unique branch of small-amplitude periodic solutions emanating from x_0. Moreover, the spatio-temporal symmetries of the bifurcation solutions are*

$$H = \Sigma_{x_0}$$

and

$$K = \ker_V(H),$$

and H acts $H-$simply on V.

The abelian Hopf $H \bmod K$ theorem gives necessary and sufficient conditions for when an $H \mathrm{mod} K$ periodic solution $x(t)$ can occur by Hopf bifurcation from a point x_0.

Theorem 2.9.2 (The Abelian Hopf $H \bmod K$ Theorem) *Let Γ be a finite abelian group acting on $\mathbb{R}^n$. There is an $H \bmod K$ periodic solution that arises by a generic Hopf bifurcation if and only if the following six conditions hold: Theorem 2.7 (a)-(d), H is an isotropy subgroup, and there exists an $H-$simple subspace V such that $K = \ker_V(H)$.*

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Chapter 3

Periodic solutions in an array of coupled FitzHugh-Nagumo cells

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Isabel S. Labouriau and Adrian C. Murza

I.S. Labouriau—Centro de Matemática da Universidade do Porto. Rua do Campo Alegre 687, 4169-007 Porto, Portugal
A.C. Murza — Departamento de Matemática, Faculdade de Ciências Universidade do Porto. Rua do Campo Alegre 687, 4169-007 Porto, Portugal

Abstract. *We analyse the dynamics of an array of N^2 identical cells coupled in the shape of a torus. Each cell is a 2-dimensional ordinary differential equation of FitzHugh-Nagumo type and the total system is $\mathbb{Z}_N \times \mathbb{Z}_N$−symmetric. The possible patterns of oscillation, compatible with the symmetry, are described. The types of patterns that effectively arise through Hopf bifurcation are shown to depend on the signs of the coupling constants, under conditions ensuring that the equations have only one equilibrium state.*

Keywords: equivariant dynamical system, ordinary differential equation, FitzHugh-Nagumo, periodic solutions, Hopf bifurcation, coupled cells

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3.1 Introduction

Hopf bifurcation has been intensively studied in equivariant dynamical systems in the recent years from both theoretical and applied points of view. Stability of equilibria, synchronization of periodic solution and in general oscillation patterns, stability of the limit cycles that arise at the bifurcation point are among the phenomena whose analysis is related to the Hopf bifurcation in these systems. Periodic solutions arising in systems with dihedral group symmetry were studied by Golubitsky et al. [10] and Swift [8], Dias and Rodrigues [2] dealt with the symmetric group, Sigrist [12] with the orthogonal group, to cite just a few of them. Dias *et al.* [1] studied periodic solutions in coupled cell systems with interior symmetries, while Dionne addresses the analysis to Hopf bifurcation in equivariant dynamical systems with wreath product [3] and direct product groups [4]. The general theory of patterns of oscillation arising in systems with abelian symmetry was developed by Filipsky and Golubitsky [5]. The dynamical behavior of 1-dimensional ordinary differential equations coupled in a square array, of arbitrary size $(2N)^2$, with the symmetry $\mathbb{Z}_N \times \mathbb{Z}_N$, was studied by Gillis and Golubitsky [6].

In this paper we use a similar idea to that of [6] to describe arrays of N^2 cells where each cell is represented by a subsystem that is a 2-dimensional differential equation of FitzHugh-Nagumo type. We are interested in the periodic solutions arising at a first Hopf bifurcation from the fully synchronised equilibrium. To each equation in the array we add a coupling term that describes how each cell is affected by its neighbours. The coupling may be associative, when it tends to reduce the difference between consecutive cells, or dissociative, when differences are increased. For associative coupling we find, not surprisingly, bifurcation into a stable periodic solution where all the cells are synchronised with identical behaviour.

When the coupling is dissociative in either one or both directions, the first Hopf bifurcation gives rise to rings of N fully synchronised cells. All the rings oscillate with the same period, with a $\frac{1}{N}$-period phase shift between rings. When there is one direction of associative coupling, the synchrony rings are organised along it. Dissociative coupling in both directions yields rings organised along the diagonal. The stability of these periodic solutions was studied numerically and were found to be unstable for small numbers of cells, stability starts to appear at $N \geqslant 11$.

For all types of coupling, there are further Hopf bifurcations, but these necessarily yield unstable solutions.

This paper is organised as follows. The equations are presented in section 3.2 together with their symmetries. Details about the action of the symmetry group $\mathbb{Z}_N \times \mathbb{Z}_N$ are summarised in section 3.3: we identify the $\mathbb{Z}_N \times \mathbb{Z}_N-$irreducible subspaces of $\mathbb{R}^{2N^2}$; the isotypic components; isotropy subgroups and their fixed point subspaces for this action. This allows, in section 3.4, the study of the Hopf bifurcation with symmetry $\mathbb{Z}_N \times \mathbb{Z}_N$, applying the abelian Hopf bifurcation theorem [5] to identify the symmetries of the branch of small-amplitude peridic solutions that may bifurcate from equilibria. In section 3.5 we derive the explicit expression of the $2N^2$ eigenvectors and eigenvalues of the system linearised about the origin. Next, in section 3.6 we perform a detailed analysis on the Hopf bifurcation by setting a parameter c to zero. In this case the FitzHugh-Nagumo equation reduce to a Van der Pol-like equation. Finally, in section 3.7, by applying the Abelian Hopf H mod K Theorem [5] we characterise the bifurcation conditions for $c > 0$ small.

3.2 Dynamics of FitzHugh-Nagumo coupled in a torus and its symmetries

The building-blocks of our square array are the following $2-$dimensional ordinary differential equations of FitzHugh-Nagumo (FHN) type

$$\begin{array}{ll} \dot{x} = x\,(a-x)\,(x-1) - y & = f_1(x,y) \\ \dot{y} = bx - cy & = f_2(x,y) \end{array} \tag{3.2.1}$$

where $a, b, c \geqslant 0$. Consider a system of N^2 such equations, coupled as a discrete torus:

$$\begin{array}{l} \dot{x}_{\alpha,\beta} = x_{\alpha,\beta}\,(a - x_{\alpha,\beta})\,(x_{\alpha,\beta} - 1) - y_{\alpha,\beta} + \gamma(x_{\alpha,\beta} - x_{\alpha+1,\beta}) + \delta(x_{\alpha,\beta} - x_{\alpha,\beta+1}) \\ \dot{y}_{\alpha,\beta} = bx_{\alpha,\beta} - cy_{\alpha,\beta} \end{array} \tag{3.2.2}$$

where $\gamma \neq \delta$ and $1 \leqslant \alpha \leqslant N$, $1 \leqslant \beta \leqslant N$, with both α and β computed $\pmod N$. When either γ or δ is negative, we say that the coupling is *associative*: the coupling term tends to reduce the difference to the neighbouring cel, otherwise we say the coupling is *dissociative*. We restrict ourselves to the case where $N \geqslant 3$ is prime.

The first step in our analysis consists in describing the symmetries of (3.2.2). Our phase space is

$$\mathbb{R}^{2N^2} = \{(x_{\alpha,\beta}, y_{\alpha,\beta}) \mid \; 1 \leqslant \alpha, \beta \leqslant N, \; x_{\alpha,\beta}, y_{\alpha,\beta} \in \mathbb{R}\}$$

and (3.2.2) is equivariant under the cyclic permutation of the columns in the squared array:

$$\gamma_1(x_{1,\beta},\dots,x_{N,\beta};y_{1,\beta},\dots,y_{N,\beta}) = (x_{2,\beta},\dots,x_{N,\beta},x_{1,\beta};y_{2,\beta},\dots,y_{N,\beta},y_{1,\beta}) \tag{3.2.3}$$

as well as under the cyclic permutation of the rows in the squared array:

$$\gamma_2(x_{\alpha,1},\dots,x_{\alpha,N};y_{\alpha,1},\dots,y_{\alpha,N}) = (x_{\alpha,2},\dots,x_{\alpha,N},x_{\alpha,1};y_{\alpha,2},\dots,y_{\alpha,N},y_{\alpha,1}). \tag{3.2.4}$$

Thus, the symmetry group of (3.2.2) is the group generated by γ_1 and γ_2 denoted $\mathbb{Z}_N \times \mathbb{Z}_N = \langle\gamma_1,\gamma_2\rangle$. We will use the notation $\gamma_1^r \cdot \gamma_2^s \in \mathbb{Z}_N \times \mathbb{Z}_N$ as $(r,s) = \gamma_1^r \cdot \gamma_2^s$. Let's refer to the system (3.2.2) in an abbreviated form $\dot{z} = f(z)$, $z = (x_{\alpha,\beta}, y_{\alpha,\beta})$ while $\lambda \in \mathbb{R}$ is a bifurcation parameter to be specified later. The compact Lie group $\mathbb{Z}_N \times \mathbb{Z}_N$ acts linearly on $\mathbb{R}^{2N^2}$ and f commutes with it (or is $\mathbb{Z}_N \times \mathbb{Z}_N-$equivariant).

We start by recalling some definitions from [10] adapted to our case.

The isotropy subgroup Σ_z of $\mathbb{Z}_N \times \mathbb{Z}_N$ at a point $z \in \mathbb{R}^{2N^2}$ is defined to be

$$\Sigma_z = \{(r,s) \in \mathbb{Z}_N \times \mathbb{Z}_N : (r,s)\cdot z = z\}.$$

Moreover, the fixed point subspace of a subgroup $\Sigma \in \mathbb{Z}_N \times \mathbb{Z}_N$ is

$$\mathrm{Fix}\,(\Sigma) = \left\{z \in \mathbb{R}^{2N^2} : (r,s)\cdot z = z,\ \forall (r,s) \in \Sigma\right\}$$

and $f\,(\mathrm{Fix}\,(\Sigma)) \subseteq \mathrm{Fix}\,(\Sigma)$.

Definition 3.2.1 *Consider a group* $\mathbf{\Gamma}$ *acting linearly on* $\mathbb{R}^n$. *Then*

1. *A subspace* $\mathbf{V} \subseteq \mathbb{R}^n$ *is said* $\mathbf{\Gamma}-$invariant, *if* $\sigma\cdot v \in \mathbf{V}$, $\forall \sigma \in \mathbf{\Gamma}$, $\forall v \in \mathbf{V}$;
2. *A subspace* $\mathbf{V} \subseteq \mathbb{R}^n$ *is said* $\mathbf{\Gamma}-$irreducible *if it is* $\mathbf{\Gamma}-$*invariant and if the only* $\mathbf{\Gamma}-$*invariant subspaces of* $\mathbf{V}$ *are* $\{\mathbf{0}\}$ *and* $\mathbf{V}$.

Definition 3.2.2 *Suppose a group* Γ *acts on two vector spaces* $\mathbf{V}$ *and* $\mathbf{W}$. *We say that* $\mathbf{V}$ *is* Γ*-isomorphic to* $\mathbf{W}$ *if there exists a linear isomorphism* $A : \mathbf{V} \longrightarrow \mathbf{W}$ *such that* $A(\sigma x) = \sigma A(x)$ *for all* $x \in \mathbf{V}$. *If* $\mathbf{V}$ *is* not Γ*-isomorphic to* $\mathbf{W}$ *we say that they are distinct representations of* Γ.

3.3 The $\mathbb{Z}_N \times \mathbb{Z}_N$ action

The action of $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_N$ is identical in the $x_{\alpha,\beta}$ and the $y_{\alpha,\beta}$ coordinates, i.e. Γ acts diagonally, $\gamma(x,y) = (\gamma x, \gamma y)$ in $\mathbb{R}^{2N^2}$ for $x, y \in \mathbb{R}^{N^2}$. Hence, instead of taking into account the whole set of $(x_{\alpha,\beta}, y_{\alpha,\beta}) \in \mathbb{R}^{2N^2}$, we will partition it into two subspaces, $\mathbb{R}^{N^2} \times \{\mathbf{0}\}$ and $\{\mathbf{0}\} \times \mathbb{R}^{N^2}$, namely $x_{\alpha,\beta} \in \mathbb{R}^{N^2}$ and $y_{\alpha,\beta} \in \mathbb{R}^{N^2}$. The action of $\mathbb{Z}_N \times \mathbb{Z}_N$ on $\mathbb{R}^{N^2}$ has been studied by Gillis and Golubitsky in [6] for $N = 2n$, we adapt their results to our case.

Let $\mathbf{k} = (k_1,k_2) \in \mathbb{Z}^2$ and consider the subspace $\mathbf{V_k} \subset \mathbb{R}^{N^2}$, where $(x_{\alpha,\beta}) \in \mathbf{V_k}$ if and only if

$$x_{\alpha,\beta} = \mathrm{Re}\left(z \exp\left[\frac{2\pi i}{N}(\alpha,\beta)\cdot\mathbf{k}\right]\right) \in \mathbb{R}^{N^2}\ z \in \mathbb{C},\ 1 \leqslant \alpha,\beta \leqslant N\,. \tag{3.3.1}$$

Proposition 3.3.1 *Consider the action of* $\mathbb{Z}_N \times \mathbb{Z}_N$ *on* $\mathbb{R}^{N^2}$ *given in* (3.2.3) *and* (3.2.4) *with* N *prime and let* I *be the set of indices* $\mathbf{k} = (k_1,k_2)$ *listed in Table 3.1. Then for* $\mathbf{k} \in I$ *we have*

1. $\dim\mathbf{V_k} = 2$ *except for* $\mathbf{k} = \mathbf{0}$, *where* $\dim\mathbf{V_0} = 1$.
2. *Each* $\mathbf{V_k}$ *defined in* (3.3.1) *is* $\mathbb{Z}_N \times \mathbb{Z}_N-$*invariant and* $\mathbb{Z}_N \times \mathbb{Z}_N-$*irreducible.*
3. *The subspaces* $\mathbf{V_k}$ *are all distinct* $\mathbb{Z}_N \times \mathbb{Z}_N$ *representations.*

4. *The group element* (r,s) *acts on* $\mathbf{V_k}$ *as a rotation:*

$$(r,s)\cdot z = \exp\left[\frac{2\pi i}{N}(r,s)\cdot\mathbf{k}\right]z.$$

5. *The subspaces* $\mathbf{V_k}$ *verify* $\bigoplus_{\mathbf{k}\in I}\mathbf{V_k} = \mathbb{R}^{N^2}$.
6. *The non-trivial isotropy subgroups for* $\mathbb{Z}_N\times\mathbb{Z}_N$ *on* $\mathbb{R}^{N^2}$ *are* $\widetilde{\mathbb{Z}}_N(r,s)$, *the subgroups generated by one element* $(r,s)\neq(0,0)$.
7. *If* $(r,s)\neq(0,0)$ *then*

$$Fix_{\mathbb{R}^{N^2}}\left(\widetilde{\mathbb{Z}}_N(r,s)\right) = \sum_{\substack{\mathbf{k}\cdot(r,s)=0 \\ (\mathrm{mod}\ N)}}\mathbf{V_k} \qquad \textit{and} \qquad \dim Fix\left(\widetilde{\mathbb{Z}}_N(r,s)\right) = N.$$

Table 3.1. Types of $\mathbb{Z}_N\times\mathbb{Z}_N-$irreducible representations in $\mathbf{V_k}\in\mathbb{R}^{N^2}$, where $\mathbf{V_k}$ is the subspace of $\mathbb{R}^{N^2}$ corresponding to $\mathbf{k}=(k_1,k_2)\in I\subset\mathbb{Z}^2$.

Type	$\dim(\mathbf{V_k})$	$\mathbf{k}$	Restrictions
(1)	1	$(0,0)$	
(2)	2	$(0,k_2)$	$1\leqslant k_2\leqslant(N-1)/2$
(3)	2	$(k_1,0)$	$1\leqslant k_1\leqslant(N-1)/2$
(4)	2	(k_1,k_1)	$1\leqslant k_1\leqslant(N-1)/2$
(5)	2	(k_1,k_2)	$1\leqslant k_2<k_1\leqslant N-1$

Proof. The arguments given in [6, Lemma 3.1] with suitable adaptations show that statements 1.–4. hold and also that the subspaces $\mathbf{V_k}$ with $\mathbf{k}\notin I$ are redundant. Since the $\mathbf{V_k}$ are all distinct irreducible representations, a calculation using 1. shows that $\sum_{\mathbf{k}\in I}\dim\mathbf{V_k}=N^2$, establishing 5.

For 6. note that since N is prime, the only non trivial subgroups of $\mathbb{Z}_N\times\mathbb{Z}_N$ are the cyclic subgroups $\widetilde{\mathbb{Z}}_N(r,s)$ generated by $(r,s)\neq(0,0)$. Each (r,s) fixes the elements of $\mathbf{V}_{(k_1,k_2)}$ when $(k_1,k_2)=(N-s,r)$. Using 4. it follows that $(r,s)\neq(0,0)$ fixes $x=(x_{\alpha,\beta})\in\mathbf{V}_{(k_1,k_2)}$ with $x\neq0$ if and only if $\exp\left[\frac{2\pi i}{N}(r,s)\cdot(k_1,k_2)\right]=1$ i.e. if and only if $(r,s)\cdot(k_1,k_2)=0 \pmod N$. Thus $\mathrm{Fix}(\mathrm{r,s})$ is the sum of all the subspaces $\mathbf{V}_{(k_1,k_2)}$ such that $(r,s)\cdot(k_1,k_2)=0 \pmod N$, it remains to compute its dimension.

Let $(k_1',k_2')=(N-s,r)$, so that $\mathbf{V}_{(k_1',k_2')}\subset\mathrm{Fix}(\mathrm{r,s})$. Then for any (k_1,k_2) we have $(r,s)\cdot(k_1,k_2)=\det\begin{bmatrix}k_1 & k_2\\ k_1' & k_2'\end{bmatrix} \pmod N$. Thus $\mathbf{V}_{(k_1,k_2)}\subset\mathrm{Fix}(\mathrm{r,s})$ if and only if $\det\begin{bmatrix}k_1 & k_2\\ k_1' & k_2'\end{bmatrix}=0 \pmod N$, and $(r,s)\in\ker\begin{bmatrix}k_1 & k_2\\ k_1' & k_2'\end{bmatrix} \pmod N$. This is equivalent to having (k_1,k_2) and (k_1',k_2') linearly dependent over $\mathbb{Z}_N$, i.e. $(k_1',k_2')=m(k_1,k_2) \pmod N$. Then $\mathrm{Fix}(r,s)=\Sigma_{m\in\mathbb{Z}_N}\mathbf{V}_{m(N-s,r)}$. Since half of the $(k_1,k_2)=m(N-s,r)$ have $k_2>k_1$, this expression adds two times the same subspace. Then, since for $(k_1,k_2)\neq(0,0)$ we have $\dim\mathbf{V}_{(k_1,k_2)}=2$ we obtain $\mathrm{dimFix}(r,s)=1+\frac{1}{2}(N-1)\cdot2=N$. ■

3.3.1 The $\mathbb{Z}_N\times\mathbb{Z}_N-$isotypic components of $\mathbb{R}^{2N^2}$

So far we have obtained the distinct $\mathbb{Z}_N\times\mathbb{Z}_N-$invariant representations of $\mathbb{R}^{N^2}$, by considering only the subspaces corresponding to the variable x. If $\mathbf{V}\subset\mathbb{R}^{N^2}$ is an irreducible subspace for the action of $\mathbb{Z}_N\times\mathbb{Z}_N$ then $\mathbf{V}\times\{\mathbf{0}\}$ and $\{\mathbf{0}\}\times\mathbf{V}$ are $\mathbb{Z}_N\times\mathbb{Z}_N-$isomorphic irreducible subspaces of $\mathbb{R}^{2N^2}$. We will use the notation $\mathbf{V}\oplus\mathbf{V}$ for the subspace $\mathbf{V}\times\mathbf{V}\subset\mathbb{R}^{2N^2}$.

Definition 3.3.1 *Suppose a group Γ acts on $\mathbb{R}^n$ and let $\mathbf{V} \subset \mathbb{R}^n$ be a Γ-irreducible subspace. The isotypic component of $\mathbb{R}^n$ corresponding to $\mathbf{V}$ is the sum of all Γ-irreducible subspaces that are Γ-isomorphic to $\mathbf{V}$.*

Once we have the $\mathbb{Z}_N \times \mathbb{Z}_N-$ireducible representations on $\mathbb{R}^{N^2}$, we can calculate the isotypic components of the representation of this group on $\mathbb{R}^{2N^2}$.

Table 3.2. Isotypic components $\mathbf{Z_k}$ of the $\mathbb{Z}_N \times \mathbb{Z}_N-$action on $\mathbb{R}^{2N^2}$, where the $\mathbf{V_k}$ are defined in (3.3.1).

$\dim(\mathbf{Z_k})$	$\mathbf{Z_k}$	Restrictions
2	$\mathbf{V}_{(0,0)} \oplus \mathbf{V}_{(0,0)}$	
4	$\mathbf{V}_{(k_1,0)} \oplus \mathbf{V}_{(k_1,0)}$	$1 \leqslant k_1 \leqslant (N-1)/2$
4	$\mathbf{V}_{(0,k_2)} \oplus \mathbf{V}_{(0,k_2)}$	$1 \leqslant k_2 \leqslant (N-1)/2$
4	$\mathbf{V}_{(k_1,k_1)} \oplus \mathbf{V}_{(k_1,k_1)}$	$1 \leqslant k_1 \leqslant (N-1)/2$
4	$\mathbf{V}_{(k_1,k_2)} \oplus \mathbf{V}_{(k_1,k_2)}$	$1 \leqslant k_2 < k_1 \leqslant N-1$

Lemma 3.3.1 *The isotypic components for the $\mathbb{Z}_N \times \mathbb{Z}_N-$representation in $\mathbb{R}^{2N^2}$ are of the form $\mathbf{Z_k} = \mathbf{V_k} \oplus \mathbf{V_k}$ given in Table 3.2.*

Proof. Since $\Gamma = \mathbb{Z}_N \times \mathbb{Z}_N$ acts diagonally, $\gamma(x,y) = (\gamma x, \gamma y)$ in $\mathbb{R}^{2N^2}$ for $x, y \in \mathbb{R}^{N^2}$, then the subspaces $\mathbf{V_k} \oplus \{\mathbf{0}\}$ and $\{\mathbf{0}\} \oplus \mathbf{V_k}$ are $\Gamma-$invariant and irreducible, by 2. of Proposition 3.3.1. If $\mathbf{k} \neq \mathbf{k}'$ then, by 3. we have that $\mathbf{V_{k'}} \oplus \{\mathbf{0}\}$ and $\{\mathbf{0}\} \oplus \mathbf{V_{k'}}$ are not $\Gamma-$isomorphic to either $\mathbf{V_k} \oplus \{\mathbf{0}\}$ or $\{\mathbf{0}\} \oplus \mathbf{V_k}$. Therefore, the only isomorphic representations are $\mathbf{V_k} \oplus \{\mathbf{0}\}$ and $\{\mathbf{0}\} \oplus \mathbf{V_k}$ for the same $\mathbf{k} \in I$ and the result follows. ■

3.4 Symmetries of generic oscillations patterns

The main goal of this section is to characterise the symmetries of periodic solutions of (3.2.2), specially those that arise at Hopf bifurcations.

Given a solution $z(t)$ with period P of a $\mathbb{Z}_N \times \mathbb{Z}_N$-equivariant differential equation $\dot{z} = f(z)$, a *spatio-temporal symmetry* of $z(t)$ is a pair (σ, θ), with $\sigma \in \mathbb{Z}_N \times \mathbb{Z}_N$ and $\theta \in \mathbb{R} \pmod{P} \sim \mathbb{S}^1$ such that $\sigma \cdot z(t) = z(t+\theta)$ for all t. The group of spatio-temporal symmetries of $z(t)$ can be identified with a pair of subgroups H and K of $\mathbb{Z}_N \times \mathbb{Z}_N$ and a homomorphism $\Theta : H \to \mathbb{S}^1$ with kernel K, where H represents the spatial parts of the spatio-temporal symmetries of $z(t)$, while K comprises the *spatial symmetries* of $z(t)$, i.e. the symmetries that fix the solution pointwise. In order to get all the spatio-temporal symmetries for solutions of (3.2.2), we use the following result of Filipitsky and Golubitsky:

Theorem 3.4.1 (abelian Hopf theorem [5]) *In systems with abelian symmetry, generically, Hopf bifurcation at a point X_0 occurs with simple eigenvalues, and there exists a unique branch of small-amplitude periodic solutions emanating from X_0. Moreover, the spatio-temporal symmetries of the bifurcating periodic solutions are $H = \Sigma_{X_0}$ and $K = \ker_V(H)$, where V is the centre subspace of the bifurcation at X_0 and H acts H-simply on V.*

Recall that a subgroup H of $\mathbb{Z}_N \times \mathbb{Z}_N$ acts *$H-$simply on a subspace* $\mathbf{V}$ if either $\mathbf{V}$ is the sum of two isomorphic H-irreducible subspaces or $\mathbf{V}$ is H-irreducible but not absolutely irreducible.

Proposition 3.4.1 *Periodic solutions of* (3.2.2) *arising through Hopf bifurcation with simple eigenvalues at an equilibrium point X_0 have the spatio-temporal symmetries of Table 3.3.*

Table 3.3. Spatio-temporal (H) and spatial (K) symmetries of (3.2.2) that may arise through Hopf bifurcation at a point X_0, with two-dimensional centre subspace $\mathbf{V}$. Here $\mathbf{V}_{\mathbf{k}^\perp}$ denotes the subspace $Fix\left(\widetilde{\mathbb{Z}}_N\left(\mathbf{k}\right)\right) \subset \mathbb{R}^{N^2}$ as in the proof of 6. of Proposition 3.3.1.

H	set containing X_0	centre subspace V	K	restrictions on $\mathbf{k}$	
Γ	$\mathbf{V_0} \oplus \mathbf{V_0}$	$V \subset \mathbf{V_0} \oplus \mathbf{V_0}$	H		
Γ	$\mathbf{V_0} \oplus \mathbf{V_0}$	$V \subsetneqq \mathbf{V_k} \oplus \mathbf{V_k}$	$\widetilde{\mathbb{Z}}_N\left(\mathbf{k}^\perp\right)$	$\mathbf{k} \neq \mathbf{0}$	
$\widetilde{\mathbb{Z}}_N\left(\mathbf{k}\right)$	$\mathbf{V}_{\mathbf{k}^\perp} \oplus \mathbf{V}_{\mathbf{k}^\perp} \backslash \{\mathbf{0}\}$	$V = \mathbf{V_0} \oplus \mathbf{V_0}$	H	$\mathbf{k} \neq \mathbf{0}$	
$\widetilde{\mathbb{Z}}_N\left(\mathbf{k}\right)$	$\mathbf{V}_{\mathbf{k}^\perp} \oplus \mathbf{V}_{\mathbf{k}^\perp} \backslash \{\mathbf{0}\}$	$V \subsetneqq \mathbf{V}_\ell \oplus \mathbf{V}_\ell$	$\mathbb{1}$	$\mathbf{k} \neq \mathbf{0}$	$\ell \cdot \mathbf{k} \neq \mathbf{0}$
$\widetilde{\mathbb{Z}}_N\left(\mathbf{k}\right)$	$\mathbf{V}_{\mathbf{k}^\perp} \oplus \mathbf{V}_{\mathbf{k}^\perp} \backslash \{\mathbf{0}\}$	$V \subsetneqq \mathbf{V}_\ell \oplus \mathbf{V}_\ell$	H	$\mathbf{k} \neq \mathbf{0}$	$\ell \cdot \mathbf{k} = \mathbf{0}$
$\mathbb{1}$	$\mathbb{R}^{2N^2} \backslash \bigcup \mathbf{V_k} \oplus \mathbf{V_k}$	$V \subset \mathbf{V_k} \oplus \mathbf{V_k}$	$\mathbb{1}$		

Proof. The proof is a direct application of Theorem 3.4.1, using the information of Section 3.3. From assertions 6. and 7. of Proposition 3.3.1, the possibilities for H are $\mathbb{1}$, $\widetilde{\mathbb{Z}}_N\left(r,s\right)$ and $\mathbb{Z}_N \times \mathbb{Z}_N$. This yields the first collumn in Table 3.3. The second collumn is obtained from the list of corresponding fixed-point subspaces.

Let V be the centre subspace at X_0. Since the eigenvalues are simple, V is two-dimensional and is contained in one of the isotypic components. Then either H acts on V by nontrivial rotations and the action is irreducible but not absolutely irreducible, or H acts trivially on $V \subset FixH$, hence V is the sum of two H-irreducible components. In any of these cases H acts H-simply on V. The possibilities, listed in Lemma 3.3.1, yield the third collumn of Table 3.3. The spatial symmetries are then obtained by checking whether V meets $Fix(H)$. ■

Hopf bifurcation with simple eigenvalues is the generic situation for systems with abelian symmetry [5, Theorem 3.1]. In the next section we will show in Theorem 3.6.1 that this is indeed the case for (3.2.2) and obtain explicit genericity conditions.

A useful general tool for identifying periodic solutions whose existence is not guaranteed by the Equivariant Hopf Theorem, is the $H \bmod K$ Theorem [9]. Although it has been shown in [9] that in general there may be periodic solutions with spatio-temporal symmetries predicted by the $H \bmod K$ Theorem that cannot be obtained from Hopf bifurcation, this is not the case here.

3.5 Linear Stability

In this section we study the stability of solutions of (3.2.2) lying in the *full synchrony subspace* $\mathbf{V}_{(0,0)} \oplus \mathbf{V}_{(0,0)} \subset \mathbb{R}^{2N^2}$. For this we choose coordinates in $\mathbb{R}^{2N^2}$ by concatenating the transposed collumns of the matrix $(x_{\alpha,\beta}, y_{\alpha,\beta})$, i.e. the coordinates are $(C_1, \ldots, C_N)^T$ where

$$C_\beta = \left(x_{1,\beta}, y_{1,\beta}, x_{2,\beta}, y_{2,\beta}, \ldots, x_{N,\beta}, y_{N,\beta}\right)^T \ . \tag{3.5.1}$$

Let $p \in \mathbf{V}_{(0,0)} \oplus \mathbf{V}_{(0,0)}$ be a point with all coordinates $(x_{\alpha,\beta}, y_{\alpha,\beta}) = (x_*, y_*)$. In these coordinates, the linearization of (3.2.2) around p is given by the $N \times N$ block circulant matrix M given by

$$M = \begin{bmatrix} A & B & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A & B & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A & B & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & A \end{bmatrix}$$

where A is an $N \times N$ block circulant matrix and B is an $N \times N$ block diagonal matrix given by

$$A = \begin{bmatrix} D & E & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D & E & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D & E & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & D \end{bmatrix} \quad B = \begin{bmatrix} F & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & F & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & F & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & F \end{bmatrix}$$

where the 2×2 matrices E and F are given by

$$E = \begin{bmatrix} -\gamma & 0 \\ 0 & 0 \end{bmatrix} F = \begin{bmatrix} -\delta & 0 \\ 0 & 0 \end{bmatrix}$$

and D is obtained from the matrix of the derivative $D(f_1, f_2)$ of (3.2.1) at (x_*, y_*) as $D = D(f_1, f_2) - E - F$. In particular, if p is the origin we have

$$D = \begin{bmatrix} d & -1 \\ b & -c \end{bmatrix} \qquad \text{with} \qquad d = -a + \delta + \gamma\,.$$

Given a vector $v \in \mathbb{C}^k$, we use the N^{th} roots of unity $\omega^r = \exp(2\pi i r/N)$ to define the vector $\Omega(r, v) \in \mathbb{C}^{kN}$ as

$$\Omega(r, v) = \left[v, \omega^r v, \omega^{2r} v, \dots, \omega^{(N-1)r} v\right]^T,\ 0 \leqslant r \leqslant N - 1.$$

The definition may be used recursively to define the vector $\Xi(r, s, v) = \Omega\left(s, \Omega(r, v)\right) \in \mathbb{C}^{k^2}$ as

$$\Xi(r, s, v) = \left[\Omega(r, v), \omega^s \Omega(r, v), \omega^{2s} \Omega(r, v), \dots, \omega^{(N-1)s} \Omega(r, v)\right]^T.$$

Theorem 3.5.1 *If $\lambda_{r,s}$ is an eigenvalue and $v \in \mathbb{C}^2$ an eigenvector of $D + \omega^r E + \omega^s F$, and if M is the linearization of* (3.2.2) *around $p \in \mathbf{V}_{(0,0)} \oplus \mathbf{V}_{(0,0)}$ then $\lambda_{r,s}$ is an eigenvalue of M with corresponding eigenvector $\Xi(r, s, v)$.*

Proof. Let us first compute the eigenvalues of the matrix A. We have, for any $v \in \mathbb{C}^2$

$$A\Omega(r, v) = \Omega\left(r, \left(D + \omega^r E\right) v\right)$$

or, in full,

$$\begin{bmatrix} D & E & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D & E & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D & E & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & D \end{bmatrix} \begin{bmatrix} v \\ \omega^r v \\ \omega^{2r} v \\ \vdots \\ \omega^{(N-1)r} v \end{bmatrix} = \begin{bmatrix} \left(D + \omega^r E\right) v \\ \omega^r \left(D + \omega^r E\right) v \\ \omega^{2r} \left(D + \omega^r E\right) v \\ \vdots \\ \omega^{(N-1)r} \left(D + \omega^r E\right) v \end{bmatrix}$$

so, if $\left(D + \omega^r E\right) v = \lambda_r v$ then $A\Omega(r, v) = \lambda_r \Omega(r, v)$.

By applying the same algorithm we can calculate the eigenvalues and eigenvectors of the matrix M. Given $u \in \mathbb{C}^{2N}$ compute

$$M\Omega(s, u) = \Omega\left(s, \left(A + \omega^s B\right) u\right)$$

or, in full

$$\begin{bmatrix} A & B & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A & B & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A & B & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & A \end{bmatrix} \begin{bmatrix} u \\ \omega^s u \\ \omega^{2s} u \\ \vdots \\ \omega^{(N-1)s} u \end{bmatrix} = \begin{bmatrix} \left(A + \omega^s B\right) u \\ \omega^s \left(A + \omega^s B\right) u \\ \omega^{2s} \left(A + \omega^s B\right) u \\ \vdots \\ \omega^{(N-1)s} \left(A + \omega^s B\right) u \end{bmatrix}$$

To complete the proof we compute $M\Xi(r,s,v) = M\Omega\left(s,\Omega(r,v)\right)$ as

$$M\Xi(r,s,v) = \Omega\left(s,\left(A+\omega^s B\right)\Omega(r,v)\right) = \Omega\left(s,A\Omega(r,v)\right) + \Omega\left(s,\omega^s B\Omega(r,v)\right).$$

Then, since B is a block diagonal matrix, then $B\Omega(r,v) = \Omega(r,Fv)$ for any $v \in \mathbb{R}^2$ and we get:

$$\begin{aligned} M\Xi(r,s,v) &= \Omega\left(s,\Omega\left(r,\left(D+\omega^r E\right)v\right)\right) + \Omega\left(s,\Omega(r,\omega^s F v)\right) \\ &= \Omega\left(s,\Omega\left(r,\left(D+\omega^r E+\omega^s F\right)v\right)\right) \end{aligned}$$

and thus

$$M\Xi(r,s,v) = \Xi\left(r,s,\left(D+\omega^r E+\omega^s F\right)v\right).$$

It follows that if $\left(D+\omega^r E+\omega^s F\right)v = \lambda_{r,s}v$ then $M\Xi(r,s,v) = \lambda_{r,s}\Xi(r,s,v)$ as we had claimed. ■

3.5.1 Form of the eigenvalues

Theorem 3.5.2 *If L is the linearisation of* (3.2.1) *around the origin and $D = L - E - F$ then the eigenvalues of $D+\omega^r E+\omega^s F$ are of the form*

$$\begin{aligned} \lambda_{(r,s)\pm} &= \frac{1}{2}\left[-(c+a)+\gamma(1-\omega^r)+\delta(1-\omega^s)\right] \\ &\pm\frac{1}{2}\sqrt{\left[(c-a)+\gamma(1-\omega^r)+\delta(1-\omega^s)\right]^2-4b} \end{aligned} \tag{3.5.2}$$

where $\sqrt{-}$ stands for the principal square root. Moreover, on the isotypic component $\mathbf{V}_{(k_1,k_2)} \oplus \mathbf{V}_{(k_1,k_2)}$, the eigenvalues of M are $\lambda_{(k_1,k_2)\pm}$ and their complex conjugates $\lambda_{(N-k_1,N-k_2)\pm}$.

Proof. It is straightforward to derive the explicit expression (3.5.2) of the eigenvalues of $D+\omega^r E+\omega^s F$; a direct calculation shows that, unless $(r,s) = (0,0)$, the complex conjugate of $\lambda_{(r,s)+}$ is not $\lambda_{(r,s)-}$, but rather $\lambda_{(N-r,N-s)+}$.

We claim that for any complex number ζ the real and imaginary parts of $\Xi(k_1,k_2,\zeta)$ lie in $\mathbf{V}_{(k_1,k_2)}$; from this it follows that the real and imaginary parts of the eigenvectors $\Xi(k_1,k_2,v)$ lie in the isotypic component $\mathbf{V}_{(k_1,k_2)} \oplus \mathbf{V}_{(k_1,k_2)}$. Since $\mathbf{V}_{(k_1,k_2)} = \mathbf{V}_{(N-k_1,N-k_2)}$, this will complete the proof that the eigenvalues $\lambda_{(k_1,k_2)\pm}$ and $\lambda_{(N-r,N-s)\pm}$ correspond to $\mathbf{V}_{(k_1,k_2)} \oplus \mathbf{V}_{(k_1,k_2)}$.

It remains to establish our claim. Using the expression (3.5.1) to write the coordinate $x_{\alpha,\beta}$ of $\Xi(k_1,k_2,\zeta)$ we obtain

$$\begin{aligned} x_{\alpha,\beta} &= \zeta\exp\left[\frac{2\pi i}{N}(\alpha-1)k_1+(\beta-1)k_2\right] = \\ &= \zeta\exp\left[\frac{2\pi i}{N}(-k_1-k_2)\right]\exp\left[\frac{2\pi i}{N}(\alpha,\beta)\cdot\mathbf{k}\right]. \end{aligned}$$

Its real and imaginary parts are of the form (3.3.1) for $z = \zeta\exp\left[\frac{2\pi i}{N}(-k_1-k_2)\right]$ and $z = -i\zeta\exp\left[\frac{2\pi i}{N}(-k_1-k_2)\right]$, respectively, and therefore lie in $\mathbf{V}_{(k_1,k_2)} \oplus \mathbf{V}_{(k_1,k_2)}$, as claimed. ■

For the mode $(r,s) = (0,0)$ the expression (3.5.2) reduces to the eigenvalues $\lambda_\pm$ of uncoupled equations (3.2.1), linearised about the origin,

$$\lambda_\pm = \frac{-c-a\pm\sqrt{(c+a)^2-4b}}{2}. \tag{3.5.3}$$

3.6 Bifurcation for $c = 0$

In this section we look at the Hopf bifurcation in the case $c = 0$, regarding a as a bifurcation parameter. The bulk of the section consists of the proof of Theorem 3.6.2 below. Since in this case the only equilibrium is the origin, only the first two rows of Table 3.3 occur.

Theorem 3.6.1 *For generic γ, δ and for $c = 0,\ b \neq 0$ all the eigenvalues of the linearization of* (3.2.2) *around the origin have multiplicity* 1.

Proof. We can write the characteristic polynomial for $L + (\omega^r - 1)E + (\omega^s - 1)F$, where L is the linearization of (3.2.1) about the origin, as

$$f(\lambda, r, s) = \lambda^2 + \lambda\left[a - \gamma\left(1 - \omega^r\right) - \delta\left(1 - \omega^s\right)\right] + b \qquad 1 \leqslant r, s \leqslant N. \tag{3.6.1}$$

We start by showing that if two of these polynomials have one root in common, then they are identical.

Indeed, let $\phi_1(\lambda)$ and $\phi_2(\lambda)$ be two polynomials of the form (3.6.1) and suppose they share one root, say $\lambda = p + iq$, while the remaining roots are $\lambda = p_1 + iq_1$ for ϕ_1 and $\lambda = p_2 + iq_2$ for ϕ_2. Since $b \neq 0$, then none of these roots is zero. Then we can write

$$\begin{aligned} \phi_j(\lambda) &= \left(\lambda - (p + iq)\right)\left(\lambda - (p_j + iq_j)\right) \\ &= \lambda^2 - \lambda\left((p + p_j) + i\left(q + q_j\right)\right) + (p + iq)\left(p_j + iq_j\right) \end{aligned}$$

and therefore $(p + iq)(p_1 + iq_1) = b = (p + iq)(p_2 + iq_2)$, so, as $p + iq \neq 0$, then $(p_1 + iq_1) = (p_2 + iq_2)$ and therefore $\phi_1(\lambda) = \phi_2(\lambda)$. Since this is valid for any pair of polynomials of the family, it only remains to show that for generic γ, δ the polynomials do not coincide.

Two polynomials $f(\lambda, r, s)$ and $f(\lambda, \tilde{r}, \tilde{s})$ of the form (3.6.1) coincide if and only if

$$\gamma\left(\omega^r - \omega^{\tilde{r}}\right) = \delta\left(\omega^{\tilde{s}} - \omega^s\right). \tag{3.6.2}$$

Thus, for (γ, δ) outside a finite number of lines defined by (3.6.2) all the eigenvalues of the linearization of (3.2.2) around the origin have multiplicity 1, as we wanted to show. ■

Theorem 3.6.2 *For $c = 0$, $b > 0$ and for any γ and δ with $\gamma\delta \neq 0$ the origin is the only equilibrium of* (3.2.2)*. For each value of γ and δ there exists $a_* \geqslant 0$ such that for $a \geqslant a_*$ the origin is asymptotically stable. The stability of the origin changes at $a = a_*$, where it undergoes a Hopf bifurcation with respect to the bifurcation parameter a, into a periodic solution. The spatial symmetries of the bifurcating solution and the values of a_* are given in Table 3.4. Moreover, if the coupling is associative, i.e, if both $\gamma < 0$ and $\delta < 0$, the bifurcating solution is stable and the bifurcation is subcritical.*

Table 3.4. Details of Hopf bifurcation on the parameter a for Theorem 3.6.2. Solutions bifurcate at $a = a_*$ (where $\theta_N = \frac{(N-1)\pi}{N}$) with spatial symmetry K and spatio-temporal symmetry $\mathbb{Z}_N \times \mathbb{Z}_N$.

sign(γ)	sign(δ)	a_*	K
-	-	0	$\mathbb{Z}_N \times \mathbb{Z}_N$
+	-	$\gamma\left(1 - \cos\theta_N\right)$	$\widetilde{\mathbb{Z}}_N\,(0, 1)$
-	+	$\delta\left(1 - \cos\theta_N\right)$	$\widetilde{\mathbb{Z}}_N\,(1, 0)$
+	+	$(\gamma + \delta)\left(1 - \cos\theta_N\right)$	$\widetilde{\mathbb{Z}}_N\left(\frac{N-1}{2}, \frac{N-1}{2}\right)$

The first step is to determine the stability of the origin. To do this, we need estimates for the real part of the eigenvalues (3.5.2). This is done in the next Lemma.

Lemma 3.6.1 *Let $A(r, s) = -a + \gamma\left(1 - \omega^r\right) + \delta\left(1 - \omega^s\right)$. For $c = 0$, $b > 0$, $\gamma\delta \neq 0$ and for all (r, s) we have* $\mathrm{Re}\ \lambda_{(r,s)-} \leqslant \frac{1}{2}\mathrm{Re}\ A(r, s)$. *If* $\mathrm{Re}\ A(r, s) \geqslant 0$ *then* $\mathrm{Re}\ \lambda_{(r,s)+} \leqslant \mathrm{Re}\ A(r, s)$, *otherwise* $\mathrm{Re}\ \lambda_{(r,s)+} < 0$.

Proof of Lemma 3.6.1. In order to evaluate the real and imaginary parts of eigenvalues $\lambda_{(r,s)\pm}$, we need to rewrite equation (3.5.2) by getting rid of the square root. For this purpose, we use a well known result from elementary algebra; we have that if $\eta = a_1 + ib_1$, where a_1 and b_1 are real, $b_1 \neq 0$, then the real and imaginary parts of $\sqrt{\eta} = \sqrt{a_1 + ib_1}$ are given by

$$\mathrm{Re}\,\sqrt{\eta} = \sqrt{\frac{|\eta| + a_1}{2}} \qquad \mathrm{Im}\,\sqrt{\eta} = \mathrm{sgn}\,(b_1)\sqrt{\frac{|\eta| - a_1}{2}}. \tag{3.6.3}$$

A direct application of (3.5.2) in Theorem 3.5.2 to the case $c = 0$ yields, if $A^2 \notin \mathbb{R}$ and for $\varepsilon_1 = \pm 1$

$$\mathrm{Re}\,\lambda_{(r,s)\varepsilon_1} = \frac{1}{2}\left(\mathrm{Re}\,A + \varepsilon_1\sqrt{\frac{|A^2 - 4b| + \mathrm{Re}\,(A^2 - 4b)}{2}}\right) \tag{3.6.4}$$

$$\mathrm{Im}\,\lambda_{(r,s)\varepsilon_1} = \frac{1}{2}\left(\mathrm{Im}\,A + \varepsilon_1\mathcal{S}\sqrt{\frac{|A^2 - 4b| - \mathrm{Re}\,(A^2 - 4b)}{2}}\right) \tag{3.6.5}$$

with $A = A(r, s)$ and $\mathcal{S} = \mathrm{sgn}\,\big(\mathrm{Im}\,\big(A^2\big)\big)$.

The statement for $\lambda_{(r,s)-}$ follows immediately from (3.6.4). For $\lambda_{(r,s)+}$, note that, for $b > 0$ and any $\eta \in \mathbb{C}$, we have

$$\left|\eta^2 - 4b\right| + \mathrm{Re}\,(\eta^2 - 4b) \leqslant 2\,(\mathrm{Re}\,\eta)^2$$

with equality holding if and only if $\mathrm{Re}\,\eta = 0$, when the expressions are identically zero. Hence, taking $\eta = A(r, s)$, we obtain from (3.6.4):

$$\mathrm{Re}\,\lambda_{(r,s)+} \leqslant \frac{1}{2}\,(\mathrm{Re}\,A(r, s) + |\mathrm{Re}\,A(r, s)|)$$

and the result follows. ■

The particular case of fully synchronised solutions in Theorem 3.6.2 is treated in the next Lemma. This case is simpler since the bifurcation takes place inside the subspace $\mathbf{V}_{(0,0)} \oplus \mathbf{V}_{(0,0)}$.

Lemma 3.6.2 *For $c = 0$, $b > 0$ and $\gamma\delta \neq 0$ the origin is the only equilibrium of* (3.2.2) *and at $a = 0$ it undergoes a Hopf bifurcation, subcritical with respect to the bifurcation parameter a, to a fully synchronised periodic solution. If both $\gamma < 0$ and $\delta < 0$, the origin is asymptotically stable for $a > 0$ and the bifurcating solution is stable. Otherwise, the periodic solution is unstable.*

Proof of Lemma 3.6.2. Inspection of (3.2.2) when $c = 0$ shows that the only equilibrium is the origin.

The restriction of (3.2.2) to the plane $\mathbf{V}_{(0,0)} \oplus \mathbf{V}_{(0,0)}$ obeys the uncoupled equations (3.2.1) whose linearisation around the origin has eigenvalues given by (3.5.3). It follows that, within $\mathbf{V}_{(0,0)} \oplus \mathbf{V}_{(0,0)}$, the origin is asymptotically stable for $a > 0$, unstable for $a < 0$. The linearisation has purely imaginary eigenvalues at $a = 0$.

Consider the positive function $\varphi(y) = \exp(-2y/b)$. Then, for the uncoupled equations (3.2.1) we get

$$\frac{\partial}{\partial x}\,(\varphi(y)\dot{x}) + \frac{\partial}{\partial y}\,(\varphi(y)\dot{y}) = \big(-3x^2 + 2ax - a\big)\,\varphi(y)$$

which is always negative if $0 \leq a \leq 3$. Hence, by Dulac's criterion, the system (3.2.1) cannot have any periodic solutions, and thus if there is a Hopf bifurcation at $a = 0$ inside the plane $\mathbf{V}_{(0,0)} \oplus \mathbf{V}_{(0,0)}$ it must be subcritical.

In order to show that indeed there is a Hopf bifurcation we apply the criterium of [11, Theorem 3.4.2] and evaluate

$$\begin{aligned} 16s^* = \;& f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ & + \frac{1}{\sqrt{b}}\,[f_{xy}\,(f_{xx} + f_{yy}) - g_{xy}\,(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}], \end{aligned}$$

where $f(x,y) = f_1(x, \sqrt{b}y) - \sqrt{b}y$, $g(x,y) = f_2(\sqrt{b}x, y) - \sqrt{b}x$ and f_{xy} denotes $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$, etc. Since, for $a = c = 0$, we have $f(x,y) = -x^3 + x^2$ and $g(x,y) = 0$, this yields $s^* = -\frac{3}{8}$. The Hopf bifurcation is not degenerate and the bifurcating periodic solution is stable within $\mathbf{V}_{(0,0)} \oplus \mathbf{V}_{(0,0)}$. Since $\partial \mathrm{Re}\ \lambda_{(0,0)\pm}/\partial a = -1/2$ the bifurcation is indeed subcritical with respect to the bifurcation parameter a.

It remains to discuss the global stability, with respect to initial conditions outside $\mathbf{V}_{(0,0)} \oplus \mathbf{V}_{(0,0)}$. If $\gamma > 0$ then from the expression (3.6.4) in the proof of Lemma 3.6.1 at $a = 0$ we obtain Re $\lambda_{(1,0)+} > 0$ and the bifurcating periodic solution is unstable. A similar argument holds for $\delta > 0$.

If both $\gamma < 0$ and $\delta < 0$, then, for $a = 0$, we get Re $A(r,s) < 0$ for $(r,s) \neq (0,0)$. Hence, by Lemma 3.6.1, all the eigenvalues $\lambda_{(r,s)\pm}$, $(r,s) \neq (0,0)$ have negative real parts and the bifurcating solution is stable. ■

Proof of Theorem 3.6.2. The case of associative coupling $\gamma < 0$ and $\delta < 0$ having been treated in Lemma 3.6.2, it remains to deal with the cases when either γ or δ is positive. Let $r_{\varepsilon_2} = \frac{N+\varepsilon_2}{2}$, $\varepsilon_2 = \pm 1$, $c = 0$, $b > 0$, $\theta_N = \frac{\pi(N-1)}{N}$, with $\sin\theta_N > 0$, $\cos\theta_N < 0$ and $\cos\theta_N < \cos\frac{2\alpha\pi}{N}$ for all $\alpha \in \mathbb{Z}$.

If $\gamma > 0$, $\delta < 0$ and $a \geqslant \gamma(1 - \cos\theta_N) = a_*$, then for all $(r,s) \neq (r_{\varepsilon_2}, 0)$ we have Re $A(r,s) \leqslant 0$, with equality only if both $a = a_*$ and $(r,s) = (r_{\varepsilon_2}, 0)$. Using (3.6.4) and (3.6.5) we get

$$A(r_{\varepsilon_2}, 0) = i\varepsilon_2\gamma\sin\theta_N;$$

$$\mathrm{Im}\lambda_{(r_{\varepsilon_2},0)\varepsilon_1} = \frac{1}{2}\left(\varepsilon_2\gamma\sin\theta_N + \varepsilon_1\sqrt{\gamma^2\sin^2\theta_N + 4b}\right)$$

with $\lambda_{(r_+,0)+} = \overline{\lambda_{(r_-,0)-}}$ and $\lambda_{(r_-,0)+} = \overline{\lambda_{(r_+,0)-}}$. In addition $\mathrm{Im}\lambda_{(r_+,0)+} > \mathrm{Im}\lambda_{(r_-,0)+} > 0$ for $b > 0$.

The results of Golubitsky and Langford [7] are always applicable to the Hopf bifurcation for $\lambda_{(r_+,0)+}$, since there are no eigenvalues of the form $k\lambda_{(r_+,0)+}$ with $k \in \mathbb{N}$. For the smaller imaginary part there may be resonances when $\lambda_{(r_+,0)+} = k\lambda_{(r_-,0)-}$ with $k \in \mathbb{N}$. Otherwise, if the other non-degeneracy conditions hold, there are two independent Hopf bifurcations at $a = a_*$ i.e. two separate solution branches that bifurcate at this point. The resonance condition may be rewritten as

$$\gamma^2 \sin\theta_N = \frac{(k-1)^2}{k} b, \quad k \in \mathbb{N},\ k \geqslant 2.$$

The bifurcating solutions are stable if and only if the branches are subcritical. The eigenspace corresponding to these branches lies in $V_{(r_\pm,0)} \oplus V_{(r_\pm,0)} \subset \mathrm{Fix}(\tilde{\mathbb{Z}}(0,1))$.

In the case $\gamma > 0$, $\delta > 0$, we have $a_* = (\gamma + \delta)(1 - \cos\theta_N)$. For $a \geqslant a_*$ and for all (r,s) we have Re $A(r,s) \leqslant 0$, and hence Re $\lambda_{(r,s)\pm} \leqslant 0$ with equality holding only when both $a = a_*$ and $\lambda_{(r_\pm,r_\pm)+}$. The eigenspace in this case lies in $\mathbf{V}_{(r_\pm,r_\pm)} \oplus \mathbf{V}_{(r_\pm,r_\pm)} \subset Fix\left(\widetilde{\mathbb{Z}}_N\left(\frac{N-1}{2}, \frac{N-1}{2}\right)\right)$. Then

$$A(r_{\varepsilon_2}, r_{\varepsilon_3}) = i(\varepsilon_2\gamma + \varepsilon_3\delta)\sin\theta_N$$

$$\lambda_{(r_{\varepsilon_2},r_{\varepsilon_3})\varepsilon_1} = \frac{1}{2}\left(A(r_{\varepsilon_2}, r_{\varepsilon_3}) + \varepsilon_1\sqrt{A^2(r_{\varepsilon_2}, r_{\varepsilon_3}) - 4b}\right),$$

with $\varepsilon_i = \pm 1, i = \{1,2,3\}$. Hence, $\mathrm{Re}\lambda_{(r_{\varepsilon_2},r_{\varepsilon_3})\varepsilon_1} = 0$ and

$$\mathrm{Im}\lambda_{(r_{\varepsilon_2},r_{\varepsilon_3})\varepsilon_1} = \frac{1}{2}\left((\varepsilon_2\gamma + \varepsilon_3\delta)\sin\theta_N + \varepsilon_1\sqrt{(\varepsilon_2\gamma + \varepsilon_3\delta)^2\sin^2\theta_N + 4b}\right)$$

with $\lambda_{(r_{-\varepsilon_2},r_{-\varepsilon_3})-} = \overline{\lambda_{(r_{\varepsilon_2},r_{\varepsilon_3})+}}$. In addition $\mathrm{Im}\lambda_{(r_{\varepsilon_2},r_{\varepsilon_3})\varepsilon_1} > 0$ when $\varepsilon_1 = +1$ with $\mathrm{Im}\lambda_{(r_+,r_+)+} > \mathrm{Im}\lambda_{(r_{\varepsilon_2},r_{\varepsilon_3})+}$ for $(\varepsilon_2, \varepsilon_3) \neq (+1,+1)$ and $b > 0$.

Hence there is a non-resonant Hopf bifurcation corresponding to $\lambda_{(r_+,r_+)+}$. As mentioned before, for the smaller imaginary parts there may be resonances when $\lambda_{(r_{\varepsilon_2},r_{\varepsilon_3})+} = k\lambda_{(r_+,r_+)+}$ with $k \in \mathbb{N}$. Otherwise, there are four independent Hopf bifurcations at $a = a_*$ if other non-degeneracy conditions hold; in this case four separate solution branches bifurcate at this point. The bifurcating solutions are stable if and only if the branches are subcritical. ■

We have checked numerically the non-degeneracy condition for bifurcation of the non-resonant branch using the formulas of Golubitsky and Langford [7]. The criticality of the bifurcation branch seems to depend on N. For $\gamma > 0$, $\delta > 0$, $N \geqslant 11$, the bifurcating solution branch seems to be always subcritical, and hence stable. For $N = 3, 5, 7$ it seems to be supercritical. If $\delta \leqslant 0$, and $N \geqslant 11$, the bifurcating branch seems to be subcritical for large values of $\gamma > 0$, supercritical otherwise.

3.7 Bifurcation for $c > 0$ small

In this section we extend the result of section 3.6 for bifiurcations at small positive values of c. We start with the case when both γ and δ are negative.

Corollary 3.7.1 *For small values of c, if $b > 0$, $\gamma < 0$ and $\delta < 0$ the origin is an equilibrium of* (3.2.2) *and there is a neighborhood of the origin containing no other equilibria. There exists $\hat{a}$ near 0 such that for $a \geqslant \hat{a}$ the origin is asymptotically stable. The stability of the origin changes at $a = \hat{a}$, where it undergoes a Hopf bifurcation, subcritical with respect to the bifurcation parameter a, into a stable periodic solution with spatial symmetries $\mathbb{Z}_N \times \mathbb{Z}_N$.*

Proof. The eigenvalues of $Df(\mathbf{0})$ are all non-zero at $c = 0$, as shown in the proof of Lemma 3.6.2. Hence $Df(\mathbf{0})$ is non-singular and the implicit function theorem ensures that, for small values of c, there is a unique equilibrium close to the origin. From the symmetry it follows that this equilibrium is the origin.

If both $\gamma < 0$ and $\delta < 0$, then it follows from the proof of Lemma 3.6.2 that for $c = 0$ the purely imaginary eigenvalues at $a = a_* = 0$ are simple. Continuity of the eigenvalues ensures the persistence of the purely imaginary pair for $c \neq 0$ at nearby values of a . Hence the corresponding eigenvectors depend smoothly on c, and the non-degeneracy conditions persist for small values of c. ■

The cases when either $\gamma > 0$ or $\delta > 0$, are treated in the next proposition.

Proposition 3.7.1 *For small values of $c > 0$, if $b > 0$ and $\gamma\delta \neq 0$ the origin is an equilibrium of* (3.2.2) *and there is a neighborhood of the origin containing no other equilibria. For $a > a_*$, where a_* has the value of Table 3.4, the origin is asymptotically stable. For almost all values of $\gamma \neq 0$ and $\delta \neq 0$, The stability of the origin changes at $a = \hat{a} < a_*$, with $\hat{a}$, near a_*, where it undergoes a non-resonant Hopf bifurcation into a periodic solution having the spatial symmetries of Table 3.4. If the bifurcation is subcritical with respect to the bifurcation parameter a, then the bifurcating periodic solution is stable.*

Proof. For small values of c, the origin is locally the only equilibrium, by the arguments given in the proof of Corollary 3.7.1. In Lemma 3.7.1 below, we show that for small $c > 0$ and $a \geq a_*$, all the eigenvalues of the linearisation have negative real parts. Hence, the origin is asymptotically stable. When a decreases from a_* the real parts of some eigenvalues change their signs. It was shown in the proof of Theorem 3.6.2 that for $c = 0$, there are several pairs of purely imaginary eigenvalues at $a = a_*$. In Lemmas 3.7.2 and 3.7.3 below, we show that, generically, for small $c > 0$, when a decreases from the value a_* of Table 3.4, the first bifurcation at $a = \hat{a} < a_*$ takes place when a single pair of eigenvalues crosses the imaginary axis at a non-resonant Hopf bifurcation. We also identify the pair of eigenvalues for which the first bifurcation takes place. ■

Lemma 3.7.1 *For small values of $c > 0$, if $b > 0$, $\gamma\delta \neq 0$, let a_* have the value of Table 3.4. If $a \geqslant a_*$ then, for all r, s, and for$\varepsilon_1 = \pm 1$, we have $\mathrm{Re}\lambda_{(r,s)\varepsilon_1} < 0$, and the origin is asymptotically stable.*

Proof. For $\varepsilon_1 = \pm 1$, the eigenvalues $\lambda_{(r,s)\epsilon_1}$ have the form

$$2\lambda_{(r,s)\epsilon_1} = A(r,s) - c + \epsilon_1 \sqrt{(A(r,s) + c)^2 - 4b}. \tag{3.7.1}$$

Using (3.6.3) and writing $A(r,s) = x + iy$, we have

$$(A+c)^2 = (c+x)^2 - y^2 + 2i(cy + xy). \tag{3.7.2}$$

Then $\mathrm{Re}\lambda_{(r,s)-} \leqslant \mathrm{Re}\lambda_{(r,s)+}$, and $\mathrm{Re}\lambda_{(r,s)+} \leqslant 0$ if and only if

$$\mathrm{Re}\sqrt{(A(r,s)+c)^2 - 4b} \leqslant (x+c)$$

This never happens if $c - x < 0$, and in this case we also have $2\mathrm{Re}\lambda_{(r,s)-} \leqslant 0$. If $c - x \geqslant 0$ let

$$p_1 = (c-x)^2 - 4xc + y^2 + 4b \quad \text{and} \quad p_2 = \left[(c+x)^2 - y^2 - 4b\right]^2 + 4y^2(c+x)^2$$

with $p_1^2 - p_2 = 16\left[-x^3c + (b + 2c^2)x^2 - (2cb + cy^2 + c^3)x + c^2b\right]$.

With this notation, $\mathrm{Re}\lambda_{(r,s)+} \leqslant 0$ if and only if $p_1 > 0$ and

$$\left[(c+x)^2 - y^2 - 4b\right]^2 + 4y^2(c+x)^2 \leqslant \left[(c-x)^2 - 4xc + y^2 + 4b\right]^2.$$

We have the following cases:

1. if $x < 0$, $c > 0$ then $p_1 > 0$ and $p_1^2 - p_2 > 0$ and so $\mathrm{Re}\lambda_{(r,s)+} < 0$;
2. if $x = 0$, $c > 0$ then $p_1 = y^2 + 4b + c^2 > 0$ and $\dfrac{p_1^2 - p_2}{16} = c^2 b > 0$ and therefore $\mathrm{Re}\lambda_{(r,s)+} < 0$;
3. at $x = c$ we have $p_1^2 - p_2 = -16c^2y^2 < 0$, so $(p_1^2 - p_2)$ changes sign for some x_*, $0 < x_* < c$.

This completes the proof, since for $a \geqslant a_*$ we have $x = \mathrm{Re}A(r,s) \leqslant 0$, as in the proof of Theorem 3.6.2. ■

Lemma 3.7.2 *For small values of $c > 0$, if $b > 0$, $\gamma > 0$, $\delta < 0$ and if $a - a* < 0$ is small, then all the eigenvalues of the linearisation of* (3.2.2) *around the origin have real parts smaller than the real part of $\lambda_{(r_+,0)+}$*

Proof. It was shown in the proof of Theorem 3.6.2 that at $c = 0$, $a = a_*$, the eigenvalues $\lambda_{(r_{\varepsilon_2},0)\varepsilon_1}$, with $\varepsilon_1 = \pm 1$with $\varepsilon_2 = \pm 1$, are purely imaginary, and all other eigenvalues have negative real parts. From the expression (3.7.1) it follows that

$$2\mathrm{Re}\lambda_{(r_{\varepsilon_2},0)\epsilon_1} = \mathrm{Re}A(r_{\varepsilon_2},0) - c + \epsilon_1 \mathrm{Re}\sqrt{(A(r_{\varepsilon_2},0)+c)^2 - 4b}$$

hence $\mathrm{Re}\lambda_{(r_{\varepsilon_2},0)-} < \mathrm{Re}\lambda_{(r_{\varepsilon_2},0)+}$. Let $\hat{a}$ be the value of a for which the pair $\lambda_{(r_+,0)+} = \overline{\lambda_{(r_-,0)+}}$ first crosses the imaginary axis. Since $\mathrm{Re}A(r_{\varepsilon_2},0)$ decreases with a, then $\hat{a} < a_*$. The estimates above show that at $a = \hat{a}$, the second pair $\lambda_{(r_+,0)-} = \overline{\lambda_{(r_-,0)-}}$ still has negative real part. For small c, the other eigenvalues still have negative real parts at $\hat{a}$, by continuity. ■

Lemma 3.7.3 *For small values of $c > 0$, if $b > 0$, $\gamma > 0$, $\delta > 0$ and if $a - a* < 0$ is small, then all the eigenvalues of the linearisation of* (3.2.2) *around the origin have real parts smaller than the real part of $\lambda_{(r_+,r_+)+}$*

Proof. As in Lemma 3.7.2 we use (3.7.1) to show that $\mathrm{Re}\lambda_{(r_{\varepsilon_2},r_{\varepsilon_3})-} < \mathrm{Re}\lambda_{(r_{\varepsilon_2},r_{\varepsilon_3})+}$ at $c > 0$. It remains to compare the real parts of the two pairs $\lambda_{(r_+,r_+)+} = \overline{\lambda_{(r_-,r_-)+}}$ and $\lambda_{(r_+,r_-)+} = \overline{\lambda_{(r_-,r_+)+}}$. To do this, we write $A(r_{\varepsilon_2}, r_{\varepsilon_3}) = x + iy$, where $x, y \in \mathbb{R}$ and obtain conditions on x and y ensuring that the eigenvalue is purely imaginary. Then we evaluate these conditions on the expressions for x and y to obtain the result.

From (3.7.1) and (3.7.2) we get that $\mathrm{Re}\lambda = 0$ if and only if

$$\left(2(c-x)^2 - X\right)^2 = X^2 + 4(x+c)^2y^2 \qquad \text{for} \qquad X = (x+c)^2 - y^2 - 4b$$

and this is equivalent to

$$(c-x)^2 = X + \frac{(c+x)^2}{(c-x)^2}y^2 = (x+c)^2 - y^2 - 4b + \frac{(c+x)^2}{(c-x)^2}y^2$$

which may be rewritten as:

$$(c-x)^2 - (x+c)^2 + 4b = y^2\left[\frac{(c+x)^2}{(c-x)^2} - 1\right].$$

This may be solved for y^2 to yield

$$y^2 = \frac{(b-cx)(c-x)^2}{cx} = \left(\frac{b}{cx} - 1\right)(c-x)^2 = \psi(x) \tag{3.7.3}$$

and note that, for $c > 0$ and if $c^2 < b$, then $\psi(x) > 0$ for $0 < x < c$, and in this interval $\psi(x)$ is monotonically decreasing.

Now consider the expressions of the real and imaginary parts of $A(r_{\varepsilon_2}, r_{\varepsilon_3})$. The real part $x(r_{\varepsilon_2}, r_{\varepsilon_3})$ satisfies

$$x(r_{\varepsilon_2}, r_{\varepsilon_3}) = -a + (\gamma+\delta)\left(1 - \cos\frac{(N-1)\pi}{N}\right) = -a + a_*$$

hence, x does not depend on ε_2 nor on ε_3, and $x > 0$ for $a < a_*$.

On the other hand, the imaginary part $y(r_{\varepsilon_2}, r_{\varepsilon_3})$ is

$$y(r_{\varepsilon_2}, r_{\varepsilon_3}) = (\varepsilon_2\gamma + \varepsilon_3\delta)\sin\frac{(N-1)\pi}{N}$$

thus y does not depend on a, and since $\sin\frac{(N-1)\pi}{N} > 0$ then,

$$|y(r_+, r_-)| < y(r_+, r_+) \qquad \text{and} \qquad y(r_+, r_+) > 0.$$

Finally, when a decreases from a_*, then x increases from zero, and hence $\psi(x)$ decreases from $+\infty$. The first value of y to satisfy (3.7.3) will be $y(r_+, r_+)$ since it has the largest absolute value. Hence the first pair of eigenvalues to cross the imaginary axis will be $\lambda_{(r_+,r_+)\varepsilon_1} = \overline{\lambda_{(r_-,r_-)\varepsilon_1}}$, as required, while the real parts of all other eigenvalues, including $\lambda_{(r_+,r_-)\varepsilon_1} = \overline{\lambda_{(r_-,r_+)\varepsilon_1}}$, are still negative. ■

Note that from Lemmas 3.7.2 and 3.7.3, it follows that the first bifurcating eigenvalue for $c > 0$ is precisely the non-resonant eigenvalue for $c = 0$, that has the largest imaginary part.

3.8 Acknowledgements

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Chapter 4

Hopf bifurcation with tetrahedral and octahedral symmetry

This chapter consists of a joint article ArXiv: 1407.2866 submitted for publication. Details of the calculations in the article are presented in the Appendix at the end of the thesis.

Isabel S. Labouriau and Adrian C. Murza

I.S. Labouriau—Centro de Matemática da Universidade do Porto. Rua do Campo Alegre 687, 4169-007 Porto, Portugal

A.C. Murza — Departamento de Matemática, Faculdade de Ciências Universidade do Porto. Rua do Campo Alegre 687, 4169-007 Porto, Portugal

Abstract. *In the study of the periodic solutions of a Γ-equivariant dynamical system, the $H \bmod K$ theorem gives all possible periodic solutions, based on the group-theoretical aspects. By contrast, the equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for each $\mathbf{C}$-axial subgroup of $\Gamma \times \mathbb{S}^1$. In this paper while characterizing the Hopf bifurcation, we identify which periodic solution types, whose existence is guaranteed by the $H \bmod K$ theorem, are obtainable by Hopf bifurcation from the origin, when the group Γ is either tetrahedral or octahedral. The two groups are isomorphic, but their representations in $\mathbb{R}^3$ and in $\mathbb{R}^6$ are not, and this changes the possible symmetries of bifurcating solutions.*

Keywords:
equivariant dynamical system, tetrahedral symmetry, periodic solutions, Hopf bifurcation

AMS 2010 Subject Classification:
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4.1 Introduction

The formalism of Γ-equivariant differential equations, ie. those equations whose associated vector field commutes with the action of a finite group Γ has been developed by Golubitsky, Stewart and Schaeffer in [4], [7] and [6]. Within this formalism, two methods for obtaining periodic solutions have been described: the $H \bmod K$ theorem [2, 6, Ch.3] and the equivariant Hopf theorem [6, 7]. The equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for all $\mathbf{C}$-axial subgroups of $\Gamma \times \mathbb{S}^1$, under some generic conditions. The $H \bmod K$ theorem offers the complete set of possible periodic solutions based exclusively on the structure of the group Γ acting on the differential equation. It also guarantees the existence of a model with this symmetry having these periodic solutions, but it is not an existence result for any specific equation.

Steady-state bifurcation problems with octahedral symmetry have been analysed by Melbourne [9] using results from singularity theory. For non-degenerate bifurcation problems equivariant with respect to the standard action on $\mathbb{R}^3$ of the octahedral group he finds three branches of symmetry-breaking steady-state bifurcations corresponding to the three maximal isotropy subgroups with one-dimensional fixed-point subspaces. Hopf bifurcation with the rotational symmetry of the tetrahedron is studied by Swift and Barany [12], motivated by problems in fluid dynamics. They find evidence of chaotic dynamics, arising from secondary bifurcations from periodic branches created at Hopf bifurcation. Generic Hopf bifurcation with the rotational symmetries of the cube is studied by Ashwin and Podvigina [1], also with the motivation of fluid dynamics.

Solutions predicted by the $H \bmod K$ theorem cannot always be obtained by a generic Hopf bifurcation from the trivial equilibrium. When the group is finite abelian, the periodic solutions whose existence is allowed by the $H \bmod K$ theorem that are realizable from the equivariant Hopf theorem are described in [3].

In this article, we pose a more specific question: which periodic solutions predicted by the $H \bmod K$ theorem are obtainable by Hopf bifurcation from the trivial steady-state when Γ is either the group $\langle \mathbb{T}, \kappa \rangle$ of symmetries of the tetrahedron or the group $\mathbb{O}$ of rotational symmetries of the cube? As abstract groups, $\langle \mathbb{T}, \kappa \rangle$ and $\mathbb{O}$ are isomorphic, but their standard representations in $\mathbb{R}^3$ and $\mathbb{C}^3$ are not. However, the representations of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ and $\mathbb{O} \times \mathbb{S}^1$ on $\mathbb{C}^3$ are isomorphic, and this is the relevant action for dealing with equivariant Hopf bifurcation. Thus it is interesting to compare results for these two groups, and we find that indeed our question has different answers for the two groups. In particular, some solutions predicted by the $H \bmod K$ theorem for $\langle \mathbb{T}, \kappa \rangle$-equivariant vector fields can only arise at a resonant Hopf bifurcation, but this is not the case for $\mathbb{O}$.

We will answer this question by finding for both groups that not all periodic solutions predicted by the $H \bmod K$ theorem occur as primary Hopf bifurcations from the trivial equilibrium. For this we analyse bifurcations taking place in 4-dimensional invariant subspaces and giving rise to periodic solutions with very small symmetry groups.

Framework of the article

The relevant actions of the groups $\langle \mathbb{T}, \kappa \rangle$ of symmetries of the tetrahedron, $\mathbb{O}$ of rotational symmetries of the cube and of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ and $\mathbb{O} \times \mathbb{S}^1$ on $\mathbb{R}^6 \sim \mathbb{C}^3$ are described in Section 4.3, after stating some preliminary results and definitions in Section 4.2. Hopf bifurcation is treated in Section 4.4, where we present the results of Ashwin and Podvigina [1] on $\mathbb{O} \times \mathbb{S}^1$, together with the formulation of the same results for the isomorphic action of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$. This includes the analysis of Hopf bifurcation inside fixed-point subspaces for submaximal isotropy subgroups, one of which we perform in more detail than in [1], giving a giving a geometric proof of the existence of four submaximal branches of periodic solutions, for some values of the parameters in a degree three normal form. Finally, we apply the $H \bmod K$ theorem in Section 4.5 where we compare the bifurcations for the two group actions.

4.2 Preliminary results and definitions

Before stating the theorem we give some definitions from [6, 7]. The reader is referred to this book for results on bifurcation with symmetry.

Let Γ be a compact Lie group. A representation of Γ on a vector space W is Γ-*simple* if either:

(a) $W \sim V \oplus V$ where V is absolutely irreducible for Γ, or

(b) W is nonabsolutely irreducible for Γ.

Let W be a Γ-simple representation and let f be a Γ-equivariant vector field in W. Then it follows [7, Ch. XVI, Lemma1.5] that if f is a Γ-equivariant vector field, and if Jacobian matrix $(df)_0$ of f evaluated at the origin has purely imaginary eigenvalues $\pm\omega i$, then in suitable coordinates $(df)_0$ has the form:

$$(df)_0 = \omega J = \omega \begin{bmatrix} 0 & -Id \\ Id & 0 \end{bmatrix}$$

where Id is the identity matrix. Consider the action of $\mathbb{S}^1$ on W given by $\theta x = e^{i\theta J}x$, A subgroup $\Sigma \subseteq \Gamma \times \mathbb{S}^1$ is $\mathbb{C}$-*axial* if Σ is an isotropy subgroup and $\dim Fix(\Sigma) = 2$.

Let $\dot{x} = f(x)$ be a Γ-equivariant differential equation with a T-periodic solution $x(t)$. We call $(\gamma,\ \theta) \in \Gamma \times \mathbb{S}^1$ a spatio-temporal symmetry of the solution $x(t)$ if $\gamma \cdot x(t+\theta) = x(t)$. A spatio-temporal symmetry of the solution $x(t)$ for which $\theta = 0$ is called a spatial symmetry, since it fixes the point $x(t)$ at every moment of time, for all t.

The main tool here will be the following theorem.

Theorem 4.2.1 (Equivariant Hopf Theorem [6,7]) *Let a compact Lie group Γ act $\Gamma-$simply, orthogonally and nontrivially on $\mathbb{R}^{2m}$. Assume that*

(a) $f : \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}^{2m}$ is Γ-equivariant. Then $f(0,\lambda) = 0$ and $(df)_{0,\lambda}$ has eigenvalues $\sigma(\lambda) \pm i\rho(\lambda)$ each of multiplicity m;

(b) $\sigma(0) = 0$ and $\rho(0) = 1$;

(c) $\sigma'(0) \neq 0$ the eigenvalue crossing condition;

(d) $\Sigma \subseteq \Gamma \times \mathbb{S}^1$ is a $\mathbb{C}$-axial subgroup.

Then there exists a unique branch of periodic solutions with period $\approx 2\pi$ emanating from the origin, with spatio-temporal symmetries Σ.

The group of all spatio-temporal symmetries of $x(t)$ is denoted $\Sigma_{x(t)} \subseteq \Gamma \times \mathbb{S}^1$. The symmetry group $\Sigma_{x(t)}$ can be identified with a pair of subgroups H and K of Γ and a homomorphism $\Phi : H \to \mathbb{S}^1$ with kernel K. We define

$$H = \{\gamma \in \Gamma : \gamma\{x(t)\} = \{x(t)\}\} \qquad K = \{\gamma \in \Gamma : \gamma x(t) = x(t)\ \forall t\}$$

where $K \subseteq \Sigma_{x(t)}$ is the subgroup of spatial symmetries of $x(t)$ and the subgroup H of Γ consists of symmetries preserving the trajectory $x(t)$ but not necessarily the points in the trajectory. We abuse notation saying that H is a group of spatio-temporal symmetries of $x(t)$. This makes sense because the groups $H \subseteq \Gamma$ and $\Sigma_{x(t)} \subseteq \Gamma\times\mathbb{S}^1$ are isomorphic; the isomorphism being the restriction to $\Sigma_{x(t)}$ of the projection of $\Gamma \times \mathbb{S}^1$ onto Γ.

Given an isotropy subgroup $\Sigma \subset \Gamma$, denote by $N(\Sigma)$ the normaliser of Σ in Γ, satisfying $N(\Sigma) = \{\gamma \in \Gamma : \gamma\Sigma = \Sigma\gamma\}$, and by L_Σ the variety $L_\Sigma = \bigcup_{\gamma\notin\Sigma} Fix(\gamma) \cap Fix(\Sigma)$.

The second important tool in this article is the following result.

Theorem 4.2.2 *(H mod K Theorem [2,6]) Let Γ be a finite group acting on $\mathbb{R}^n$. There is a periodic solution to some Γ-equivariant system of ODEs on $\mathbb{R}^n$ with spatial symmetries K and spatio-temporal symmetries H if and only if the following conditions hold:*

(a) H/K is cyclic;

(b) K is an isotropy subgroup;

(c) dim $Fix(K) \geqslant 2$. If dim $Fix(K) = 2$, then either $H = K$ or $H = N(K)$;

(d) H fixes a connected component of $\mathrm{Fix(K)\backslash L_K}$.

Moreover, if $(a) - (d)$ hold, the system can be chosen so that the periodic solution is stable.

When $H/K \sim \mathbb{Z}_m$, the periodic solution $x(t)$ is called either a standing wave or (usually for $m \geqslant 3$) a discrete rotating wave; and when $H/K \sim \mathbb{S}^1$ it is called a rotating wave [6, page 64]. Here all rotating waves are discrete.

4.3 Group actions

Our aim in this article is to compare the bifurcation of periodic solutions for generic differential equations equivariant under two different representations of the same group. In this section we describe the two representations.

4.3.1 Symmetries of the tetrahedron

The group $\mathbb{T}$ of rotational symmetries of the tetrahedron [12] has order 12. Its action on $\mathbb{R}^3$ is generated by two rotations R and C of orders 2 and 3, respectively, and given by

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Next we want to augment the group $\mathbb{T}$ with a reflection, given by

$$\kappa = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

to form $\langle \mathbb{T}, \kappa \rangle$, the full group of symmetries of the thetrahedron, that has order 24. We obtain an action on $\mathbb{R}^6$ by identifying $\mathbb{R}^6 \equiv \mathbb{C}^3$ and taking the same matrices as generators.

The representation of $\langle \mathbb{T}, \kappa \rangle$ on $\mathbb{C}^3$ is $\langle \mathbb{T}, \kappa \rangle$-simple. The isotropy lattice of $\langle \mathbb{T}, \kappa \rangle$ is shown in Figure 4.3.1.1.

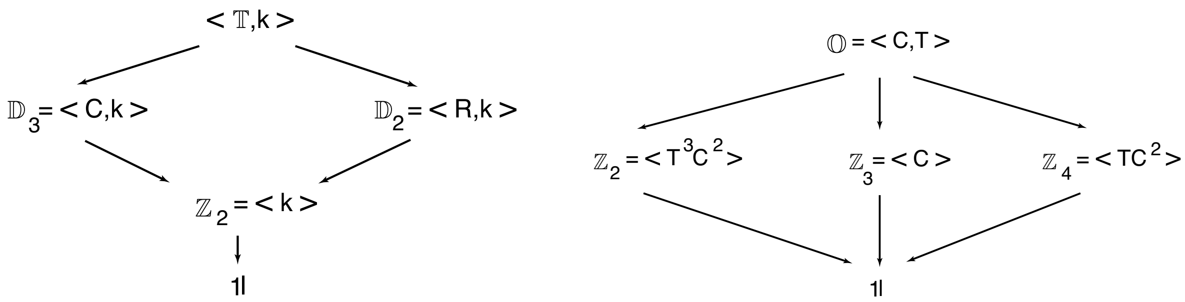

Figure 4.3.1.1. Isotropy lattices for the groups $\langle \mathbb{T}, \kappa \rangle$ of symmetries of the tetrahedron (left) and $\mathbb{O}$ of rotational symmetries of the cube (right).

4.3.2 Symmetries of the cube

The action on $\mathbb{R}^3$ of the group $\mathbb{O}$ of rotational symmetries of the cube is generated by the rotation C of order 3, with the matrix above, and by the rotation T of order 4

$$T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As in the case of the symmetries of the tetrahedron, we obtain an action of $\mathbb{O}$ on $\mathbb{R}^6$ by identifying $\mathbb{R}^6 \equiv \mathbb{C}^3$ and using the same matrices.

As abstract groups, $\langle \mathbb{T}, \kappa \rangle$ and $\mathbb{O}$ are isomorphic, the isomorphism maps C into itself and the rotation T of order 4 in $\mathbb{O}$ into the rotation-reflection $C^2 R\kappa$ in $\langle \mathbb{T}, \kappa \rangle$. However, the two representations are not isomorphic, as can be seen comparing their isotropy lattices in Figure 4.3.1.1.

4.3.3 Adding $\mathbb{S}^1$

The corresponding actions of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ and of $\mathbb{O} \times \mathbb{S}^1$ on $\mathbb{C}^3$ are obtained by adding the elements $e^{i\theta} \cdot \mathrm{Id}$, $\theta \in (0, 2\pi)$ to the group. Note that with this action the elements of $\mathbb{S}^1$ commute with those of $\langle \mathbb{T}, \kappa \rangle$ and of $\mathbb{O}$. The representations $\mathbb{O} \times \mathbb{S}^1$ and $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ are isomorphic, the isomorphism being given by:

$$C \mapsto C \qquad T \mapsto e^{i\pi} C^2 R\kappa \qquad e^{i\theta} \mapsto e^{i\theta}.$$

4.4 Hopf bifurcation

The first step in studying Γ-equivariant Hopf bifurcation is to obtain the $\mathbb{C}$-axial subgroups of $\Gamma \times \mathbb{S}^1$. Isotropy subgroups of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1 \sim \mathbb{O} \times \mathbb{S}^1$ are listed in Table 4.1.

Table 4.1. Isotropy subgroups and corresponding types of solutions, fixed-point subspaces for the action of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1 \sim \mathbb{O} \times \mathbb{S}^1$ on $\mathbb{C}^3$, where $\omega = e^{2\pi i/3}$.

Index	Name	Isotropy subgroup	Generators in $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$	Generators in $\mathbb{O} \times \mathbb{S}^1$	Fixed-point subspace	dim
(a)	Origin	$\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$	$\{C, R, \kappa, e^{i\theta}\}$	$\{C, T, e^{i\theta}\}$	$\{(0,0,0)\}$	0
(b)	Pure mode	$\mathbb{D}_4$	$\{e^{\pi i} C^2 RC, \kappa\}$	$\{TC^2, e^{\pi i} T^2\}$	$\{(z,0,0)\}$	2
(c)	Standing wave	$\mathbb{D}_3$	$\{C, \kappa\}$	$\{C, e^{\pi i} T^2 C^2 T\}$	$\{(z,z,z)\}$	2
(d)	Rotating wave	$\mathbb{Z}_3$	$\{\bar{\omega} C\}$	$\{\bar{\omega} C\}$	$\{(z, \omega z, \omega^2 z)\}$	2
(e)	Standing wave	$\mathbb{D}_2$	$\{e^{\pi i} R, \kappa\}$	$\{e^{\pi i} TC^2TC^2, T^3C^2\}$	$\{(0,z,z)\}$	2
(f)	Rotating wave	$\mathbb{Z}_4$	$\{e^{\pi i/2} C^2 \mathrm{R} \kappa\}$	$\{e^{-\pi i/2} T\}$	$\{(z, iz, 0)\}$	2
(g)	2-Sphere solutions	$\mathbb{Z}_2$	$\{e^{\pi i} R\}$	$\{e^{\pi i} TC^2TC^2\}$	$\{(0, z_1, z_2)\}$	4
(h)	2-Sphere solutions	$\mathbb{Z}_2$	$\{\kappa\}$	$\{e^{\pi i} T^2 C^2 TC\}$	$\{(z_1, z_2, z_2)\}$	4
(i)	General solutions	$\mathbb{1}$	$\{Id\}$	$\{Id\}$	$\{(z_1, z_2, z_3)\}$	6

The normal form for a $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$-equivariant vector field truncated to the cubic order is

$$\begin{cases} \dot{z}_1 = z_1 \left(\lambda + \gamma\left(|z_1|^2 + |z_2|^2 + |z_3|^2\right) + \alpha\left(|z_2|^2 + |z_3|^2\right)\right) + \bar{z}_1 \beta\left(z_2^2 + z_3^2\right) = P(z_1, z_2, z_3) \\ \dot{z}_2 = z_2 \left(\lambda + \gamma\left(|z_1|^2 + |z_2|^2 + |z_3|^2\right) + \alpha\left(|z_1|^2 + |z_3|^2\right)\right) + \bar{z}_2 \beta\left(z_1^2 + z_3^2\right) = Q(z_1, z_2, z_3) \\ \dot{z}_3 = z_3 \left(\lambda + \gamma\left(|z_1|^2 + |z_2|^2 + |z_3|^2\right) + \alpha\left(|z_1|^2 + |z_2|^2\right)\right) + \bar{z}_3 \beta\left(z_1^2 + z_2^2\right) = R(z_1, z_2, z_3) \end{cases} \tag{4.4.1}$$

where $\alpha, \beta, \gamma, \lambda$ are all complex coefficients. This normal form is the same used in [12] for the $\mathbb{T} \times \mathbb{S}^1$ action, except that the extra symmetry κ forces some of the coefficients to be equal. The normal form (4.4.1) is slightly different, but equivalent, to the one given in [1]. As in [1, 12], the origin is always an equilibrium of (4.4.1) and it undergoes a Hopf bifurcation when λ crosses the imaginary axis. By the Equivariant Hopf Theorem, this generates several branches of periodic solutions, corresponding to the $\mathbb{C}$-axial subgroups of $\mathbb{T} \times \mathbb{S}^1$.

Under additional conditions on the parameters in (4.4.1) there may be other periodic solution branches arising through Hopf bifurcation outside the fixed-point subspaces for $\mathbb{C}$-axial subgroups. These have been analysed in [1], we proceed to describe them briefly, with some additional information from [10].

4.4.1 Submaximal branches in $\{(z_1, z_2, 0)\}$

As a fixed-point subspace for the $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ action, this subspace is conjugated to $Fix\left(\mathbb{Z}_2\left(e^{\pi i} R\right)\right)$, (see Table 4.1) the conjugacy being realised by C^2. The same subspace appears as $Fix\left(\mathbb{Z}_2\left(e^{\pi i} T^2\right)\right)$, under the $\mathbb{O} \times \mathbb{S}^1$ action, and is conjugated to $Fix\left(\mathbb{Z}_2\left(e^{\pi i} TC^2TC^2\right)\right)$ in Table 4.1. It contains the fixed-point subspaces

$\{(z,0,0)\}$, $\{(z,z,0)\}$, $\{(iz,z,0)\}$, corresponding to $\mathbb{C}$-axial subgroups, as well as a conjugate copy of each one of them.

Solution branches of (4.4.1) in the fixed-point subspace $\{(z_1,z_2,0)\}$ have been analysed in [1, 10, 11]. Restricting the normal form (4.4.1) to this subspace and eliminating solutions that lie in the two-dimensional fixed-point subspaces, one finds that for $\beta \neq 0$ there is a solution branch with no additional symmetries, if and only if both $|\alpha/\beta| > 1$ and $-1 < \mathcal{R}e\,(\alpha/\beta) < 1$ hold. These solutions lie in the subspaces $\{(\xi z, z, 0)\}$, with $\xi = re^{i\phi}$, where

$$\cos(2\phi) = -\mathcal{R}e\,(\alpha/\beta) \quad \sin(2\phi) = \pm\sqrt{1-(\mathcal{R}e\,(\alpha/\beta))^2} \quad r^2 = \frac{\mathcal{I}m\,(\alpha/\beta)+\sin(2\phi)}{\mathcal{I}m\,(\alpha/\beta)-\sin(2\phi)}. \tag{4.4.2}$$

The submaximal branch of periodic solutions connects all the maximal branches that lie in the subspace $\{(z_1,z_2,0)\}$. This can be deduced from the expressions (4.4.2), as we proceed to explain, and as illustrated in Figure 4.4.1.1.

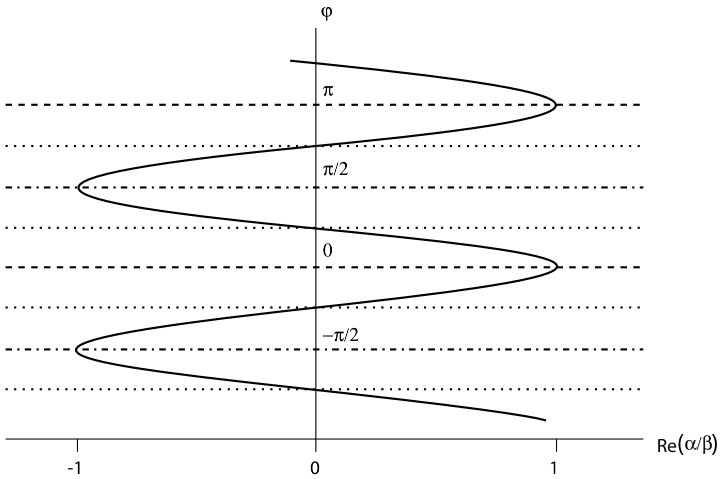

Figure 4.4.1.1. Submaximal branch in the subspace $\{(z_1,z_2,0)\}$ and its connection to the maximal branches. Dashed lines stand for the branches in $\{(\pm z,z,0)\}$ and dot-dash for the branches in $\{(\pm iz,z,0)\}$. Bifurcation into the branches that lie in $\{(0,z,0)\}$ and $\{(z,0,0)\}$, represented by the dotted lines, only occur when both $|\mathcal{R}e\,(\alpha/\beta)| = 0$ and $|\mathcal{I}m\,(\alpha/\beta)| = 1$.

When $\mathcal{R}e\,(\alpha/\beta) = +1$ we get $\xi = \pm i$ and the submaximal branches lie in the subspaces $\{(\pm iz,z,0)\}$. Since the submaximal branches only exist for $\mathcal{R}e\,(\alpha/\beta) \leq 1$, then, as $\mathcal{R}e\,(\alpha/\beta)$ increases to $+1$, pairs of submaximal branches coalesce into the subspaces $\{(\pm iz,z,0)\}$, as in Figure 4.4.1.1.

Similarly, for $\mathcal{R}e\,(\alpha/\beta) = -1$ we have $\xi = \pm 1$. As $\mathcal{R}e\,(\alpha/\beta)$ decreases to -1, the submaximal branches coalesce into the subspace$\{(\pm z,z,0)\}$, at a pitchfork.

To see what happens at $|\alpha/\beta| = 1$, we start with $|\alpha/\beta| > 1$. The submaximal branch exists when $|\mathcal{R}e\,(\alpha/\beta)| < 1$ and hence $|\mathcal{I}m\,(\alpha/\beta)| > 1$. As $|\alpha/\beta|$ decreases, when we reach the value 1 it must be with $|\mathcal{I}m\,(\alpha/\beta)| = 1$ and therefore $0 = |\mathcal{R}e\,(\alpha/\beta)| = -\cos(2\phi)$, hence $\sin(2\phi) = \pm 1$. The expression for r^2 in (4.4.2) shows that in this case either r tends to 0 or r tends to ∞ and pairs of submaximal branches come together at the subspaces $\{(0,z,0)\}$ and $\{(z,0,0)\}$. Figure 4.4.1.1 is misleading for this bifurcation, because $0 = |\mathcal{R}e\,(\alpha/\beta)| = -\cos(2\phi)$ does not in itself imply that the solution lies in one of the subspaces $\{(0,z,0)\}$, $\{(z,0,0)\}$. Indeed, it is possible to have $\cos(2\phi) = 0$ and $\sin(2\phi) = \pm 1$ with $|\alpha/\beta| > 1$. In this case, the right hand side of the expression for r^2 in (4.4.2) is positive, and the solution branches are of the form $(re^{i\phi}z,z,0)\}$, for some $r > 0$ and with $\phi = (2k+1)\pi/4$, $k \in \{1,2,3,4\}$.

From the expression for r^2 in (4.4.2) it follows that the condition $r = 1$ at the submaximal branch only occurs when $\sin(2\phi) = 0$. This is satisfied at the values of α and β for which the submaximal branch bifurcates from one of the $\mathbb{C}$-axial subspaces, as we have already seen. This implies that the submaximal solution branches have no additional symmetry, because, for any γ in $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ or $\mathbb{O} \times \mathbb{S}^1$, the norm of the first coordinate of $\gamma \cdot (\xi z, z, 0)$ is either $|z|$, or $r|z|$, or zero. If $r \neq 1$ then the only possible symmetries are those that fix the subspace $\{(z_1, z_2, 0)\}$.

4.4.2 Submaximal branches in $\{(z_1, z_1, z_2)\}$

As a fixed-point subspace for the $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ action, this subspace is conjugated to $Fix\,(\mathbb{Z}_2\,(\kappa))$, and, under the $\mathbb{O} \times \mathbb{S}^1$ action, as $Fix\,\left(\mathbb{Z}_2\,\left(e^{\pi i} T^2 C^2 T C\right)\right)$ (see Table 4.1) the conjugacy being realised by C^2. It contains the fixed-point subspaces $\{(z, z, 0)\}$, $\{(0, 0, z)\}$, $\{(z, z, z)\}$, corresponding to $\mathbb{C}$-axial subgroups, as well as a conjugate copy of each one of them. In what follows, we give a geometric construction for finding solution branches of (4.4.1) in the fixed-point subspace $\{(z_1, z_1, z_2)\}$, completing the description given in [1]. In particular, we show that the existence of these branches only depends on the parameters α and β in the normal form, and that there are parameter values for which three submaximal solution branches coexist.

Restricting the normal form (4.4.1) to this subspace and eliminating some solutions that lie in the two-dimensional fixed-point subspaces, one finds that for $\beta \neq 0$ there is a solution branch through the point $(z, z, \xi z)$ with $\xi = re^{-i\psi}$, if and only if

$$\beta - \alpha + r^2 \alpha + \beta \left(r^2 e^{-2i\psi} - 2e^{2i\psi} \right) = 0. \tag{4.4.3}$$

For $(x, y) = (\cos 2\psi, \sin 2\psi)$, this is equivalent to the conditions (4.4.4) (4.4.5) below:

$$\mathcal{R}(x, y) = (x - x_0)^2 - (y - y_0)^2 + K_r > 0 \tag{4.4.4}$$

$$K_r = \left[(3\,\mathcal{I}m\,(\alpha/\beta))^2 - (\mathcal{R}e\,(\alpha/\beta) - 1)^2 + 12\,\mathcal{R}e\,(\alpha/\beta) \right] /16$$

and

$$\mathcal{I}(x, y) = (x - x_0)(y - y_0) + K_i = 0 \qquad K_i = \frac{3}{16} \mathcal{I}m\,(\alpha/\beta)\,[\mathcal{R}e\,(\alpha/\beta) + 1] \tag{4.4.5}$$

where

$$x_0 = (1 - 3\mathcal{R}e\,(\alpha/\beta))/4 \qquad y_0 = \mathcal{I}m\,(\alpha/\beta)/4 \quad \text{and} \quad r^2 = \mathcal{R}(x, y) / \left| \alpha + \beta e^{-2i\psi} \right|^2. \tag{4.4.6}$$

Thus, solution branches will correspond to points (x, y) where the (possibly degenerate) hyperbola $\mathcal{I}(x, y) = 0$ intersects the unit circle, subject to the condition $\mathcal{R}(x, y) > 0$. There may be up to four intersection points. By inspection we find that $(x, y) = (1, 0)$ is always an intersection point, where either $\psi = 0$ or $\psi = \pi$. Substituting into (4.4.3), it follows that $r = 1$. These solutions corresponds to the two-dimensional fixed point subspace $\{(z, z, z)\}$ and its conjugate $\{(z, z, -z)\}$.

Solutions that lie in the fixed-point subspace $\{(z, z, 0)\}$ correspond to $r = 0$ and hence to $\mathcal{R}(x, y) = 0$. The subspace $\{(0, 0, z)\}$ corresponds to $r \to \infty$ in (4.4.3), i.e. to $e^{2i\psi} = (1 - \alpha/\beta)/2$, hence $\mathcal{I}(x, y) = 0$ if and only if either $\mathcal{I}m\,(\alpha/\beta) = 0$ or $\mathcal{R}e\,(\alpha/\beta) = 3/2$.

Generically, the fact that $(1, 0)$ is an intersection point implies that $\mathcal{I}(x, y) = 0$ meets the circle on at least one more point. We proceed to describe some of the situations that may arise.

When $\mathcal{R}e\,(\alpha/\beta) = -1$, we have $K_i = 0$ and hence $\mathcal{I}(x, y) = 0$ consists of the two lines $x = 1$ and $y = y_0$, shown in Figure 4.4.2.1. The line $x = 1$ is tangent to the unit circle, hence, when $|\mathcal{I}m\,(\alpha/\beta)| \leq 4$, the intersection of $\mathcal{I}(x, y) = 0$ and the circle consists of three points, except when $\mathcal{I}m\,(\alpha/\beta) = 0$ (Figure 4.4.2.1, left) and in the limit cases $\mathcal{I}m\,(\alpha/\beta) = \pm 4$, when two solutions come together at a saddle-node. The other case when $\mathcal{I}(x, y) = 0$ consists of two lines parallel to the axes occurs when $\mathcal{I}m\,(\alpha/\beta) = 0$, when solutions lie in the subspace $\{(0, 0, z)\}$.

In the general case, when $\mathcal{R}e\,(\alpha/\beta) \neq -1$, $\mathcal{I}m\,(\alpha/\beta) \neq 0$ (Figure 4.4.2.2), the curve $\mathcal{I}(x, y) = 0$ is a hyperbola intersecting the unit circle at $(x, y) = (1, 0)$ and on one to three other points. Again, these other

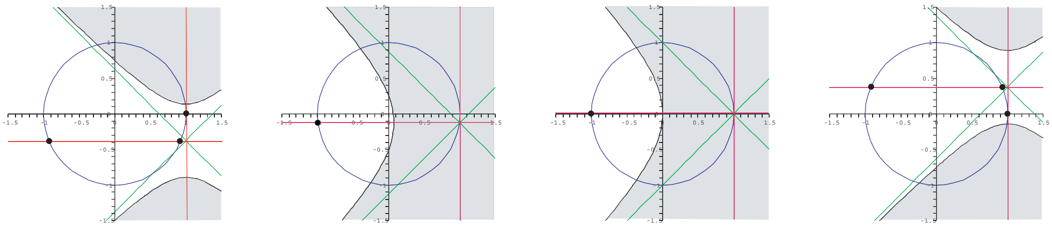

Figure 4.4.2.1. For $\mathcal{R}e\,(\alpha/\beta) = -1$ and $|\mathcal{I}m\,(\alpha/\beta)| \leq 4$, the set $\mathcal{I}(x,y) = 0$ consists of two lines (red), meeting the unit circle (blue) at two or three points that correspond to periodic solution branches (black dots) if $\mathcal{R}(x,y) > 0$ (outside gray area). Green lines are asymptotes to $\mathcal{R}(x,y) = 0$. Graphs, from left to right are for $\mathcal{I}m\,(\alpha/\beta) = -3/2; -1/2; 0; +3/2$.

intersections may correspond to submaximal branches or not, depending on the sign of $\mathcal{R}(x,y)$. Examples with one, three, and four submaximal branches are shown in Figure 4.4.2.2. These branches bifurcate at the fixed-point subspace $\{(z,z,0)\}$, when the intersection meets the line $\mathcal{R}(x,y) = 0$, or from the subspaces $\{(z,z,\pm z)\}$. A pair of branches may also terminate at a saddle-node bifurcation.

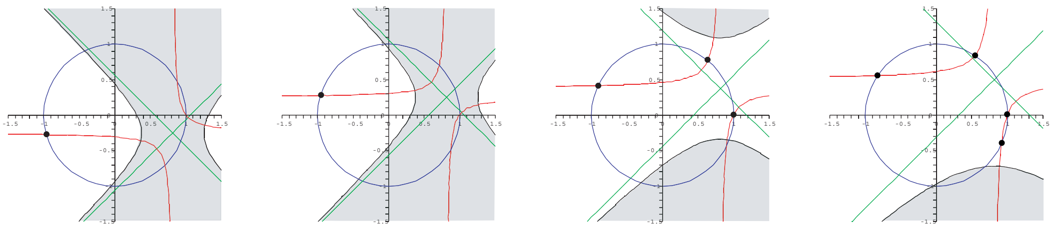

Figure 4.4.2.2. Generically the curve $\mathcal{I}(x,y) = 0$ is a hyperbola (red curves) shown here for $\mathcal{R}e\,(\alpha/\beta) = -3/4$ and, from left to right, for $\mathcal{I}m\,(\alpha/\beta) = -1; +1; +3/2; +2$. The hyperbola intersects the unit circle (blue) at up to four points, that will correspond to periodic solution branches (black dots) if $\mathcal{R}(x,y) > 0$ (outside gray area). Green lines are asymptotes to $\mathcal{R}(x,y) = 0$.

4.5 Spatio-temporal symmetries

The $H \bmod K$ theorem [2, 6] states necessary and sufficient conditions for the existence of a Γ-equivariant differential equation having a periodic solution with specified spatial symmetries $K \subset \Gamma$ and spatio-temporal symmetries $H \subset \Gamma$, as explained in Section 4.2.

For a given $\Gamma-$equivariant differential equation, the H mod K Theorem gives necessary conditions on the symmetries of periodic solutions. Not all these solutions arise by a Hopf bifurcation from the trivial equilibrium — we call this a *primary Hopf bifurcation*. In this section we address the question of determining which periodic solution types, whose existence is guaranteed by the $H \bmod K$ theorem, are obtainable at primary Hopf bifurcations, when the symmetry group is either $\langle \mathbb{T}, \kappa \rangle$ or $\mathbb{O}$.

The first step in answering this question is the next lemma:

Lemma 4.5.1 *Pairs of subgroups H, K of symmetries of periodic solutions arising through a primary Hopf bifurcation for $\Gamma = \langle \mathbb{T}, \kappa \rangle$ are given in Table 4.2 and for $\Gamma = \mathbb{O}$ in Table 4.3.*

Table 4.2. Spatio-temporal symmetries of solutions arising through primary Hopf bifurcation from the trivial equilibrium and number of branches, for the action of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ on $\mathbb{C}^3$. The index refers to Table 4.1. Subgroups of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ below the dividing line are not $\mathbb{C}$-axial.

index	subgroup $\Sigma \subset \langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ generators	Spatio-temporal symmetries H generators	Spatial symmetries K generators	number of branches
(b)	$\{e^{\pi i}C^2RC, \kappa\}$	$\{C^2RC, \kappa\}$	$\{R, \kappa\}$	3
(c)	$\{C, \kappa\}$	$\{C, \kappa\}$	$\{C, \kappa\}$	4
(d)	$\{e^{-2\pi i/3}C\}$	$\{C\}$	$\mathbb{1}$	8
(e)	$\{e^{\pi i}R, \kappa\}$	$\{R, \kappa\}$	$\{\kappa\}$	6
(f)	$\{e^{\pi i/2}C^2R\kappa\}$	$\{C^2R\kappa\}$	$\mathbb{1}$	6
(g)	$\{e^{\pi i}R\}$	$\{R\}$	$\mathbb{1}$	12
(h)	$\{\kappa\}$	$\{\kappa\}$	$\{\kappa\}$	12

Table 4.3. Spatio-temporal symmetries of solutions arising through primary Hopf bifurcation from the trivial equilibrium and number of branches, for the action of $\mathbb{O} \times \mathbb{S}^1$ on $\mathbb{C}^3$. The index refers to Table 4.1. Subgroups of $\mathbb{O} \times \mathbb{S}^1$ below the dividing line are not $\mathbb{C}$-axial.

index	subgroup $\Sigma \subset \mathbb{O} \times \mathbb{S}^1$ generators	Spatio-temporal symmetries H generators	Spatial symmetries K generators	number of branches
(b)	$\{TC^2, e^{\pi i}T^2\}$	$\{TC^2, T^2\}$	$\{TC^2\}$	3
(c)	$\{C, e^{\pi i}T^2C^2T\}$	$\{C, T^2C^2T\}$	$\{C\}$	4
(d)	$\{e^{-2\pi i/3}C\}$	$\{C\}$	$\mathbb{1}$	8
(e)	$\{e^{\pi i}TC^2TC^2, T^3C^2\}$	$\{TC^2TC^2, T^3C^2\}$	$\{T^3C^2\}$	6
(f)	$\{e^{-\pi i/2}T\}$	$\{T\}$	$\mathbb{1}$	6
(g)	$\{e^{\pi i}TC^2TC^2\}$	$\{TC^2TC^2\}$	$\mathbb{1}$	12
(h)	$\{e^{\pi i}T^2C^2TC\}$	$\{T^2C^2TC\}$	$\mathbb{1}$	12

Proof. The symmetries corresponding to the $\mathbb{C}$-axial subgroups of $\Gamma \times \mathbb{S}^1$ provide the first five rows of Tables 4.2 and 4.3. The last two rows correspond to the submaximal branches found in 4.4.1 and 4.4.2 above. ■

The next step is to identify the subgroups corresponding to Theorem 4.2.2.

Lemma 4.5.2 *Pairs of subgroups H, K satisfying conditions $(a)-(d)$ of Theorem 4.2.2 are given in Table 4.4 for $\Gamma = \langle \mathbb{T}, \kappa \rangle$ and in Table 4.5 for $\Gamma = \mathbb{O}$.*

Proof. Conditions $(a) - (d)$ of Theorem 4.2.2 are immediate for the isotropy subgroups with two-dimensional fixed-point subspaces. For $\Gamma = \mathbb{O}$, these are all the non-trivial subgroups. For $\Gamma = \langle \mathbb{T}, \kappa \rangle$, condition (d) has to be verified for $K = \mathbb{Z}_2$, that has a four-dimensional fixed-point subspace. In this case $L_K = \{(z, z, z)\} \cup \{(-z, z, z)\} \cup \{(z, 0, 0)\}$, and since $\dim Fix(\mathbb{Z}_2(\kappa)) = 4$, and L_K consists of two-dimensional subspaces, it follows that $Fix(K) \backslash L_K$ is connected and condition (d) follows.

For $K = \mathbb{1}$ we have that L_K is the union of a finite number of subspaces of dimensions 2 and 4, and again we have condition (d) because $Fix(K) \backslash L_K = \mathbb{C}^3 \backslash L_K$ is connected. In this case, condition (a) is the only restriction and H may be any cyclic subgroup of Γ. ■

Of all the subgroups of $\langle \mathbb{T}, \kappa \rangle$ of order two that appear as H in a pair $H \sim \mathbb{Z}_n, K = \mathbb{1}$ in Table 4.4, only $H \sim \mathbb{Z}_3(C)$ is an isotropy subgroup, as can be seen in Figure 4.3.1.1. The two subgroups of $\langle \mathbb{T}, \kappa \rangle$ of order

Table 4.4. Possible pairs H, K for Theorem 4.2.2 in the action of $\langle\mathbb{T},\kappa\rangle$ on $\mathbb{C}^3$.

K	Generators of K	H	Generators of H	$Fix(K)$	dim
$\mathbb{D}_2$	$\{R,\kappa\}$	K $N(K)=\mathbb{D}_4$	$\{R,\kappa\}$ $\{C^2RC,\kappa\}$	$\{(z,0,0)\}$	2
$\mathbb{D}_3$	$\{C,\kappa\}$	$N(K)=K$	$\{C,\kappa\}$	$\{(z,z,z)\}$	2
$\mathbb{Z}_2$	$\{\kappa\}$	K $N(K)=\mathbb{D}_2$	$\{\kappa\}$ $\{R,\kappa\}$	$\{(z_1,z_2,z_2)\}$	4
$\mathbb{1}$	$\{Id\}$	$\mathbb{Z}_4$ $\mathbb{Z}_3$ $\mathbb{Z}_2$ $\mathbb{Z}_2$ K	$\{C^2R\kappa\}$ $\{C\}$ $\{R\}$ $\{\kappa\}$ $\{Id\}$	$\mathbb{C}^3$	6

Table 4.5. Possible pairs H, K for Theorem 4.2.2 in the action of $\mathbb{O}$ on $\mathbb{C}^3$.

K	Generators of K	H	Generators of H	$Fix(K)$	dim
$\mathbb{Z}_4$	$\{TC^2\}$	K $N(K)=\mathbb{D}_4$	$\{TC^2\}$ $\{T^2,TC^2\}$	$\{(z,0,0)\}$	2
$\mathbb{Z}_3$	$\{C\}$	K $N(K)=\mathbb{D}_3$	$\{C\}$ $\{C,T^2C^2T\}$	$\{(z,z,z)\}$	2
$\mathbb{Z}_2$	$\{T^3C^2\}$	K $N(K)=\mathbb{D}_2$	$\{T^3C^2\}$ $\{TC^2TC^2,T^3C^2\}$	$\{(0,z,z)\}$	2
$\mathbb{1}$	$\{Id\}$	$\mathbb{Z}_4$ $\mathbb{Z}_3$ $\mathbb{Z}_2$ $\mathbb{Z}_2$ K	$\{T\}$ $\{C\}$ $\{TC^2TC^2\}$ $\{T^2C^2TC\}$ $\{Id\}$	$\mathbb{C}^3$	6

two, $H=\mathbb{Z}_2(\kappa)$ and $H=\mathbb{Z}_2(R)$ are not conjugated, since κ fixes a four dimensional subspace, whereas R fixes a subspace of dimension two. In contrast, all but one of the cyclic subgroups of $\mathbb{O}$ are isotropy subgroups, as can be seen comparing Table 4.5 to Figure 4.3.1.1, and noting that $\mathbb{Z}_4(T)$ is conjugated to $\mathbb{Z}_4(TC^2)$ and that $\mathbb{Z}_2(T^2C^2TC)$ is conjugated to $\mathbb{Z}_2(T^3C^2)$. This will have a marked effect on the primary Hopf bifurcations.

Proposition 4.5.1 *For the representation of the group $\langle\mathbb{T},\kappa\rangle$ on $\mathbb{R}^6\sim\mathbb{C}^3$, all pairs of subgroups H,K satisfying the conditions of the H mod K Theorem, with $H\neq\mathbb{1}$, occur as spatio-temporal symmetries of periodic solutions arising through a primary Hopf bifurcation, except for:*

1. *the pair $H=K=\mathbb{D}_2$, generated by $\{R,\kappa\}$;*
2. *the pair $H=\mathbb{Z}_2(\kappa)$, $K=\mathbb{1}$.*

Proof. The result follows by inspection of Tables 4.2 and 4.4, we discuss here why these pairs do not arise in a primary Hopf bifurcation. Case (1) refers to a non-trivial isotropy subgroup $K\subset\langle\mathbb{T},\kappa\rangle$ for which $N(K)\neq K$. For the group $\langle\mathbb{T},\kappa\rangle$, there are two non-trivial isotropy subgroups in this situation, as can be seen in Table 4.4. The first one, $K=\mathbb{D}_2$, is not an isotropy subgroup of $\langle\mathbb{T},\kappa\rangle\times\mathbb{S}^1$, so the pair $H=K=\mathbb{D}_2$ does not occur in Table 4.2 as a Hopf bifurcation from the trivial solution in a normal form with symmetry $\langle\mathbb{T},\kappa\rangle$.

The second subgroup, $K = \mathbb{Z}_2(\kappa)$ occurs as an isotropy subgroup of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ with four-dimensional fixed-point subspace. As we have seen in 4.4.2, in this subspace there are periodic solutions with $K = H = \mathbb{Z}_2(\kappa)$ arising through a Hopf bifurcation with submaximal symmetry. On the other hand, the normaliser of $\mathbb{Z}_2(\kappa)$ corresponds to a $\mathbb{C}$-axial subgroup of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$, so there is a Hopf bifurcation from the trivial solution with $H = N(K)$.

Case (2) concerns the situation when $K = \mathbb{1}$. All the cyclic subgroups $H \subset \langle \mathbb{T}, \kappa \rangle$, with the exception of $\mathbb{Z}_2(\kappa)$, are the projection into $\langle \mathbb{T}, \kappa \rangle$ of cyclic isotropy subgroups of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$, so they correspond to primary Hopf bifurcations. ■

Another reason why the pair $H = K = \mathbb{D}_2$ does not occur at a primary Hopf bifurcation is the following: a non-trivial periodic solution in $Fix(\mathbb{D}_2)$ has the form $X(t) = (z(t), 0, 0)$. Then $Y(t) = C^2RCX(t) = -X(t)$ is also a solution contained in the same plane. If the origin is inside $X(t)$ then the curves $Y(t)$ and $X(t)$ must intersect, so they coincide as curves, and this means that C^2RC is a spatio-temporal symmetry of $X(t)$. Hence, if $H = K = \mathbb{D}_2$, then $X(t)$ cannot encircle the origin, and hence it cannot arise from a Hopf bifurcation from the trivial equilibrium. Of course it may originate at a Hopf bifurcation from another equilibrium. The argument does not apply to the other subgroup K with $N(K) \neq K$, because in this case $\dim Fix(K) = 4$ and indeed, for some values of the parameters in the normal form, there are primary Hopf bifurcations into solutions whith $H = K = \mathbb{Z}_2(\kappa)$.

The second case in Proposition 4.5.1 is more interesting: if $X(t) = (z_1(t), z_2(t), z_3(t))$ is a 2π-periodic solution with $H = \mathbb{Z}_2(\kappa)$, $K = \mathbb{1}$, then for some $\theta \neq 0 \pmod{2\pi}$ and for all t we have $z_1(t+\theta) = z_1(t)$ and $z_3(t+\theta) = z_2(t) = z_2(t+2\theta)$. If $z_2(t)$ and $z_3(t)$ are identically zero, then $X(t) \in Fix(\mathbb{D}_2)$ and hence $K = \mathbb{D}_2$, contradicting our assumption. If $z_2(t)$ and $z_3(t)$ are non-zero, then $\theta = \pi$ and in then $RX(t) = (z_1(t), -z_2(t), -z_3(t))$ is also a solution whose trajectory may not intersect $\{X(t)\}$, since $R \notin H$. The possibilities are then of the form $X(t) = (z_1(t), z_2(t), z_2(t+\pi))$ with a π-periodic z_1. If $z_1(t) \equiv 0$, then $X(t)$ cannot be obtained from the $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ action, because in this case it would be $X(t) \in Fix(\mathbb{Z}_2(e^{\pi i}R))$, hence $R \in H$. The only other possibility is to have all coordinates of $X(t)$ not zero, with $z_2(t)$ and $z_3(t)$ having twice the period of $z_1(t)$, so if they do arise at a Hopf bifurcation, it will be at a 2-1 resonance. This last situation does not occur for the group $\mathbb{O}$, as can be seen in the next result.

Proposition 4.5.2 *For the representation of the group $\mathbb{O}$ on $\mathbb{R}^6 \sim \mathbb{C}^3$, all pairs of subgroups H, K satisfying the conditions of the $H \bmod K$ Theorem, with $H \neq \mathbb{1}$, occur as spatio-temporal symmetries of periodic solutions arising through a primary Hopf bifurcation, except for the following pairs:*

1. *$H = K = \mathbb{Z}_4$, generated by $\{TC^2\}$;*
2. *$H = K = \mathbb{Z}_3$ generated by $\{C\}$;*
3. *$H = K = \mathbb{Z}_2$ generated by $\{T^3C^2\}$.*

Proof. As in Proposition 4.5.1, the result follows comparing Tables 4.3 and 4.5.

All the cyclic subgroups $H \subset \mathbb{O}$ are the projection into $\mathbb{O}$ of cyclic isotropy subgroups Σ of $\mathbb{O} \times \mathbb{S}^1$, so the pairs $H, K = \mathbb{Z}_n, \mathbb{1}$ correspond to primary Hopf bifurcations: for $\mathbb{Z}_4$ and $\mathbb{Z}_3$ the subgroup $\Sigma \subset \mathbb{O} \times \mathbb{S}^1$ is $\mathbb{C}$-axial, whereas for the subgroups of $\mathbb{O}$ of order two, the subspace $Fix(\Sigma)$ is four-dimensional and has been treated in 4.4.1 and 4.4.2 above.

All the non-trivial isotropy subgroups K of $\mathbb{O}$ satisfy $N(K) \neq K$, so they are candidates for cases where $H = K$ does not occur. This is indeed the case, since they are not isotropy subgroups of $\mathbb{O} \times \mathbb{S}^1$. ■

The pairs H, K in Proposition 4.5.2, that are not symmetries of solutions arising through primary Hopf bifurcation, are of the form $H = K$, where there would be only spatial symmetries. Here, $\dim Fix(K) = 2$ for all cases. As remarked after the proof of Proposition 4.5.1, the origin cannot lie inside a closed trajectory with these symmetries, hence they cannot arise at a primary Hopf bifurcation.

4.6 Acknowledgements

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Chapter 5

Limit cycles for a class of quintic $\mathbb{Z}_6$-equivariant systems without infinite critical points

This chapter consists of a joint article accepted for publication by the Bulletin of the Belgian Mathematical Society.

M.J. Álvarez, Isabel S. Labouriau and Adrian C. Murza

M.J. Álvarez, Departament de Matemàtiques i Informàtica, Universitat de les Illes Balears, 07122, Palma de Mallorca, Spain
I.S. Labouriau, Centro de Matemática da Universidade do Porto. Rua do Campo Alegre 687, 4169-007 Porto, Portugal
A.C. Murza, Centro de Matemática da Universidade do Porto. Rua do Campo Alegre 687, 4169-007 Porto, Portugal

Abstract. *We analyze the dynamics of a class of $\mathbb{Z}_6$-equivariant systems of the form $\dot{z} = pz^2\bar{z} + sz^3\bar{z}^2 - \bar{z}^5$, where z is complex, the time t is real, while p and s are complex parameters. This study is the natural continuation of a previous work (M.J. Álvarez, A. Gasull, R. Prohens, Proc. Am. Math. Soc.* ***136****, (2008), 1035–1043) on the normal form of $\mathbb{Z}_4$-equivariant systems. Our study uses the reduction of the equation to an Abel one, and provide criteria for proving in some cases uniqueness and hyperbolicity of the limit cycle surrounding either 1, 7 or 13 critical points, the origin being always one of these points.*

Keywords:
Planar autonomous ordinary differential equations, symmetric polinomial systems, limit cycles

AMS 2010 Subject Classification:
Primary: 34C07, 34C14; Secondary: 34C23, 37C27

5.1 Introduction and main results

Hilbert XVI^{th} problem represents one of the open question in mathematics and it has produced an impressive amount of publications throughout the last century. The study of this problem in the context of equivariant dynamical systems is a relatively new branch of analysis and is based on the development within the last twenty years of the theory of Golubitsky, Stewart and Schaeffer in [9, 10]. Other authors [6] have specifically considered this theory when studying the limit cycles and related phenomena in systems with symmetry. Roughly speaking the presence of symmetry may complicate the bifurcation analysis because it often forces eigenvalues of high multiplicity. This is not the case of planar systems; on the contrary, it simplifies the analysis because of the reduction to isotypic components. More precisely it allows us to reduce the bifurcation analysis to a region of the complex plane.

In this paper we analyze the $\mathbb{Z}_6$-equivariant system

$$\dot{z} = \frac{dz}{dt} = (p_1 + ip_2)z^2\bar{z} + (s_1 + is_2)z^3\bar{z}^2 - \bar{z}^5 = f(z), \tag{5.1.1}$$

where $p_1, p_2, s_1, s_2 \in \mathbb{R}$.

The general form of the $\mathbb{Z}_q$-equivariant equation is

$$\dot{z} = zA(|z|^2) + B\bar{z}^{q-1} + O(|z|^{q+1}),$$

where A is a polynomial of degree $[(q-1)/2]$. The study of this class of equations is developed in several books, see [3, 6], when the resonances are strong, *i.e.* $q < 4$ or weak $q > 4$. The special case $q = 4$ is also treated in several other articles, see [1, 6, 13]. In these mentioned works it is said that the weak resonances are easier to study than the other cases, as the equivariant term $\bar{z}^{q-1}$ is not dominant with respect to the function on $\bar{z}^2$. This is true if the interest lies in obtaining a bifurcation diagram near the origin, but it is no longer true if the analysis is global and involves the study of limit cycles. This is the goal of the present work: studying the global phase portrait of system (5.1.1) paying special attention to the existence, location and uniqueness of limit cycles surrounding 1, 7 or 13 critical points. As far as we know this is the first work in which the existence of limit cycles is studied for this kind of systems.

The main result of our paper is the following.

Theorem 5.1.1 *Consider equation* (5.1.1) *with* $p_2 \neq 0$, $|s_2| > 1$ *and define the quantities:*

$$\Sigma_A^- = \frac{p_2 s_1 s_2 - \sqrt{p_2^2(s_1^2 + s_2^2 - 1)}}{s_2^2 - 1}, \qquad \Sigma_A^+ = \frac{p_2 s_1 s_2 + \sqrt{p_2^2(s_1^2 + s_2^2 - 1)}}{s_2^2 - 1}.$$

Then, the following statements are true:

(a) If one of the conditions

$$(i) \quad p_1 \notin \left(\Sigma_A^-, \Sigma_A^+\right), \qquad (ii) \quad p_1 \notin \left(\frac{\Sigma_A^-}{2}, \frac{\Sigma_A^+}{2}\right)$$

is satisfied, then equation (5.1.1) *has at most one limit cycle surrounding the origin. Furthermore, when the limit cycle exists it is hyperbolic.*

(b) There are equations (5.1.1) *under condition* (ii) *having exactly one limit cycle surrounding either* 1, 7 *or* 13 *critical points, and equations* (5.1.1) *under condition* (i) *having exactly one hyperbolic limit cycle surrounding either* 7 *critical points if* $p_1 \neq \Sigma_A^{\pm}$*, or the only critical point if* $p_1 = \Sigma_A^{\pm}$*.*

The conditions (i) and (ii) of Theorem 5.1.1 hold except when p_1 lies in $\left(\Sigma_A^-, \Sigma_A^+\right) \cap \left(\frac{\Sigma_A^-}{2}, \frac{\Sigma_A^+}{2}\right)$. The intersection of the two intervals is often empty, but this is not always the case as the example of Figure 5.1.0.1 shows. Examples of the relative positions of the two intervals are also given in Figure 5.3.0.2 below.

Our strategy for proving Theorem 5.1.1 will be to transform the system (5.1.1) into a scalar Abel equation and to study it. Conditions (i) and (ii) define regions where one of the functions in the Abel equation does not change sign. Since these functions correspond to derivatives of the Poincaré return map, this imposes an upper bound on the number of limit cycles. When either p_1 lies in the intersection of the two intervals, or $|s_2| \leq 1$, this analysis is not conclusive, and the study of the equations would require other methods. The qualitative meaning

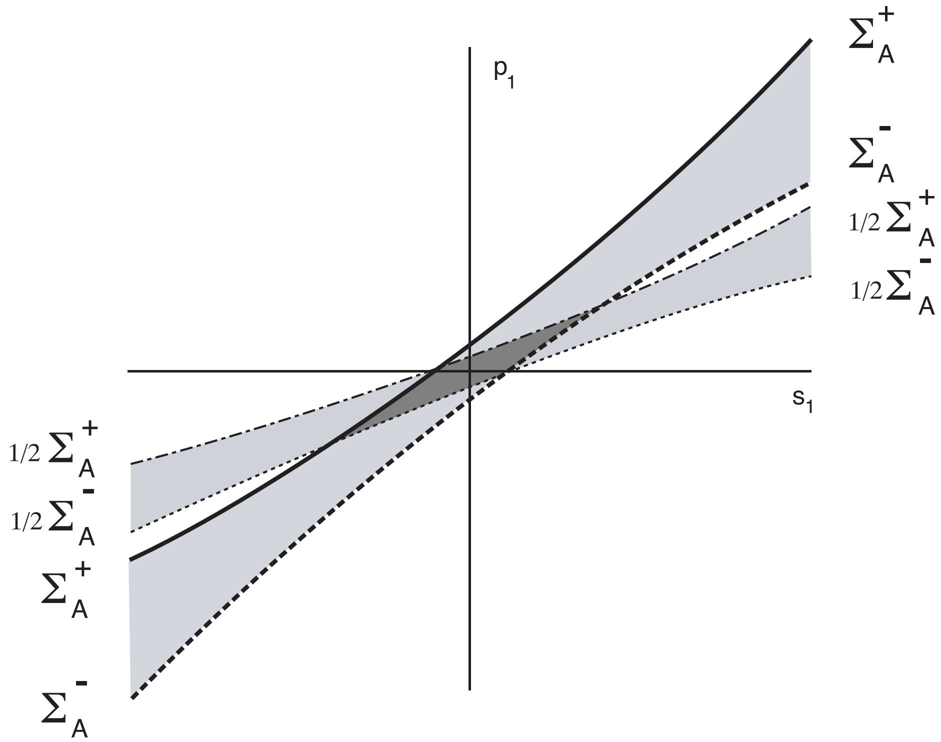

Figure 5.1.0.1. Equation (5.1.1) has at most one limit cycle surrounding the origin for (s_1, p_1) outside the dark intersection of the two shaded areas, when $p_2 = 1$ and $s_2 = 4$. Solid line is Σ_A^+, dashed line stands for Σ_A^-, dotted line corresponds to $\Sigma_A^-/2$ and dashed-dotted to $\Sigma_A^+/2$. For p_1 outside the interval $\left(\Sigma_A^-, \Sigma_A^+\right)$ there are at most 7 equilibria, but when p_1 lies in that interval, there may be 13 equilibria surrounded by a limit cycle.

of conditions of statement (a) of the previous theorem is also briefly explained and illustrated in Remark 5.3.3 and in Figure 5.3.0.2 below.

The paper is organized as follows. In Section 5.2 we state some preliminary results while in Section 5.3 the study of the critical points is performed. Section 5.4 is entirely devoted to the proof of the main theorem of the paper.

5.2 Preliminary results

We start by obtaining the symmetries of (5.1.1). Following [9, 10], a system of differential equations $dx/dt = f(x)$ is said to be Γ-equivariant if it commutes with the group action of Γ, ie. $f(\gamma x) = \gamma f(x),\ \forall \gamma \in \Gamma$. Here $\Gamma = \mathbb{Z}_6$ with the standard action on $\mathbb{C}$ generated by $\gamma_1 = \exp(2\pi i/6)$ acting by complex multiplication. Applying this concept to equation (5.1.1) we have the following result.

Proposition 5.2.1 *Equation* (5.1.1) *is* $\mathbb{Z}_6$*-equivariant.*

Proof. Let $\gamma_k = \exp(2\pi i k/6),\ k = 0, \ldots, 5$. Then a simple calculation shows that $f(\gamma_k z) = \gamma_k f(z)$. This is true because the monomials in z, $\bar{z}$ appearing in the expression of f are the following: $\bar{z}^5$, which is γ_k-equivariant, and monomials of the form $z^{\ell+1}\bar{z}^{\ell}$, that are $\mathbb{Z}_n$-equivariant for all n. ■

Equation (5.1.1) represents a perturbation of a Hamiltonian one and in the following we identify conditions that some parameters have to fulfill in order to obtain this Hamiltonian. We have the following result.

Theorem 5.2.1 *The Hamiltonian part of $\mathbb{Z}_6$-equivariant equation* (5.1.1) *is*

$$\dot{z} = i(p_2 + s_2 z\bar{z})z^2\bar{z} - \bar{z}^5.$$

Proof. An equation $\dot{z} = F(z, \bar{z})$ is Hamiltonian if $\frac{\partial F}{\partial z} + \frac{\partial \bar{F}}{\partial \bar{z}} = 0$. For equation (5.1.1) we have

$$\frac{\partial F}{\partial z} = 2(p_1 + ip_2)z\bar{z} + 3(s_1 + is_2)z^2\bar{z}^2$$

$$\frac{\partial \bar{F}}{\partial \bar{z}} = 2(p_1 - ip_2)z\bar{z} + 3(s_1 - is_2)z^2\bar{z}^2$$

and consequently it is Hamiltonian if and only if $p_1 = s_1 = 0$. ■

As we have briefly said in the introduction, we reduce the study of system (5.1.1) to the analysis of a scalar equation of Abel type. The first step consists in converting equation (5.1.1) from cartesian into polar coordinates.

Lemma 5.2.1 *The study of periodic orbits of equation* (5.1.1) *that surround the origin, for $p_2 \neq 0$, reduces to the study of non contractible solutions that satisfy $x(0) = x(2\pi)$ of the Abel equation*

$$\frac{dx}{d\theta} = A(\theta)x^3 + B(\theta)x^2 + C(\theta)x \tag{5.2.1}$$

where

$$\begin{aligned} A(\theta) &= \frac{2}{p_2}\left(p_1 - p_2 s_1 s_2 + p_1 s_2^2 + (2p_1 s_2 - p_2 s_1)\sin(6\theta)\right) + \\ &\quad + \frac{2}{p_2}\left((p_2 \sin(6\theta) - p_1\cos(6\theta) + p_2 s_2)\cos(6\theta)\right), \\ B(\theta) &= \frac{2}{p_2}\left(p_2 s_1 - 2p_1 s_2 - p_2\cos(6\theta) - 2p_1\sin(6\theta)\right), \\ C(\theta) &= \frac{2p_1}{p_2}. \end{aligned} \tag{5.2.2}$$

Proof. Using the change of variables

$$z = \sqrt{r}(\cos(\theta) + i\sin(\theta))$$

and the time rescaling, $\frac{dt}{ds} = r$, it follows that the solutions of equation (5.1.1) are equivalent to those of the polar system

$$\begin{cases} \dot{r} = 2rp_1 + 2r^2\left(s_1 - \cos(6\theta)\right) \\ \dot{\theta} = p_2 + r\left(s_2 + \sin(6\theta)\right) \end{cases}. \tag{5.2.3}$$

From equation (5.2.3) we obtain

$$\frac{dr}{d\theta} = \frac{2rp_1 + 2r^2\left(s_1 - \cos(6\theta)\right)}{p_2 + r\left(s_2 + \sin(6\theta)\right)}.$$

Then we apply the Cherkas transformation $x = \dfrac{r}{p_2 + r\left(s_2 + \sin(6\theta)\right)}$, see [5], to get the scalar equation (5.2.1). Obviously the limit cycles that surround the origin of equation (5.1.1) are transformed into non contractible periodic orbits of equation (5.2.1), as they cannot intersect the set $\{\dot{\theta} = 0\}$. For more details see [7].

■

As we have already mentioned in the introductory section, our goal in this work is to apply the methodology developed in [1] to study conditions for existence, location and unicity of the limit cycles surrounding $1,\ 7$ and 13 critical points.

A natural way for proving the existence of a limit cycle is to show that, in the Poncaré compactification, infinity has no critical points and both infinity and the origin have the same stability. Therefore, we would like to find the sets of parameters for which these conditions are satisfied. In the following lemma we determine the stability of infinity.

Lemma 5.2.2 *Consider equation* (5.1.1) *in the Poincaré compactification of the plane. Then:*

(i) There are no critical points at infinity if and only if $|s_2| > 1$;

(ii) When $s_2 > 1$, *infinity is an attractor (resp. a repellor) when* $s_1 > 0$ *(resp.* $s_1 < 0$*) and the opposite when* $s_2 < -1$.

Proof. The proof follows the same steps as the Lemma 2.2 in [1]. After the change of variable $R = 1/r$ in system (5.2.3) and reparametrization $\dfrac{dt}{ds} = R$, we get the system

$$\begin{cases} R' = \frac{dR}{ds} = -2R(s_1 - \cos(6\theta)) - 2p_1R^2, \\ \\ \theta' = \frac{d\theta}{ds} = s_2 + \sin(6\theta) + p_2R, \end{cases}$$

giving the invariant set $\{R = 0\}$, that corresponds to the infinity of system (5.2.3). Consequently, it has no critical points at infinity if and only if $|s_2| > 1$. To compute the stability of infinity in this case, we follow [11] and study the stability of $\{R = 0\}$ in the system above. This stability is given by the sign of

$$\int_0^{2\pi} \frac{-2(s_1 - \cos 6\theta)}{s_2 + \sin(6\theta)} d\theta = \frac{-\mathrm{sgn}(s_2)4\pi s_1}{\sqrt{s_2^2 - 1}},$$

and the result follows. ■

5.3 Analysis of the critical points

In this section we are going to analyze which conditions must be satisfied to ensure that equation (5.2.3) has one, seven or thirteen critical points. Obviously, the origin of the system is always a critical point. We are going to prove that it is *monodromic*: there is no trajectory of the differential equations that approaches the critical point with a definite limit direction.

For this purpose, let us define the generalized Lyapunov constants. Consider the solution of the following scalar equation

$$\frac{dr}{d\theta} = \sum_{i=1}^{\infty} R_i(\theta)\, r^i, \tag{5.3.1}$$

where $R_i(\theta),\ i \geqslant 1$ are $T-$periodic functions. To define the generalized Lyapunov constants, consider the solution of (5.3.1) that for $\theta = 0$ passes through ρ. It may be written as

$$r(\theta, \rho) = \sum_{i=1}^{\infty} u_i(\theta)\rho^i$$

with $u_1(0) = 1,\ u_k(0) = 0, \forall\, k \geqslant 2$. Hence, the return map of this solution is given by the series

$$\Pi(\rho) = \sum_{i=1}^{\infty} u_i(T)\rho^i.$$

For a given system, in order to determine the stability of a solution, the only significant term in the return map is the first nonvanishing term that makes it differ from the identity map. Moreover, this term will determine the stability of this solution. On the other hand, if we consider a family of systems depending on parameters, each of the $u_i(T)$ depends on these parameters. We will call k^{th} generalized Lyapunov constant $V_k = u_k(T)$ the value of this expression assuming $u_1(T) = 1, u_2(T) = \ldots, = u_{k-1}(T) = 0$.

Lemma 5.3.1 *The origin of system* (5.2.3) *is monodromic, if* $p_2 \neq 0$*. Moreover, its stability if given by the sign of* p_1*, if it is not zero, and by the sign of* s_1 *if* $p_1 = 0$*.*

Proof. To prove that the origin is monodromic we calculate the arriving directions of the flow to the origin, see [2, Chapter IX] for more details. Concretely if we write system (5.2.3) in cartesian coordinates we get

$$\begin{aligned} \dot{x} = P(x,y) = &\; x^5(-1+s_1) - s_2x^4y + 2(5+s_1)x^3y^2 + (-5+s_1)xy^4 \\ &+ p_1x(x^2+y^2) - y^3(p_2+s_2y^2) - x^2y(p_2+2s_2y^2) \\ \dot{y} = Q(x,y) = &\; p_2x^3 + s_2x^5 + p_1x^2y + 5x^4y + s_1x^4y + p_2xy^2 \\ &+ 2s_2x^3y^2 + p_1y^3 - 10x^2y^3 + 2s_1x^2y^3 + s_2xy^4 + y^5 + s_1y^5 \end{aligned} \tag{5.3.2}$$

Then any solutions arriving at the origin will be tangent to the directions θ that are the zeros of $r\dot{\theta} = R(x,y) = xQ(x,y) - yP(x,y) = -p_2(x^2+y^2)^2$, which is always different from zero. It follows that the origin is either a focus or a center.

To know the stability of the origin, we compute the two first Lyapunov constants. As it is well-known, the two first Lyapunov constants of an Abel equation are given by

$$V_1 = \exp\left(\int_0^{2\pi} C(\theta)d\theta\right) - 1, \qquad V_2 = \int_0^{2\pi} B(\theta)d\theta.$$

Applying this to equation (5.2.1) with the expressions given in (5.2.2) we get the following result:

$$V_1 = \exp\left(4\pi\frac{p_1}{p_2}\right) - 1,$$

and if $V_1 = 0$, then $V_2 = 4\pi s_1$, and we get the result. ■

Remark 5.3.1 *It is clear that* $(V_1, V_2) = (0,0)$ *if and only if* $(p_1, s_1) = (0,0)$ *i.e. equation* (5.1.1) *is hamiltonian (see Theorem 5.2.1). Consequently, as the origin remains being monodromic, in this case it is a center.*

In the next result we study the equilibria of equation (5.2.3) with $r \neq 0$. They will be the non-zero critical points of the system.

Lemma 5.3.2 *Let* $-\pi/6 < \theta < \pi/6$*. Then the equilibria of system* (5.2.3) *with* $r \neq 0$ *are given by:*

$$r = \frac{-p_2}{s_2 + \sin(2\theta_\pm)}, \qquad \theta_\pm = \frac{1}{3}\arctan(\Delta_\pm), \tag{5.3.3}$$

where $\Delta_\pm = \dfrac{p_1 \pm u}{p_2 - p_1s_2 + p_2s_1}$ *and* $u = \sqrt{p_1^2 + p_2^2 - (p_1s_2 - p_2s_1)^2}$.

Proof. Let $-\pi/6 < \theta < \pi/6$. To compute the critical points of system (5.2.3), we have to solve the following nonlinear system:

$$\begin{aligned} 0 &= 2rp_1 + 2r^2(s_1 - \cos(6\theta)) \\ 0 &= p_2 + r(s_2 + \sin(6\theta)). \end{aligned} \tag{5.3.4}$$

Let $x = 6\theta$ and $t = \tan\frac{x}{2}$, so $t = \tan 3\theta$. Then, doing some simple computations one gets

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}. \tag{5.3.5}$$

Eliminating r from equations (5.3.4) and using the previous formulas we get

$$(-p_2 + p_1 s_2 - p_2 s_1)t^2 + 2p_1 t + p_2 + p_1 s_2 - p_2 s_1 = 0.$$

Solving the previous equation for t, yields the result.

Finally, consider the interval $-\pi/6 \leqslant \theta < \pi/6$, let $x = 6\theta$ and $\tau = \cot\frac{x}{2}$, so $\tau = \cot 3\theta$. The same expressions (5.3.5) for $\sin x$ and $\cos x$ hold if t is replaced by $\tau = 1/t$, hence the rest of the proof is applicable. ■

We will prove now that simultaneous equilibria of the type $(r,\theta) = (r,0)$, $(r,\theta) = (\tilde{r}, \pi/6)$ are not possible.

Lemma 5.3.3 *If* $|s_2| > 1$, *then there are no parameter values for which there are simultaneous equilibria of system* (5.2.3) *for* $\theta = 0$ *and* $\theta = \pi/6$ *different from the origin.*

Proof. If we solve

$$\begin{cases} 0 = p_1 + r(s_1 - \cos 6\theta) \\ 0 = p_2 + r(s_2 + \sin 6\theta) \end{cases}$$

for $\theta = 0$, we get $p_1 s_2 = p_2(s_1 - 1)$ with the restriction $\operatorname{sign} p_2 = -\operatorname{sign} s_2$, to have r well defined in the second equation. On the other hand, solving the same system for $\theta = \pi/6$, we get $p_1 s_2 = p_2(s_1+1)$ with the same restriction $\operatorname{sign} p_2 = -\operatorname{sign} s_2$. This means $p_2 = 0$ that implies $r = 0$ and the result follows. ■

In the following we sumarize the conditions that parameters have to fulfill in order that system (5.2.3) has exactly one, seven or thirteen critical points (see figure 5.3.0.1).

Lemma 5.3.4 *Consider system* (5.2.3) *with* $|s_2| > 1$. *If* $s_2 p_2 \geq 0$, *then the only equilibrium is the origin. If* $s_2 p_2 < 0$ *then the number of equilibria of system* (5.2.3) *is determined by the quadratic form:*

$$\mathcal{Q}(p_1,p_2) = p_1^2 + p_2^2 - (p_1 s_2 - p_2 s_1)^2 = (1 - s_2^2)p_1^2 + (1 - s_1^2)p_2^2 + 2 s_1 s_2 p_1 p_2 \tag{5.3.6}$$

Concretely:

1. *exactly one equilibrium (the origin) if* $\mathcal{Q}(p_1,p_2) < 0$*;*
2. *exactly seven equilibria (the origin and one non-degenerate saddle-node per sextant) if* $\mathcal{Q}(p_1,p_2) = 0$*;*
3. *exactly thirteen equilibria (the origin and two per sextant, a saddle and a node) if* $\mathcal{Q}(p_1,p_2) > 0$.

Proof. We will do the proof for the case $s_2 > 1$, the other case being analogous.

There will exist more critical points different from the origin if and only if the formulas (5.3.3) given in Lemma 5.3.2 are realizable with $r > 0$. This will not occur either when the expression for r is negative (that corresponds to the condition $p_2 \geq 0$) or when the discriminant in $\Delta_\pm$ is negative (that corresponds to $\mathcal{Q}(p_1,p_2) < 0$).

In order to have exactly six more critical points, two conditions have to be satisfied: $p_2 < 0$ to ensure positive values of r, as $s_2 > 1$, and the quantities $\Delta_\pm$ have to coincide, *i.e.* the discriminant u in Lemma 5.3.2 has to be zero, that is the condition (2) given in the statement of the lemma. Hence, $r_+ = r_-$ and $\theta_+ = \theta_-$.

We prove now that these additional critical points are saddle-nodes. By symmetry, we only need to prove it for one of them. The Jacobian matrix of system (5.2.3) evaluated at (r_+, θ_+) is

$$J_{(r_+,\theta_+)} = \begin{pmatrix} 2p_1 + 4r_+(s_1 - \cos(6\theta_+) & 12 r_+^2 \sin(6\theta_+) \\ s_2 + \sin(6\theta_+) & 6 r_+ \cos(6\theta_+) \end{pmatrix}. \tag{5.3.7}$$

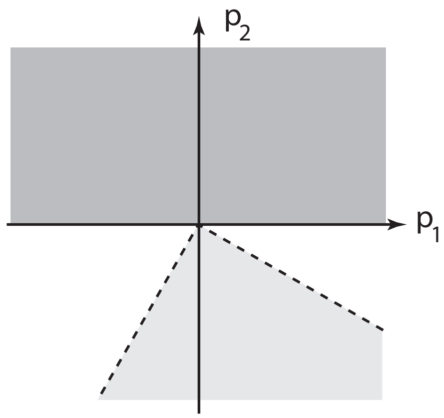

Figure 5.3.0.1. Bifurcation diagram for equilibria of equation (5.2.3) with $s_2 > 1$ on the (p_1, p_2) plane. For (p_1, p_2) in the shaded regions, the only equilibrium is the origin. In the white region there are two other equilibria, a saddle and a node, in each sextant. On the dotted line, the saddle and the node come together into a saddle-node in each sextant. When p_2 tends to 0 in the white region, the two equilibria tend to the origin. For $s_2 < -1$, the diagram is obtained by reflecting on the p_1 axis.

Evaluating the Jacobian matrix (5.3.7) at the concrete expression (5.3.3) of the critical point and taking into account the condition $p_1^2 + p_2^2 = (p_1 s_2 - p_2 s_1)^2$, the eigenvalues of the matrix are

$$\lambda_1 = 0 \qquad\qquad \lambda_2 = 2p_1 - \frac{6p_2(p_2 + p_2 s_1 - p_1 s_2)}{(p_2 s_2 + p_1)(1 + s_1) - p_1 s_2^2}.$$

Therefore, (r_+, θ_+) has a zero eigenvalue. To show that these critical points are saddle-nodes we use the following reasoning. It is well–known, see for instance [2], that the sum of the indices of all critical points contained in the interior of a limit cycle of a planar system is $+1$. As under our hypothesis the infinity does not have critical points on it, it is a limit cycle of the system and it has seven singularities in its interior: the origin, that is a focus and hence has index +1, and 6 more critical points, all of the same type because of the symmetry. Consequently, the index of these critical points must be 0. As we have proved that they are semi-hyperbolic critical points then they must be saddle-nodes.

In order that equation (5.2.3) has exactly thirteen critical points it is enough that $r > 0$ (*i.e.* $p_2 < 0$) and the discriminant in $\Delta_\pm$ of Lemma 5.3.2 is positive, that is the condition (3) of the statement.

To get the stability of the twelve critical points we evaluate the Jacobian matrix (5.3.7) at these critical points, and taking into account the condition $p_1^2 + p_2^2 > (p_1 s_2 - p_2 s_1)^2$, the eigenvalues of the critical points (r_+, θ_+) are $\lambda_{1,2} = R_+ \pm S_+$, while the eigenvalues of (r_-, θ_-) are $\alpha_{1,2} = R_- \pm S_-$, where

$$R_\pm = \frac{p_1^2 s_1 \pm 3p_2^2 s_1 \mp 2p_1 p_2 s_2 + 2p_1 u}{\mp p_1 s_1 \mp p_2 s_2 + u},$$

$$S_\pm = \frac{\sqrt{48(p_1^2 + p_2^2)u(u \mp p_1 s_1 \mp p_2 s_2) + (2p_1^2 s_1 + 6p_2^2 s_1 - 4p_1 p_2 s_1 2 \pm 4p_1 u)^2}}{p_1 \mp p_1 s_1 \mp p_2 s_2}.$$

In the following we will show that one of the critical points has index $+1$, while the other is a saddle.

Doing some computations one gets that the product of the eigenvalues of (r_+, θ_+) and (r_-, θ_-) are, respectively

$$R_+^2 - S_+^2 = \frac{-12(p_1^2+p_2^2)u}{u - p_1 s_1 - p_2 s_2}$$

$$R_-^2 - S_-^2 = \frac{-12(p_1^2+p_2^2)u}{u + p_1 s_1 + p_2 s_2}.$$

The numerator of both expressions is negative.

If $p_1 s_1 + p_2 s_2 > 0$ then $S_-^2 > 0$ and $R_-^2 - S_-^2 < 0$. Consequently the critical point (r_-, θ_-) is a saddle. On the other hand, the denominator of $R_+^2 - S_+^2$ is negative. This is true as

$$\begin{aligned} & u - p_1 s_1 - p_2 s_2 < 0 \iff u^2 < (p_1 s_1 + p_2 s_2)^2 \iff \\ & \iff p_1^2 + p_2^2 - p_2^2 s_1^2 - p_1^2 s_2^2 + 2p_1 p_2 s_1 s_2 < p_1^2 s_1^2 + p_2^2 s_2^2 + 2p_1 p_2 s_1 s_2 \\ & \iff 0 < (p_1^2 + p_2^2)(s_1^2 + s_2^2 - 1), \end{aligned}$$

that is always true. Hence, $R_+^2 - S_+^2 > 0$ and (r_+, θ_+) has index $+1$.

If $p_1 s_1 + p_2 s_2 < 0$, doing similar reasonings one gets that the critical point (r_+, θ_+) is a saddle while (r_-, θ_-) has index $+1$. ■

Note that $\mathcal{Q}$ is a quadratic form on p_1, p_2, and its determinant $1 - s_1^2 - s_2^2$, is negative if $s_2 > 1$. Hence, for each choice of s_1, s_2 with $s_2 > 1$, the points where $\mathcal{Q}(p_1, p_2)$ is negative lie on two sectors, delimited by the two perpendicular lines where $\mathcal{Q}(p_1, p_2) = 0$. Since $\mathcal{Q}(0, p_2) = (1 - s_2^2)p_2^2 < 0$ for $s_2 > 1$, then the sectors where there are two equilibria in each sextant do not include the p_2 axis, as in Figure 5.3.0.1.

Lemma 5.3.5 *Consider $|s_2| > 1$ and define the following two numbers:*

$$\Sigma_A^- = \frac{p_2 s_1 s_2 - \sqrt{p_2^2\left(s_1^2 + s_2^2 - 1\right)}}{s_2^2 - 1}, \quad \Sigma_A^+ = \frac{p_2 s_1 s_2 + \sqrt{p_2^2\left(s_1^2 + s_2^2 - 1\right)}}{s_2^2 - 1}.$$

Let $A(\theta)$ be the function given in Lemma 5.2.1. Then the function $A(\theta)$ changes sign if and only if $p_1 \in \left(\Sigma_A^-, \Sigma_A^+\right)$.

Proof. Writing $x = \sin(6\theta),\ y = \cos(6\theta)$, the function $A(\theta)$ in (5.2.2) becomes

$$A(x,y) = \frac{2}{p_2}\left(p_1 - p_2 s_1 s_2 + p_1 s_2^2 + (2p_1 s_2 - p_2 s_1)x + (p_2 x - p_1 y + p_2 s_2)y\right).$$

Next we solve the set of equations

$$\begin{aligned} A(x,y) &= 0, \\ x^2 + y^2 &= 1. \end{aligned}$$

to get the solutions

$$\begin{aligned} x_1 &= -s_2,\ y_1 = \sqrt{1 - s_2^2} \\ x_2 &= -s_2,\ y_2 = -\sqrt{1 - s_2^2} \end{aligned}$$

$$x_\pm = \frac{p_1 p_2 s_1 - p_1^2 s_2 \pm p_2\sqrt{p_1^2 + p_2^2 - (p_2 s_1 - p_1 s_2)^2}}{p_1^2 + p_2^2}$$

$$y_\pm = \frac{p_2^2 s_1 - p_1 p_2 s_2 \mp p_1\sqrt{p_1^2 + p_2^2 - (p_2 s_1 - p_1 s_2)^2}}{p_1^2 + p_2^2}$$

Observe now the the two first pairs of solutions $(x_1, y_1), (x_2, y_2)$ cannot be solutions of our equation $A(\theta) = 0$ since $x = \sin(6\theta) = -s_2 < -1$.

On the other hand, if we look for the intervals where the function $A(\theta)$ does not change sign we have two possibilities: either $|x_\pm| > 1$ (and consequently $|y_\pm| > 1$) or the discriminant of $x_\pm$, $\Delta = p_1^2 + p_2^2 - (p_2 s_1 - p_1 s_2)^2$ is negative or zero. In the case $\Delta < 0$, the solutions will be complex non-real, and in the second case $\Delta = 0$, the function will have a zero but a double one, and it will not change its sign.

The first possibility turns out to be impossible in our region of parameters. The second possibility leads to the region $p_1 \in \mathcal{R} \setminus (\Sigma_A^+, \Sigma_A^-)$. ■

Lemma 5.3.6 *Consider $|s_2| > 1$ and define the following two numbers:*

$$\Sigma_B^\pm = \frac{\Sigma_A^\pm}{2}.$$

Let $B(\theta)$ be the function given in Lemma 5.2.1. Then the function $B(\theta)$ changes sign if and only if $p_1 \in \left(\Sigma_B^-, \Sigma_B^+\right)$.

Proof. By direct computations with the same substitution as the one in the proof of the previous lemma, $x = \sin(6\theta), y = \cos(6\theta)$, we get that the zeroes of the system

$$\begin{aligned} B(x, y) &= 0, \\ x^2 + y^2 &= 1, \end{aligned}$$

are

$$\begin{aligned} x_\pm &= \frac{2p_1p_2s_1 - 4p_1^2 s_2 \pm p_2\sqrt{4p_1^2 + p_2^2 - (p_2s_1 - 2p_1s_2)^2}}{4p_1^2 + p_2^2}, \\ y_\pm &= \frac{p_2^2 s_1 - 2p_1p_2s_2 \mp 2p_1\sqrt{4p_1^2 + p_2^2 - (p_2s_1 - 2p_1s_2)^2}}{4p_1^2 + p_2^2}. \end{aligned}$$

Applying arguments similar to those in the previous proof, we get that the function $B(\theta)$ will not change sign if and only if $p_1 \notin (\Sigma_B^+, \Sigma_B^-)$.

■

Remark 5.3.2
- *In general systems there are examples for which the function $A(\theta)$ changes sign but $B(\theta)$ does not and vice-versa.*
- *The condition given in Lemma 5.3.5 under which the function $A(\theta)$ does not change sign is closely related to the number of critical points of system* (5.2.3)*. More precisely, the condition $\Delta < 0$ in the proof of Lemma 5.3.5 is the same as the condition for existence of a unique critical point in system* (5.2.3)*, while the condition $\Delta = 0$, together with $p_2 < 0$, is equivalent to the existence of exactly 7 critical points.*

In the following we will need some results on Abel equations proved in [12] and [8]. We sumarize them in a theorem.

Theorem 5.3.1 *Consider the Abel equation* (5.2.1) *and assume that either $A(\theta) \not\equiv 0$ or $B(\theta) \not\equiv 0$ does not change sign. Then it has at most three solutions satisfying $x(0) = x(2\pi)$, taking into account their multiplicities.*

Remark 5.3.3 *Condition (a) ot Theorem 5.1.1 implies that one of the functions $A(\theta), B(\theta)$ of the Abel equation does not change sign. If condition (i) is satisfied then $A(\theta)$ does not change sign and, as the third derivative of the Poincaré return map of the Abel equation is this function $A(\theta)$ then, the Abel equation can have at most 3*

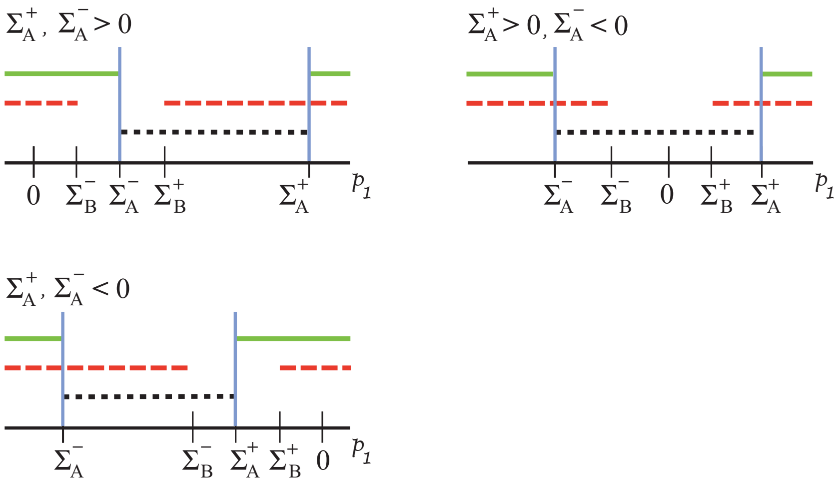

Figure 5.3.0.2. A sketch of intervals where the conditions of Theorem 5.1.1 are satisfied, with $s_2 > 1, p_2 < 0$. The result follows from the relationship between the function $A(\theta)$ and $B(\theta)$ changing sign and the number of critical points of (5.2.3). The green solid line indicates where the function $A(\theta)$ does not change sign, the red dashed one where $B(\theta)$ does not change sign, while the dotted line represents the interval where 13 critical points exist. The blue vertical lines (that correspond to $\Sigma_A^{\pm}$) stand for 7 critical points. There is another possibility, not shown here, that the two intervals are disjoint, see Figure 5.1.0.1.

limit cycles. In our case, one of them is the origin and another one is infinity. Consequently only one non-trivial limit cycle can exist, both for the scalar equation and for the planar system.

If condition (ii) is verified then it is the function $B(\theta)$ that does not change sign and, by a change of coordinates, one can get another scalar equation for which the third derivative of the Poincaré map is the function $B(\theta)$, getting the same conclusions as before.

5.4 Limit cycles

Proof of Theorem 5.1.1. We first define the function $c(\theta) = s_2 + \sin(6\theta)$ and the set $\Theta := \{(r,\theta) : \dot{\theta} = p_2 + (s_2 + \sin(6\theta))r = 0\}$. Since $|s_2| > 1$, we have $c(\theta) \neq 0$, $\forall \theta \in [0, 2\pi]$.

(a) Let's assume first that condition (i) is satisfied. By Lemma 5.2.1, we reduce the study of the periodic orbits of equation (5.1.1) to the analysis of the non contractible periodic orbits of the Abel equation (5.2.1). Since $p_1 \notin \left(\Sigma_A^-, \Sigma_A^+\right)$, by condition (i) of Lemma 5.3.5, we know that function $A(\theta)$ in the Abel equation does not change sign. Hence, from Theorem 5.3.1, the maximum number of solutions satisfying $x(0) = x(2\pi)$ in system (5.2.1) taking into account their multiplicities, is three. One of them is trivially $x = 0$. Since $c(\theta) \neq 0$, by simple calculations we can prove that the curve $x = 1/c(\theta)$ is a second solution satisfying this condition. As shown in [1], undoing the Cherkas transformation we get that $x = 1/c(\theta)$ is mapped into infinity of the differential equation. Then, by Lemma 5.2.1, the maximum number of limit cycles of equation (5.1.1) is one. Moreover, from the same lemma it follows that the limit cycle is hyperbolic. From the symmetry, it follows that a unique limit cycle must surround the origin.

(b) We follow the same analysis method as in [1]. When $p_1 > \Sigma_A^+$, by Lemma 5.3.1, both the origin and infinity in the Poincaré compactification are repelors. In particular the origin is an unstable focus. On the other hand, from Lemma 5.3.5, $A(\theta)$ does not vanish when $p_1 > \Sigma_A^+$, and the origin is the unique critical point. It is easy to see that, since $p_2 A(\theta) > 0$, then the exterior of the closed curve Θ is positively invariant and therefore, by applying the Poincaré-Bendixson Theorem and part (a) of this theorem, there is exactly one hyperbolic limit cycle surrounding the curve Θ. Moreover, this limit cycle is stable.

When $p_1 = \Sigma_A^+$, six more semi-elementary critical points appear and they are located on Θ. They are saddle-nodes as shown in Lemma 5.3.4. We will show that at this value of p_1 the periodic orbit still exists and

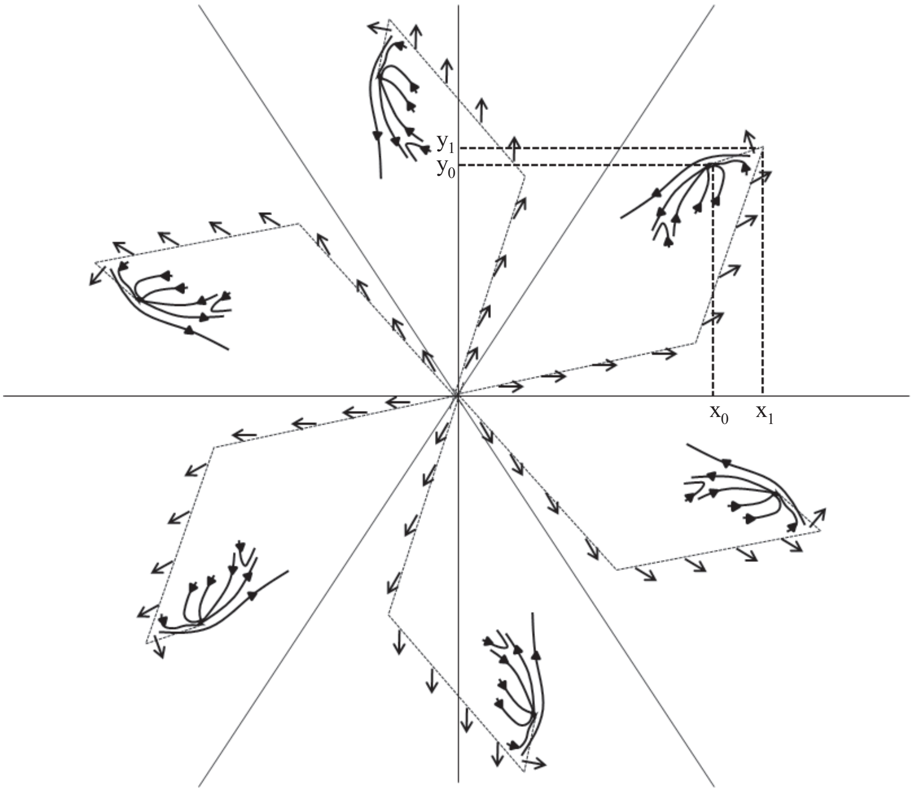

Figure 5.4.0.1. The polygonal curve with no contact with the flow of the differential equation and the separatrices of the saddle-nodes of system (5.3.4).

it surrounds the seven critical points. We will prove this by constructing a polygonal line with no contact with the flow of the differential equation. On the polygonal, the vector field points outside, and consequently, as the infinity is a repelor, the ω-limit set of the unstable separatrices of the saddle-nodes must be a limit cycle surrounding Θ, see Figure 5.4.0.1 and Example 5.4.1 below.

The $\mathbb{Z}_6-$equivariance of system (5.1.1) allows us to study the flow in only one sextant of the phase space in cartesian coordinates, the behaviour in the rest of the phase space being identical. The polygonal line will join the origin to one of the saddle-nodes as in Figure 5.4.0.1.

We explain the construction of the polygonal line when $0 < s_1 \leq 1$, $s_2 > 1$ and $p_2 < 0$. In this case, $p_1 = \Sigma_A^+$ implies that $p_1 > 0$ and, in the notation of Lemma 5.3.2, we have $u = 0$. Hence

$$\frac{1}{\Delta_\pm} = \frac{(s_2^2 - 1)(1 + s_1)}{s_1 s_2 - \sqrt{s_1^2 + s_2^2 - 1}} - s_2 < -1$$

and using the expression in Lemma 5.3.2, we find that the angular coordinate θ_0 of the saddle-node satisfies $-1 < \tan 3\theta_0 < 0$, and therefore $\pi/4 < \theta_0 < \pi/3$.

The first segment in the polygonal is the line $\theta = \pi/4$. Using the expression (5.2.3) we obtain that on this line $\dot{\theta} = p_2 + r(s_2 - 1)$ which is negative for r between 0 and $r_1 = -p2/(s_2 - 1) > 0$. Thus, the vector field is transverse to this segment and points away from the saddle-node on it.

Another segment in the polygonal is obtained using the eigenvector v corresponding to the non-zero eigenvalue of the saddle-node z_0. The line $z_0 + tv$ is the tangent to the separatrix of the hyperbolic region of the saddle-node. Let t_0 be the first positive value of t for which the vector field fails to be transverse to this line.

If the lines $z_0 + tv$ and $\theta = \pi/4$ cross at a point with $0 < t < t_0$ and with $0 < r < r_1$, then the polygonal consists of the two segments. If this is not the case, then the segment joining $z_0 + t_0 v$ to $r_1\left(\sqrt{2}/2, \sqrt{2}/2\right)$ will also be transverse to the vector field, and the three segments will form the desired polygonal.

By using the same arguments presented above and the fact that infinity is a repelor, it follows by the Poincaré-Bendixson Theorem that the only possible ω-limit for the unstable separatrix of the saddle-node is a periodic orbit which has to surround the six saddle-nodes, see again Figure 5.4.0.1.

If $p_1 = \Sigma_A^+$ then the function $A(\theta)$ does not change sign; therefore the arguments presented in part (a) of the proof of this theorem assure the hyperbolicity of the limit cycle.

When we move p_1 towards zero but still very close to Σ_A^+, then $B(\theta)$ is strictly positive because $\Sigma_A^+ > \Sigma_B^+$ and there are 13 critical points as shown in Lemma 5.3.4: the origin (which is a focus), six saddles and six critical points of index $+1$ on Θ. Applying one more time part (a) of the proof of this theorem, we know that the maximum number of limit cycles surrounding the origin is one. If $p_1 = \Sigma_A^+$ the limit cycle is hyperbolic and it still exists for the mentioned value of p_1. Then we have the vector field with $B(\theta)$ not changing sign, 12 non-zero critical points and a limit cycle which surrounds them together with the origin. ∎

Example 5.4.1 *As an example of the construction of the polygonal line in the proof of Theorem 5.1.1 we present a particular case, done with Maple. Let's fix parameters* $p_2 = -1,\ s_1 = -0.5,\ s_2 = 1.2$. *With these values of the parameters, we have* $\Sigma_A^- = -0.52423$, $\Sigma_A^+ = 3.25151$.

To construct the polygonal line we work in cartesian coordinates. A key point in the process of identifying the three segments of the polygonal line is knowing explicitly the eigenvector corresponding to the non-zero eigenvalue of the saddle-node $(x_0, y_0) = (1.358, 1.5)$. *This eigenvector is* $v = (-0.8594, -0.5114)$ *so the slope of the tangent to the hyperbolic direction of the saddle-node is* $0.5114/0.8594$ *and the straight line of this slope passing through the saddle-node is* $R \equiv \{y = 1.5 + \dfrac{0.5114}{0.8594}(x - 1.358)\}$. *The tangent to the separatrix of the hyperbolic region of the saddle-node is locally transverse to this line. The scalar product of the vector field associated to equation* (5.3.4) *with the normal vector to* R, $(0.5114, -0.8594)$, *is given by the equation*

$$-2.39191647949065x^5 + 2.34410741916533x^4 + 4.86235167862649x^3 -$$
$$2.71272659052423x^2 - 2.33924612305747x - 0.92289951077311$$

when evaluated on the straight line R; *its unique real root is* $x \simeq -1.1737$ *and the flow is transversal from inside out through* R *for any* $x > -1.1737$.

Let us now define the polygonal line as

$$(x(t), y(t)) = \begin{cases} (t,t) & \text{if } 0 \leqslant t < 2\sqrt{\frac{1}{2.8}}, \\ (t, 1.5(t - 2\sqrt{\frac{1}{3}}) + 2\sqrt{\frac{1}{2.8}}) & \text{if } 2\sqrt{\frac{1}{2.8}} \leqslant t < 1.425, \\ ((t-2)\dfrac{x_1 - x_0}{x_1 - 2} + x_0, (t-2)\dfrac{y_1 - y_0}{y_1 - 2} + y_0) & \text{if } 1.425 \leqslant t < 2. \end{cases}$$

Let's now consider the point $(x_1, y_1) = (1.4250, 1.5399)$, *which stands at the intersection between the second and third segments of the polygonal line. The scalar product between the normal to each segment and the flow of the differential equation is negative, when evaluated on the corresponding segments. To show the calculation, we will exemplify it for the first segment, the remaining cases being treated similarly.*

We elected the segment of the line $L \equiv \{y = x\}$ *and when substituting it into the system in cartesian coordinates* (5.3.2) *we obtain*

$$\begin{cases} \dot{x} = 4x^5(s_1 - s_2 + 1) + 2x^3(p_1 - p_2) \\ \dot{y} = 4x^5(4s_1 + 5s_2 - 4) + 2x^3(p_1 + p_2). \end{cases}$$

A normal vector to the line L *is* $(-1, 1)$ *and the scalar product of* $(\dot{x}, \dot{y})$ *with* $(-1, 1)$ *yields* $f(x) = x^3(4p_2 + x^2(9s_2 - 8))$. *Solving this last equation leads to* $f(x) < 0$ *for* $-2\sqrt{\frac{-p_2}{9s_2 - 8}} \leqslant x \leqslant 2\sqrt{\frac{-p_2}{9s_2 - 8}}$. *So we choose*

$0 \leqslant x \leqslant 2\sqrt{\frac{-p_2}{9s_2-8}}$, *and we get that this scalar product is negative in the region where the polygonal line is defined as* (t, t).

By using the same arguments presented above it follows that the only possible ω*-limit for the unstable separatrix of the saddle-node is a periodic orbit which has to surround the six saddle-nodes, see again Figure 5.4.0.1.*

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Chapter 6

Conclusions

In the paper ”Periodic solutions in an array of coupled FitzHugh-Nagumo cells” we use a similar idea to that of [2] to describe arrays of N^2 cells where each cell is represented by a subsystem that is a 2-dimensional differential equation of FitzHugh-Nagumo type. We are interested in the periodic solutions arising at a first Hopf bifurcation from the fully synchronised equilibrium. To each equation in the array we add a coupling term that describes how each cell is affected by its neighbours. The coupling may be associative, when it tends to reduce the difference between consecutive cells, or dissociative, when differences are increased. For associative coupling we find, not surprisingly, bifurcation into a stable periodic solution where all the cells are synchronised with identical behaviour.

When the coupling is dissociative in either one or both directions, the first Hopf bifurcation gives rise to rings of N fully synchronised cells. All the rings oscillate with the same period, with a $\frac{1}{N}$-period phase shift between rings. When there is one direction of associative coupling, the synchrony rings are organised along it. Dissociative coupling in both directions yields rings organised along the diagonal. The stability of these periodic solutions was studied numerically and were found to be unstable for small numbers of cells, stability starts to appear at $N \geqslant 11$. For all types of coupling, there are further Hopf bifurcations, but these necessarily yield unstable solutions.

One of the main goals for the future, is to prove the existence of multifrequencies in an array formed by two identical tori with bidirectional coupling. The idea is constructing a set of multifrequency patterns of interest and analyzing the action the symmetry group of the network has on the constructed set. We have evidence that when the symmetry group of a single torus acts on the traveling waves, it leaves them unchanged; by the contrary, when it acts on the in-phase oscillations, they are shifted in time by ϕ. This proved that one of the possible patterns of oscillations of the network formed by the two coupled tori is represented by traveling waves produced by one torus and in-phase oscillations at $N-$times the frequency of the traveling waves, showed by the other torus. It has to be emphasized, however, that the constructed set is only one of the possible sets of patterns of oscillations of the network.

We would like, based on the general theory [3], [4], to investigate the possible patterns of oscillations of an arbitrary number of tori with different inter-tori couplings. Among the possible inter-tori architectures, we will consider all-to-all and nearest-neighbor couplings. Other possible toric architectures involve solid tori or toric surfaces as described above, but with different types of interneuronal couplings, such as bidirectional, in which case the $\mathbb{Z}_N$ group is replaced by $\mathbb{D}_N$. In addition, we won't limit ourselves to the torus or torus-like architectures. Other types of topologies will be investigated as well. As an example, we have in mind non-orientable surfaces as suggested in [5].

The equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for all $\mathbf{C}$-axial subgroups of $\Gamma \times \mathbb{S}^1$, under some generic conditions. The $H \bmod K$ theorem offers the complete set of possible periodic solutions based exclusively on the structure of the group Γ acting on the differential equation. It also guarantees the existence of a model with this symmetry having these periodic solutions, but it is not an existence result for any specific equation.

In our second article, "Hopf bifurcation with tetrahedral and octahedral symmetry", we pose the question: which periodic solutions predicted by the $H \bmod K$ theorem are obtainable by Hopf bifurcation from the trivial steady-state when Γ is either the group $\langle \mathbb{T}, \kappa \rangle$ of symmetries of the tetrahedron or the group $\mathbb{O}$ of rotational symmetries of the cube. As abstract groups, $\langle \mathbb{T}, \kappa \rangle$ and $\mathbb{O}$ are isomorphic, but their standard representations in $\mathbb{R}^3$ and $\mathbb{C}^3$ are not and this changes the possible symmetries of bifurcating solutions. However, the representations of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ and $\mathbb{O} \times \mathbb{S}^1$ on $\mathbb{C}^3$ are isomorphic, and this is the relevant action for dealing with equivariant Hopf bifurcation. Thus it is interesting to compare results for these two groups, and we find that indeed our question has different answers for the two groups.

We identify which periodic solution types, whose existence is guaranteed by the $H \bmod K$ theorem, are obtainable by Hopf bifurcation from the origin, when the group Γ is either tetrahedral or octahedral. In particular, some solutions predicted by the $H \bmod K$ theorem for $\langle \mathbb{T}, \kappa \rangle$-equivariant vector fields can only arise at a resonant Hopf bifurcation, but this is not the case for $\mathbb{O}$.
The same question has been addressed for abelian groups in the article "The Abelian Hopf $H \bmod K$ Theorem" [1] by Filipski and Golubitsky, but in general it remains unanswered. We know the candidates for pairs $H,\ K$ of subgroups that do not arise as spatio-temporal symmetries after a primary Hopf bifurcation: they are, firstly,pairs with $H = K$ where the normalizer $N_\Gamma(K) \neq K$, and secondly, pairs with $K = \mathbb{1}$ and H is cyclic. However, our examples show that both situations may occur at a primary Hopf bifurcation, hence it would be interesting to obtain a complete characterisation.
In the paper "Limit cycles for a class of quintic $\mathbb{Z}_6$-equivariant systems without infinite critical points" we analyze a $\mathbb{Z}_6$-equivariant polynomial system in the plane.

We studied the global phase portrait of

$$\dot{z} = \frac{dz}{dt} = (p_1 + ip_2)z^2\bar{z} + (s_1 + is_2)z^3\bar{z}^2 - \bar{z}^5 = f(z),$$

paying special attention to the existence, location and uniqueness of limit cycles surrounding 1, 7 or 13 critical points. We proved that if with $p_2 \neq 0$, $|s_2| > 1$ and:

$$\Sigma_A^- = \frac{p_2 s_1 s_2 - \sqrt{p_2^2(s_1^2 + s_2^2 - 1)}}{s_2^2 - 1}, \qquad \Sigma_A^+ = \frac{p_2 s_1 s_2 + \sqrt{p_2^2(s_1^2 + s_2^2 - 1)}}{s_2^2 - 1},$$

then, the following statements are true:

(a) If one of the conditions

$$(i) \quad p_1 \notin \left(\Sigma_A^-, \Sigma_A^+\right), \qquad (ii) \quad p_1 \notin \left(\frac{\Sigma_A^-}{2}, \frac{\Sigma_A^+}{2}\right)$$

is satisfied, then the equation has at most one limit cycle surrounding the origin. Furthermore, when the limit cycle exists it is hyperbolic.

(b) There are parameter values under condition (ii) for which the equation has exactly one limit cycle surrounding either $1,\ 7$ or 13 critical points. There are also parameter values satisfying condition (i) where the equation has exactly one hyperbolic limit cycle surrounding either 7 critical points if $p_1 \neq \Sigma_A^\pm$, or the only critical point if $p_1 = \Sigma_A^\pm$.

Many of the results obtained can be extended to a $\mathbb{Z}_{2n}-$equivariant family of polynomial vector fields of degree $2n - 1$. The attempt at a complete extension of the results to this new situation is a possible direction for future work.

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Chapter 7

Appendix to Chapter 4

We present here some details of the calculations of the article ”Hopf bifurcation with tetrahedral and octahedral symmetry” that appears as Chapter 4 in this thesis.

7.1 Elements of the group

The group $\mathbb{T}$ of rotational symmetries of the tetrahedron [1] has order 12. Its action on $\mathbb{R}^6 \equiv \mathbb{C}^3$ is generated by

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{7.1.1}$$

The remaining 10 elements of the group are identity plus the following

$$C^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad RC = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \quad CR = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \tag{7.1.2}$$

$$RCR = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \quad RC^2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad C^2R = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \tag{7.1.3}$$

$$CRC = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad CRC^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C^2RC = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{7.1.4}$$

Next we want to ”augment” the group $\mathbb{T}$ with a reflection, given by

$$\kappa = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{7.1.5}$$

to form $\langle \mathbb{T}, \kappa \rangle$, the full group of symmetries of the thetrahedron.

The remaining 11 elements of the $\langle \mathbb{T}, \kappa \rangle$ are:

$$R\kappa = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \quad C\kappa = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{7.1.6}$$

$$C^2\kappa = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad RC\kappa = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad CR\kappa = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{7.1.7}$$

$$RCR\kappa = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad RC^2\kappa = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad C^2R\kappa = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{7.1.8}$$

$$CRC\kappa = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad CRC^2\kappa = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad C^2RC\kappa = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{7.1.9}$$

The 11 elements are obtained by direct calculation. Any other products yield the same elements.

The 23 elements of $\mathbb{O}$ (besides the identity) are:

$$T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$T^3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad C^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$TC = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad TC^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad T^2C = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$T^2C^2 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad T^3C = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad T^3C^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$TC^2TC^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad T^2C^2TC^2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad T^3C^2TC^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

$$T^2CTC^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad TCT^2C^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad CT^2C^2TC^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

$$C^2T^2C^2TC^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad TCT^2C^2TC^2 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad TC^2T^2C^2TC^2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

$$T^2CT^2C^2TC^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad T^2C^2T^2C^2TC^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad T^3CT^2C^2TC^2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

The following equations are satisfied:

(a)

$$TC^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = (CT^3)^3$$

(b)

$$TC^2TC^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = (CT^3)^2$$

(c)

$$T^2CT^2C^2TC^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = CT^3$$

(d)

$$TCT^2C^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = C^2TC^2.$$

(e)

$$T^2C^2T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \in N(\Sigma),\ where\ \Sigma = \mathbb{Z}_3(C)$$

(f)

$$T^2C^2TC = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = CT^2C^2TC^2.$$

Lemma 7.1.1 *The representation of $\langle \mathbb{T}, \kappa \rangle$ on $\mathbb{C}^3$ is $\langle \mathbb{T}, \kappa \rangle$-simple.*

Proof. To see that the representation is $\langle \mathbb{T}, \kappa \rangle-$simple, let $\mathbb{C}^3 = \{(z_1, z_2, z_3), z_j \in \mathbb{C}\}$. Moreover, let us consider

$$V = \{(x_1, x_2, x_3), x_j \in \mathbb{R}\} \qquad \text{and} \qquad W = \{(ix_1, ix_2, ix_3), x_j \in \mathbb{R}\},$$

where both V and W are $\langle \mathbb{T}, \kappa \rangle-$invariant. Therefore $\mathbb{C}^3 = V \oplus W$ is the direct sum of two isomorphic absolutely irreducible representations.

It remains to show that these representations are absolutely irreducible. To prove that these representations are absolutely irreducible we show that the only 3×3 matrices commuting with every element of the group $\langle \mathbb{T}, \kappa \rangle$, are scalar multiples of identity.

Let's start with R :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = \begin{bmatrix} a & b & c \\ -d & -e & -f \\ -g & -h & -j \end{bmatrix},$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} a & -b & -c \\ d & -e & -f \\ g & -h & -j \end{bmatrix}.$$

From these two equations we have $b = c = d = g = 0$.

Next we take C^2RC and we have

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & j \end{bmatrix} = \begin{bmatrix} -a & 0 & 0 \\ 0 & e & f \\ 0 & -h & -j \end{bmatrix},$$

Table 7.1. Solution types, isotropy subgroups, their generators and fixed-point subspaces for the action of $\mathbb{T}\times\mathbb{S}^1$ on $\mathbb{C}^3$ from [1], where $\omega = e^{2\pi i/3}$ and $\sigma = e^{\pi i}$, from Swift and Barany [1].

Name	Isotropy subgroup	Generators	Fixed-point subspace
(a) Origin	$\mathbb{T}\times\mathbb{S}^1$		$\{(0,0,0)\}$
(b) Pure mode	$\mathbb{D}_2$	$R, \sigma CRC^2$	$\{(z,0,0)\}$
(c) Standing wave	$\mathbb{Z}_3(0)$	C	$\{(z,z,z)\}$
(d) Rotating wave 1	$\mathbb{Z}_3(1)$	ωC	$\{(z,\omega z,\bar{\omega}z)\}$
(e) Rotating wave 2	$\mathbb{Z}_3(2)$	$\bar{\omega}C$	$\{(z,\bar{\omega}z,\omega z)\}$
(f) 2-Sphere solutions	$\mathbb{Z}_2$	σCRC^2	$\{(z_1,z_2,0)\}$
(g) General solutions	$\mathbb{1}$		$\{(z_1,z_2,z_3)\}$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & j \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -a & 0 & 0 \\ 0 & e & -f \\ 0 & h & -j \end{bmatrix}$$

and therefore $f = h = 0$.

Next we take $C^2R\kappa$ and compute

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & j \end{bmatrix} = \begin{bmatrix} 0 & -e & 0 \\ a & 0 & 0 \\ 0 & 0 & -j \end{bmatrix},$$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & j \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -a & 0 \\ e & 0 & 0 \\ 0 & 0 & -j \end{bmatrix}$$

which yields $a = e$.

Finally we take $CR\kappa$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & j \end{bmatrix} = \begin{bmatrix} 0 & 0 & -j \\ 0 & -a & 0 \\ a & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & j \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a \\ 0 & -a & 0 \\ j & 0 & 0 \end{bmatrix}$$

which yields $a = j$. Therefore the only matrix commuting with $\langle\mathbb{T},\kappa\rangle$ is the a multiple of identity and the representation of $\langle\mathbb{T},\kappa\rangle$ is absolutely irreducible. ■

The isotropy subgroups of $\mathbb{T}\times\mathbb{S}^1$ have been obtained by Swift and Barany [1] and are listed in Table 7.1.

Proposition 7.1.1 *For a generic $\langle\mathbb{T},\kappa\rangle$-equivariant vector field, the solution types, the spatio-temporal symmetries H, spatial symmetries K and the number of the branches of periodic solutions of* (4.4.1) *undergoing a Hopf bifurcation as λ crosses the imaginary axis, are those given in Table 7.2.*

Proof. Periodic solution branches arising through a Hopf bifurcation lie in the fixed-point subspaces for the $\mathbb{C}$-axial subgroups $\Sigma \subset \langle\mathbb{T},\kappa\rangle\times\mathbb{S}^1$, listed in the first three columns of Table 7.2. Identifying $\langle\mathbb{T},\kappa\rangle\times\{Id\}$ with $\langle\mathbb{T},\kappa\rangle$, these branches have spatial symmetries in $K \sim \Sigma\cap\langle\mathbb{T},\kappa\rangle$, and spatio-temporal symmetries in

$$H = \{\gamma \in \langle\mathbb{T},\kappa\rangle \quad e^{\theta i}\gamma \in \Sigma \quad \text{for some } \theta\}.$$

Table 7.2. Solution types, $\mathbb{C}$-axial subgroups, spatio-temporal symmetries H generators, spatio-temporal symmetries K generators and number of branches for the action of $\langle\mathbb{T},\kappa\rangle \times \mathbb{S}^1$ on $\mathbb{C}^3$.

index	solution's name	$\mathbb{C}$-axial subroup Σ generators	Spatio-temporal symmetries H generators	Spatial symmetries K generators	number of branches
(b)	pure mode	$\{e^{\pi i}C^2RC\kappa,\kappa\}$	$\{C^2RC\kappa,\kappa\}$	$\{R,\kappa\}$	3
(c)	standing waves	$\{C,\kappa\}$	$\{C,\kappa\}$	$\{C,\kappa\}$	4
(d)	discrete rotating waves	$\{e^{-2\pi i/3}C\}$	$\{C\}$	$\mathbb{1}$	8
(e)	standing waves	$\{e^{\pi i}CRC^2,\kappa C\}$	$\{R,\kappa\}$	$\{\kappa\}$	6
(f)	discrete rotating waves	$\{e^{\pi i/2}C^2R\kappa\}$	$\{C^2R\kappa\}$	$\mathbb{1}$	6

Table 7.3. Solution types, $\mathbb{C}$-axial subgroups, spatio-temporal symmetries H generators, spatio-temporal symmetries K generators and number of branches for the action of $\mathbb{O}\times\mathbb{S}^1$ on $\mathbb{C}^3$.

index	solution's name	$\mathbb{C}$-axial subroup Σ generators	Spatio-temporal symmetries H generators	Spatial symmetries K generators	number of branches
(b')	discrete rotating waves	$\{TC^2,e^{\pi i}T^2\}$	$\{TC^2,T^2\}$	$\{TC^2\}$	3
(c')	standing waves	$\{C,e^{\pi i}T^2C^2TC^2\}$	$\{C,T^2C^2TC^2\}$	$\{C\}$	4
(d')	discrete rotating waves	$\{e^{-2\pi i/3}C\}$	$\{C\}$	$\mathbb{1}$	8
(e')	pure modes	$\{TCT^2C^2,e^{\pi i}T^2\}$	$\{TCT^2C^2,T^2\}$	$\{TCT^2C^2\}$	6
(f')	standing waves	$\{e^{-\pi i/2}T\}$	$\{T\}$	$\mathbb{1}$	6

The last column of Table 7.2 gives the number different copies of each branch created by $\langle\mathbb{T},\kappa\rangle$, obtained as we proceed to explain.

Let N_H be the normalizer of H in $\langle\mathbb{T},\kappa\rangle$. Then $N_H = \{\gamma\in\langle\mathbb{T},\kappa\rangle : \gamma(Fix(H))\subset Fix(H)\}$. Copies of the branch inside $V = Fix(H)$ would arise through the action of N_H. Since Σ acts on $Fix(H)$ by a phase shift, copies by Σ of these branches correspond to the same periodic trajectory. Then the total number of different copies of the branch inside $Fix(H)$ is $|N_H|/|H|$.

The total number of branches is then the number of branches in $Fix(H)$ multiplied by the number of subspaces conjugate to $Fix(H)$, given by the number of different cosets $\gamma\cdot\Sigma = \{\gamma\cdot\delta : \delta\in\Sigma\}$, $\gamma\in\langle\mathbb{T},\kappa\rangle$. This last number is $|\langle\mathbb{T},\kappa\rangle|/|N_H|$. We explain case (d) in detail, the other cases are similar.

In case (d), we have $H=\{\mathbb{1},C,C^2\}$ and $Fix(\Sigma)=\{(z,e^{2\pi i/3}z,e^{-2\pi i/3}z)\}=V$, i.e. $H\cdot V\subset V$. Since C is not the identity in V, because of the phase shift, it follows that there are 3 copies of each branch in V. Since $|\langle\mathbb{T},\kappa\rangle|=24$ and $|H|=3$ in case (d) of Table 7.3, it follows that there are 8 different cosets of H. ■

7.2 Analysis in $Fix\ \mathbb{Z}_2(e^{\pi i}CRC^2)$: 2-sphere solutions

In the following we work in the fow-invariant subspace $\{(z_1,z_2,0)\}$. For the group $\langle\mathbb{T},\kappa\rangle$ this corresponds to $Fix(\mathbb{Z}_2)=Fix(\{e^{\pi i}CRC^2\})$ (row (g) in Table 4.1). The aim of this section is to prove the following result, valid for both groups $\langle\mathbb{T},\kappa\rangle$ and $\mathbb{O}$, but formulated here for $\langle\mathbb{T},\kappa\rangle$:

Proposition 7.2.1 *In the subspace $Fix\ \mathbb{Z}_2(e^{\pi i}CRC^2)$, there always exist solutions of* (4.4.1) *in:*

(b) $Fix\,(\mathbb{D}_4) = Fix\left(\{e^{\pi i}C^2RC\kappa,\kappa\}\right) = \{(z,0,0)\}$;

(b') $C^2\cdot Fix\,(\mathbb{D}_4) = Fix\left(\{e^{\pi i}RC\kappa, C\kappa\}\right) = \{(0,z,0)\}$;

(e) $Fix\left(\mathbb{Z}_2\times\{\tilde{\kappa}\}\right)=Fix\left(\{e^{\pi i}C\mathrm{RC}^2\}\times\{\tilde{\kappa}\}\right)=\{(z,z,0)\}$;

(e') $R\cdot Fix\left(\mathbb{Z}_2\times\{\tilde{\kappa}\}\right)=R\cdot Fix\left(\{e^{\pi i}C\mathrm{RC}^2\}\times\{\tilde{\kappa}\}\right)=\{(z,-z,0)\}$;

(f) $Fix\left(\left\{e^{\pi i}C\mathrm{RC}^2\right\}\times\left\{e^{3\pi i/2}C^2R\kappa\right\}\right)=\{(-iz,z,0)\}$;

(f') $Fix\left(\left\{e^{\pi i}C\mathrm{RC}^2\right\}\times\left\{e^{\pi i/2}C^2R\kappa\right\}\right)=\{(iz,z,0)\}$.

In addition for $\beta\neq 0$, if $\left|\dfrac{\alpha}{\beta}\right|\geq 1$ and $A=\dfrac{\alpha_i\beta_i+\alpha_r\beta_r}{\beta_i^2+\beta_r^2}=\mathcal{R}e\left(\dfrac{\alpha}{\beta}\right)\in[-1,1]$ then there are solutions of the form $(\xi z,z,0)$, with $\xi=re^{i\phi}$, where

$$\cos(2\phi)=-A \qquad \textit{and} \qquad r^2=\frac{\mathcal{I}m\left(\dfrac{\alpha}{\beta}\right)+\sin(2\phi)}{\mathcal{I}m\left(\dfrac{\alpha}{\beta}\right)-\sin(2\phi)}.$$

These solutions have no additional symmetry, except if either $A=0$ or $|A|=1$ when we have a pitchfork bifurcation on the parameter A, from the symmetric branches described above.

The solution branches (b) and (b') are conjugated by C, (e) and (e') are conjugated by R and (f) and (f') by C^2RC. These had been already obtained directly from the equivariant Hopf Theorem.

Proof. Let us write

$$\begin{aligned} z_1&=r_1e^{i\phi_1},\\ z_2&=r_2e^{i\phi_2}, \end{aligned} \tag{7.2.1}$$

while $r_3=0$. We look for equilibria of system (4.4.1) so we have

$$\begin{cases} 0=r_1e^{i\phi_1}\left(\lambda+\gamma\left(r_1^2+r_2^2\right)+\alpha r_2^2\right)+\beta r_1e^{-i\phi_1}r_2^2e^{2i\phi_2}\\ \\ 0=r_2e^{i\phi_2}\left(\lambda+\gamma\left(r_1^2+r_2^2\right)+\alpha r_1^2\right)+\beta r_2e^{-i\phi_2}r_1^2e^{2i\phi_1} \end{cases} \tag{7.2.2}$$

Equations (7.2.2) have the trivial solution $r_1=r_2=0$. They also have a symmetric solution $r_2=0$ and $r_1^2=-\lambda/\gamma$ for any ϕ_1, corresponding to case (b), as well as the conjugate solution $r_1=0$ and $r_2^2=-\lambda/\gamma$ for any ϕ_2, that corresponds to case (b'). In order to look for other solutions we divide the first equation by r_1 and the second by r_2. Multiplying the first equation by $e^{i\phi_1}$ and the second by $e^{i\phi_2}$ yields

$$\begin{cases} e^{2i\phi_1}\left(\lambda+\gamma\left(r_1^2+r_2^2\right)+\alpha r_2^2\right)=-\beta r_2^2e^{2i\phi_2}\\ \\ e^{2i\phi_2}\left(\lambda+\gamma\left(r_1^2+r_2^2\right)+\alpha r_1^2\right)=-\beta r_1^2e^{2i\phi_1} \end{cases} \tag{7.2.3}$$

We then write $\phi=\phi_2-\phi_1$, divide the first equation in (7.2.3) by $e^{2i\phi_1}$ and the second equation by $e^{2i\phi_2}$ to get

$$\begin{cases} \lambda+\gamma\left(r_1^2+r_2^2\right)+\alpha r_2^2=-\beta r_2^2e^{2i\phi}\\ \\ e^{2i\phi}\left(\lambda+\gamma\left(r_1^2+r_2^2\right)+\alpha r_1^2\right)=-\beta r_1^2 \end{cases} \tag{7.2.4}$$

We next divide the second equation in (7.2.4) by $e^{2i\phi}$ to get

$$\begin{cases} \lambda+\gamma\left(r_1^2+r_2^2\right)+\alpha r_2^2=-\beta r_2^2e^{2i\phi}\\ \\ \lambda+\gamma\left(r_1^2+r_2^2\right)+\alpha r_1^2=-\beta r_1^2e^{-2i\phi} \end{cases} \tag{7.2.5}$$

From (7.2.5) we obtain

$$-\alpha r_1^2-\beta r_1^2e^{-2i\phi}+\alpha r_2^2+\beta r_2^2e^{2i\phi}=0 \tag{7.2.6}$$

and therefore when considering the real parts α_r and β_r and the imaginary parts α_i and β_i of α and β respectively, and expanding the exponentials, we get

$$\begin{aligned} -\alpha_r r_1^2 - i\alpha_i r_1^2 - \beta_r r_1^2(\cos 2\phi - i\sin 2\phi) - i\beta_i r_1^2(\cos 2\phi - i\sin 2\phi) + \\ \alpha_r r_2^2 + i\alpha_i r_2^2 + \beta_r r_2^2(\cos 2\phi + i\sin 2\phi) + i\beta_i r_2^2(\cos 2\phi + i\sin 2\phi) = 0. \end{aligned} \tag{7.2.7}$$

We put the condition that real and imaginary parts in (7.2.6) equal zero and obtain

$$\begin{cases} -\alpha_r r_1^2 - \beta_r r_1^2 \cos 2\phi - \beta_i r_1^2 \sin 2\phi + \alpha_r r_2^2 + \beta_r r_2^2 \cos 2\phi - \beta_i r_2^2 \sin 2\phi = 0 \\ -\alpha_i r_1^2 + \beta_r r_1^2 \sin 2\phi - \beta_i r_1^2 \cos 2\phi + \alpha_i r_2^2 + \beta_r r_2^2 \sin 2\phi + \beta_i r_2^2 \cos 2\phi = 0. \end{cases} \tag{7.2.8}$$

From the first equation in (7.2.7) we have

$$r^2 = \frac{r_1^2}{r_2^2} = \frac{-\alpha_r - \beta_r \cos 2\phi + \beta_i \sin 2\phi}{-\alpha_r - \beta_r \cos 2\phi - \beta_i \sin 2\phi} = \frac{\mathcal{R}e\left(\alpha + \beta e^{2i\phi}\right)}{\mathcal{R}e\left(\alpha + \beta e^{-2i\phi}\right)} \tag{7.2.9}$$

and from the second

$$r^2 = \frac{r_1^2}{r_2^2} = \frac{-\alpha_i - \beta_r \sin 2\phi - \beta_i \cos 2\phi}{-\alpha_i + \beta_r \sin 2\phi - \beta_i \cos 2\phi} = \frac{\mathcal{I}m\left(\alpha + \beta e^{2i\phi}\right)}{\mathcal{I}m\left(\alpha + \beta e^{-2i\phi}\right)}. \tag{7.2.10}$$

We then equate (7.2.9) and (7.2.10) and then solve for ϕ, the solutions being

$$\phi = \left\{0, \pi, \frac{\pi}{2}, \frac{-\pi}{2}, \frac{1}{2}\arccos(-A), -\frac{1}{2}\arccos(-A)\right\}$$

where

$$A = \frac{\alpha_i\beta_i + \alpha_r\beta_r}{\beta_i^2 + \beta_r^2} = \mathcal{R}e\left(\frac{\alpha}{\beta}\right). \tag{7.2.11}$$

We now classify the types of solutions depending on ϕ.

(b) If $r_2 = 0$ and $r_1^2 = -\lambda/\gamma$ then for any ϕ_1 the solutions lie on the subspace $\{(z,0,0)\}$.
(b') If $r_1 = 0$ and $r_2^2 = -\lambda/\gamma$ then for any ϕ_2 the solutions lie on the subspace $\{(0,z,0)\}$.
(e) If $\phi = 0$, then $\phi_1 = \phi_2,\ r_1 = r_2$ and the solutions belong to the subspace $(\{z,z,0)\}$.
(e') If $\phi = \pi$, then $\phi_2 = \pi + \phi_1,\ r_1 = r_2$ and solutions lie on the subspace $\{(z,-z,0)\}$.
(f) If $\phi = -\frac{\pi}{2}$, then $\phi_2 = \phi_1 - \frac{\pi}{2},\ r_1 = r_2$ and solutions lie on the subspace $\{(-iz,z,0)\}$.
(f') If $\phi = \frac{\pi}{2}$, then $\phi_2 = \phi_1 + \frac{\pi}{2},\ r_1 = r_2$ and solutions lie on the subspace $\{(iz,z,0)\}$.
(g) If $|A| \le 1$ then there are solutions $\phi = \pm\frac{1}{2}\arccos(-A)$, discussed below.

Now we discuss the case (g). Substituting $\cos(\phi) = -A = -\mathcal{R}e\left(\frac{\alpha}{\beta}\right)$ into either (7.2.9) or (7.2.10) we obtain

$$r^2 = \frac{\mathcal{I}m\left(\frac{\alpha}{\beta}\right) + \sin(2\phi)}{\mathcal{I}m\left(\frac{\alpha}{\beta}\right) - \sin(2\phi)}. \tag{7.2.12}$$

The right hand side of (7.2.12) is positive if and only if $|\sin(2\phi)| \le \left|\mathcal{I}m\left(\frac{\alpha}{\beta}\right)\right|$, or equivalently,

$$1 - \left(\mathcal{R}e\left(\frac{\alpha}{\beta}\right)\right)^2 = \sin^2(2\phi) \le \left(\mathcal{I}m\left(\frac{\alpha}{\beta}\right)\right)^2 \quad \Leftrightarrow \quad 1 \le \left|\frac{\alpha}{\beta}\right|^2. \tag{7.2.13}$$

From (7.2.1) we have $z_1 = \xi z_2$ where $\xi = re^{i\phi}$. Therefore, solutions belong to the subspace $\{(z_1, z_2, 0)\} = \{(\xi z_2, z_2, 0)\}$. Hence there are two solutions with $\cos(2\phi) = -A$ that come together into one of the solutions $(e)-(f')$ when $A = \pm 1$. The pitchfork bifurcations corresponding to cases (e), (e') correspond to $A = -1$, are supercritical, and have the symmetry $\tilde{\kappa}$. The pitchfork bifurcations corresponding to cases (f), (f') correspond to $A = 1$ are subcritical, and have the symmetry $e^{-\pi i/2}C^2RC\kappa$.

Next we want to know if the Hopf bifurcation occurs as a primary or a secondary bifurcation. Suppose there is a primary Hopf bifurcation in $\lambda = 0 + i\lambda_i$, $\lambda_i \neq 0$. We distinguish two cases:

(a) $A = \pm 1$, $r_1^2 = r_2^2$ and $\phi = \{0, \pi\}$;

(b) $A = \pm 1$, $r_1^2 = r_2^2$ and $\phi = \{-\pi/2, \pi/2\}$.

Case (a). From equation (7.2.5) we have

$$r_2^2 = \frac{-\lambda}{2\gamma + \alpha + \beta} = \frac{-\lambda(2\bar{\gamma} + \bar{\alpha} + \bar{\beta})}{|2\gamma + \alpha + \beta|^2} \geqslant 0, \tag{7.2.14}$$

with $\lambda(2\bar{\gamma} + \bar{\alpha} + \bar{\beta}) \in \mathbb{R}^-$.

We have

$$\lambda(2\gamma + \alpha + \beta) = i\lambda_i(2\gamma_r + \alpha_r + \beta_r) - \lambda_i(2\gamma_i + \alpha_i + \beta_i). \tag{7.2.15}$$

In addition, since $\lambda_i \neq 0$ we have

$$2\gamma_r + \alpha_r + \beta_r = 0$$

and, if $\lambda_i > 0$,

$$2\gamma_i + \alpha_i + \beta_i \leqslant 0.$$

Then, from equation (7.2.14) we obtain

$$r_2^2 = \frac{-\lambda(2\bar{\gamma} + \bar{\alpha} + \bar{\beta})}{|2\gamma + \alpha + \beta|^2} = \frac{-\lambda_i(2\gamma_i + \alpha_i + \beta_i)}{|2\gamma + \alpha + \beta|^2}. \tag{7.2.16}$$

If $r_2^2 = 0$ then $2\gamma_i + \alpha_i + \beta_i = 0$ but in this case $|2\gamma + \alpha + \beta|^2 = 0$ which means $\lambda = 0$. In consequence the branch arises at a secondary bifurcation.

Case (b) is analogous, replacing β by $-\beta$.

In the following result we resume the conditions under which the subspace $\{(z_1 = z_2 \dfrac{r_1}{r_2} e^{i\phi}, z_2, 0)\}$ is invariant under the flow of (4.4.1). ■

Proposition 7.2.2 *The subspace* $\{(\xi z_2, z_2, 0)\}$ *is invariant by the flow of* (4.4.1) *if and only if either*

(a) $\beta = 0$ *or*

(b) $\xi = e^{\pi i/2}$.

Proof. Let

$$\mathcal{X} = P(z_1, z_2, z_3)\frac{\partial}{\partial z_1} + Q(z_1, z_2, z_3)\frac{\partial}{\partial z_2} + R(z_1, z_2, z_3)\frac{\partial}{\partial z_3}$$

be the vector field of (4.4.1), and the algebraic surface

$$h(z_1, z_2, z_3) = z_1 - \xi z_2 = 0$$

representing the subspace $\{(\xi z_2, z_2, 0)\}$. Then h is invariant under $\mathcal{X}$ if

$$\mathcal{X}h = P(z_1, z_2, z_3)\frac{\partial h}{\partial z_1} + Q(z_1, z_2, z_3)\frac{\partial h}{\partial z_2} + R(z_1, z_2, z_3)\frac{\partial h}{\partial z_3} = K(z_1, z_2, z_3)h$$

for some polynomial K. From equation (4.4.1) it follows that

$$K = \lambda + \gamma\left(|z_1|^2 + |z_2|^2 + |z_3|^2\right)$$

and this value of K is valid if and only if

$$\begin{cases} \alpha = \beta = 0 \text{ or} \\ \xi = e^{\pi i/2}. \end{cases} \tag{7.2.17}$$

Indeed, from equation (4.4.1) we have

$$\mathcal{X}h = Kh + R, \tag{7.2.18}$$

where

$$R = \alpha(z_1|z_2|^2 - \xi z_2|z_1|^2) + \beta(\bar{z}_1 z_2^2 - \xi \bar{z}_2 z_1^2). \tag{7.2.19}$$

Since $z_1 = \xi z_2$, from equation (7.2.19) $R = 0$ reduces to $\beta z_2|z_2|^2(\bar{\xi} - \xi^3) = 0$; this means $\xi = e^{\pi i/2}$. This corresponds to the case (f) in Proposition 7.2.1. In addition $\hat{\xi} = e^{3\pi i/2}$ corresponds to the case (f') in Proposition 7.2.1. From Table 4.1, the group that fixes $\{(\xi z_2, z_2, 0)\}$ is $\left\{e^{-\pi i/2}C^2\mathrm{R}\kappa\right\}$. ■ It remains to show that the solutions with $A \neq \pm 1$ have no more symmetry. This is done in Lemma 7.2.1 below.

Lemma 7.2.1 *The solutions with $\phi \neq \frac{k\pi}{2}$ for integer k have no more symmetry besides $\mathbb{Z}_2 = (e^{\pi i}CRC^2)$.*

Proof. From (7.1.1)-(7.1.9) we selected the following candidates for fixing $\xi z_2, z_2$: $\mathbb{1}$, R, CRC^2, C^2RC, $C^2\kappa$, $CRC\kappa$, $C^2R\kappa$, $RC^2\kappa$. So we compute:

$$\mathbb{1} \cdot (\xi z_2, z_2, 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \xi z_2 \\ z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \xi z_2 \\ z_2 \\ 0 \end{bmatrix} \tag{7.2.20}$$

$$R \cdot (\xi z_2, z_2, 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \xi z_2 \\ z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \xi z_2 \\ -z_2 \\ 0 \end{bmatrix} \tag{7.2.21}$$

$$CRC^2 \cdot (\xi z_2, z_2, 0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \xi z_2 \\ z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\xi z_2 \\ -z_2 \\ 0 \end{bmatrix} \tag{7.2.22}$$

$$C^2RC \cdot (\xi z_2, z_2, 0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \xi z_2 \\ z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\xi z_2 \\ z_2 \\ 0 \end{bmatrix} \tag{7.2.23}$$

$$C^2\kappa \cdot (\xi z_2, z_2, 0) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \xi z_2 \\ z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} z_2 \\ \xi z_2 \\ 0 \end{bmatrix} \tag{7.2.24}$$

$$CRC\kappa \cdot (\xi z_2, z_2, 0) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \xi z_2 \\ z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -z_2 \\ -\xi z_2 \\ 0 \end{bmatrix} \tag{7.2.25}$$

$$C^2R\kappa \cdot (\xi z_2, z_2, 0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \xi z_2 \\ z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -z_2 \\ \xi z_2 \\ 0 \end{bmatrix} \tag{7.2.26}$$

$$RC^2\kappa \cdot (\xi z_2, z_2, 0) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \xi z_2 \\ z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\xi z_2 \\ 0 \end{bmatrix}. \tag{7.2.27}$$

It is clear from these calculations that besides $\mathbb{1}$, the only subgroup that fixes $\{(\xi z_2, z_2, 0)\}$ is $e^{i\pi}CRC^2$. ■

7.3 Analysis in $\mathrm{Fix}\ \mathbb{Z}_2(\tilde{\kappa})$: $2-$sphere solutions

In the following we work in Fix $\mathbb{Z}_2(\tilde{\kappa}) = \{(z_1, z_1, z_3)\}$ (row (h) in Table 4.1). This is a subgroup of the following isotropy subgroups: $\{e^{\pi i} \circ \mathrm{CRC}^2\} \times \{\tilde{\kappa}\}$ (row (e) in Table 4.1), of $\{CRC^2, \sigma R, \tilde{\kappa}\}$ (conjugated to row (b) in Table 4.1), and of $\mathbb{Z}_3(0) \times \{\tilde{\kappa}\}$ (conjugated to row (c) in Table 4.1). Hence, by Lemma 7.1.1, we already know that there are solutions in the subspaces $\{(z, z, 0)\}$, $\{(0, 0, z)\}$ and $\{(z, z, z)\}$.

Consider the following quantities as coefficients of a quadratic form:

$$\begin{aligned}
A &= -\alpha_i\beta_r + \alpha_r\beta_i = \mathcal{I}m\left(\alpha\bar{\beta}\right) \\
C &= -\beta_r^2 + 3\alpha_i\beta_i + 3\alpha_r\beta_r - \beta_i^2 = 3\mathcal{R}e\left(\alpha\bar{\beta}\right) - |\beta|^2 \\
F &= 4(\beta_r^2 + \beta_i^2) = 4\,|\beta|^2 \\
A_1 &= (\alpha_i\beta_r - \alpha_r\beta_i)^2 = A^2 \\
B_1 &= \alpha_i\beta_i - \alpha_r\beta_r = \mathcal{R}e\left(\alpha\bar{\beta}\right).
\end{aligned} \tag{7.3.1}$$

Proposition 7.3.1 *In the subspace* Fix $\mathbb{Z}_2(\tilde{\kappa})$*, there exist solutions to* (4.4.1) *in:*

- *(b)* $\mathrm{Fix}\left(CRC^2, \sigma R, \tilde{\kappa}\right) = \{(0, 0, z)\}$;
- *(c)* $\mathrm{Fix}\left(\mathbb{Z}_3(0) \times \{\kappa\}\right) = \mathrm{Fix}\left(\{C\} \times \{\kappa\}\right) = \{(z, z, z)\}$;
- *(e)* $\mathrm{Fix}\left(\left\{e^{\pi i} \circ \mathrm{CRC}^2\right\} \times \{\tilde{\kappa}\}\right) = \{(z, z, 0)\}$.

Moreover, there is always a branch with no additional symmetry. If $A^2 + C^2 > F^2$ then this branch is unique. Otherwise, there may be up to three additional branches.

Proof. The items $(b) - (e)$ are deduced directly as particular subspaces of $\{(z_1, z_1, z_2)\}$ in Table 4.1. Since $z_1 = z_2$ we can write the normal form as:

$$\left\{\begin{aligned}
\dot{z}_1 &= z_1\left[\lambda + (2\gamma + \alpha + \beta)\,|z_1|^2 + (\gamma + \alpha)\,|z_3|^2\right] + \beta\bar{z}_1 z_3^2 \\
\dot{z}_3 &= z_3\left[\lambda + (2\gamma + 2\alpha)\,|z_1|^2 + \gamma|z_3|^2\right] + 2\beta z_1^2\bar{z}_3
\end{aligned}\right. \tag{7.3.2}$$

and $z_1 = r_1e^{i\phi_1}$ and $z_3 = r_3e^{i\phi_3}$. We look for equilibria of system (7.3.2) so we have

$$\left\{\begin{aligned}
0 &= r_1e^{i\phi_1}\left[\lambda + (2\gamma + \alpha + \beta)\,r_1^2 + (\gamma + \alpha)\,r_3^2\right] + \beta r_1r_3^2e^{i(2\phi_3 - \phi_1)} \\
0 &= r_3e^{i\phi_3}\left[\lambda + (2\gamma + 2\alpha)\,r_1^2 + \gamma r_3^2\right] + 2\beta r_1^2r_3e^{i(2\phi_1 - \phi_3)}
\end{aligned}\right. \tag{7.3.3}$$

We divide the first equation in (7.3.3) by r_1 and the second by r_3 to get

$$\left\{\begin{aligned}
0 &= e^{i\phi_1}\left[\lambda + (2\gamma + \alpha + \beta)\,r_1^2 + (\gamma + \alpha)\,r_3^2\right] + \beta r_3^2e^{i(2\phi_3 - \phi_1)} \\
0 &= e^{i\phi_3}\left[\lambda + (2\gamma + 2\alpha)\,r_1^2 + \gamma r_3^2\right] + 2\beta r_1^2e^{i(2\phi_1 - \phi_3)}.
\end{aligned}\right. \tag{7.3.4}$$

This eliminates the solutions $r_1 = 0$, which corresponds to case (b) and $r_3 = 0$, which corresponds to case (e), that we already knew about.

Multiplying the first equation by $e^{-i\phi_1}$, the second by $e^{-i\phi_3}$ and calling $\phi = 2(\phi_3 - \phi_1)$ in (7.3.4) we get

$$\begin{cases} 0 = \lambda + (2\gamma + \alpha + \beta)\, r_1^2 + (\gamma + \alpha)\, r_3^2 + \beta r_3^2 e^{i\phi} \\ 0 = \lambda + (2\gamma + 2\alpha)\, r_1^2 + \gamma r_3^2 + 2\beta r_1^2 e^{-i\phi} \end{cases} \tag{7.3.5}$$

From (7.3.5) we obtain

$$(-\alpha + \beta) r_1^2 + \alpha r_3^2 + \beta r_3^2 e^{i\phi} - 2\beta r_1^2 e^{-i\phi} = 0. \tag{7.3.6}$$

Note that equation (7.3.6) does not depend on λ nor on γ. Once solutions r_1, r_2, ϕ to (7.3.6) are found for given α and β, they can be substituted in (7.3.5) to yield a relation between λ and γ for which the solution exists.

When considering the real parts α_r and β_r and the imaginary parts α_i and β_i of α and β respectively, and expanding the exponentials in (7.3.6) , we get

$$\begin{aligned} (-i\alpha_i - \alpha_r + i\beta_i + \beta_r) r_1^2 + (i\alpha_i + \alpha_r) r_3^2 + (i\beta_i + \beta_r) r_3^2 (\cos\phi + i \sin\phi) \\ -2(i\beta_i + \beta_r) r_1^2 (\cos\phi - i\sin\phi) = 0. \end{aligned} \tag{7.3.7}$$

We put the condition that real and imaginary parts in (7.3.7) equal zero and obtain

$$\begin{cases} 0 = (-\alpha_r + \beta_r) r_1^2 + \alpha_r r_3^2 + r_3^2(\beta_r \cos\phi - \beta_i \sin\phi) - 2r_1^2(\beta_r \cos\phi + \beta_i \sin\phi) \\ 0 = (-\alpha_i + \beta_i) r_1^2 + \alpha_i r_3^2 + r_3^2(\beta_i \cos\phi + \beta_r \sin\phi) - 2r_1^2(\beta_i \cos\phi - \beta_r \sin\phi), \end{cases} \tag{7.3.8}$$

that is

$$\begin{cases} r_1^2(\alpha_r - \beta_r + 2\beta_r \cos\phi + 2\beta_i \sin\phi) = r_3^2(\alpha_r + \beta_r \cos\phi - \beta_i \sin\phi) \\ r_1^2(2\beta_i \cos\phi - 2\beta_r \sin\phi + \alpha_i - \beta_i) = r_3^2(\alpha_i + \beta_i \cos\phi + \beta_r \sin\phi). \end{cases} \tag{7.3.9}$$

Dividing the two equations in (7.3.9) yields

$$\frac{\alpha_r - \beta_r + 2\beta_r \cos\phi + 2\beta_i \sin\phi}{2\beta_i \cos\phi - 2\beta_r \sin\phi + \alpha_i - \beta_i} = \frac{\alpha_r + \beta_r \cos\phi - \beta_i \sin\phi}{\alpha_i + \beta_i \cos\phi + \beta_r \sin\phi} \tag{7.3.10}$$

and also

$$\frac{r_3^2}{r_1^2} = \frac{\alpha_r - \beta_r + 2\beta_r \cos\phi + 2\beta_i \sin\phi}{\alpha_i + \beta_i \cos\phi + \beta_r \sin\phi} = \frac{2\beta_i \cos\phi - 2\beta_r \sin\phi + \alpha_i - \beta_i}{\alpha_r + \beta_r \cos\phi - \beta_i \sin\phi} > 0. \tag{7.3.11}$$

If we consider $(x, y) = (\cos\phi, \sin\phi)$, then solving equation (7.3.10) is equivalent to solving a quadratic equation on (x, y) and intersecting the solution set with the unitary circle. The proof is completed in Proposition 7.3.2 below.

■

Then we have

Proposition 7.3.2 *Equilibria to system* (4.4.1) *are found at the intersection of the unit circle with the hyperbola*

$$A - Ax + Cy + Fxy = 0, \tag{7.3.12}$$

and we have

(1) The hyperbola (7.3.12) *always intersects the unit circle at* $(x, y) = (1, 0)$*; this corresponds to the in-phase solutions* $z_1 = z_3$ *and it has the symmetry described in case* (c)*.*

(2) *The second intersection point corresponds to a phase difference* $\phi = \arccos(x_2)$*, where* x_2 *is given in* (7.4.2).

(3) *The second intersection point* (x_2, y_2), (7.4.2) *corresponds to a branch with no additional symmetry.*

(4) *If* $A^2 + C^2 > F^2$ *and* $F \neq -C$ *there are no more intersection points between the unit circle and the hyperbola.*

Proof. Since the quadratic part of (7.3.12) is Fxy, then, if $\beta \neq 0$, equation (7.3.12) defines either a hyperbola or two lines (a degenerate hyperbola). We want to find the possible intersections of the hyperbola and the circle $1 = x^2 + y^2$.

Let

$$g(x, y) = F(x - x_0)(y - y_0) - k = Fx_0y_0 - k - Fx_0y - Fy_0x + Fxy.$$

Then we have $x_0 = -\frac{C}{F}$, $y_0 = \frac{A}{F}$ and $k = -A(\frac{C}{F} + 1)$. If $C = -F$ then equation (7.3.12) defines two lines given by $x = x_0 = 1$ and by $y = y_0 = \frac{A}{F}$, respectively.

By inspection we find that $(x, y) = (1, 0)$ is always an intersection point. Substituting into (7.3.6), it follows that $r_1 = r_3$. This solution with no phase difference, corresponds to the fixed point subspace $\{(z, z, z)\}$, that we already knew about. Generically, the fact that $(1, 0)$ is an intersection point implies that the hyperbola and the circle meet on at least one more point. A second intersection point, found with Mathematica, is given in Appendix 7.4.

To check when there is no other intersection point besides $(1, 0)$, let $g(x, y) = x^2 + y^2 - 1$ and $h(x, y) = Fxy + Cy - Ax - A$. Since at $(1, 0)$ we have $Dh(1, 0) = (-A, F + C)$ and $Dg(1, 0) = (2, 0)$, then the curve $h(x, y) = 0$ is tangent to the unit circle at $(1, 0)$ if and only if $F = -C$, and in this case $h(x, y) = 0$ on the line $x = 1$. It follows that if $F \neq -C$ there are at least two intersection points for the two curves. A second intersection point can be found in equation (7.4.2) in Appendix A.

Note that if the point (x_0, y_0) lies outside the unit circle and if $F \neq -C$, then the hyperbola meets the unit circle at exactly two points. The condition for this is $A^2 + C^2 > F^2$. This means that there is only one branch of periodic solutions not corresponding to $\mathbb{C}-$axial subgroups. If (x_0, y_0) is inside the circle then there may be up to 4 intersection points. The other intersection points, besides $(1, 0)$, correspond to branches with no additional symmetry. Indeed, any additional symmetry would have to correspond to one of the subgroups conjugated to either (b), (c) or (e) in Table 4.1. We have obtained case (c) through the solution $(1, 0)$ above. The other two cases appear when $r_1 = 0$ and $r_3 = 0$ and were eliminated from equation (7.3.4). It remains to see if these symmetries occur in the additional branches.

In the case (b) $r_1 = 0$, $r_3 \neq 0$ the isotropy subgroup is $\mathbb{D}_2 \times \{\kappa\}$ and solutions belong to the subspace $(z, 0, 0)$. Then from equation (7.3.9) we have

$$\begin{cases} r_3^2(\alpha_r + \beta_r \cos\phi - \beta_i \sin\phi = 0) \\ r_3^2(\alpha_i + \beta_i \cos\phi - \beta_r \sin\phi = 0) \end{cases} \tag{7.3.13}$$

If we write $x = \cos\phi$, $y = \sin\phi$ then we obtain

$$\begin{cases} \alpha_r + \beta_r x - \beta_i y = 0 \\ \alpha_i + \beta_i x - \beta_r y = 0 \end{cases} \tag{7.3.14}$$

and the solution has the form of perpendicular straight lines

$$\begin{cases} y = \dfrac{\beta_r}{\beta_i} x + \dfrac{\alpha_r}{\beta_i} \\ y = -\dfrac{\beta_i}{\beta_r} x - \dfrac{\alpha_i}{\beta_r} \end{cases} \tag{7.3.15}$$

In the case (e) $r_3 = 0,\ r_1 \neq 0$ the isotropy subgroup is $\left\{e^{\pi i} \circ C\mathrm{RC}^2\right\} \times \left\{\tilde{\kappa}\right\}$ and solutions belong to the subspace $\{(z, z, 0)\}$. Then from equation (7.3.9) we have

$$\begin{cases} r_1^2(\alpha_r - \beta_r + 2\beta_r \cos\phi + 2\beta_i \sin\phi = 0) \\ \\ r_1^2(\alpha_i - \beta_i + 2\beta_i \cos\phi - 2\beta_r \sin\phi = 0) \end{cases} \tag{7.3.16}$$

If we write $x = \cos\phi$, $y = \sin\phi$ then we obtain

$$\begin{cases} \alpha_r - \beta_r + 2\beta_r x + 2\beta_i y = 0 \\ \\ \alpha_i - \beta_i + 2\beta_i x - 2\beta_r y = 0 \end{cases} \tag{7.3.17}$$

and the solution has the form of perpendicular straight lines

$$\begin{cases} y = -\dfrac{\beta_r}{\beta_i}x + \dfrac{\beta_r}{2\beta_i} - \dfrac{\alpha_r}{2\beta_i} \\ \\ y = \dfrac{\beta_i}{\beta_r}x + \dfrac{\beta_i}{2\beta_r} + \dfrac{\alpha_i}{2\beta_r} \end{cases} \tag{7.3.18}$$

Generically, the intersection point will not lie in the unit circle and hence there exists no more symmetry. ■

7.4 Second intersection point of the hyperbolas of Section 7.3

Next we want to find the possible intersections of the hyperbola and the circle defined in (7.4.1).

$$\begin{aligned} 0 &= A + Bx + Cy + Dx^2 + Ey^2 + Fxy = p_1(x, y) \\ 1 &= x^2 + y^2 \end{aligned} \tag{7.4.1}$$

By inspection we find that $(x_1, y_1) = (1, 0)$ is always an intersection point. To find the second intersection point, we use Mathematica. We define $u = (w + v)^{1/3}$

where

$w = -10368\alpha_i^2\alpha_r\beta_i^9 - 128\alpha_r^3\beta_i^9 - 34560\alpha_i\alpha_r\beta_i^{10} - 24192\alpha_r\beta_i^{11} + 10368\alpha_i^3\beta_i^8\beta_r - 20352\alpha_i\alpha_r^2\beta_i^8\beta_r +$
$34560\alpha_i^2\beta_i^9\beta_r - 34560\alpha_r^2\beta_i^9\beta_r + 24192\alpha_i\beta_i^{10}\beta_r - 10752\alpha_i^2\alpha_r\beta_i^7\beta_r^2 - 10752\alpha_r^3\beta_i^7\beta_r^2 - 103680\alpha_i\alpha_r\beta_i^8\beta_r^2 -$
$120960\alpha_r\beta_i^9\beta_r^2 + 31232\alpha_i^3\beta_i^6\beta_r^3 - 50688\alpha_i\alpha_r^2\beta_i^6\beta_r^3 + 138240\alpha_i^2\beta_i^7\beta_r^3 - 138240\alpha_r^2\beta_i^7\beta_r^3 + 120960\alpha_i\beta_i^8\beta_r^3 +$
$29952\alpha_i^2\alpha_r\beta_i^5\beta_r^4 - 31488\alpha_r^3\beta_i^5\beta_r^4 - 69120\alpha_i\alpha_r\beta_i^6\beta_r^4 - 241920\alpha_r\beta_i^7\beta_r^4 + 31488\alpha_i^3\beta_i^4\beta_r^5 - 29952\alpha_i\alpha_r^2\beta_i^4\beta_r^5 +$
$207360\alpha_i^2\beta_i^5\beta_r^5 - 207360\alpha_r^2\beta_i^5\beta_r^5 + 241920\alpha_i\beta_i^6\beta_r^5 + 50688\alpha_i^2\alpha_r\beta_i^3\beta_r^6 - 31232\alpha_r^3\beta_i^3\beta_r^6 + 69120\alpha_i\alpha_r\beta_i^4\beta_r^6 -$
$241920\alpha_r\beta_i^5\beta_r^6 + 10752\alpha_i^3\beta_i^2\beta_r^7 + 10752\alpha_i\alpha_r^2\beta_i^2\beta_r^7 + 138240\alpha_i^2\beta_i^3\beta_r^7 - 138240\alpha_r^2\beta_i^3\beta_r^7 + 241920\alpha_i\beta_i^4\beta_r^7 +$
$20352\alpha_i^2\alpha_r\beta_i\beta_r^8 - 10368\alpha_r^3\beta_i\beta_r^8 + 103680\alpha_i\alpha_r\beta_i^2\beta_r^8 - 120960\alpha_r\beta_i^3\beta_r^8 + 128\alpha_i^3\beta_r^9 + 10368\alpha_i\alpha_r^2\beta_r^9 + 34560\alpha_i^2\beta_i\beta_r^9 -$
$34560\alpha_r^2\beta_i\beta_r^9 + 120960\alpha_i\beta_i^2\beta_r^9 + 34560\alpha_i\alpha_r\beta_r^{10} - 24192\alpha_r\beta_i\beta_r^{10} + 24192\alpha_i\beta_r^{11}$,

$v = (((-10368\alpha_i^2\alpha_r\beta_i^9 - 128\alpha_r^3\beta_i^9 - 34560\alpha_i\alpha_r\beta_i^{10} - 24192\alpha_r\beta_i^{11} + 10368\alpha_i^3\beta_i^8\beta_r - 20352\alpha_i\alpha_r^2\beta_i^8\beta_r +$
$34560\alpha_i^2\beta_i^9\beta_r - 34560\alpha_r^2\beta_i^9\beta_r + 24192\alpha_i\beta_i^{10}\beta_r - 10752\alpha_i^2\alpha_r\beta_i^7\beta_r^2 - 10752\alpha_r^3\beta_i^7\beta_r^2 - 103680\alpha_i\alpha_r\beta_i^8\beta_r^2 -$
$120960\alpha_r\beta_i^9\beta_r^2 + 31232\alpha_i^3\beta_i^6\beta_r^3 - 50688\alpha_i\alpha_r^2\beta_i^6\beta_r^3 + 138240\alpha_i^2\beta_i^7\beta_r^3 - 138240\alpha_r^2\beta_i^7\beta_r^3 + 120960\alpha_i\beta_i^8\beta_r^3 +$
$29952\alpha_i^2\alpha_r\beta_i^5\beta_r^4 - 31488\alpha_r^3\beta_i^5\beta_r^4 - 69120\alpha_i\alpha_r\beta_i^6\beta_r^4 - 241920\alpha_r\beta_i^7\beta_r^4 + 31488\alpha_i^3\beta_i^4\beta_r^5 - 29952\alpha_i\alpha_r^2\beta_i^4\beta_r^5 +$
$207360\alpha_i^2\beta_i^5\beta_r^5 - 207360\alpha_r^2\beta_i^5\beta_r^5 + 241920\alpha_i\beta_i^6\beta_r^5 + 50688\alpha_i^2\alpha_r\beta_i^3\beta_r^6 - 31232\alpha_r^3\beta_i^3\beta_r^6 + 69120\alpha_i\alpha_r\beta_i^4\beta_r^6 -$
$241920\alpha_r\beta_i^5\beta_r^6 + 10752\alpha_i^3\beta_i^2\beta_r^7 + 10752\alpha_i\alpha_r^2\beta_i^2\beta_r^7 + 138240\alpha_i^2\beta_i^3\beta_r^7 - 138240\alpha_r^2\beta_i^3\beta_r^7 + 241920\alpha_i\beta_i^4\beta_r^7 +$
$20352\alpha_i^2\alpha_r\beta_i\beta_r^8 - 10368\alpha_r^3\beta_i\beta_r^8 + 103680\alpha_i\alpha_r\beta_i^2\beta_r^8 - 120960\alpha_r\beta_i^3\beta_r^8 + 128\alpha_i^3\beta_r^9 + 10368\alpha_i\alpha_r^2\beta_r^9 + 34560\alpha_i^2\beta_i\beta_r^9 -$
$34560\alpha_r^2\beta_i\beta_r^9 + 120960\alpha_i\beta_i^2\beta_r^9 + 34560\alpha_i\alpha_r\beta_r^{10} - 24192\alpha_r\beta_i\beta_r^{10} + 24192\alpha_i\beta_r^{11})^2 + 4((-64(\alpha_r\beta_i^3 - \alpha_i\beta_i^2\beta_r +$

$\alpha_r\beta_i\beta_r^2 - \beta_r^3)^2 + 48(9\alpha_i^2\beta_i^2 + \alpha_r^2\beta_r^2 - 6\alpha_i\beta_i^3 - 15\beta_i^4 + 16\alpha_i\alpha_r\beta_r - 6\alpha_r\beta_i^2 + \alpha_i^2\beta_r^2 + 9\alpha_r^2\beta_r^2 - 6\alpha_i\beta_i\beta_r^2 - 30\beta_i^2\beta_r^2 - 6\alpha_r\beta_r^3 - 15\beta_r^4)(\beta_i^4 + 2\beta_i^2\beta_r^2 + \beta_r^4))^{2/3})$,

$u_1 = 1/(48 \cdot 2^{1/3}(\beta_i^4 + 2\beta_i^2\beta_r^2 + \beta_r^4))$,

$u_2 = (\alpha_r\beta_i^3 - \alpha_i\beta_i^2\beta_r + \alpha_r\beta_i\beta_r^2 - \alpha_i\beta_r^3)/(6(\beta_i^4 + 2\beta_i^2\beta_r^2 + \beta_r^4))$,

$u_3 = u_2 - (-64(\alpha_r\beta_i^3 - \alpha_i\beta_i^2\beta_r + \alpha_r\beta_i\beta_r^2 - \beta_r^3)^2 + 48(9\alpha_i^2\beta_i^2 + \alpha_r^2\beta_r^2 - 6\alpha_i\beta_i^3 - 15\beta_i^4 + 16\alpha_i\alpha_r\beta_r - 6\alpha_r\beta_i^2 + \alpha_i^2\beta_r^2 + 9\alpha_r^2\beta_r^2 - 6\alpha_i\beta_i\beta_r^2 - 30\beta_i^2\beta_r^2 - 6\alpha_r\beta_r^3 - 15\beta_r^4)(\beta_i^4 + 2\beta_i^2\beta_r^2 + \beta_r^4))/(24 \cdot 2^{2/3}(\beta_i^4 + 2\beta_i^2\beta_r^2 + \beta_r^4)u) + u_1 u)$,

$u_4 = u_2 - u_3$,

$u_5 = 3\alpha_i\alpha_r\beta_i^2 + 3\alpha_r\beta_i^3 + 3\alpha_r^2\beta_i\beta_r - 3\alpha_i\beta_i^2\beta_r - 3\alpha_i\alpha_r\beta_r^2 + 3\alpha_r\beta_i\beta_r^2 - 3\alpha_i\beta_r^3$.

The second intersection point of the circle and the hyperbola in (7.4.1) is

$$\begin{aligned}(x_2, y_2) = (&(u_5 + 9\alpha_i^2\beta i^2 u_3 - 6\alpha_i\beta i^3 u_3 - 15\beta i^4 u_3 + 18\alpha_i\alpha_r\beta_i\beta_r u_3 \\ &-6\alpha_r\beta_i^2\beta_r u_3 + 9\alpha_r^2\beta_r^2 u_3 - 6\alpha_i\beta_i\beta_r^2 u_3 - 30\beta_i^2\beta_r^2 u_3 - 6\alpha_r\beta_r^3 u_3 - 15\beta_r^4 u_3 \\ &-4\alpha_r\beta_i^3 u_4^2 + 4\alpha_i\beta_i^2\beta_r u_4^2 - 4\alpha_r\beta_i\beta_r^2 u_4^2 + 4\alpha_i\beta_r^3 u_4^2 + 16\beta_i^4 u_4^3 + 32\beta_i^2\beta_r^2 u_4^3 + \\ &16\beta_r^4 u_4^3)/u_5, u_4/(24 \cdot 2^{2/3}(\beta_i^4 + 2\beta_i^2\beta_r^2 + \beta_r^4)(w + (w + 4u_4^3)^{1/2})^{1/3}) + \\ &u_1(w + (w + 4u_4^3)^{1/2})^{1/3}).\end{aligned} \tag{7.4.2}$$

7.5 The H mod K theorem

We present here some intermediate calculations used to obtain the results in Section 4.5.

The subgroup N_H and the fixed-point subspace $Fix(\Sigma)$, corresponding to the $\mathbb{C}$-axial subgroups $\mathbb{D}_4$, $\mathbb{Z}_3(0) \times \{\kappa\} = \{C\} \times \{\kappa\}$, $\mathbb{Z}_3(1) = \{e^{-2\pi i/3}C\}$ and $\{e^{\pi i}C\mathrm{R}C^2\} \times \{C^2\kappa\}$, cases $(b) - (e)$ in Table 4.1, respectively. We use the notation $W = CRC^2\kappa$.

index	N_H	Fixed-point-subspace
(b)	$\{\mathbb{1}, W, W^2, W^3, W\kappa, W^2\kappa, W^3\kappa, \kappa\}$	$\{(z, 0, 0)\}$
(c)	$\{\mathbb{1},\ C,\ C^2,\ \kappa,\ C\kappa,\ C^2\kappa\}$	$\{(z, z, z)\}$
(d)	$\{\mathbb{1},\ C,\ C^2\}$	$\{(z, e^{2\pi i/3}z, e^{-2\pi i/3}z)\}$
(e)	$\{\mathbb{1},\ CRC^2,\ C^2\kappa,\ CRC\kappa\}$	$\{(z, z, 0)\}$

Isotropy subgroups, their generators and dimension of the fixed-point subspaces for the action of $\langle\mathbb{T}, \kappa\rangle$ on $\mathbb{C}^3$.

Isotropy subgroup	Generators	Fix(Σ)	dim	Generators of $N(\Sigma)$
$\langle\mathbb{T}, \kappa\rangle$	$\{R, C, \kappa\}$	$\{(0, 0, 0)\}$	0	$\{R, C, \kappa\}$
$\mathbb{Z}_2(R) \times \mathbb{Z}_2(\kappa)$	$\{R, \kappa\}$	$\{(z, 0, 0)\}$	2	$\{R, \kappa, CRC^2, C^2RC\}$
$\mathbb{Z}_3(0) \times \mathbb{Z}_2(\kappa)$	$\{C, \kappa\}$	$\{(z, z, z)\}$	2	$\{C, \kappa\}$
$\mathbb{Z}_2(\kappa)$	$\{\kappa\}$	$\{(z_1, z_2, z_2)\}$	4	$\{\kappa, R\}$
$\mathbb{1}$	$\mathbb{1}$	$\mathbb{C}^3$	6	$\mathbb{1}$

Possible pairs H, K for Theorem 4.2.2 in the action of $\langle \mathbb{T}, \kappa \rangle$ on $\mathbb{C}^3$, with $\dim Fix(K) = 2$ or 4.

Isotropy subgroup K	Generators of K	H	Generators of H	$Fix(K)$	dim
$\mathbb{Z}_2(R) \times \mathbb{Z}_2(\kappa)$	$\{R, \kappa\}$	K	$\{R, \kappa\}$	$\{(z, 0, 0)\}$	2
$\mathbb{Z}_2(R) \times \mathbb{Z}_2(\kappa)$	$\{R, \kappa\}$	$N(K)$	$\{CRC^2\kappa, \kappa\}$	$\{(z, 0, 0)\}$	2
$\mathbb{Z}_3(0) \times \mathbb{Z}_2(\kappa)$	$\{C, \kappa\}$	$N(K) = K$	$\{C, \kappa\}$	$\{(z, z, z)\}$	2
$\mathbb{Z}_2(\kappa)$	$\{\kappa\}$	$N(K)$	$\{\kappa, R\}$	$\{(z_1, z_2, z_2)\}$	4
$\mathbb{Z}_2(\kappa)$	$\{\kappa\}$	K	$\{\kappa\}$	$\{(z_1, z_2, z_2)\}$	4

Isotropy subgroups and their normalizers for the action of $\mathbb{O}$ on $\mathbb{C}^3$.

Isotropy subgroup	$N(\Sigma)$
$\mathbb{Z}_2(TCT^2C^2)$	$\{Id, T^2, T^2C^2TC^2, TCT^2C^2\}$
$\mathbb{Z}_3(C)$	$\{Id, C, C^2, T^2C^2TC^2, C^2T^2C^2TC^2, CT^2C^2TC^2\}$
$\mathbb{Z}_4(CT^3)$	$\{Id, T^2, TC^2, T^3C^2, TC^2TC^2, T^3C^2TC^2, CT^2C^2TC^2, CT^3\}$

Isotropy subgroups, their generators and dimension of the fixed-point subspaces for the action of $\mathbb{O}$ on $\mathbb{C}^3$.

Isotropy subgroup	Generators	Fix(Σ)	dim	Generators of $N(\Sigma)$
$\mathbb{O}$	$\{T, C\}$	$\{(0, 0, 0)\}$	0	$\{T, C\}$
$\mathbb{Z}_2(TCT^2C^2)$	$\{TCT^2C^2\}$	$\{(z, z, 0)\}$	2	$\{T^2, TCT^2C^2\}$
$\mathbb{Z}_3(C)$	$\{C\}$	$\{(z, z, z)\}$	2	$\{C, T^2C^2T\}$
$\mathbb{Z}_4(CT^3)$	$\{CT^3\}$	$\{(z, 0, 0)\}$	2	$\{T^2, CT^3\}$
$\mathbb{1}$	Id	$\mathbb{C}^3$	6	Id

Possible pairs H, K for Theorem 4.2.2 in the action of $\mathbb{O}$ on $\mathbb{C}^3$, with $\dim Fix(K) = 2$.

Isotropy subgroup K	Generators of K	H	Generators of H	$Fix(K)$	dim
$\mathbb{Z}_2(TCT^2C^2)$	$\{TCT^2C^2\}$	K	$\{TCT^2C^2\}$	$\{(z, z, 0)\}$	2
$\mathbb{Z}_2(TCT^2C^2)$	$\{TCT^2C^2\}$	$N(K)$	$\{T^2, TCT^2C^2\}$	$\{(z, z, 0)\}$	2
$\mathbb{Z}_3(C)$	$\{C\}$	K	$\{C\}$	$\{(z, z, z)\}$	2
$\mathbb{Z}_3(C)$	$\{C\}$	$N(K)$	$\{C, T^2C^2T\}$	$\{(z, z, z)\}$	2
$\mathbb{Z}_4(CT^3)$	$\{CT^3\}$	K	$\{CT^3\}$	$\{(z, 0, 0)\}$	2
$\mathbb{Z}_4(CT^3)$	$\{CT^3\}$	$N(K)$	$\{T^2, CT^3\}$	$\{(z, 0, 0)\}$	2

Bibliography

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