Mathematical problems of hydro-electric power stations management

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To my parents and sister
for their support and encouragement

and to José Pedro
for his endless love and
for making me laugh countless times.

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Abstract

This work analyses a model for a hydro-electric system where some power stations have reversible turbines. The goal is to optimize the profit of power production while meeting the system's associated restrictions.
This problem is formulated as an optimal control problem. A further analysis of such formulation leads us to write the problem as a particular case of an abstract problem of minimizing a quadratic non-definite functional, subject to linear and cone constraints. The abstract formulation highlighted a relevant feature of our hydro-electric problem: the minimum does not corresponds to an isolated minimizer.
For the abstract problem, and motivated by the particular case, new sufficient conditions of optimality are deduced for local minimizers and for local directional minimizers. A main feature of these new sufficient conditions is that they cover the case when the minimizer is not isolated. We also directly deduce sufficient conditions of optimality for the problem of hydro-electric system.

The cases of systems with one and two power stations are analyzed in detail, applying numerical and analytical tools. Existence results, necessary and our new sufficient conditions, as well as the particular properties of the problems are used, leading to a comprehensive analysis of the solution of the problem.

Since the objective function associated to the problem is nonconvex, several local minimums may exist. Global optimization methods are necessary to get a global optimal solution. In this work two different approaches, involving a Chen-Burer algorithm and a projection estimation refinement method, are discussed and compared.

Resumo

Neste trabalho, consideramos um modelo para um sistema hidroelétrico onde algumas das centrais possuem turbinas reversíveis. O objectivo é optimizar o lucro da produção de energia, tendo em consideração as restrições associadas ao sistema.
Este problema pode ser formulado como um problema de controlo ótimo. Uma análise mais aprofundada desta formulação leva-nos a escrever o problema como um caso particular de um problema abstrato de minimização de uma funcional quadrática não definida, sujeita a restrições lineares e cónicas.
Para este problema abstrato e motivados pelo caso particular, deduzimos novas condições suficientes de otimalidade para minimizantes locais e minimizantes direcionais locais. Uma particularidade destas novas condições suficientes é que cobrem o caso de minimizantes não isolados, presentes no problema do sistema hidroelétrico. Para este problema deduzimos diretamente condições suficientes de otimalidade.

Os casos particulares de sistemas com uma e duas estações hidroelétricas são analisados em detalhe, aplicando ferramentas numéricas e analíticas. Resultados de existência, condições necessárias e as novas condições suficientes, bem como propriedades particulares do problema, permitem uma análise mais completa da solução ótima do problema.

Como a função objectivo associada ao problema é não convexa, podem existir vários mínimos locais. Métodos de optimização global são considerados para se obter uma solução ótima global. Duas abordagens diferentes, envolvendo um algoritmo de Chen-Burer e um método de estimação de projeção, são discutidas e comparadas.

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Nomenclature

OCP	Optimal control problem
NCO	Necessary conditions of optimality
PMP	Pontryagin maximum principle
B	Closed unit ball in $\mathbb{R}^n$ centered in the origin
B_X	Closed unit ball in a space X, centered in the origin
$cl\ S$	Closure of a set S
$supp\ \mu$	Support of a measure μ
$d(A, B)$	Hausdorff distance between sets A and B
$\lvert \cdot \rvert$	Euclidean norm in $\mathbb{R}^n$
$\lVert \cdot \rVert_X$	Norm in the space X
$Gr f$	Graph of a function f
$S^{\perp}$	Orthogonal complement of a subspace S
B^*	Adjoint of a linear operator B
Y^*	Dual space of a normed linear space Y
$\ker L$	Kernel of L
C^1	Space of continuously differentiable functions
$L_1([a, b], \mathbb{R}^p)$	Space of integrable functions $f : [a, b] \to \mathbb{R}^p$, with norm $\lVert f \rVert_{L^1} = \int_a^b \lvert f \rvert dt$
$L_2([a, b], \mathbb{R})$	Space of measurable functions $f : [a, b] \to \mathbb{R}$, whose square is integrable, with norm $\lVert f \rVert_{L^2} = \left(\int_a^b \lvert f \rvert^2 dt \right)^{1/2}$

$L_\infty([a,b],\mathbb{R})$	Space of measurable functions $f : [a,b] \to \mathbb{R}$, whose essential supremum is finite, with norm $\|f\|_{L^\infty} = \underset{t\in[a,b]}{ess\ sup}\,\|f(t)\|$
$C([a,b],\mathbb{R})$	Space of continuous functions $f : [a,b] \to \mathbb{R}$
$\mathscr{L} \times \mathscr{B}$	Product σ-algebra generated by the Lebesgue subsets $\mathscr{L}$ and the Borel subsets of $\mathbb{R}$.
$BV([a,b],\mathbb{R})$	Space of functions $f : [a,b] \to \mathbb{R}$ of bounded variation
$AC([a,b],\mathbb{R})$	Space of absolutely continuous functions $f : [a,b] \to \mathbb{R}$

Chapter 1

Introduction

Water is becoming a scarce and valuable resource, with population and consumption rising, and the concern for a good use of it is more evident nowadays. The way that we use water to produce energy must be effective and efficient to maximize its benefit. The management of interconnected reservoir systems in a river is of particular importance if there is also the possibility of reusing the water in a situation of drought. This may be implemented in modern reversible hydro-electric power stations, which are associated with reservoirs in a cascade structure, where it is possible both to turbine water to downstream to produce electric power and to pump from downstream to refill an upstream reservoir.

Hydro-electric systems with multi-reservoirs in cascade have attracted the attention of many researchers in different contexts (see, for example, [24] and references therein). Different methods were applied and/or developed to solve the problem of management of these systems. Optimization and predictive control techniques (see, e.g. [28, 36]) as well as stochastic approaches (see, e.g. [25]), considering uncertainties in power prices and water inflows, are frequently used. In this work we take a deterministic model and we use optimal control techniques. Optimization of quadratic non-definite functions subject to linear and cone constraints will also be under attention.

In an academic training in REN - Redes Energéticas Nacionais a model was developed to represent a real hydro-electric system, taking into account the physical and technical characteristics of the system. Associated to this model, it was considered as objective, the maximization of the selling profit of electric energy production. The problem was considered in the framework of optimal control theory and it was numerically solved. Several numerical simulations addressing different scenarios were undertaken. Results of this work were published in [23, 41, 42].

The purpose of this thesis is to analyze the problem addressed in the above work, with more detail and using analytical optimal control tools. Due to the complexity of the model involved, it was decided to consider a simplified version of it. This simplified version intends to retain the main characteristics of the original model. It is still a model for a hydro-electric system where some power stations are equipped with reversible turbines.

The problem is analyzed in the framework of optimal control theory. The fluxes of water to turbine or pump on each power station are associated to control variables and the water volumes in reservoirs are the state variables. The objective is to find the turbined/pumped water flows and the corresponding volumes in the reservoirs that maximize the profit of selling the energy produced by the system. It is still a challenging problem since besides the constraints on the control, it also involves pure state constraints. Furthermore, the cost function is non-convex which contributes to increase the complexity of the problem.

To carry out an analytical study of the problem, we start by getting some knowledge of its optimal trajectory profile. This is achieved with the use of an optimization software, which allows us to obtain a numerical solution for the optimal control problem.

To validate the numerical solution, necessary and sufficient conditions of optimality are called for. In some cases it is observed that the numerical solution has an irregular behavior over a certain timeframe, making us wary of the possibility that the solution for the problem may be not unique. In fact, we conclude to be in presence of an infinity of minimizers leading to the same objective value. The minimizer is not an isolated minimizer. Classical sufficient conditions of optimality involve hypothesis which are not satisfied by non isolated local minimizers. Our problem can be interpreted as an infinite dimensional optimization problem with a non convex quadratic functional subject to linear and cone constraints. Quadratic forms play an important role in the calculus of Variations (see, e.g., [2, 22]) and particular attention is paid to the Legendre condition, a condition associated to the second variation of the functional to be optimized and involving derivatives of order two with respect to the derivative argument. Here we have a quadratic form depending on the trajectories only and not on their derivatives. However both the trajectories and the derivatives are subject to some geometric constraints. The theory of infinite-dimensional quadratic functionals is closely related to second-order sufficient optimality conditions for optimal control problems, and have been studied by many authors (see, for example, [2, 21, 22, 33, 37]). In general, these results are deduced under very restrictive hypotheses which hardly can be verified in many problems. Motivated by our problem and in particular by the presence of non isolated minimizers we deduce new sufficient conditions of optimality for an abstract problem with quadratic functional subject to linear and cone constraints. Local minimizers and also directional minimizers are under attention. To our knowledge the treatment of sufficient conditions for directional minimizers is new.

Since the cost function associated to our problem is a non-convex function there may exist several local minima with different cost values. Global optimization methods are necessary to obtain a global solution. Here, we present two different approaches with the purpose of finding the global solution for the problem under consideration. These approaches are based on the Chen-Burer algorithm (see [12] and references therein) and on a projection estimation refinement method (see [7]).

1.1 Outline of the Thesis

This thesis is organized as follows:

Chapter 2 is dedicated to a description of the problem. A generic complex model for a hydro-electric system, in optimal control context, is presented. We identify the variables, the constraints and the objective function. The model presented is then simplified and a new formulation for the hydro-electric system problem is considered. This simplified model will be the focus of this thesis. Special cases are formulated: a system with one power station and with two power stations in cascade.

The classical optimal control problem and some fundamental and basic tools for its treatment are presented in chapter 3, namely, existence results and necessary conditions of optimality.

From mathematical point of view, the problem associated with the hydro-electric power system can be seen as a problem of minimizing a quadratic non-definite functional subject to linear and cone constraints. A key feature of this problem is the possible existence of minimizers which are not isolated. In chapter 4 new sufficient conditions of optimality, for local minimizers and also for directional minimizers, are proposed. These conditions cover the case of non-isolated minimizers. The results of this chapter constitute the main contribution of the thesis. They are published in [17, 18].

In chapter 5 we analyse with detail particular cases of the problems presented in chapter 2. Taking into account the profile of numerical solutions, obtained with available software, we apply some mathematical tools to validate such solutions. More specifically, results on existence of solution and necessary conditions of optimality

in the form of the maximum principle of Pontryagin (PMP) are applied. The new sufficient conditions of optimality, developed in chapter 4, are then considered and the proof that the solution found is in fact a local minimum follows.
Here, we illustrate the importance of combining numerical and theoretical tools to solve an optimization problem. In many cases, the numerical methods alone do not give assurance that the obtained candidate is a solution to the problem. However, those methods supported by analytical results, sufficient conditions and analysis of particular properties of the problem, can be all together essential for a rigorous treatment of the problem.

In chapter 6 numerical methods with the goal of achieving global solutions, are under attention. After discretization of our problem, we obtain a problem of minimization of an indefinite quadratic form subject to linear constraints, to which we apply two different approaches. In the first approach, the Chen-Burer algorithm is directly applied and we obtain a candidate for the global solution of the problem.
In the second approach, we use a specific structure of the cost function and construct a projection of the set of feasible solutions on a relevant subspace, reducing in this way the dimension of the problem. The Chen-Burer algorithm is then applied to the projected low-dimensional problem, solving it faster than in the first approach. In the end of the chapter, we discuss and compare the two approaches. A brief overview of these results are published in [8], and they will appear with more detail in a paper submitted for publication ([9]).

Finally, chapter 7 concludes this thesis, with a summary of the main contributions of this work and a description of suggestions for future research.

1.2 Main contributions

- A model, which intends to keep some main characteristics of a hydro-electric system is created. Such model is analyzed in the context of optimal control theory;

- Optimal solutions for the problem of maximizing the profit of energy sale, considering systems with one and two power stations in cascade, are obtained;

- The optimal control problem of hydro-electric power production is treated with analytical tools;

- Sufficient conditions of optimality for the abstract problem of minimizing an infinite-dimensional quadratic functional subject to linear and cone constraints are derived. Such conditions are deduced for local minimizers and local directional minimizers;
- Sufficient conditions are derived for the problem of management of hydro-electric system;
- Periodicity properties of the solution for the case of periodic price, are investigated;
- For the system with one power station, it is proven that the presence of a reversible turbine always improves the profit;
- A global solution to the problem of hydro-electric power production is obtained, using a new and faster method. Such method focuses on global optimization techniques and on projection estimation refinement methods, which ultimately seeks to reduce the dimension of the problem.

Chapter 2

Problem Presentation

A control problem for a cascade of hydro-electric power stations is presented in this chapter. Several technical and physical constraints are associated to the hydro-electric system. This turns the problem very complicated and a simplified model, which intends to keep the main characteristics of the original problem, is defined. This model is examined for the cases of a system with one power station and two power stations in cascade.

2.1 Hydro-electric resources in cascade

When observing hydro-electric systems, we can distinguish power stations with different configurations. To characterize a hydro-electric power station it is necessary to identify:

- storage capacity. In our case, we consider reservoirs, i.e., natural or artificial water storages which in general are used to regularize flows, produce energy, supply water, etc;

- hydraulic configuration. The stations can be in a cascade system or they may be hydraulically independent from each other. In a cascade, a set of hydro-electric power stations are connected through a net of water flows. The stations are both hydraulically and electrically connected;

- the following characteristics of the station:

  ◇ turbining - the power station can convert kinetic into mechanical energy through the fall of water which activates the turbines and converts that energy into electric energy;

  ◇ pumping - the power station has the possibility to reverse the turbine and pump water from a downstream to an up-stream reservoir, increasing in this way the volume of the reservoir upstream;

  ◇ discharging - this ability allows to ensure the security of the reservoirs. When the water level is in the limit of the reservoir, the station releases water without producing energy and without costs or earnings.

In this work we focus on a cascade where some of the hydro-electric power stations can have reversible turbines. Discharging is not considered.

2.2 The Model

In this section, we present a model (based on [23,41,42]) for a cascade of hydro-electric power stations where some of the stations have reversible turbines.
The dynamic behavior of the system is introduced in the model through differential equations. These equations relate two types of variables:

- the volumes of each reservoir, $V_i(t)$. These are the state variables;
- the water flows, $u_i(t)$. These are the control variables, which have a direct impact on the process, affecting the state variables in a prefixed way.

Here, i is the index that identifies the reservoir and t is the instant of time.

The following picture illustrates two reservoirs associated to a hypothetical system of hydro-electric power stations in cascade.

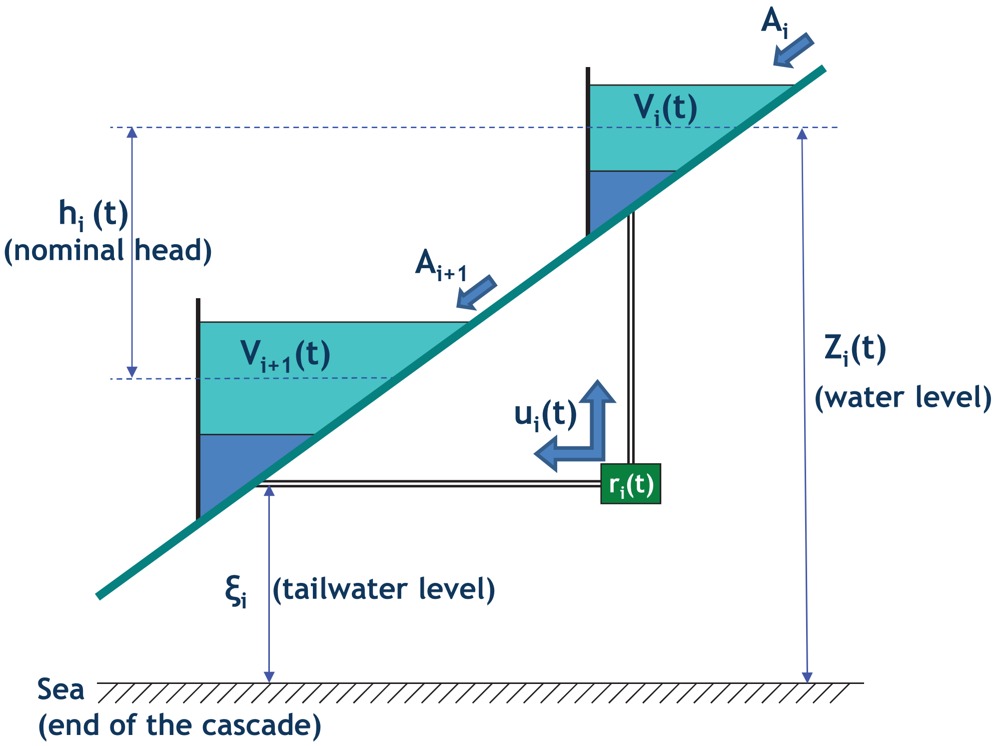


Figure 2.1: Generic cascade mechanism considering reversible turbines

At instant t, the hydro-electric power station i can be described in terms of the following variables:

$Z_i(t)$ – water level in reservoir i;

$h_i(t)$ – head in reservoir i;

$V_i(t)$ – volume of water in reservoir i;

A_i – incoming flow to reservoir i (e.g. rain);

$u_i(t)$ – water flow in station i, from pumping or turbining action.

The water level in the reservoir i, $Z_i(t)$ is given by the following expression:

$$Z_i(t) = Z_i^0 + \alpha_i \left(\frac{V_i(t)}{V_i^0} - 1 \right)^{\beta_i},$$

where

V_i^0 – minimum water volume in reservoir i;

Z_i^0 – nominal water level (meters above sea level) in reservoir i;

$\alpha_i,\ \beta_i$ – positive parameters.

At instant t and for each reservoir i, the following dynamic equation is taken to be valid

$$\dot{V}_i(t) = A_i - u_i(t) + \sum_{m \in M_i} u_m(t),$$

where M_i is the set of indices for upstream reservoirs immediately before reservoir i.

The variation of the water volume in a reservoir depends of the incoming flow and depends on the total amount of water turbined or pumped. This means that the stations depend on each other. So, all the decisions taken for one reservoir shall be taken under a global vision, since it has impact along the cascade. This turns the analysis of the problem quite difficult.

The hydro-electric power station is subject to physical and technical limitations (constraints or restrictions). The set of constraints defines the domain of admissible controls for the problem.
In each reservoir, the water storage must be between a minimum and a maximum level,

$$Z_i^{min} \leq Z_i(t) \leq Z_i^{max},$$

and the water flow must be between a minimum and a maximum value

$$\zeta_i \left(h_i(t) - h_i^0 \right) - u_i^{0P} \leq u_i(t) \leq u_i^{0T} \left(\frac{h_i(t)}{h_i^0} \right)^{1/2},$$

where

u_i^{0T} – amount of nominal turbined water in the station i;

u_i^{0P} – amount of nominal pumped water in the station i;

ζ_i – pumping coefficient in the station i;

h_i^0 – nominal heads of the reservoir i.

In particular, when the power station i only turbines, the minimum value allowed for u_i is zero.

Given a system of hydro-electric power stations, we have as objective to manage the energetic resources that exist in an efficient way, bearing in mind the price of energy during the time horizon considered. The quality and performance of the decisions undertaken are measured by an objective function. The objective is to maximize the profit obtained with buying/selling the electricity consumed/produced. Such objective function is represented by:

$$\max \int_0^T c(t) \left(\sum_{i \in I} r_i(t) \right) dt,$$

where

I – total number of reservoirs in the cascade;

$c(t)$ –economic value of electric energy in the market at instant t;

T – length of the time horizon;

$r_i(t)$ – value of power produced/consumed by the station i

$r_i(t)$ is described by

$$r_i(t) = \begin{cases} 9.8 * u_i(t) * \left(h_i(t) - \Delta h_i^T(t)\right) * \mu_i^T * (1 - \phi_i^{prog}) * (1 - \phi_i^{cint}) * (1 - \phi_i^{fort}) & \text{if } u_i(t) \geq 0 \\ 9.8 * u_i(t) * \left(h_i(t) - \Delta h_i^P(t)\right) * 1/\mu_i^P * (1 - \phi_i^{prog}) * (1 - \phi_i^{cint}) * (1 - \phi_i^{fort}) & \text{if } u_i(t) < 0. \end{cases}$$

Here

$$\Delta h_i^T(t) = \Delta h_i^{0T} \left(\frac{u_i(t)}{u_i^{0T}} \right)^2, \qquad \Delta h_i^P(t) = \Delta h_i^{0P} \left(\frac{u_i(t)}{u_i^{0P}} \right)^2$$

and

Δh_i^{0T} – head loss in reservoir i at instant t for turbining;

Δh_i^{0P} – head loss in reservoir i at instant t for pumping;

μ_i^T – global efficiency of the reservoir i, when it turbines;

μ_i^P – global efficiency of the reservoir i, when it pumps;

ϕ_i^{prog}, ϕ_i^{fort}, ϕ_i^{cint} – rates of availability and maintenance.

A sequence of decisions determined essentially by choices of $u_i(\cdot)$, that optimize the objective function, satisfying the constraints of the problem, is called an optimal control policy.

2.2.1 Example - Problem with 2 reservoirs in cascade

The next picture (Fig. 2.2) illustrate a problem with 2 reservoirs.

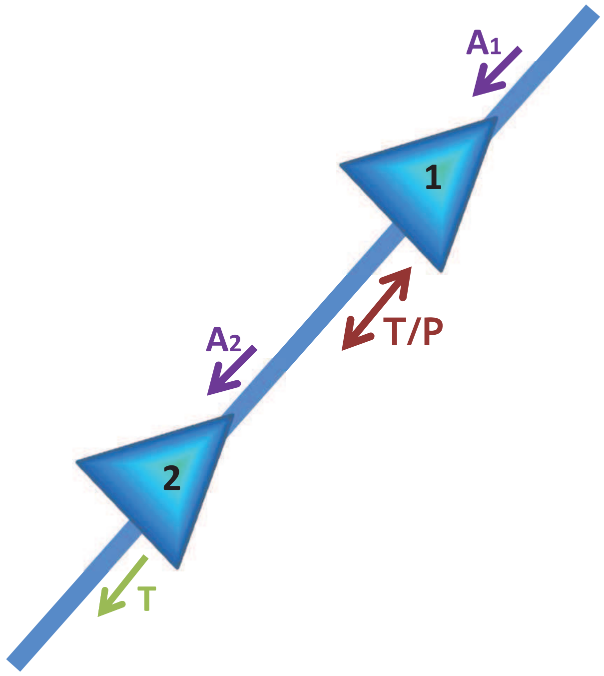


Figure 2.2: Scheme of a cascade with two hydro-electric power stations

Each station has a volume associated, $V_i(t)$, $i = 1, 2$, as well as a turbined or pumped water flow, $u_i(t)$, $i = 1, 2$.
Only the station 1 is reversible, it has the ability to turbine downstream, with energy production and consequent profit for the company, as well as, the ability to pump water upstream, resulting power consumption and cost for the company.
The station 2 has only the ability to turbine downstream.

The respective optimal control problem can be written in the form:

$$
\begin{aligned}
\max \quad & \int_{t=0}^{T} c(t) \left(\sum_{i=1}^{2} r_i(t) \right) \\
\text{s.t.} \quad & \dot{V}_1(t) = A_1 - u_1(t), \\
& \dot{V}_2(t) = A_2 + u_1(t) - u_2(t), \\
& V_i(0) = V_i(T), \qquad i = 1, 2 \\
& Z_i(t) = Z_i^0 + \alpha_i \left(\frac{V_i(t)}{V_i^0} - 1 \right)^{\beta_i}, \qquad i = 1, 2 \\
& h_1(t) = Z_1(t) - \max \left\{ Z_2(t), \xi_1 \right\}, \\
& h_2(t) = Z_2(t) - \xi_2, \\
& \zeta_1 \left(h_1(t) - h_1^0 \right) - u_1^{0P} \leq u_1(t) \leq u_1^{0T} \left(\frac{h_1(t)}{h_1^0} \right)^{\frac{1}{2}}, \\
& 0 \leq u_2(t) \leq u_2^{0T} \left(\frac{h_2(t)}{h_2^0} \right)^{\frac{1}{2}}, \\
& Z_i^{min} \leq Z_i(t) \leq Z_i^{max}, \qquad i = 1, 2
\end{aligned}
$$

where $t \in [0, T]$.

In attachments A and B, we can find a table with the meaning of all the variables and parameters of this model, as well as, an example of possible data that can be used here.

2.3 Simplified Model

The model presented in the last section revealed very difficult to deal with, using analytic tools. The treatment of the problem with such a complex objective function and time-dependent control constraints is a very hard task to accomplish. Because of that we start by considering a simplification of that model. We try to maintain the main characteristics of the original model. The objective function is simplified and the upper and lower bounds for the controls will now be fixed.

The new model has shown to be a very stimulating problem from mathematical point of view, and until the end of this work we do not come back to the original one.

An outline of the simplified model is now described.

For a cascade of N hydro-electric power stations, the dynamics of water volumes, $V_k(t)$, $k = \overline{1,N}$, are described by the following control system

$$\dot{V}_k(t) = A_k - u_k(t) + \sum_{m \in M(k)} u_m, \quad k = \overline{1,N}, \tag{2.1}$$

where $M(k)$ is the set of indices of reservoirs upstream from the reservoir k (immediately up).

Set $V(\cdot) = (V_1(\cdot), \dots, V_N(\cdot))$ and $u(\cdot) = (u_1(\cdot), \dots, u_N(\cdot))$. The controls $u(t) = (u_1(t), \dots, u_N(t))$ are the turbined/pumped flows of water for reservoirs $\overline{1,N}$ at time t, and constants A_k are the incoming flows, $k = \overline{1,N}$. $V(t) = (V_1(t), \dots, V_N(t))$ constitute the state variables.
The equation (2.1) is called water balance equation (see, e.g., [34]).

The state variables and the control variables satisfy the following technical constraints:

$$V_k(0) = V_k(T), \qquad V_k(t) \in [V_k^m, V_k^M], \qquad u_k(t) \in [u_k^m, u_k^M].$$

The constants V_k^m and V_k^M, $k = \overline{1,N}$, stand for the minimum and maximum water volumes imposed; the constants u_k^m and u_k^M, $k = \overline{1,N}$, are the imposed minimum and maximum turbined/pumped water flows.
The equality $V_k(0) = V_k(T)$ is called periodic constraint and it ensures, in particular, that the reservoir k does not spend all the water on the period $[0,T]$. Also, and under similar conditions, on a period of time that would follow, the optimal solution would repeat itself.

The objective is to find optimal controls $\hat{u}_k(\cdot) \in L_\infty([0,T], \mathbb{R})$, space of measurable function $u : [0,T] \to \mathbb{R}$, essentially bounded, and respective volumes $\hat{V}_k(\cdot) \in AC([0,T], \mathbb{R})$, space of absolutely continuous function $V : [0,T] \to \mathbb{R}$, which maximize the profit of selling energy. The objective function is given by:

$$J(u(\cdot), V(\cdot)) = \sum_{k=1}^{N} \int_0^T c(t)\ u_k(t) \left(\frac{V_k(t)}{S_k} + H_k - \frac{V_{j(k)}(t)}{S_{j(k)}} - H_{j(k)} \right) dt. \tag{2.2}$$

where $j(k)$ is the index associated to the (unique) downstream reservoir, which receives water from reservoir k.

Here, $c(\cdot)$ is the price of the energy and the expression multiplied by $c(t)$ represents potential energy. H_k and S_k, $k = \overline{1,N}$, are the liquid surface elevation and the area associated to each reservoir k. For simplicity, it is assumed that the reservoirs have cylindric form and that the gravity constant is equal to one. It is also assumed that all the potential energy is converted into electric energy. The following picture illustrates the case of 2 power stations.

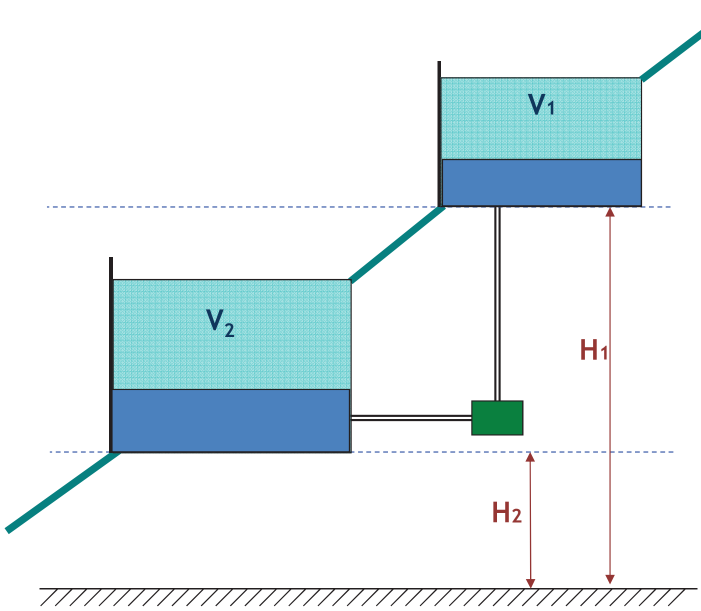


Figure 2.3: System with two power stations

The model of section 2.2 and the new one must be close to each other, and for this, the parameters (u_k^m, u_k^M, H_k, S_k) will be reasonably chosen (for details see appendix C).

Using (2.1), the objective function (2.2) can be equivalently written as

$$\sum_{k=1}^{N} \int_0^T c(t) \left(-\dot{V}_k(t) + A_k + \sum_{m \in M(k)} u_m(t) \right) \times \left(\frac{V_k(t)}{S_k} + H_k - \frac{V_{j(k)}(t)}{S_{j(k)}} - H_{j(k)} \right) dt. \quad (2.3)$$

Lemma 2.3.1. The following equality holds:

$$\int_0^T \sum_{k=1}^N c(t) \left(\frac{V_k(t)}{S_k} \sum_{m\in M(k)} u_m(t) - u_k(t) \frac{V_{j(k)}(t)}{S_{j(k)}} \right) dt = 0.$$

Proof. If $m \in M(k)$, then $k = j(m)$. Moreover $j(k)$ is empty or has only one element. Therefore we have

$$\int_0^T \sum_{k=1}^N c(t) \left(\frac{V_k(t)}{S_k} \sum_{m\in M(k)} u_m(t) - u_k(t) \frac{V_{j(k)}(t)}{S_{j(k)}} \right) dt =$$

$$= \int_0^T c(t) \left(\sum_{m=1}^N \sum_{k=j(m)} \frac{V_k(t)}{S_k} u_m(t) - \sum_{k=1}^N \frac{V_{j(k)}(t)}{S_{j(k)}} u_k(t) \right) dt$$

$$= \int_0^T c(t) \left(\sum_{k=1}^N \sum_{m=j(k)} \frac{V_m(t)}{S_m} u_k(t) - \sum_{k=1}^N \frac{V_{j(k)}(t)}{S_{j(k)}} u_k(t) \right) dt$$

$$= \int_0^T c(t) \left(\sum_{k=1}^N \frac{V_{j(k)}(t)}{S_{j(k)}} u_k(t) - \sum_{k=1}^N \frac{V_{j(k)}(t)}{S_{j(k)}} u_k(t) \right) dt \;=\; 0.$$

This completes the proof. ■

The notation $BV([0,T],\mathbb{R})$ is used for the space of bounded variation functions $f : [0,T] \to \mathbb{R}$. We assume that $c(\cdot) \in BV([0,T],\mathbb{R}), c(\cdot)$ is right-continuous and $c(0) = c(T)$.

Without changing the notation for the objective function J, we convert the maximization problem into a minimization one. Integrating (2.3) by parts and using Lemma 2.3.1, we obtain the following problem:

$$(P) \qquad \min \quad J(u(\cdot),V(\cdot)) = -\sum_{k=1}^N \Bigg[\frac{A_k}{S_k} \int_0^T c(t)V_k(t)dt + \frac{1}{2S_k} \int_{]0,T]} V_k^2(t)\, dc(t)$$
$$+ \left(H_k - H_{j(k)}\right) \int_{]0,T]} \left(V_k(t) + \sum_{m\in\mathscr{M}(k)} V_m(t) \right) dc(t)$$
$$+ \left(H_k - H_{j(k)}\right) \left(A_k - \sum_{m\in\mathscr{M}(k)} A_m \right) \int_0^T c(t)dt \Bigg],$$

$$
\begin{aligned}
\text{s.t.} \qquad \qquad \dot{V}_k(t) =& A_k - u_k(t) + \sum_{m \in M(k)} u_m(t), \quad a.e.\ t \in [0, T] \\
V_k(0) =& V_k(T), \\
V_k(t) \in& [V_k^m, V_k^M], \quad \forall\, t \in [0, T] \\
u_k(t) \in& [u_k^m, u_k^M], \quad a.e.\ t \in [0, T], \quad k = \overline{1, N}.
\end{aligned}
$$

Here, $\mathscr{M}(k)$ is the set containing the indices corresponding to *all* upstream reservoirs appearing in cascade before reservoir k.

2.3.1 Particular cases

In this work, we analyze with detail two particular cases, of one and two power stations in cascade.
Consider the case of a system with one power station with reversible turbines. The Alqueva dam in Guadiana, a river in south of Portugal, is an example of that. The Alqueva dam constitutes one of the largest dams and artificial lakes, $250km$, in Western Europe.

Figure 2.4 illustrates the case of a unique station which is reversible.

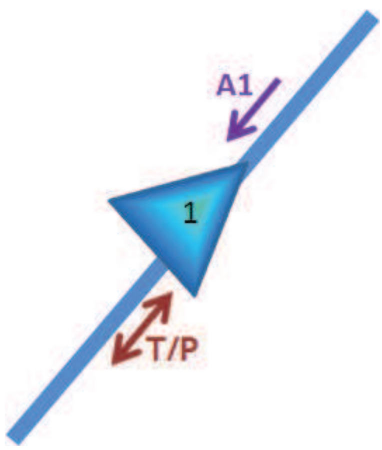


Figure 2.4: Scheme of a system with 1 power station

For this particular case problem (P) takes the form:

$$
\begin{aligned}
(P_1) \qquad \min \quad & -\left[\frac{A_1}{S_1}\int_0^T c(t)V_1(t)dt + H_1\int_{]0,T]} V_1(t)dc(t) + \frac{1}{2S_1}\int_{]0,T]} V_1^2(t)dc(t)\right. \\
& \left. + A_1H_1\int_0^T c(t)dt\right], \\
\text{s.t.} \quad & \dot{V}_1(t) = A_1 - u_1(t), \quad a.e.\ t \in [0,T] \\
& V_1(0) = V_1(T), \\
& V_1^m \le V_1(t) \le V_1^M \quad \forall\, t \in [0,T], \\
& u_1^m \le u_1(t) \le u_1^M, \quad a.e.\ t \in [0,T].
\end{aligned}
$$

Take the time horizon divided into two intervals, $[0,\tau]$ and $[\tau,T]$. The price $c(t)$ is defined as

$$
c(t) = \begin{cases} c_1, & t \in [0,\tau[\cup\{T\} \\ c_2, & t \in [\tau,T[, \end{cases} \tag{2.4}
$$

where $c_1,\ c_2$ are positive constants.

This price function reflects a two different demand periods. A high price for the energy corresponds to a high demand of energy and inversely, a low price corresponds to a low demand of energy.
With this particular price function, problem (P_1) can be written as :

$$
\begin{aligned}
(P_{1C}) \qquad \min \quad & -\frac{A_1c_1}{S_1}\int_0^\tau V_1(t)dt - \frac{A_1c_2}{S_1}\int_\tau^T V_1(t)dt + \frac{c_2-c_1}{2S_1}V_1^2(0) - \frac{c_2-c_1}{2S_1}V_1^2(\tau) \\
& + H_1(c_2-c_1)V_1(0) - H_1(c_2-c_1)V_1(\tau) - A_1H_1(\tau c_1 + (T-\tau)c_2), \\
\text{s.t.} \quad & \dot{V}_1(t) = A_1 - u_1(t),\ a.e.\ t \in [0,T] \\
& V_1(0) = V_1(T), \\
& V_1(t) \in [V_1^m, V_1^M],\ \forall\, t \in [0,T] \\
& u_1(t) \in [u_1^m, u_1^M],\ a.e.\ t \in [0,T].
\end{aligned}
$$

By state augmentation techniques, we can additionally reformulate the above problem into Mayer form (see [10]). In fact, if

$$
\begin{aligned}
W_1(t) =& V_1(t+\tau),\ t \in [0,\tau],\\
Z_1(t) =& \int_0^t V_1(t)dt,\ t \in [0,\tau],\\
Z_2(t) =& \int_0^t W_1(t)dt,\ t \in [0,\tau],
\end{aligned}
\tag{2.5}
$$

the problem (P_{1C}) can be equivalently written as:

$$
\begin{aligned}
(P_{1M}) \quad \min \quad & -\frac{A_1c_1}{S_1}Z_1(\tau) - \frac{A_1c_2}{S_1}Z_2(\tau) + \frac{c_2-c_1}{2S_1}V_1^2(0) - \frac{c_2-c_1}{2S_1}W_1^2(0)\\
& + H_1(c_2-c_1)V_1(0) - H_1(c_2-c_1)W_1(0) - A_1H_1(\tau c_1 + (T-\tau)c_2),\\
\text{s.t.} \quad & \dot{V}_1(t) = A_1 - u_1(t),\ a.e.\ t \in [0,\tau],\\
& \dot{W}_1(t) = A_1 - w_1(t),\ a.e.\ t \in [0,\tau]\\
& \dot{Z}_1(t) = V_1(t),\ a.e.\ t \in [0,\tau]\\
& \dot{Z}_2(t) = W_1(t),\ a.e.\ t \in [0,\tau]\\
& V_1(0) = W_1(\tau),\\
& V_1(\tau) = W_1(0),\\
& Z_1(0) = Z_2(0) = 0,\\
& V_1(t), W_1(t) \in [V_1^m, V_1^M],\ \forall t \in [0,\tau]\\
& u_1(t), w_1(t) \in [u_1^m, u_1^M],\ a.e.\ t \in [0,\tau].
\end{aligned}
\tag{2.6}
$$

The case of a system with 2 reservoirs in cascade is similar. Take $A_2 = 0$. We have in this case, the following formulation for problem (P):

$$
\begin{aligned}
(P_2) \quad \min \quad & -\left[\frac{A_1}{S_1}\int_0^T c(t)V_1(t)dt + (H_1 - H_2)\int_0^T V_1(t)dc(t)\right. \\
& + \frac{1}{2S_1}\int_0^T V_1^2(t)dc(t) + H_2\int_0^T (V_2(t) + V_1(t))\, dc(t) \\
& \left. + \frac{1}{2S_2}\int_0^T V_2^2(t)dc(t) + A_1H_1\int_0^T c(t)dt\right], \\
\text{s.t.} \quad & \dot{V}_1(t) = A_1 - u_1(t),\ a.e.\ t \in [0,T] \\
& \dot{V}_2(t) = u_1(t) - u_2(t),\ a.e.\ t \in [0,T] \\
& V_i(0) = V_i(T), \\
& V_i(t) \in [V_i^m, V_i^M],\ \forall\ t \in [0,T] \\
& u_i(t) \in [u_i^m, u_i^M],\ a.e.\ t \in [0,T], \quad \text{for } i = 1,2.
\end{aligned}
$$

Considering the price function $c(t)$ given by (2.4), we can write problem (P_2) as:

$$
\begin{aligned}
(P_{2C}) \quad \min \quad & -\frac{A_1c_1}{s_1}\int_0^\tau V_1(t)dt - \frac{A_1c_2}{s_1}\int_\tau^T V_1(t)dt + H_1(c_2 - c_1)V_1(0) + \frac{c_2 - c_1}{2s_1}V_1^2(0) \\
& - H_1(c_2 - c_1)V_1(\tau) - \frac{c_2 - c_1}{2s_1}V_1^2(\tau) + H_2(c_2 - c_1)V_2(0) + \frac{c_2 - c_1}{2s_2}V_2^2(0) \\
& - H_2(c_2 - c_1)V_2(\tau) - \frac{c_2 - c_1}{2s_2}V_2^2(\tau) - A_1H_1(\tau c_1 + (T - \tau)c_2), \\
\text{s.t.} \quad & \dot{V}_1(t) = A_1 - u_1(t),\ a.e.\ t \in [0,T] \qquad (2.7) \\
& \dot{V}_2(t) = u_1(t) - u_2(t),\ a.e.\ t \in [0,T] \\
& V_i(0) = V_i(T), \\
& V_i(t) \in [V_i^m, V_i^M],\ \forall\ t \in [0,T] \\
& u_i(t) \in [u_i^m, u_i^M],\ a.e.\ t \in [0,T], \quad \text{for } i = 1,2.
\end{aligned}
$$

Based on similar definitions for $W_i(t)$ and $Z_i(t),\ i = 1, 2$ (see (2.5)), the optimal control problem (P_{2C}) can be rewritten in the Mayer form:

$$
\begin{aligned}
(P_{2M}) \quad \min \quad & -\frac{A_1 c_1}{s_1} Z_1(\tau) - \frac{A_1 c_2}{s_1} Z_2(\tau) + H_1(c_2 - c_1)V_1(0) + \frac{c_2 - c_1}{2s_1} V_1^2(0) \\
& - H_1(c_2 - c_1)W_1(0) - \frac{c_2 - c_1}{2s_1} W_1^2(0) + H_2(c_2 - c_1)V_2(0) + \frac{c_2 - c_1}{2s_2} V_2^2(0) \\
& - H_2(c_2 - c_1)W_2(0) - \frac{c_2 - c_1}{2s_2} W_2^2(0) - A_1 H_1(\tau c_1 + (T - \tau)c_2), \\
s.t. \quad & \dot{V}_1(t) = A_1 - u_1(t),\ a.e.\ t \in [0, \tau] \\
& \dot{W}_1(t) = A_1 - w_1(t),\ a.e.\ t \in [0, \tau] \\
& \dot{V}_2(t) = u_1(t) - u_2(t),\ a.e.\ t \in [0, \tau] \\
& \dot{W}_2(t) = w_1(t) - w_2(t),\ a.e.\ t \in [0, \tau] \\
& \dot{Z}_1(t) = V_1(t),\ a.e.\ t \in [0, \tau] \\
& \dot{Z}_2(t) = W_1(t),\ a.e.\ t \in [0, \tau] \\
& V_i(0) = W_i(\tau), \\
& V_i(\tau) = W_i(0), \\
& Z_i(0) = 0, \\
& V_i(t), W_i(t) \in [V_i^m, V_i^M],\ \forall t \in [0, \tau] \\
& u_i(t), w_i(t) \in [u_i^m, u_i^M],\ a.e.\ t \in [0, \tau], \qquad \text{for } i = 1, 2.
\end{aligned}
\tag{2.8}
$$

These different formulations for the same problem will be used in the thesis according to convenience. The Mayer form (2.6) and (2.8), for instance, is necessary to apply the software we use to get the numerical results.

Chapter 3

Background notes

The optimal control theory emerge in the early to mid 1950's, in response to several engineering and economic problems. This theory grew rapidly and nowadays it is recognized as an important tool for the treatment of problems that occur in such diverse fields as medicine, ecology, economics and electric power production.

The optimal control theory can be seen as a generalization of the calculus of variations. It gives mathematical methods to derive control policies for a given system, in such a way that a certain optimality criterion is achieved.
This theory is grounded on two main ideas. The dynamic programming with the associated optimality principle, introduced by Bellman, and the maximum principle introduced by Pontryagin and his collaborators ([38]).
The maximum principle of Pontryagin (PMP) which is seen by many authors as the main result in optimal control theory, provides a set of necessary conditions for local optimality. In general, these conditions are not sufficient. If an existence theorem is applied and guarantees that a solution exists, then all the candidates that satisfy the necessary conditions could be compared and the optimal global solution could be chosen. However, to obtain all the candidates from the necessary conditions, can be a hard or even impossible task. Such necessary conditions can be very complex, essentially when constraints on the state and control are involved. To guarantee that a candidate is in fact an optimal solution, at least in a local sense, sufficient conditions of optimality can be of particular relevance.

In this chapter, we will present a classic optimal control problem and we will discuss some fundamental and basic tools for the problem with state constraints. For a more extensive study about these topics see [3, 10, 14, 15, 20, 22, 26, 31, 32, 44, 45, 47, 48].

3.1 Optimal Control Problem

Optimal control problems can appear with different formulations, depending on the form of the cost function, the time domain (continuous/discrete), the type of constraints and the type of variables.
Here, we don't give an exhaustive list of all the possibilities but we focus on the problem we are interested in. For more details see [10], [11] and [26].

An optimal control problem requires:

- a mathematical model of the system to be controlled;
- a cost function in a certain form (Mayer, Lagrange or Bolza);
- the specification of all constraints to be satisfied by states and controls;
- the specification of all boundary conditions on states;
- the statement of what variables are free.

In control theory a main object is a dynamic system that we consider here to be given by ordinary differential equations:

$$\dot{x} = f(t, x(t), u(t)) \quad a.e.\ t \in [S, T], \tag{3.1}$$

where $f : [S, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. The control function $u(\cdot) : [S, T] \to \mathbb{R}^m$ is usually subject to some constraint

$$u(t) \in U(t), \text{ a.e. } t \in [S, T] \qquad (U(t) \subseteq \mathbb{R}^m).$$

The state variable $x(\cdot)$ is an absolutely continuous function, with values in $\mathbb{R}^n$. Constraints on this variable can also be considered, for example, constraints at the initial time and at the final time (endpoint constraints)

$$(x(S), x(T)) \in C,$$

and/or pure state constraints

$$h(x(t)) \leq 0, \quad \forall\ t \in [S, T].$$

The optimal control theory is a very useful tool to deal with continuous time optimization problems of the following form:

$$
\begin{aligned}
(OCP) \qquad & \min \quad J(x(\cdot),u(\cdot)) = g(x(S),x(T)) + \int_S^T L(t,x(t),u(t))dt, \\
& \text{s.t.} \quad \dot{x} = f(t,x(t),u(t)), \quad a.e.\ t \in [S,T] \\
& \qquad\quad u(.) \in U(t), \quad a.e.\ t \in [S,T] \\
& \qquad\quad (x(S),x(T)) \in C, \\
& \qquad\quad h(x(t)) \leq 0, \quad \forall\, t \in [S,T],
\end{aligned}
$$

where $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f : [S,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$. U is a multifunction with $U(t) \subset \mathbb{R}^m,\ t \in [S,T]$ and $C \subset \mathbb{R}^n \times \mathbb{R}^n$ is a closed set.

The objective functional (performance criterion) may be specified in Bolza form:

$$J = g(x(S),x(T)) + \int_S^T L(t,x(t),u(t))dt,$$

or in Mayer form:

$$J = g(x(S),x(T)),$$

or in Lagrange form:

$$J = \int_S^T L(t,x(t),u(t))dt.$$

Mayer and Bolza problem formulations are theoretically equivalent.
We can recast Bolza form into Mayer form by means of a process called *state augmentation*. An additional state variable x_l is defined, the augmented state variable being now $\tilde{x} = (x_l, x)$. By introducing an additional differential equation

$$\dot{x}_l(t) = L(t,x(t),u(t)) \qquad x_l(S) = 0$$

it is possible to replace the integral term in the cost by $x_l(T)$.

Feasible pair or feasible process

A control function $u(\cdot)$ such that $u(t) \in U(t),\ a.e.\ t \in [S,T]$ is called *feasible* for problem (OCP), if the response $x(\cdot, x_0, u(\cdot))$, solution of (3.1) when $x(S) = x_0$, is defined on the interval $S \leq t \leq T$, and $u(\cdot)$ and $x(\cdot, x_0, u(\cdot))$ satisfy all the constraints of the problem in this time interval. The pair $(u(\cdot), x(\cdot))$ is then called a *feasible pair* or *feasible process*.

Optimal solution

$(\hat{u}, \hat{x})$ is an optimal solution for problem (OCP) if $(\hat{u}, \hat{x})$ is an admissible process that minimizes the cost over all admissible processes.

Strong local minimum

An admissible process $(\hat{u}, \hat{x})$ is called a strong local minimum for the optimal control problem if,

$$\exists \epsilon > 0,\ \forall (u, x) \ \text{ admissible, } \|x - \hat{x}\| < \epsilon \Rightarrow J(\hat{x}(\cdot), \hat{u}(\cdot)) \leq J(x(\cdot), u(\cdot)).$$

3.1.1 Existence of solution

The following theorem is adapted from [26] (*Theorem 4* and *Corollary 1* of chapter 4.2) for the case of fixed time interval.
We denote by C^1 the space of continuously differentiable functions.

Theorem 3.1.1.

Consider the nonlinear process in $\mathbb{R}^n$

$$\dot{x} = f(t, x, u), \qquad \textit{where} \quad f : \mathbb{R}^{1+n+m} \to \mathbb{R}^n \quad \textit{is} \quad C^1.$$

The data are as follows:

1. *The initial and final sets* X_0 *and* X_1 *are fixed, nonempty, compact sets in* $\mathbb{R}^n$*.*

2. *The control restraint set* $\Omega(t, x)$ *is a nonempty, compact set, varying continuously in* $\mathbb{R}^m$ *for* $(t, x) \in [S, T] \times \mathbb{R}^n$*.*

3. *The state constraints are (possibly vacuous)* $h^1(x) \geq 0,\ h^2(x) \geq 0,\ \cdots, h^r(x) \geq 0$*, a finite or infinite family of constraints, where* $h^1,\ h^2,\ \cdots, h^r$ *are real continuous functions on* $\mathbb{R}^n$*.*

4. *The family* $\mathscr{F}$ *of admissible controllers consists of all measurable functions* $u : [S, T] \to \mathbb{R}^m$ *such that each* $u(t)$ *has a response* $x(t)$ *on* $S \leq t \leq T$ *steering* $x(S) \in X_0$ *to* $x(T) \in X_1$ *and* $u(t) \in \Omega(t, x(t)),\ h^1(x(t)) \geq 0,\ h^2(x(t)) \geq 0,\ \cdots, h^r(x(t)) \geq 0$.

5. *The cost for each $u \in \mathscr{F}$ is*

$$C(u) = g(x(T)) + \int_S^T f^0(t, x(t), u(t))dt + \max_{S \le t \le T} \gamma(x(t))$$

where $f^0 : \mathbb{R}^{1+n+m} \to \mathbb{R}$ is a C^1 function, and $g(x)$ and $\gamma(x)$ are continuous in $\mathbb{R}^n$.

Assume

(a) *The family $\mathscr{F}$ of admissible controllers is not empty.*
(b) *There exists an uniform bound, $|x(t)| \le b$ on $[S, T]$ for all responses $x(t)$ to controllers $u \in \mathscr{F}$.*
(c) *The extended velocity set*

$$\hat{V}(t, x) = \{(f^0(t, x, u), f(t, x, u))|u \in \Omega\}$$

is convex in $\mathbb{R}^{1+n}$ for each fixed (t, x).

Then there exists an optimal controller $\hat{u}(t)$, $S \le t \le T$, in $\mathscr{F}$, minimizing $C(u)$.

This result is easily generalized for the case where g depends also on $x(S)$. In this case, we can reformulate the problem in such a way that $g(x(S), x(T))$ will depend only on the final state. For that, it is enough to add a new state variable $z \in \mathbb{R}^n$, such that $\dot{z}(t) = 0$ and define new initial and final sets

$$\bar{X}_0 = \{(x, z) : x \in X_0, x = z\} \subset X_0 \times X_0$$

$$\bar{X}_1 = \{(x, z) : x \in X_1, z \in X_0\}$$

Observe that $g(z(T), x(T)) = g(z(S), x(T)) = g(x(S), x(T))$.

3.1.2 Pontryagin's maximum principle for OCP with state constraints

The existence theorem of the previous section may guarantee that an optimal solution exists. In that case, the optimal process is among all the processes that satisfy the necessary conditions of optimality. Theorem 3.1.2 of the next page presents the Maximum Principle for the following optimal control problem (P_G) with pure state constraints.

$$(P_G)\begin{cases}\text{Minimize} & g(x(S),x(T))\\ \text{subject to} & \\ & \dot{x}(t)=f(t,x(t),u(t)),\ \text{a.e. } t\in[S,T]\\ & h_i(t,x(t))\leq 0,\ \forall\, t\in[S,T],\ i=\overline{1,r}\\ & u(t)\in U(t),\ \text{a.e. } t\in[S,T]\\ & (x(S),x(T))\in C,\end{cases}$$

where $g:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, $f:[S,T]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$, and $h_i:[S,T]\times\mathbb{R}^n\to\mathbb{R}$, $U:[S,T]\rightsquigarrow\mathbb{R}^m$ is a multifunction and $C\subset\mathbb{R}^n\times\mathbb{R}^n$ is a closed set.

Since the optimal control problem (P_G) includes pure state constraints, it requires the introduction of multipliers that are Borel measures. The norm of a measure μ is denoted by $\|\mu\|_{T.V.}=\int_{[S,T]}\mu(dt)$ where $T.V.$ means the total variation of the measure. The support of a measure μ, denoted by $supp\{\mu\}$, is the smallest closed subset $A\subset[S,T]$, such that for all relatively open subsets $B\subset[S,T]\setminus A$, we have $\mu(B)=0$. For more details about this topics see [19, 39].

Denote by H the unmaximized Hamiltonian:

$$H(t,x,p,u)=p\cdot f(t,x,u).$$

Theorem 3.1.2 that follows is a version of *Theorem 9.5.1* from R.B. Vinter [47], for the case where g and h are continuously differentiable and C is convex.
Let $C\subseteq\mathbb{R}^n$ be a nonempty, convex set, and $c\in C$. Then the *Normal Cone* to C at c, denoted by $N_C(c)$, is defined as

$$N_C(c)=\{p\in\mathbb{R}^n:\langle p,x-c\rangle\leq 0,\ \forall x\in C\}.$$

For details, see [43].

We use the notation GrU for the graph of a set-valued map U and B for the closed unit ball in $\mathbb{R}^n$ centered in the origin. The product $\mathscr{L}\times\mathscr{B}$ denotes the σ-algebra generated by the Lebesgue subsets $\mathscr{L}$ of $[a; b]$ and the Borel subsets of $\mathbb{R}$.

Theorem 3.1.2.

Let $(\hat{u}, \hat{x})$ be a local minimizer for (P_G). Assume that, for some $\delta > 0$, the hypotheses (H1) to (H4) are satisfied, namely:

H1: *$f(\cdot, x, \cdot)$ is $\mathscr{L} \times \mathscr{B}$ measurable for fixed x. There exists a Borel measurable function $k(\cdot, \cdot) : [S, T] \times \mathbb{R}^m \to \mathbb{R}$ such that $k(t, u(t))$ is integrable and*

$$|f(t, x, u) - f(t, x', u)| \leq k(t, u)|x - x'|$$

$$for\ all\ x, x' \in \hat{x}(t) + \delta B, u \in U(t), a.e.;$$

H2: *$Gr\ U$ is $\mathscr{L} \times \mathscr{B}$ measurable;*

H3: *g is continuously differentiable on $(\hat{x}(S), \hat{x}(T)) + \delta B$;*

H4: *h is continuously differentiable and there exists $K_i > 0$ such that*

$$|h_i(t, x) - h_i(t, x')| \leq K_i|x - x'|\ for\ all\ x, x' \in \hat{x}(t) + \delta B,\ t \in [S, T].$$

Assume furthermore that

S1: *$f(t, \cdot, u)$ is continuously differentiable on $\hat{x}(t) + \delta B$, for all $u \in U(t),\ a.e. t \in [S, T]$*

Then, there exist $p \in AC([S, T], \mathbb{R}^n)$, $\lambda \geq 0$, and a nonnegative Borel measure μ satisfying

(i) $(p, \mu, \lambda) \neq 0$,

(ii) the adjoint equation:

$$-\dot{p}(t) = H_x(t, \hat{x}(t), q(t), \hat{u}(t)) \quad a.e.$$

(iii) the transversality condition:

$$(p(S), -q(T)) \in \lambda \nabla g(\hat{x}(S), \hat{x}(T)) + N_C(\hat{x}(S), \hat{x}(T)),$$

(iv) the Weierstrass condition:

$$H(t, \hat{x}(t), q(t), \hat{u}(t)) = \max_{u \in U(t)} H(t, \hat{x}(t), q(t), u) \quad a.e.$$

(v) $supp\{\mu_i\} \subset I_i(\hat{x})$,

$$\text{where } q(t) = \begin{cases} p(t) + \int_{[S,t[} \sum_{i=1}^{r} \nabla_x h_r(t,x)\mu_r(ds) & t \in [S,T[\\ p(T) + \int_{[S,T]} \sum_{i=1}^{r} \nabla_x h_i(t,x)\mu_i(ds) & t = T \end{cases}$$

and $I_i(\bar{x}) = \{t \in [S,T] : h_i(t, \bar{x}(t)) = 0\}$

The function p is called the adjoint function and λ the cost multiplier.

Remark 1:
It is a simple matter to see that we may assume that $\lambda \in \{0, 1\}$ (If (p, λ, μ) serves as a set of multipliers then, for any $a > 0$, $(ap, a\lambda, a\mu)$ also serves).

Remark 2:
If problem (P_G) is in Bolza form, i.e., the cost function contains an added integral term $\int_S^T L(t, x(t), u(t))dt$, then the necessary conditions take the same form, applied with H replaced by $H(t, x, p, u) = p \cdot f(t, x, u) - \lambda L(t, x(t), u(t))$.

This can easily be deduced using a state augmentation technique. Define $\dot{z}(t) = L(t, x(t), u(t))$, $z(0) = 0$, add this new variable and constraint to the problem and express the integral term on the cost as $z(T)$. Application of the Theorem 3.1.2 to this reformulation gives the result.

Chapter 4

Sufficient conditions

In this section we propose a set of sufficient conditions of optimality for the control problem of a cascade of hydro-electric power stations set out in section 2.3. Its abstract formulation leads us to the consideration of infinite-dimensional quadratic functionals subject to linear and cone constraints. Classical sufficient conditions of optimality are validated under hypotheses that fail when local minimizers are not isolated. Here, sufficient conditions of optimality for *local minimizers* and also for *directional minimizers* are established for the abstract problem. These conditions depart from classical ones since they apply to non-isolated local minimizers if some additional conditions are satisfied. To our knowledge the treatment of sufficient conditions for *directional minimizers* is new.

Taking advantage of the particular structure of the problem of hydro-electric systems under consideration, we then deduce sufficient conditions of optimality for that problem.

We shall use the following notations: the closure of a set A, the orthogonal complement of a subspace S, the adjoint of a linear operator B and the dual space of a normed linear space Y are denoted by $\mathrm{cl}A$, $S^{\perp}$, B^* and Y^*, respectively. If L is a linear map, then $\ker L$ stands for the kernel of L; $L_2([0,T],\mathbb{R})$ represents the space of measurable functions $f:[0,T]\to\mathbb{R}$, whose square is integrable. Given a normed linear space X, B_X represents the closed unit ball centered in the origin of the space.

4.1 Local minima of quadratic functionals subject to cone constraints

Let X be a Hilbert space, Y and Z be normed spaces, $K \subset Z$ be a closed and convex cone, $V : X \to X$, $A : X \to Y$, and $C : X \to Z$, be linear bounded operators, and $v \in X$ be a vector. The operator V is symmetric. Consider the following minimization problem (Q):

$$(Q) \qquad \min \; J(x) = \frac{1}{2}\langle x, Vx\rangle + \langle v, x\rangle, \quad \text{s.t. } \; x \in \Omega = \{x \in X \mid Ax = 0, \;\; -Cx \in K\}.$$

We say that $\hat{x} \in \Omega$ is a (local) *minimizer* point for problem (Q) iff there exists $\epsilon > 0$ such that $J(x) \geq J(\hat{x})$, for all $x \in \Omega \cap (\hat{x} + \epsilon B_X)$.

We will also use the concept of *directional minimizer*. The point $\hat{x} \in \Omega$ is a *directional minimizer* point iff, for all w satisfying $\hat{x} + hw \in \Omega$, for all $h \in [0, h_w]$ where h_w is some positive constant, there exists $\epsilon_w > 0$ such that $J(\hat{x} + hw) \geq J(\hat{x})$, for all $h \in [0, \epsilon_w]$.

Our aim is to deduce sufficient conditions assuring that zero is a local minimizer for problem (Q). Set $L = \ker A \cap \ker C$. Assume the two following hypotheses:

(H1) $\langle p, Vp\rangle \geq 0$, $\forall\, p \in L$;

(H2) there exist $y^* \in Y^*$ and $z^* \in Z^*$ such that $A^*y^* + C^*z^* + v = 0$.

Note that the classical sufficient conditions of optimality in the classic Lagrangian theory and in the general mathematical programming problem (see, e.g., [22, 35]) impose the inequality $\langle p, Vp\rangle \geq (\text{const})\|p\|^2$, $p \in L$. This guarantees that zero is an isolated local minimizer. In this work, we deal with possibly non-isolated minima and, as a consequence, we need a weaker condition. The non-negativity of the quadratic functional $\langle p, Vp\rangle$ on the subspace L alone does not guarantee that zero is a local minimum, and we shall also assume that one of the following *additional* conditions is satisfied:

(C_0) $\langle z^*, Cq\rangle < 0$, for all q such that $q \in L^\perp \cap \ker A$, $-Cq \in K$ and $q \neq 0$;

(C_γ) there exists $\gamma > 0$ such that $\langle z^*, Cq\rangle \leq -\gamma\|q\|$, for all q satisfying $q \in L^\perp \cap \ker A$ and $-Cq \in K$.

Lemma 4.1.1. Assume (H1) and (H2). Let $x = p + q \in \ker A$, where $p \in L$ and $q \in L^{\perp}$. Then, the following inequality holds:

$$J(x) \geq -\langle z^*, Cq\rangle + \langle q, Vp\rangle + \frac{1}{2}\langle q, Vq\rangle.$$

Proof. Indeed, we have

$$J(x) = \frac{1}{2}\langle x, Vx\rangle + \langle v, x\rangle.$$

From this and condition (H2) we obtain

$$J(x) = -\langle z^*, Cq\rangle + \langle q, Vp\rangle + \frac{1}{2}\langle p, Vp\rangle + \frac{1}{2}\langle q, Vq\rangle.$$

Condition (H1) implies

$$J(x) \geq -\langle z^*, Cq\rangle + \langle q, Vp\rangle + \frac{1}{2}\langle q, Vq\rangle.$$

This completes the proof. ∎

Proposition 4.1.1. Assume (H1) and (H2). In addition,

(a) if condition (C_0) is satisfied, then, for all $x \in \ker A$ and $-Cx \in K$, there exists $\epsilon > 0$ such that the inequality $J(tx) \geq 0$ holds, whenever $t \in [0, \epsilon]$ (i.e., 0 is a local directional minimizer for (Q)).

(b) if condition (C_γ) is satisfied, then, for all $x \in \ker A$ and $-Cx \in K$, there exists $\epsilon > 0$ such that the inequality $J(x) \geq 0$ holds, whenever $\|x\| \leq \epsilon$ (i.e., 0 is a local minimizer for (Q)).

Proof. Write $x = p + q$, where $p \in L$ and $q \in L^{\perp}$. Since x and p are in $\ker A$, also $q \in \ker A$. From Lemma 4.1.1 and condition (C_0) we have

$$J(tx) \geq -t\langle z^*, Cq\rangle + t^2\langle q, Vp\rangle + \frac{t^2}{2}\langle q, Vq\rangle > 0,$$

whenever $t > 0$ is sufficiently small. This completes the proof of (a).

From Lemma 4.1.1 and condition (C_γ) we have

$$J(x) \geq -\langle z^*, Cq\rangle + \langle q, Vp\rangle + \frac{1}{2}\langle q, Vq\rangle \geq \|q\| \left(\gamma - \|V\| \left(\|p\| + \frac{\|q\|}{2}\right)\right) > 0,$$

whenever $\|x\|$ is sufficiently small. This completes the proof of (b). ■

Example 4.1.1. There are directional minimizers that are not minimizers.

Indeed, let $X = L_2([0,1],\mathbb{R})$ and consider the problem

$$\min\ J(x(\cdot)) = -\int_0^1 x(s)ds - \int_0^1 x^2(s)ds, \qquad \text{s.t.}\ \ x(s) \le 0.$$

Let $Z = X$, $K = \{z(\cdot) \in L_2([0,1],\mathbb{R}) \mid z(s) \ge 0,\ s \in [0,1]\}$, $A = 0$, $C = I$, and $L = \{0\}$. Condition (C_0) is satisfied with $z^* \equiv 1$. By Proposition 4.1.1, zero is a local directional minimizer. Consider the sequence

$$x_n(s) = \begin{cases} -\sqrt{n}, & s \in [0,1/n] \\ 0, & s \in]1/n,1]. \end{cases}$$

Obviously, $\|x_n(\cdot)\|_{L_2} = 1$ and $J(tx_n(\cdot)) = t\left(\frac{1}{\sqrt{n}} - t\right) \ge 0$, only if $t \le \frac{1}{\sqrt{n}}$. ■

Lemma 4.1.2. Let $M \subset X$ be a closed subspace, and let $N \subset X$ be a finite-dimensional subspace. Then $\dim\big(M \cap \big(M^\perp + N\big)\big) < +\infty$.

Proof. Let $N = \mathrm{Lin}\{e_1,\dots,e_n\}$. Denote by $\pi_M(y)$ the orthogonal projection of $y \in X$ onto M. Set $p_i = \pi_M(e_i)$, $i = \overline{1,n}$. Consider $x \in M \cap \big(M^\perp + N\big)$. Then, there exist $\xi_1,\dots,\xi_n \in \mathbb{R}$ and $q \in M^\perp$ such that

$$x = q + \sum_{i=1}^n \xi_i e_i.$$

Since

$$x = \pi_M(x) = \sum_{i=1}^n \xi_i \pi_M(e_i) = \sum_{i=1}^n \xi_i p_i,$$

we see that any $x \in M \cap \big(M^\perp + N\big)$ is a linear combination of vectors $p_1,\dots,p_n$. ■

Proposition 4.1.2. Assume in problem (Q) that $Z = \mathbb{R}^n$, $K = \mathbb{R}^n_+$, and $z^* > 0$. Then, condition (C_γ) is satisfied.

Proof. Since $(\ker C)^\perp = \mathrm{im}C^*$ is a finite-dimensional subspace, from Lemma 4.1.2 we see that the subspace

$$\ker A \cap L^\perp = \ker A \cap \mathrm{cl}\big((\ker A)^\perp + (\ker C)^\perp\big) = \ker A \cap \big((\ker A)^\perp + (\ker C)^\perp\big)$$

is finite-dimensional. Suppose that there exists a sequence $q_j \in \ker A \cap L^\perp$, $-Cq_j \in K$, $|q_j| = 1$ such that $\langle z^*, Cq_j\rangle \uparrow 0$. Without any loss of generality, q_j converges to a vector q_0. Obviously $q_0 \in \ker A \cap L^\perp$, $-Cq_0 \in K$, $\langle z^*, Cq_0\rangle = 0$, and $|q_0| = 1$. Since $z^* > 0$, we have $Cq_0 = 0$, i.e., $q_0 \in \ker A \cap \ker C = L$. Therefore we have $q_0 \in L \cap L^\perp = \{0\}$, a contradiction. ■

Return to problem (Q) in its more general setting. Denote by K^* the conjugate cone of cone K. Consider functionals $z_j^* \in K^*$, $j = \overline{1,n}$. Set $\xi_j(x) = \langle C^* z_j^*, x\rangle$ and

$$\xi(x) = (\xi_1(x), \ldots, \xi_n(x)).$$

Assume that $Y = \mathbb{R}^m$ and that the problem has now the following *special* form:

$$\begin{aligned} &\min\ J(x) = \frac{1}{2}\langle \xi(x), V\xi(x)\rangle + \langle v, \xi(x)\rangle,\\ &\text{s.t. } A\xi(x) = 0,\\ &\qquad -Cx \in K. \end{aligned}$$

Here $V : \mathbb{R}^n \to \mathbb{R}^n$, $A : \mathbb{R}^n \to \mathbb{R}^m$ and $v \in \mathbb{R}^n$.

Consider an auxiliary finite-dimensional *majorant* problem

$$\min\ J(\xi) = \frac{1}{2}\langle \xi, V\xi\rangle + \langle v, \xi\rangle, \quad \text{s.t.} \quad A\xi = 0 \text{ and } \xi \le 0.$$

Proposition 4.1.3. Assume that the following conditions are satisfied:

1. $\langle p, Vp\rangle \ge 0$, $p \in \ker A$,
2. there exist $y^* \in \mathbb{R}^m$ and $z^* \in \mathbb{R}^n$ such that $A^*y^* + z^* + v = 0$ and $z^* > 0$.

Then, $\xi = 0$ is a local minimizer for the auxiliary problem, and $x = 0$ is a local minimizer for the original problem.

Proof. From Proposition 4.1.2 we see that $\xi = 0$ is a local minimizer for the auxiliary problem. Let x be an admissible point for the original problem. If the norm of x is sufficiently small, then $|\xi(x)|$ is also small. Moreover, the inclusion $-Cx \in K$, implies the inequality $\xi(x) \le 0$. Therefore $\xi(x)$ is an admissible point for the auxiliary problem and $J(x) \ge 0$. ■

Example 4.1.2. Consider the problem

$$\min \int_0^1 x(t)dt - \left(\int_0^1 x(t)dt\right)^2 \qquad \text{s.t. } x(t) \geq 0.$$

Zero is a local minimizer.

Indeed, set $X = Z = L_2([0,1],\mathbb{R})$, $C = I$, $z^* = 1$, and

$$\xi(x) = \int_0^1 x(t)dt.$$

Obviously $\xi = 0$ is a solution to the problem

$$\min \ \xi - \xi^2, \qquad \text{s.t. } \xi \geq 0.$$

■

We shall deal now with the problem

$$\min \ J(x) + \langle g, x\rangle, \qquad \text{s.t. } x \in \Omega,$$

where $g \in X$.

Proposition 4.1.4. Assume that there exist $\epsilon > 0$, $y^* \in Y^*$ and $z^* \in K^*$ such that

1. $A^*y^* + C^*z^* + g = 0$,
2. $J(x) \geq 0$, $x \in \Omega \cap \epsilon B_X$.

Then, $J(x) + \langle g, x\rangle \geq 0$ whenever $x \in \Omega \cap \epsilon B_X$.

Proof. Indeed, if $x \in \Omega \cap \epsilon B_X$, then we have

$$J(x) + \langle g, x\rangle \geq \langle g, x\rangle = -\langle Cx, z^*\rangle \geq 0.$$

This completes the proof. ■

The second condition in Proposition 4.1.4 can be deduced from Proposition 4.1.3, for example. Indeed, consider the problem

$$\begin{aligned} \min \ I(x) &= \frac{1}{2}\langle \xi(x), V\xi(x)\rangle + \langle v, \xi(x)\rangle + \langle g, x\rangle, \\ \text{s.t. } \Lambda\xi(x) &= 0, \ Ax = 0 \ \text{ and } \ -Cx \in K. \end{aligned}$$

The following result is an immediate consequence of Propositions 4.1.3 and 4.1.4

Proposition 4.1.5. Assume that there exist $\epsilon > 0$, $y^* \in Y^*$, and $z^* \in K^*$ such that

1. $A^*y^* + C^*z^* + g = 0$,
2. conditions of Proposition 4.1.3 are satisfied.

Then, $I(x) \geq 0$ whenever $x \in \Omega \cap \epsilon B_X$.

The following example shows that, if the second condition of Proposition 4.1.3 is not satisfied, then there can exist local directional minimizers that are not local minimizers.

Example 4.1.3. Consider the problem

$$\begin{aligned} \min \ &\int_0^1 \phi(s)ds - \phi^2(1), \\ &\dot{\phi} = x, \quad \phi(0) = 0 \ \text{ and } \ \phi(s) \geq 0. \end{aligned}$$

Zero is a local directional minimizer, but not a local minimizer.

The problem can be written in the following form:

$$\begin{aligned} \min \ J(x(\cdot)) = &\int_0^1 \int_0^s x(r)drds - \left(\int_0^1 x(s)ds\right)^2, \\ &\int_0^s x(r)dr \geq 0. \end{aligned}$$

Here, $X = L_2([0,1],\mathbb{R})$, $Z = C([0,1],\mathbb{R})$, $Z^* = BV([0,1],\mathbb{R})$, $A = 0$, $C = -\int_0^s$, $K = \{z(\cdot) \in C([0,1],\mathbb{R}) \mid z(s) \geq 0, \ s \in [0,1]\}$, $L = \{0\}$, and condition (C_0) is satisfied with $z^* = d\mu(s) = ds$. Indeed, if

$$\int_0^s q(r)dr \geq 0 \quad \text{and} \quad q(\cdot) \neq 0, \quad \text{then} \quad \exists\, s \in [0,1] \ : \ \int_0^s q(r)dr > 0,$$

because

$$\int_0^s q(r)dr = 0, \quad s \in [0,1],$$

implies $q(\cdot) = 0$. By Proposition 4.1.1, zero is a local directional minimizer. Consider the sequence

$$x_n(s) = \begin{cases} 0, & s \in [0, 1-1/n[\\ \sqrt{n}, & s \in [1-1/n, 1]. \end{cases}$$

Obviously, $\|x_n(\cdot)\|_{L_2} = 1$ and $J(tx_n(\cdot)) = \dfrac{t}{n}\left(\dfrac{1}{2\sqrt{n}} - t\right) \geq 0$, only if $t \leq \dfrac{1}{2\sqrt{n}}$. ■

Zero is a local directional minimizer for this problem but it is not a local minimizer.

In the next section the optimal control problem (P) presented in section 2.3 is considered, and sufficient conditions for local directional minimizers and also for local minimizers are derived.

4.2 Sufficient Conditions of optimality for problem (P)

Motivated by the previous considerations, we deduce sufficient conditions of optimality for the problem of hydro-electric power stations.

Recall the formulation of problem (P):

$$\begin{aligned}
(P) \quad \min \quad J(u(\cdot), V(\cdot)) = - \sum_{k=1}^{N} \Bigg[& \frac{A_k}{S_k} \int_0^T c(t) V_k(t) dt + \frac{1}{2S_k} \int_{]0,T]} V_k^2(t)\, dc(t) \\
& + \left(H_k - H_{j(k)}\right) \int_{]0,T]} \left(V_k(t) + \sum_{m \in \mathscr{M}(k)} V_m(t) \right) dc(t) \Bigg] \\
& + \left(H_k - H_{j(k)}\right) \left(A_k - \sum_{m \in \mathscr{M}(k)} A_m \right) \int_0^T c(t) dt \Bigg],
\end{aligned}$$

$$\begin{aligned}
\text{s.t.} \quad & \dot{V}_k(t) = A_k - u_k(t) + \sum_{m \in M(k)} u_m(t), \ a.e.\ t \in [0,T] \\
& V_k(0) = V_k(T), \\
& V_k(t) \in [V_k^m, V_k^M], \ \forall\, t \in [0,T] \\
& u_k(t) \in [u_k^m, u_k^M], \ a.e.\ t \in [0,T].
\end{aligned}$$

First, we deduce sufficient conditions for a local directional minimizer.

Theorem 4.2.1. *Let* $(\hat{u}_k(\cdot), \hat{V}_k(\cdot))$, $k = \overline{1, N}$, *be a control process. Assume that the following conditions are satisfied:*

1. *there exist right-continuous functions* $p_k(\cdot) \in BV([0,T], \mathbb{R})$ *and* $\eta_k(\cdot) \in BV([0,T], \mathbb{R})$, $k = \overline{1, N}$, *satisfying*

$$dp_k(t) = -\frac{A_k}{S_k} c(t)dt - \left(H_k - H_{j(k)}\right) dc(t) - \sum_{l \in \mathscr{M}^{-1}(k)} \left(H_l - H_{j(l)}\right) dc(t) - \frac{\hat{V}_k(t)}{S_k} dc(t) + d\eta_k(t), \tag{4.1}$$

$$p_k(0) = p_k(T);$$

2. *the equality*

$$\max_{\substack{u_k \in [u_k^m, u_k^M], \\ k = \overline{1,N}}} \sum_{k=1}^{N} p_k(t) \left(-u_k + \sum_{m \in M(k)} u_m \right) = \sum_{k=1}^{N} p_k(t) \left(-\hat{u}_k(t) + \sum_{m \in M(k)} \hat{u}_m(t) \right)$$

   holds;

3. *the functions* $\eta_k(\cdot)$, $k = \overline{1, N}$, *satisfy the inequalities*

$$\begin{aligned} d\eta_k(t) &\leq 0, \quad \text{if} \quad \hat{V}_k(t) = V_k^m; \\ d\eta_k(t) &\geq 0, \quad \text{if} \quad \hat{V}_k(t) = V_k^M; \\ d\eta_k(t) &= 0, \quad \text{if} \quad \hat{V}_k(t) \in \left] V_k^m, V_k^M \right[; \end{aligned}$$

4. *if* $dc(t) > 0$, *then the functions* $\eta_k(\cdot)$, $k = \overline{1, N}$, *satisfy the inequalities*

$$d\eta_k(t) < 0, \quad \text{if} \quad \hat{V}_k(t) = V_k^m; \qquad d\eta_k(t) > 0, \quad \text{if} \quad \hat{V}_k(t) = V_k^M;$$

5. *if* $\hat{V}_k(t) \in \left] V_k^m, V_k^M \right[$ *for some* k, *then* $dc(t) \leq 0$.

Then, $J(\hat{u}(\cdot) + h\bar{u}(\cdot), \hat{V}(\cdot) + h\bar{V}(\cdot)) \geq J(\hat{u}(\cdot), \hat{V}(\cdot))$, *whenever* $(\hat{u}_k(\cdot) + \bar{u}_k(\cdot), \hat{V}_k(\cdot) + \bar{V}_k(\cdot))$, $k = \overline{1, N}$, *is an admissible process and* $h > 0$ *is sufficiently small.*

Note. In equation (4.1) we use the notation $d\nu(t) = f(t)\, d\varphi(t)$ to express the relationship $\nu(t) - \nu(0) = \int_{]0,t]} f(t)\, d\varphi(t)$, this integral being a Lebesgue-Stieltjes integral.

Furthermore, in conditions *3. - 5.* the inequality $d\nu(t) \geq 0$ ($\leq 0, = 0$), for $t \in E$, means that $\int_E f(t)d\nu(t) \geq 0$ ($\leq 0, = 0$) for every non-negative continuous function f.

Proof. Let $(\hat{u}_k(\cdot) + \bar{u}_k(\cdot), \hat{V}_k(\cdot) + \bar{V}_k(\cdot))$, $k = \overline{1, N}$, be an admissible process and let $h > 0$ be sufficiently small. Then, we have

$$\begin{aligned}
\Delta J =& J(\hat{u}(\cdot) + h\bar{u}(\cdot), \hat{V}(\cdot) + h\bar{V}(\cdot)) - J(\hat{u}(\cdot), \hat{V}(\cdot)) \\
=& -\sum_{k=1}^{N} \Bigg[h \int_0^T \frac{A_k}{S_k} c(t)\bar{V}_k(t)\, dt + h \int_{]0,T]} \left(\left(H_k - H_{j(k)}\right) + \frac{\hat{V}_k(t)}{S_k} \right) \bar{V}_k(t)\, dc(t) \\
&+ h \sum_{m \in \mathscr{M}(k)} \left(H_k - H_{j(k)}\right) \int_{]0,T]} \bar{V}_m(t)\, dc(t) + \frac{h^2}{2S_k} \int_{]0,T]} \bar{V}_k^2(t)\, dc(t) \Bigg].
\end{aligned}$$

Using (4.1) we get

$$\begin{aligned}
\Delta J =& \sum_{k=1}^{N} \Bigg[h \Bigg(\int_{]0,T]} \bar{V}_k(t)\, dp_k(t) + \sum_{l \in \mathscr{M}^{-1}(k)} \left(H_l - H_{j(l)}\right) \int_{]0,T]} \bar{V}_k(t)\, dc(t) - \int_{]0,T]} \bar{V}_k(t) d\eta_k(t) \Bigg) \\
&- h \sum_{m \in \mathscr{M}(k)} \left(H_k - H_{j(k)}\right) \int_{]0,T]} \bar{V}_m(t)\, dc(t) - \frac{h^2}{2S_k} \int_{]0,T]} \bar{V}_k^2(t)\, dc(t) \Bigg].
\end{aligned}$$

Observe that

$$\sum_{k=1}^{N} \sum_{l \in \mathscr{M}^{-1}(k)} = \sum_{l=1}^{N} \sum_{k \in \mathscr{M}(l)}.$$

From this we obtain

$$\Delta J = \sum_{k=1}^{N} \left[h \left(\int_{]0,T]} \bar{V}_k(t)\, dp_k(t) - \int_{]0,T]} \bar{V}_k(t)\, d\eta_k(t) \right) - \frac{h^2}{2S_k} \int_{]0,T]} \bar{V}_k^2(t)\, dc(t) \right].$$

Integrating by parts and using periodicity conditions we get

$$\Delta J = \sum_{k=1}^{N} \left[h \int_0^T p_k(t) \left(\bar{u}_k(t) - \sum_{m \in M(k)} \bar{u}_m(t) \right) dt - h \int_{]0,T]} \bar{V}_k(t)\, d\eta_k(t) - \frac{h^2}{2S_k} \int_{]0,T]} \bar{V}_k^2(t)\, dc(t) \right].$$

Taking into account conditions *2.-5.* of the theorem we get $\Delta J \geq 0$.
This completes the proof. ■

Under some additional conditions on the structure of the problem we can prove sufficient conditions for local minima. Consider a partition of the interval $[0, T]$, $0 = \tau_0 < \tau_1 < \ldots < \tau_Q = T$. Assume that the price is a piecewise constant function:

$$c(t) = c_q, \quad t \in [\tau_q, \tau_{q+1}[\,, \;\; q = \overline{0, Q-1}.$$

(We set $c_Q = c_0$.)

Theorem 4.2.2. *Let $(\hat{u}_k(\cdot), \hat{V}_k(\cdot))$, $k = \overline{1, N}$, be a control process. Assume that the following conditions are satisfied:*

1. *there exist right-continuous functions $p_k(\cdot) \in BV([0, T], \mathbb{R})$ and piecewise absolutely continuous functions $\eta_k(\cdot)$, $k = \overline{1, N}$, satisfying*

$$\begin{aligned} dp_k(t) = &-\frac{A_k}{S_k} c(t)\, dt - \left(H_k - H_{j(k)}\right) dc(t) \\ &- \sum_{l \in \mathscr{M}^{-1}(k)} \left(H_l - H_{j(l)}\right) dc(t) - \frac{\hat{V}_k(t)}{S_k} dc(t) + d\eta_k, \end{aligned}$$

$$p_k(0) = p_k(T),$$

$$\eta_k(t) = \nu_k(t) + \sum_{q=1}^{Q} \Delta\eta_k(\tau_q) H(t - \tau_q),$$

where $\nu_k(\cdot) \in AC([0, T], \mathbb{R})$, $\Delta\eta_k(\tau_q)$ are constants, and $H(\cdot)$ stands for the Heaviside step function;

2. *the equality*

$$\max_{\substack{u_k \in [u_k^m, u_k^M], \\ k=\overline{1,N}}} \sum_{k=1}^{N} p_k(t) \left(-u_k + \sum_{m \in M(k)} u_m\right) = \sum_{k=1}^{N} p_k(t) \left(-\hat{u}_k(t) + \sum_{m \in M(k)} \hat{u}_m(t)\right)$$

holds;

3. *the functions $\nu_k(\cdot)$, $k = \overline{1, N}$, satisfy the inequalities*

$$d\nu_k(t) \le 0, \;\; \text{if} \quad \hat{V}_k(t) = V_k^m, \qquad d\nu_k(t) \ge 0, \;\; \text{if} \quad \hat{V}_k(t) = V_k^M;$$

$$d\nu_k(t) = 0, \;\; \text{if} \quad \hat{V}_k(t) \in \left]V_k^m, V_k^M\right[,$$

4. *if $c_{q-1} < c_q$, for some $q = \overline{0, Q}$ then, for all $k = \overline{1, N}$, the inequalities*

$$\Delta\eta_k(\tau_q) < 0, \;\; \text{if} \quad \hat{V}_k(\tau_q) = V_k^m \quad \text{and} \quad \Delta\eta_k(\tau_q) > 0, \;\; \text{if} \quad \hat{V}_k(\tau_q) = V_k^M,$$

$$\Delta\eta_k(\tau_q) = 0, \quad if \quad \hat{V}_k(t) \in \left]V_k^m, V_k^M\right[, \quad hold;$$

5. *if* $\hat{V}_k(t) \in \left]V_k^m, V_k^M\right[$ *for some* k, *then* $dc(t) \le 0$.

Then, $J(\hat{u}(\cdot) + \bar{u}(\cdot), \hat{V}(\cdot) + \bar{V}(\cdot)) \ge J(\hat{u}(\cdot), \hat{V}(\cdot))$ *wherever* $(\hat{u}_k(\cdot) + \bar{u}_k(\cdot), \hat{V}_k(\cdot) + \bar{V}_k(\cdot))$, $k = \overline{1, N}$, *is an admissible process and* $\max_{q=\overline{0,Q}, k=\overline{1,N}} |\bar{V}_k(\tau_q)|$ *is sufficiently small.*

Observe that the conclusion of this theorem allow us to say that $(\hat{u}(\cdot), \hat{V}(\cdot))$ is a local minimizer for the problem.

Proof. Let $(\hat{u}_k(\cdot) + \bar{u}_k(\cdot), \hat{V}_k(\cdot) + \bar{V}_k(\cdot))$, $k = \overline{1, N}$, be an admissible process. Arguing as in the proof of the previous theorem we get

$$\Delta J = \sum_{k=1}^{N} \left[\int_0^T p_k(t) \left(\bar{u}_k(t) - \sum_{m \in \mathscr{M}(k)} \bar{u}_m(t) \right) dt - \int_{]0,T]} \bar{V}_k(t)\, d\nu_k(t) \right.$$
$$\left. - \sum_{q=0}^{Q} \bar{V}_k(\tau_q) \Delta\eta_k(\tau_q) - \frac{1}{2S_k} \sum_{q=0}^{Q} (c_q - c_{q-1})\, \bar{V}_k^2(\tau_q) \right] \ge 0, \tag{4.2}$$

whenever $\max_{q=\overline{0,Q}, k=\overline{1,N}} |\bar{V}_k(\tau_q)|$ is sufficiently small. ■

These sufficient conditions are formulated in terms of bounded variation functions $p_k(\cdot)$. These functions are strictly connected with multipliers $q_k(\cdot)$ of the necessary conditions of section 3.1.2. Moreover, $q(\cdot)$ absorbs in its definition the measure terms (compare (iv) and (v) of Theorem 3.1.2 with *2.* of Theorem 4.2.1 or Theorem 4.2.2).

Assume that the price $c(t)$ is a T-periodic function. We shall show that the T-periodic extension of a local optimal process on the interval $[0, T]$ to the interval $[0, ST]$ is a local optimal process on this interval. Let $S > 1$ be an integer. Consider the problem

$$(P_S) \quad \min \quad J(u(\cdot), V(\cdot)) = - \sum_{k=1}^{N} \left[\frac{A_k}{S_k} \int_0^{ST} c(t) V_k(t) dt + \frac{1}{2S_k} \int_{]0,ST]} V_k^2(t), dc(t) \right.$$
$$\left. + \left(H_k - H_{j(k)}\right) \int_{]0,ST]} \left(V_k(t) + \sum_{m \in \mathscr{M}(k)} V_m(t) \right) dc(t) \right],$$

$$\text{s.t.} \quad \dot{V}_k(t) = A_k - u_k(t) + \sum_{m \in M(k)} u_m(t), \ a.e. \ t \in [0, ST]$$

$$V_k(0) = V_k(ST),$$

$$V_k(t) \in \left[V_k^m, V_k^M\right], \ \forall \ t \in [0, ST]$$

$$u_k(t) \in \left[u_k^m, u_k^M\right], \ a.e. \ t \in [0, ST], \quad k = \overline{1, N}.$$

Theorem 4.2.3. *Let $(\hat{u}_k(\cdot), \hat{V}_k(\cdot))$, $k = \overline{1, N}$, be a control process satisfying conditions of Theorem 4.2.1 (Theorem 4.2.2) on the interval $[0, T]$. Then, its T-periodic continuation to the interval $[0, ST]$ is a local directional minimizer (local minimizer) for the above problem (P_S).*

Proof. (local directional minimizer)
Take the T-periodic functions on the interval $[0, ST]$

$$V_k^*(t) = \begin{cases} \hat{V}_k(t), & t \in [0, T] \\ \hat{V}_k(t - sT), & t \in]sT, (s+1)T], \end{cases} \qquad u_k^*(t) = \begin{cases} \hat{u}_k(t), & t \in [0, T] \\ \hat{u}_k(t - sT), & t \in]sT, (s+1)T], \end{cases}$$

where $s = \overline{1, S-1}$, $k = \overline{1, N}$.
By hypothesis,

$$c(t) = c(t - sT), \quad \forall \ t \in]sT, (s+1)T], \ s = \overline{1, S-1}$$

. Define the multipliers

$$p_k^*(t) = \begin{cases} \hat{p}_k(t), & t \in [0, T] \\ \hat{p}_k(t - sT), & t \in]sT, (s+1)T], \end{cases}$$

$$\eta_k^*(t) = \begin{cases} \hat{\eta}_k(t), & t \in [0, T] \\ \hat{\eta}_k(t - sT) + \hat{\eta}_k(sT) - \hat{\eta}_k(0), & t \in]sT, (s+1)T], \end{cases}$$

for $s = \overline{1, S-1}$, $k = \overline{1, N}$.

To prove that $(u_k^*(\cdot), V_k^*(\cdot))$, $k = \overline{1, N}$ is a local directional minimizer for problem (P_S), we check that all the conditions of the Theorem 4.2.1 are verified, with multipliers p_k^*, η_k^*. Let us start with condition *1.*. By construction, $p_k^*(\cdot)$ and $\eta_k^*(\cdot)$ are right continuous functions and belong to $BV([0,T],\mathbb{R})$. Also, for $t \in]sT,(s+1)T[$, with $s=\overline{0,S-1}$,

$$\begin{aligned}
dp_k^*(t) =& d\hat{p}_k(t-sT)\\
=& -\frac{A_k}{S_k}c(t-sT)dt - \left(H_k - H_{j(k)}\right)dc(t-sT) - \sum_{l\in\mathscr{M}^{-1}(k)}(H_l - H_{j(l)})dc(t-sT)\\
&- \frac{\hat{V}_k(t-sT)}{S_k}dc(t-sT) + d\hat{\eta}_k(t-sT)\\
=& -\frac{A_k}{S_k}c(t)dt - \left(H_k - H_{j(k)}\right)dc(t) - \sum_{l\in\mathscr{M}^{-1}(k)}(H_l - H_{j(l)})dc(t) - \frac{V_k^*(t)}{S_k}dc(t)\\
&+ d\eta_k^*(t)
\end{aligned}$$

Also,

$$p_k^*(ST) = \hat{p}_k(T) = \hat{p}_k(0) = p_k^*(0).$$

Condition *1.* is verified.

Now, let us consider condition *2.*, i.e.,

$$\max_{\substack{u_k\in\left[u_k^m,u_k^M\right]\\ k=\overline{1,N}}} \sum_{k=1}^N p_k^*(t)\left(-u_k + \sum_{m\in M(k)} u_m\right) = \sum_{k=1}^N p_k^*(t)\left(-u_k^*(t) + \sum_{m\in M(k)} u_m^*(t)\right),$$

On the interval $[0,T]$ this conditions is satisfied because $p_k^*(t) = \hat{p}_k(t)$, $u_k^*(t) = \hat{u}_k(t)$ and $u_m^*(t) = \hat{u}_m(t)$, $\forall t\in[0,T]$. On the interval $[sT,(s+1)T[$, for $s=\overline{1,(s-1)T}$, it comes

$$\begin{aligned}
\max_{\substack{u_k\in[u_k^m,u_k^M]\\ k=\overline{1,N}}} \sum_{k=1}^N p_k^*(t)\Bigg(-u_k + \sum_{m\in M(k)} u_m\Bigg) &= \max_{\substack{u_k\in[u_k^m,u_k^M]\\ k=\overline{1,N}}} \sum_{k=1}^N \hat{p}_k(t-sT)\Bigg(-u_k + \sum_{m\in M(k)} u_m\Bigg)\\
&= \sum_{k=1}^N \hat{p}_k(t-sT)\Bigg(-\hat{u}_k(t-sT)\\
&\quad + \sum_{m\in M(k)} \hat{u}_m(t-sT)\Bigg)\\
&= \sum_{k=1}^N p_k^*(t)\Bigg(-u_k^*(t) + \sum_{m\in M(k)} u_m^*(t)\Bigg)
\end{aligned}$$

and condition *2.* is verified.

In $[sT,(s+1)T]$, if $V_k^*(t)=\hat{V}_k(t-sT)=V_k^m$ then $d\hat{\eta}(t-sT)\leq 0$ and $d\hat{\eta}(t-sT)=d\eta_k^*(t)\leq 0$.

Similar arguments can be applied when $V_k^*(t)=V_k^M$ and $V_k^*\in]V_k^m,V_k^M[$, so condition *3.* follows.

Conditions *4.* and *5.* can be deduced also easily since $dc(t)=dc(t-sT)$.
In this way, we prove Theorem 4.2.3 for the case of local directional minimizer. ■

The part of proof concerning local minimizer follows with similar arguments.

Chapter 5

Examples

In this chapter we analyze the two problems, (P_1) and (P_2) presented in section 2.3. We search for a solution, using available software. Taking into account the profile of the solution given by the numerical tools, we apply some mathematical tools to validate such solution. More specifically, we use existence results to guarantee that an optimal solution exists. After that, we verify that the conditions of maximum principle of Pontryagin for optimal control problem with state constraints, presented in section 3.1.2, are satisfied. We then apply the new sufficient conditions of optimality of section 4.2, and in this way the numerical solution is completely validated. For the particular case of a system with one power station, we also prove that the use of reversible turbines always improve the profit.

5.1 System with 1 reservoir

Consider problem (P_1), for the case with one power station, presented in section 2.3.1, and its reformulation (P_{1M}) as a Mayer problem (2.6). To help the reading, problem (P_{1M}) is rewritten below.

$$
\begin{aligned}
(P_{1M}) \quad \min \quad & -\frac{A_1 c_1}{S_1} Z_1(\tau) - \frac{A_1 c_2}{S_1} Z_2(\tau) + \frac{c_2 - c_1}{2S_1} V_1^2(0) - \frac{c_2 - c_1}{2S_1} W_1^2(0) \\
& + H_1(c_2 - c_1) V_1(0) - H_1(c_2 - c_1) W_1(0) - A_1 H_1(\tau c_1 + (T - \tau) c_2),
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.} \quad & \dot{V}_1(t) = A_1 - u_1(t), \ \ a.e. \ t \in [0,\tau] \\
& \dot{W}_1(t) = A_1 - w_1(t), \ \ a.e. \ t \in [0,\tau] \\
& \dot{Z}_1(t) = V_1(t), \ \ a.e. \ t \in [0,\tau] \\
& \dot{Z}_2(t) = W_1(t), \ \ a.e. \ t \in [0,\tau] \\
& V_1(0) = W_1(\tau), \\
& V_1(\tau) = W_1(0), \\
& Z_1(0) = Z_2(0) = 0, \\
& V_1(t), W_1(t) \in [V_1^m, V_1^M], \ \ \forall t \in [0,\tau] \\
& u_1(t), w_1(t) \in [u_1^m, u_1^M], \ \ a.e. \ t \in [0,\tau].
\end{aligned}
\tag{5.1}
$$

Here, and on what follows, we assume that $u_1^m < A_1 < u_1^M$. We start by proving the existence of solution for this problem.

5.1.1 Existence of solution

The existence theorem, Theorem 3.1.1, presented in section 3.1.1, and the comments that follow such theorem, can easily be applied to problem (5.1).
The constraints $V_1(0) = W_1(\tau)$, $V_1(\tau) = W_1(0)$ are not addressed in Theorem 3.1.1. However, to have uncoupled initial and final constraint sets, we can replace the above constraints by new variables and constraints in the following way:

$$
\begin{aligned}
\dot{Y}_1(t) &= 0 & \qquad \dot{Y}_2(t) &= 0 \\
Y_1(0) &= V_1(0) & \qquad Y_2(0) &= W_1(0) \\
Y_1(\tau) &= W_1(\tau) & \qquad Y_2(\tau) &= V_1(\tau)
\end{aligned}
$$

The endpoint constraints can now be expressed as $(V_1(0), W_1(0), Z_1(0), Z_2(0), Y_1(0), Y_2(0)) \in X_0$ and $(V_1(\tau), W_1(\tau), Z_1(\tau), Z_2(\tau), Y_1(\tau), Y_2(\tau)) \in X_1$,
where
$X_0 = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 : \ V_1^m \leq x_1 \leq V_1^M, \ V_1^m \leq x_2 \leq V_1^M, \ x_1 = x_5, \ x_2 = x_6, \ x_3 = x_4 = 0\}$ and

$X_1 = \{(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{R}^6 : \; V_1^m \leq y_1 \leq V_1^M, \; V_1^m \leq y_2 \leq V_1^M, \; y_1 = y_6, \; y_2 = y_5, L_1 \leq y_3 \leq L_2, L_1 \leq y_4 \leq L_2\}$.

Constants L_1 and L_2 are lower and upper bounds for the objective function when $(Z_1(\tau), Z_2(\tau)) \in [\tau V_1^m, \tau V_1^M] \times [\tau V_1^m, \tau V_1^M]$. X_0 and X_1 are non-empty compact sets.

The control constraint set, $\Omega = [u_1^m, u_1^M] \times [u_1^m, u_1^M] \subset \mathbb{R}^2$ is also a non empty compact set.

The functions associated to the pure state constraints can be defined as

$$h_1(V_1, W_1) = V_1^M - V_1,$$

$$h_2(V_1, W_1) = V_1 - V_1^m,$$

$$h_3(V_1, W_1) = V_1^M - W_1,$$

$$h_4(V_1, W_1) = W_1 - V_1^m.$$

All the functions $h_i(V_1, W_1)$, for $i = \overline{1,4}$ are real continuous functions.

The cost function is a particular case of

$$C(u) = g(x(S), x(T)) + \int_S^T f^0(t, x(t), u(t))dt + \max_{S \geq t \geq T} \gamma(x(t))$$

where $S = 0$, $x = (V_1, W_1, Z_1, Z_2, Y_1, Y_2)$, $f^0 = \gamma = 0$ and $g(x(S), x(T))$ is the objective function of (P_{1M}), a continuous function in $\mathbb{R}^6 \times \mathbb{R}^6$.

The family of admissible controllers is not empty, since $u_1(t) = w_1(t) = A_1, \; \forall t \in [0, \tau]$ and $V_1(0) = W_1(0) = V_1^M$, leads to an admissible process for the problem.

The existence of a uniform upper bound for all admissible trajectories is an immediate consequence of the constraints $V_1(t), W_1(t) \in [V_1^m, V_1^M], \forall t \in [0, T]$.

Since $f^0(t, x, u) = 0$, the extended velocity set is

$$F(x, t) = \{(f(t, x, u), 0) | u \in \Omega\}.$$

For each fixed (t, x) the components of $f(t, x, u) = (A_1 - u_1, A_1 - w_1, V_1, W_1, 0, 0)$ depend linearly on u and Ω is convex. So, the extended velocity set, $F(t, x)$, is convex for each (t, x).

In conclusion, there exists an optimal control $\hat{u}(t)$, $0 \leq t \leq \tau$, minimizing $C(u)$ among all admissible controls.

5.1.2 Numerical Results

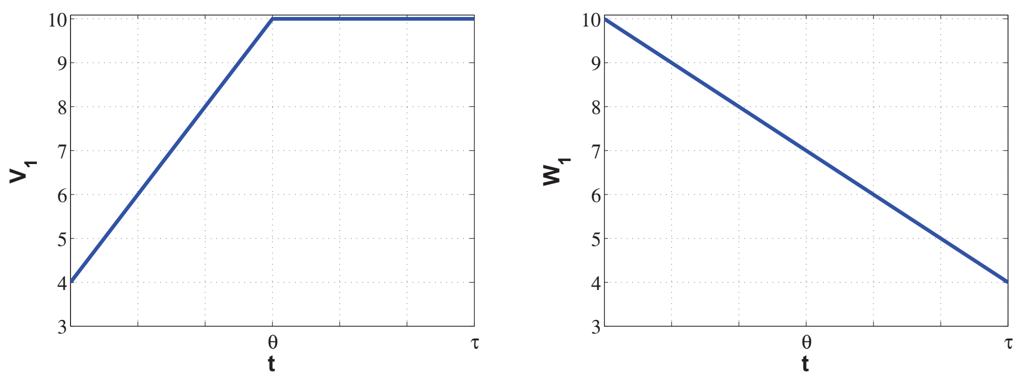
Problem (5.1) was numerically treated, for particular data, with some software and interfaces. It was used the optimization package from book [45], as well as, the interface ICLOCS [16] (`http://www.ee.ic.ac.uk/ICLOCS/`) and AMPL (`http://www.ampl.com/`) which use IPOPT (a software package for large-scale nonlinear optimization). All these softwares are designed to find (local) solutions of mathematical optimization problems.

The following data was used:

$$\begin{array}{lllll} u_1^m = -1 & u_1^M = 2 & V_1^m = 3 & V_1^M = 10 & A_1 = 1 \\ c_1 = 2 & c_2 = 5 & H_1 = 3 & S_1 = 100 & \tau = 6 \end{array} \tag{5.2}$$

Figure 5.1 shows the numerical solution obtained for this problem, with the optimization package from [45].

This software is configured to solve optimal control problems in a regime of dialogue with the computer. The user has the possibility to select the optimization methods during the optimization process (e.g. simplex, conjugated gradient, Newton's method), the possibility of choosing a penalty coefficient (to obtain a feasible solution this coefficient must be large enough) and the numerical precision of the selected method.



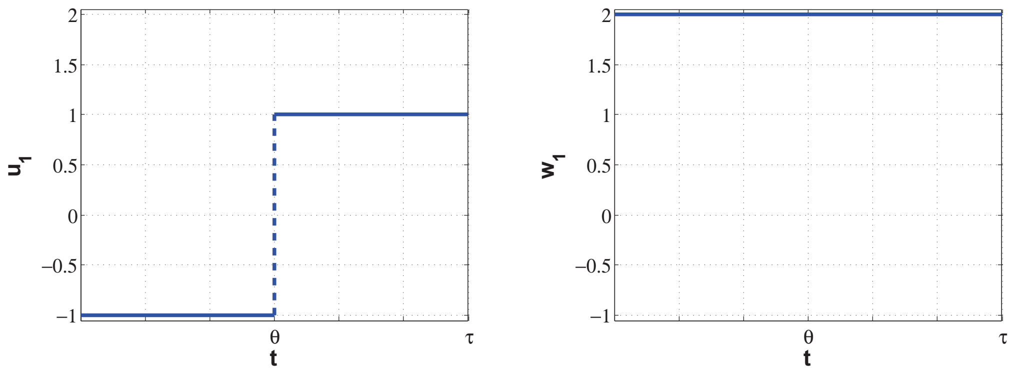


Figure 5.1: Numerical solution

As we can see in the picture, $V_1(t)$ reaches the boundary at an instant $t = \theta$ and stays there until the end. The trajectory $W_1(t)$ starts in V_1^M and decreases until attaining the value $V_1(0)$. The control $u_1(t)$ pumps until $t = \theta$, being responsible for the increase of $V_1(t)$. After $t = \theta$, it takes the value A_1 and $V_1(t)$ lies on the boundary. $w_1(t)$ is equal to the maximum value allowed in all the interval $[0, \tau]$.

5.1.3 Necessary conditions of optimality

In a first step, and to consolidate the numeric results described in the previous section, necessary conditions of optimality were used. These conditions, only necessary, do not ensure that the given solution is optimal, they merely give more confidence to that statement. In its turn, the sufficient conditions which we deal with in the next section, validate completely the solution as locally optimal. So, the results of this section do not contribute for the final conclusion of optimality, once sufficient conditions are successfully applied. They are presented as an illustrative way of working out the information contained in the necessary conditions.

The numerical solution can be expressed as:

$$\hat{V}_1(t) = \begin{cases} V_1(0) + (A_1 - u_1^m)t, & t \in [0, \theta[\\ V_1^M, & t \in [\theta, \tau], \end{cases} \qquad \hat{W}_1(t) = V_1^M + (A - u_1^M)t, \ \forall t \in [0, \tau],$$

$$\hat{u}_1(t) = \begin{cases} u_1^m, & t \in [0, \theta[\\ A_1, & t \in [\theta, \tau], \end{cases} \qquad \hat{w}_1(t) = u_1^M, \ \forall t \in [0, \tau].$$

Using the fact that

$$\hat{V}_1(\theta) = V_1^M = \hat{V}_1(0) + (A_1 - u_1^m)\theta$$
$$\hat{V}_1(0) = \hat{W}_1(\tau) = V_1^M + (A_1 - u_1^M)\tau$$

we get

$$V_1^M = V_1^M + (A_1 - u_1^M)\tau + (A_1 - u_1^m)\theta \;\Rightarrow\; \theta = \frac{\tau(u_1^M - A_1)}{A_1 - u_1^m}.$$

We now check that the NCO as stated in Theorem 3.1.2 are satisfied with a set of multipliers in normal form, i.e. with $\lambda = 1$. In this case, condition (i) of this theorem is trivially satisfied.

Observe that

$(V_1(0), W_1(0), Z_1(0), Z_2(0), V_1(\tau), W_1(\tau), Z_1(\tau), Z_2(\tau)) \in C = \{(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) : y_1 = y_6, y_2 = y_5 \text{ and } y_3 = y_4 = 0\}$ and

$N_C = \{(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) : y_1 = -y_6,\; y_2 = -y_5 \text{ and } y_7 = y_8 = 0\}$.

According to the NCO there exist absolutely continuous functions p_i and regular Borel measures $\mu_i,\; i = \overline{1,4}$ such that

(ii) $$p_1(t) = p_1(0) - p_3(0)t$$
$$p_2(t) = p_2(0) - p_4(0)t$$
$$p_3(t) = p_3(0)$$
$$p_4(t) = p_4(0)$$

(iii) $$p_1(0) = p_2(\tau) + \mu_4\{[0,\tau]\} + H_1(c_2 - c_1) + \frac{c_2 - c_1}{S_1}\hat{V}_1(0)$$
$$p_2(0) = p_1(\tau) + \mu_2\{[0,\tau]\}[-H_1(c_2 - c_1) - \frac{c_2 - c_1}{S_1}\hat{W}_1(0)$$
$$p_3(0) = \frac{c_1 A_1}{S_1}$$
$$p_4(0) = \frac{c_2 A_1}{S_1}$$

(iv)

$$H(\hat{V}_1(t), \hat{W}_1(t), q(t), \hat{u}_1(t), \hat{w}_1(t)) =$$
$$= \max_{u_1, w_1 \in U} \left(\frac{c_1 A_1}{S_1}t - p_1(0) - \mu_2\{[0,t[\}\right) u_1(t) + \left(\frac{c_2 A_1}{S_1}t - p_2(0) - \mu_4\{0\}\right) w_1(t)$$

(v)

$$\mu_1([0,\tau]) = 0 \qquad \mu_2([0,\tau]) = \mu_2([\theta,\tau])$$
$$\mu_3([0,\tau]) = 0 \qquad \mu_4([0,\tau]) = \mu_4\{0\}.$$

The analysis of these conditions is made successively on subintervals that make up the whole interval $[0,\tau]$.

- In $[0,\theta[$:

  In this interval we can write

$$H(\hat{V}_1(t), \hat{W}_1(t), q(t), \hat{u}_1(t), \hat{w}_1(t)) =$$
$$= \max_{u_1,w_1\in U} \left(\frac{c_1A_1}{S_1}t - p_1(0)\right) u_1(t) + \left(\frac{c_2A_1}{S_1}t - p_2(0) - \mu_4\{0\}\right) w_1(t).$$

  We also have $\hat{u}_1(t) = u_1^m$ and $\hat{w}_1(t) = u_1^M$. So, we deduce that

$$\frac{c_1A_1}{S_1}t - p_1(0) \le 0,\ \forall t\in[0,\theta[\ \Rightarrow\ p_1(0) \ge \frac{c_1A_1}{S_1}\theta \tag{5.3}$$

$$\frac{c_2A_1}{S_1}t - p_2(0) - \mu_4\{0\} \ge 0\ \Rightarrow\ \frac{c_2A_1}{S_1}t - p_2(0) \ge \mu_4\{0\} \ge 0\quad \forall t\in[0,\theta[$$
$$\Rightarrow\ p_2(0) \le 0 \tag{5.4}$$

- In $[\theta,\tau[$:

  In this interval we can write

$$H(\hat{V}_1(t), \hat{W}_1(t), q(t), \hat{u}_1(t), \hat{w}_1(t)) =$$
$$= \max_{u_1,w_1\in U} \left(\frac{c_1A_1}{S_1}t - p_1(0) - \mu_2\{[\theta,t[\}\right) u_1 + \left(\frac{c_2A_1}{S_1}t - p_2(0) - \mu_4\{0\}\right) w_1.$$

  Since $\hat{u}_1(t) = A_1$, we deduce that

$$\frac{c_1A_1}{S_1}t - p_1(0) - \mu_2\{[\theta,t[\} = 0.$$

Taking the limit, when $t \to \tau^-$, it comes

$$\frac{A_1 c_1}{S_1}\tau - p_1(0) = \mu_2\{[\theta, \tau[\}. \tag{5.5}$$

From (ii), (iii) and (5.4), we can write:

$$p_2(0) = p_1(0) - \frac{A_1 c_1}{S_1}\tau + \mu_2\{[\theta, \tau]\} - H_1(c_2 - c_1) - \frac{c_2 - c_1}{S_1}V_1^M \leq 0. \tag{5.6}$$

Using the equality (5.5) and (5.6), it comes

$$\mu_2\{[\theta, \tau[\} + \mu_2\{\tau\} = \frac{A_1 c_1}{S_1}\tau - p_1(0) + \mu_2\{\tau\} = p_2(0) - p_1(0) + \frac{A_1 c_1}{S_1}\tau + H_1(c_2 - c_1) + \frac{c_2 - c_1}{S_1}V_1^M$$

$$\Rightarrow \mu_2\{\tau\} = p_2(0) + H_1(c_2 - c_1) + \frac{c_2 - c_1}{S_1}V_1^M. \tag{5.7}$$

Also, from (ii) and (iii), we obtain

$$p_1(0) = p_2(0) - \frac{A_1 c_2}{S_1}\tau + \mu_4\{0\} + H_1(c_2 - c_1) + \frac{c_2 - c_1}{S_1}\hat{V}_1(0). \tag{5.8}$$

Using (5.7) and (5.8) we get

$$\mu_2\{\tau\} + \mu_4\{0\} = \frac{A_1 c_2}{S_1}\tau + \frac{c_2 - c_1}{S_1}(V_1^M - \hat{V}_1(0)) + p_1(0) > 0$$

$\Rightarrow \mu_2\{\tau\}$ and $\mu_4\{0\}$ not simultaneously null.

Assuming that μ_2 does not have an atom in $t = \tau$ ($\mu_2\{\tau\} = 0$), we obtain

$$\mu_4\{0\} = \frac{A_1 \tau c_2}{S_1} + p_1(0) + \frac{c_2 - c_1}{S_1}(V_1^M - \hat{V}_1(0))$$

and

$$p_2(0) = -H_1(c_2 - c_1) - \frac{c_2 - c_1}{S_1}V_1^M.$$

From (5.4), we know that $-p_2(0) \geq \mu_4\{0\}$. Then

$$p_1(0) \leq -\frac{A_1 c_2 \tau}{S_1} + H_1(c_2 - c_1) + \frac{c_2 - c_1}{S_1}\hat{V}_1(0). \tag{5.9}$$

Joining the information from (5.3) and (5.9), we can write

$$p_1(0) \in \left[\frac{A_1\theta c_1}{S_1}; \quad -\frac{A_1 c_2 \tau}{S_1} + H_1(c_2 - c_1) + \frac{c_2 - c_1}{S_1}\hat{V}_1(0) \right].$$

Taking $p_1(0) = \dfrac{A_1\theta c_1}{S_1}$, we find a set of multipliers that respect the optimality necessary conditions:

$\lambda = 1$

$\mu_1\{[0,\tau]\} \equiv \mu_3\{[0,\tau]\} \equiv 0$

$$\mu_2\{[0,t[\} \equiv \begin{cases} 0, & t \in [0,\theta[\\ \dfrac{A_1 c_1}{S_1}(t-\theta), & t \in [\theta,\tau] \end{cases}$$

$$\mu_4\{[0,t[\} \equiv \frac{A_1\tau c_2}{S_1} + \frac{A_1\theta c_1}{S_1} + \frac{c_2 - c_1}{S_1}(V_1^M - \hat{V}_1(0)), \qquad \forall t \in [0,\tau]$$

$$p_1(t) = \frac{A_1 c_1}{S_1}(\theta - t); \qquad p_2(t) = -H_1(c_2 - c_1) - \frac{c_2 - c_1}{S_1}V_1^M - \frac{c_2 A_1 t}{S_1}$$

$$p_3(t) = \frac{c_1 A_1 t}{S_1}; \qquad p_4(t) = \frac{c_2 A_1 t}{S_1}$$

where $\quad \hat{V}_1(0) = V_1^M + (A_1 - u_1^M)\tau \quad$ and $\quad \theta = \dfrac{(u_1^M - A_1)}{A_1 - u_1^m}\tau.$

Observe that such set of multipliers was found with no particularization of values for the parameters. This means that if $c_2 > c_1$, $\hat{V}_1(0)$ is admissible and $0 < \theta < \tau$, there is always a set of multipliers satisfying the NCO for the admissible solution with the profile we considered and θ defined as above. In particular for data (5.2) it comes $\hat{V}_1(0) = 4$ and $\theta = 3$.

5.1.4 Sufficient conditions of optimality

In chapter 4, sufficient conditions of optimality (Theorem 4.2.2) are developed for the optimal control problem (P) presented in section 2.3, with N hydro-electric power stations in cascade. Here we specify such conditions for our problem with 1 reservoir and the price $c(t)$ given by (2.4). We verify that these conditions are satisfied for the reference pair $(\hat{u}(\cdot), \hat{V}(\cdot))$ of the previous sections. The formulation under consideration is now (P_1) (equivalent form to (5.1)) rewritten below

$$
\begin{aligned}
(P_1) \qquad \min \quad & -\left[\frac{A_1}{S_1}\int_0^T c(t)V_1(t)dt + H_1\int_{]0,T]} V_1(t)dc(t) + \frac{1}{2S_1}\int_{]0,T]} V_1^2(t)dc(t)\right], \\
\text{s.t.} \quad & \dot{V}_1(t) = A_1 - u_1(t), \ \ a.e. \ t \in [0,T] \\
& V_1(0) = V_1(T), \\
& V_1^m \le V_1(t) \le V_1^M, \ \ \forall\, t \in [0,T] \\
& u_1^m \le u_1(t) \le u_1^M, \ \ a.e. \ t \in [0,T].
\end{aligned}
$$

We prove that under some assumptions on the data, the profile of the optimal trajectory, found by numerical results for particular data (see Figure 5.1), is maintained in a more general setting.

Theorem 5.1.1. *Let $V_1^m < V_1^M - \theta(A_1 - u_1^m)$, where*

$$\theta := \frac{u_1^M - A_1}{A_1 - u_1^m}(T - \tau)$$

and $p_\tau \le 0$, where

$$p_\tau = (c_1 - c_2)\left(H_1 + \frac{V_1^M}{S_1}\right) + \frac{c_2 u_1^M}{S_1}(T - \tau) + \frac{\theta c_1 u_1^m}{S_1}.$$

Assume that $\theta < \tau$. Then, the process $(\hat{u}_1(\cdot), \hat{V}_1(\cdot))$, where

$$\hat{u}_1(t) = \begin{cases} u_1^m, & t \in [0,\theta] \\ A_1, & t \in]\theta,\tau] \\ u_1^M, & t \in]\tau, T] \end{cases} \quad \text{and } \hat{V}_1(t) = \begin{cases} V_1^M + (t-\theta)\left(A_1 - u_1^m\right), & t \in [0,\theta] \\ V_1^M, & t \in]\theta,\tau] \\ V_1^M + (t-\tau)\left(A_1 - u_1^M\right), & t \in]\tau, T] \end{cases}$$

is locally optimal.

Proof. Consider the functions

$$\eta(t) = \begin{cases} 0, & t \in [0,\theta[\\ \dfrac{c_1(t-\theta)A_1}{S_1}, & t \in [\theta,\tau[\\ \dfrac{c_1(\tau-\theta)A_1}{S_1} + \Delta\eta, & t \in [\tau,T] \end{cases} \quad \text{and} \quad p(t) = \begin{cases} \dfrac{(\theta-t)c_1A_1}{S_1}, & t \in [0,\theta[\\ 0, & t \in [\theta,\tau[\\ p_\tau - \dfrac{(t-\tau)c_2A_1}{S_1}, & t \in [\tau,T[\\ \dfrac{\theta c_1 A_1}{S_1}, & t = T. \end{cases}$$

where

$$\Delta\eta = \theta\frac{c_1A_1}{S_1} + (T-\tau)\frac{c_2A_1}{S_1} + \frac{c_2-c_1}{S_1}(\hat{V}_1(\tau)-\hat{V}_1(0)).$$

If $p_\tau \leq 0$, then, since $\Delta\eta > 0$, the conditions of Theorem 4.2.2 are satisfied. $(\hat{u}_1(\cdot),\hat{V}_1(\cdot))$ is a local minimizer for the problem. Note that, in this case, from (4.2) we have

$$\Delta J = J(\hat{u}(\cdot)+\bar{u}(\cdot),\hat{V}(\cdot)+\bar{V}(\cdot)) - J(\hat{u}(\cdot),\hat{V}(\cdot)) \geq -\Delta\eta\bar{V}_1(\tau) - \frac{c_2-c_1}{2S_1}\bar{V}_1^2(\tau) \geq 0,$$

whenever $\hat{V}_1(\cdot)+\bar{V}_1(\cdot)$ is admissible and $\bar{V}_1(\tau)$ is sufficiently small. □

Note that, the statement of the last theorem is made without particularization of the values for the parameters. It suffices that the assumptions of the theorem are satisfied, to validate such optimal solution profile.

5.1.5 Pumping action

In this section we use the equivalent formulation (P_{1C}) (see section 2.3.1) of the problem (5.1) where the objective function takes the form:

$$\begin{aligned} J(u_1(\cdot),V_1(\cdot)) = {} & H_1(c_2-c_1)V_1(0) + \frac{1}{2S_1}(c_2-c_1)\,(V_1(0))^2 - \frac{c_1A_1}{S_1}\int_0^\tau V_1(t)\,dt \\ & - \frac{c_2A_1}{S_1}\int_\tau^T V_1(t)\,dt + H_1(c_1-c_2)V_1(\tau) + \frac{1}{2S_1}(c_1-c_2)V_1^2(\tau). \end{aligned}$$

We are assuming that the affluence A_1 satisfies $u_1^m < A_1 < u_1^M$. For the case $A_1 = u_1^M$, it is easy to conclude that the optimal solution is $\hat{V}_1(t) = V_1^M,\ \forall t$. In this case no pumping occurs. It would be expected that for values of A_1 very close to u_1^M, $A_1 < u_1^M$, no pumping would occur also. Here we show that it is not the case. We show that the use of reversible turbines always improves the profit. This is established in Theorem

5.1.2. Before that two helping lemmas are deduced.
It is assumed that $u_1^m < 0 < A_1 < u_1^M$ and τ, c_1, c_2 are constants such that $\tau \in]0, T[$ and $c_1 < c_2$.

Lemma 5.1.1. *If $(\hat{u}_1(\cdot), \hat{V}_1(\cdot))$ is an optimal process for problem (P_{1C}) and $\hat{V}_1(0) = V_1^M$, then $\hat{V}_1(t) = V_1^M$, for all $t \in [0, T]$.*

Proof. Let $(\hat{u}_1(\cdot), \hat{V}_1(\cdot))$ be an optimal process for problem (P_{1C}) and $\hat{V}_1(0) = V_1^M$. Then, for every admissible process $(u_1(\cdot), V_1(\cdot))$ satisfying $V_1(0) = V_1^M$ we have,

$$\begin{aligned}
J(u_1(\cdot), V_1(\cdot)) &= H_1(c_2 - c_1)V_1^M + \frac{1}{2S_1}(c_2 - c_1)\left(V_1^M\right)^2 - \frac{c_1 A_1}{S_1}\int_0^\tau V_1(t)\,dt \\
&\quad - \frac{c_2 A_1}{S_1}\int_\tau^T V_1(t)\,dt + H_1(c_1 - c_2)V_1(\tau) + \frac{1}{2S_1}(c_1 - c_2)\left(V_1(\tau)\right)^2 \\
&\geq H_1(c_2 - c_1)V_1^M + \frac{1}{2S_1}(c_2 - c_1)\left(V_1^M\right)^2 - \frac{c_1 A_1}{S_1}\int_0^\tau V_1^M\,dt - \frac{c_2 A_1}{S_1}\int_\tau^T V_1^M\,dt \\
&\quad + H_1(c_1 - c_2)V_1^M + \frac{1}{2S_1}(c_1 - c_2)\left(V_1^M\right)^2.
\end{aligned}$$

If $V_1(t) < V_1^M$ on some subset, then the above inequality is strict. Since the inequality is still valid for $(u_1(\cdot), V_1(\cdot)) = (\hat{u}_1(\cdot), \hat{V}_1(\cdot))$ and this process is optimal, we obtain $\hat{V}_1(t) = V_1^M$. □

Lemma 5.1.2. *The optimal process $(\hat{u}_1(\cdot), \hat{V}_1(\cdot))$ for problem (P_{1C}) satisfies $\hat{V}_1(0) < V_1^M$.*

Proof. Assume that $\hat{V}_1(0) = V_1^M$. Then, by Lemma 5.1.1 we have $\hat{V}_1(t) \equiv V_1^M$. Moreover, $\hat{u}_1(t) \equiv A_1$.

Consider the family of processes $(u_y(\cdot), V_y(\cdot))$, defined as

$$u_y(t) := \begin{cases} u_1^m, & \text{if } t \in [0, \alpha[\\ A_1, & \text{if } t \in [\alpha, \beta[\\ u_1^M, & \text{if } t \in [\beta, T[\end{cases} \quad \text{and } V_y(t) := \begin{cases} y + (A_1 - u_1^m)t, & \text{if } t \in [0, \alpha[\\ V_1^M, & \text{if } t \in [\alpha, \beta[\\ V_1^M + (A_1 - u_1^M)(t - \beta), & \text{if } t \in [\beta, T[\end{cases}$$

with

$$\alpha := \frac{V_1^M - y}{A_1 - u_1^m} \quad \text{and} \quad \beta := T - \frac{V_1^M - y}{u_1^M - A_1},$$

where $V_1^M - y > 0$ is small enough to satisfy $y > V_1^m$ and $\alpha < \tau < \beta$.

Then, we have

$$
\begin{aligned}
&J\left(u_y\left(\cdot\right), V_y\left(\cdot\right)\right) - J\left(\hat{u}_1\left(\cdot\right), \hat{V}_1\left(\cdot\right)\right) = \\
=& -\frac{c_1 A_1}{S_1}\left(y\alpha + \left(A_1 - u_1^m\right)\frac{\alpha^2}{2} - V_1^M\alpha\right) - \frac{c_2 A_1}{S_1}\left(\left(A_1 - u_1^M\right)\frac{\left(T-\beta\right)^2}{2}\right) \\
&+ H_1\left(c_2 - c_1\right)\left(y - V_1^M\right) + \frac{1}{2S_1}\left(c_2 - c_1\right)\left(y^2 - \left(V_1^M\right)^2\right) \\
=& \frac{c_1 A_1}{2S_1}\alpha\left(V_1^M - y\right) - \frac{c_2 A_1}{S_1}\left(\left(A_1 - u_1^M\right)\frac{\left(T-\beta\right)^2}{2}\right) + H_1\left(c_2 - c_1\right)\left(y - V_1^M\right) \\
&+ \frac{1}{2S_1}\left(c_2 - c_1\right)\left(y^2 - \left(V_1^M\right)^2\right) \\
=& \frac{c_1 A_1}{2S_1}\frac{\left(V_1^M - y\right)^2}{A_1 - u_1^m} + \frac{c_2 A_1}{2S_1}\frac{\left(V_1^M - y\right)^2}{u_1^M - A_1} + H_1\left(c_2 - c_1\right)\left(y - V_1^M\right) \\
&+ \frac{1}{2S_1}\left(c_2 - c_1\right)\left(y^2 - \left(V_1^M\right)^2\right) \\
=& \left(V_1^M - y\right)G\left(y\right),
\end{aligned}
$$

where

$$G(y) := \frac{c_1 A_1}{2S_1}\frac{V_1^M - y}{A_1 - u_1^m} + \frac{c_2 A_1}{2S_1}\frac{V_1^M - y}{u_1^M - A_1} - H_1(c_2 - c_1) - \frac{1}{2S_1}(c_2 - c_1)(y + V_1^M).$$

Since $G(y)$ is linear in y and $G(V_1^M) = -H_1(c_2 - c_1) - \frac{1}{S_1}(c_2 - c_1)V_1^M < 0$, we have

$$J(u_y(\cdot), V_y(\cdot)) < J(\hat{u}_1(\cdot), \hat{V}_1(\cdot))$$

whenever $y < V_1^M$ is close to V_1^M, a contradiction. □

Theorem 5.1.2. *Let $(\hat{u}_1(\cdot), \hat{V}_1(\cdot))$ be an optimal process for problem (5.1). Then, $\hat{u}_1(t) < 0$ on some non null measure set.*

Proof. By Lemma 5.1.2 we have $\hat{V}_1(0) < V_1^M$. Consider the set of admissible trajectories $V_1(\cdot)$, satisfying $V_1(0) = \hat{V}_1(0)$. The associated cost is

$$J(u_1(\cdot),V_1(\cdot)) = H_1(c_2-c_1)\hat{V}_1(0) + \frac{1}{2S_1}(c_2-c_1)\hat{V}_1^2(0) - \frac{c_1A_1}{S_1}\int_0^\tau V_1(t)\,dt \\ - \frac{c_2A_1}{S_1}\int_\tau^T V_1(t)\,dt + H_1(c_1-c_2)V_1(\tau) + \frac{1}{2S_1}(c_1-c_2)V_1^2(\tau).$$

Suppose that $\hat{u}_1(t) \geq 0$, $a.e.\, t \in [0,T]$. Take $\theta = \min\{s \mid \hat{V}_1(s) = \max_{t\in[0,\tau]} \hat{V}_1(t)\}$. Since $c_2 > c_1$, we have $\max_{t\in[0,T]} \hat{V}_1(t) \geq \hat{V}_1(0)$. Consider the process $(\tilde{u}_1(\cdot), \tilde{V}_1(\cdot))$, with $\tilde{V}_1(0) = \hat{V}_1(0)$ and

$$\tilde{u}_1(t) := \begin{cases} u_1^m, & \text{if } t \in [0,\alpha[\\ A_1, & \text{if } t \in [\alpha,\theta[\\ \hat{u}_1(t), & \text{if } t \in [\theta,T]. \end{cases}$$

Here, α is chosen to satisfy $\tilde{V}_1(\alpha) = \hat{V}_1(\theta)$. Therefore, we have $A_1\theta - \int_0^\theta \hat{u}_1(\tau)\,d\tau = \alpha(A_1 - u_1^m)$. Hence,

$$\alpha = \frac{A_1\theta - \int_0^\theta \hat{u}_1(\tau)\,d\tau}{A_1 - u_1^m} \leq \frac{A_1\theta}{A_1 - u_1^m} < \theta.$$

Since $\hat{u}_1(t) \geq 0$, by definition of $\tilde{u}_1$ we have $\tilde{V}_1(t) \geq \hat{V}_1(t)$ on $[0,\theta]$, and $\tilde{V}_1(t) = \hat{V}_1(t)$ on $[\theta,T]$. Comparing cost functions we obtain

$$J(\tilde{u}_1(\cdot),\tilde{V}_1(\cdot)) - J(\hat{u}_1(\cdot),\hat{V}_1(\cdot)) = -\frac{c_1A_1}{S_1}\int_0^\alpha (\tilde{V}_1(t) - \hat{V}_1(t))\,dt - \frac{c_1A_1}{S_1}\int_\alpha^\theta (\hat{V}_1(\theta) - \hat{V}_1(t))dt < 0,$$

a contradiction. $\square$

Thus, the use of reversible turbines always improves the profit.

The following picture illustrates this fact. Take the data given in (5.2). We have $\hat{V}_1(0) = 4$ and $\theta = 3$. The condition $p_\tau < 0$ is satisfied and the optimal process, as was already shown, has the following profile (see Fig. 5.2).

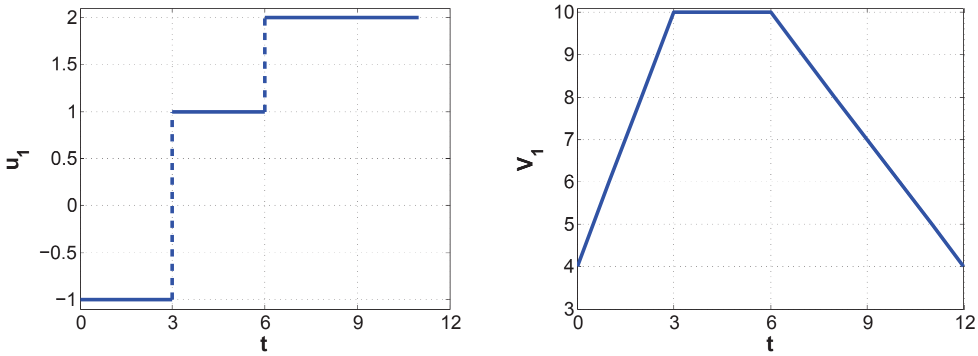


Figure 5.2: Optimal process - $u_1^m < 0$

Note that $\hat{u}_1(t) = -1,\ t \in [0, 3]$, i.e., water is pumped on this interval. In this way, the station accumulates water when the price is low to be used when the price is higher.

This solution is optimal when compared to any particular admissible solutions satisfying $u_1(t) \geq 0,\ \forall t$.

Consider the case $u_1^m = 0$. Fig. 5.3 shows the numerical optimal solution obtained.

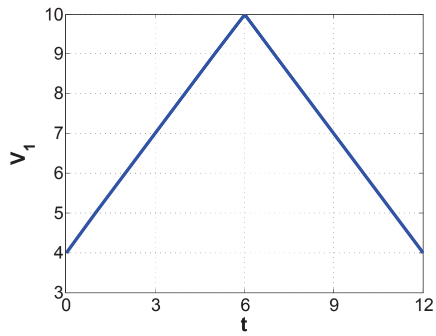


Figure 5.3: Optimal process - $u_1^m = 0$

In this case, we have

$$\hat{u}_1(t) = \begin{cases} 0, & t \in]0, \tau] \\ u_1^M, & t \in]\tau, T] \end{cases} \quad \text{and} \quad \hat{V}_1(t) = \begin{cases} V_1(0) + A_1 t, & t \in [0, \tau] \\ V_1^M + (t - \tau)(A_1 - u_1^M), & t \in]\tau, T]. \end{cases}$$

Table 5.1 shows the cost for both situations, $u_1^m < 0$ and $u_1^m = 0$.

	Case $u_1^m = 0$	Case $u_1^m = -1$
Cost	-58.2	-58.4

Table 5.1: Costs obtained with and without reversible turbines

The values in the table represent a profit of 58.2€ and 58.4€, respectively. We can say that pumping water brings benefits for the management of the cascade since a better profit is obtained.

Remark:
From the periodic condition, $V_1(0) = V_1(T)$, and taking into account that $V_1(0) = V_1^M - A_1\tau$, we obtain the condition $A_1 = \dfrac{u_1^M(T-\tau)}{T}$. This equality must be satisfied in the case of $u_1^m = 0$ to give rise to a solution profile like in Fig. 5.3. For the data values used, (5.2), such condition is fulfilled.

5.2 System with 2 reservoirs

In this section we analyze the problem (P_2), with 2 reservoirs in cascade, presented in section 2.3. The existence Theorem (3.1.1) can be applied to this more complex problem, and following the same line of arguments used in section (5.1.1), we get existence of optimal solution for (P_2). Similarly and based on the numerical solution, we will verify that the necessary conditions and the sufficient condition are satisfied for this problem, when considering some particular data.
Problem (P_2) in Mayer form can be written as follows.

$$
\begin{aligned}
(P_{2M})\ \min\ J(u(\cdot),V(\cdot)) = & -\frac{A_1c_1}{s_1}Z_1(\tau) - \frac{A_1c_2}{s_1}Z_2(\tau) + H_1(c_2-c_1)V_1(0) + \frac{c_2-c_1}{2s_1}V_1^2(0) \\
& - H_1(c_2-c_1)W_1(0) - \frac{c_2-c_1}{2s_1}W_1^2(0) + H_2(c_2-c_1)V_2(0) + \frac{c_2-c_1}{2s_2}V_2^2(0) \\
& - H_2(c_2-c_1)W_2(0) - \frac{c_2-c_1}{2s_2}W_2^2(0) - A_1H_1(\tau c_1 + (T-\tau)c_2), \\
\text{s.t.} \quad & \dot V_1(t) = A_1 - u_1(t),\ a.e.\ t\in[0,\tau] \\
& \dot W_1(t) = A_1 - w_1(t),\ a.e.\ t\in[0,\tau] \\
& \dot V_2(t) = u_1(t) - u_2(t),\ a.e.\ t\in[0,\tau] \\
& \dot W_2(t) = w_1(t) - w_2(t),\ a.e.\ t\in[0,\tau] \\
& \dot Z_1(t) = V_1(t),\ a.e.\ t\in[0,\tau] \\
& \dot Z_2(t) = W_1(t),\ a.e.\ t\in[0,\tau] \\
& V_i(0) = W_i(\tau), \\
& V_i(\tau) = W_i(0), \\
& Z_i(0) = 0, \\
& V_i(t), W_i(t) \in [V_i^m, V_i^M],\ \forall t\in[0,\tau] \\
& u_i(t), w_i(t) \in [u_i^m, u_i^M],\ a.e.\ t\in[0,\tau], \qquad \text{for } i=1,2
\end{aligned}
\tag{5.10}
$$

where $u=(u_1,u_2,w_1,w_2)$ and $V=(V_1,V_2,W_1,W_2,Z_1,Z_2)$.

Continuity of the involved functions and convexity and compactness properties are still present here. So existence of optimal solution follows.
Two sets of data were considered for problem (5.10). Two different profiles were obtained and analyzed.

5.2.1 Case study 1

We start by getting a numerical solution using the software and interfaces referred before.

5.2.1.1 Numerical Results

For the particular data (5.11) below, we obtain the numerical solution for problem (5.10), presented in Fig. 5.4 - 5.5. Here $\tau = T/2$.
Data:

$$\begin{array}{cccccccc} u_1^m\text{=-1} & u_2^m\text{=0} & V_1^m\text{=2} & V_2^m\text{=0} & c_1\text{=2} & H_1\text{=3} & S_1\text{=1} & \text{A=1} \\ u_1^M\text{=2} & u_2^M\text{=2} & V_1^M\text{=10} & V_2^M\text{=2} & c_2\text{=20} & H_2\text{=1} & S_2\text{=0.5} & \text{T=12} \end{array} \tag{5.11}$$

Figures 5.4 and 5.5 represent the numerical solution (trajectories and controls).

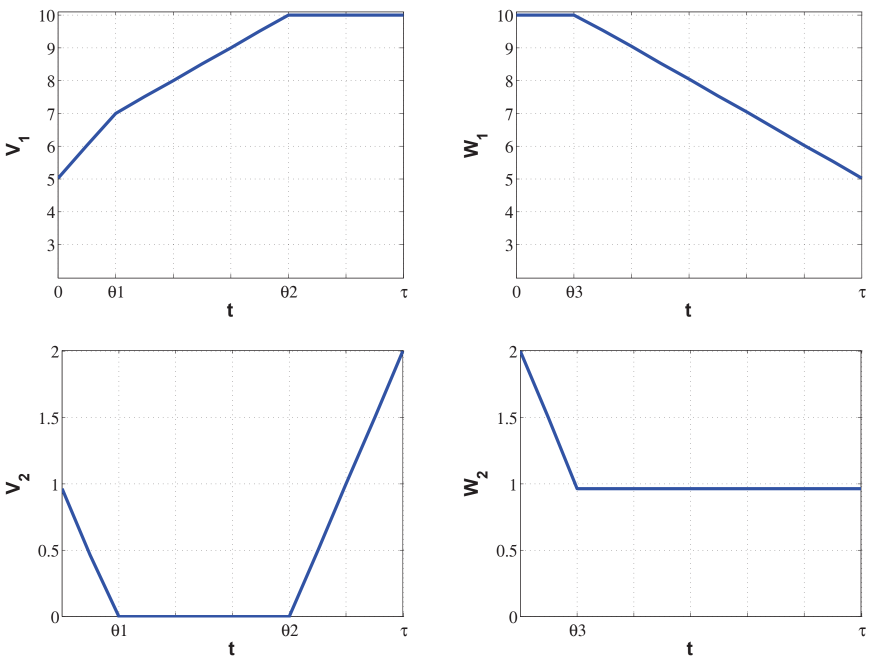


Figure 5.4: Numerical results - trajectories

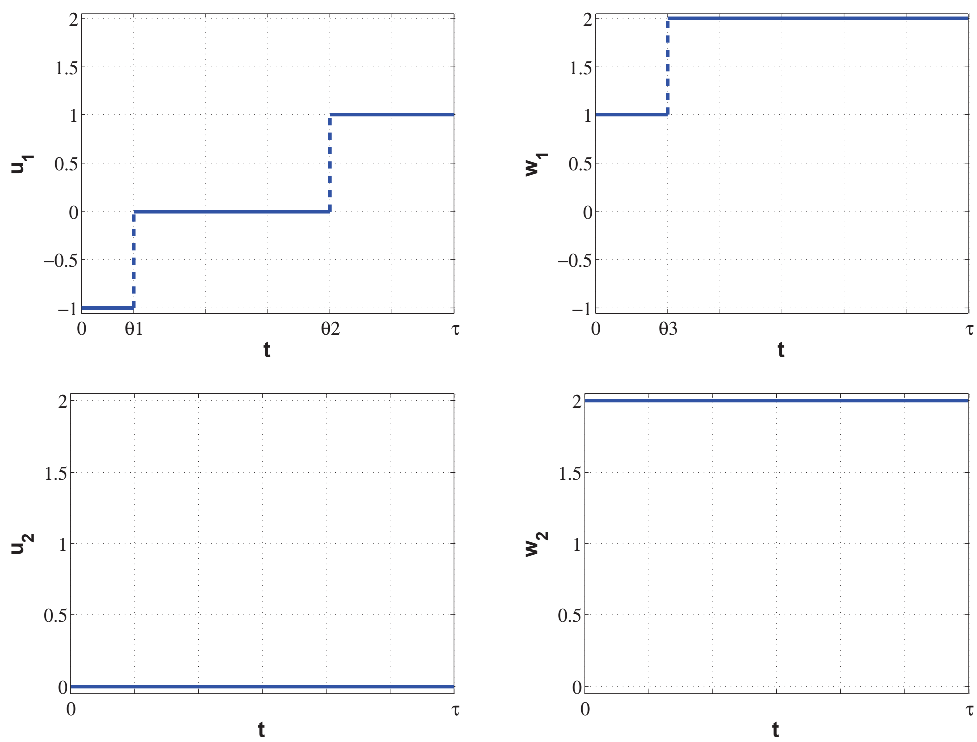


Figure 5.5: Numerical results - controls

The controls functions u_1, w_1, u_2, and w_2 are responsible for the behavior of the trajectories V_1, W_1, V_2, and W_2. As we can observe in the pictures, the station 1 pumps in the beginning of the time interval, causing the increase of the volume V_1. During the interval $[\theta_1, \theta_2]$, the station one is inactive, and after $t = \theta_2$ it turbines a constant amount A_1 of water. If such wouldn't happen, the volume V_1 would exceed the maximum allowable or would contribute to a less favorable cost. Note that V_1 and W_1 touch the upper bounds in $[\theta_2, \tau]$ and $[0, \theta_3]$, respectively. On the other hand, V_2 attains the lower boundary in $[\theta_1, \theta_2]$ and the upper boundary only at time $t = \tau$. The behavior of the state variable V_2, is the reflection of the control policy u_1, since u_2 is equal to zero in all the time interval. The state W_2 is equal to V_2^M only at time $t = 0$.

5.2.1.2 Necessary conditions of optimality

Based on the previous numerical results we make an analysis of the necessary conditions of optimality. All that was written in the introduction of section 5.1.3, regarding necessary conditions applies here too. Theorem 3.1.2 is used considering previous data (5.11) and the formulation (5.10) of (P_2). For shortness, we write $H(t)$ when H is calculated on the reference trajectory.

The maximum principle establishes that, if $((\hat{u}_1,\hat{w}_1,\hat{u}_2,\hat{w}_2),(\hat{V}_1,\hat{W}_1,\hat{V}_2,\hat{W}_2,\hat{Z}_1,\hat{Z}_2))$ is an optimal process, then there exist absolutely continuous functions $p_i,\ i=\overline{1,6}$, nonnegative Borel measures $\mu_i,\ i=\overline{1,8}$ and a real number λ such that

(i) $(p,\ \mu,\ \lambda)\neq(0,\ 0,\ 0)$
where $p=(p_1,p_2,p_3,p_4,p_5,p_6)$ and $\mu=(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7,\mu_8)$

(ii) $(\dot{p}_1,\dot{p}_2,\dot{p}_3,\dot{p}_4,\dot{p}_5,\dot{p}_6)=-(q_5,q_6,0,0,0,0)$

(iii)
$$p_1(0)=q_2(T)+\lambda(c_2-c_1)\left(H_1+\frac{\hat{V}_1(0)}{S_1}\right)$$
$$p_2(0)=q_1(T)-\lambda(c_2-c_1)\left(H_1+\frac{\hat{V}_1(0)}{S_1}\right)$$
$$p_3(0)=q_4(T)+\lambda(c_2-c_1)\left(H_2+\frac{\hat{V}_2(0)}{S_2}\right)$$
$$p_4(0)=q_3(T)-\lambda(c_2-c_1)\left(H_2+\frac{\hat{V}_2(0)}{S_2}\right)$$
$$p_5(0),\ p_6(0)\in\mathbb{R}$$
$$q_5(T)=\lambda\frac{A_1c_1}{S_1}$$
$$q_6(T)=\lambda\frac{A_1c_2}{S_2}$$

(iv)

$$\begin{aligned}H(t)=\max_{u_1,w_1,u_2,w_2\in U}&q_1(t)(A-u_1)+q_2(t)(A-w_1)\\&+q_3(t)(u_1-u_2)+q_4(t)(w_1-w_2)\end{aligned}$$

(v) $supp\ \{\mu_i\}\subset I_i$
where $I_i=\{t\in[0,\tau]\colon h_i(\hat{V}_i(t),\hat{W}_i(t))=0\}$

$$\begin{aligned}h_1(V_1)=&V_1-V_1^M & h_5(V_2)=&V_2-V_2^M\\h_2(V_1)=&V_1^m-V_1 & h_6(V_2)=&V_2^m-V_2\\h_3(W_1)=&W_1-V_1^M & h_7(W_2)=&W_2-V_2^M\\h_4(W_1)=&V_1^m-W_1 & h_8(W_2)=&V_2^m-W_2\end{aligned}$$

and
$q_1(t)=p_1(t)-\mu_1\{[0,t[\}+\mu_2\{[0,t[\}$

$$\begin{aligned}
q_2(t) &= p_2(t) - \mu_3\{[0,t[\} + \mu_4\{[0,t[\} \\
q_3(t) &= p_3(t) - \mu_5\{[0,t[\} + \mu_6\{[0,t[\} \\
q_4(t) &= p_4(t) - \mu_7\{[0,t[\} + \mu_8\{[0,t[\} \\
q_5(t) &= p_5(t) \\
q_6(t) &= p_6(t), \qquad\qquad \forall t \in [0,\tau]
\end{aligned}$$

Using the data (5.11), we get the following information based on the analysis of the necessary conditions.

(ii)
$$\begin{aligned}
p_1(t) &= p_1(0) - 2\lambda t \\
p_2(t) &= p_2(0) - 20\lambda t \\
p_3(t) &= p_3(0) \\
p_4(t) &= p_4(0) \\
p_5(t) &= 2\lambda \\
p_6(t) &= 20\lambda
\end{aligned}$$

(iii)
$$p_2(0) = p_1(0) - 12\lambda - \mu_1\{[0,\tau]\} + \mu_2\{[0,\tau]\} - 18\lambda(3 + \hat{V}_1(\tau))$$
$$p_4(0) = p_3(0) - \mu_5\{[0,\tau]\} + \mu_6\{[0,\tau]\} - 18\lambda(1 + 2\hat{V}_2(\tau))$$
$$36\lambda(\hat{V}_2(0) - \hat{V}_2(\tau)) - \mu_7\{[0,\tau]\} + \mu_8\{[0,\tau]\} - \mu_5\{[0,\tau]\} + \mu_6\{[0,\tau]\} = 0$$
$$-132\lambda + 18\lambda(\hat{V}_1(0) - \hat{V}_1(\tau)) - \mu_3\{[0,\tau]\} + \mu_4\{[0,\tau]\} - \mu_1\{[0,\tau]\} + \mu_2\{[0,\tau]\} = 0$$

(iv)

$$\begin{aligned}
H(t) = \max_{\substack{u_1,w_1,\\ u_2,w_2\in U}} \; & (-p_1(0) + 2\lambda t + p_3(0) + \mu_1\{[0,t[\} - \mu_2\{[0,t[\} - \mu_5\{[0,t[\} + \mu_6\{[0,t[\})\, u_1(t) \\
& + (-p_2(0) + 20\lambda t + \mu_3\{[0,t[\} - \mu_4\{[0,t[\} + p_4(0) - \mu_7\{[0,t[\} + \mu_8\{[0,t[\})\, w_1(t) \\
& + (-p_3(0) + \mu_5\{[0,t[\} - \mu_6\{[0,t[\})\, u_2(t) \\
& + (-p_4(0) + \mu_7\{[0,t[\} - \mu_8\{[0,t[\})\, w_2(t)
\end{aligned}$$

where $U = [u_1^m, u_1^M]$.

In Figure 5.4 the numerical optimal trajectories $\hat{V}_1$, $\hat{W}_1$ and $\hat{V}_2$, $\hat{W}_2$, are presented. We define some instants that are crucial: θ_1, θ_2 and θ_3. These instants coincide with changes of the controls behavior.

According to the numerical solution we have:

$$\hat{u}_1(t) = \begin{cases} -1, & t \in [0, \theta_1[\\ 0, & t \in [\theta_1, \theta_2[\\ 1, & t \in [\theta_2, \tau], \end{cases} \qquad \hat{w}_1(t) = \begin{cases} 1, & t \in [0, \theta_3[\\ 2, & t \in [\theta_3, \tau], \end{cases}$$

$$\hat{u}_2(t) = 0, \; t \in [0, \tau], \qquad \hat{w}_2(t) = 2, \; t \in [0, \tau].$$

So, we can define the state variables as

$$\hat{V}_1(t) = \begin{cases} \hat{V}_1(0) + 2t, & t \in [0, \theta_1[\\ \hat{V}_1(0) + \theta_1 + t, & t \in [\theta_1, \theta_2[\\ V_1^M, & t \in [\theta_2, \tau[, \end{cases} \qquad \hat{W}_1(t) = \begin{cases} V_1^M, & t \in [0, \theta_3[\\ V_1^M + \theta_3 - t, & t \in [\theta_3, \tau], \end{cases}$$

$$\hat{V}_2(t) = \begin{cases} \hat{V}_2(0) - t, & t \in [0, \theta_1[\\ V_2^m, & t \in [\theta_1, \theta_2[\\ V_2^m - \theta_2 + t, & t \in [\theta_2, \tau[, \end{cases} \qquad \hat{W}_2(t) = \begin{cases} \hat{W}_2(0) - t, & t \in [0, \theta_3[\\ \hat{W}_2(0) - \theta_3, & t \in [\theta_3, \tau]. \end{cases}$$

From the continuity of $\hat{V}_1(t)$ and $\hat{V}_2(t)$, it comes

$$\hat{V}_1(0) + \theta_1 + \theta_2 = V_1^M$$

and

$$\hat{V}_2(0) - \theta_1 = V_2^m = 0 \Rightarrow \hat{V}_2(0) = \theta_1.$$

We proceed with the analysis on main subintervals of the interval $[0, \tau]$:
Observe that the measures μ_1, μ_3 and μ_7 are null since the corresponding state constraints are never active. The measure μ_6 is concentrated at $t = \tau$

- $t \in [0, \theta_1[$:
  Here $\mu_2 = \mu_5 = 0$.
  Also,

$$H(t) = \max_{\substack{u_i, w_i \in U \\ i=1,2}} (-p_1(0) + 2t + p_3(0))\, u_1 + (-p_3(0))\, u_2 + (-p_4(0) - \mu_8\{0\})\, w_2$$
$$+ (-p_2(0) + 20t - \mu_4\{[0, t[\} + p_4(0) + \mu_8\{0\})\, w_1$$

$*$ $\bar{u}_1(t) = -1$, then $-p_1(0) + 2t + p_3(0) \leq 0 \Rightarrow -p_1(0) + p_3(0) < 0$.
$*$ $\bar{w}_1(t) = 1$, then $-p_2(0) + 20t - \mu_4\{[0,t[\} + p_4(0) + \mu_8\{0\} = 0$.
$*$ $\bar{u}_2(t) = 0$, then $p_3(0) \geq 0$ ($\Rightarrow p_1(0) > 0$).
$*$ $\bar{w}_2(t) = 2$, then $p_4(0) + \mu_8\{0\} \leq 0 \Rightarrow p_4(0) \leq 0$.

- $t \in [\theta_1, \theta_3[$:

Here $\mu_2 = 0$ on every subset of $[0, \theta_3[$, $\mu_5\{[0,t[\} = \mu_5\{[\theta_1, t[\}$ and $\mu_8\{[0,t[\} = \mu_8\{0\}$.

$$
\begin{aligned}
H(t) = \max_{\substack{u_i, w_i \in U \\ i=1,2}} & \left(-p_1(0) + 2t + p_3(0) - \mu_5\{[\theta_1, t[\}\right) u_1 + \left(-p_3(0) + \mu_5\{[\theta_1, t[\}\right) u_2 \\
& + \left(-p_2(0) + 20t - \mu_4\{[0,t[\} + p_4(0) + \mu_8\{0\}\right) w_1 + \left(-p_4(0) - \mu_8\{0\}\right) w_2
\end{aligned}
$$

$*$ $\bar{u}_1(t) = 0$, then $\mu_5\{[\theta_1, t[\} = -p_1(0) + 2t + p_3(0) \geq 0$.
$*$ $\bar{w}_1(t) = 1$, then $-p_2(0) + 20t - \mu_4\{[0,t[\} + p_4(0) + \mu_8\{0\} = 0$.
$*$ $\bar{u}_2(t) = 0$, then $-p_3(0) + \mu_5\{[\theta_1, t[\} \leq 0 \Leftrightarrow$
$-p_3(0) - p_1(0) + 2t + p_3(0) \leq 0 \Rightarrow p_1(0) \geq 2\theta_3$.
$*$ $\bar{w}_2(t) = 2$, then $p_4(0) + \mu_8\{0\} \leq 0 \Rightarrow p_4(0) \leq 0$.

Since $p_3(0) - p_1(0) + 2t \leq 0,\ \forall\, t \in [0, \theta_1[$ and $p_3(0) - p_1(0) + 2t \geq 0,\ \forall\, t \in [\theta_1, \theta_2[$ then $p_3(0) - p_1(0) + 2\theta_1 = 0 \Leftrightarrow p_3(0) - p_1(0) = -2\theta_1$.

- $t \in [\theta_3, \theta_2[$:

$\mu_2 = 0$ on every subset of $[0, \theta_2[$, $\mu_4\{[0,t[\} = \mu_4\{[0, \theta_3]\}$, $\mu_5\{[0,t[\} = \mu_5\{[\theta_1, t[\}$, $\mu_8\{[0,t[\} = \mu_8\{0\}$.

$$
\begin{aligned}
H(t) = \max_{\substack{u_i, w_i \in U \\ i=1,2}} & \left(-p_1(0) + 2t + p_3(0) - \mu_5\{[\theta_1, t[\}\right) u_1 + \left(-p_3(0) + \mu_5\{[\theta_1, t[\}\right) u_2 \\
& + \left(-p_2(0) + 20t - \mu_4\{[0,\theta_3]\} + p_4(0) + \mu_8\{0\}\right) w_1 + \left(-p_4(0) - \mu_8\{0\}\right) w_2
\end{aligned}
$$

$*$ $\bar{u}_1(t) = 0$, then $\mu_5\{[\theta_1, t[\} = -p_1(0) + 2t + p_3(0) \Rightarrow \mu_5\{[\theta_1, t[\} = 2(t - \theta_1)$.
$*$ $\bar{w}_1(t) = 2$, then $-p_2(0) + 20t - \mu_4\{[0,\theta_3]\} + p_4(0) + \mu_8\{0\} \geq 0$.

$*$ $\bar{u}_2(t) = 0$, then $p_3(0) \geq \mu_5\{[\theta_1, t[\} = 2(t - \theta_1)$.
$*$ $\bar{w}_2(t) = 2$, then $p_4(0) + \mu_8\{0\} \leq 0 \Rightarrow p_4(0) \leq 0$.

- $t \in [\theta_2, \tau[$:

$\mu_2\{[0,t[\} = \mu_2\{[\theta_2, t]\}$,
$\mu_4\{[0,t[\} = \mu_4\{[0,\theta_3]\}$, $\mu_5\{[0,t[\} = \mu_5\{[\theta_1,\theta_2[\}$ and $\mu_8\{[0,t[\} = \mu_8\{0\}$

$$\begin{aligned} H(t) = \max_{\substack{u_i, w_i \in U \\ i=1,2}} & \left(-p_1(0) + 2t + p_3(0) - \mu_2\{[\theta_2, t[\} - \mu_5\{[\theta_1, \theta_2]\}\right) u_1 \\ & + \left(-p_2(0) + 20t - \mu_4\{[0,\theta_3]\} + p_4(0) + \mu_8\{0\}\right) w_1 \\ & + \left(-p_3(0) + \mu_5\{[\theta_1,\theta_2]\}\right) u_2 + \left(-p_4(0) - \mu_8\{0\}\right) w_2 \end{aligned}$$

$*$ $\bar{u}_1(t) = 1$, then $-p_1(0) + 2t + p_3(0) - \mu_2\{[\theta_2, t[\} - \mu_5\{[\theta_1, \theta_2]\} = 0$.
$*$ $\bar{w}_1(t) = 2$, then $-p_2(0) + 20t - \mu_4\{[0,\theta_3]\} + p_4(0) + \mu_8\{0\} \geq 0$.
$*$ $\bar{u}_2(t) = 0$, then $p_3(0) \geq \mu_5\{[\theta_1, \theta_2]\}$.
$*$ $\bar{w}_2(t) = 2$, then $p_4(0) + \mu_8\{0\} \leq 0 \Rightarrow p_4(0) \leq 0$.

As $\mu_2\{[\theta_2, t[\} = -p_1(0)+2t+p_3(0)-\mu_5\{[\theta_1,\theta_2[\}-\mu_5\{\theta_2\}$, $\mu_5\{[\theta_1,\theta_2[\} = 2(\theta_2-\theta_1)$ and $p_3(0)-p_1(0) = -2\theta_1$, then we have $\mu_2\{[\theta_2, t[\} = 2t-2\theta_1-2(\theta_2-\theta_1)-\mu_5\{\theta_2\} = 2(t-\theta_2) - \mu_5\{\theta_2\}$.
Applying limits when $t \downarrow \theta_2$ $(t \to \theta_2,\ t > \theta_2)$, it comes $\mu_2\{\theta_2\} = -\mu_5\{\theta_2\} \Rightarrow \mu_5\{\theta_2\} = 0$, $\mu_2\{\theta_2\} = 0$.

From the periodic constraints, we have

$$\begin{cases} \hat{V}_1(0) = \hat{W}_1(\tau) \\ \hat{V}_1(\tau) = \hat{W}_1(0) \\ \hat{V}_2(0) = \hat{W}_2(\tau) \\ \hat{V}_2(\tau) = \hat{W}_2(0) \end{cases} \Leftrightarrow \begin{cases} 10 - \theta_1 - \theta_2 = 4 + \theta_3 \\ 10 = 10 \\ \theta_1 = \hat{W}_2(0) - \theta_3 \\ 6 - \theta_2 = \hat{W}_2(0) \end{cases}$$

Since $\hat{W}_2(t)$ starts on the lower boundary, $\hat{W}_2(0) = 2$ and we have

$$\theta_2 = 4 \qquad \theta_1 + \theta_3 = 2.$$

The information above and transversality conditions lead to:

$$\begin{gathered}
\mu_6\{\tau\} - 16 - 38\theta_1 + \mu_8\{0\} = 0 \\
p_2(0) = p_1(0) + \mu_2\{[\theta_2, \tau]\} - 246 \\
\mu_8\{0\} + \mu_6\{\tau\} = 80 - 38\theta_1 \\
p_4(0) = p_1(0) - 98 + \mu_6\{\tau\}
\end{gathered}$$

And we conclude that $\theta_1 = 16/19$, $\theta_3 = 22/19$ and $p_2(0) = p_1(0) - 242$.

Take $\mu_6\{\tau\} = 0$ and $p_1(0) = 8$. We obtain a possible set of multipliers that satisfy all the necessary conditions of optimality.

The trajectories $\hat{V}_1,\ \hat{V}_2$ can now be completely defined on the whole interval $[0, T]$ (remember that $\hat{W}_1(t) = \hat{V}_1(6+t)$ and $\hat{W}_2(t) = \hat{V}_2(6+t)$):

$$\hat{V}_1(t) = \begin{cases} \frac{98}{19} + 2t, & t \in [0, \theta_1[\\ \frac{114}{19} + t, & t \in [\theta_1, \theta_2[\\ 10, & t \in [\theta_2, 6+\theta_3[\\ 16 + \frac{22}{19} - t, & t \in [6+\theta_3, 12], \end{cases} \qquad \hat{V}_2(t) = \begin{cases} \frac{16}{19} - t, & t \in [0, \theta_1[\\ 0, & t \in [\theta_1, \theta_2[\\ t - 4, & t \in [\theta_2, 6[\\ 8 - t, & t \in [6, 6+\theta_3[\\ \frac{16}{19}, & t \in [6+\theta_3, 12]. \end{cases}$$

The values of $\theta_1,\ \theta_2$ and θ_3 were determined exactly:

$$\theta_1 = \frac{16}{19}, \quad \theta_2 = 4, \quad \theta_3 = 22/19.$$

The set of multipliers can be defined as:

$$\mu_1, \mu_3, \mu_6, \mu_7 \equiv 0 \qquad \mu_2\{[\theta_2, t[\} = 2(t - \theta_2) \qquad \mu_8\{0\} = 48\delta_{\{0\}}$$

$$\mu_4\{[0, t[\} = \begin{cases} 192 + 20t, & t < \theta_3 \\ 192 + 20\theta_3, & t \geq \theta_3, \end{cases} \qquad \mu_5\{[\theta_1, t[\} = \begin{cases} 0, & t \leq \theta_1 \\ 2(t - \theta_1), & \theta_1 < t < \theta_2 \\ 2(\theta_2 - \theta_1), & t \geq \theta_2. \end{cases}$$

$$\lambda = 1$$

$$\begin{array}{llll} p_1(t) = 8 - 2t & p_2(t) = -234 - 20t & p_3(t) = 8 - \frac{32}{19} & \\ p_4(t) = -90 & p_5(t) = 2 & p_6(t) = 20, & t \in [0, 6]. \end{array}$$

Here $\delta_{\{0\}}$ denotes the unit measure concentrated at $t = 0$.

To completely validate $(\hat{u}, \hat{V})$ as a local minimizer, we analyze sufficient conditions of optimality.

5.2.1.3 Sufficient conditions of optimality

We specify the conditions of Theorem 4.2.2 for our problem with 2 reservoirs and we verify that these conditions are satisfied for the reference pair $(\hat{u}(\cdot), \hat{V}(\cdot))$ displayed at the end of last section. To do that, the formulation (P_2) will be under consideration:

$$
\begin{array}{lll}
(P_2) & \min & -\left[\dfrac{A_1}{S_1}\displaystyle\int_0^T c(t)V_1(t)dt + (H_1 - H_2)\int_{]0,T]} V_1(t)dc(t)\right. \\
 & & \displaystyle + \frac{1}{2S_1}\int_{]0,T]} V_1^2(t)dc(t) + H_2 \int_{]0,T]} (V_2(t) + V_1(t))\, dc(t) \\
 & & \left.\displaystyle + \frac{1}{2S_2}\int_{]0,T]} V_2^2(t)dc(t)\right], \\
 & \text{s.t.} & \dot{V}_1(t) = A_1 - u_1(t),\ a.e.\ t \in [0, T] \\
 & & \dot{V}_2(t) = u_1(t) - u_2(t),\ a.e.\ t \in [0, T] \\
 & & V_i(0) = V_i(T), \\
 & & V_i^m \le V_i(t) \le V_i^M,\ \forall t \in [0, T] \\
 & & u_i^m \le u_i(t) \le u_i^M,\ a.e.\ t \in [0, T]
\end{array}
$$

for $i = 1, 2$ and $c(t)$ given by (2.4) with $\tau = T/2$.

Theorem 5.2.1. *Let $(\hat{u}_k(\cdot), \hat{V}_k(\cdot))$, $k = \overline{1, 2}$, be a control process. Assume that the following conditions are satisfied:*

1. *there exist right continuous functions $p_1(\cdot), p_2(\cdot) \in BV([0, T], R)$ and piecewise absolutely continuous functions $\eta_k(\cdot)$, $k = \overline{1, 2}$, satisfying*

$$
dp_1(t) = -\frac{A_1}{S_1}c(t)\, dt - H_1\, dc(t) - \frac{\hat{V}_1(t)}{S_1}dc(t) + d\eta_1,
$$

$$dp_2(t) = -H_2\, dc(t) - \frac{\hat{V}_2(t)}{S_2} dc(t) + d\eta_2,$$

$$p_1(0) = p_1(T), \qquad p_2(0) = p_2(T),$$

$$\eta_1(t) = \nu_1(t) + \Delta\eta_1(\tau)H(t-\tau) + \Delta\eta_1(T)H(t-T),$$

$$\eta_2(t) = \nu_2(t) + \Delta\eta_2(\tau)H(t-\tau) + \Delta\eta_2(T)H(t-T),$$

where $\nu_1(\cdot), \nu_2(\cdot) \in AC([0,T], \mathbb{R})$, $\Delta\eta_i(\tau)$, $\Delta\eta_i(T)$, $i = 1,2$ *are constants, and* $H(\cdot)$ *stands for the Heaviside step function;*

2. *the equality*

$$\max_{u_k \in [u_k^m, u_k^M],\ k=\overline{1,2}} u_1(-p_1(t)+p_2(t)) + u_2(-p_2(t)) = \hat{u}_1(-p_1(t)+p_2(t)) + \hat{u}_2(-p_2(t)),$$

   holds a.e. $t \in [0,T]$*;*

3. *the functions* $\nu_1(\cdot)$, $\nu_2(\cdot)$, *satisfy the inequalities*

$$\begin{aligned} d\nu_k(t) &\leq 0, \quad \text{if} \quad \hat{V}_k(t) = V_k^m \\ d\nu_k(t) &\geq 0, \quad \text{if} \quad \hat{V}_k(t) = V_k^M \\ d\nu_k(t) &= 0, \quad \text{if} \quad \hat{V}_k(t) \in]V_k^m, V_k^M[\end{aligned}$$

   for $k = 1, 2$*;*

4. *if* $c_1 < c_2$, *then for all* $k = \overline{1,2}$ *and* $\tau_q \in \{\tau, T\}$, *the inequalities*

$$\begin{aligned} \Delta\eta_k(\tau_q) &< 0, \quad \text{if} \quad \hat{V}_k(\tau_q) = V_k^m, \\ \Delta\eta_k(\tau_q) &> 0, \quad \text{if} \quad \hat{V}_k(\tau_q) = V_k^M \ \text{ and} \\ \Delta\eta_k(\tau_q) &= 0, \quad \text{if} \quad \hat{V}_k(\tau_q) \in]V_k^m, V_k^M[\ \text{ hold;} \end{aligned}$$

5. *if* $\hat{V}_k(t) \in]V_k^m, V_k^M[$ *for some* k, *then* $dc(t) \leq 0$.

Then

$$J(\hat{u}(\cdot) + \bar{u}(\cdot), \hat{V}(\cdot) + \bar{V}(\cdot)) \geq J(\hat{u}(\cdot), \hat{V}(\cdot))$$

wherever $(\hat{u}_k(\cdot)+\bar{u}_k(\cdot), \hat{V}_k(\cdot)+\bar{V}_k(\cdot))$, $k = \overline{1,2}$, *is an admissible process and* $\max_{k=\overline{1,2}} |\bar{V}_k(\tau)|$ *is sufficiently small.*

Taking into account the position of $\hat{V}_1(t)$ and $\hat{V}_2(t)$ relative to the boundary of admissible volume sets, we can write

$$\int_{]0,T]} d\eta_1(t) = \int_{[\theta_2,\tau+\theta_3]} d\eta_1(t) \qquad \text{and} \qquad \int_{]0,T]} d\eta_2(t) = \int_{[\theta_1,\theta_2]} d\eta_2(t) + \Delta\eta_2\{\tau\}$$

From *1.* of Theorem 5.2.1, we have

$$\begin{aligned} p_1(t) =& p_1(0) - \int_{]0,t]} \frac{A_1 c(t)}{S_1} dt + \int_{]0,t]} d\eta_1(\tau) - \int_{]0,t]} \left(H_1 + \frac{\hat{V}_1(t)}{S_1} \right) dc(t) \\ p_2(t) =& p_2(0) - \int_{]0,t]} \left(H_2 + \frac{\hat{V}_2(t)}{S_2} \right) dc(t) + \int_{]0,t]} d\eta_2(\tau) \end{aligned}$$

and from the periodicity condition $p_1(0) = p_1(T)$ we obtain

$$\begin{aligned} \int_{]0,T]} d\eta_1(t) =& \frac{A_1 T(c_1 + c_2)}{2S_1} + \frac{(c_2 - c_1)}{S_1} \left(V_1^M - \hat{V}_1(T) \right) \\ \int_{]0,T]} d\eta_2(t) =& \frac{(c_2 - c_1)}{S_2} (V_2^M - \hat{V}_2(T)) \end{aligned}$$

Analysis of condition *2.* leads to

$$p_2(0) \geq 0 \qquad \text{and} \qquad p_1(0) \geq \frac{A_1 c_1 \theta_1}{S_1} + p_2(0) \geq 0 \tag{5.12}$$

$$\int_{[\theta_1,t]} d\eta_2(\tau) = p_1(0) - \frac{A_1 c_1}{S_1} t - p_2(0) \leq 0 \quad \text{and} \quad \int_{[\theta_1,t]} d\eta_2(\tau) \geq -p_2(0), \quad t \in [\theta_1, \theta_2[\tag{5.13}$$

From the last two inequalities, we get $p_1(0) \geq \frac{A_1 c_1}{S_1} t$.

Considering the first inequality of (5.13), taking the limit when $t \downarrow \theta_1$ we obtain

$$p_1(0) \leq \frac{A_1 c_1 \theta_1}{S_1} + p_2(0) \tag{5.14}$$

From (5.12) and (5.14), it comes

$$p_1(0) = \frac{A_1 c_1 \theta_1}{S_1} + p_2(0) \tag{5.15}$$

and with (5.13) we conclude

$$\int_{[\theta_1,t]} d\eta_2(t) = \frac{A_1 c_1}{S_1} (\theta_1 - t), \quad t \in [\theta_1, \theta_2[. \tag{5.16}$$

Also from condition *2.* and (5.15), we can write

$$\int_{[\theta_2,t]} d\eta_1(t) = \frac{A_1c_1}{S_1}(t-\theta_1) + \int_{[\theta_1,\theta_2]} d\eta_2(t) \leq 0, \quad t \in [\theta_2, \tau[. \tag{5.17}$$

From 5.16 and 5.17 it comes

$$\int_{[\theta_2,t]} d\eta_1(t) = \frac{A_1c_1}{S_1}(t-\theta_2) \quad t \in [\theta_2, \tau[.$$

Condition *2.* conveys that for $t \in [\tau, \theta_3[$,

$$\int_{[\theta_2,t]} d\eta_1(t) = \frac{A_1c_1}{S_1}(\tau-\theta_1)+\frac{A_1c_2}{S_1}(t-\tau)+\left(H_1+\frac{V_1^M}{S_1}-H_2-\frac{\hat{V}_2(T)}{S_2}\right)(c_2-c_1) \tag{5.18}$$

$$p_2(0) \leq \left(H_2+\frac{\hat{V}_2(T)}{S_2}\right)(c_2-c_1) \tag{5.19}$$

$$-\frac{A_1c_1\theta_1}{S_1}+\frac{A_1c_2}{S_1}(t-T)+\left(H_1+\frac{\hat{V}_1(T)}{S_1}-H_2-\frac{\hat{V}_2(T)}{S_2}\right)(c_2-c_1) \geq 0, \quad t \in [\theta_3, T[\tag{5.20}$$

$$p_1(0)+\frac{A_1c_2}{S_1}(t-T)+\left(H_1+\frac{\hat{V}_1(T)}{S_1}\right)(c_2-c_1) \geq 0, \quad t \in [\theta_3, T[. \tag{5.21}$$

The above information allow us to express p_1, p_2, $\int_{[0,t[} d\eta_1(t), \int_{[0,t[} d\eta_2(t)$ as follows:

$$p_1(t) = \begin{cases} p_1(0) - \dfrac{A_1c_1}{S_1}t, \ t \in [0,\theta_2[\\ p_1(0) - \dfrac{A_1c_1\theta_2}{S_1}, \ t \in [\theta_2,\tau[\\ p_1(0) - \dfrac{A_1c_1\theta_1}{S_1} - (c_2-c_1)\left(H_2+\dfrac{\hat{V}_2(T)}{S_2}\right), \ t \in [\tau,\theta_3[\\ p_1(0) - \dfrac{A_1c_2\theta_1}{S_1} - (c_2-c_1)\left(H_1+\dfrac{\hat{V}_1(T)}{S_1}\right), \ t \in [\theta_3,T[\end{cases}$$

$$p_2(t) = \begin{cases} p_1(0) - \dfrac{A_1c_1\theta_1}{S_1}, \ t \in [0,\theta_1[\\ p_1(0) - \dfrac{A_1c_1}{S_1}t, \ t \in [\theta_1,\theta_2[\\ p_1(0) - \dfrac{A_1c_1}{S_1}\theta_2, \ t \in [\theta_2,\tau[\\ p_1(0) - \dfrac{A_1c_1\theta_1}{S_1} - (c_2-c_1)\left(H_2 + \dfrac{\hat{V}_2(T)}{S_2}\right), \ t \in [\tau, T[\end{cases}$$

$$\int_{[0,t]} d\eta_1(t) = \begin{cases} 0, \ t \in [0,\theta_2[\\ \dfrac{A_1c_1}{S_1}(t-\theta_2), \ t \in [\theta_2,\tau[\\ \dfrac{A_1c_1}{S_1}(\tau-\theta_1) + \dfrac{A_1c_2}{S_1}(t-\tau) + (c_2-c_1)\left(H_1 - H_2 + \dfrac{V_1^M}{S_1} - \dfrac{\hat{V}_2(T)}{S_2}\right), \\ \ t \in [\tau,\theta_3[\\ \dfrac{A_1T}{2S_1}(c_1+c_2) + (c_2-c_1)\left(\dfrac{V_1^M - \hat{V}_1(T)}{S_1}\right), \ t \geq \theta_3 \end{cases}$$

and

$$\int_{[0,t]} d\eta_2(t) = \begin{cases} 0, \ t \in [0,\theta_1[\\ \dfrac{A_1c_1}{S_1}(\theta_1 - t), \ t \in [\theta_1,\theta_2[\\ \dfrac{A_1c_1}{S_1}(\theta_1-\theta_2), \ t \in [\theta_2,\tau[\\ (c_2-c_1)\left(\dfrac{V_2^M - \hat{V}_2(T)}{S_2}\right), \ t \geq \tau. \end{cases}$$

Assuming that $p_1(0) = \dfrac{A_1c_1\theta_2}{S_1}$ and using the data (5.11) we can write:

$\lambda = 1$

$$p_1(t) = \begin{cases} 8-2t, \ t \in [0,\theta_2[\\ 0, \ t \in [\theta_2,\tau[\\ -10-38\theta_1, \ t \in [\tau,\theta_3[\\ 86-20t+18\theta_1, \ t \in [\theta_3,T[\end{cases} \qquad p_2(t) = \begin{cases} 8-2\theta_1, \ t \in [0,\theta_1[\\ 8-2t, \ t \in [\theta_1,\theta_2[\\ 0, \ t \in [\theta_2,\tau[\\ -10-38\theta_1, \ t \in [\tau,T[\end{cases}$$

$$\int_{[0,t[} d\eta_1(t) = \begin{cases} 0, \ t \in [0, \theta_2[\\ 2t - 8, \ t \in [\theta_2, \tau[\\ -38\theta_1 + 20t + 108, \ t \in [\tau, \theta_3[\\ 204 + 18\theta_1, \ t \geq \theta_3 \end{cases} \qquad \int_{[0,t[} d\eta_2(t) = \begin{cases} 0, \ t \in [0, \theta_1[\\ 2(\theta_1 - t), \ t \in [\theta_1, \theta_2[\\ 2(\theta_1 - 4), \ t \in [\theta_2, \tau[\\ 72 - 18\theta_1, \ t \geq \tau \end{cases}$$

where $\theta_1 = 16/19$, $\theta_2 = 4$, $\theta_3 = 8 - \theta_1$ $\hat{V}_1(0) = 6 - \theta_1$ and $\hat{V}_2(0) = 6 - \theta_1$.

In $[0, \theta_2[\cup]\theta_3, T]$, $\hat{V}_1(t) \in]V_1^m, V_1^M[$. Also in $[0, \theta_1[\cup]\theta_2, \tau[\cup]\tau, T]$, $V_2(t) \in]V_2^m, V_2^M[$ and in these intervals $d\nu_1(t) = 0$ and $d\nu_2(t) = 0$ respectively. So condition *3.* is satisfied.

In $[\theta_2, \theta_3]$, $V_1(t) = V_1^M$ and $d\nu_1(t) \geq 0$. In $[\theta_1, \theta_2]$ we have $V_2(t) = V_2^m$ and $d\nu_2(t) \leq 0$. Then (3) is verified.

Furthermore, $\hat{V}_1(\tau) = V_1^M$ and $\Delta\eta_1(\tau) = 176 > 0$. Also $\hat{V}_2(\tau) = V_2^M$ and $\Delta\eta_2(\tau) = 59 > 0$. So condition 4. is also verified.

Condition *5.* is obviously true, since $V_1(t) \in]V_1^m, V_1^M[$ in $[0, \theta_2[\cup]\theta_3, T]$, $V_2(t) \in]V_2^m, V_2^M[$ in $[0, \theta_1[\bigcup]\theta_2, \tau[\bigcup]\tau, T[$ and in these intervals $dc(t) = 0$.

We conclude that $(\hat{u}, \hat{V})$ is in fact a local minimizer for the problem.

5.2.2 Case study 2

Problem (P_2) is here considered with objective function in original form (2.2):

$$\begin{aligned} \min \quad & \sum_{k=1}^{N} \int_0^T -c(t) \left[u_1(t) \left(\frac{V_1(t)}{S_1} + H_1 - \frac{V_2(t)}{S_2} - H_2 \right) + u_2(t) \left(\frac{V_2(t)}{S_2} + H_2 \right) \right], \\ \text{s.t.} \quad & \dot{V}_1(t) = A_1 - u_1(t), \ a.e. \ t \in [0, T] \\ & \dot{V}_2(t) = u_1(t) - u_2(t), \ a.e. \ t \in [0, T] \\ & V_i(0) = V_i(T), \\ & V_i(t) \in [V_i^m, V_i^M], \ \forall \ t \in [0, T] \\ & u_i(t) \in [u_i^m, u_i^M], \ a.e. \ t \in [0, T], \quad \text{for } i = 1, 2. \end{aligned} \tag{5.22}$$

The analysis made in this section is published in [40].

5.2.2.1 Numerical Results

Problem (5.22) is analyzed considering now the following data:

$$\begin{array}{llllll} u_1^m = -0.351 & u_2^M = 0.8316 & V_2^m = 48.3 & H_2 = 1 & c_1 = 2 & T = 24 \\ u_1^M = 0.44496 & V_1^m = 86.7 & V_2^M = 66 & S_2 = 44.5 & c_2 = 20 & \\ u_2^m = 0 & V_1^M = 147 & H_1 = 3 & S_1 = 81.7 & A = 0.158 & \end{array} \tag{5.23}$$

We take once more $\tau = T/2$.
Figure 5.6 represents the numerical solution.
The picture represents V_1, V_2, u_1, u_2 on the whole interval $[0, T]$. We didn't split the interval on two as in case study 1.

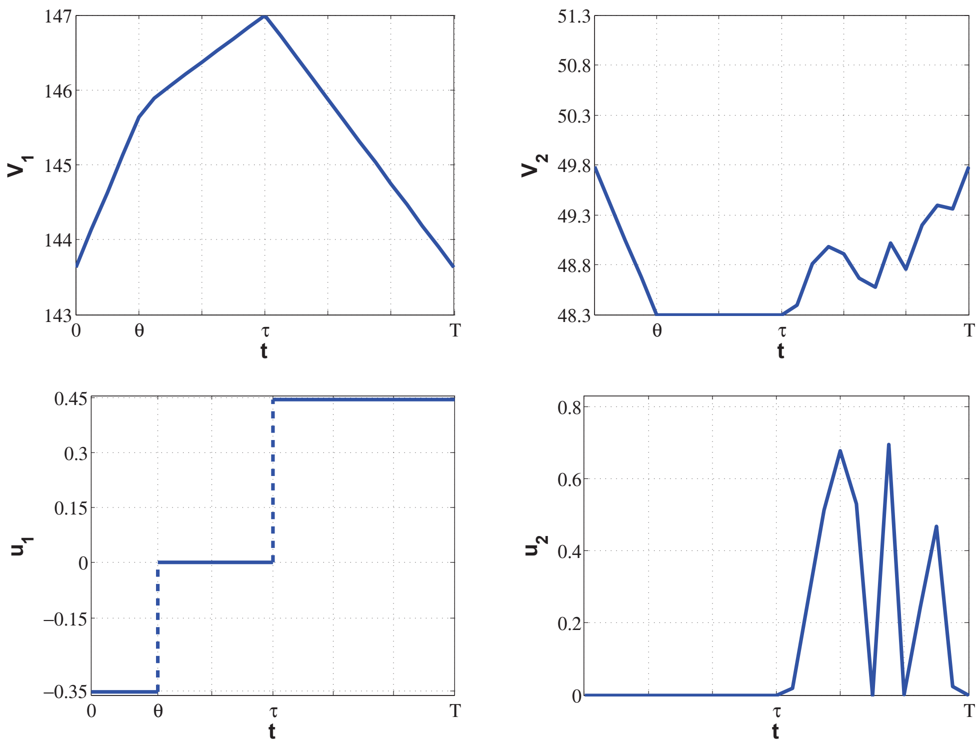


Figure 5.6: Numerical results - real data

The profile of the optimal solution is different from case 1. Now, the trajectory $\hat{V}_1(\cdot)$ only touches the upper boundary at instant $t = \tau$. $\hat{V}_2(\cdot)$ doesn't attain the maximum value allowed and it has an irregular behavior in the second part of the interval ($[\tau, T]$), which is a consequence of the irregular behavior of the control function $u_2(\cdot)$.

It is clear from the objective function written as in (5.10) (equivalent form for (5.22)) that changes on the trajectory $\hat{V}_2(t)$ that maintain the values of $\hat{V}_2(0)(=\hat{V}_2(T))$ and $\hat{V}_2(\tau)$, do not affect the cost. The optimal trajectory $\hat{V}_2(\cdot)$ on the interval $[\tau, T]$ is not unique (*a more detailed analysis will be presented in the next subsection*). This explains the irregular behavior of $\hat{u}_2$ and $\hat{V}_2$ on Figure 5.6.

5.2.2.2 Infinity of solutions

Let us see that the optimal trajectory $\hat{V}_2(t)$ for problem (P_2) with data (5.23) is not unique. This explains the irregular behavior of $\hat{u}_2$ and $\hat{V}_2$ on Figure 5.6. Indeed, using the fact that $u_1(t) = u_1^M,\ t \in [\tau, T]$ and $V_1(\tau) = V_1^M,\ V_2(\tau) = V_2^m$, it is possible to write any admissible trajectory $(V_1(\cdot), V_2(\cdot))$ as:

$$\begin{aligned}
V_1(T) =& V_1(\tau) - \int_\tau^T (A - u_1^M)dt, \text{ for } t \in [\tau, T] \\
=& V_1^M - (A - u_1^M)\tau \\
V_2(T) =& V_2(\tau) - \int_\tau^T (u_1^M - u_2(t))dt, \text{ for } t \in [\tau, T] \\
=& V_2^m - u_1^M \tau + \int_\tau^T u_2(t)dt
\end{aligned}$$

Observe that the cost function in (5.10) only depends on $V_2(\tau)$ and $V_2(0)(= V_2(T))$. The values $\hat{V}_2(\tau)$ and $\hat{V}_2(0)$ can be attained by more than one $V_2(\cdot)$, while $(u_2(\cdot), V_2(\cdot))$ remains admissible.
Take $u_2(t) = const = \bar{u}_2$, for $t \in [\tau, T]$. If

$$\bar{u}_2 = \frac{1}{T - \tau}\left(\hat{V}_2(T) - V_2^m + \tau u_1^M\right) \tag{5.24}$$

then $V_2(T) = \hat{V}_2(T)$.
For our particular data (5.23), this value $\bar{u}_2$ belongs to the interval $]u_2^m, u_2^M[$ and $V_2(\cdot)$ is admissible. This gives rise to another optimal solution for the problem.

Furthermore, we can show that there is an infinity of solutions to the problem. Take the above $\bar{u}_2$ which is admissible. Define a piecewise constant function $u_2(t)$ such that

$$u_2(t) = \begin{cases} \alpha, \ t \in [\tau, \gamma[\\ \beta, \ t \in [\gamma, T] \end{cases}$$

with $\gamma, \ \alpha, \ \beta$ constants, and

$$\int_\tau^T u_2(t)dt = \bar{u}_2\tau,$$

where $\bar{u}_2$ is defined in (5.24).
Then

$$\begin{aligned} \int_\tau^T u_2(t)dt = & \int_\tau^\gamma \alpha dt + \int_\gamma^T \beta dt \\ = & \alpha\gamma + \beta(\tau - \gamma) \\ = & \bar{u}_2\tau \end{aligned}$$

$$\gamma = \frac{T(\bar{u}_2 - \beta)}{2(\alpha - \beta)}$$

This equation has an infinity of solutions (γ, α, β) with $\gamma \in]\tau, T[, \ \alpha, \ \beta \in [u_2^m, u_2^M]$ and $V_2(t)$ admissible.
The cost function keeps the same value for all control functions $u_2(t)$ defined in such way.

5.2.2.3 Necessary conditions of optimality

Observing the previous numerical results, we can write:

$$\hat{u}_1(t) = \begin{cases} u_1^m, \ \text{for } t \in [0, \theta[\\ 0, \ \text{for } t \in [\theta, \tau[\\ u_1^M, \ \text{for } t \in [\tau, T], \end{cases} \qquad \hat{u}_2(t) = \begin{cases} 0, \ \text{for } t \in [0, \tau[\\ w_2(t), \ \text{for } t \in [\tau, T] \end{cases} \tag{5.25}$$

where $\theta \in]0, \tau[$ is the instant of change of control behavior.
Since $\hat{V}_2(\cdot)$ is not unique and $\hat{u}_2(\cdot)$ has an irregular behavior, we don't fix the value of the control on the interval $[\tau, T]$.

Also,

$$V_1(\tau) = V_1^M \quad \text{and} \quad V_2(\theta) = V_2^m \tag{5.26}$$

and

$$\hat{V}_1(t) = \begin{cases} \hat{V}_1(0) + (A_1 - u_1^m)t, \ t \in [0, \theta[\\ \hat{V}_1(\theta) + A_1(t-\theta), \ t \in [\theta, \tau[\\ V_1^M + (A_1 - u_1^M)(t-\tau), \ t \in [\tau, T], \end{cases} \qquad \hat{V}_2(t) = \begin{cases} \hat{V}_2(0) + u_1^m t, \ t \in [0, \theta[\\ V_2^m, \ t \in [\theta, \tau[\\ V_2^m + u_1^M(t-\tau) - w(t), \ t \in [\tau, T], \end{cases}$$

where $w(t) = \displaystyle\int_\tau^t w_2(s)ds$ and $w_2(s) \in [u_2^m, u_2^M]$.

From (5.26) and description of $\hat{V}_1, \ \hat{V}_2$ above we get

$$\hat{V}_1(0) = V_1^M + u_1^m\theta - A_1\tau \qquad \text{and} \qquad \hat{V}_2(0) = V_2^m - u_1^m\theta. \tag{5.27}$$

Also, from the periodic constraints $V_i(0) = V_i(T), \ i = 1, 2$ we can deduce that

$$\theta = \frac{\tau(2A_1 - u_1^M)}{u_1^m} \qquad \text{and} \qquad \int_\tau^T w_2(s)ds = A_1 T. \tag{5.28}$$

Applying necessary conditions we get (see formulation (5.22)):

(i) $(p, \ \mu, \ \lambda) \neq (0, \ 0, \ 0)$, $\qquad p = (p_1, \ p_2), \quad \mu = (\mu_1, \ \mu_2, \ \mu_3, \ \mu_4)$,

(ii) $p_1(t) = p_1(0) - \dfrac{\lambda}{S_1}\displaystyle\int_0^t c(t)u_1(s)ds,$

$p_2(t) = p_2(0) + \dfrac{\lambda}{S_2}\displaystyle\int_0^t c(t)(u_1(s) - u_2(s))ds,$

(iii) $p_1(0) = p_1(T) - \mu_1\{[0, T]\} + \mu_2\{[0, T]\},$

$p_2(0) = p_2(T) - \mu_3\{[0, T]\} + \mu_4\{[0, T]\}.$

(iv) The following maximum is attained when $u_1 = \hat{u}_1(t)$ and $u_2 = \hat{u}_2(t), \ a.e. \ t \in$

$[0,T]$:

$$\max_{\substack{u_i \in U\\ i=1,2}} u_1\Bigg(-p_1(t)+\mu_1\{[0,t[\}-\mu_2\{[0,t[\}+p_2(t)-\mu_3\{[0,t[\}+\mu_4\{[0,t[\}$$
$$+\lambda c(t)\left(\frac{\hat{V}_1(t)}{S_1}+H_1-H_2-\frac{\hat{V}_2(t)}{S_2}\right)\Bigg)$$
$$+u_2\Bigg(-p_2(t)+\mu_3\{[0,t[\}-\mu_4\{[0,t[\}+\lambda c(t)\left(\frac{\hat{V}_2(t)}{S_2}+H_2\right)\Bigg)$$

(v) *supp* $\{\mu_i\}\subset I_i,\ i=\overline{1,4}$, where

$$I_1=\{t\in[0,T]\colon V_1(t)=V_1^m\}\qquad I_2=\{t\in[0,T]\colon V_1(t)=V_1^M\}$$
$$I_3=\{t\in[0,T]\colon V_2(t)=V_2^m\}\qquad I_4=\{t\in[0,T]\colon V_2(t)=V_2^M\}$$

where $p_1,\ p_2$ are absolutely continuous functions, $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4$ are nonnegative Borel measures and λ is a non-negative real number (we will assume that $\lambda=1$).

Observe that $\hat{V}_1(t)\neq V_1^m$ and $\hat{V}_2(t)\neq V_2^M,\ \forall t$, so we have $\mu_1\equiv\mu_4\equiv 0$. Since $\hat{V}_1(t)=V_1^M$ only at $t=\tau$, we can write $\mu_2\{[0,T]\}=\mu_2\{\tau\}$. From $(ii),(iii)$ and (v) we get

$$\mu_2\{[0,T]\}=\frac{c_1u_1^m\theta}{S_1}+\frac{c_2u_1^M\tau}{S_1},\ \mu_3\{[0,T]\}=\frac{c_1u_1^m\theta}{S_2}+\frac{c_2u_1^M\tau}{S_2}-\frac{c_2}{S_2}\int_\tau^T w_2(s)ds. \tag{5.29}$$

Information expressed by condition (iv) is analyzed separately on $[0,\theta[,\ [\theta,\tau[$ and $[\tau,T]$. The signals of coefficients of $u_1,\ u_2$ in (iv) determines the values of $\hat{u}_1$ and $\hat{u}_2$. We are departing from assumed $\hat{u}_1$ and $\hat{u}_2$, so, we obtain in this case those signals. On what follows we use (ii) and write the expressions in terms of $p_1(0)$ and $p_2(0)$.

To make the reading easier we define $\alpha=\left(\dfrac{\hat{V}_1(0)}{S_1}+H_1-\dfrac{\hat{V}_2(0)}{S_2}-H_2\right)$.

- On $[0,\theta[$:

$\mu_2\{[0,\theta[\}=\mu_3\{[0,\theta[\}=0$. As $\hat{u}_1(t)=u_1^m$ and $\hat{u}_2(t)=u_2^m=0$, we have

$$\frac{A_1c_1\theta}{S_1}-p_1(0)+p_2(0)+c_1\alpha\leq 0\qquad\qquad -p_2(0)+\frac{c_1\hat{V}_2(0)}{S_2}+c_1H_2\leq 0. \tag{5.30}$$

• On $[\theta, \tau[$:

$\mu_2\{[0,\tau[\} = 0$. As $\hat{u}_1(t) = 0 \in]u_1^m, u_1^M[$ and $\hat{u}_2(t) = u_2^m$ we get

$$\frac{A_1 c_1 t}{S_1} - p_1(0) + p_2(0) + c_1\alpha = \mu_3\{[\theta, t[\} \geq 0. \tag{5.31}$$

Taking the limit when $t \downarrow \theta$ $(t \to \theta,\ t > \theta)$ in the above expression, we obtain

$$\frac{A_1 c_1 \theta}{S_1} - p_1(0) + p_2(0) + c_1\alpha \geq 0. \tag{5.32}$$

From (5.30) and (5.32) we have

$$\frac{A_1 c_1 \theta}{S_1} - p_1(0) + p_2(0) + c_1\alpha = 0. \tag{5.33}$$

Consider now $\hat{u}_2(t) = 0 = u_2^m$. From (iv), (5.31) and (5.33), we get

$$\frac{A_1 c_1 t}{S_1} - p_1(0) + c_1\left(\frac{\hat{V}_1(0)}{S_1} + H_1\right) \leq 0.$$

The function on the left-hand side of the last inequality is increasing and continuous. The inequality is satisfied $\forall\ t \in [\theta, \tau[$ if and only if it is satisfied for $t = \tau$. We can write

$$p_1(0) \geq c_1\left(\frac{A_1\tau}{S_1} + \frac{\hat{V}_1(0)}{S_1} + H_1\right). \tag{5.34}$$

• On $[\tau, T[$:

$\mu_2\{[0,T]\} = \mu_2\{\tau\}$; $\hat{u}_1(t) = u_1^M$. After some calculus, we conclude from (iv)

$$-p_1(0) + \frac{c_2}{S_1}\left(-u_1^M\tau - u_1^m\theta + A_1 t\right) + p_2(0) + \frac{c_2}{S_2}\left(A_1 T - u_1^M\tau - u_1^m\theta\right) + c_2\alpha \geq 0. \tag{5.35}$$

Choosing $\hat{V}_2(t)$ and $\hat{u}_2(t)$ admissible, such that $\hat{V}_2(t) \in]V_2^m, V_2^M[$ and $\hat{u}_2(t) \in]u_2^m, u_2^M[$, $\forall\, t[\tau, T]$ we obtain $\mu_3\{[0,T]\} = \mu_3\{[\theta,\tau]\}$. Figure 5.6 shows one such trajectory. We get, also from (iv),

$$p_2(0) = \frac{c_2}{S_2}\left(u_1^M\tau - A_1 T + \hat{V}_2(0) + u_1^m\theta\right) + c_2 H_2. \tag{5.36}$$

Now, looking at the conclusions above, we can easily obtain:

- $\hat{V}_1(0),\ \hat{V}_2(0)$ and θ written in terms of data of the problem (see (5.27) and (5.28))

- $p_2(0)$ written in terms of the data of the problem (see (5.36))
- $p_1(0)$ (see (5.33))
- equations defining $\mu_3\{[0,t[\} = \mu_3\{[\theta,t[\}$ and $\mu_3\{[0,T]\}$ (see (5.31) and (5.29))
- equation defining $\mu_2\{[0,T]\} = \mu_2\{\tau\}$ (see (5.29))

Using the data of the problem and the above information we can write

$$\hat{V}_1(t)=\begin{cases} 143.56+0.51t, \ t\in[0,\theta[\\ 145.1+0.16t, \ t\in[\theta,\tau[\\ 147-0.287(t-12), \ t\in[\tau,T], \end{cases} \qquad \hat{V}_2(t)=\begin{cases} 49.85-0.351t, \ t\in[0,\theta[\\ 48.3, \ t\in[\theta,\tau[\\ 48.3+0.44496(t-12)-w(t), \ t\in[\tau,T], \end{cases}$$

where $w(t) = \int\limits_{\tau}^{t} w_2(s)ds$ and $\theta = 4.4$.
The multipliers p_1 and p_2 are now completely determined. The inequalities (5.34) and (5.35) must also be satisfied. For data (5.23), p_1 and p_2 can be defined as

$$p_1(t) = \begin{cases} 47.69+0.0086t, \ t\in[0,\theta[\\ 47.73, \ t\in[\theta,\tau[\\ 49.04-0.11t, \ t\in[\tau,T], \end{cases} \qquad p_2(t) = \begin{cases} 42.4-0.016t, \ t\in[0,\theta[\\ 42.33, \ t\in[\theta,\tau[\\ 39.93+0.19t-0.45w(t), \ t\in[\tau,T[\\ 43.03, \ t=T \end{cases}$$

and we can write

$$\mu_2\{[0,t[\} = \begin{cases} 0, \ t\in[0,\tau[\cup]\tau,T] \\ 1.27, \ t=\tau, \end{cases} \qquad \mu_3\{[0,t[\} = \begin{cases} 0, \ t\in[0,\theta[\cup]\tau,T] \\ 0.004t-0.02, \ t\in[\theta,\tau[\\ 0.63, \ t=\tau, \end{cases}$$

where $w(T) = \displaystyle\int_{\tau}^{T} w_2(s)ds = 3.79, \qquad V_1(0) = 143.56 \qquad$ and $\qquad V_2(0) = 49.85.$

Figure 5.7 overlaps the numerical solution with the analytical solution. These solutions are the same with exception of $V_2(t)$ on $[\tau,T]$. As outlined before, the solution is not unique on that set.

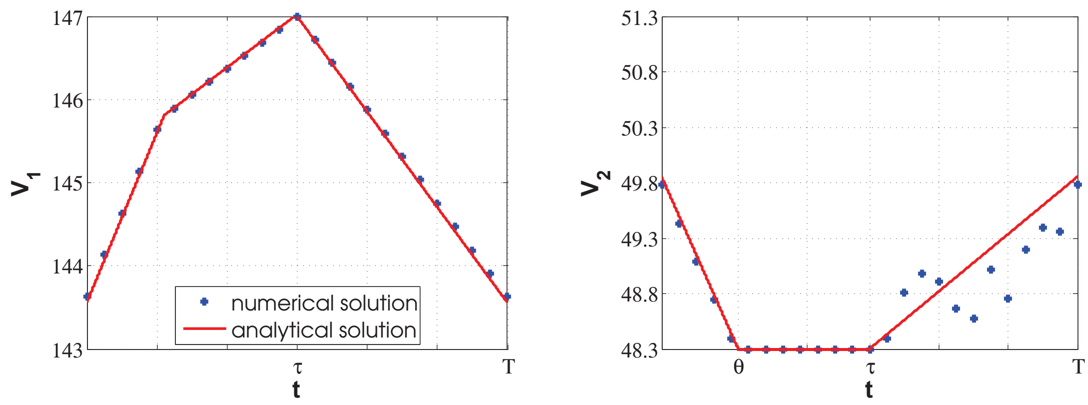


Figure 5.7: Numerical results vs. analytical results

5.2.2.4 Sufficient conditions of optimality

We now prove that the sufficient conditions of optimality stated in Theorem 5.2.1 are satisfied by some set of multipliers when the reference process is $(\hat{u}(\cdot), \hat{V}(\cdot))$ of the previous section. The formulation under consideration is (P_2) (see section 5.2.1.3).

Taking into account the profile of $\hat{V}_1(t)$ and $\hat{V}_2(t)$, we can write

$$\int_{[0,T]} d\eta_1(t) = \Delta\eta_1(\tau) \qquad \text{and} \qquad \int_{[0,T]} d\eta_2(t) = \int_{[\theta,a]} d\eta_2(t).$$

From (5.27) and (5.28), $\theta = 4.4$, $V_1(0) = 143.56$ and $V_2(0) = 49.85$.

Now, from condition *1.*

$$p_1(t) = p_1(0) - \frac{A_1}{S_1}\int_0^t c(t)dt + \int_{]0,t]} d\eta_1(t) - \int_{]0,t]} \left(H_1 + \frac{\hat{V}_1(t)}{S_1}\right) dc(t),$$

$$p_2(t) = p_2(0) - \int_{]0,t]} \left(H_2 + \frac{\hat{V}_2(t)}{S_2}\right) dc(t) + \int_{]0,t]} d\eta_2(t)$$

and from the periodicity condition $p_1(0) = p_1(T)$ we deduce

$$\int_{]0,T]} d\eta_1(t) = \frac{A_1\tau(c_1 + c_2)}{S_1} + \frac{(c_2 - c_1)}{S_1}\left(V_1^M - \hat{V}_1(T)\right),$$

$$\int_{]0,T]} d\eta_2(t) = \frac{(c_2 - c_1)}{S_2}(V_2^m - \hat{V}_2(T)).$$

Analysis of condition *2.* and *3.* of Theorem 5.2.1 leads to

$$p_1(0) \geq \frac{A_1 c_1 t}{S_1} + p_2(0) \geq 0 \qquad a.e.\ t \in [0, \theta[\qquad \text{and} \qquad p_2(0) \geq 0 \tag{5.37}$$

$$\int_{[\theta,t]} d\eta_2(s) = p_1(0) - p_2(0) - \frac{A_1 c_1 t}{S_1} \leq 0 \qquad a.e.\ t \in [\theta, \tau[\tag{5.38}$$

$$\int_{]0,t]} d\eta_1(s) = 0 \qquad p_2(0) \geq -\int_{[\theta,t]} d\eta_2(s) \qquad a.e.\ t \in [\theta, \tau[$$

From (5.37) and (5.38) we obtain $p_1(0) = p_2(0) + \frac{A_1 c_1 \theta}{S_1}$.

From conditions *3.* and *4.* of Theorem 5.2.1 we have

$$\int_{]0,t]} d\eta_1(s) = \Delta\eta_1(\tau) \ \text{ and } \int_{[\theta,t]} d\eta_2(s) \leq 0,\ \forall t \in [\tau, T[.$$

Due to the infinity of solutions for $V_2(t)$ on $[\tau, T]$, we can write, for $t \in [\tau, T[$

$$-p_2(t) = 0 \Rightarrow p_2(0) = (c_2 - c_1)\left(H_2 + \frac{V_2^m - u_1^m \theta}{S_2}\right)$$

$$-p_1(0) + p_2(0) + (c_2 - c_1)\left(H_1 + \frac{\hat{V}_1(T)}{S_1} - H_2 - \frac{\hat{V}_2(T)}{S_2}\right) + \frac{A_1 c_2 (t - T)}{S_1} \geq 0.$$

Working the above information and the data of the problem we can write:

$\lambda = 1$ and

$$p_1(t) = \begin{cases} 38.18 + 0.0039t,\ t \in [0, \tau[\\ -0.04t - 46.52,\ t \in [\tau, T[\\ 38.18,\ t = T, \end{cases} \qquad p_2(t) = \begin{cases} 38.16,\ t \in [0, \theta[\\ 38.18 - 0.004t,\ t \in [\theta, \tau[\\ 0,\ t \in [\tau, T[\\ 38.16,\ t = T, \end{cases}$$

$$\int_{]0,t]} d\eta_1(s) = \begin{cases} 0,\ t < \tau \\ 1.27,\ t \geq \tau, \end{cases} \qquad \int_{]0,t]} d\eta_2(s) = \begin{cases} 0,\ t < \theta \\ -0.004t + 0.02,\ t \in [\theta, \tau[\\ -0.63,\ t \geq \tau. \end{cases}$$

These multipliers were calculated taking into consideration conditions *1.* and *2.* of Theorem 4.2.2 essentially. Let us check that conditions *3.*, *4.* and *5.* are fully accomplished.

In $[0, \tau[\bigcup]\tau, T]$, $\hat{V}_1(t) \in]V_1^m, V_1^M[$ and we have $d\nu_1(t) = 0$. For $t = \tau$, $\hat{V}_1(\tau) = V_1^M$ and $d\nu_1(t) \geq 0$. Also $\hat{V}_2(t) \in]V_2^m, V_2^M[$ on $[0, \theta[\bigcup]\tau, T]$, and on that set $d\nu_2(t) = 0$. When $t \in [\theta, \tau]$, we have $\hat{V}_2(t) = V_2^m$ and $d\nu_2(t) \leq 0$. So we can claim that condition *3.* is verified.
Furthermore, $\hat{V}_1(\tau) = V_1^M$ and $\Delta\eta_1(\tau) = 1.27 > 0$. Also, since $\hat{V}_2(\tau) = V_2^m$ and $\Delta\eta_2(\tau) = -0.63 - (0.02 - 0.004 * 12) < 0$, condition *4.* is verified.
Condition *5.* is obviously satisfied. Observe that $\hat{V}_1(t) \in]V_1^m, V_1^M[$ in $[0, \tau[\cup]\tau, T]$ and $\hat{V}_2(t) \in]V_2^m, V_2^M[$ in $[0, \theta[\cup]\tau, T]$ and in these intervals $dc(t) = 0$.

We conclude that $(\hat{u}, \hat{V})$ is in fact a local minimizer for the problem.

Chapter 6

Global Solution

In this chapter we look for the global solution of the problem with 2 reservoirs and cost function given by (2.4). The formulation under consideration is (P_{2C}) given in (2.7). We look for an admissible process that optimizes the cost function over all admissible processes for the problem.
The issue here is the maximization of an indefinite quadratic form subject to linear constraints. The nonconvexity of the cost function enables the existence of several local minima and the application of global optimization methods is relevant to obtain the global optimal solution. We adopt two different numerical approaches that focus on global optimization techniques, the Chen-Burer algorithm and the projection estimation refinement method (PER method), used to reduce the dimension of the problem. Results and execution time of the two procedures are compared.

6.1 Discretized Problem

We consider the formulation (P_{2C}) presented in section 2.3.1 with $\tau = T/2$. Let N be an even natural number. In this analysis, we are assuming that $N = T = 24$. A discretization of the problem is undertaken and we get

$$
\begin{aligned}
\min \quad & -\frac{A_1 c_1}{s_1}\sum_{k=0}^{\frac{N}{2}-1} V_1(k) - \frac{A_1 c_2}{s_1}\sum_{k=\frac{N}{2}}^{N-1} V_1(k) + H_1(c_2-c_1)V_1(0) + \frac{c_2-c_1}{2S_1}V_1^2(0) \\
& - H_1(c_2-c_1)V_1\left(\frac{N}{2}\right) - \frac{c_2-c_1}{2S_1}V_1^2\left(\frac{N}{2}\right) + H_2(c_2-c_1)V_2(0) + \frac{c_2-c_1}{2S_2}V_2^2(0) \\
& - H_2(c_2-c_1)V_2\left(\frac{N}{2}\right) - \frac{c_2-c_1}{2S_2}V_2^2\left(\frac{N}{2}\right), \\
\text{s.t.} \quad & V_1(k+1) = V_1(k) + A_1 - u_1(k), \qquad (6.1) \\
& V_2(k+1) = V_2(k) + u_1(k) - u_2(k), \\
& V_i(0) = V_i(N), \\
& V_i(k) \in [V_i^m, V_i^M], \\
& u_i(k) \in [u_i^m, u_i^M], \quad \text{for } i = 1, 2.
\end{aligned}
$$

Define new variables x and y in the following way:

$$
x = \left[V_1(0), V_1\left(\frac{N}{2}\right), V_2(0), V_2\left(\frac{N}{2}\right)\right], \tag{6.2}
$$

and

$$
\begin{aligned}
y = \Bigg[& V_1(1), \cdots, V_1\left(\frac{N}{2}-1\right), V_1\left(\frac{N}{2}+1\right), \cdots, V_1(N-1), \\
& V_2(1), \cdots, V_2\left(\frac{N}{2}-1\right), V_2\left(\frac{N}{2}+1\right), \cdots, V_2(N-1)\Bigg].
\end{aligned} \tag{6.3}
$$

The cost function can now be expressed as

$$
I(x,y) = \langle a, x\rangle + \langle b, y\rangle + \langle x, Qx\rangle, \tag{6.4}
$$

where a and b are appropriate vectors, gathering the linear part of the cost, relative to x and y respectively. Q is an appropriate matrix representing the quadratic part of the cost function. More precisely, $Q,\ a$ and b are given by

$$Q = \begin{pmatrix} \frac{c_2 - c_1}{2S_1} & 0 & 0 & 0 \\ 0 & -\frac{c_2 - c_1}{2S_1} & 0 & 0 \\ 0 & 0 & \frac{c_2 - c_1}{2S_2} & 0 \\ 0 & 0 & 0 & -\frac{c_2 - c_1}{2S_2} \end{pmatrix} \tag{6.5}$$

$$a = \left[H_1(c_2 - c_1) - \frac{A_1 c_1}{S_1}, -H_1(c_2 - c_1) - \frac{A_1 c_2}{S_1}, H_2(c_2 - c_1), -H_2(c_2 - c_1) \right] \tag{6.6}$$

$$b = \left[\overbrace{-\frac{A_1 c_1}{S_1}, \cdots, -\frac{A_1 c_1}{S_1}}^{\frac{N}{2}\text{-1 components}}, \overbrace{-\frac{A_1 c_2}{S_1}, \cdots, -\frac{A_1 c_2}{S_1}}^{\frac{N}{2}\text{-1 components}}, \overbrace{0, \cdots, 0,}^{\frac{N}{2}\text{-1 components}} \overbrace{0, \cdots, 0}^{\frac{N}{2}\text{-1 components}} \right]. \tag{6.7}$$

The constraints of the problem are translated into

$$V_i(k) \in [V_i^m, V_i^M],\ \text{for } k = 0, \cdots, N-1 \text{ and } i = 1, 2,$$

$$V_1(N-1) + A_1 - V_1(0) \in [u_1^m, u_1^M], \tag{6.8}$$

$$V_2(N-1) + V_1(N-1) + A_1 - V_1(0) - V_2(0) \in [u_2^m, u_2^M], \tag{6.9}$$

and for $k = 0, \cdots, N-2$

$$V_1(k) + A_1 - V_1(k+1) \in [u_1^m, u_1^M], \tag{6.10}$$

$$V_2(k) + V_1(k) + A_1 - V_1(k+1) - V_2(k+1) \in [u_2^m, u_2^M]. \tag{6.11}$$

The expressions in (6.8 - 6.11) are equal to $u_1(N-1), u_2(N-1), u_1(k)$ and $u_1(k)$, respectively. Observe that we do not consider $V_i(N),\ i = 1, 2$, as variables. In fact since $V_i(N) = V_i(T) = V_i(0)$, we only need to ensure the admissibility of control action to go from $V_i(N-1)$ to $V_i(N) = V_i(0)$.

This is guaranteed by equations (6.8) and (6.9).

6.2 Chen-Burer Algorithm

The algorithm introduced by Jieqiu Chen and Samuel Burer (see [12] and reference therein) aims the optimization of nonconvex quadratic programming problems with linear and bounded constraints. It is a global optimization algorithm which involves two main components. The first one is a finite branch-and-bound (B&B) scheme, in which branching is based on the first-order Karush-Kuhn-Tucker conditions (KKT conditions). The other is a polyhedral-semidefinite relaxation that is applied at each node of the B&B tree. Such relaxation is derived from completely positive and doubly nonnegative programs (see [4–6]). One of the advantages of this method is that the B&B tree is finite. Other, is that we can develop stronger relaxations for the problem. The implementation of the global optimization solver for quadratic problems use the same syntax of the local optimization routine `quadprog` of the Matlab and requires an external linear programming solver.

Chen-Burer algorithm addresses problems of the form

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}x^T H x + f^T x \\ \text{s.t.} \quad & Ax \leq b \\ & A_{eq}x = b_{eq} \\ & LB \leq x \leq UB \end{aligned} \tag{6.12}$$

where $x \in \mathbb{R}^n$ is the variable and $H \in \mathbb{R}^{n\times n}$, $f \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $A_{eq} \in \mathbb{R}^{m_{eq}\times n}$, $b_{eq} \in \mathbb{R}^{m_{eq}}$, $LB \in \mathbb{R}^n$ and $UB \in \mathbb{R}^n$ are parameters. Here, H is a symmetric and generally not positive semidefinite matrix. The admissible set must have an interior point and A_{eq} must be of full row rank.
Components of the vectors LB and UB are allowed to be infinite, but the feasible set of the problem must be bounded. The algorithm is of free use and it is available at `http://dollar.biz.uiowa.edu/~sburer`.

6.3 Numerical Results

Problem (6.1) is considered, with data given by (5.23). Two different approaches are discussed and compared. In the first approach, we directly apply the Chen-

Burer algorithm. In the second one we use the specific structure of the cost function already outlined in (6.4), to reduce the dimension of the problem. A projection of the set of feasible solutions onto a subspace of the new cost function arguments is taken. Then the Chen-Burer algorithm is applied to the projected low-dimensional problem. The solution obtained is then used to construct an approximate solution to the original discrete problem via a simple convex programming problem. Such approximate solution is finally used as an initial guess for local optimization software.

6.3.1 1$^{\text{st}}$ Approach

Direct application of Chen-Burer algorithm to the original discretized problem (6.1), with $N = 24$, gives the optimal trajectory shown in Fig. 6.1. Observe that this problem involves 48 variables, 96 inequality constraints and 96 boxing constraints. By rearranging the position of variables, the resultant global solution can be rewritten as $(\hat{x}, \hat{y})$ where

$$\hat{x} = [143.64,\ 147,\ 49.76,\ 48.3].$$

Using relationship between $(\hat{x}, \hat{y})$ and $\hat{V}_i,\ i = 1, 2$ (see (6.2) and (6.3) and also (6.8)-(6.11)) we obtain the following pictures.

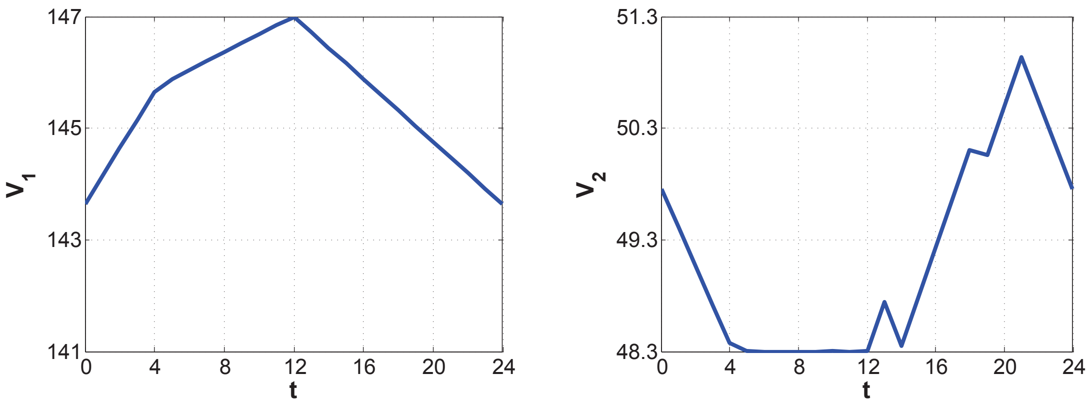


Figure 6.1: Trajectories associated to the global solution

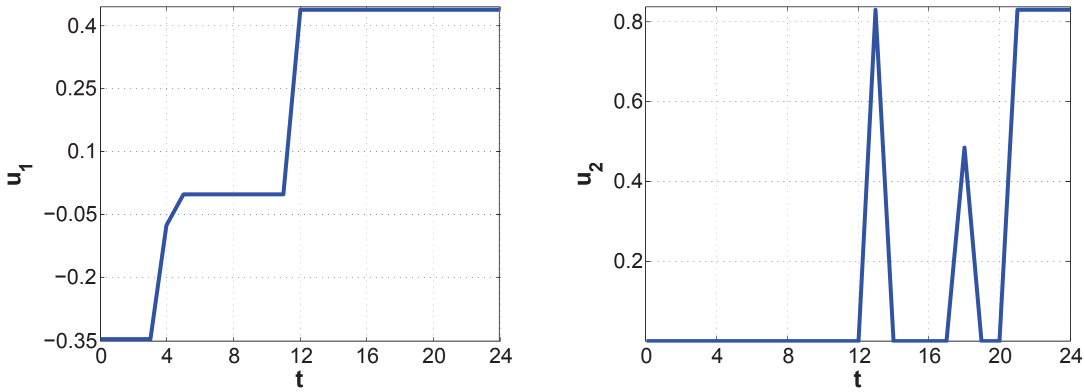


Figure 6.2: Control functions associated to the global solution

The cost value associated to this global solution is 308.918 and the execution time is 24 hours. This execution time is too long. In the next section we take another approach to solve this same problem.

6.3.2 2nd Approach

On a second approach, we start by reducing the dimension of the problem with the help of an algorithm, based on the *projection estimation refinement method* (PER) from [7]. This method approximates the orthogonal projection P of a polytope X onto a subspace, by a sequence of polytopes $P^0, P^1, ..., P^k, ...$ that tend to P, and with $P^k \subset P$ for all k. The number of vertices of the polytopes P_k increase by one at each iteration. A new polytope is constructed on the basis of the previous one by means of computing support functions for the projection P and Fourier-Motzkin convolution method (see [46]). In [13], a robust algorithm for solving this problem was proposed.

For approximating polyhedra, two descriptions are constructed simultaneously, one as a set of its vertices that belong to the boundary of P and the other as the solution set of the system of linear inequalities

$$P^k = \{x \in \mathbb{R}^q : \langle c_j, x \rangle \leq d_j,\ j = 1, 2, \cdots, N\}$$

where $c_j \in \mathbb{R}^q$ and $d_j \in \mathbb{R}$.

Figure 6.3 describes two iterations in the constructive process of the polytopes. The ●-dots on the left picture are the candidates to include in the next step. These are the most distant points of the feasible set, in the criteria space, from the convex set planes. They are found through maximization in directions orthogonal to the convex hull planes. In Fig. 6.3 (right), the most distant new point ●-dot is included into the convex hull planes.

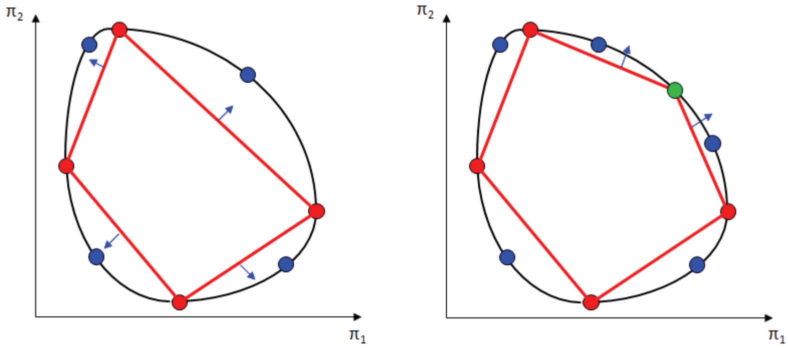


Figure 6.3: (Left) 1st iteration; (Right) 2nd iteration

If inequalities of internal approximating sets and the values of the corresponding support functions are known, it is easy to find external approximating sets $\bar{P}^0, \bar{P}^1, ..., \bar{P}^k$, which contain the projection P, i.e., $P^k \subset P \subset \bar{P}^k$ for all k.

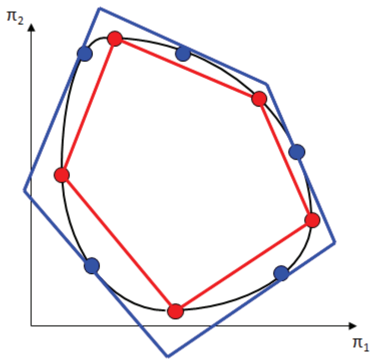


Figure 6.4: Internal estimation (convex hull of vertices) and external estimation (described by support-planes)

The objective is to find a pair $(P^k, \bar{P}^k)$, such that $d(P^k, \bar{P}^k) \leq \epsilon$, where $d(P^k, \bar{P}^k)$ is the Hausdorff distance between the sets P^k and $\bar{P}^k$ and ϵ is a given precision.

Computational details and a discussion of these techniques for polyhedral approximation can be found in [29, 30].
Returning to our discretized problem, recall that the cost function is

$$I(x, y) = \langle a, x \rangle + \langle b, y \rangle + \langle x, Qx \rangle \to \min,$$

where $Q,\ a$ and b are defined in (6.5),(6.6) and (6.7). We remind that

$$x = \left[V_1(0), V_1\left(\frac{N}{2}\right), V_2(0), V_2\left(\frac{N}{2}\right)\right], \quad \text{and}$$

$$\begin{aligned} y = \Big[& V_1(1), \cdots, V_1\left(\frac{N}{2}-1\right), V_1\left(\frac{N}{2}+1\right), \cdots, V_1(N-1), \\ & V_2(1), \cdots, V_2\left(\frac{N}{2}-1\right), V_2\left(\frac{N}{2}+1\right), \cdots, V_2(N-1)\Big]. \end{aligned}$$

Define a new variable

$$z = \langle b, y \rangle.$$

Once it is known the value of z, the value of $V_1(1)$ can be calculated as:

$$V_1(1) = -\left(\frac{s_1}{A_1 c_1} z + V_1(2) + \ldots + V_1\left(\frac{N}{2}-1\right) + \frac{c_2}{c_1}\left(V_1\left(\frac{N}{2}+1\right) + \ldots + V_1(N-1)\right)\right).$$

The cost function is expressed in terms of x and z as

$$\langle \bar{a}, \bar{x} \rangle + \langle \bar{x}, \bar{Q}\bar{x} \rangle \to \min, \tag{6.13}$$

where $\bar{x} = (x, z)$, $\bar{a} = (a, 1)$, and $\bar{Q} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$.

The projection of the set of feasible solutions onto the subspace of variables

$$(V_1(0), V_1(N/2), V_2(0), V_2(N/2), z)$$

is constructed using PER method. Taking this projection as admissible set and cost function (6.13), we obtain an optimization problem in $\mathbb{R}^5$.

The application of the Chen-Burer algorithm to this new simple problem, leads to a solution $(\hat{x}, \hat{z})$. A simpler convex programming problem can then be applied and an approximate solution $(\hat{x}, \hat{y})$ to the original discrete problem is obtained. Finally, this approximate solution is used as an initial guess in a local optimization software, from which we get a global solution for the original problem. This approach was followed for the discretized problem (6.1). A feasible set for the projected problem (exterior approximation with 15 inequalities and 10 boxing constraints) is calculated, using the PER method. The Chen-Burer algorithm is then applied to the low dimension problem and the solution obtained is

$$\hat{\bar{x}} = (\hat{x}, \hat{z}) = [140.66,\ 147,\ 48.30,\ 49.16,\ \ -68.18].$$

An approximate solution to the original discretized problem is calculated by solving the following convex quadratic programming problem with the function `QuadProg` from the Matlab:

$$\begin{aligned} \text{minimize} \quad & \|\Pi(w) - \hat{x}\|^2, \\ \text{s.t.} \quad & Aw \leq b, \\ & A_{eq}w = b_{eq}, \\ & LB \leq w \leq UB, \end{aligned}$$

$$\begin{aligned} \text{where} \qquad & w = (V_1(0), V_1(1), \cdots, V_1(N-1), V_2(0), V_2(1), \cdots, V_2(N-1)), \\ & \Pi(w) = (V_1(0),\ V_1(N/2),\ V_2(0),\ V_2(N/2)), \\ & \hat{x} = (\hat{V}_1(0),\ \hat{V}_1(N/2),\ \hat{V}_2(0),\ \hat{V}_2(N/2)). \end{aligned}$$

Figure 6.5 shows the resultant solution, trajectories and control variables.

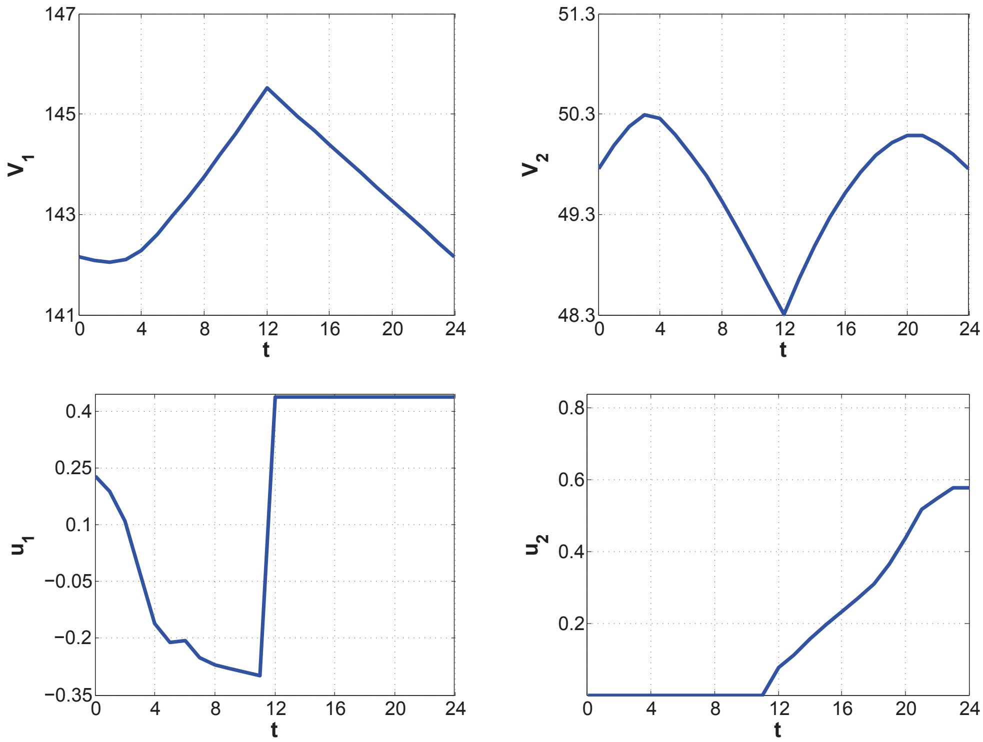


Figure 6.5: Approximate solution

The controls, together with $\hat{x}$, are then used as an initial guess to apply the optimization package from [45]. The final result is presented in Fig. 6.6

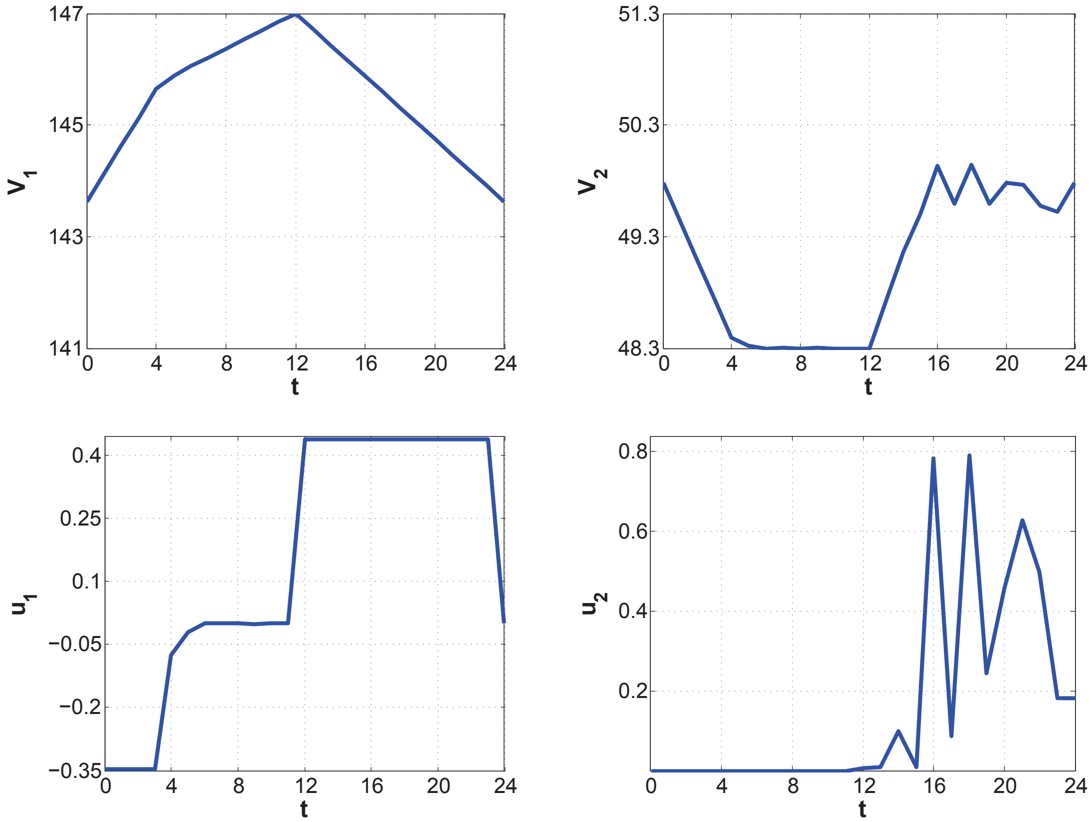


Figure 6.6: Final results with new approach

The cost obtained with the two approaches is almost the same, as we can see in table 6.1.

	1st approach	**2nd approach**
	• Chen - Burer Algorithm (directly)	• PER • Chen - Burer Algorithm • QuadProg • Local optimization
Cost	-308.9	-308.6
Total time execution	24 hours	1.48 min

Table 6.1: Comparison of methods

By carrying out this analysis, we may conclude that the 2^{nd} approach has a shorter runtime than the 1^{st} approach. This difference is quite significant so, the use of the 2^{nd} approach was quite rewarding.

The short gap between the two approaches is negligible. The 2^{nd} approach use several different numerical methods, it accumulates numerical errors from each of them, which may explain the difference of the two cost values.

Chapter 7

Conclusions

In this work, an optimal control problem for a model of hydro-electric power stations in cascade where some of the stations have reversible turbines, is analyzed. The objective was to optimize the profit of power production. In mathematical terms, this is a problem of minimizing an infinite-dimensional quadratic non-definite functional subject to linear and cone constraints. The presence of state constraints and the nonconvexity of the cost function contribute to an increased complexity of the problem. A characteristic feature of these problems is that it is possible that the minimizer is not isolated. In such cases sufficient conditions for optimality are much more effective than necessary conditions. Traditional sufficient conditions do not address non-isolated minimizers. Thus, new sufficient conditions for optimality are deduced for an abstract problem with mathematical structure as referred above. Sufficient conditions of optimality for the problem with N power stations and a periodic price were then directly derived. This showed that a periodic extension of the optimal solution on the period is optimal on the whole interval. This is an important result since it simplifies the elaboration of the management strategy. The new sufficient conditions are an important contribution from this thesis. They are originally deduced for an abstract problem of minimizing a quadratic non-definite functional subject to linear and cone constraints. The developed results are expected to be of interest to classes of problems other than those considered in this thesis. This will be considered in future work.

Examples of hydro-electric power stations in cascade are analyzed. Numerical optimization is performed, using available software. An analytical analysis of the problem is undertaken to validate the numerical solution. Taking into account the profile of the numerical solution, the sufficient conditions of optimality are proved to be met by

this solution. It was demonstrated that, in the framework of our model, and in the case of one power station, the use of reversible turbines always improves the profit.
It would be of interest to extend the analysis made in these simple cases of one and two power stations, to the more general problem of several power stations. It is still expected that, under some conditions on the data of the problem, the use of reversible turbines always improves the profit.

The nonconvexity of the cost function enables the existence of several local minima and global optimization methods are of particular relevance. Two different numerical approaches were considered. These approaches focus on global optimization techniques (Chen-Burer algorithm) and on the projection estimation refinement method (PER method) to reduce the dimension of the problem. Results and execution time for the two procedures were compared, and it was concluded that the 2^{nd} approach is more efficient and less time consuming.

Our models are meant to capture the essential features of real life problems in hydroelectric systems. By increasing its complexity, it is possible to achieve models that reflect more closely reality. This is a subject open to being further developed in the future.

Appendix A

Notations

Variables

$u_i(t)$ -	turbined/pumped flows of water for reservoir i at time t (hm^3/h)
$V_i(t)$ -	water volume in the reservoir i at time t (hm^3)
$Z_i(t)$ -	water level in reservoirs i at time t (m)
$h_i(t)$ -	heads (differences in water levels) at time t (m)

Hourly data

A_i -	incoming flows at time t (hm^3/h)
$c(t)$ -	price function of selling energy (€/MWh)

Constants

u_i^{0T} -	nominal turbined water volumes(m^3/s)
u_i^{0P} -	nominal pumped water volumes(m^3/s)
V_i^0 -	minimal water volume in reservoir, i (hm^3)
Z_i^{max} -	maximal water level (meters above sea level) in reservoir i (m)
Z_i^{min} -	minimal water level (meters above sea level) in reservoir i (m)
Z_i^0 -	nominal water level (meters above sea level) in reservoirs i (m)
h_i^0 -	nominal heads (m)
Δh_i^{0T} -	nominal head loss in case of turbining (m)
Δh_i^{0B} -	nominal head loss in case of pumping (m)
$\alpha_i,\ \beta_i,\ \zeta_i,\ \xi_i$ -	positive technical parameters
μ_i^T -	global income constant of the station i when it is turbining
μ_i^B -	global income constant of the station i when it is pumping
$\Phi_i^{prog},\ \Phi_i^{fort},\ \Phi_i^{cint}$ -	probability parameters associated to forced stop of station i

Appendix B

Data for the original model

Constants	Reservoir 1	Reservoir 2
Z_i^{max}	450.000	231.000
Z_i^{min}	435.900	226.100
V_i^M	147.000	66.000
V_i^m	86.700	48.300
u_i^{0T}	120.000	220.000
u_i^{0P}	96.000	—
V_i^0	48.900	48.000
Z_i^0	420.000	226.000
h_i^0	211.000	66.000
Δh_i^{0T}	3.000	2.000
Δh_i^{0P}	1.900	—
α_i	1.396	0.299
β_i	0.669	0.975
ζ_i	0.250	—
ξ_i	226.100	158.100
μ_i^T/μ_i^P	94.000	85.500
Φ_i^{prog}	0.087	0.087
Φ_i^{fort}	0.013	0.013
Φ_i^{cint}	0.015	0.015
A_i	0.158	0

Appendix C

Adjustment of parameters

The model presented in section 2.2.1 is complex and a simplified model (P_2) (see section 2.3) was considered for the problem of hydro-electric power production. Can we compare or confront the results of both models? To do that the parameters of the simplified model must be chosen with some criterion and linked in someway with the original ones. Here, we take as reference the data for the original problem presented in Appendix B.

Consider to adapt the control constraints. In the original model they are

$$\zeta_i(h_i(t) - h_i^0) - u_i^{0P} \leq u_i(t) \leq u_i^{0T}\sqrt{\frac{h_i(t)}{h_i^0}},\ i = 1, 2.$$

In the simplified model, we take boxing control constraints. Our option was to take intervals that contains the above ones.

$$\zeta_i(h_i^{min} - h_i^0) - u_i^{0P} \leq u_i(t) \leq u_i^{0T}\sqrt{\frac{h_i^{max}}{h_i^0}}.$$

Defining

$$h_1^{min} = 205\ (m), \quad h_2^{min} = 68\ (m) \quad h_1^{max} = 224\ (m), \quad h_2^{max} = 73\ (m),$$

and converting units ($m^3/s \to hm^3/h$), we obtain

$$-97.5 \leq u_1(t) \leq 123.6 \ (m^3/s) \quad \rightarrow \quad -0.351 \leq u_1(t) \leq 0.44496 \ (hm^3/h)$$

$$0 \leq u_2(t) \leq 231 \ (m^3/s) \quad \rightarrow \quad 0 \leq u_2(t) \leq 0.8316 \ (hm^3/h).$$

We keep the original values for V_i^M and V_i^m.

$$86.7 \leq V_1(t) \leq 147 \ (hm^3) \qquad 48.3 \leq V_2(t) \leq 66 \ (hm^3)$$

$$A_1 = 0.158 \ (hm^3).$$

To determine the parameters S_1, S_2, H_1 and H_2 for (P_2), we use the following expressions (see Figure 2.3)

$$Z_1 = H_1 + \frac{V_1}{S_1}, \qquad Z_2 = H_2 + \frac{V_2}{S_2}. \tag{C.1}$$

As we should ensure that

$$Z_1^{min} > H_1 > Z_2^{max} > Z_2^{min} > H_2 \Leftrightarrow 435.9 > H_1 > 231 > 226 > H_2,$$

then we take $H_1 = 300 \ m = 3 \ hm$ and $H_2 = 100 \ m = 1 \ hm$, as acceptable parameters. From (C.1), we take out the values of S_1 and S_2, obtaining $S_1 = 81.7 \ hm^2$ and $S_2 = 44.5 \ hm^2$.

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