Hitchin Pairs for the Indefinite Unitary Group

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To Ali and Sana

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Abstract

Let X be a closed Riemann surface of genus $g \geq 2$ and let K be the canonical bundle over X. A *quiver* Q is a directed graph specified by a set of vertices Q_0, a set of arrows Q_1 and head and tail maps $h, t : Q_1 \to Q_0$. A twisted *Q-bundle* is a twisted representation of Q in the category of coherent sheaves on X.

Let L be a line bundle on X. An L-twisted $\mathrm{U}(p,q)$-Higgs bundle is a special class of twisted quiver bundles for a quiver Q which contains an oriented cycle. So an L-twisted $\mathrm{U}(p,q)$-Higgs bundle is a quadruple $E = (V, W, \beta, \gamma)$, where V and W are holomorphic vector bundles on X of ranks p and q respectively,

$$\beta : W \to V \otimes L,$$
$$\gamma : V \to W \otimes L,$$

are holomorphic bundle maps. Note that if $L = K$ these are $\mathrm{U}(p,q)$-Higgs bundles. There is a usual slope stability condition depending on a real parameter α, and the value which is relevant for $\mathrm{U}(p,q)$-Higgs bundles is $\alpha = 0$.

The tuple $t := (p, q, a, b) = (\mathrm{rk}(V), \mathrm{rk}(W), \deg(V), \deg(W))$ will be referred to as the type of L-twisted $\mathrm{U}(p,q)$-Higgs bundle $E = (V, W, \beta, \gamma)$. For a fixed type $t = (p, q, a, b)$, we denote by $\mathcal{M}_\alpha(t)$ the moduli space of α-semistable L-twisted $\mathrm{U}(p,q)$-Higgs bundles of type t.

Our main object of study is that of the moduli space of twisted $\mathrm{U}(p,q)$-Higgs bundles. Here we examine how these moduli spaces change when the parameter crosses a critical value. We define *flip loci* in the stable loci $\mathcal{S}_{\alpha_c^\pm} \subset \mathcal{M}_{\alpha_c^\pm}^s(t)$ by the condition that the points in $\mathcal{S}_{\alpha_c^+}$ represent twisted $\mathrm{U}(p,q)$-Higgs bundles which are stable for the values above the critical value α_c but unstable for the values below α_c.

The main result we obtain is that the codimention of $\mathcal{S}_{\alpha_c^\pm}$ is strictly positive with a certain condition on the stability parameter, from which the birationality of the moduli spaces is immediate. A twisted $\mathrm{U}(p,q)$-Higgs bundle $E \in \mathcal{S}_{\alpha_c^\pm}$ is strictly α_c-semistable and to it we can assign a (unique up to isomorphism) Jordan-Hölder graded object which is the sum of stable objects of type t_i such that $t = \sum_i t_i$. The local structure of the moduli at the point determined by E can then be studied using this graded object. In particular, to estimate bounds on the codimension of $\mathcal{S}_{\alpha_c^\pm}$, it suffices to study the homological algebra of twisted $\mathrm{U}(p,q)$-Higgs bundles by considering a hypercohomological Euler characteristic $\chi(t_j, t_i)$ of a two term complex known as Hom-*complex*.

Concretely, we associate a Q-bundle to the Hom-complex of twisted $\mathrm{U}(p,q)$-Higgs bundles such that that a solution to the vortex equations on the twisted $\mathrm{U}(p,q)$-Higgs bundle induces a solution to a

natural Kähler-Einstein type equation on the associated Q-bundle. Using this method, we can then prove that $\chi(t_j, t_i) \leq 1 - g$, under a certain condition on the stability parameter.

Note that when one of the Higgs fields vanishes in twisted $U(p,q)$-Higgs bundles, one obtains a holomorphic triples. From this point of view, our results can be seen as a generalization of well known results for such triples.

Resumo

Seja X uma superfície de Riemann de género $g \geq 2$. Um *quiver* é um grafo orientado determinado por conjuntos Q_0 de vértices e Q_1 de flechas, juntamente com mapas ponta e cauda $h, t : Q_1 \to Q_0$. Um Q-fibrado torcido é uma representação torcida de Q na categoria de feixes coerentes sobre X.

Seja L um fibrado em rectas sobre X. Os $\mathrm{U}(p,q)$-fibrados de Higgs torcidos com L constituem uma classe especial de Q-fibrados trocidos para um quiver Q que contém um ciclo, nomeadamente, um tal fibrado de Higgs é uma escolha de $E = (V, W, \beta, \gamma)$ onde V e W são fibrados vectoriais holomorfos sobre X de dimensões p e q respectivamente, e onde

$$\beta : W \to V \otimes L \qquad \gamma : V \to W \otimes L$$

são mapas lineares holomorfos; note-se que no caso $L = K$ referimo-nos a estes Q-fibrados simplesmente como $\mathrm{U}(p,q)$-fibrados de Higgs. É possível definir uma condição de estabilidade declive para estes objectos, que depende da escolha de um parâmetro real α, sendo que no caso Higgs bundles usuais o valor relevante é $\alpha = 0$.

Referiremos-nos ao vector $t := (p, q, a, b) = (\mathrm{rk}(V), \mathrm{rk}(W), \deg(V), \deg(W))$ como o tipo do Q-fibrado $E = (V, W, \beta, \gamma)$ e denotaremos por $\mathcal{M}_\alpha(t)$ o espaço de moduli de $\mathrm{U}(p,q)$-fibrados de Higgs que são α-semiestáveis e de tipo t.

O nosso objecto de estudo é especificamente o espaço de $\mathrm{U}(p,q)$-fibrados de Higgs torcidos. Analisamos como estes espaços se alteram quando o parâmetro de estabilidade atravessa um valor crítico. Definimos uma *região de* flip $\mathcal{S}_{\alpha_c^\pm} \subset \mathcal{M}_{\alpha_c^\pm}^s(t)$ dentro do conjunto de pontos estáveis pela condição de que os pontos de $\mathcal{S}_{\alpha_c^\pm}$ representem fibrados que são estáveis para os valores acima do valor crítico α_c, mas instáveis para os valores abaixo. O resultado central é a demonstração de que a codimensão de $\mathcal{S}_{\alpha_c^\pm}$ é estritamente positiva sob certas condições no parâmetro de estabilidade, donde facilmente se conclui a equivalência birracional dos espaços de paraâmetros.

Um $\mathrm{U}(p,q)$-fibrado de Higgs torcido $E \in \mathcal{S}_{\alpha_c^\pm}$ é (estritamente) semiestável e portanto podemos fazer-lhe corresponder (de maneira única a menos de isomorfismo) um objecto graduado de Jordan-Hölder constituído por uma soma directa de $\mathrm{U}(p,q)$-fibrado de Higgs torcidos estáveis de tipo t_i tal que $t = \sum t_i$. A estrutura local do espaço de parâmetros no ponto determinado por E pode então ser estudada através deste objecto. Em particular, para se conseguem estimativas sobre a codimensão de $\mathcal{S}_{\alpha_c^\pm}$, é suficiente estudar a algebra homológica dos $\mathrm{U}(p,q)$-fibrados de Higgs torcidos considerando uma característica de Euler hipercohomologica $\chi(t_i, t_j)$ correspondente a um complexo de dois termos conhecido como complexo-Hom.

Concretamente, associamos um Q-fibrado ao complexo Hom de $\mathrm{U}(p,q)$-fibrados de Higgs torcidos tal que as soluções das equações-vórtice sobre estes fibrados de Higgs induzem soluções para

certas equações naturais de tipo Kähler-Einstein associadas ao Q-fibrado. Usando estas construções, conseguimos uma estimativa $\chi(t_j, t_i) \leq 1 - g$ sob certas condições no parâmtro de estabilidade.

Note-se que quando um dos campos de Higgs é zero, o nosso caso reduz-se áquele de triplos holomorfos e portanto os nossos resultados podem ser vistos como generalizações de resultados bem conhecidos sobre tais objectos.

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Chapter 1

Introduction

Let X be a Riemann surface of genus $g \geq 2$ and let $K = T^*X^{0,1}$ be the canonical line bundle of X.

A *quiver* is a directed graph, specified by a set of vertices Q_0 and a set of arrows Q_1, together with head and tail maps $h, t : Q_1 \to Q_0$. We shall only consider finite quivers, i.e. quivers for which Q_0 and Q_1 are finite. Given a quiver and a closed Riemann surface of genus $g \geq 2$, a *quiver bundle* is a collection of holomorphic vector bundles indexed by the vertices of the graph $\{V_i\}_{i \in Q_0}$ and morphisms indexed by the arrows of the graph $\{\varphi_a : V_{ta} \to V_{ha}\}_{a \in Q_1}$. The *type* of Q-bundle $\mathcal{E}$ is given by

$$t(E) = (\mathrm{rk}(V_i); \deg(V_i))_{i \in Q_0},$$

where $\mathrm{rk}(V_i)$ and $\deg(V_i)$ are the rank and degree of V_i, respectively. Note that this is independent of φ. A twisted Q-bundle is given by in addition specifying a line bundle M_a for each arrow, so the maps φ_a should go $\varphi_a : V_{ta} \otimes M_a \to V_{ha}$. The stability condition stays the same in chambers but wall-crossing phenomena arise and can be used in the study of the moduli spaces. An early spectacular success for this approach is Thaddeus' proof of the rank two Verlinde formula [38], using Bradlow pairs [6]. Triples are Q-bundles for a quiver with two vertices and a single arrow connecting them. Moduli spaces of triples have been studied extensively, using wall-crossing techniques, without being exhaustive, we mention [7, 8], where connectedness and irreducibility results for triples were studied, and the later work [28–32] of Muñoz and others on finer topological invariants, such as Hodge numbers. More generally, chains are Q-bundles for a quiver of type A_n. Chains have also been studied from a similar point of view; we mention here the work of Alvaréz-Consul–García-Prada–Schmitt [**?**], García-Prada–Heinloth–Schmitt [18] and García-Prada–Heinloth [17].

One of the main objectives of this thesis is to initiate a study of properties of moduli of Q-bundles when Q has oriented cycles.

L. Álvarez Cónsul and O. García-Prada, in [2], introduced a stability criterion for twisted Q-bundles. It depends on some real numbers α_i for each $i \in Q_0$. The $\alpha = (\alpha_i)$-slope of a Q-bundle $\mathcal{E} = (V, \varphi)$ is by definition

$$\mu_\alpha(\mathcal{E}) = \frac{\sum \big(\alpha_i \mathrm{rk}(V_i) + \deg(V_i) \big)}{\sum \mathrm{rk}(V_i)}.$$

Note that this only depends on the topological types of the bundles. A Q-bundle is α-semistable if $\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E})$ for any invariant Q-subbundle $\mathcal{F}$. Furthermore, $\mathcal{E}$ is α-stable if we have strict

inequality whenever $\mathcal{F}$ is a proper Q-subbundle. Finally, $\mathcal{E}$ is said to be α-polystable if it is a direct sum of α-stable bundles of the same α-slope. The moduli spaces of Q-bundles have been constructed by Schmitt using Geometric Invariant Theory in [33, 35]. More precisely, he constructed the moduli spaces of coherent $\mathcal{O}_X$-modules on a smooth projective variety X [34**?**]. The construction is similar to that of the moduli spaces of vector bundles over a smooth projective algebraic curve, in [27].

There is a *Hitchin-Kobayashi correspondence*, proved by L. Álvarez Cónsul and O. García-Prada, between stable holomorphic Q-bundles and solutions to natural gauge theoretic quiver vortex equations (generalizing the Hitchin equations),

$$\sqrt{-1}\Lambda F(V_i) + \sum_{i=ha} \varphi_a \varphi_a^* - \sum_{i=ta} \varphi_a^* \varphi_a = \tau_i \mathrm{Id}_{V_i}$$

for each $i \in Q_0$ such that $V_i \neq 0$, where $F(V_i)$ is the curvature of the Chern connection associated to the metric H_i on the holomorphic vector bundle V_i.

Our aim is to investigate the situation when Q has oriented cycles. Since the number of effective stability parameters is one less than the number of vertices of the quiver, in order to encounter wall crossing phenomena, we are led to considering the following quiver as the simplest non-trivial case:

(1.1)

When the twisting is by K, they are called $\mathrm{U}(p,q)$-Higgs bundles. Thus a $\mathrm{U}(p,q)$-Higgs bundle is a quadruple $E = (V, W, \beta, \gamma)$, where V and W are vector bundles of rank p and q, respectively, and each arrow is twisted by the canonical bundle K of X, meaning that

$$\beta : W \to V \otimes K, \; \gamma : V \to W \otimes K.$$

The stability notion for Q-bundles for the quiver (1.1) depends on a real parameter α and the value which is relevant for $\mathrm{U}(p,q)$-Higgs bundles is $\alpha = 0$. From the point of view of quiver bundles it is natural to allow for twisting of the maps β and γ by an arbitrary line bundle L on X, rather than just the canonical bundle K. The type of E is $t = (p, q, a, b) = (\mathrm{rk}(V), \mathrm{rk}(W), \deg(V), \deg(W))$. We denote by $\mathcal{M}_\alpha(t)$ the moduli space of α-semistable L-twisted $\mathrm{U}(p,q)$-Higgs bundles of type t. The parameter α is constrained to lie in an interval $\alpha_m \leq \alpha \leq \alpha_M$ (with $\alpha_M = \infty$ and $\alpha_m = -\infty$ if $p = q$) and the stability condition changes at a discrete set of critical values α_c for α.

It is also relevant to mention that these can be seen as a special case of *the G-Higgs bundles* defined by Hitchin in [24], where G is a real form of a complex reductive Lie group. Such objects provide a natural generalization of holomorphic vector bundles, which correspond to the case $G = \mathrm{U}(n)$ and zero Higgs field. A G-Higgs bundle over X is a pair (E, φ) consisting of a holomorphic $H^{\mathbb{C}}$-bundle E where $H \subset G$ is a maximal compact subgroup of G and $H^{\mathbb{C}} \subset G^{\mathbb{C}}$ their complexifications, and a section φ, called the Higgs field, is a global holomorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$, where $\mathfrak{m}$ is the complement of $\mathfrak{h}$ in the Cartan decomposition with respect to the non-degenerate $\mathrm{Ad}(G)$-invariant bilinear B on $\mathfrak{g}$.

The moduli space of polystable G-Higgs bundles over a compact Riemann surface X can be identified through non-abelian Hodge theory [12, 13, 24, 36, 37] with *the character variety* for representations of the fundamental group (see [15] for the Hitchin-Kobayashi correspondence for

G-Higgs bundles). The character variety is the quotient space of reductive representations of the fundamental group of *X* in *G*, modulo the action of *G* by conjugation.

Quiver bundles also arise naturally in the study of Higgs bundles because the fixed points of the natural $\mathbb{C}^*$-action on the Higgs bundle moduli space, given by scalar multiplication on the Higgs field, are quiver bundles for a quiver of type A_n. These can also be viewed as critical points for the Bott–Morse function obtained as a moment map for the action of $S^1 \subset \mathbb{C}^*$, where the function is given by

$$f : \mathcal{M}_\alpha(t) \longrightarrow \mathbb{R} \qquad (1.2)$$
$$E \mapsto \frac{1}{2\pi}(\|\beta\|_{L^2}^2 + \|\gamma\|_{L^2}^2)$$

where $\|\gamma\|_{L^2}^2 := i \int_X \operatorname{tr}(\gamma\gamma^*)$. Quiver bundles for a quiver of type A_n are known as *holomorphic chains* [1] and, in the case $n = 2$, as *holomorphic triples* [9]. The stability condition for Higgs bundles corresponds to a particular value of the stability parameter for the chains.

Moduli of holomorphic triples and their parameter dependence were studied in [9], where a birational description of the moduli was obtained for $\alpha \geq 2g - 2$. Álvarez-Cónsul and García-Prada in [?] characterize a region where the moduli spaces of chains of a given type are birationally equivalent, similarly to the case of triples, the stability parameters α_i must satisfy $\alpha_i - \alpha_{i-1} \geq 2g - 2$. Note that holomorphic triples can be viewed as $\mathrm{U}(p,q)$-Higgs bundles with $\gamma = 0$.

One of the main methods employed in [9] is the careful study of the variation of the moduli spaces with the parameter α, together with a study of the moduli space for extreme parameter values. The argument involves the *deformation complex* of a triple. This complex is a two-term complex of sheaves, which can actually be introduced in the generality of arbitrary Q-bundles. Triples form the local minima of the above mentioned Bott–Morse function, defined in (1.2), on the moduli space of $\mathrm{U}(p,q)$-Higgs bundles, and so those results could be used in [8] to prove connectedness of moduli of $\mathrm{U}(p,q)$-Higgs bundles.

In this thesis we study the variation with α of the moduli space of L-twisted $\mathrm{U}(p,q)$-Higgs bundles with a view to obtaining birationality results. Up until now generalizations of the work on triples in [9], such as [?], have focused on chains of length greater than 2. Thus one relevant new feature of our results is that they apply to Q-bundles for a quiver Q which contains an oriented cycle. Our main result is that the moduli spaces are birational for a certain range of the parameter. More precisely

Theorem. *Fix a type $t = (p,q,a,b)$. Let α_c be a critical value and $\varepsilon > 0$ be small enough that α_c is the only critical value between $\alpha_c^- = \alpha_c - \varepsilon$ and $\alpha_c^+ = \alpha_c + \varepsilon$. If either one of the following conditions holds:*

(1) $-\deg(L) < a/p - b/q < 0$ *and* $\alpha_c \in [0, \frac{2pq}{pq-q^2+p+q}(b/q - a/p - \deg(L)) + \deg(L))$.

(2) $0 < a/p - b/q < \deg(L)$ *and* $\alpha_c \in (\frac{2pq}{pq-p^2+p+q}(b/q - a/p + \deg(L)) - \deg(L), 0]$.

Then then the moduli spaces $\mathcal{M}_{\alpha_c^-}(t)$ and $\mathcal{M}_{\alpha_c^+}(t)$ are birationally equivalent.

Under suitable co-primality conditions on the topological invariants (p,q,a,b) we also have results for the full moduli spaces $\mathcal{M}_\alpha(t)$; we refer to Theorem 3.6.3 below for the precise result.

A systematic study of U(p,q)-Higgs bundles was carried out in [8], based on results for holomorphic triples from [9]. In particular, it was shown that the moduli space of U(p,q)-Higgs bundles is irreducible (again under suitable co-primality conditions). Using these results, we deduce the following corollary to our main theorem (see Theorem 3.6.5 below).

Theorem. *Fix a type $t = (p,q,a,b)$. Suppose that $(p+q,a+b) = 1$ and $L = K$. If either one of the following conditions holds:*

(1) $a/p - b/q > -\deg(L)$, $q \leq p$ and $0 \leq \alpha_c^{\pm} < \alpha_0$,

(2) $a/p - b/q < \deg(L)$, $p \leq q$ and $\alpha_0' < \alpha_c^{\pm} \leq 0$.

Then the full moduli spaces $\mathcal{M}_\alpha(t)$ is irreducible.

This thesis is organized as follows:

In the first chapter we collect some general facts. Section 2.1 is an introduction to quiver bundles. In Section 2.2 we recall stability of quiver bundles depending on the real parameters and describe stability of the dual of a Q-bundle. Hitchin-Kobayashi correspondence for quiver bundles is reviewed in Section 2.3. Section 2.4 is a introduction to deformation theory, and we prove vanishing of hypercohomology in degree two, for some special case. In Section 2.5 we study the infinitesimal deformation of Q-bundle and show that it is canonically isomorphic to the first hypercohomology group of the Hom-complex, defined in Section 2.4. Section 2.7 is the specialization of the previous sections for twisted U(p,q)-Higgs bundles and in Section 2.8 we give a brief introduction to the holomorphic chains.

In chapter 3 we study the moduli spaces of twisted U(p,q)-Higgs bundles. Section 3.1 and Section 3.2 contain a proof of a generalization of a well-known Milnor-Wood type inequality and in it we analyze how the α-stability condition constrains the parameter range for fixed type $t = (p,q,a,b)$. Also we give a description of the critical values determined by the kernels and prove injectivity of β, γ for special values of parameter. In Section 3.3 we give a description of moduli spaces for large α and also we prove that for some values of α the α-stability of twisted U(p,q)-Higgs bundles are related to the stability of the associated Higgs bundle. Section 3.4 contains a proof of vanishing of hypercohomology in degree two, which is essential in our approach to the study of how the moduli spaces of twisted U(p,q)-Higgs bundles vary with the parameter. Using this earlier result we prove the smoothness of the moduli spaces, the main result stated in this section. In Section 3.5 we study the variations in the moduli spaces for fixed type $t = (p,q,a,b)$ and different values of α. In order to analyze the differences between moduli spaces when the stability parameter crosses a critical value, we study the homological algebra of twisted U(p,q)-Higgs bundles. We estimate the Euler characteristics $\chi(E'',E')$ when E'' and E' are polystable with the same α-slope, which is the main theorem stated in Subsection 3.5.2. Finally in Section 3.6 we give a proof of birationality of moduli spaces of twisted U(p,q)-Higgs bundles.

In Chapter 4 we study the deformation theory of holomorphic chains. We have to introduce an augmented version of the deformation complex, which is actually a Q-bundle. The main theorem mentioned in Section 4.1.2 is that given a solution to the vortex equations on a holomorphic chains, this result produces a solution to a natural Kähler-Einstein type equation on the associated Q-bundle. This gives a generalization of a well-known result for triples, [9]. In the case of triples, one can introduce

an augmented version of the deformation complex, which is actually a length three holomorphic chain. In [9, Lemma 4.3] it is shown that, given a solution to the vortex equations on a holomorphic triple, this result produces a solution on the augmented deformation complex. Using this generalization we simplified the proof of a result in [**?**], more precisely, the estimation of the hypercohomology Euler characteristic of deformation complex, as it was of crucial significance in order to show birationality.

Chapter 2

Preliminaries

In this chapter we introduce the main objects of our study. In particular, we give a brief introduction to gauge theory, stability and moduli for twisted quiver bundles. We recall (from [2] and [20]) vortex equations for twisted quiver bundles and their link with a stability condition.

2.1 Basic Definitions

First we discuss some of the basic differential geometry of complex vector bundles. Good references for this part are Kobayashi's book [26] and Griffiths and Harris [21].

Let X be a closed Riemann surface of genus $g > 1$. Algebraically a Riemann surface is the same as a non-singular complex projective curve. Throughout this chapter, given a smooth complex vector bundle M on X, $\mathcal{A}^k(M)$ (resp. $\mathcal{A}^{i,j}(M)$) is the complex vector space of smooth M-valued k-forms (resp. (i,j)-forms) on X, ω is a fixed Kähler form on X, which is the negative imaginary part of a fixed Hermitian metric on X, and $\Lambda : \mathcal{A}^{i,j}(M) \to \mathcal{A}^{i-1,j-1}(M)$ is contraction with ω.

Definition 2.1.1. A *connection* ∇ on $M \to X$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{A}^o(M) \to \mathcal{A}^1(M)$$

satisfying the Leibniz rule: $\nabla(fs) = df \otimes s + f\nabla s$, for a function $f \in C^\infty(M)$ and a section s, where d is the de Rham operator on M. Locally

$$\nabla = d + A$$

where the matrix $A = (A_{ij})$ is the connection matrix with A_{ij} is a 1-form on X.

Given a Hermitian metric H we call a connection *unitary* (we will always denote it by A or d_A) if it preserves H, i.e.

$$d\langle s_1, s_2\rangle = \langle As_1, s_2\rangle + \langle s_1, As_2\rangle.$$

Definition 2.1.2. The *curvature* of a connection is

$$F_\nabla = \nabla^2 : \mathcal{A}^0(M) \to \mathcal{A}^2(M)$$

where we are extending ∇ to n-forms in $\mathcal{A}^n(M)$ in the obvious way. If $\mathfrak{g}_M$ denotes the bundle of skew-hermitian endomorphisms of M and $\mathfrak{g}_M^{\mathbb{C}}$ its complexification, then $F_A \in \mathcal{A}^2(\mathfrak{g}_M)$ for a unitary

connection, and $F_\nabla \in \mathcal{A}^2(\mathfrak{g}_M^\mathbb{C})$ in general. Moreover, if the local expression of ∇ is $d + A$, the local expression of F_∇ is $dA + A^2$.

Definition 2.1.3. A $\bar{\partial}$-operator on a smooth complex vector bundle $M \to X$ is a $\mathbb{C}$-linear operator

$$\bar{\partial}_M : \mathcal{A}^0(M) \to \mathcal{A}^{0,1}(M)$$

satisfying the Leibniz rule.

Indeed, if $\{s_i\}$ be a local holomorphic frame, then the Leibniz rule uniquely determines the $\bar{\partial}_M$-operator on the underlying complex vector bundle. Conversely, since there is no integrability condition on Riemann surfaces, given a $\bar{\partial}_M$-operator as defined above one can always find local holomorphic frames (see [3]). Holomorphic vector bundles over X correspond to complex algebraic vector bundles.

Note that a connection always induces a $\bar{\partial}$-operator by taking its $(0,1)$ part. Conversely, a $\bar{\partial}$-operator gives a unique unitary connection, called the *Chern connection*, which we will denote by $\bar{\partial}_A = (\bar{\partial}_M, H)$.

Definition 2.1.4. The *complex gauge group* is defined by

$$\mathcal{G}^\mathbb{C} = \{ g : M \to M \mid g \text{ is a } \mathbb{C}^\infty \text{ bundle isomorphism} \}.$$

2.2 Quiver bundles

2.2.1 Quivers

A *quiver Q* is a directed graph specified by a set of vertices Q_0, a set of arrows Q_1 and head and tail maps $h, t : Q_1 \to Q_0$. We shall assume that Q is finite.

2.2.2 Twisted quiver sheaves and bundles

Let Q be a quiver and $M = \{M_a\}_{a \in Q_1}$ be a collection of finite rank locally free sheaves of $\mathcal{O}_X$-modules.

It is to be noted that we do not distinguish vector bundles and locally free finite rank sheaves.

Definition 2.2.1. An *M-twisted Q-sheaf* on X is a pair $\mathcal{E} = (V, \varphi)$, where V is a collection of coherent sheaves V_i on X, for each $i \in Q_0$, and φ is a collection of morphisms $\varphi_a : V_{ta} \otimes M_a \to V_{ha}$, for each $a \in Q_1$.

A morphism between twisted Q-sheaves (V, φ) and (W, ψ) on X is given by a collection of morphisms $f_i : V_i \to W_i$, for each $i \in Q_0$, such that the diagrams

$$
\begin{array}{ccc}
V_{ta} \otimes M_a & \xrightarrow{\varphi_a} & V_{ha} \\
\downarrow{\scriptstyle f_{ta} \otimes 1} & & \downarrow{\scriptstyle f_{ha}} \\
W_{ta} \otimes M_a & \xrightarrow{\psi_a} & W_{ha}
\end{array}
$$

commutes for every $a \in Q_1$.

In this way the M-twisted Q-sheaves form an Abelian category. The notions of Q-subsheaves, quotient Q-sheaves, as well as simple Q-sheaves are defined in the obvious way.

A *holomorphic M-twisted Q-bundle* is an M-twisted Q-sheaf $\mathcal{E} = (V, \varphi)$ such that the sheaf V_i is a holomorphic vector bundle, for each $i \in Q_0$. Let $\mathcal{E} = (V, \varphi)$ be a Q-bundle, the subbundles $(0,0)$ and $\mathcal{E}$ itself are called the *trivial subbundles*. The *type* of Q-bundle $\mathcal{E}$ is given by the tuple

$$t(E) = (\mathrm{rk}(V_i); \deg(V_i))_{i \in Q_0},$$

where $\mathrm{rk}(V_i)$ and $\deg(V_i))$ are the rank and degree of V_i, respectively. Note that this is independent of φ.

2.3 Stability via a slope function

A notion of stability for Q-bundles, depending on real parameters α_i, has been introduced by A. King (1993). L. Álvarez Cónsul and O. García-Prada established a *Hitchin-Kobayashi correspondence*. This is a bijective correspondence between gauge equivalence classes of solutions to the so-called *Q-vortex equations* and isomorphism classes of poly-stable Q-bundles, which generalizes the classical Narasimhan-Seshadri theorem for vector bundles.

Fix a tuple $\alpha = (\alpha_i) \in \mathbb{R}^{|Q_0|}$ of real numbers. For a non-zero Q-bundle $\mathcal{E} = (V, \varphi)$, the associated α-*slope* is defined as

$$\mu_\alpha(\mathcal{E}) = \frac{\sum\limits_{i \in Q_0} \left(\alpha_i \, \mathrm{rk}(V_i) + \deg(V_i) \right)}{\sum\limits_{i \in Q_0} \mathrm{rk}(V_i)}.$$

Definition 2.3.1. A Q-bundle $\mathcal{E} = (V, \varphi)$ is said to be α-(semi)stable if for all non-trivial Q-subbundle $\mathcal{F}$ of $\mathcal{E}$,

$$\mu_\alpha(\mathcal{F}) < (\leq) \mu_\alpha(\mathcal{E}).$$

An α-polystable Q-bundle is a finite direct sum of α-stable Q-bundles, all of them with the same α-slope.

A Q-bundle $\mathcal{E}$ is strictly α-semistable if it is α-semistable and there is a proper (non-trivial) subbundle $\mathcal{F} \subset \mathcal{E}$ such that $\mu_\alpha(\mathcal{F}) = \mu_\alpha(\mathcal{E})$.

Stability exhibits the following properties:

(1) If we translate the stability parameter $\alpha = (\alpha_i)_{i \in Q_0}$ by a global constant $c \in \mathbb{R}$, obtaining $\alpha' = (\alpha_i')_{i \in Q_0}$, with $\alpha_i' = \alpha_i + c$, then $\mu_{\alpha'}(\mathcal{E}) = \mu_\alpha(\mathcal{E}) - c$. Hence the stability condition does not change under global translations. So we may assume that $\alpha_0 = 0$.

(2) As usual with stability criteria, in Definition 2.3.1, to check α-stability of a Q-bundle $\mathcal{E} = (V, \varphi)$, it suffices to check $\mu_\alpha(\mathcal{F}) < \mu_\alpha(\mathcal{E})$ for the proper Q-subbundles $\mathcal{F} = (W_i)$, such that $W_i \subset V_i$ is saturated, i.e. such that the quotient V_i/W_i is torsion-free, for each $i \in Q_0$. That subbundle is saturated means that it is obtained by taking the kernel of the induced map $V_i \to Q/\mathrm{Tor}(Q) \to 0$, where $Q = V_i/W_i$.

(3) If (V, φ) is stable and $\lambda \in \mathbb{C}^*$, then $(V, \lambda \varphi)$ is stable.

The following is a well-known fact, e.g., see [35, Exercise 2.5.6.6]. Consider a strictly α-semistable (semistable but not stable) Q-bundle $\mathcal{E} = (V, \varphi)$. As it is not α-stable, $\mathcal{E}$ admits a Q-subbundle $\mathcal{F} \subset \mathcal{E}$ of the same α-slope which is preserved by φ. If $\mathcal{F}$ is a Q-subbundle of $\mathcal{E}$ of least rank and same α-slope which is preserved by φ, it follows that $\mathcal{F}$ is α-stable and hence the induced pair $(\mathcal{F}, \varphi)$ is stable. Then, by induction one obtains a flag of Q-subbundles

$$\mathcal{F}_0 = 0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \mathcal{E}$$

where $\mu_\alpha(\mathcal{F}_i/\mathcal{F}_{i-1}) = \mu_\alpha(\mathcal{E})$ for $1 \leq i \leq m$, and where the induced Q-bundles $(F_i/F_{i-1}, \varphi_i)$ are α-stable. This is the *Jordan-Hölder filtration* of $\mathcal{E}$, and it is not unique. However, the *associated graded object*

$$\mathrm{Gr}(\mathcal{E}, \varphi) := \bigoplus_{i=1}^{m} (\mathcal{F}_i/\mathcal{F}_{i-1}, \varphi_i)$$

is unique up to isomorphism.

Definition 2.3.2. Two semi-stable Q-bundles $\mathcal{E} = (V, \varphi)$ and $\mathcal{E}' = (W, \psi)$ are said to be *S-equivalent* if $\mathrm{Gr}(V, \varphi) \cong \mathrm{Gr}(W, \psi)$.

Each *S*-equivalence class contains a unique polystable representative.

Remark 2.3.3. If a Q-bundle $\mathcal{E} = (V, \varphi)$ is stable, then the induced Jordan-Hölder filtration is trivial and therefore the isomorphism class of the graded object is the isomorphism class of the original Q-bundle.

2.3.1 The dual Q-bundle

A Q-bundle $\mathcal{E} = (V, \varphi)$ has a corresponding dual Q-bundle $\mathcal{E}^* := (V^*, \varphi^*)$, where V_i^* is the dual of V_i and φ_a^* is the transpose of φ_a. The following result relates stability parameter of a Q-bundle and its dual.

Lemma 2.3.4. *$\mathcal{E}$ is $\alpha = (\alpha_i)_{i \in Q_0}$-(semi)stable, if and only if $\mathcal{E}^*$ is $-\alpha = (-\alpha_i)_{i \in Q_0}$-(semi)stable.*

Proof. First note that there is a one-to-one correspondence between Q-subbundles of $\mathcal{E}^*$ and quotient bundles of $\mathcal{E}$. Now we can conclude the result using the following equality

$$\mu_{-\alpha}(\mathcal{E}^*) = -\mu_\alpha(\mathcal{E}).$$

$\square$

2.3.2 Moduli spaces of twisted Q-bundles

The moduli spaces of twisted Q-bundles have been constructed by Schmitt using Geometric Invariant Theory in [33, 35]. These moduli spaces depend on the real parameters α_i, for all $i \in Q_0$.

Fix a type $t = (r_i; d_i)_{i \in Q_0}$. Denote by $\mathcal{M}_\alpha(t)$ the moduli space of α-polystable twisted Q-bundles of type t and the moduli space of α-stable Q-bundles by $\mathcal{M}_\alpha(t)^s \subset \mathcal{M}_\alpha(t)$.

2.4   The Gauge equations

The gauge equations will also depend on a fixed collection q of Hermitian metrics q_a on M_a, for each $a \in Q_1$, which we fix once and for all. Let $\mathcal{E} = (V, \varphi)$ be a M-twisted Q-bundle on X. A Hermitian metric on $\mathcal{E}$ is a collection H of Hermitian metrics H_i on V_i, for each $i \in Q_0$ with $V_i \neq 0$. To define the gauge equations on $\mathcal{E}$, we note that $\varphi_a : V_{ta} \otimes M_a \to V_{ha}$ has a smooth adjoint morphism $\varphi_a^* : V_{ha} \to V_{ta} \otimes M_a$ with respect to the Hermitian metrics $H_{ta} \otimes q_a$ on $V_{ta} \otimes M_a$ and H_{ha} on V_{ha}, for each $a \in Q_1$, so it makes sense to consider the compositions $\varphi_a \circ \varphi_a^*$ and $\varphi_a^* \circ \varphi_a$. The following definitions are found in [2].

Remark 2.4.1. Let α be the stability parameter. Define τ be collections of real numbers τ_i, for which

$$\tau_i = \mu_\alpha(\mathcal{E}) - \alpha_i, \ i \in Q_0. \tag{2.4.1}$$

Then α can be recovered from τ as following

$$\alpha_i = \tau_0 - \tau_i, \ i \in Q_0. \tag{2.4.2}$$

Definition 2.4.2. A Hermitian metric H satisfies the *quiver τ-vortex equations* if

$$\sqrt{-1}\Lambda F(V_i) + \sum_{i=ha} \varphi_a \varphi_a^* - \sum_{i=ta} \varphi_a^* \varphi_a = \tau_i \mathrm{Id}_{V_i} \tag{2.4.3}$$

for each $i \in Q_0$ such that $V_i \neq 0$, where $F(V_i)$ is the curvature of the Chern connection associated to the metric H_i on the holomorphic vector bundle V_i.

The following is the *Hitchin-Kobayashi correspondence* between the twisted quiver vortex equations and the stability condition for holomorphic twisted quiver bundles, given in [2, Theorem 3.1]:

Proposition 2.4.3. *A holomorphic Q-bundle $\mathcal{E}$ is α-polystable if and only if it admits a Hermitian metric H satisfying the quiver τ-vortex equations* (2.4.3), *where α and τ are related by* (2.4.1).

2.5   Deformation Theory of Q-bundles

Definition 2.5.1. Let $\mathcal{E} = (V, \varphi)$ and $\mathcal{F} = (W, \psi)$ be M-twisted Q-bundles. We introduce the following notation:

$$\mathcal{H}om^0 = \bigoplus_{i \in Q_0} \mathrm{Hom}(V_i, W_i),$$

$$\mathcal{H}om^1 = \bigoplus_{a \in Q_1} \mathrm{Hom}(V_{ta} \otimes M_a, W_{ha}).$$

With this notation we consider the complex of sheaves

$$\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{F}) : \ \mathcal{H}om^0 \xrightarrow{b} \mathcal{H}om^1, \tag{2.5.1}$$

defined by

$$b(f_i)_{i \in Q_0} = \sum_{a \in Q_1} b_a(f_{ta}, f_{ha}), \text{ for } f_i \in \text{Hom}(V_i, W_i) \tag{2.5.2}$$

where

$$b_a : \text{Hom}(V_{ta}, W_{ta}) \oplus \text{Hom}(V_{ha}, W_{ha}) \to \text{Hom}(V_{ta} \otimes M_a, W_{ha}) \hookrightarrow \mathcal{H}om^1 \tag{2.5.3}$$

is given by

$$b_a(f_{ta}, f_{ha}) = \left(f_{ha} \circ \varphi_a - \psi_a \circ (f_{ta} \otimes \text{Id}_{M_a}) \right).$$

The complex $\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{F})$ is called Hom-*complex* for Q-bundles, defined in [20].

Proposition 2.5.2. *[20, Theorem 4.1 and Theorem 5.1] Let $\mathcal{E} = (V, \varphi)$ and $\mathcal{F} = (W, \psi)$ be M-twisted Q-bundles, then there are natural isomorphisms*

$$\text{Hom}(\mathcal{E}, \mathcal{F}) \cong \mathbb{H}^0(\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{F})),$$
$$\text{Ext}^1(\mathcal{E}, \mathcal{F}) \cong \mathbb{H}^1(\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{F})),$$

and a long exact sequence associated to the complex $\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{F})$:

$$0 \longrightarrow \mathbb{H}^0(\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{F})) \longrightarrow H^0(\mathcal{H}om^0) \longrightarrow H^0(\mathcal{H}om^1) \longrightarrow \mathbb{H}^1(\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{F})) \tag{2.5.4}$$
$$\longrightarrow H^1(\mathcal{H}om^0) \longrightarrow H^1(\mathcal{H}om^1) \longrightarrow \mathbb{H}^2(\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{F})) \longrightarrow 0. \tag{2.5.5}$$

Definition 2.5.3. We denote by $\chi(\mathcal{E}, \mathcal{F})$ the hypercohomology Euler characteristic for the complex $\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{F})$, so we have the following

$$\chi(\mathcal{E}, \mathcal{F}) = \dim \mathbb{H}^0(\mathcal{H}om^\bullet) - \dim \mathbb{H}^1(\mathcal{H}om^\bullet) + \dim \mathbb{H}^2(\mathcal{H}om^\bullet).$$

As an immediate consequence from the long exact sequence (2.5.4) and the Riemann-Roch formula we can obtain the following.

Proposition 2.5.4. *For any Q-bundles $\mathcal{E}$ and $\mathcal{F}$ we have*

$$\chi(\mathcal{E}, \mathcal{F}) = \chi(\mathcal{H}om^0) - \chi(\mathcal{H}om^1) \tag{2.5.6}$$
$$= (1 - g)(\text{rk}(\mathcal{H}om^0) - \text{rk}(\mathcal{H}om^1)) + \deg(\mathcal{H}om^0) - \deg(\mathcal{H}om^1) \tag{2.5.7}$$

The previous proposition shows that $\chi(\mathcal{E}, \mathcal{F})$ only depends on the types $t' = t(\mathcal{E})$ and $t = t(\mathcal{F})$ so we may use the notation

$$\chi(t', t) := \chi(\mathcal{E}, \mathcal{F}).$$

We recall the type of a Q-bundle $\mathcal{E} = (V, \varphi)$ is given by $t(\mathcal{E}) = (\text{rk}(V_i); \deg(V_i))_{i \in Q_0}$.

Lemma 2.5.5. *For any extensions $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of holomorphic Q-bundles,*

$$\chi(\mathcal{E}, \mathcal{E}) = \chi(\mathcal{E}', \mathcal{E}') + \chi(\mathcal{E}'', \mathcal{E}'') + \chi(\mathcal{E}'', \mathcal{E}') + \chi(\mathcal{E}', \mathcal{E}'').$$

Proof. Since the Euler characteristic is topological, we may assume that $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$. Then it is clear that the Hom-complexes satisfy:

$$\mathcal{E}nd^\bullet(\mathcal{E}) = \mathcal{H}om^\bullet(\mathcal{E}', \mathcal{E}') \oplus \mathcal{H}om^\bullet(\mathcal{E}'', \mathcal{E}'') \oplus \mathcal{H}om^\bullet(\mathcal{E}'', \mathcal{E}') \oplus \mathcal{H}om^\bullet(\mathcal{E}', \mathcal{E}''),$$

from which the result is immediate. $\qquad\square$

2.5.1 Vanishing of hypercohomology in degrees zero and two

The following is the analogous result for semistable vector bundles, given the identification of $\mathbb{H}^0(\mathcal{H}om^\bullet(\mathcal{E}', \mathcal{E}))$ with $\mathrm{Hom}(\mathcal{E}', \mathcal{E})$.

Proposition 2.5.6. *Let $\mathcal{E}'$ and $\mathcal{E}$ are α-semistable Q-bundles.*

(1) If $\mu_\alpha(\mathcal{E}) < \mu_\alpha(\mathcal{E}')$, then $\mathbb{H}^0(\mathcal{H}om^\bullet(\mathcal{E}', \mathcal{E})) = 0$.

(2) If $\mu_\alpha(\mathcal{E}') = \mu_\alpha(\mathcal{E})$, and $\mathcal{E}'$ is α-stable, then

$$\mathbb{H}^0(\mathcal{H}om^\bullet(\mathcal{E}', \mathcal{E})) \cong \begin{cases} 0 & \text{if } \mathcal{E} \not\cong \mathcal{E}' \\ \mathbb{C} & \text{if } \mathcal{E} \cong \mathcal{E}'. \end{cases}$$

For any Q-bundle $\mathcal{E} = (V, \varphi)$ we can associate a $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle for $n = \sum_{i \in Q_0} \mathrm{rk}(V_i)$ as follows:

$$\widetilde{E} = \left(\bigoplus_{i \in Q_0} V_i, \varphi = \begin{array}{cc} & \cdots \quad V_{ha} \quad \cdots \\ \begin{matrix} \vdots \\ V_{ta} \\ \vdots \end{matrix} & \begin{pmatrix} & & \\ & \varphi_a & \\ & & \end{pmatrix} \end{array} \right).$$

In the following we assume that the Q-bundles are twisted with a fixed line bundle L for each arrow.

Proposition 2.5.7. *Let $\mathcal{E}'$ and $\mathcal{E}$ be L-twisted α-semistable Q-bundles and let $\deg(L) \leq 2 - 2g$. If $\mathcal{E}'$ and $\mathcal{E}$ be stable as $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles and $\mu_\alpha(\mathcal{E}) = \mu_\alpha(\mathcal{E}')$, then $\mathbb{H}^2(\mathcal{H}om^\bullet(\mathcal{E}', \mathcal{E})) = 0$.*

Proof. From the long exact sequence (2.5.4) it is clear that $\mathbb{H}^2(\mathcal{H}om^\bullet(\mathcal{E}', \mathcal{E})) = 0$ if and only if the map $H^1(\mathcal{H}om^0) \to H^1(\mathcal{H}om^1)$ is surjective. By Serre duality this is equivalent to the injectivity of the map

$$H^0(\mathcal{H}om^{1*} \otimes K) \to H^0(\mathcal{H}om^{0*} \otimes K).$$

Since $\mathcal{E}'$ and $\mathcal{E}$ are stable as $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles, using the previous proposition, they are simple. Thus,

$$\ker(H^0(\mathrm{Hom}(\mathcal{E}, \mathcal{E}')) \to H^0(\mathrm{Hom}(\mathcal{E}, \mathcal{E}') \otimes L^*)) = \begin{cases} 0 & \text{if } \mathcal{E} \not\cong \mathcal{E}' \\ \mathbb{C} & \text{if } \mathcal{E} \cong \mathcal{E}'. \end{cases} \tag{2.5.8}$$

Note that, if we consider $\mathcal{E}'$ and $\mathcal{E}$ as $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle, then $\mathrm{Hom}(\mathcal{E}, \mathcal{E}') = \mathcal{H}om^{0*} \oplus (\mathcal{H}om^{1*} \otimes L^*)$. Using this observation and the fact that the map $\mathrm{Hom}(E, E') \to \mathrm{Hom}(E, E') \otimes L^*$ interchanges

these two summands, we obtain in either case of (2.5.8)

$$\ker(H^0(\mathcal{H}om^{1*} \otimes L^*) \to H^0(\mathcal{H}om^{0*} \otimes L^*)) = 0.$$

If $L^* = K$ the above gives the result. Otherwise as $L^* \otimes K$ is negative, consequently $\ker(H^0(\mathcal{H}om^{1*} \otimes K) \to H^0(\mathcal{H}om^{0*} \otimes K)) = 0.$ □

2.6 Infinitesimal deformations

In this section we are interested in an infinitesimal study of deformations of the Q-bundle $(\mathcal{E}, \varphi)$. We show that the infinitesimal deformation space of Q-bundle is canonically isomorphic to the first hypercohomology group of Hom-complex. The proof can be done in two ways, firstly using Čech cohomology to represent an infinitesimal deformation of the Q-bundle as an object over $\mathrm{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2))$, and secondly by analytical approach of Dolbeault resolution and differential geometry. This result was certainly expected to hold, and various special cases have appeared in the literature but a proof covering the full generality of Q-bundles has not appeared. This follows a standard method like in the work of Biswas-Ramanan [5].

We have the following basic observation, [22, *II* Ex. 2.8]:

Proposition 2.6.1. *Let X be a scheme over field $\mathbb{C}$, and $x \in X$ a point with residue field k. Then the tangent space to X at x can be naturally described as the set of maps $\mathrm{Spec}\,\mathbb{C}[\varepsilon]/\varepsilon^2 \to X$ with set-theoretic image x.*

To understand the tangent space of the moduli space of quivers at a point $\mathcal{E} = (V, \varphi)$, by the above proposition, we are thus naturally let to consider families of the appropriate type over $\mathrm{Spec}\,\mathbb{C}[\varepsilon]/\varepsilon^2$.

Proposition 2.6.2. *Let $\mathcal{E} = (V, \varphi)$ be a Q-bundle. The space of infinitesimal deformations of $\mathcal{E}$ is canonically isomorphic to the first hypercohomology $\mathbb{H}^1(\mathcal{E}nd^\bullet(\mathcal{E}))$, where $\mathcal{E}nd^\bullet(\mathcal{E})$ is the Hom-complex defined in* (2.5.1)

Proof. (Using Čech cohomology) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a finite covering of X by affine open sets. Let $D_\varepsilon = \mathbb{C}[\varepsilon]/(\varepsilon^2)$. As the only open set in $\mathrm{Spec}(D_\varepsilon)$ containing (ε) is $\mathrm{Spec}(D_\varepsilon)$ itself, so any bundle $\mathbb{V}$ on $\mathrm{Spec}(D_\varepsilon) \times X$ can be trivialized on $\{\mathrm{Spec}(D_\varepsilon) \times U_i\}_{i \in I}$.

We may consider the Čech resolution of the complex $\mathcal{E}nd^\bullet(\mathcal{E})$ as follows:

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\Big\uparrow & & \Big\uparrow \\
\check{C}^1\big(\bigoplus_{i \in Q_0} \mathrm{End}(V_i)\big) & \xrightarrow{d^1} & \check{C}^1\big(\bigoplus_{a \in Q_1} \mathrm{Hom}(V_{ta}, V_{ha})\big) \\
\Big\uparrow{\scriptstyle \delta} & & \Big\uparrow{\scriptstyle \delta} \\
\check{C}^0\big(\bigoplus_{i \in Q_0} \mathrm{End}(V_i)\big) & \xrightarrow{d^0} & \check{C}^0\big(\bigoplus_{a \in Q_1} \mathrm{Hom}(V_{ta}, V_{ha})\big)
\end{array}
$$

So the first hypercohomology will be obtained from the following double complex,

$$\check{C}^0\big(\bigoplus_{i\in Q_0}\text{End}(V_i)\big)\big) \xrightarrow{\underline{d}^0} \check{C}^1\big(\bigoplus_{i\in Q_0}\text{End}(V_i)\big)\big)\oplus\check{C}^0\big(\bigoplus_{a\in Q_1}\text{Hom}(V_{ta},V_{ha})\big) \xrightarrow{\underline{d}^1} \check{C}^1\big(\bigoplus_{a\in Q_1}\text{Hom}(V_{ta},V_{ha})\big)$$

Hence $\mathbb{H}^1(\mathcal{H}om^\bullet(\mathcal{E},\mathcal{E})) = Z/B$, where Z consist of pairs $(\{s_{i,j}^k\}_{k\in Q_0}, \{\psi_i^a\}_{a\in Q_1})$, $s_{i,j}^k \in \Gamma(U_{i,j},\text{End}(V_k))$ and $\psi_i^a \in \Gamma(U_i,\text{Hom}(V_{ta},V_{ha}))$, such that

$$s_{i,j}^k + s_{j,l}^k = s_{i,l}^k \tag{2.6.1}$$

$$\psi_i^a - \psi_j^a = s_{i,j}^{ha} \circ \varphi_{a_j} - \varphi_{a_j}\circ s_{i,j}^{ta}, \text{ where } \varphi_{a_i} := \varphi_a|_{U_i}. \tag{2.6.2}$$

And, $B = \text{im}(\underline{d}^0)$ is the subspace of Z. Starting with an element of Z, we shall construct a bundle $\mathbb{V}_k$ on $X\times D_\varepsilon$ and $\underline{\varphi}_a \in \Gamma(X\times D_\varepsilon,\text{Hom}(\mathbb{V}_{ta},\mathbb{V}_{ha}))$ with isomorphisms $\mathbb{V}_k|_{X\times\{p_0\}} \to V_k$ and $\underline{\varphi}_a|_{X\times\{p_0\}} = \varphi_a$, where p_0 is the closed point. Take the bundle $\overline{V}_{k_i} := \pi^*(V_k|_{U_i})$ on $U_i\times D_\varepsilon$ for every i, where $\pi: U_i\times D_\varepsilon \to U_i$. Now we may identify with $\overline{V}_{k_{ij}}$ the restrictions of $\overline{V}_{k_i}$ and $\overline{V}_{k_j}$ to $U_{ij}\times D_\varepsilon$ by means of the isomorphism $1+s_{i,j}^k.\varepsilon$. Using (2.6.1) one can easily check the compatibility condition of these isomorphisms,

$$(1+s_{i,j}^k\varepsilon)\circ(1+s_{j,l}^k\varepsilon) = 1+s_{i,l}^k\varepsilon.$$

Therefore one can glue all the $\overline{V}_{k_i}$ together and obtain $\mathbb{V}_k \to X\times D_\varepsilon$. Next we show that $\mathbb{V}_k|_{X\times p_0} \cong V_k$. Let $i: X\times p_0 \hookrightarrow X\times D_\varepsilon$, then $i^*\mathbb{V}_k = \frac{\bigsqcup i^*\overline{V}_{k_i}}{\sim}$ with

$$i^*\overline{V}_k|_{U_{i,j}} \underset{i^*(1+s_{i,j}^k.\varepsilon)}{\longrightarrow} i^*\overline{V}_k|_{U_{i,j}},$$

which means $i^*\mathbb{V}_k = V_k$, since $i^*(1+s_{i,j}^k.\varepsilon) = \text{Id}$.
On $U_i\times D_\varepsilon$, we have

$$\underline{\varphi}_{a_i} = \varphi_{a_i} + \psi_i^a.\varepsilon \in \Gamma(U_i\times D_\varepsilon,\text{Hom}(\mathbb{V}_{ta},\mathbb{V}_{ha})).$$

We claim that these sections of $\text{Hom}(\mathbb{V}_{ta},\mathbb{V}_{ha})$ on $U_i\times D_\varepsilon$ patch together to give a global section. Indeed we have to show that over $U_{i,j}\times D_\varepsilon$ the following identity holds,

$$g_{i,j}\underline{\varphi}_{a_j} = \underline{\varphi}_{a_i}$$

where,

$$g_{i,j} = (1-(s_{i,j}^{ta})^*.\varepsilon)(1+(s_{i,j}^{ha}).\varepsilon) = (1+(s_{i,j}^{ha}-(s_{i,j}^{ta})^*).\varepsilon)$$

But the equality follows from the cocycle condition (2.6.2):

$$\begin{aligned}
g_{i,j}\underline{\varphi}_{a_j} &= (1+(s_{i,j}^{ha}-(s_{i,j}^{ta})^*).\varepsilon)(\varphi_{a_j}+\psi_j^a.\varepsilon) \\
&= \varphi_{a_j}+(s_{i,j}^{ha}\circ\varphi_{a_j}-\varphi_{a_j}\circ s_{i,j}^{ta}+\psi_j^a).\varepsilon \\
&= \varphi_{a_i}+\psi_i^a.\varepsilon \\
&= \underline{\varphi}_{a_i}.
\end{aligned}$$

Hence the above construction associate an infinitesimal deformation of $E = (V_i, \varphi_a)_{i \in Q_0, a \in Q_1}$ to $((s_{i,j}^k)_{k \in Q_0}, (\psi_i^a)_{a \in Q_1})$. Suppose that $((s_{i,j}^k)_{k \in Q_0}, (\psi_i^a)_{a \in Q_1}) \in B$, therefore

$$s_{i,j}^k = s_i^k - s_j^k,$$
$$\psi_i^a = s_i^{ha} \circ \varphi_{a_i} - \varphi_{a_i} \circ s_i^{ta}$$

Then $\mathbb{V}_k|_{U_{i,j} \times D_\varepsilon} \xrightarrow[1 + (s_i^k - s_j^k).\varepsilon]{\cong} \mathbb{V}_k|_{U_{i,j} \times D_\varepsilon}$. Hence if we consider the automorphism $1 + s_i^k.\varepsilon$ of $\mathbb{V}_k|_{U_{i,j} \times D_\varepsilon}$ then the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{V}_k|_{U_{i,j} \times D_\varepsilon} & \xrightarrow{1 + s_i^k.\varepsilon} & \mathbb{V}_k|_{U_{i,j} \times D_\varepsilon} \\
{\scriptstyle 1 + s_{i,j}^k.\varepsilon} \downarrow & & \downarrow {\scriptstyle \mathrm{id}} \\
\mathbb{V}_k|_{U_{i,j} \times D_\varepsilon} & \xrightarrow{1 + s_j^k.\varepsilon} & \mathbb{V}_k|_{U_{i,j} \times D_\varepsilon}
\end{array}
$$

This shows that, if $((s_{i,j}^k)_{k \in Q_0}, (\psi_i^a)_{a \in Q_1}) \in B$, then the construction will be a bundle isomorphic to $\pi_2^*(V_k)$ globally, where $\pi_2 : X \times D_\varepsilon \longrightarrow X$. Moreover we have

$$
\begin{aligned}
(1 - (s_i^{ha} - s_i^{ta}).\varepsilon)(\varphi_{a_i} + \psi_i^a.\varepsilon) &= \varphi_{a_i} + (\psi_i^a - \underbrace{(s_i^{ha} \circ \varphi_a - s_i^{ta} \circ \varphi_a)}_{\psi_i^a})\varepsilon \\
&= \varphi_{a_i}
\end{aligned}
$$

Thus the associated $\underline{\varphi_a}$ is isomorphic to the trivial one, $\pi_2^*(\varphi_a)$. Therefore if $((s_{i,j}^k)_{k \in Q_0}, (\psi_i^a)_{a \in Q_1}) \in B$ then the associated Q-bundle $(\mathbb{V}_k, \underline{\varphi_a})_{k \in Q_0, a \in Q_1}$ is isomorphic to the trivial one. Thus we have given a well-defined map from $\mathbb{H}^1(\mathcal{H}om^\bullet(\mathcal{E}, \mathcal{E}))$ into the space of infinitesimal deformations.

Now we want to define a map from the set of infinitesimal deformations of $\mathcal{E} = (V_i, \varphi_a)_{i \in Q_0, a \in Q_1}$ into $\mathbb{H}^1(\mathcal{E}nd^\bullet(\mathcal{E}))$. Suppose that $(\mathbb{V}_k, \underline{\varphi_a})_{i \in Q_0, a \in Q_1}$ is an infinitesimal deformation of $\mathcal{E}$. Using the fact that U_i is affine one can see that the bundle $\mathbb{V}_{k_i} = \mathbb{V}_k|_{U_i \times D_\varepsilon}$ is the pull-back of V_k on U_i. Therefore $\mathbb{V}_k$ is obtained by glueing $\mathbb{V}_{k_i}[\varepsilon]$ and $\mathbb{V}_{k_j}[\varepsilon]$ over $U_{i,j}[\varepsilon]$ by using some automorphism of $\mathbb{V}_{k_{i,j}}[\varepsilon]$. The automorphism of $\mathbb{V}_{k_{i,j}}[\varepsilon]$ is of the form $1 + g_{i,j}^k \varepsilon$, where

$$g_{i,j}^k : V_{k_i}|_{U_{i,j}} \xrightarrow{\cong} V_{k_j}|_{U_{i,j}}$$

satisfying the condition $g_{i,j}^k + g_{j,l}^k = g_{i,l}^k$ on $U_{i,j,k}$.

Also it can be seen that the homomorphism $\underline{\varphi_a}$ is given by $\varphi_a + \psi_i^a \varepsilon$ with $\psi_i^a \in \Gamma(U_i, \mathrm{Hom}(V_{ta}, V_{ha}))$ satisfying the compatibility condition on $U_{i,j}$, since $\underline{\varphi_a}$ is a global homomorphism. Thus we have

$$(1 + (g_{i,j}^{ha} - (g_{i,j}^{ta})^t).\varepsilon)(\varphi_a + \psi_i^a \varepsilon) = (\varphi_a + \psi_j \varepsilon).$$

This implies that $d^1(g_{i,j}^k) = \psi_i^a - \psi_j^a$ and hence $((g_{i,j}^k), (\psi_i^a))_{k \in Q_0, a \in Q_1} \in Z$. This gives a map from the set of infinitesimal deformations to $\mathbb{H}^1(\mathcal{E}nd^\bullet(\mathcal{E}))$. It is easy to see that the two maps are inverses of each other. $\qquad\square$

Here is the second proof by using the Dolbeault cohomology:

Proof. Let $\mathcal{D}^{0,1}(V_i)$ be the set of all $\bar{\partial}_i$ operators on V_i. Suppose

$$\mathcal{C} := \{\, (\bar{\partial}_i, \beta_a)_{i \in Q_0, a \in Q_1} \mid \bar{\partial}_{ta,ha}(\beta_a) = 0 \,\} \subset \prod_{i \in Q_0} \mathcal{D}^{0,1}(V_i) \times \prod_{a \in Q_1} \mathcal{A}^0(\operatorname{Hom}(V_{ta}, V_{ha}))$$

where $\bar{\partial}_{ta,ha} : \mathcal{A}^0(\operatorname{Hom}(V_{ta}, V_{ha})) \to \mathcal{A}^{0,1}(\operatorname{Hom}(V_{ta}, V_{ha}))$ is the induced $\bar{\partial}$ operator given by

$$\bar{\partial}_{ta,ha}(\eta) = \bar{\partial}_{ha} \circ \eta - \eta \circ \bar{\partial}_{ta}, \text{ for all } \eta \in \mathcal{A}^0(\operatorname{Hom}(V_{ta}, V_{ha})).$$

Let $\operatorname{Aut}_{C^\infty}(V_i)$ be the set of all C^∞ bundle isomorphisms. Define $\mathcal{G}^{\mathbb{C}} := \prod_i \operatorname{Aut}_{\mathbb{C}^\infty}(V_i)$. The group $\mathcal{G}^{\mathbb{C}}$ acts on $\prod_{i \in Q_0} \mathcal{D}^{0,1}(V_i) \times \prod_{a \in Q_1} \mathcal{A}^0(\operatorname{Hom}(V_{ta}, V_{ha}))$ by

$$(g_i).((\bar{\partial}_i)_{i \in Q_0}, (\beta_a)_{a \in Q_1}) = ((g_i \circ \bar{\partial}_i \circ g_i^{-1})_{i \in Q_0}, (g_{ha} \circ \beta_a \circ g_{ta}^{-1})_{a \in Q_1}).$$

We consider $\mathcal{M} = \mathcal{C}^s / \mathcal{G}^{\mathbb{C}} = \{\, (\bar{\partial}_i, \beta_a) \mid \bar{\partial}_{ta,ha}(\beta_a) = 0 \text{ and } (V_i, \beta_a) \text{ is stable} \,\}/\mathcal{G}^{\mathbb{C}}$. Strictly speaking one should use appropriate Sobolev completions as in Atiyah and Bott [4] see, for example, Hausel and Thaddeuse [23] for the case of Higgs bundles.

Now, we linearize the equation $\bar{\partial}_{ta,ha}(\beta_a) = 0$ as follows:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big(\bar{\partial}_{A_{ta}+t.\dot{A}_{ta}, A_{ha}+t.\dot{A}_{ha}}(\beta_a + t.\dot{\varphi}_a)\big)\big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\underbrace{\bar{\partial}_{A_{ta}+t.\dot{A}_{ta}}}_{\bar{\partial}_{A_{ta}}+t.\dot{A}_{ta}} \circ (\beta_a + t.\dot{\varphi}_a) - (\beta_a + t.\dot{\varphi}_a) \circ \underbrace{\bar{\partial}_{A_{ha}+t.\dot{A}_{ha}}}_{\bar{\partial}_{A_{ha}}+t.\dot{A}_{ha}}\big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big(\underbrace{\bar{\partial}_{A_{ha}\circ\beta_a - \beta_a\bar{\partial}_{A_{ta}}}}_{0} + \\
&\quad t.(\dot{A}_{ha}\beta_a - \beta_a.\dot{A}_{ta} + \bar{\partial}_{A_{ha}} \circ \dot{\varphi}_a - \dot{\varphi}_a \circ \bar{\partial}_{A_t a}) + t^2.(\dots)\big)\big|_{t=0} \\
&= \dot{A}_{ha} \circ \beta_a - \beta_a \circ \dot{A}_{ta} + \bar{\partial}_{ta,ha}(\dot{\beta}_a),
\end{aligned}
$$

therefore we have $[\dot{A}_a, \beta_a] + \bar{\partial}_{ta,ha}(\dot{\beta}_a) = 0$. Similarly, the linearization of the action will be as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(g(t).(\bar{\partial}_i, \beta_a)\big) = (-\bar{\partial}_i \psi_i, [\psi, \beta_a])$$

where $\psi = \dot{g} = \frac{\mathrm{d}}{\mathrm{d}t} g(t)\big|_{t=0} \in \prod \mathcal{A}^0(\operatorname{End}(V_i))$, for $g(t) \in \mathcal{G}^{\mathbb{C}}$, and $[\psi, \beta_a] = \psi_{ha} \circ \beta_a - \beta_a \circ \psi_{ta}$.

Define

$$\prod_{i \in Q_0} \mathcal{D}^{0,1}(V_i) \times \prod_{a \in Q_1} \mathcal{A}^0(\operatorname{Hom}(V_{ta}, V_{ha})) \xrightarrow{F} \prod \mathcal{A}^{0,1}(\operatorname{Hom}(V_{ta}, V_{ha}))$$

by

$$F((\bar{\partial}_i)_{i \in Q_0}, (\beta_a)_{a \in Q_1}) = (\bar{\partial}_{ta,ha}(\beta_a)).$$

Note that $\mathcal{C} = F^{-1}(0)$ and therefore $\ker(dF) = T\mathcal{C}$. Deformation complex of $((\bar{\partial}_i)_{i \in Q_0}, (\beta_a)_{a \in Q_1})$:

$$\prod_{a \in Q_1} \mathcal{A}^0(\operatorname{End}(V_i)) \to \prod_{i \in Q_0} \mathcal{A}^{0,1}(V_i) \times \prod_{a \in Q_1} \mathcal{A}^0(\operatorname{Hom}(V_{ta}, V_{ha})) \to \prod \mathcal{A}^{0,1}(\operatorname{Hom}(V_{ta}, V_{ha}))$$

where

$$\psi \mapsto (\bar{\partial}_i(\psi_i), [\psi, \beta_a]) \text{ and } (\dot{\alpha}_i, \dot{\phi}_a) \mapsto \bar{\partial}_{ta,ha}(\dot{\phi}_a) + [\dot{\alpha}, \beta_a].$$

Using

$$T_{[(\bar{\partial}_i, \beta_a)]}\mathcal{M} \cong T_{(\bar{\partial}_i, \beta_a)}\mathcal{C}/T_{(\bar{\partial}_i, \beta_a)}(\mathcal{G}^{\mathbb{C}}.(\bar{\partial}_i, \beta_a))$$

we conclude that $T_{[(\bar{\partial}_i, \beta_a)]}\mathcal{M}$ is isomorphic with the first cohomology group of the deformation complex.

Now consider the Dolbeault resolution of the complex $\mathcal{E}nd^{\bullet}(\mathcal{E})$ as follows:

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow \bar{\partial} & & \uparrow \bar{\partial} \\
\mathcal{A}^{0,1}(\bigoplus_{i \in Q_0} \mathrm{End}(V_i)) & \xrightarrow{d^1} & \mathcal{A}^{0,1}(\bigoplus_{a \in Q_1} \mathrm{Hom}(V_{ta}, V_{ha})) \\
\uparrow \bar{\partial} & & \uparrow \bar{\partial} \\
\mathcal{A}^{0}(\bigoplus_{i \in Q_0} \mathrm{End}(V_i)) & \xrightarrow{d^0} & \mathcal{A}^{0}(\bigoplus_{a \in Q_1} \mathrm{Hom}(V_{ta}, V_{ha}))
\end{array}$$

So the first hypercohomology will be obtained from the following double complex

$$\mathcal{A}^{0}(\bigoplus_{i \in Q_0} \mathrm{End}(V_i)) \longrightarrow \mathcal{A}^{0,1}(\bigoplus_{i \in Q_0} \mathrm{End}(V_i))) \oplus \mathcal{A}^{0}(\bigoplus_{a \in Q_1} \mathrm{Hom}(V_{ta}, V_{ha})) \longrightarrow \mathcal{A}^{0,1}(\bigoplus_{a \in Q_1} \mathrm{Hom}(V_{ta}, V_{ha}))$$

Note the Dolbealt resolution of the complex $\mathcal{E}nd^{\bullet}(\mathcal{E})$ is the deformation complex and hence

$$T_{[(\bar{\partial}_i, \beta_a)]}\mathcal{M} \cong \mathbb{H}^1(\mathcal{E}nd^{\bullet}(\mathcal{E})).$$

$\square$

2.7   Twisted $\mathrm{U}(p,q)$-Higgs bundles

An important example of twisted Q-bundles, which is our main object study in this paper, is that of twisted $\mathrm{U}(p,q)$-Higgs bundles on X given in the following. It is to be noted that twisted $\mathrm{U}(p,q)$-Higgs bundles in our study is twisted with the same line bundle for each arrow.

Definition 2.7.1. Let L be a line bundle on X. An *L-twisted* $\mathrm{U}(p,q)$-*Higgs bundle* is a twisted Q-bundle for the quiver

$$V \quad \rightleftharpoons \quad W$$
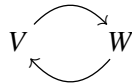
each arrow twisted by L, s.t. $\mathrm{rk}(V) = p$ and $\mathrm{rk}(W) = q$. Thus L-twisted $\mathrm{U}(p,q)$-Higgs bundle is a quadruple $E = (V, W, \beta, \gamma)$, where V and W are holomorphic vector bundles on X of ranks p and q respectively, and

$$\beta : W \longrightarrow V \otimes L,$$

$$\gamma : V \longrightarrow W \otimes L,$$

are holomorphic maps. The *type* of a twisted U(p,q)-Higgs bundle $E = (V, W, \beta, \gamma)$ is defined by a tuple of integers $t(E) := (p, q, a, b)$ determined by ranks and degrees of V and W, respectively.

Note that K-twisted U(p,q)-Higgs bundles can be seen as a special case of G-Higgs bundles ([25], see also [8, 10, 15, 19]), where G is a real form of a complex reductive Lie group.

2.7.1 Gauge Equations

For this L twisted quiver bundle one can consider the general quiver equations as defined in 2.4.3. Fixing a Hermitian metric on X, compatible with its Riemann surface structure, since $\dim_{\mathbb{C}} X = 1$, this metric will be Kähler, and so, there is a Kähler form ω that we can choose such that:

$$\int_X \omega = 2\pi.$$

Let $\tau = (\tau_1, \tau_2)$ be a pair of real numbers. A Hermitian metric H satisfies the *L-twisted quiver τ-vortex equations* on twisted U(p,q)-Higgs bundle E if

$$\sqrt{-1}\Lambda F_{H_V} + \beta\beta^* - \gamma^*\gamma = \tau_1 \operatorname{Id}_V, \tag{2.7.1}$$

$$\sqrt{-1}\Lambda F_{H_W} + \gamma\gamma^* - \beta^*\beta = \tau_2 \operatorname{Id}_W. \tag{2.7.2}$$

where F_{H_V} and F_{H_W} are the curvature of the Chern connections associated to the metrics H_V and H_W, respectively.

Remark 2.7.2.   (i) If a holomorphic twisted U(p,q)-bundle E admits a Hermitian metric satisfying the τ-vortex equations, then taking traces in (2.7.1), summing for V and W, and integrating over X, we see that the parameters τ_1 and τ_2 are constrained by $p\tau_1 + q\tau_2 = \deg(V) + \deg(W)$.

   (ii) If $L = K$ the equations are conformally invariant and so depend only on the Riemann surface structure on X. In this case they are *the Hitchin equations* for the U(p,q)-Higgs bundle, see [15].

Let $E = (V, W, \beta, \gamma)$ be a twisted U(p,q)-Higgs bundle, and α be a real number; α is called the *stability parameter*. The definitions of the previous section specialize as follows. The α-*slope* of E is defined to be

$$\mu_\alpha(E) = \mu(E) + \alpha\frac{p}{p+q},$$

where $\mu(E) := \mu(V \oplus W)$. A twisted U$(p,q)$-bundle E is α-*semistable* if, for every proper (non-trivial) subobject $F \subset E$,

$$\mu_\alpha(F) \leq \mu_\alpha(E).$$

Further, E is α-*stable* if this inequality is always strict. A twisted U(p,q)-bundle is called α-*polystable* if it is the direct sum of α-stable twisted U(p,q)-Higgs bundles of the same α-slope.

The value $\alpha = 0$ is the relevant value for the non-abelian Hodge theorem which identifies the moduli space of Higgs bundles with the character variety for representations of the fundamental group.

Remark 2.7.3. The stability can be defined using quotients as for vector bundles. Note that for any subobject $E' = (V', W')$ we obtain an induced quotient bundle $E/E' = (V/V', W/W', \overline{\beta}, \overline{\gamma})$ and E is α-(semi)stable if $\mu_\alpha(E/E')(\geq) > \mu_\alpha(E)$.

The following is a special case of the Hitchin-Kobayashi correspondence between the twisted quiver vortex equations and the stability condition for holomorphic twisted quiver bundles, stated in Proposition 2.4.3.

Proposition 2.7.4. *A solution to* (2.7.1) *exists if and only if E is α-polystable for $\alpha = \tau_2 - \tau_1$.*

2.8 Moduli Space of twisted $\mathrm{U}(p,q)$-Higgs bundles

The moduli space of $\mathrm{U}(p,q)$-Higgs bundles, in particular its connectedness properties, was studied in [8]. Those results apply to $L = K$ and $\alpha = 2g - 2$. This study in turn was based on a study of holomorphic triples, given in [9]. The moduli spaces of twisted $\mathrm{U}(p,q)$-Higgs bundles come in families parametrized by a parameter α. Moreover, the moduli spaces changes at a certain discrete set of critical values. For more details see Section 3.1.3.

Fix a type $t = (p, q, a, b)$. We denote the moduli space of α-semistable twisted $\mathrm{U}(p,q)$-Higgs bundles $E = (V, W, \beta, \gamma)$ which have the type $t(E) = (p, q, a, b)$, where $a = \deg(V)$ and $b = \deg(W)$, by

$$\mathcal{M}_\alpha(t) = \mathcal{M}_\alpha(p, q, a, b),$$

and the moduli space of α-stable twisted $\mathrm{U}(p,q)$-Higgs bundles by $\mathcal{M}_\alpha^s(t)$.

2.9 Holomorphic triples

Triples are Q-bundles for a quiver with two vertices and a single arrow connecting them. Moduli spaces of triples have been studied extensively, using wall-crossing techniques. We mention [9], where connectedness and irreducibility results for triples were studied.

Here we briefly recall the relevant definitions, see also [7, 14].

Definition 2.9.1. A holomorphic triple on X, $T = (E_1, E_2, \varphi)$, consists of two holomorphic vector bundles E_1 and E_2 on X and a holomorphic map $\varphi : E_2 \to E_1$. Denoting the ranks E_1 and E_2 by n_1 and n_2, and their degrees by d_1 and d_2, we refer to (n_1, n_2, d_1, d_2) as the type of the triple. A homomorphism from $T' = (E_1, E_2, \varphi')$ to $T = (E_1, E_2, \varphi)$ is a commutative diagram

$$
\begin{array}{ccc}
E_2' & \xrightarrow{\varphi'} & E_1' \\
\downarrow & & \downarrow \\
E_2 & \xrightarrow{\varphi} & E_1
\end{array}
$$

$T' = (E_1', E_2', \varphi')$ is a subtriple of $T = (E_1, E_2, \varphi)$ if the homomorphisms of sheaves $E_1' \to E_1$ and $E_2' \to E_2$ are injective.

For any $\alpha \in \mathbb{R}$ the α-slope of T is defined by,

$$\mu_\alpha(T) = \frac{d_1 + d_2 + n_2\alpha}{n_1 + n_2}$$

Definition 2.9.2. The triple $T = (E_1, E_2, \varphi)$ is α-(semi)stable if $\mu_\alpha(T')(\leq) < \mu_\alpha(T)$ for any proper subtriple $T' = (E_1, E_2')$. A triple is called α-polystable if it is the direct sum of α-stable triples of the same α-slope. It is strictly α-semistable (polystable) if it is α-semistable (polystable) but not α-stable.

We denote the moduli space of isomorphism classes of α-polystable triples of type $t = (n_1, n_2, d_1, d_2)$ by

$$\mathcal{N}_\alpha(t) = \mathcal{N}_\alpha(n_1, n_2, d_1, d_2).$$

The isomorphism classes of α-stable triples form a subspace which we denoted by $\mathcal{N}_\alpha^s(t)$.

Let $\mu_i = \mu(E_i) = \frac{d_i}{n_i}$. We define

$$\alpha_m = \mu_1 - \mu_2,$$
$$\alpha_M = \left(1 + \frac{n_1 + n_2}{|n_1 - n_2|}\right), \ n_1 \neq n_2.$$

Proposition 2.9.3 (([7, 14])). *The moduli space $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ is a complex analytic variety, which is projective when α is rational. A necessary condition for $\mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$ to be nonempty is*

$$\begin{cases} 0 \leq \alpha_m \leq \alpha \leq \alpha_M & \text{if } n_1 \neq n_2, \\ 0 \leq \alpha_m \leq \alpha & \text{if } n_1 = n_2 \end{cases}$$

We get the following from the results in [9].

Theorem 2.9.4. *(1) A triple $T = (E_1, E_2, \varphi)$ of type t is α_m-polystable if and only if $\varphi = 0$ and E_1 and E_2 are polystable. We thus have $\mathcal{N}_{\alpha_m}(t) \cong M(n_1, d_1) \times M(n_2, d_2)$. where, $M(n, d)$ denotes the moduli space of polystable bundles of rank n and degree d. In particular, $\mathcal{N}_{\alpha_m}(n_1, n_2, d_1, d_2)$ is nonempty and irreducible.*

(2) If $\alpha > \alpha_m$ is any value such that $2g - 2 \leq \alpha$ (and $\alpha < \alpha_M$ if $n_1 \neq n_2$) then $\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)$ is nonempty and irreducible. Moreover:

- *If $n_1 = n_2 = n$ then $\mathcal{N}_\alpha(n, n, d_1, d_2)$ is birationally equivalent to a $\mathbb{P}^N$-fibration over $M^s(n, d_2) \times \text{Sym}^{d_1 - d_2}(X)$, where $M^s(n, d_2)$ denotes the subspace of stable bundles of type (n, d_2), $\text{Sym}^{d_1 - d_2}(X)$ is the symmetric product, and the fiber dimension is $N = n(d_1 - d_2) - 1$.*

- *If $n_1 > n_2$ then $\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)$ is birationally equivalent to a $\mathbb{P}^N$-fibration over $M^s(n_1 - n_2, d_1 - d_2) \times M^s(n_2, d_2)$, where the fiber dimension is $N = n_2 d_1 - n_1 d_2 + n_2(n_1 - n_2)(g - 1) - 1$. The birational equivalence is an isomorphism if $\text{GCD}(n_1 - n_2, d_1 - d_2) = 1$ and $\text{GCD}(n_2, d_2) = 1$.*

- *If $n_1 < n_2$ then $\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)$ is birationally equivalent to a $\mathbb{P}^N$-fibration over $M^s(n_2 - n_1, d_2 - d_1) \times M^s(n_1, d_1)$, where the fiber dimension is $N = n_2 d_1 - n_1 d_2 + n_1(n_2 - n_1)(g - 1) - 1$. The birational equivalence is an isomorphism if $\mathrm{GCD}(n_2 - n_1, d_2 - d_1) = 1$ and $\mathrm{GCD}(n_1, d_1) = 1$. In particular, if $n_1 \neq n_2$ then $\mathcal{N}_\alpha^s(n_1, n_2, d_1, d_2)$ is a smooth manifold of dimension $(g - 1)(n_1^2 + n_2^2 - n_1 n_2) - n_1 d_2 + n_2 d_1 + 1$.*

(3) *If $n_1 \neq n_2$ then $\mathcal{N}_{\alpha_M}(n_1, n_2, d_1, d_2)$ is nonempty and irreducible. Moreover*

$$\mathcal{N}_{\alpha_M}(n_1, n_2, d_1, d_2) \cong \begin{cases} M(n_2, d_2) \times M(n_1 - n_2, d_1 - d_2) & \text{if } n_1 > n_2 \\ M(n_1, d_1) \times M(n_2 - n_1, d_2 - d_1) & \text{if } n_1 < n_2. \end{cases}$$

2.10 Holomorphic chains

Another important class of Q-bundles is *holomorphic chains*: in this case the quiver is a linear unbroken string, so a length $m + 1$ chain is a collection $(E_i, \varphi_i)_{i=0}^m$, where $\varphi_i : E_i \to E_{i-1}$ is a holomorphic map between holomorphic bundles. A notion of stability for $(m + 1)$- chains, depending on m real parameters α_i, has been introduced in [1] and moduli spaces have been constructed in [33]. The variation of the moduli spaces of holomorphic chains with stability parameter, and in particular characterization of the parameter region where the moduli spaces are birationally equivalent, was studied in [**?**].

Definition 2.10.1. A holomorphic $(m + 1)$-chain is a diagram

$$C : \quad E_m \xrightarrow{\varphi_m} E_{m-1} \xrightarrow{\varphi_{m-1}} \cdots \xrightarrow{\varphi_2} E_1 \xrightarrow{\varphi_1} E_0$$

where each E_i is a holomorphic vector bundle and $\varphi_i : E_i \longrightarrow E_{i-1}$ is a holomorphic map. The tuple $t = (\mathrm{rk}(E_0), \ldots, \mathrm{rk}(E_m); \deg(E_0), \ldots, \deg(E_m))$ will be referred as the *type* of the chain C.

Let $\alpha = (\alpha_0, \cdots, \alpha_m) \in \mathbb{R}^{m+1}$. The α-slope of a chain C of type $t = (r_0, \cdots, r_m; d_0, \cdots, d_m)$ is defined by the fraction

$$\mu_\alpha(C) = \frac{\sum_{i=0}^m (d_i + \alpha_i r_i)}{\sum_{i=0}^m r_i}$$

Definition 2.10.2. A holomorphic chain C is α-(semi)stable if

$$\mu_\alpha(C') < (\leq) \mu_\alpha(C)$$

is verified for any non-trivial subchain $C' = (F_j, j = 0, \ldots, m)$ of C. We call a chain C α-polystable, if it is written as a direct sum $C = C_1 \oplus \cdots \oplus C_t$ where C_k is an α-stable holomorphic chain with $\mu_\alpha(C_k) = \mu_\alpha(C)$, $k = 1, \ldots, t$.

Remark 2.10.3. A subsheaf of a vector bundle over M is a vector bundle. The subsheaf is not a subbundle, but it is contained in a subbundle of the same rank who has the maximal degree containing the subsheaf. Thus semistability can be checked by subchains composed of subbundles.

Definition 2.10.4. A region $R(t)(R^s(t))$ in $\mathbb{R}^m$ is the collection of α such that there exists an α-semistable (α-stable) holomorphic chain of type t. A chamber is a locally closed subset of $R(t)$ where the α-stability condition is independent of α.

The following is a chain version of *Jordan-Hölder Theorem*.

Let C be a α-semistable holomorphic chain, then C has a Jordan-Hölder filtration

$$0 = C_0 \subset C_1 \subset \ldots \subset C_n = C$$

such that C_i/C_{i-1} is a α-stable chain with $\mu_\alpha(C_i/C_{i-1}) = \mu_\alpha(C)$ for $0 \le i \le n$.

2.10.1 Vortex equations and Hitchin-Kobayashi correspondence

Natural gauge-theoretic equations on a holomorphic chain $C = (E_0, \cdots, E_m; \varphi_0, \cdots, \varphi_m)$ are as follows. Define $\tau = (\tau_0, \cdots, \tau_m) \in \mathbb{R}^{m+1}$ by

$$\tau_j = \mu_\alpha(C) - \alpha_j, \quad j = 0, \cdots, m, \tag{2.10.1}$$

with the convention $\alpha_0 = 0$. Therefore

$$\alpha_j = \tau_0 - \tau_j, \quad j = 0, \cdots, m.$$

The *vortex equations* for Hermitian metrics on $E_0, \cdots, E_m$, are

$$\sqrt{-1}\Lambda F(E_i) + \varphi_{i+1}\varphi_{i+1}^* - \varphi_i^*\varphi_i = \tau_i \mathrm{Id}_{E_i}, \quad j = 0, \cdots, m. \tag{2.10.2}$$

Where $F(E_i)$ is the curvature of the Chern connection on E_i, Λ is the contraction operator with respect to a fixed Kähler form ω on X, such that $\mathrm{vol}(X) = 2\pi$, and φ_i^* is the adjoint of φ_i.

One has the following.

Proposition 2.10.5. *[? , Theorem* 2.15] *A holomorphic chain C is α-polystable if and only if the vortex equations have a solution, where α and τ are related by* (2.10.1).

2.10.2 The birationality region

Given two holomorphic chains C' and C'', we define a 2-term complex, as in (2.5.1),

$$F^\bullet(C'', C') : \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \tag{2.10.3}$$

with terms

$$\mathcal{F}^0 = \bigoplus_{j-i=0} \mathrm{Hom}(E_j'', E_i'), \quad \mathcal{F}^1 = \bigoplus_{j-i=1} \mathrm{Hom}(E_j'', E_i'),$$

and the map d is defined by

$$d(g_0, \ldots, g_m) = (g_{i-1} \circ \phi_i'' - \phi_i' \circ g_i), \text{ for } g_i \in \mathrm{Hom}(E_i'', E_i').$$

The complex $F^\bullet(C'', C')$ is called *deformation complex*.

Definition 2.10.6. Fix a type $t = (r_j, j = 0, \cdots, m; d_j, j = 0, \cdots, m)$. The region $\widetilde{R}(t) \subset \mathbb{R}^{m+1}$ is the set of points α such that for all types $t' = (r'_j, j = 0, \cdots, m; d'_j, j = 0, \cdots, m)$ and $t'' = (r''_j, j = 0, \cdots, m; d''_j, j = 0, \cdots, m)$, with $t' + t'' = t$ and $\mu_\alpha(t') = \mu_\alpha(t'')$, and for all linear chains C' and C'' with dimension vectors $r' = (r'_j, j = 0, \cdots, m)$ and $r'' = (r''_j, j = 0, \cdots, m)$, respectively, the map b of (2.10.3) is not an isomorphism. The set $R_{2g-2} \subset \mathbb{R}^{m+1}$ is the set of all $\alpha = (\alpha_0, \cdots, \alpha_m)$ such that $\alpha_i - \alpha_{i-1} \geq 2g - 2$ for all $i = 1, \cdots, m$.

In [33, Theorem 4.7, Theorem 4.20] the following are proved:

Theorem 2.10.7. *Let C' and C'' be non-zero holomorphic chains of types t' and t'', respectively, and let $\alpha \in \mathbb{R}^{m+1}$. Suppose that the following conditions hold:*

- *C' and C'' be α-polystable with $\mu_\alpha(C') = \mu_\alpha(C'')$.*

- *$\alpha \in \widetilde{R}(t) \cap R_{2g-2}$.*

Then $\chi(C'', C') \leq 1 - g$.

Theorem 2.10.8. *Let ω be a wall contained in the region $\widetilde{R}(t) \cap R_{2g-2}$. Then $\mathcal{M}^s_{\alpha_\omega^+}$ and $\mathcal{M}^s_{\alpha_\omega^-}$ are birationally equivalent.*

Chapter 3

Twisted $\mathrm{U}(p,q)$-Higgs bundles

3.1 Twisted $\mathrm{U}(p,q)$-Higgs bundles

Recall that the unitary group $\mathrm{U}(p,q)$ of signature (p,q) is a non-compact real form of $\mathrm{GL}(p+q,\mathbb{C})$, and has maximal compact subgroup $H = \mathrm{U}(p) \times \mathrm{U}(q)$.

Let L be a line bundle on X. An L-twisted $\mathrm{U}(p,q)$-Higgs bundle is a twisted Q-bundle for the quiver with two vertices and one arrow in each direction between them.

We recall that a twisted $\mathrm{U}(p,q)$-Higgs bundle is a quadruple $E = (V,W,\beta,\gamma)$, where V and W are holomorphic vector bundles on X of ranks p and q respectively, and

$$\beta : W \longrightarrow V \otimes L,$$

$$\gamma : V \longrightarrow W \otimes L,$$

are holomorphic maps.

Let $\tau = (\tau_1, \tau_2)$ be a pair of real numbers. A Hermitian metric H satisfies the *L-twisted quiver τ-vortex equations* on twisted $\mathrm{U}(p,q)$-Higgs bundle E if

$$\sqrt{-1}\Lambda F_{H_V} + \beta\beta^* - \gamma^*\gamma = \tau_1 \operatorname{Id}_V, \tag{3.1.1}$$

$$\sqrt{-1}\Lambda F_{H_W} + \gamma\gamma^* - \beta^*\beta = \tau_2 \operatorname{Id}_W. \tag{3.1.2}$$

where F_{H_V} and F_{H_W} are the curvature of the Chern connections associated to the metrics H_V and H_W, respectively.

Let $E = (V,W,\beta,\gamma)$ be a twisted $\mathrm{U}(p,q)$-Higgs bundle, and α be a real number; α is called the *stability parameter*. The α-*slope* of E is defined to be

$$\mu_\alpha(E) = \mu(E) + \alpha\frac{p}{p+q}.$$

A twisted $\mathrm{U}(p,q)$-bundle E is α-*semistable* if, for every proper (non-trivial) subobject $F \subset E$,

$$\mu_\alpha(F) \le \mu_\alpha(E).$$

Further, E is α-*stable* if this inequality is always strict. A twisted U(p,q)-bundle is called α-*polystable* if it is the direct sum of α-stable twisted U(p,q)-Higgs bundles of the same α-slope.

Example 3.1.1. Choose a square root $K^{\frac{1}{2}}$ of the canonical bundle K, and a section ω of K^2. Consider $E = (K^{\frac{1}{2}}, K^{-\frac{1}{2}}, 1, \omega)$. Note that, since $K^{\frac{1}{2}}$ is not invariant, there are no invariant subobjects of positive degree and hence E is a α stable U$(1,1)$-Higgs bundle, for $\alpha \geq 2 - 2g$.

3.1.1 Critical values

A twisted U(p,q)-Higgs bundle E is strictly α-semistable (α-semistable but not α-stable) if and only if there is a proper subobject $F = (V', W')$ such that $\mu_\alpha(F) = \mu_\alpha(E)$, i.e.,

$$\mu(V' \oplus W') + \alpha \frac{p'}{p' + q'} = \mu(V \oplus W) + \alpha \frac{p}{p + q}.$$

The case in which the terms containing α drop from the above equality and E is strictly α-semistable for all values of α, i.e.,

$$\frac{p}{p + q} = \frac{p'}{p' + q'}, \text{ and}$$

$$\mu(V \oplus W) = \mu(V' \oplus W')$$

is called α-*independent strict semistablity*.

Definition 3.1.2. For a fixed type (p,q,a,b) a value of α is called a *critical value* if there exist integers p', q', a' and b' such that $\frac{p'}{p'+q'} \neq \frac{p}{p+q}$ but $\frac{a'+b'}{p'+q'} + \alpha \frac{p'}{p'+q'} = \frac{a+b}{p+q} + \alpha \frac{p}{p+q}$, with $0 \leq p' \leq p$, $0 \leq q' \leq q$, $(p',q',a',b') \neq (p,q,a,b)$ and $(p',q') \neq (0,0)$. We say that α is *generic* if it is not critical.

Lemma 3.1.3. *We have the following situations in which strict α-independent semistability can not occur:*

(i) [9, Lemma 2.7], if there is an integer m such that $\mathrm{GCD}(p+q, d_1 + d_2 - mp) = 1$.

(ii) If $\mathrm{GCD}(p,q) = 1$, for $p \neq q$.

Proof. To prove (ii), assume on the contrary that $E = (V, W, \beta, \gamma)$ is a α-semistable twisted U(p,q)-Higgs bundle with a proper subobject $E' = (V', W')$ such that

$$\mu(V' \oplus W') + \alpha \frac{p'}{p' + q'} = \mu(V \oplus W) + \alpha \frac{p}{p + q} \text{ and}$$

$$\frac{p'}{p' + q'} = \frac{p}{p + q}, \tag{3.1.3}$$

where p' and q' are the ranks of V' and W' respectively. Since E' is proper, either $p' < p$ or $q' < q$ and then the equality (3.1.3) contradicts that p and q are coprime. $\square$

3.1.2   Relation with triples

For any twisted U(p,q)-Higgs bundle $E = (V,W,\beta,\gamma)$ with $\beta = 0$ we can always associate a twisted triple T defined by $T = (W \otimes L, V, \gamma)$. The following result relates the stability conditions for the corresponding triples and that for twisted U(p,q)-Higgs bundles.

Lemma 3.1.4. *Let $E = (V,W,\beta,\gamma)$ be a twisted* U(p,q)*-Higgs bundle, then the following hold:*

(i) If $\beta = 0$ then E is α-semistable if and only if the corresponding triple is $\alpha + \deg(L)$-semistable.

(ii) If $\gamma = 0$ then E is α-semistable if and only if the corresponding triple is $-\alpha + \deg(L)$-semistable.

Proof. Part (i). Let $T = (E_1, E_2, \phi)$ be the triple corresponding to the U(p,q)-Higgs bundle E. Thus $E_1 = W \otimes L$ and $E_2 = V$ and, hence,

$$\deg(E_1) = \deg(W) + q\deg(L).$$

Since $q = \mathrm{rk}(E_1)$ and $p = \mathrm{rk}(E_2)$ it follows that

$$\mu_{\alpha+\deg(L)}(T) = \mu_\alpha(E) + \deg(L) \tag{3.1.4}$$

Clearly there is a correspondence between invariant subtriples $T' = (E_1', E_2')$ and invariant subobjects of E. Now, it follows from (3.1.4) that $\mu_\alpha(E') \leq \mu_\alpha(E)$ if and only if $\mu_{\alpha'}(T') \leq \mu_{\alpha'}(T)$ for $\alpha' = \alpha + \deg(L)$.

   Part (ii). From the definition of slope stability and Remark 2.3 it is clear that $E = (V,W,\beta,\gamma)$ is α-semistable if and only if $E' = (W,V,\gamma,\beta)$ is $-\alpha$-semistable. Hence the result follows using part (i).   □

Remark 3.1.5. From the above lemma one can see that the stability parameter of the corresponding triples is the translation of the stability parameter for that twisted U(p,q)-Higgs bundles. Recall (from [9]) that the range of stability parameter for twisted U(p,q)-Higgs bundles with $\beta = 0$ is $\alpha_m \leq \alpha \leq \alpha_M$, where $\alpha_m = \mu(W) - \mu(V) + \deg(L)$ and $\alpha_M = \frac{|q-p|+p+q}{|q-p|}\alpha_m$.

3.1.3   Deformation Theory of twisted U(p,q)-Higgs bundles

The result of this section are the specialization to twisted U(p,q)-Higgs bundles.

Definition 3.1.6. Let $E = (V,W,\beta,\gamma)$ and $E' = (V',W',\beta',\gamma')$ be L-twisted twisted U(p,q)-Higgs bundles. We introduce the following notation:

$$\mathcal{H}om^0 = \mathrm{Hom}(V',V) \oplus \mathrm{Hom}(W',W),$$
$$\mathcal{H}om^1 = \mathrm{Hom}(V',W \otimes L) \oplus \mathrm{Hom}(W',V \otimes L).$$

With this notation we consider the complex of sheaves

$$\mathcal{H}om^\bullet(E',E): \ \mathcal{H}om^0 \xrightarrow{a_0} \mathcal{H}om^1 \tag{3.1.5}$$

defined by

$$a_0(f_1, f_2) = \big(\phi_a(f_1, f_2), \phi_b(f_1, f_2)\big), \text{ for } (f_1, f_2) \in \mathcal{H}om^0$$

where

$$\phi_a : \mathcal{H}om^0 \to \mathrm{Hom}(V', W \otimes L) \hookrightarrow \mathcal{H}om^1 \text{ and } \phi_b : \mathcal{H}om^0 \to \mathrm{Hom}(W', V \otimes L) \hookrightarrow \mathcal{H}om^1$$

are given by

$$\phi_a(f_1, f_2) = (f_2 \otimes Id_L) \circ \gamma' - \gamma \circ f_1),$$
$$\phi_b(f_1, f_2) = (f_1 \otimes Id_L) \circ \beta' - \beta \circ f_2).$$

The complex $\mathcal{H}om^\bullet(E', E)$ is called the Hom-*complex*. This is a special case of the Hom-complex for Q-bundles defined in the chapter 1, and also for G-Higgs bundles (in the case $L = K$ see e.g, [5]).

The following proposition follows from [20, Theorem 4.1 and Theorem 5.1].

Proposition 3.1.7. *Let E and E' be twisted* U(p,q)*-Higgs bundles, then there are natural isomorphisms*

$$\mathrm{Hom}(E', E) \cong \mathbb{H}^0(\mathcal{H}om^\bullet(E', E))$$
$$\mathrm{Ext}^1(E', E) \cong \mathbb{H}^1(\mathcal{H}om^\bullet(E', E))$$

and a long exact sequence associated to the complex $\mathcal{H}om^\bullet(E', E)$:

$$0 \longrightarrow \mathbb{H}^0(\mathcal{H}om^\bullet(E', E)) \longrightarrow H^0(\mathcal{H}om^0) \longrightarrow H^0(\mathcal{H}om^1) \longrightarrow \mathbb{H}^1(\mathcal{H}om^\bullet(E', E)) \qquad (3.1.6)$$
$$\longrightarrow H^1(\mathcal{H}om^0) \longrightarrow H^1(\mathcal{H}om^1) \longrightarrow \mathbb{H}^2(\mathcal{H}om^\bullet(E', E)) \longrightarrow 0.$$

When $E = E'$, we have $\mathrm{End}(E) = \mathrm{Hom}(E, E) \cong \mathbb{H}^0(\mathcal{H}om^\bullet(E, E))$.

Definition 3.1.8. We denote by $\chi(E', E)$ the hypercohomology Euler characteristic for the complex $\mathcal{H}om^\bullet(E', E)$, i.e.

$$\chi(E', E) = \dim \mathbb{H}^0(\mathcal{H}om^\bullet(E', E)) - \dim \mathbb{H}^1(\mathcal{H}om^\bullet(E', E)) + \dim \mathbb{H}^2(\mathcal{H}om^\bullet(E', E)).$$

Remark 3.1.9. Suppose that $E = E' \oplus E''$. Then it is clear that the Hom-complexes satisfy:

$$\mathcal{H}om^\bullet(E, E) = \mathcal{H}om^\bullet(E', E') \oplus \mathcal{H}om^\bullet(E'', E'') \oplus \mathcal{H}om^\bullet(E'', E') \oplus \mathcal{H}om^\bullet(E', E''),$$

and so the hypercohomology groups have an analogous direct sum decomposition.

Lemma 3.1.10. *For any extension* $0 \to E' \to E \to E'' \to 0$ *of twisted* U(p,q)*-Higgs bundles,*

$$\chi(E, E) = \chi(E', E') + \chi(E'', E'') + \chi(E'', E') + \chi(E', E'').$$

Proof. Since the Euler characteristic is topological, we may assume that $E = E' \oplus E''$. Now the result is immediate in view of Remark 3.1.9. $\qquad\square$

As an immediate consequence of the long exact sequence (3.1.6) and the Riemann-Roch formula we can obtain the following.

Proposition 3.1.11. *For any twisted* U(p,q)*-Higgs bundles* E' *and* E *we have*

$$
\begin{aligned}
\chi(E',E) &= \chi(\mathcal{H}om^0) - \chi(\mathcal{H}om^1) \\
&= (1-g)(\mathrm{rk}(\mathcal{H}om^0) - \mathrm{rk}(\mathcal{H}om^1)) + \deg(\mathcal{H}om^0) - \deg(\mathcal{H}om^1) \\
&= (1-g)\big(p'p + q'q - p'q - q'p\big) + (q'-p')(\deg(W) - \deg(V)) + \\
&\quad (q-p)(\deg(V') - \deg(W')) - (pq' + p'q)\deg(L)
\end{aligned}
$$

Recall the type $t(E) = (p,q,a,b)$, where $a = \deg(V)$, $b = \deg(W)$. The previous proposition shows that $\chi(E',E)$ only depends on the types $t' = t(E')$ and $t = t(E)$ of E' and E, respectively, so we may use the notation

$$
\chi(t',t) := \chi(E',E).
$$

Lemma 3.1.12. *For any extension* $0 \to E' \to E \to E'' \to 0$ *of twisted* U(p,q)*-Higgs bundles,*

$$
\chi(E,E) = \chi(E',E') + \chi(E'',E'') + \chi(E'',E') + \chi(E',E'').
$$

Given the identification of $\mathbb{H}^0(\mathcal{H}om^\bullet(E',E))$ with $\mathrm{Hom}(E',E)$, by Proposition 3.1.7, the following is the direct analogues of the corresponding result for semistable vector bundles.

Proposition 3.1.13. *Let* E *and* E' *be* α*-semistable twisted* U(p,q)*-Higgs bundles.*

(1) If $\mu_\alpha(E) < \mu_\alpha(E')$*, then* $\mathbb{H}^0(\mathcal{H}om^\bullet(E',E)) = 0$.

(2) If $\mu_\alpha(E') = \mu_\alpha(E)$*, and* E' *is* α*-stable, then*

$$
\mathbb{H}^0(\mathcal{H}om^\bullet(E',E)) \cong \begin{cases} 0 & \text{if } E \not\cong E' \\ \mathbb{C} & \text{if } E \cong E'. \end{cases}
$$

Definition 3.1.14. A twisted U(p,q)-Higgs bundle $E = (V,W,\varphi = \beta + \gamma)$ is *infinitesimally simple* if $\mathrm{End}(E) \cong \mathbb{C}$ and it is *simple* if $\mathrm{Aut}(E) \cong \mathbb{C}^*$, where $\mathrm{Aut}(E)$ denotes the automorphism group of E.

Lemma 3.1.15. *Let* (V,W,β,γ) *be a twisted* U(p,q)*-Higgs bundle. If* (V,W,β,γ) *is infinitesimally simple then it is simple. Hence if* (V,W,β,γ) *is* α*-stable then it is simple.*

Proof. Following is the usual argument for vector bundles. The first statement follows from definition and the fact that 0 is not an automorphism. The second one is a consequence of Proposition 3.1.7 and Proposition 3.1.13. $\qquad\square$

Proposition 3.1.16. *Let* $E = (V,W,\beta,\gamma)$ *be an* α*-stable twisted* U(p,q)*-Higgs bundle of type* $t = (p,q,a,b)$.

(1) *The space of infinitesimal deformations of E is isomorphic to the first hypercohomology group* $\mathbb{H}^1(\mathcal{H}om^\bullet(E,E))$.

(2) *If* $\mathbb{H}^2(\mathcal{H}om^\bullet(E,E)) = 0$, *then the moduli space* $\mathcal{M}_\alpha^s(t)$ *is smooth in a neighborhood of the point defined by E and*

$$\dim \mathcal{M}_\alpha^s(t) = \dim \mathbb{H}^1(\mathcal{H}om^\bullet(E',E))$$
$$= 1 - \chi(E,E) = (g-1)(q-p)^2 + 2pq\deg(L) + 1.$$

Proof. Statement (1) follows from [5, Theorem 2.3]. (2) Analogous to Proposition 2.14 of [11]. $\square$

3.2   Consequences of stability

3.2.1   Bounds on the topological invariants and Milnor–Wood inequality

In this section we explore the constraints imposed by stability on the topological invariants of U(p,q)-Higgs bundles and on the stability parameter α.

Proposition 3.2.1. *Let* $E = (V,W,\beta,\gamma)$ *be a* α-*semistable L-twisted* U(p,q)-*Higgs bundle. Then*

$$2p\big(\mu(V) - \mu_\alpha(E)\big) \le \mathrm{rk}(\gamma)\deg(L) + \alpha(\mathrm{rk}(\gamma) - 2p) \tag{3.2.1}$$

$$2q\big(\mu(W) - \mu_\alpha(E)\big) \le \mathrm{rk}(\beta)\deg(L) - \mathrm{rk}(\beta)\alpha \tag{3.2.2}$$

Moreover, if $\deg(L) + \alpha > 0$ *and equality holds in* (3.2.5) *then either E is strictly semistable or* $p = q$ *and* γ *is an isomorphism* $\gamma: V \xrightarrow{\cong} W \otimes L$. *Similarly, if* $\deg(L) - \alpha > 0$ *and equality holds in* (3.2.6) *then either E is strictly semistable or* $p = q$ *and* β *is an isomorphism* $\beta: W \xrightarrow{\cong} V \otimes L$.

Proof. We adapt as follows the argument given in the proof of [8, Lemma 3.24]. We shall only prove (3.2.1), since the same argument apply to β proves (3.2.2).
If $\gamma = 0$ applying α-semistability condition on V gives (3.2.1). Therefore we may assume $\gamma \ne 0$. Consider invariants subobjects $E_N = (\ker(\gamma),0)$ and $E_I = (V,\mathrm{im}(\gamma) \otimes L^{-1})$. The α-semistability inequality for E_N and E_I yields that

$$\frac{\deg(\ker(\gamma))}{\mathrm{rk}(\ker(\gamma))} + \alpha \le \mu_\alpha(E),$$

$$\frac{\deg(V) + \deg(I)}{p + \mathrm{rk}(\gamma)} + \alpha\frac{p}{p + \mathrm{rk}(\gamma)} \le \mu_\alpha(E).$$

By adding the above inequalities and using $\mathrm{rk}(\ker(\gamma)) + \mathrm{rk}(\mathrm{im}(\gamma)) = p$ we obtain

$$\deg(\ker(\gamma)) + \deg(\mathrm{im}(\gamma) \otimes L^{-1}) + \deg(V) + \alpha\big(p + \mathrm{rk}(\ker(\gamma))\big) \le 2p\mu_\alpha(E) \tag{3.2.3}$$

Now, since $\deg\big((V/\ker(\gamma))^* \otimes \text{im}(\gamma)\big) \geq 0$,

$$\deg(\ker(\gamma)) + \deg(\text{im}(\gamma)) \geq \deg(V). \tag{3.2.4}$$

From (3.2.3) and (3.2.4) we get

$$2\deg(V) - \text{rk}(\gamma)\deg(L) + \alpha\big(p + \text{rk}(\ker(\gamma))\big) \leq 2p\mu_\alpha(E).$$

which proves (3.2.1). $\qquad\qquad\square$

Proposition 3.2.2. *Let $E = (V, W, \beta, \gamma)$ be a α-semistable twisted $\text{U}(p,q)$-Higgs bundle. Then*

$$\frac{2pq}{p+q}\big(\mu(V) - \mu(W)\big) \leq \text{rk}(\gamma)\deg(L) + \alpha\big(\text{rk}(\gamma) - \frac{2pq}{p+q}\big) \tag{3.2.5}$$

$$\frac{2pq}{p+q}\big(\mu(W) - \mu(V)\big) \leq \text{rk}(\beta)\deg(L) + \alpha\big(\frac{2pq}{p+q} - \text{rk}(\beta)\big) \tag{3.2.6}$$

Moreover, if $\deg(L) + \alpha > 0$ and equality holds in (3.2.5) then either E is strictly semistable or $p = q$ and γ is an isomorphism $\gamma: V \xrightarrow{\cong} W \otimes L$. Similarly, if $\deg(L) - \alpha > 0$ and equality holds in (3.2.6) then either E is strictly semistable or $p = q$ and β is an isomorphism $\beta: W \xrightarrow{\cong} V \otimes L$.

Proof. The result follows immediately using the previous proposition and following identities:

$$\mu(V) - \mu_\alpha(E) = \frac{q}{p+q}\big(\mu(V) - \mu(W)\big) - \alpha\frac{p}{p+q},$$
$$\mu(W) - \mu_\alpha(E) = \frac{p}{p+q}\big(\mu(W) - \mu(V)\big) - \alpha\frac{p}{p+q}.$$

The statement about equality for $\deg(L) - \alpha > 0$ also follows as in loc. cit. $\qquad\square$

By analogy with the case of $\text{U}(p,q)$-Higgs bundles we make the following definition.

Definition 3.2.3. The *Toledo invariant* of a twisted $\text{U}(p,q)$-Higgs bundle $E = (V, W, \beta, \gamma)$ is

$$\tau(E) = 2\frac{q\deg(V) - p\deg(W)}{p+q} = \frac{2pq}{p+q}\big(\mu(V) - \mu(W)\big).$$

With this definition we can write (3.2.5) and (3.2.6) as

$$\tau(E) \leq \text{rk}(\gamma)\deg(L) + \alpha\big(\text{rk}(\gamma) - \frac{2pq}{p+q}\big) \tag{3.2.7}$$

$$-\tau(E) \leq \text{rk}(\beta)\deg(L) + \alpha\big(\frac{2pq}{p+q} - \text{rk}(\beta)\big) \tag{3.2.8}$$

The following is the analogue of the Milnor–Wood inequality for $\text{U}(p,q)$-Higgs bundles ([8, Corollary 3.27]). When $L = K$, it is a special case of a general result of Biquard–García-Prada–Rubio [? , Theorem 4.5], which is valid for G-Higgs bundles for any semisimple G of Hermitian type.

Proposition 3.2.4. *Let $E = (V, W, \beta, \gamma)$ be an α-semistable twisted* U(p,q)*-Higgs bundle. Then the following inequality holds:*

$$-\mathrm{rk}(\beta)\deg(L) + \alpha\left(\mathrm{rk}(\beta) - \frac{2pq}{p+q}\right) \le \tau(E) \le \mathrm{rk}(\gamma)\deg(L) + \alpha\left(\mathrm{rk}(\gamma) - \frac{2pq}{p+q}\right).$$

Proof. In view of the definition of $\tau(E)$, we can write (3.2.5) and (3.2.6) as

$$\tau(E) \le \mathrm{rk}(\gamma)\deg(L) + \alpha\left(\mathrm{rk}(\gamma) - \frac{2pq}{p+q}\right), \tag{3.2.9}$$

$$-\tau(E) \le \mathrm{rk}(\beta)\deg(L) + \alpha\left(\frac{2pq}{p+q} - \mathrm{rk}(\beta)\right) \tag{3.2.10}$$

from which the result is immediate. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

When equality holds in the Milnor–Wood inequality, more information on the maps β and γ can be obtained from Proposition 3.2.2. In this respect we have the following result.

Proposition 3.2.5. *Let $E = (V, W, \beta, \gamma)$ be an α-semistable twisted* U(p,q)*-Higgs bundle.*

1. *Assume that $\alpha > -\deg(L)$. Then*

$$\tau(E) \le \min\{p,q\}\left(\deg(L) - \alpha\frac{|p-q|}{p+q}\right)$$

 and if equality holds then $p \le q$ and γ is an isomorphism onto its image.

2. *Assume that $\alpha \le -\deg(L)$. Then*
$$\tau(E) \le -\alpha\frac{2pq}{p+q}$$

 and if equality holds and $\alpha < -\deg(L)$ then $\gamma = 0$.

3. *Assume that $\alpha < \deg(L)$. Then*

$$\tau(E) \ge \min\{p,q\}\left(-\alpha\frac{|p-q|}{p+q} - \deg(L)\right)$$

 and if equality holds then $q \le p$ and β is an isomorphism onto its image.

4. *Assume that $\alpha \ge \deg(L)$. Then*
$$\tau(E) \ge -\alpha\frac{2pq}{p+q}$$

 and if equality holds and $\alpha > \deg(L)$ then $\beta = 0$.

Proof. We rewrite (3.2.9) as $\tau(E) \le \mathrm{rk}(\gamma)(\deg(L) + \alpha) - \alpha\frac{2pq}{p+q}$. Then (1) and (2) are immediate from Proposition 3.2.2. Similarly, (3) and (4) follow rewriting (3.2.10) as $\tau(E) \ge \mathrm{rk}(\beta)(\alpha - \deg(L)) - \alpha\frac{2pq}{p+q}$. $\qquad\square$

In the case when $|\alpha| < \deg(L)$ we can write the inequality of the preceding proposition in a more suggestive manner as follows.

Corollary 3.2.6. *Assume that* $|\alpha| < \deg(L)$ *and let* E *be an* α*-semistable twisted* $U(p,q)$*-Higgs bundle. Then*

$$|\tau(E)| \le \min\{p,q\}(\deg(L) - \alpha \frac{|p-q|}{p+q}).$$

Remark 3.2.7. In the cases of Proposition 3.2.5 when one of the Higgs fields β and γ is an isomorphism onto its image, it is natural to explore rigidity phenomena for twisted $U(p,q)$-Hitchin pairs, along the lines of [8] (for $U(p,q)$-Higgs bundles) and Biquard–García-Prada–Rubio [?] (for parameter dependent G-Higgs bundles when G is Hermitian of tube type). This line of enquiry will be pursued elsewhere.

3.2.2   Range for the stability parameter

In the following we determine a range for the stability parameter whenever $p \ne q$. We denote the minimum and the maximum value for α by α_m and α_M, respectively.

Proposition 3.2.8. *Assume* $p \ne q$ *and let* E *be a* α*-semistable twisted* $U(p,q)$*-Higgs bundle. Then* $\alpha_m \le \alpha \le \alpha_M$, *where*

$$\alpha_m = \begin{cases} -\dfrac{2\max\{p,q\}}{|q-p|}(\mu(V) - \mu(W)) - \dfrac{p+q}{|q-p|}\deg(L) & \text{if } \mu(V) - \mu(W) > -\deg(L), \\ -(\mu(V) - \mu(W)) & \text{if } \mu(V) - \mu(W) \le -\deg(L), \end{cases}$$

and

$$\alpha_M = \begin{cases} -\dfrac{2\max\{p,q\}}{|q-p|}(\mu(V) - \mu(W)) + \dfrac{p+q}{|q-p|}\deg(L) & \text{if } \mu(V) - \mu(W) < \deg(L), \\ -(\mu(V) - \mu(W)) & \text{if } \mu(V) - \mu(W) \ge \deg(L). \end{cases}$$

Proof. First we determine α_M. Using (3.2.9) we get

$$\alpha(\frac{2pq}{p+q} - \mathrm{rk}(\gamma)) \le \mathrm{rk}(\gamma)\deg(L) - \tau(E)$$

since $p \ne q$ therefore $\dfrac{2pq}{p+q} - \mathrm{rk}(\gamma) > 0$. Hence the above inequality yields

$$\alpha \le \frac{p+q}{2pq - (p+q)\mathrm{rk}(\gamma)}(\mathrm{rk}(\gamma)\deg(L) - \tau(E)).$$

In order to find the upper bound, study monotonicity of $f(r) = \dfrac{rd - \tau}{c - r}$, where $c = \dfrac{2pq}{p+q}$, $d = \deg(L)$ and $r \in [0, \min\{p,q\}]$. Then we obtain the following:

(a) If $\deg(L) = \mu(V) - \mu(W)$ then f is constant and

$$\alpha \le \mu(W) - \mu(V).$$

(b) If $\deg(L) > \mu(V) - \mu(W)$ then f is increasing so

$$\alpha \le \frac{p+q}{|q-p|}\Big(\deg(L) - \frac{\tau(E)}{\min\{p,q\}}\Big) = \frac{p+q}{|q-p|}\deg(L) - \frac{2\max\{p,q\}}{|q-p|}(\mu(V) - \mu(W))$$

and, if equality holds then $\mathrm{rk}(\gamma) = \min\{p,q\}$.

(c) If $\deg(L) < \mu(V) - \mu(W)$ then f is decreasing so

$$\alpha \le \mu(W) - \mu(V)$$

and, if equality holds then $\gamma = 0$.

Now we determine the lower bound α_m. Inequality (3.2.10) yields

$$\alpha \ge \frac{\mathrm{rk}(\beta)\deg(L) + \tau(E)}{\mathrm{rk}(\beta) - \dfrac{2pq}{p+q}}.$$

Similarly, by studying the monotoniciy of $g(r) = \dfrac{rd + \tau}{r - c}$, we obtain the following:

(a)′ If $\mu(V) - \mu(W) = -\deg(L)$ then g is constant and

$$\alpha \ge \mu(W) - \mu(V).$$

(b)′ If $\mu(V) - \mu(W) < -\deg(L)$ then g is increasing, so

$$\alpha \ge \mu(W) - \mu(V),$$

and, if equality holds then $\beta = 0$.

(c)′ If $\mu(V) - \mu(W) > -\deg(L)$ then g is decreasing, so

$$\alpha \ge -\frac{p+q}{|q-p|}\Big(\deg(L) + \frac{\tau(E)}{\min\{p,q\}}\Big) = -\frac{p+q}{|q-p|}\deg(L) - \frac{2\max\{p,q\}}{|q-p|}(\mu(V) - \mu(W)),$$

and, if equality holds then $\mathrm{rk}(\beta) = \min\{p,q\}$.

Note that if $\mu(V) - \mu(W) \ge 0$ then $\mu(V) - \mu(W) \ge -\deg(L)$, and if $\mu(V) - \mu(W) \le 0$ then $\mu(V) - \mu(W) < \deg(L)$. Hence the result follows. $\qquad\square$

Remark 3.2.9. The preceding proof gives the following additional information when α equals one of the extreme values α_m and α_M:

- if $\mu(V) - \mu(W) < \deg(L)$ and $\alpha = \alpha_M$ then $\mathrm{rk}(\gamma) = \min\{p,q\}$;

- if $\mu(V) - \mu(W) > \deg(L)$ and $\alpha = \alpha_M$ then $\gamma = 0$;

- if $\mu(V) - \mu(W) > -\deg(L)$ and $\alpha = \alpha_m$ then $\mathrm{rk}(\beta) = \min\{p,q\}$, and

- if $\mu(V) - \mu(W) < -\deg(L)$ and $\alpha = \alpha_m$ then $\beta = 0$.

The following corollary is relevant because $\alpha = 0$ is the value of stability parameter for which the Non-abelian Hodge Theorem gives the correspondence between $U(p,q)$-Higgs bundles and representations of the fundamental group of X.

Corollary 3.2.10. *With the notation of Proposition 3.2.8, the inequality $\alpha_M \geq 0$ holds if and only if $\tau(E) \leq \min\{p,q\}\deg(L)$ and the inequality $\alpha_m \leq 0$ holds if and only if $\tau(E) \geq -\min\{p,q\}\deg(L)$. Thus $0 \in [\alpha_m, \alpha_M]$ if and only if $|\tau(E)| \leq \min\{p,q\}\deg(L)$.*

Proof. Immediate from Proposition 3.2.8. $\qquad\square$

Remark 3.2.11. Note that the condition $|\tau(E)| \leq \min\{p,q\}\deg(L)$ is stronger than the condition $|\mu(V) - \mu(W)| \leq \deg(L)$.

Proof. Immediate from Proposition 3.2.8. $\qquad\square$

3.2.3 Parameters forcing special properties of the Higgs fields

In this section we use a variation on the preceding arguments to find a parameter range where β and γ have special properties. Assume that the twisted $U(p,q)$-Higgs bundle $E = (V, W, \beta, \gamma)$ has type (p, q, a, b).

For the following proposition it is convenient to introduce the following notation. For $0 \leq i < q \leq p$, let

$$\alpha_i = \frac{2pq}{q(p-q) + (i+1)(p+q)}\big(\mu(W) - \mu(V) - \deg(L)\big) + \deg(L),$$

and for $0 \leq j < p \leq q$, let

$$\alpha'_j = \frac{2pq}{p(q-p) + (j+1)(p+q)}\big(\mu(W) - \mu(V) + \deg(L)\big) - \deg(L).$$

Proposition 3.2.12. *Let $E = (V, W, \beta, \gamma)$ be an α-semistable twisted $U(p,q)$-Higgs bundle. Then we have the following:*

(i) *Assume that $p \geq q$ and $\mu(V) - \mu(W) > -\deg(L)$. If $\alpha < \alpha_{i-1}$ then $\mathrm{rk}(\ker(\beta)) < i$. In particular β is injective whenever*

$$\alpha < \alpha_0 = \frac{2pq}{pq - q^2 + p + q}\big(\mu(W) - \mu(V) - \deg(L)\big) + \deg(L).$$

(ii) *Assume that $p \geq q$ and $\mu(V) - \mu(W) < -\deg(L)$. If $\alpha < \alpha_{i-1}$ then $\mathrm{rk}(\ker(\beta)) > i$. In particular β is zero whenever*

$$\alpha < \alpha_{q-2} = \frac{2pq}{2pq - p - q}\big(\mu(W) - \mu(V) - \deg(L)\big) + \deg(L).$$

(iii) *Assume that $p \leq q$ and $\mu(V) - \mu(W) < \deg(L)$. If $\alpha > \alpha'_j$ then $\mathrm{rk}(\ker(\gamma)) < j$. In particular γ is injective whenever*

$$\alpha > \alpha'_0 = \frac{2pq}{pq - p^2 + p + q}\left(\mu(W) - \mu(V) + \deg(L)\right) - \deg(L).$$

(iv) *Assume that $p \leq q$ and $\mu(V) - \mu(W) > \deg(L)$. If $\alpha > \alpha'_j$ then $\mathrm{rk}(\ker(\gamma)) > j$. In particular γ is zero whenever*

$$\alpha > \alpha'_{p-2} = \frac{2pq}{2pq - p - q}\left(\mu(W) - \mu(V) + \deg(L)\right) - \deg(L).$$

Proof. We shall only prove parts (i) and (ii). One can deduce the other parts in a similar way. Suppose that $\mathrm{rk}(\ker(\beta)) = n > 0$. The inequality (3.2.6) yields

$$\alpha \geq \frac{2pq}{n(p+q) + q(p-q)}\left(\mu(W) - \mu(V) - \deg(L)\right) + \deg(L) = \alpha_{n-1}.$$

Now suppose $\mu(W) - \mu(V) - \deg(L) < 0$, then α_i increases with i and so, if $n \geq i$ then $\alpha \geq \alpha_{i-1}$. Hence, if $\alpha < i - 1$ then $n < i$. In particular, if $\alpha < \alpha_0$ then β is injective, which gives part (i).

On the other hand, if $\mu(W) - \mu(V) - \deg(L) > 0$, then α_i decreases with i and so, if $n \leq i$ then $\alpha \geq \alpha_{i-1}$. Hence, if $\alpha < \alpha_{i-1}$ then $n > i$. In particular, if $\alpha < \alpha_{q-2}$ then β is zero, proving part (ii). $\square$

Corollary 3.2.13. *Let $E = (V, W, \beta, \gamma)$ be an α-semistable twisted U(p,q)-Higgs bundle. Then we have the following:*

(i) *If $p \geq q$ and $\mu(W) - \mu(V) > -\deg(L)$ then γ is surjective whenever*

$$\alpha > \alpha_t := \frac{2pq}{pq - q^2 + p + q}\left(\mu(W) - \mu(V) + \deg(L)\right) - \deg(L).$$

(ii) *If $p \leq q$ and $\mu(W) - \mu(V) < \deg(L)$ then β is surjective whenever*

$$\alpha < \alpha'_t := \frac{2pq}{pq - p^2 + p + q}\left(\mu(W) - \mu(V) - \deg(L)\right) + \deg(L)$$

Proof. Associated to $E = (V, W, \beta, \gamma)$ there is a dual L-twisted U(p,q)-Higgs bundle $E^* = (V^*, W^*, \gamma^*, \beta^*)$. Clearly there is a one-to-one correspondence between subobjects of E and quotients of E^*, and $\mu_{-\alpha}(E) = -\mu_\alpha(E^*)$. Therefore α-stability of E^* is equivalent to $-\alpha$-stability of E. Using Proposition 3.2.12 we can find a range for the stability parameter of E^* where β^* and γ^* are injective. Hence the result follows by relating the stability parameter of E and E^*. $\square$

Remark 3.2.14. We have the following additional information:

(1) In the case $q = 1$ we have $\alpha_0 = \alpha'_0 = \mu(W) - \mu(V)$.

(2) $\alpha_0 > 0$ iff $\tau(E) < -(q-1)\deg(L)$.

(3) $\alpha'_0 < 0$ iff $\tau(E) > (p-1)\deg(L)$.

The following results shows that the bounds in Proposition 3.2.12 are meaningful in view of the bounds for α of Proposition 3.2.8.

Proposition 3.2.15. *Let α_0 and α_0' be given in Proposition 3.2.12. Then the following holds.*

(*i*) *Assume that $p > q$. If $\mu(V) - \mu(W) > -\deg(L)$ then $\alpha_0 > \alpha_m$, and if $\mu(V) - \mu(W) < -\deg(L)$ then $\alpha_{q-2} > \alpha_m$.*

(*ii*) *Assume that $p < q$. If $\mu(V) - \mu(W) < \deg(L)$ then $\alpha_0' < \alpha_M$, and if $\mu(V) - \mu(W) > \deg(L)$ then $\alpha_{p-2}' < \alpha_M$.*

Proof. For (i), using $\mu(V) - \mu(W) > -\deg(L)$ we get

$$\alpha_0 - \alpha_m = \big(\mu(V) - \mu(W)\big)\Big(\frac{-2pq}{q(p-q)+p+q} + \frac{2p}{p-q}\Big)$$
$$+ \deg(L)\Big(\frac{-2pq}{q(p-q)+p+q} + 1 + \frac{p+q}{p-q}\Big)$$
$$> \deg(L)\Big(-\frac{2p}{p-q} + 1 + \frac{p+q}{p-q}\Big) = 0,$$

where we have used that $p > q$ makes the term which multiplies $\mu(V) - \mu(W)$ positive. Thus $\alpha_0 > \alpha_m$. Moreover, when $\mu(V) - \mu(W) < -\deg(L)$ and $p > q$, we have $\alpha_m = \alpha_{q-1} < \alpha_{q-2}$ (cf. the proof of Proposition 3.2.12). This finishes the proof of (i).

For (ii), using $\mu(V) - \mu(W) < \deg(L)$ we obtain the following

$$\alpha_M - \alpha_0' = \big(\mu(V) - \mu(W)\big)\Big(\frac{-2q}{q-p} + \frac{2pq}{p(q-p)+p+q}\Big)$$
$$+ \deg(L)\Big(\frac{p+q}{q-p} - \frac{2pq}{p(q-p)+p+q} + 1\Big)$$
$$> \deg(L)\Big(-\frac{2q}{q-p} + 1 + \frac{p+q}{q-p}\Big) = 0,$$

where we have used that $p < q$ makes the term which multiplies $\mu(V) - \mu(W)$ negative. Hence $\alpha_0' < \alpha_M$. Moreover, when $\mu(V) - \mu(W) > \deg(L)$ and $p < q$, we have $\alpha_M = \alpha_{p-1}' > \alpha_{p-2}'$ (again, cf. the proof of Proposition 3.2.12). This finishes the proof of (ii). □

3.2.4 The comparison between twisted $U(p,q)$-Higgs bundles and $GL(p+q,\mathbb{C})$-Higgs bundles

We will use this comparison for vanishing of hypercohomology in degree two. We recall (from [24] and the references therein) the following about $GL(n,\mathbb{C})$-Higgs bundles.

A $GL(n,\mathbb{C})$-Higgs bundle on X is a pair (E, ϕ), where E is a rank n holomorphic vector bundle over X and $\phi \in H^0(\mathrm{End}(E) \otimes K)$ is a holomorphic endomorphism of E twisted by the canonical bundle K of X. More generally, replacing K by an arbitrary line bundle on X, we obtain the notion of an *L*-twisted $GL(n,\mathbb{C})$-*Higgs pair* or *Hitchin pair* on X.

A $GL(n,\mathbb{C})$-Higgs bundle (E, ϕ) is stable if the slope stability condition

$$\mu(E') < \mu(E) \tag{3.2.11}$$

holds for all proper ϕ-invariant subobjects E' of E. Semistability is defined by replacing the above strict inequality with a weak inequality. A Higgs bundle is called polystable if it is the direct sum of stable Higgs bundles with the same slope.

Note that for any twisted U(p,q)-Higgs bundle we can associate a GL$(p+q,\mathbb{C})$-Higgs bundle defined by taking $\widetilde{E} = V \oplus W$ and $\phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$.

The following result is reminiscent of Theorem 3.26 of [16], which is a result for Sp$(2n,\mathbb{R})$-Higgs bundles. The corresponding result for 0-semistable U(p,q)-Higgs bundles can be found in the appendix to the first preprint version of [11] and the proof given there easily adapts to the present situation. We include it here for the convenience of the reader.

Recall from Proposition 3.2.12 that for $p = q$,

$$\alpha_0 = p\big(\mu(W) - \mu(V) - \deg(L)\big) + \deg(L), \tag{3.2.12}$$

$$\alpha_0' = p\big(\mu(W) - \mu(V) + \deg(L)\big) - \deg(L). \tag{3.2.13}$$

Proposition 3.2.16. *Let $E = (V,W,\beta,\gamma)$ be an α-semistable twisted U(p,q)-Higgs bundle such that $p = q$. Suppose that one of the following conditions holds:*

1. *$\mu(V) - \mu(W) > -\deg(L)$ and $0 \leq \alpha < \alpha_0$.*

2. *$\mu(V) - \mu(W) < \deg(L)$ and $\alpha_0' < \alpha \leq 0$.*

Then the associated GL$(2p,\mathbb{C})$-Higgs bundle $\widetilde{E}$ is semistable. Moreover α-stability of E implies stability of $\widetilde{E}$ unless there is an isomorphism $f : V \to W$ such that $\beta f = f^{-1}\gamma$. In this case $(\widetilde{E},\phi)$ is polystable and decomposes as

$$(\widetilde{E},\phi) = (\widetilde{E}_1,\phi_1) \oplus (\widetilde{E}_2,\phi_2)$$

where each summand is a stable GL$(p,\mathbb{C})$-Higgs bundle isomorphic to $(V,\beta f)$.

Proof. Let $\widetilde{E}'$ be an invariant subobject of $\widetilde{E}$. Let $\pi_1 : \widetilde{E} \to V$ and $\pi_2 : \widetilde{E}' \to W$ be the projections on V and W. Taking the kernels and images of the projections, we get the following short exact sequences of vector bundles

$$0 \to W'' \to \widetilde{E}' \to V' \to 0,$$
$$0 \to V'' \to \widetilde{E}' \to W' \to 0. \tag{3.2.14}$$

we can then deduce that

$$\deg W'' + \deg V' = \deg \widetilde{E}' = \deg V'' + \deg W'$$
$$q'' + p' = \mathrm{rk}\,\widetilde{E}' = p'' + q' \tag{3.2.15}$$

where q'', q', p'' and p' denote the rank of W'', W', V'' and V', respectively. We claim that (V',W') and (V'',W'') are ϕ-invariant and, therefore, define U(p,p)-subobject of E. First, let $x_1 \in V'$. We can write $x_1 = \pi_1(x)$ for some $x = x_1 + x_2$ in $\widetilde{E}'$, therefore

$$\phi(x) = \phi(x_1) + \phi(x_2).$$

It follows that $\phi(x_1) \in W$ and $\phi(x_2) \in V$. Then $\pi_1(\phi(x)) = \phi(x_2) \in V$ and $\pi_2(\phi(x)) = \phi(x_1) \in W$. But $\phi(x) \in \widetilde{E}'$ because $\widetilde{E}'$ is ϕ-invariant, and hence $\phi(x_2) \in V'$ and $\phi(x_1) \in W'$. The α-semistability conditions applied to (V', W') and (V'', W'') imply

$$\deg V' + \deg W' \leq \mu(E)(p' + q') + \frac{q' - p'}{2}\alpha \tag{3.2.16}$$

$$\deg V'' + \deg W'' \leq \mu(E)(p'' + q'') + \frac{q'' - p''}{2}\alpha \tag{3.2.17}$$

Adding these two inequalities and using (3.2.15), we get

$$\mu(\widetilde{E}') \leq \mu(\widetilde{E}) + \frac{q' - p' + q'' - p''}{2(p' + p'' + q' + q'')}\alpha = \mu(\widetilde{E}) + \frac{q' - p'}{p' + p'' + q' + q''}\alpha \tag{3.2.18}$$

From Proposition 3.2.12 we obtain the injectivity of β and γ by using the hypothesizes (1) and (2), respectively. Injectivity of β and γ yield $q' \leq p'$ and $q' \geq p'$, respectively. Hence, in both cases of hypotheses, $(q' - p')\alpha$ is negative. Therefore (3.2.18) proves that $\widetilde{E}$ is semistable. Suppose E is α-stable therefore by the above argument $\widetilde{E}$ is semistable and it is stable if (3.2.18) is strict for all non-trivial subobjects $\widetilde{E}' \subset \widetilde{E}$. The equality holds in (3.2.18) if it holds in both (3.2.16) and (3.2.17). Since E is α-stable the only condition such that a non-trivial subobject $\widetilde{E}' \subset \widetilde{E}$ can yield equality in (3.2.18) is that

$$V' \oplus W' = V \oplus W \text{ and } V'' \oplus W''.$$

In this case from (3.2.14) we obtain isomorphisms $E' \to V$ and $E' \to W$. Therefore, combining these, we get an isomorphism $f : V \to W$ such that $\beta f = f^{-1}\gamma$. Hence if there no such an isomorphism between V and W then $(\widetilde{E}, \phi)$ is α-stable. Now suppose that there exists such an isomorphism $f : V \to W$, define

$$(\widetilde{E}_1, \phi_1) = (\{(v, f(v)) \in \widetilde{E} | v \in V\}, \phi|_{\widetilde{E}_1})$$
$$(\widetilde{E}_2, \phi_2) = (\{(v, -f(v)) \in \widetilde{E} | v \in V\}, \phi|_{\widetilde{E}_2})$$

the face $\beta f = f^{-1}\gamma$ implies that (E_i, ϕ_i), $i = 1, 2$, define $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles isomorphic to $(V, \beta f)$. We have

$$(\widetilde{E}, \phi) = (\widetilde{E}_1, \phi_1) \oplus (\widetilde{E}_2, \phi_2).$$

with

$$\mu(\widetilde{E}_1) = \mu(\widetilde{E}) = \mu(\widetilde{E}_2).$$

To show that each summand is a stable $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle, note that any non-trivial subobject $\widetilde{E}'$ of $\widetilde{E}_i$ is a subobject of $\widetilde{E}$ and hence $\mu(\widetilde{E}') < \mu(\widetilde{E}) = \mu(\widetilde{E}_i)$. $\qquad\square$

Remark 3.2.17. We can also conclude from the proof of the above proposition that twisted $\mathrm{U}(p, q)$-Higgs bundle is stable (for $\alpha = 0$) if and only if the associated $\mathrm{GL}(p + q, \mathbb{C})$-Higgs bundle is stable, unless there is an isomorphism $f : V \to W$ such that $\beta f = f^{-1}\gamma$.

3.3 Moduli Space of twisted U(p,q)-Higgs bundles

Fix the type $t = (p,q,a,b)$. Recall that we denote the moduli space of α-polystable twisted U(p,q)-Higgs bundles with the given type by

$$\mathcal{M}_\alpha(t) = \mathcal{M}_\alpha(p,q,a,b),$$

and the moduli space of α-stable twisted U(p,q)-Higgs bundle by $\mathcal{M}_\alpha^s(t) \subset \mathcal{M}_\alpha(t)$. In order to study smoothness of the moduli space we investigate vanishing of the second hypercohomology group of the deformation complex (cf. Proposition 3.1.16). This vanishing is not guaranteed by α-stability for $\alpha \neq 0$, in contrast to the case of triples, where vanishing is guaranteed for $\alpha > 0$.

3.4 Description of Moduli spaces for α_M

Throughout this section we assume that the twisted U(p,q)-Higgs bundle $E = (V,W,\beta,\gamma)$ has type $t = (p,q,a,b)$. Let $\varepsilon > 0$ be such that there is no critical value in $(\alpha_M - \varepsilon, \alpha_M)$. We shall refer E is α_M^--semistable if it is α-semistable twisted U(p,q)-Higgs bundle for which $\alpha \in (\alpha_M - \varepsilon, \alpha_M)$.

Remark 3.4.1. If $E = (V,W,\beta,\gamma)$ is α_M^--semistable then, by Proposition 3.2.12, γ is injective, since α is at least as big as α_0.

Proposition 3.4.2. *Let $E = (V,W,\beta,\gamma)$ be a twisted U(p,q)-Higgs bundle of type t such that $p \leq q$ and $\mu(V) - \mu(W) < \deg(L)$. If E is α_M^--semistable then $E_I = (V, \mathrm{im}(\gamma) \otimes L^{-1}, \beta, \gamma)$ is α_M-semistable. In particular α_M-semistablity of E_I is equivalent to α_M-semistability of E.*

Proof. Using injectivity of γ we can obtain the following

$$\mu_\alpha(E) = \mu_\alpha(E_I) + \frac{q}{p+q}\big(\mu(W) - \mu(V)\big) + \frac{p-q}{2(p+q)}\alpha + \frac{1}{2}\deg(L)$$

Since $\alpha_M = \frac{2q}{q-p}\big(\mu(W) - \mu(V) + \frac{p+q}{2q}\deg(L)\big)$ therefore

$$\mu_\alpha(E) = \mu_\alpha(E_I) + \frac{q-p}{2(p+q)}\big(\alpha_M - \alpha\big). \tag{3.4.1}$$

Now, suppose $E' \subset E_I$ be a subobject of E_I. Applying α-semistablity condition of E on E', since E' can be consider as subobject of E, and using (3.4.1) we get

$$\mu_\alpha(E') - \mu_\alpha(E_I) \leq \frac{q-p}{2(p+q)}\big(\alpha_M - \alpha\big).$$

Therefore $\mu_{\alpha_M}(E') \leq \mu_{\alpha_M}(E_I)$, by taking the limit $\alpha \to \alpha_M$, which implies that E_I is α_M-semistable. $\square$

Proposition 3.4.3. *Let $E = (V,W,\beta,\gamma)$ be twisted U(p,q)-Higgs bundle such that we have the following extension*

$$0 \longrightarrow V \xrightarrow{\gamma} W \otimes L \to F \otimes L \to 0$$

with F locally free. Then there is an $\varepsilon \in (0, \frac{2}{(q-p)(q-p-1)^2})$ such that α_M^--semistability of E implies semistability of F. In the converse direction, if V is semistable and F is stable then E is α_M^--stable.

Proof. Let $F' = \frac{W'}{\gamma(V) \otimes L^{-1}}$ be a subbundle of F. Let $E' = (V, W', \beta, \gamma)$, $n = \text{rk}(F)$, $n' = \text{rk}(F')$ and $q' = \text{rk}(W')$. Using

$$q = n + p \qquad\qquad (3.4.2)$$
$$q' = n' + p$$
$$q\mu(W) = n\mu(F) + p\mu(V) - p\deg(L)$$
$$q'\mu(W') = n'\mu(F') + p\mu(V) - p\deg(L),$$

we get

$$\mu(F') - \mu(F) = \frac{2p}{2p+n}\left[\frac{(2p+n)(2p+n')}{2pn'}(\mu_\alpha(E') - \mu_\alpha(E)) - \frac{n-n'}{2n'}(\alpha - 2(\mu_F - \mu_1) - \deg(L))\right].$$

Since $\alpha_M = \frac{2q}{q-p}\left(\mu(W) - \mu(V) + \frac{p+q}{2q}\deg(L)\right) = 2(\mu(F) - \mu(V)) + \deg(L)$, therefore we get

$$\mu(F') - \mu(F) = \frac{2p}{2p+n}\left(\frac{(2p+n)(2p+n')}{2pn'}(\mu_\alpha(E') - \mu_\alpha(E)) + \left(\frac{n-n'}{n'}\right)\frac{\varepsilon}{2}\right) \qquad (3.4.3)$$

$$(3.4.4)$$

Applying α-semistability condition on E' we obtain

$$\mu(F') - \mu(F) \le \frac{2p}{2p+n}\left(\frac{n-n'}{n'}\frac{\varepsilon}{2}\right)$$

if we take $\varepsilon < \frac{1}{n(n-1)^2}$, then

$$\mu(F') - \mu(F) < \frac{1}{n(n-1)}.$$

But above inequality is equivalent to $\mu(F') - \mu(F) \le 0$, since $\mu(F)$ and $\mu(F')$ are rational numbers with denominator n and $n'(\le (n-1))$ respectively.

Conversely, suppose on contrary for given $\varepsilon > 0$, E' be the destabilizing subobject of E. Now if $E' = (V, W', \beta', \gamma')$ with $W' \subset W$ then by considering $F' := \frac{W'}{\gamma(V) \otimes L^{-1}}$ and from (3.4.3) we have

$$\mu(F') - \mu(F) > 0,$$

contradicting the stability of F. On the other hand $E' = (V', W', \beta', \gamma')$ with $V' \subset V$ and $W' \subset W$ defines a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V & \overset{\gamma}{\longrightarrow} & W \otimes L & \longrightarrow & F \otimes L & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & V' & \longrightarrow & W' \otimes L & \longrightarrow & F' \otimes L & \longrightarrow & 0
\end{array}
$$

with $F' \subset F$. It is straight forward to work out that

$$(2p' + n')\left(\mu_{\alpha_M^-}(E') - \mu_{\alpha_M^-}(E)\right) + \frac{p'n}{2p+n}\varepsilon = 2p'(\mu(V') - \mu(V)) + n'(\mu(F') - \mu(F)) + \frac{pn'}{2p+n}\varepsilon.$$

The above equality along with the semistability of V yield

$$\mu(F) - \mu(F') \leq \frac{pn'}{2p+n}\varepsilon < \frac{1}{n(n-1)},$$

implies $\mu(F) - \mu(F') \leq 0$, which is a contradiction. $\qquad\square$

Lemma 3.4.4. *Let $p = q$ and $a - b = p \deg(L)$, and let $\alpha \geq 0$. Then*

$$\mathcal{M}_\alpha(t) \cong \mathcal{M}_{L^2}((p,a)).$$

Where $\mathcal{M}_{L^2}((p,a))$ is the moduli space of semistable L^2-twisted Higgs bundles.

Proof. Let $E = (V, W, \beta, \gamma)$ be a α-semistable L-twisted U(p,p)-Higgs bundle. From hypotheses and (3.2.1) it follows that $\gamma : V \longrightarrow W \otimes L$ is an isomorphism. We can then compose $\gamma : V \longrightarrow W \otimes L$ with $\beta \otimes Id_L : W \otimes L \longrightarrow V \otimes L^2$ to get a L^2-twisted Higgs pair (V, θ_V), $\theta_V : V \longrightarrow V \otimes L^2$. Conversely given an isomorphism $\gamma : V \longrightarrow W \otimes L$ we can recover β from θ_V. We claim that α-semistability of E is equivalent to semistability of twisted Higgs pair (V, θ_V). We first prove that the α-semistability of E implies semistability of (V, θ_V).

Suppose that E is α-semistable. Let $V' \subset V$ be a θ-invariant subobject of V. Then $E' = (V', \gamma(V') \otimes L^{-1})$ defines an invariant subobject of E. From α-semistability of E we have

$$\mu(V' \oplus (\gamma(V') \otimes L^{-1})) + \frac{1}{2}\alpha \leq \mu(V \oplus W) + \frac{1}{2}\alpha. \tag{3.4.5}$$

Note that $\mu(\gamma(V') \otimes L^{-1}) = \mu(V') - \deg(L)$ and $\mu(W) = \mu(V) - \deg(L)$. Hence using (3.4.5) and these observations we get

$$\mu(V') \leq \mu(V),$$

which implies that (V, θ_V) is semistable.

We now prove that the semistability of (V, θ_V) implies α-semistablity of (V, W, β, γ). Suppose that (V, θ_V) is semistable. Let $E' = (V', W')$ be an invariant subobject of E. Then V' and $\gamma^{-1}(W' \otimes L)$ are θ_V-invariant subobjects of V and hence satisfy the semistability inequality. Moreover, since γ is an

isomorphism, it follows that $\text{rk}(W') \geq \text{rk}(V')$ and hence

$$\mu(V' \oplus W') \leq \mu(V) - \frac{q'}{p'+q'} \deg(L)$$
$$\leq \mu(V) - \frac{1}{2} \deg(L) = \mu(V \oplus W),$$

where p' and q' are ranks of V' and W', respectively. Therefore we have

$$\mu_\alpha(E') = \mu(V' \oplus W') + \alpha \frac{p'}{p'+q'}$$
$$\leq \mu(V \oplus W) + \frac{1}{2}\alpha = \mu_\alpha(E)$$

This proves that E is α-semistable. $\qquad\qquad\square$

Proposition 3.4.5. *Let $\mathcal{M}_{\alpha_M}(t)$ be the moduli space of α_M-semistable twisted $\text{U}(p,q)$-bundles of type $t = (p,q,a,b)$ such that $p \leq q$ and $a/p - b/q < \deg(L)$. Then*

$$\mathcal{M}_{\alpha_M}(t) \cong \mathcal{M}_{L^2}(p,a) \times \mathcal{N}(q-p).$$

Where $\mathcal{M}_{L^2}(p,a)$ and $\mathcal{N}(q-p)$ are the Moduli spaces of semistable L^2-twisted Higgs pairs and moduli space of semistable vector bundles of rank $q-p$, respectively.

Proof. Let $E = (V,W,\beta,\gamma)$ be a α_M-semistable twisted $\text{U}(p,q)$-Higgs bundle. Associated to E define a twisted $\text{U}(p,p)$-Higgs bundle by $E_I := (V, \gamma(V) \otimes L^{-1}, \beta, \gamma)$ and a vector bundle of rank $q-p$ by $F := \frac{W}{\gamma(V) \otimes L^{-1}}$. From Proposition 3.4.2 and Proposition 3.4.3 we get

$$E_I \in \mathcal{M}_{\alpha_M}(t'), \ F \in \mathcal{N}(q-p)$$

where $t' = (p,p,a,a-p\deg(L))$. Therefore we obtain an isomorphism

$$\mathcal{M}_{\alpha_M}(t) \to \mathcal{M}_{\alpha_M}(t') \times \mathcal{N}(q-p).$$

Hence result follows using the previous lemma. $\qquad\qquad\square$

3.4.1   Vanishing of hypercohomology in two

In order to study smoothness of the moduli space we investigate vanishing of the second hypercohomology group of the deformation complex (cf. Proposition 3.1.16). This vanishing will also play an important role in the analysis of the flip loci in Section 3.5. We note that vanishing is not guaranteed by α-stability for $\alpha \neq 0$, in contrast to the case of triples (and chains), where vanishing is guaranteed for $\alpha > 0$.

By using the obvious symmetry of the quiver interchanging the vertices we can associate to a $\text{U}(p,q)$-Higgs bundle a $\text{U}(q,p)$-Higgs bundle. The following proposition is immediate.

Proposition 3.4.6. *Let $E = (V, W, \beta, \gamma)$ be a U(p,q)-Higgs bundle and let $\sigma(E) = (W, V, \gamma, \beta)$ be the associated U(q,p)-Higgs bundle. Then E is α-stable if and only if $\sigma(E)$ is $-\alpha$-stable, and similarly for poly- and semi-stability.* □

The next result uses this construction and Serre duality to identify the second hypercohomology of the Hom-complex with the dual of a zeroth hypercohomology group.

Lemma 3.4.7. *Let $E = (V, W, \beta, \gamma)$ and $E' = (V', W', \beta', \gamma')$ be L-twisted U(p,q)-Higgs bundles and let $E'' = \sigma(E') \otimes L^{-1}K = (W' \otimes L^{-1}K, V \otimes L^{-1}K, \gamma \otimes 1, \beta \otimes 1)$. Then*

$$\mathbb{H}^2(\mathcal{H}om^\bullet(E', E)) \cong \mathbb{H}^0(\mathcal{H}om^\bullet(E, E''))^*.$$

Proof. By Serre duality for hypercohomology

$$\mathbb{H}^2(\mathcal{H}om^\bullet(E', E)) \cong \mathbb{H}^0(\mathcal{H}om^{\bullet^\vee}(E', E) \otimes K)^*$$

where the dual complex twisted by K is

$$\mathcal{H}om^{\bullet^\vee}(E', E) \otimes K : \left(\mathrm{Hom}(V, W' \otimes L^{-1}) \oplus \mathrm{Hom}(W, V' \otimes L^{-1}) \right) \otimes K$$
$$\rightarrow \left(\mathrm{Hom}(V, V') \oplus \mathrm{Hom}(W, W') \right) \otimes K.$$

One easily checks that the differentials correspond, so that

$$\mathcal{H}om^{\bullet^\vee}(E', E) \otimes K \cong \mathcal{H}om^\bullet(E, E'').$$

This completes the proof. □

Lemma 3.4.8. *Let $E = (V, W, \beta, \gamma)$ and $E' = (V', W', \beta', \gamma')$ be α-semistable L-twisted U(p,q)-Higgs bundles and as above let $E'' = \sigma(E') \otimes L^{-1}K = (W' \otimes L^{-1}K, V' \otimes L^{-1}K, \gamma' \otimes 1, \beta' \otimes 1)$. Let $f \in \mathbb{H}^0(\mathcal{H}om^\bullet(E, E''))$ viewed as morphism $f: E \to E''$ and write $\lambda(f) = \frac{\mathrm{rk}(f(V))}{\mathrm{rk}(f(V)) + \mathrm{rk}(f(W))}$. Then, if $f \neq 0$, the inequality*

$$\alpha(2\lambda(f) - 1) + 2g - 2 - \deg(L) \geq 0 \tag{3.4.6}$$

holds. Moreover, if E and E'' are α-stable, then strict inequality holds unless $f: E \xrightarrow{\cong} E''$ is an isomorphism.

Proof. Write $N = \ker(f) \subset E$ and $I = \mathrm{im}(f) \subset E''$. Then α-semistability of E implies that $\mu_\alpha(N) \leq \mu_\alpha(E)$, which is equivalent to

$$\mu_\alpha(I) \geq \mu_\alpha(E); \tag{3.4.7}$$

note that this also holds if $N = 0$, since then $I \cong E$. Moreover, by Proposition 3.4.6, E'' is $-\alpha$-semistable and so $\mu_{-\alpha}(I) \leq \mu_{-\alpha}(E'')$. This, using that $\mu_{-\alpha}(I) = \mu_\alpha(I) - 2\alpha\lambda(f)$ and $\mu_{-\alpha}(E'') = \mu_\alpha(E) - \alpha + (2g - 2 - \deg(L))$, is equivalent to

$$\mu_\alpha(I) \leq \mu_\alpha(E) + 2\alpha\lambda(f) - \alpha + 2g - 2 - \deg(L). \tag{3.4.8}$$

Combining (3.4.7) and (3.4.8) gives the result. The statement about strict inequality is easy. □

The following is our first main result on vanishing of $\mathbb{H}^2$. It should be compared with [9, Proposition 3.6]. The reason why extra conditions are required for the vanishing is essentially that the "total Higgs field" $\beta + \gamma \in H^0(\mathrm{End}(V \oplus W) \otimes L)$ is not nilpotent, contrary to the case of triples.

Proposition 3.4.9. *Let $E = (V,W,\beta,\gamma)$ and $E' = (V',W',\beta',\gamma')$ be L-twisted $\mathrm{U}(p,q)$-Higgs bundles of types $t(E) = (p,q,a,b)$ and $t(E') = (p',q',a',b')$, respectively. Assume that E and E' are α-semistable with $\mu_\alpha(E) = \mu_\alpha(E')$. Let $E'' = \sigma(E') \otimes L^{-1}K$. Assume that one of the following hypotheses hold:*

(A) $\deg(L) > 2g - 2$;

(B) $\deg(L) = 2g - 2$, both E and E' are α-stable and there is no isomorphism $f : E \xrightarrow{\cong} E''$.

Then $\mathbb{H}^2(\mathcal{H}om^\bullet(E',E)) = 0$ if one of the following additional conditions holds:

(1) $\alpha = 0$;

(2) $\alpha > 0$ and either β' is injective or β is surjective;

(3) $\alpha < 0$ and either γ' is injective or γ is surjective.

Proof. Suppose first that $\alpha = 0$. Then either of the conditions (A) and (B) guarantee that strict inequality holds in (3.4.6). Hence Lemmas 3.4.7 and 3.4.8 imply the stated vanishing of $\mathbb{H}^2$.

Now suppose that $\beta' : W' \to V' \otimes L$ is injective. If $f : E \to E''$ is non-zero then, since f is a morphism of twisted $\mathrm{U}(p,q)$-Higgs bundles, we have $\mathrm{rk}(f(W)) \geq \mathrm{rk}(f(V))$. Hence $\lambda(f) = \frac{\mathrm{rk}(f(V))}{\mathrm{rk}(f(V)) + \mathrm{rk}(f(W))}$ satisfies $\lambda(f) \leq 1/2$. If additionally $\alpha > 0$, it follows that $\alpha(2\lambda(f) - 1) \leq 0$ which contradicts Lemma 3.4.8 under either of the conditions (A) and (B). Therefore there are no non-zero morphisms $f : E \to E''$ and so Lemma 3.4.7 implies vanishing of $\mathbb{H}^2(\mathcal{H}om^\bullet(E',E))$.

We have deduced vanishing of $\mathbb{H}^2$ under the conditions $\alpha > 0$ and β' injective. The remaining conditions in (2) and (3) for vanishing of $\mathbb{H}^2$ can now be deduced by using symmetry arguments as follows.

Suppose first that $\alpha < 0$ and γ' is injective. Then, using Proposition 3.4.6 $(\sigma(E'), \sigma(E))$ is a pair of twisted $\mathrm{U}(p,q)$-Higgs bundles which are $-\alpha$-semistable and such that the β-map (which is $\sigma(\gamma')$) of $\sigma(E')$ is injective. Observe that

$$\mathcal{H}om^\bullet(\sigma(E'), \sigma(E)) \cong \mathcal{H}om^\bullet(E', E).$$

Hence, noting that $-\alpha > 0$, the conclusion follows from the previous case.

Next suppose that $\alpha < 0$ and γ is surjective. Then of dual $\mathrm{U}(p,q)$-Higgs bundles (E^*, E'^*) is a pair of twisted $\mathrm{U}(p,q)$-Higgs bundles which are $-\alpha$-semistable and such that the β-map (which is γ^*) of E^* is injective. Observe that

$$\mathcal{H}om^\bullet(E^*, E'^*) \cong \mathcal{H}om^\bullet(E', E).$$

Hence again the conclusion follows from the previous case.

The final case, $\alpha > 0$ and β surjective, follows in a similar way, combining the two previous constructions. $\qquad\square$

In the case when $q = 1$ we also have the following result on vanishing of the second hypercohomology of the deformation complex.

Proposition 3.4.10. *Let E be an α-semistable L-twisted* U$(p,1)$*-Higgs bundle with $p \geq 2$. Assume that* $\deg(L) > 2g - 2$. *Then* $\mathbb{H}^2(\mathcal{E}nd^\bullet(E)) = 0$ *for all α in the range*

$$p(\mu(V) - \mu(W)) - (p+1)(\deg(L) - 2g + 2) < \alpha < p(\mu(V) - \mu(W)) + (p+1)(\deg(L) - 2g + 2).$$

Proof. Assume first that $\alpha \geq 0$. Note that an isomorphism as in (B) of the hypothesis of Proposition 3.4.9 cannot exist when $p \neq q$. Hence the proposition immediate gives the result if $\alpha = 0$. Moreover, if $\beta \neq 0$, then it is injective, and hence $\mathbb{H}^2(\mathcal{H}om^\bullet(E', E)) = 0$ by (2) of the proposition. We may thus assume that $\beta = 0$ and consider the L-twisted triple $E_T : V \xrightarrow{\gamma} W \otimes L$. We have that

$$\mathbb{H}^2(\mathcal{E}nd^\bullet(E)) = \mathbb{H}^2(\mathcal{E}nd^\bullet(E_T)) \oplus H^1(\mathrm{Hom}(W, V) \otimes L),$$

where $\mathcal{E}nd^\bullet(E_T)$ is the deformation complex of the triple. The vanishing of $\mathbb{H}^2(\mathcal{E}nd^\bullet(E_T))$ for an α-semistable triple when $\alpha > 0$ is well known[1] (cf. [9]). Hence it remains to show that $H^1(\mathrm{Hom}(W, V) \otimes L) = 0$ which, by Serre duality, is equivalent to the vanishing

$$H^0(\mathrm{Hom}(V, W) \otimes L^{-1}K) = 0.$$

So assume we have a non-zero $f : V \to W \otimes L^{-1}K$. Then f induces as non-zero map of line bundles $f : V/\ker(f) \to W \otimes L^{-1}K$ and hence

$$\deg(W) - \deg(L) + 2g - 2 \geq \deg(V) - \deg(\ker(f)). \tag{3.4.9}$$

On the other hand, since $\beta = 0$ we can consider the subobject $(\ker(f), W, 0, \gamma)$ of E and hence, by α-semistability,

$$\mu_\alpha(\ker(f) \oplus W) \leq \mu_\alpha(V \oplus W)$$
$$\Longleftrightarrow (p+1)\deg(\ker(f)) + \deg(W) \leq p\deg(V) + \alpha, \tag{3.4.10}$$

where we have used that $\mathrm{rk}(\ker(f)) = p - 1$ and $\mathrm{rk}(W) = 1$. Now combining (3.4.9) and (3.4.10) we obtain

$$\alpha \geq p(\mu(V) - \mu(W)) + (p+1)(\deg(L) - 2g + 2).$$

This establishes the vanishing of $\mathbb{H}^2$ for α in the range

$$0 \leq \alpha < p(\mu(V) - \mu(W)) + (p+1)(\deg(L) - 2g + 2).$$

On the other hand, if $\alpha \leq 0$, applying the preceding result to the dual twisted U(p,q)-Higgs bundle $(V^*, W^*, \gamma^*, \beta^*)$ gives vanishing of $\mathbb{H}^2$ for α in the range

$$0 \geq \alpha > p(\mu(V) - \mu(W)) - (p+1)(\deg(L) - 2g + 2).$$

---

[1]Note that the stability parameter for the corresponding untwisted triple as considered in [9] is $\alpha + \deg(L)$.

This finishes the proof. $\square$

In general the preceding proposition does not guarantee vanishing of $\mathbb{H}^2$ for all values of the parameter α. But for some values of the topological invariants, the upper bound of the preceding proposition is actually larger than the maximal value for the parameter α. More precisely, we have the following result.lt.

Proposition 3.4.11. *Under the hypothesis of the previous proposition, with additionally assumption we have the following*

(1) If $p\big(\mu(V) - \mu(W)\big) > 2g - 2 - (p-2)\big(\deg(L) - (2g-2)\big)$ then $\mathbb{H}^2(\mathcal{E}nd^\bullet(E)) = 0$ for all $\alpha \geq 0$

(2) If $p\big(\mu(V) - \mu(W)\big) < -2g + 2 + (p-2)\big(\deg(L) - (2g-2)\big)$ then $\mathbb{H}^2(\mathcal{E}nd^\bullet(E)) = 0$ for all $\alpha \leq 0$

Proof. The upper and lower bound for α given in Proposition 3.2.8 is, in this case

$$\alpha_M = -\frac{2p}{p-1}\big(\mu(V) - \mu(W)\big) + \frac{p+1}{p-1}\deg(L),$$

$$\alpha_m = -\frac{2p}{p-1}\big(\mu(V) - \mu(W)\big) - \frac{p+1}{p-1}\deg(L).$$

It is simple to check that the inequalities of the statements are equivalent to α_M being less than the upper bound and α_m being bigger than the lower bound for α of Proposition 3.4.10 . $\square$

Proposition 3.4.12. *Let $E = (V, W, \beta, \gamma)$ be an α-stable L-twisted $\mathrm{U}(p,q)$-Higgs bundle with $p = q$. Let α_0 and α_0' be given by (3.2.12) and (3.2.13), respectively. Suppose that there is no isomorphism $f : V \to W$ such that $\beta f = f^{-1}\gamma$ and that one of the following conditions holds:*

    1. $\mu(V) - \mu(W) > -\deg(L)$ and $0 \leq \alpha < \alpha_0$,

    2. $\mu(V) - \mu(W) < \deg(L)$ and $\alpha_0' < \alpha \leq 0$.

Then $\mathbb{H}^2(\mathcal{E}nd^\bullet(E)) = 0$.

Proof. From Proposition 3.2.16 we have that the corresponding $\mathrm{GL}(p+q, \mathbb{C})$-Higgs bundle $\tilde{E}$ is stable. Hence $\mathbb{H}^2$ of the deformation complex of $\tilde{E}$ is isomorphic to $\mathbb{C}$, corresponding to central endomorphisms, and so the part of this $\mathbb{H}^2$ which corresponds to $\mathbb{H}^2(\mathcal{E}nd^\bullet(E))$ vanishes. $\square$

The following trivial observation is sometimes useful.

Proposition 3.4.13. *Let E and E' be L-twisted $\mathrm{U}(p,q)$-Higgs bundles such that $\mathbb{H}^2(\mathcal{E}nd^\bullet(E \oplus E')) = 0$. Then*

$$\mathbb{H}^2(\mathcal{H}om^\bullet(E', E)) = \mathbb{H}^2(\mathcal{H}om^\bullet(E, E')) = 0.$$

Proof. Immediate in view of Remark 3.1.9. $\square$

We can summarise our main results on vanishing of $\mathbb{H}^2$ as follows.

Lemma 3.4.14. *Fix a type $t = (p,q,a,b)$ and let E be an α-semistable L-twisted U(p,q)-Higgs bundle of type t with $\deg(L) \geq 2g-2$. If $\deg(L) = 2g-2$ assume moreover that E is α-stable. If either one of the following conditions holds:*

(1) *$q = 1$, $p \geq 2$ and $p(\mu(V) - \mu(W)) - (p+1)(\deg(L) - 2g + 2) < \alpha < p(\mu(V) - \mu(W)) + (p + 1)(\deg(L) - 2g + 2)$,*

(2) *$a/p - b/q > -\deg(L)$ and $0 \leq \alpha < \frac{2pq}{\min\{p,q\}|p-q|+p+q}\big(b/q - a/p - \deg(L)\big) + \deg(L)$,*

(3) *$a/p - b/q < \deg(L)$ and $\frac{2pq}{\min\{p,q\}|p-q|+p+q}(b/q - a/p + \deg(L)) - \deg(L) < \alpha \leq 0$.*

Then $\mathbb{H}^2(\mathcal{E}nd^\bullet(E))$ vanishes.

Proof. For part (1), use Proposition 3.4.10. The other parts follow from Proposition 3.2.12, Corollary 3.2.13, and Proposition 3.4.9. $\qquad\qquad\square$

Proposition 3.4.15. *Fix a type $t = (p,q,a,b)$. If either one of the following conditions holds:*

(1) *$q = 1$, $p \geq 2$ and $p(\mu(V) - \mu(W)) - (p+1)(\deg(L) - 2g + 2) < \alpha < p(\mu(V) - \mu(W)) + (p + 1)(\deg(L) - 2g + 2)$,*

(2) *$a/p - b/q > -\deg(L)$ and $0 \leq \alpha_c < \frac{2pq}{\min\{p,q\}|p-q|+p+q}\big(b/q - a/p - \deg(L)\big) + \deg(L)$,*

(3) *$a/p - b/q < \deg(L)$ and $\frac{2pq}{\min\{p,q\}|p-q|+p+q}(b/q - a/p + \deg(L)) - \deg(L) < \alpha_c \leq 0$.*

Then the moduli space $\mathcal{M}_\alpha^s(t)$ is smooth.

Proof. Combine Lemma 3.4.14 and Proposition 3.1.16. $\qquad\qquad\square$

3.5   Crossing critical values

3.5.1   Flip loci

In this section we study the variations in the moduli spaces $\mathcal{M}_\alpha^s(t)$, for fixed type $t = (p,q,a,b)$ and different values of α. Here we are using a method similar to the one for chains given in [**?**].

Let α_c be a critical value. We adopt the following notation:

$$\alpha_c^+ = \alpha_c + \varepsilon, \ \alpha_c^- = \alpha_c - \varepsilon,$$

where $\varepsilon > 0$ is small enough so that α_c is the only critical value in the interval (α_c^-, α_c^+). We begin with a set-theoretic description of the differences between two spaces $\mathcal{M}_{\alpha_c^+}$ and $\mathcal{M}_{\alpha_c^-}$.

Definition 3.5.1. We define *flip loci* $\mathcal{S}_{\alpha_c^\pm} \subset \mathcal{M}_{\alpha_c^\pm}^s$ by the conditions that the points in $\mathcal{S}_{\alpha_c^+}$ represent twisted U(p,q)-Higgs bundles which are α_c^+-stable but α_c^--unstable.

The following is immediate.

Lemma 3.5.2. *In the above notation (as sets):*

$$\mathcal{M}_{\alpha_c^+}^s - \mathcal{S}_{\alpha_c^+} = \mathcal{M}_{\alpha_c}^s = \mathcal{M}_{\alpha_c^-}^s - \mathcal{S}_{\alpha_c^-}.$$

A twisted $U(p,q)$-Higgs bundle $E \in \mathcal{S}_{\alpha_c^\pm}$ is strictly α_c-semistable and so we can use the Jordan-Hölder filtrations of E in order to estimate the codimension of $\mathcal{S}_{\alpha_c^\pm}$ in $\mathcal{M}_{\alpha_c^\pm}$.

The following is an analogue for twisted $U(p,q)$-Higgs bundles of [**?** , Proposition 4.3], which is a result for cha

Proposition 3.5.3. *Fix a type $t = (p,q,a,b)$. Let α_c be a critical value and let $\mathcal{S}$ be a family of α_c-semistable twisted $U(p,q)$-Higgs bundles E of type t, all of them pairwise non-isomorphic, and whose Jordan-Hölder filtrations have an associated graded of the form $\mathrm{Gr}(E) = \bigoplus_{i=1}^m Q_i$, with Q_i twisted $U(p,q)$-Higgs bundle of type t_i. If either one of the following conditions holds:*

(1) *$q = 1$, $p \geq 2$ and $p(a/p - b/q) - \deg(L)(p+1) < \alpha_c < p(a/p - b/q) + \deg(L)(p+1)$,*

(2) *$a/p - b/q > -\deg(L)$ and $0 \leq \alpha_c < \frac{2pq}{\min\{p,q\}|p-q|+p+q}\left(b/q - a/p - \deg(L)\right) + \deg(L)$,*

(3) *$a/p - b/q < \deg(L)$ and $\frac{2pq}{\min\{p,q\}|p-q|+p+q}\left(b/q - a/p + \deg(L)\right) - \deg(L) < \alpha_c \leq 0$.*

Then

$$\dim \mathcal{S} \leq -\sum_{i \leq j} \chi(t_j, t_i) - \frac{m(m-3)}{2}. \tag{3.5.1}$$

Proof. Suppose $m = 2$ then, from the way we defined $\mathcal{S}$, there exists a surjective canonical map

$$i : \mathcal{S} \to \mathcal{M}_{\alpha_c}^s(t_1) \times \mathcal{M}_{\alpha_c}^s(t_2)$$

with $i^{-1}(Q_1, Q_2) \cong \mathbb{P}(\mathrm{Ext}^1(Q_2, Q_1))$, where $\mathbb{P}(\mathrm{Ext}^1(Q_2, Q_1))$ parametrizes equivalence classes of extensions

$$0 \to Q_1 \to E \to Q_2 \to 0.$$

Notice that Q_1 and Q_2 satisfy the hypothesis of Proposition 3.4.9 and therefore, cf. Proposition 3.1.11, $\dim(\mathbb{P}\mathrm{Ext}^1(Q_2, Q_1))$ is constant as Q_1 and Q_2 vary in their corresponding moduli spaces. Hence, we obtain

$$\dim \mathcal{S} \leq \dim \mathcal{M}_{\alpha_c}^s(t_1) + \dim \mathcal{M}_{\alpha_c}^s(t_2) + \dim \mathbb{P}(\mathrm{Ext}^1(Q_2, Q_1)).$$

Consequently by induction on m, we have

$$\dim \mathcal{S} \leq \sum_{1 \leq i \leq m} \dim \mathcal{M}_{\alpha_c}^s(t_i) + \sum_{1 \leq i < j \leq m} \dim \mathbb{P}(\mathrm{Ext}^1(Q_j, Q_i)).$$

We claim that $\mathbb{H}^2(\mathcal{H}om^\bullet(Q_j, Q_i)) = 0$, therefore by Proposition 3.1.16 $\dim \mathcal{M}_{\alpha_c}^s(t_i) = 1 - \chi(t_i, t_i)$ and hence the result follows by using $\dim \mathrm{Ext}^1(Q_j, Q_i) = -\chi(t_j, t_i)$; note that we may assume that Q_j and Q_i are not isomorphic, since this is true outside a subspace of positive codimension in $\mathcal{S}$ (cf. the proof of [**?** , Proposition 4.3]).

Now we prove the claim. From the extension $0 \to Q_i \to E \to Q_j \to 0$ we get that the vanishing of $\mathbb{H}^2(\mathcal{E}nd^\bullet(E))$ implies that $\mathbb{H}^2(\mathcal{H}om^\bullet(Q_j, Q_i)) = 0$: This is true because from such an extension we get a short exact sequence of complexes

$$0 \to \mathcal{N}^\bullet \to \mathcal{E}nd^\bullet(E) \to \mathcal{H}om^\bullet(Q_j, Q_i) \to 0$$

where $\mathcal{N}^\bullet = \ker\big(\mathcal{E}nd^\bullet(E) \to \mathcal{H}om^\bullet(Q_j,Q_i)\big)$, and therefore there is a associated long exact sequence

$$\cdots \longrightarrow \mathbb{H}^2(\mathcal{N}) \longrightarrow \mathbb{H}^2(\mathcal{E}nd^\bullet(E)) \longrightarrow \mathbb{H}^2(\mathcal{H}om(Q_j,Q_i)) \longrightarrow 0$$

from which it is clear that the vanishing of $\mathbb{H}^2(\mathcal{E}nd^\bullet(E))$ implies the vanishing of $\mathbb{H}^2(\mathcal{H}om^\bullet(Q_j,Q_i))$. Using Proposition 3.4.9, in either case of hypothesis, we obtain $\mathbb{H}^2(\mathcal{E}nd^\bullet(E)) = 0$ and consequently $\mathbb{H}^2(\mathcal{H}om^\bullet(Q_j,Q_i)) = 0$, for each $i \le j$. $\qquad\qquad\square$

In order to show that the flip loci $\mathcal{S}_{\alpha_c^\pm}$ has positive codimension we need to bound the values of $\chi(t_i,t_j)$ in (3.5.1). This is what we do next.

3.5.2   Bound for χ

Here we consider a Q-bundle associated to the complex $\mathcal{H}om^\bullet(E',E)$ and construct a solution to the vortex equations on this Q-bundle from solutions on E' and E. The quiver Q is the following:

The construction generalizes the one of [9] Lemma 4.2.

The Q-bundle associated to $\mathcal{H}om^\bullet(E',E)$

Let $E = (V,W,\beta,\gamma)$ and $E' = (V',W',\beta',\gamma')$ be L-twisted U(p,q)-Higgs bundles. Let us consider the following twisted Q-bundle $\widetilde{E}$ (the morphisms are twisted by L for each arrow):

$$\mathrm{Hom}(W',V) \quad \mathrm{Hom}(V',V) \oplus \mathrm{Hom}(W',W) \quad \mathrm{Hom}(V',W) \qquad (3.5.2)$$

with arrows ϕ_d, ϕ_c (top) and ϕ_b, ϕ_a (bottom).

where

$$\phi_a(f_1,f_2) = (f_2 \otimes 1_L) \circ \gamma' - \gamma \circ f_1 \,,$$
$$\phi_b(f_1,f_2) = (f_1 \otimes 1_L) \circ \beta' - \beta \circ f_2 \,,$$
$$\phi_c(g) = (\beta \circ g, (g \otimes 1_L) \circ \beta') \,,$$
$$\phi_d(h) = ((h \otimes 1_L) \circ \gamma', \gamma \circ h).$$

We will write briefly as $\widetilde{E}$

$$\mathcal{H}om^{12} \overset{\phi_d}{\underset{\phi_b}{\rightleftarrows}}_L \mathcal{H}om^0 \overset{\phi_c}{\underset{\phi_a}{\rightleftarrows}}_L \mathcal{H}om^{11}$$

note that $\mathcal{H}om^1 = C_{11} \oplus C_{12}$ and $a_0 = (\varphi_a, \varphi_b)$, where $a_0 : \mathcal{H}om^0 \to \mathcal{H}om^1$ is the Hom-complex (3.1.5).

In this section, by using Theorem 2.7.4, we prove that if E' and E are α-polystable then $\widetilde{E}$ is α-polystable for a suitable choice of α.

Lemma 3.5.4. *Let E and E' be holomorphic twisted* $\mathrm{U}(p,q)$*-bundles and suppose we have solutions to the* (τ_1, τ_2)*-vortex equations on E and the* (τ_1', τ_2')*-vortex equations on E' such that $\tau_1 - \tau_1' = \tau_2 - \tau_2'$. Then the induced Hermitian metric on the Q-bundle $\widetilde{E}$ satisfies the vortex equations*

$$
\begin{aligned}
\sqrt{-1} \wedge F(\mathcal{H}om^{12}) + \phi_b \phi_b^* - \phi_d^* \phi_d &= \widetilde{\tau}_2 \mathrm{Id}_{\mathcal{H}om^{12}}, \\
\sqrt{-1} \wedge F(\mathcal{H}om^0) + \phi_c \phi_{\mathcal{H}om}^* + \phi_d \phi_d^* - \phi_a^* \phi_a - \phi_b^* \phi_b &= \widetilde{\tau}_1 \mathrm{Id}_{\mathcal{H}om^0}, \\
\sqrt{-1} \wedge F(\mathcal{H}om^{11}) + \phi_a \phi_a^* - \phi_{\mathcal{H}om}^* \phi_c &= \widetilde{\tau}_0 \mathrm{Id}_{\mathcal{H}om^{11}}.
\end{aligned}
$$

For $\tau = \left(\widetilde{\tau}_0, \widetilde{\tau}_1, \widetilde{\tau}_2 \right)$ given by

$$
\begin{aligned}
\widetilde{\tau}_0 &= \tau_2 - \tau_1', \\
\widetilde{\tau}_1 &= \tau_1 - \tau_1' = \tau_2 - \tau_2', \\
\widetilde{\tau}_2 &= \tau_1 - \tau_2'.
\end{aligned}
$$

Proof. The vortex equations for E and E' are

$$
\begin{aligned}
\sqrt{-1} \wedge F(V) + \beta\beta^* - \gamma^*\gamma &= \tau_1 \mathrm{Id}_V, \\
\sqrt{-1} \wedge F(W) + \gamma\gamma^* - \beta^*\beta &= \tau_2 \mathrm{Id}_W, \\
\sqrt{-1} \wedge F(V') + \beta'\beta'^* - \gamma'^*\gamma' &= \tau_1' \mathrm{Id}_{V'}, \\
\sqrt{-1} \wedge F(W') + \gamma'\gamma'^* - \beta'^*\beta' &= \tau_2' \mathrm{Id}_{W'}.
\end{aligned}
$$

We have $F(\mathcal{H}om^0)(\psi, \eta) = (F(V) \circ \psi - \psi \circ F(V'), F(W) \circ \eta - \eta \circ F(W'))$. Now we calculate ϕ_a^* and ϕ_b^*: for $(f_1, f_2) \in \mathcal{H}om^0$, $g \in \mathcal{H}om^{11}$ and $h \in \mathcal{H}om^{12}$,

$$
\begin{aligned}
\langle \phi_a^*(g), (f_1, f_2) \rangle_{\mathcal{H}om^0} &= \langle g, \phi_a((f_1, f_2)) \rangle_{\mathcal{H}om^{11}} \\
&= \langle g, (f_2 \otimes 1_L) \circ \gamma' - \gamma \circ f_1 \rangle_{\mathcal{H}om^{11}} \\
&= \langle g, (f_2 \otimes 1_L) \circ \gamma' \rangle_{C_{11}} - \langle g, \gamma \circ f_1 \rangle_{\mathcal{H}om^{11}} \\
&= \langle (g \circ \gamma'^*) \otimes 1_{L^*}, f_2 \rangle_{\mathrm{Hom}(W',W)} + \langle -\gamma^* \circ g, f_1 \rangle_{\mathrm{Hom}(V',V)} \\
&= \langle \left(-\gamma^* \circ g, (g \circ \gamma'^*) \otimes 1_{L^*} \right), (f_1, f_2) \rangle_{\mathcal{H}om^0}
\end{aligned}
$$

and

$$
\begin{aligned}
\langle \phi_b^*(h), (f_1, f_2) \rangle_{\mathcal{H}om^0} &= \langle h, \phi_b((f_1, f_2)) \rangle_{\mathcal{H}om^{12}} \\
&= \langle h, (f_1 \otimes 1_L) \circ \beta' - \beta \circ f_2 \rangle_{\mathcal{H}om^{12}} \\
&= \langle (h \circ \beta'^*) \otimes 1_{L^*}, f_1 \rangle_{\mathrm{Hom}(V',V)} - \langle \beta^* \circ h, f_2 \rangle_{\mathrm{Hom}(W',W)} \\
&= \langle \left((h \circ \beta'^*) \otimes 1_{L^*}, -\beta^* \circ h \right), (f_1, f_2) \rangle_{\mathcal{H}om^0}
\end{aligned}
$$

Hence,

$$\phi_a^*(g) = (-\gamma^* \circ g, (g \circ \gamma'^*) \otimes 1_{L^*}),$$
$$\phi_b^*(h) = ((h \circ \beta'^*) \otimes 1_{L^*}, -\beta^* \circ h).$$

By the similar calculation as above, we have

$$\phi_c^*(f_1, f_2) = (f_2 \circ \beta'^*) \otimes 1_{L^*} - \beta^* \circ f_1,$$
$$\phi_d^*(f_1, f_2) = (f_1 \circ \gamma'^*) \otimes 1_{L^*} - \gamma^* \circ f_2.$$

Let $g \in \mathcal{H}om^{11}$ and $h \in \mathcal{H}om^{12}$, then we have:

$$\phi_c^* \phi_c(g) = \phi_c^*(\beta \circ g, (g \otimes 1_L) \circ \beta')$$
$$= \beta^* \beta \circ g + g \circ \beta' \beta'^*.$$

$$\phi_d^* \phi_d(h) = \phi_d^*((h \otimes 1_L) \circ \gamma', \gamma \circ h)$$
$$= h \circ \gamma' \gamma'^* - \gamma^* \gamma \circ h.$$

and

$$\phi_b \phi_b^*(h) = \phi_b(h \circ \beta'^* \otimes 1_{L^*}, \beta^* \circ h)$$
$$= h \circ \beta'^* \beta' - \beta \beta^*.$$

$$\phi_a \phi_a^*(g) = \phi_a(g \circ \gamma'^* \otimes 1_{L^*}, -\gamma^* \circ g)$$
$$= g \circ \gamma'^* \gamma' + \gamma \gamma^* \circ g.$$

Thus,

$$\phi_b \phi_b^* - \phi_d^* \phi_d(h) = h \circ \beta'^* \beta' - \beta \beta^* \circ h - h \circ \gamma' \gamma'^* + \gamma^* \gamma \circ h$$
$$\phi_a \phi_a^* - \phi_c^* \phi_c(g) = g \circ \gamma'^* \gamma' + \gamma \gamma^* \circ g - \beta^* \beta \circ g - g \circ \beta' \beta'^*.$$

Hence for $g \in \mathcal{H}om^{11}$ and $h \in \mathcal{H}om^{12}$ we have,

$$(\sqrt{-1} \wedge F(\mathcal{H}om^{11}) + \phi_a \phi_a^* - \phi_c^* \phi_c)(g) = \sqrt{-1} \wedge (F(W) \circ g - g \circ F(V')) + \phi_a \phi_a^* - \phi_c^* \phi_c(g)$$
$$= (\sqrt{-1} \wedge F(W) + \gamma \gamma^* - \beta^* \beta) \circ g +$$
$$g \circ (-\sqrt{-1} \wedge F(V') + \gamma'^* \gamma' - \beta' \beta'^*)$$
$$= \tau_2 \mathrm{Id}_W \circ g - g \circ \tau_1' \mathrm{Id}_{V'}$$
$$= (\tau_2 - \tau_1')g$$

$$
\begin{aligned}
\left(\sqrt{-1} \wedge F\left(\mathcal{H}om^{12}\right) + \phi_b \phi_b^* - \phi_d^* \phi_d\right)(h) &= \sqrt{-1} \wedge \left(\otimes F(V) \circ h - h \circ F(W')\right) + \phi_b \phi_b^* - \phi_d^* \phi_d(h) \\
&= \left(\sqrt{-1} \wedge F(V) + \gamma^* \gamma - \beta \beta^*\right) \circ h + \\
&\quad h \circ \left(-\sqrt{-1} \wedge F(W') + \beta'^* \beta' - \gamma' \gamma'^*\right) \\
&= \tau_1 \mathrm{Id}_V \circ h - h \circ \tau_2' \mathrm{Id}_{W'} \\
&= (\tau_1 - \tau_2')h.
\end{aligned}
$$

Similarly for $(f_1, f_2) \in \mathcal{H}om^0$ we have,

$$
\begin{aligned}
\phi_c \phi_c^*(f_1, f_2) &= \phi_c\left((f_2 \circ \beta'^*) \otimes 1_{L^*} - \beta^* \circ f_1\right) \\
&= \left(\beta \beta^* \circ f_1 - \beta \circ (f_2 \circ \beta'^* \otimes 1_{L^*}), f_2 \circ \beta'^* \beta' - (\beta^* \circ f_1 \otimes 1_L) \otimes \beta'\right)
\end{aligned}
$$

$$
\begin{aligned}
\phi_d \phi_d^*(f_1, f_2) &= \phi_d\left((f_1 \circ \gamma'^*) \otimes 1_{L^*} - \gamma^* \circ f_2\right) \\
&= \left(f_1 \circ \gamma'^* \gamma' - \gamma^* \circ f_2 \otimes 1_L \circ \gamma', \gamma \circ (f_1 \circ \gamma'^* \otimes 1_{L^*}) - \gamma \gamma^* \circ f_2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_a^* \phi_a(f_1, f_2) &= \phi_a^*\left(f_2 \otimes 1_L \circ \gamma' - \gamma \circ f_1\right) \\
&= \left(-\gamma^* \circ f_2 \otimes 1_L \circ \gamma' + \gamma^* \gamma \circ f_1, (f_2 \circ \gamma' \gamma'^* - \gamma \circ f_1 \circ \gamma'^* \otimes 1_{L^*})\right)
\end{aligned}
$$

$$
\begin{aligned}
\phi_b^* \phi_b(f_1, f_2) &= \phi_b^*\left(f_1 \otimes 1_L \circ \beta' - \beta \circ f_2\right) \\
&= \left(f_1 \circ \beta' \beta'^* - \beta \circ f_2 \circ \beta'^* \otimes 1_{L^*}, \beta^* \circ f_1 \otimes 1_L \circ \beta' - \beta^* \beta \circ f_2\right)
\end{aligned}
$$

So,

$$
\begin{aligned}
\left(\phi_c \phi_c^* + \phi_d \phi_d^* - \phi_a^* \phi_a - \phi_b^* \phi_b\right)(f_1, f_2) &= \left(\beta \beta^* \circ f_1 + f_1 \circ \gamma'^* \gamma' - \gamma^* \gamma \circ f_1 - f_1 \circ \beta' \beta'^*, \right. \\
&\quad \left. f_2 \circ \beta'^* \beta' - \gamma \gamma^* \circ f_2 - f_2 \circ \gamma' \gamma'^* + \beta^* \beta \circ f_2\right)
\end{aligned}
$$

Hence we have,

$$
\begin{aligned}
&\left(\sqrt{-1} \wedge F\left(\mathcal{H}om^0\right) + \phi_c \phi_c^* + \phi_d \phi_d^* - \phi_a^* \phi_a - \phi_b^* \phi_b\right)(f_1, f_2) \\
&= \left(\sqrt{-1} \wedge (F(V) \circ f_1 - f_1 \circ F(V')), \sqrt{-1} \wedge (F(W) \circ f_2 - f_2 \circ F(W'))\right) + \\
&\quad \left(\beta \beta^* \circ f_1 + f_1 \circ \gamma'^* \gamma' - \gamma^* \gamma \circ f_1 - f_1 \circ \beta' \beta'^*, f_2 \circ \beta'^* \beta' - \gamma \gamma^* \circ f_2 - f_2 \circ \gamma' \gamma'^* + \beta^* \beta \circ f_2\right) \\
&= \left(\left(\sqrt{-1} \wedge F(V) + \beta \beta^* - \gamma^* \gamma\right) \circ f_1 + f_1 \circ \left(-\sqrt{-1} \wedge F(V') + \gamma'^* \gamma' - \beta' \beta'^*\right), \right. \\
&\quad \left. \left(\sqrt{-1} \wedge F(W) + \gamma \gamma^* - \beta^* \beta\right) \circ f_2 + f_2 \circ \left(-\sqrt{-1} \wedge F(W') + \beta'^* \beta' - \gamma' \gamma'^*\right)\right) \\
&= \left((\tau_1 - \tau_1') f_1, (\tau_2 - \tau_2') f_2\right).
\end{aligned}
$$

The proof will be completed by using $\tau_1 - \tau_1' = \tau_2 - \tau_2'$. $\qquad \square$

Theorem 3.5.5. *Let E and E' be α-polystable twisted $U(p,q)$-Higgs bundles. Then Q-bundle $\widetilde{E}$ is α-polystable for $\boldsymbol{\alpha} = (\alpha, 2\alpha)$.*

Proof. Since E and E' are α-polystable, from Theorem 2.7.4 follows that they support solutions to the (τ_1, τ_2)- and (τ_1', τ_2')-vortex equations where $\alpha = \tau_2 - \tau_1 = \tau_2' - \tau_1'$. Using Lemma 3.5.4 it follows that the Q-bundle $\widetilde{E}$ admits a Hermitian metric such that vortex equations are satisfied for $\tau = (\tau_2 - \tau_1', \tau_2 - \tau_2', \tau_1 - \tau_2')$. Now from Theorem 4.0.8 we get that $\widetilde{E}$ is α-polystable for

$$\alpha_1 = \tau_2 - \tau_1' - \tau_2 + \tau_2' = \alpha,$$
$$\alpha_2 = \tau_2 - \tau_1' - \tau_1 + \tau_2' = 2\alpha.$$

$\square$

Bound for $\chi(E', E)$

We are using the method in [9] and we start with some lemmas needed to estimate $\chi(E', E)$.

Lemma 3.5.6. *Let E and E' be α-polystable twisted U(p,q)-Higgs bundles. Let $\mathcal{H}om^\bullet(E', E)$ be the deformation complex of E and E', as in* (3.1.6). *Then the following inequalities hold.*

$$\deg(\ker(a_0)) \leq \mathrm{rk}(\ker(a_0))\big(\mu_\alpha(E') - \mu_\alpha(E)\big) \tag{3.5.3}$$

$$\deg(\mathrm{im}(a_0)) \leq \big(\mathrm{rk}(\mathcal{H}om^1) - \mathrm{rk}(\mathrm{im}(a_0))\big)\big(\mu_\alpha(E) - \mu_\alpha(E') - \deg(L)\big) - \tag{3.5.4}$$
$$\alpha\big(\mathrm{rk}(\mathcal{H}om^1) - \mathrm{rk}(\mathrm{im}(a_0)) - 2\,\mathrm{rk}(\mathrm{coker}(\phi_b))\big) + \deg(\mathcal{H}om^1).$$

Proof. Assume that $\mathrm{rk}(\ker(a_0)) > 0$ as if it is zero then (3.5.3) is obvious. It follows from Proposition 3.5.5 that the Q-bundle $\widetilde{E}$ is $\alpha = (\alpha, 2\alpha)$-polystable. We can define a subobject of $\widetilde{E}$ by

$$\mathcal{K}: 0 \qquad \ker(a_0) \qquad 0.$$

It follows from the α-polystability that

$$\mu_\alpha(\mathcal{K}) = \mu(\ker(a_0)) + \alpha \quad \leq \quad \mu_\alpha(\widetilde{E}) = \mu_\alpha(E') - \mu_\alpha(E) + \alpha.$$

Thus we have

$$\mu(\ker(a_0)) \leq \mu_\alpha(E') - \mu_\alpha(E),$$

which is equivalent to (3.5.3). The second inequality is obvious when $\mathrm{rk}(\mathrm{im}(a_0)) = \mathrm{rk}(\mathcal{H}om^1)$. We thus assume $\mathrm{rk}(\mathrm{im}(a_0)) < \mathrm{rk}(\mathcal{H}om^1)$. We define a quotient of the bundle $\widetilde{E}$ by

$$\mathcal{Q}: \mathrm{coker}(\phi_b) \otimes L^{-1} \qquad 0 \qquad \mathrm{coker}(\phi_a) \otimes L^{-1}$$

we may take the saturation if they are not torsion free, now by the α-polystability of $\widetilde{E}$ we have

$$\mu_\alpha(\mathcal{Q}) = \mu(\mathcal{Q}) + 2\alpha \frac{\text{rk}(\text{coker}(\phi_b))}{\text{rk}(\text{coker}(\phi_a)) + \text{rk}(\text{coker}(\phi_b))} \geq \mu_\alpha(\widetilde{E}) = \mu_\alpha(E') - \mu_\alpha(E) + \alpha. \quad (3.5.5)$$

Note that $\mu(\mathcal{Q}) = \mu(\text{coker}(a_0)) - \deg(L)$. This and (3.5.5) together with the fact that

$$\mu(\text{coker}(a_0)) \leq \frac{\deg(\mathcal{H}om^1) - \deg(\text{im}(a_0))}{\text{rk}(\mathcal{H}om^1) - \text{rk}(\text{im}(a_0))},$$

leads us to (3.5.4). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.5.7. *Let $E = (V, W, \beta, \gamma)$ and $E' = (V', W', \beta', \gamma')$ are non-zero twisted $U(p,q)$-Higgs bundles of types $t = (p,q,a,b)$ and $t' = (p',q',a',b')$ such that $p' - q'$ and $p - q$ have the same sign. Suppose that the following conditions hold*

- *$-\deg(L) \leq \alpha \leq \deg(L)$ and $\deg(L) \geq 2g - 2$.*

- *E and E' are α-polystable with $\mu_\alpha(E) = \mu_\alpha(E')$,*

- *the map a_0 is not an isomorphism.*

Then

$$\chi(E', E) \leq 1 - g,$$

if the map a_0 is not generically an isomorphism, otherwise $\chi(E', E) < 0$.

Proof. By the estimates (3.5.3) and (3.5.4), we obtain

$$\deg(\ker(a_0)) + \deg(\text{im}(a_0)) \leq \big(\mu_\alpha(E') - \mu_\alpha(E)\big)\big(\text{rk}(\ker(a_0)) + \text{rk}(\text{im}(a_0)) - \text{rk}(\mathcal{H}om^1)\big) -$$
$$\alpha\big(\text{rk}(\text{coker}(\phi_a)) - \text{rk}(\text{coker}(\phi_a))\big) - \deg(L)\big(\text{rk}(\mathcal{H}om^1) - \text{rk}(\text{im}(a_0))\big) + \deg(\mathcal{H}om^1).$$

As $\mu_\alpha(E) = \mu_\alpha(E')$ we deduce

$$\deg(\mathcal{H}om^0) - \deg(\mathcal{H}om^1) \leq -\alpha\big(\text{rk}(\text{coker}(\phi_a)) - \text{rk}(\text{coker}(\phi_a))\big) - \deg(L)\big(\text{rk}\,\text{coker}(\phi_a) + \text{rk}\,\text{coker}(\phi_a)\big)$$

$$\deg(\mathcal{H}om^0) - \deg(\mathcal{H}om^1) \leq \begin{cases} -\deg(L)\,\text{rk}(\text{coker}(\phi_b)) & \text{if } -\deg(L) \leq \alpha \leq 0 \\ -\deg(L)\,\text{rk}(\text{coker}(\phi_a)) & \text{if } 0 \leq \alpha \leq \deg(L). \end{cases} \quad (3.5.6)$$

On the other hand we have

$$\chi(E', E) = (1 - g)\big(\text{rk}(\mathcal{H}om^0) - \text{rk}(\mathcal{H}om^1)\big) + \deg(\mathcal{H}om^0) - \deg(\mathcal{H}om^1).$$

Combining (3.5.6) with the above equality, we get

$$\chi(E', E) \leq \begin{cases} (1 - g)\big(\text{rk}(\mathcal{H}om^0) - \text{rk}(\mathcal{H}om^1) + 2\,\text{rk}(\text{coker}(\phi_b))\big) & \text{if } -\deg(L) \leq \alpha \leq 0 \\ (1 - g)\big(\text{rk}(\mathcal{H}om^0) - \text{rk}(\mathcal{H}om^1) + 2\,\text{rk}(\text{coker}(\phi_a))\big) & \text{if } 0 \leq \alpha \leq \deg(L). \end{cases} \quad (3.5.7)$$

Note that $\mathrm{rk}(\mathcal{H}om^0) \geq \mathrm{rk}(\mathcal{H}om^1)$. If a_0 is not generically an isomorphism then either cases of (3.5.7) implies $\chi(E',E) \leq (1-g)$. Otherwise,

$$\chi(E',E) = \deg(\mathcal{H}om^0) - \deg(\mathcal{H}om^1) < 0$$

since equality happens only if a_0 is an isomorphism. □

The following is a situation when the map $a_0 : \mathcal{H}om^0 \to \mathcal{H}om^1$ of (3.1.5) can not be an isomorphism.

Proposition 3.5.8. *Let γ' is not injective and γ is not surjective. Then the map a_0 is not an isomorphism.*

Proof. By hypothesis $\mathrm{coker}(\gamma)$ and $\ker(\gamma')$ both are non-zero. Choose a complement to $\mathrm{im}(\gamma)$ in $W \otimes L$ so that

$$W \otimes L = \mathrm{im}(\gamma) \oplus \mathrm{im}(\gamma)^\perp$$

There is an inclusion

$$\mathrm{Hom}(\ker(\gamma'), \mathrm{im}(\gamma)^\perp) \hookrightarrow \mathcal{H}om^1$$

Let $(f_1,f_2) \in \mathcal{H}om^0$ and $x \in \ker(\gamma')$, then

$$a_0(f_1,f_2)(x,0) = (f_2 \otimes 1_L(\gamma'(x)) - \gamma(f_1(x)),0) = (-\gamma(f_1(x)),0)$$

which belongs to $\mathrm{Hom}(\ker(\gamma'), \mathrm{im}(\gamma))$. Hence $\mathrm{im}(a_0)$ and $\mathrm{Hom}(\ker(\gamma'), \mathrm{im}(\gamma))$ have trivial intersection and therefore a_0 can not be an isomorphism. □

Remark 3.5.9. There is also a similar result when β' is not injective and β is not surjective. In the case $p = q$ the above result implies that if the map a_0 of the complex $\mathcal{E}nd^\bullet(E)$ is an isomorphism then β and γ both are isomorphism which is not possible because these maps are twisted with a positive degree line bundle.

3.6 Birationality of moduli spaces

Let α_c, α_c^+ and α_c^- be defined as in Section 3.5.1, where $\varepsilon > 0$ is small enough so that α_c is the only critical value in the interval (α_c^-, α_c^+). Fix a type $t = (p,q,a,b)$.

Proposition 3.6.1. *Let α_c be a critical value for twisted U(p,q)-Higgs bundles of type $t = (a,b,p,q)$. If either one of the following conditions holds:*

(1) $a/p - b/q > -\deg(L)$, $q \leq p$ and $0 \leq \alpha_c^\pm < \frac{2pq}{pq-q^2+p+q}\big(b/q - a/p - \deg(L)\big) + \deg(L)$,

(2) $a/p - b/q < \deg(L)$, $p \leq q$ and $\frac{2pq}{pq-p^2+p+q}(b/q - a/p + \deg(L)) - \deg(L) < \alpha_c^\pm \leq 0$.

Then the codimension of the flip loci $\mathcal{S}_{\alpha_c^\pm} \subset \mathcal{M}_{\alpha_c^\pm}^s(t)$ is strictly positive.

Proof. From Propositions 3.4.15 and 3.1.16, $\mathcal{M}^s_{\alpha_c^\pm}$ is smooth of dimension $1 - \chi(t,t)$. Hence, using that by Lemma 3.1.12 $\chi(t,t) = \sum\limits_{1 \leq i,j \leq m} \chi(t_i,t_i)$, we have

$$\begin{aligned}
\operatorname{codim} \mathcal{S}_{\alpha_c^\pm} &= \dim \mathcal{M}^s_{\alpha_c^\pm}(t) - \dim \mathcal{S}_{\alpha_c^\pm} \\
&= 1 - \chi(t,t) - \dim \mathcal{S}_{\alpha_c^\pm} \\
&= 1 - \sum_{i,j} \chi(t_i,t_j) - \dim \mathcal{S}_{\alpha_c^\pm},
\end{aligned}$$

where t_i, t_i and m occur in $\operatorname{Gr}(E) = \bigoplus_{i=1}^m Q_i$ coming from a α_c-Jordan-Hölder filtration of E. Now using the inequality (3.5.1) we get that the codimension of the strictly semistable locus is at least

$$\begin{aligned}
&\min\{1 - \sum_{i,j} \chi(t_i,t_j) + \sum_{i \leq j} \chi(t_j,t_i) + \frac{m(m-3)}{2}\} \\
&= \min\{-\sum_{j<i} \chi(t_j,t_i) + \frac{m(m-3)+2}{2}\},
\end{aligned}$$

where the minimum is taken over all t_i and m. Now we show that Q_i and Q_j satisfy the hypotheses of Lemma 3.5.7. Hypotheses (1) and (2) imply that β and γ are injective, respectively. Therefore in both cases $p_j - q_j$ and $p_i - q_i$ have the same sign, for all i, j. Note that there are some i and j such that the map a_0 of the Hom-complex $\mathcal{H}om^\bullet(Q_j,Q_i)$ is not an isomorphism, since otherwise $\mathcal{E}nd^\bullet(E)$ will be an isomorphism which is not possible. This is because for $p \neq q$ we have $\operatorname{rk}(\mathcal{H}om^0) = p^2 + q^2 > 2pq = \operatorname{rk}(\mathcal{H}om^1)$ which implies that the map a_0 can not be an isomorphism, and for $p = q$ it can be an isomorphism only if β and γ both are isomorphisms but this is not possible since these maps are twisted with a degree positive line bundle, see Remark 3.5.9.

Hence we have that $-\chi(t_j,t_i) > 0$ and therefore

$$\operatorname{codim} \mathcal{S}_{\alpha_c^\pm} > \min\{\frac{m(m-3)+2}{2}\}.$$

Clearly, the minimum is attained when $m = 2$ giving the result. $\qquad\square$

Remark 3.6.2. For $q = 1$, one might have hoped to obtain a stronger result in Proposition 3.6.1, based on (1) of Proposition 3.5.3. The problem is that we also need to satisfy the hypotheses of Lemma 3.5.7 and this requires injectivity of β or γ.

From Proposition 3.6.1 we immediately obtain the following.

Theorem 3.6.3. *Fix a type $t = (p,q,a,b)$. Let α_c be a critical value. If either one of the following conditions holds:*

(1) $a/p - b/q > -\deg(L)$, $q \leq p$ and $0 \leq \alpha_c^\pm < \frac{2pq}{pq-q^2+p+q}\left(b/q - a/p - \deg(L)\right) + \deg(L)$,

(2) $a/p - b/q < \deg(L)$, $p \leq q$ and $\frac{2pq}{pq-p^2+p+q}(b/q - a/p + \deg(L)) - \deg(L) < \alpha_c^\pm \leq 0$.

Then the moduli spaces $\mathcal{M}^s_{\alpha_c^-}(t)$ and $\mathcal{M}^s_{\alpha_c^+}(t)$ are birationally equivalent. In particular, if either of the conditions of Lemma 3.1.3 holds then the moduli spaces $\mathcal{M}_{\alpha_c^-}(t)$ and $\mathcal{M}_{\alpha_c^+}(t)$ are birationally equivalent.

Remark 3.6.4. In view of Remark 3.2.14, non-emptiness of the intervals for $\alpha_c^{\pm}$ in the preceding theorem bounds the Toledo invariant. Thus the ranges for the Toledo invariant $\tau = \frac{2pq}{p+q}(a/p - b/q)$ for which the statement of the theorem is meaningful are:

(1) $-\frac{2pq}{p+q}\deg(L) < \tau < -(q-1)\deg(L))$;

(2) $(p-1)\deg(L) < \tau < \frac{2pq}{p+q}\deg(L)$.

Note that in case (1) we have $q \le p$ and hence $-\frac{2pq}{p+q}\deg(L) \le -q\deg(L)$, while in case (2) we have $p \le q$ and hence $p\deg(L) \le \frac{2pq}{p+q}\deg(L)$.

Finally we have the following corollary.

Theorem 3.6.5. *Let $L = K$ and fix a type $t = (p,q,a,b)$. Suppose that $(p+q, a+b) = 1$ and that $\tau = \frac{2pq}{p+q}(a/p - b/q)$ satisfies $|\tau| \le \min\{p,q\}(2g-2)$. Suppose that either one of the following conditions holds:*

(1) $a/p - b/q > -(2g-2)$, $q \le p$ and $0 \le \alpha < \frac{2pq}{pq-q^2+p+q}\big(b/q - a/p - (2g-2)\big) + 2g - 2$,

(2) $a/p - b/q < 2g - 2$, $p \le q$ and $\frac{2pq}{pq-p^2+p+q}\big(b/q - a/p + 2g - 2\big) - (2g-2) < \alpha \le 0$.

Then the moduli space $\mathcal{M}_\alpha(t)$ is irreducible.

Proof. Recall that the value of the parameter for which the non-abelian Hodge Theorem applies is $\alpha = 0$. Thus, using [8, Theorem 6.5], the moduli space $\mathcal{M}_0(t)$ is irreducible and non-empty (both the co-primality condition and the bound on the Toledo invariant are needed for this). Hence the result follows from Theorem 3.6.3. □

Remark 3.6.6. Note that unless $p = q$, the conditions on $a/b - b/q$ in the preceding theorem are guaranteed by the hypothesis $|\tau| \le \min\{p,q\}(2g-2)$ (cf. Remark 3.6.4).

Remark 3.6.7. In the non-coprime case it is known from [8] that the closure of the stable locus in $\mathcal{M}_0(t)$ is connected (however, irreducibility is still an open question). Thus, in the non-coprime case, the closure of the stable locus of $\mathcal{M}_\alpha(t)$ is connected under the remaining hypotheses of the preceding theorem.

Chapter 4

Holomorphic Chains

Deformation theory is essential to study the variation of the moduli spaces of α-semistable holomorphic chains as the parameter changes. In this chapter we study the deformation theory of holomorphic chains .

Recall that a *holomorphic $(m+1)$-chain* on a compact Riemann surface of genus $g \geq 2$ is a diagram

$$C: E_m \xrightarrow{\phi_m} E_{m-1} \xrightarrow{\phi_{m-1}} \cdots \xrightarrow{\phi_2} E_1 \xrightarrow{\phi_1} E_0,$$

where each E_i is a holomorphic vector bundle and $\phi_i : E_i \longrightarrow E_{i-1}$ is a holomorphic map.

The tuple $t := (r_0, \ldots, r_m; d_0, \ldots, d_m)$, with $r_i = \text{rk}(E_i)$ and $d_i = \deg(E_i)$, will be referred as the *type* of the chain C.

Let $\alpha = (\alpha_0, \ldots, \alpha_m) \in \mathbb{R}^{m+1}$. The α-slope of a chain C of type $t = (r_0, \ldots, r_m; d_0, \ldots, d_m)$ is defined by the fraction

$$\mu_\alpha(C) := \frac{\sum_{i=0}^m \left(r_i \alpha_i + d_i \right)}{\sum_{i=0}^m r_i}.$$

A holomorphic $(m+1)$-chain C is said to be α-stable (semistable), if the inequality

$$\mu_\alpha(C') < (\leq) \mu_\alpha(C)$$

is verified for any non-trivial subchain C' of C. A chain C is called α-*polystable* if it is the direct sum of α-stable chains of the same α-slope.

Let α be the stability parameter. Define $\tau = (\tau_0, \ldots, \tau_m) \in \mathbb{R}^{m+1}$ by

$$\tau_i = \mu_\alpha(C) - \alpha_i, \ i = 0, \ldots, m, \tag{4.0.1}$$

with the convention $\alpha_0 = 0$, using the Remark 2.3. Then α can be recovered from τ by

$$\alpha_i = \tau_0 - \tau_i, \ i = 0, \ldots, m. \tag{4.0.2}$$

A Hermitian metric satisfies the chains τ-vortex equations if

$$\sqrt{-1} \Lambda F(E_i) + \phi_{i+1} \phi_{i+1}^* - \phi_i^* \phi_i = \tau_i Id_{E_i}, \ i = 0, \ldots, m \tag{4.0.3}$$

where $F(E_i)$ is the curvature of the Hermitian connection on E_i, Λ is contraction with Kähler form and $vol(X) = 2\pi$.

One has the Hitchin-Kobayashi correspondence for holomorphic chains as follows:

Theorem 4.0.8. *[14, Theorem 3.4] A holomorphic chain C is α-polystable if and only if the τ-vortex equations have a solution, where α and τ are related by* (4.0.2).

4.1   Extensions and deformations of chains

In this section we study the deformation theory of holomorphic chains. The infinitesimal deformations of holomorphic chains are given by the first hypercohomology group of a certain complex of sheaves associated to the holomorphic chains, called *deformation complex.*

Throughout this section we fix a stability parameter $\alpha = (\alpha_i, i = 0,\ldots,m)$ and two holomorphic chains C'' and C', of types t' and t'' respectively, given by

$$C' : E'_m \xrightarrow{\phi'_m} E'_{m-1} \xrightarrow{\phi'_{m-1}} \cdots \xrightarrow{\phi'_2} E'_1 \xrightarrow{\phi'_1} E'_0,$$

$$C'' : E''_m \xrightarrow{\phi''_m} E''_{m-1} \xrightarrow{\phi''_{m-1}} \cdots \xrightarrow{\phi''_2} E''_1 \xrightarrow{\phi''_1} E''_0.$$

Let $\mathrm{Hom}(C'',C')$ denote the linear space of homomorphisms from C'' to C', and let $\mathrm{Ext}^1(C'',C')$ denote the linear space of equivalence classes of extensions of the form

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0,$$

where by this we mean a commutative diagram

4.1.1   Deformation Complex

Let C' and C'' be two holomorphic chains. Let $F^\bullet(C'',C') : \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1$ be defined as in (2.10.3).

The following proposition analyzes $\mathrm{Hom}(C'',C')$ and $\mathrm{Ext}^1(C'',C')$ by using the hypercohomology groups of deformation complex.

Proposition 4.1.1. *[? , Proposition 3.1.] There are natural isomorphisms*

$$\mathrm{Hom}(C'',C') \cong \mathbb{H}^0(F^\bullet(C'',C')),$$

$$\mathrm{Ext}^1(C'',C') \cong \mathbb{H}^1(F^\bullet(C'',C')),$$

and a long exact sequence associated to the complex $F^\bullet(C'',C')$:

$$0 \longrightarrow \mathbb{H}^0(F^\bullet(C'',C')) \longrightarrow H^0(\mathcal{F}^0) \longrightarrow H^0(\mathcal{F}^1) \longrightarrow \mathbb{H}^1(F^\bullet(C'',C')) \qquad (4.1.1)$$

$$\longrightarrow H^1(\mathcal{F}^0) \longrightarrow H^1(\mathcal{F}^1) \longrightarrow \mathbb{H}^2(F^\bullet(C'',C')) \longrightarrow 0. \qquad (4.1.2)$$

4.1.2 The associated quiver to the deformation complex of chains

In this subsection we introduce a Q-bundle, associated to the deformation complex, and show that given a solution to the vortex equations on a holomorphic chains, produces a solution on the corresponding quiver bundle.

The associated quiver

Let us consider the following quiver

$$\bigoplus_{j-i=-m} \mathrm{Hom}(E''_j, E'_i) \xrightarrow{\phi_{-m}} \cdots \xrightarrow{\phi_{-1}} \bigoplus_{j-i=0} \mathrm{Hom}(E''_j, E'_i) \xrightarrow{\phi_0} \bigoplus_{j-i=1} \mathrm{Hom}(E''_j, E'_i) \xrightarrow{\phi_1}$$

$$\cdots \xrightarrow{\phi_m} \bigoplus_{j-i=m} \mathrm{Hom}(E''_j, E'_i) \quad (4.1.3)$$

where for any k, $0 \leq k \leq m$, the maps ϕ_k and ϕ_{-k} are defined as follows

$$\phi_{-k}(g_i) = \sum_{i=0}^{m-k} a_{k_i}(g_i), \text{ for } g_i \in \mathrm{Hom}(E''_i, E'_{i+k}),$$

$$\phi_k(f_i) = \sum_{i=0}^{m-k} a_{-k_i}(f_i, f_{i+1}), \text{ for } f_i \in \mathrm{Hom}(E''_{i+k}, E'_i),$$

with

$$\phi_{-k_i}: \mathrm{Hom}(E''_i, E'_{i+k}) \to \mathrm{Hom}(E''_i, E'_{i+k-1}) \oplus \mathrm{Hom}(E''_{i+1}, E'_{i+k}) \hookrightarrow \bigoplus_{j-i=-(k-1)} \mathrm{Hom}(E''_j, E'_i),$$

$$\phi_{k_i}: \mathrm{Hom}(E''_{i+k}, E'_i) \oplus \mathrm{Hom}(E''_{i+k+1}, E'_{i+1}) \to \mathrm{Hom}(E''_{i+k+1}, E'_i) \hookrightarrow \bigoplus_{j-i=k+1} \mathrm{Hom}(E''_j, E'_i),$$

given by

$$\phi_{-k_i}(g_i) = g_i \circ \phi''_{i+1} - \phi'_{i+k} \circ g_i,$$

$$\phi_{k_i}(f_i, f_{i+1}) = f_i \circ \phi''_{i+k+1} - \phi'_{i+1} \circ f_{i+1}.$$

Note that the middle two terms $\bigoplus_{j-i=0} \mathrm{Hom}(E''_j, \otimes E'_i) \xrightarrow{\phi_0} \bigoplus_{j-i=0} \mathrm{Hom}(E''_j, \otimes E'_i)$ coincide with the deformation complex of chains C'' and C', defined in (2.10.3).

Above construction corresponds to the following quiver, which we will denote by $\widetilde{F^\bullet}(C'',C')$.

$$(4.1.4)$$

Lemma 4.1.2. *Let C' and C'' be holomorphic chains and suppose we have solutions to the $(\tau'_0, \ldots, \tau'_m)$-vortex equations on C' and the $(\tau''_0, \ldots, \tau''_m)$-vortex equation on C''. Then the induced Hermitian metric on the defined quiver in (4.1.4) satisfies the quiver $\widetilde{\tau}$-vortex equations, where $\widetilde{\tau} = (\tau'_i - \tau''_{i+k}, \tau'_{i+k} - \tau''_i, 0 \le k \le m, 0 \le i \le m-k)$.*

Proof. We shall only show that the induced Hermitian metric satisfies the equation at $\mathrm{Hom}(E''_{i+k}, E'_i)$, for $0 \le k \le m$, $0 \le i \le m-k$.

The vortex equations for C' and C'' are

$$\sqrt{-1} \wedge F(E'_i) + \phi'_{i+1} \phi'^*_{i+1} - \phi'^*_i \phi'_i = \tau'_i \, \mathrm{Id}_{E'_i}, \ i = 0, \ldots, m$$

$$\sqrt{-1} \wedge F(E''_i) + \phi''_{i+1} \phi''^*_{i+1} - \phi''^*_i \phi''_i = \tau''_i \, \mathrm{Id}_{E''_i}, \ i = 0, \ldots, m.$$

It should be noted that $F\left(\mathrm{Hom}(E''_{i+k}, E'_i)\right)(f) = F(E'_i) \circ f - f \circ F(E''_{i+k}))$. Now from the quiver (4.1.4) at $\mathrm{Hom}(E''_{i+k}, E'_i)$ we have

A straightforward calculation gives the following

$$\phi_a^*(g) = g \circ \phi_{i+k+1}^{''\,*},$$
$$\phi_b^*(h) = \phi_i^{'\,*} \circ h,$$
$$\phi_c^*(f) = f \circ \phi_{i+k}^{''\,*},$$
$$\phi_d^*(f) = \phi_{i+1}^{'\,*} \circ f.$$

Therefore

$$
\begin{aligned}
\left(\phi_c \phi_c^* + \phi_d \phi_d^* - \phi_a^* \phi_a - \phi_b^* \phi_b\right)(f) &= \phi_c(f \circ \phi_{i+k}^{''\,*}) + \phi_d(\phi_{i+1}^{'\,*} \circ f) - \phi_a^*(f \circ \phi_{i+k+1}'') - \phi_b^*(\phi_i' \circ f) \\
&= f \circ \phi_{i+k}^{''\,*} \circ \phi_{i+k}'' + \phi_{i+1}' \circ \phi_{i+1}^{'\,*} \circ f - f \circ \phi_{i+k+1}'' \phi_{i+k+1}^{''\,*} - \phi_i^{'\,*} \phi_i' \circ f
\end{aligned}
$$

Hence for $f \in \mathrm{Hom}(E_{i+k}'', E_i')$ we have,

$$
\begin{aligned}
&\left(\sqrt{-1} \wedge F\left(\mathrm{Hom}(E_{i+k}'', E_i')\right) + \phi_c \phi_c^* + \phi_d \phi_d^* - \phi_a^* \phi_a - \phi_b^* \phi_b\right)(f) \\
&= \left(\left(\sqrt{-1} \wedge F(E_i') + \phi_{i+1}'' \phi_{i+1}^{''\,*} - \phi_i^{''\,*} \phi_i''\right) \circ f - f \circ \left(\sqrt{-1} \wedge F(E_{i+k}'') - \phi_{i+k}' \phi_{i+k}^{'\,*} + \phi_{i+k+1}^{'\,*} \phi_{i+k+1}'\right)\right) \\
&= (\tau_i' - \tau_{i+k}'')f.
\end{aligned}
$$

Similarly the induced Hermitian metrics satisfy the equation at $\mathrm{Hom}(E_i'', E_{i+k}')$, for $0 \le k \le m$ and $0 \le i \le m-k$. Hence we conclude the proof of lemma. $\qquad\square$

Theorem 4.1.3. *Let C' and C'' be $\alpha' = (\alpha_1', \ldots, \alpha_m')$- and $\alpha'' = (\alpha_1'', \ldots, \alpha_m'')$-polystable chains, respectively. Then the Q-bundle $\widetilde{F}^\bullet(C'', C')$, defined in (4.1.4) is $\widetilde{\alpha} = (\widetilde{\alpha}_{-k_i}, \widetilde{\alpha}_{k_i}, 0 \le k \le m, 0 \le i \le m-k)$-polystable for*

$$
\begin{aligned}
\widetilde{\alpha}_{-k_i} &= \alpha_m'' + \alpha_i' - \alpha_{i+k}'' \\
\widetilde{\alpha}_{k_i} &= \alpha_m'' + \alpha_{i+k}' - \alpha_i''
\end{aligned}
$$

Proof. Since the chains C' and C'' respectively are α'- and α''-polystable, it follows from Proposition 4.0.8 that $(\tau_0', \ldots, \tau_m')$- and $(\tau_0'', \ldots, \tau_m'')$-vortex equations have a solution, respectively. Hence by the previous lemma the associated Q-bundle $\widetilde{F}^\bullet(C'', C')$ supports a solution to the quiver vortex equations for

$$
\begin{aligned}
\widetilde{\tau}_{k_i} &= \tau_{i+k}' - \tau_i'', \text{ for } 0 \le i \le m-k,\ 0 \le k \le m \\
\widetilde{\tau}_{-k_i} &= \tau_i' - \tau_{i+k}'', \text{ for } 0 \le i \le m-k,\ 0 < k \le m.
\end{aligned}
$$

Therefore the Hitchin-Kobayashi correspondence for quiver bundles implies that $\widetilde{F}^\bullet(C'', C')$ is $\widetilde{\alpha}$-polystable for

$$
\begin{aligned}
\widetilde{\alpha}_{-k_i} &= \tau_0' - \tau_m'' - (\tau_{i+k}' - \tau_i'') = \tau_0' - \tau_{i+k}' + \tau_0'' - \tau_m'' + \tau_i'' - \tau_0'' = \alpha_m'' + \alpha_{i+k}' - \alpha_i'' \\
\widetilde{\alpha}_{k_i} &= \tau_0' - \tau_m'' - (\tau_i' - \tau_{i+k}'') = \tau_0' - \tau_i' + \tau_0'' - \tau_m'' + \tau_{i+k}'' - \tau_0'' = \alpha_m'' + \alpha_i' - \alpha_{i+k}''
\end{aligned}
$$

$\qquad\square$

We consider the following quiver obtained through the direct sum of terms for $k = 0$ of (4.1.4)

$$\begin{array}{c}
\text{Hom}(E_0'', E_1) \qquad\qquad \text{Hom}(E_1'', E_0') \\
\vdots \qquad\qquad\qquad \vdots \\
\text{Hom}(E_0'', E_{m-1}') \quad \text{Hom}(E_{i-1}'', E_i') \qquad \text{Hom}(E_i'', E_{i-1}') \quad \text{Hom}(E_{m-1}'', E_0') \\
\text{Hom}(E_0'', E_m') \quad \cdots \quad \bigoplus_{j-i=0} \text{Hom}(E_j'', E_i') \quad \cdots \quad \text{Hom}(E_m'', E_0') \\
\text{Hom}(E_1'', E_m') \quad \text{Hom}(E_i'', E_{i+1}') \qquad \text{Hom}(E_{i+1}'', E_i') \quad \text{Hom}(E_m'', E_1') \\
\vdots \qquad\qquad\qquad \vdots \\
\text{Hom}(E_{m-1}'', E_m') \qquad\qquad \text{Hom}(E_m'', E_{m-1}')
\end{array}$$

(4.1.5)

Analogue to Theorem 4.1.3 we have:

Theorem 4.1.4. *Let C' and C'' be $\alpha = (\alpha_1, \ldots, \alpha_m)$-polystable chains. Then the Q-bundle 4.1.5 is $\widetilde{\alpha} = (\widetilde{\alpha}_{-k_i}, \widetilde{\alpha}_{k_i}, 0 \leq k \leq m, 0 \leq i \leq m - k)$-polystable for*

$$\begin{aligned}
\widetilde{\alpha}_{k_i} &= \alpha_m + \alpha_i - \alpha_{i+k}, \quad \text{for all } 0 < k \leq m \\
\widetilde{\alpha}_{-k_i} &= \alpha_m + \alpha_{i+k} - \alpha_i \quad \text{for all } 0 < k \leq m \\
\widetilde{\alpha}_0 &= \alpha_m.
\end{aligned}$$

We simplified the proof of the following result in [**?**].

Proposition 4.1.5. *Let $C' = (E_0', \cdots, E_m'; \varphi_0', \cdots, \varphi_m')$ and $C'' = (E_0'', \cdots, E_m''; \varphi_0'', \cdots, \varphi_m'')$ are α-polystable holomorphic chains and $\alpha_{i+1} - \alpha_i \geq 2g - 2$ for all $i = 1, \cdots, m$. Let $F^\bullet(C'', C')$ be the complex defined in (2.10.3). Then following inequalities hold.*

$$\mu(\ker(d)) \leq \mu_\alpha(C'') - \mu_\alpha(C'), \tag{4.1.6}$$

$$\mu(\text{coker}(d)) \geq \mu_\alpha(C'') - \mu_\alpha(C') + 2g - 2. \tag{4.1.7}$$
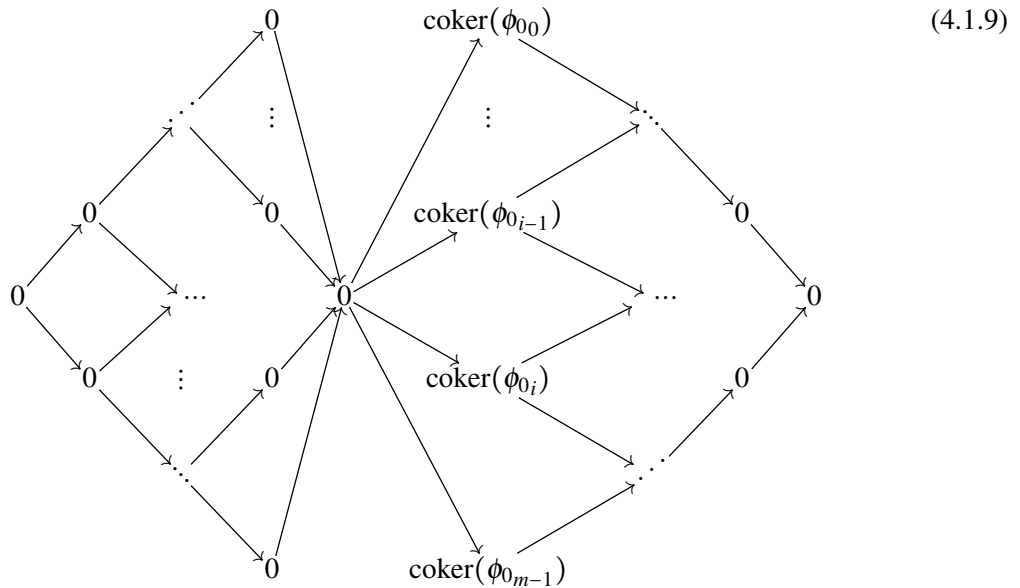
Proof. Suppose that $\mathrm{rk}(\ker(d)) > 0$ since if it is zero then the first inequality is obvious. We can define a Q-subbundle of Q-bundle (4.1.5) by

$$(4.1.8)$$
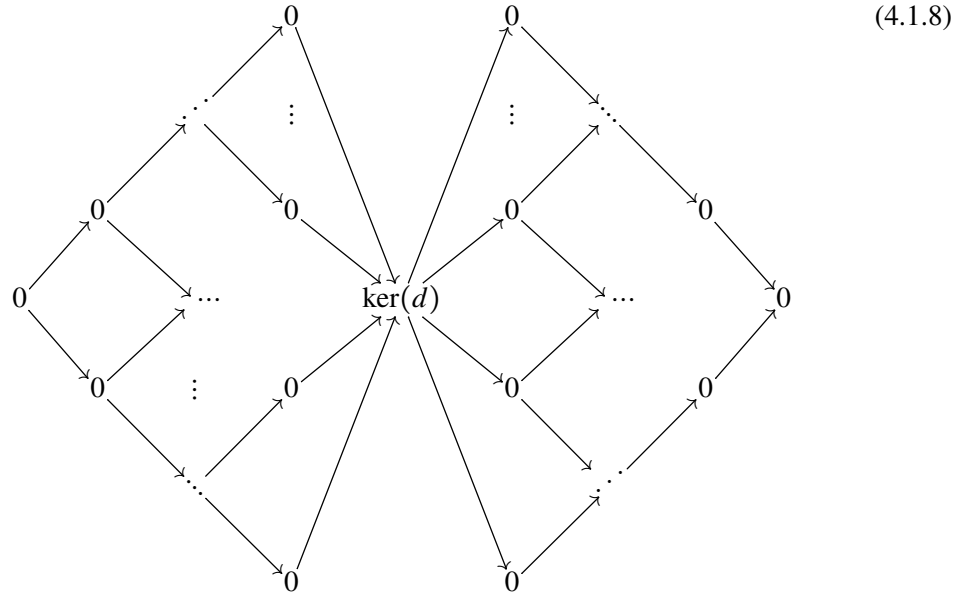
It follows from the $\widetilde{\alpha}$-semistability of Q-bundle (4.1.5) that

$$\mu(\ker(d)) + \alpha_m \le \mu_\alpha(C'') - \mu_\alpha(C') + \alpha_m,$$

giving (4.1.6). To prove (4.1.7) we consider a quotient of Q-bundle (4.1.5) by

$$(4.1.9)$$

$\widetilde{\alpha}$-smistability condition applied to the above quotient bundle yields

$$\mu(\mathrm{coker}(\phi_0)) + \alpha_m + \frac{\sum_{i=0}^{m-1}(\alpha_i - \alpha_{i+1})\,\mathrm{rk}(\mathrm{coker}(\phi_{0_i}))}{\sum_{i=0}^{m-1}\mathrm{rk}(\mathrm{coker}(\phi_{0_i}))} \ge \mu_\alpha(C'') - \mu_\alpha(C') + \alpha_m$$

Note that $\phi_0 = d$. So, the above implies

$$\mu(\text{coker}(d)) \geq \mu_\alpha(C'') - \mu_\alpha(C') + \frac{\sum_{i=0}^{m-1}(\alpha_{i+1} - \alpha_i)\,\text{rk}(\text{coker}(\phi_{0_i}))}{\sum_{i=0}^{m-1}\text{rk}(\text{coker}(\phi_{0_i})}$$

$$\geq \mu_\alpha(C'') - \mu_\alpha(C') + 2g - 2,$$

which imply (4.1.7). □

References

[1] Álvarez-Cónsul, L. and García-Prada, O. (2001). Dimensional reduction,-equivariant bundles and stable holomorphic chains. *International Journal of Mathematics*, 12(02):159–201.

[2] Álvarez-Cónsul, L. and García-Prada, O. (2003). Hitchin–kobayashi correspondence, quivers, and vortices. *Communications in mathematical physics*, 238(1-2):1–33.

[3] Atiyah, M. F. and Bott, R. (1983a). The yang-mills equations over riemann surfaces. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 308(1505):523–615.

[4] Atiyah, M. F. and Bott, R. (1983b). The yang-mills equations over riemann surfaces. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 308(1505):523–615.

[5] Biswas, I. and Ramanan, S. (1994). An infinitesimal study of the moduli of hitchin pairs. *Journal of the London Mathematical Society*, 49(2):219–231.

[6] Bradlow, S. B. et al. (1991). Special metrics and stability for holomorphic bundles with global sections. *Journal of Differential Geometry*, 33(1):169–213.

[7] Bradlow, S. B. and Garcia-Prada, O. (1996). Stable triples, equivariant bundles and dimensional reduction. *Mathematische Annalen*, 304(1):225–252.

[8] Bradlow, S. B., García-Prada, O., and Gothen, P. B. (2003). Surface group representations and U(p,q)-Higgs bundles. *J. Differential Geom.*, 64(1):111–170.

[9] Bradlow, S. B., García-Prada, O., and Gothen, P. B. (2004). Moduli spaces of holomorphic triples over compact Riemann surfaces. *Math. Ann.*, 328(1-2):299–351.

[10] Bradlow, S. B., García-Prada, O., and Gothen, P. B. (2006). Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces. *Geom. Dedicata*, 122:185–213.

[11] Bradlow, S. B., Garcia-Prada, O., and Gothen, P. B. (2013). Higgs bundles for the non-compact dual of the special orthogonal group. *Geometriae Dedicata*, pages 1–48.

[12] Corlette, K. (1988). Flat bundles with canonical metrics. *J. Diff. Geom*, 28:361–382.

[13] Donaldson, S. K. (1987). Twisted harmonic maps and the self-duality equations. *Proceedings of the London Mathematical Society*, 55(1):127–131.

[14] García-Prada, O. (1994). Dimensional reduction of stable bundles, vortices and stable pairs. *International Journal of mathematics*, 5(01):1–52.

[15] Garcia-Prada, O., Gothen, P. B., et al. (2009). The hitchin-kobayashi correspondence, higgs pairs and surface group representations. *arXiv preprint arXiv:0909.4487*.

[16] García-Prada, O., Gothen, P. B., and i Riera, I. M. (2013a). Higgs bundles and surface group representations in the real symplectic group. *Journal of Topology*, 6(1):64–118.

[17] García-Prada, O., Heinloth, J., et al. (2013b). The y-genus of the moduli space of PGL_n-higgs bundles on a curve (for degree coprime to n). *Duke Mathematical Journal*, 162(14):2731–2749.

[18] García-Prada, O., Heinloth, J., and Schmitt, A. (2014). On the motives of moduli of chains and Higgs bundles. *J. Eur. Math. Soc. (JEMS)*, 16(12):2617–2668.

[19] Gothen, P. B. (2014). Representations of surface groups and higgs bundles. *Moduli Spaces*, 411:151.

[20] Gothen, P. B. and King, A. D. (2005). Homological algebra of twisted quiver bundles. *Journal of the London Mathematical Society*, 71(1):85–99.

[21] Griffiths, P. and Harris, J. (2014). *Principles of algebraic geometry*. John Wiley & Sons.

[22] Hartshorne, R. (1977). *Algebraic geometry*, volume 52. Springer Science & Business Media.

[23] Hausel, T. and Thaddeus, M. (2004). Generators for the cohomology ring of the moduli space of rank 2 higgs bundles. *Proceedings of the London Mathematical society*, 88(03):632–658.

[24] Hitchin, N. J. (1987). The self-duality equations on a riemann surface. *Proc. London Math. Soc*, 55(3):59–126.

[25] Hitchin, N. J. (1992). Lie groups and Teichmüller space. *Topology*, 31(3):449–473.

[26] Kobayashi, S. (2014). *Differential geometry of complex vector bundles*, volume 15. Princeton University Press.

[27] Le Potier, J. (1997). *Lectures on vector bundles*. Cambridge University Press.

[28] Muñoz, V. (2008). Hodge polynomials of the moduli spaces of rank 3 pairs. *Geometriae Dedicata*, 136(1):17–46.

[29] Muñoz, V. (2009). Torelli theorem for the moduli spaces of pairs. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 146, pages 675–693. Cambridge Univ Press.

[30] Muñoz, V. (2010). Hodge structures of the moduli spaces of pairs. *International Journal of Mathematics*, 21(11):1505–1529.

[31] Munoz, V., Ortega, D., and VAZquez-Gallo, M.-J. (2007). Hodge polynomials of the moduli spaces of pairs. *International Journal of Mathematics*, 18(06):695–721.

[32] Muñoz, V., Ortega, D., and Vázquez-Gallo, M.-J. (2009). Hodge polynomials of the moduli spaces of triples of rank (2, 2). *The Quarterly Journal of Mathematics*, 60(2):235–272.

[33] Schmitt, A. (2005). Moduli for decorated tuples of sheaves and representation spaces for quivers. In *Proceedings of the Indian Academy of Sciences-Mathematical Sciences*, volume 115, pages 15–49. Springer.

[34] Schmitt, A. H. (2003). Moduli problems of sheaves associated with oriented trees. *Algebras and Representation theory*, 6(1):1–32.

[35] Schmitt, A. H. (2008). Geometric invariant theory and decorated principal bundles, zurich lectures in advanced mathematics. *European Mathematical Society, Zurich*.

[36] Simpson, C. T. (1988). Constructing variations of hodge structure using yang-mills theory and applications to uniformization. *Journal of the American Mathematical Society*, 1(4):867–918.

[37] Simpson, C. T. (1992). Higgs bundles and local systems. *Publications Mathématiques de l'IHÉS*, 75(1):5–95.

[38] Thaddeus, M. (1994). Stable pairs, linear systems and the verlinde formula. *Invent. Math.*, 117(1):317–353.