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Bifurcation of Projected Patterns

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Abstract

This thesis is related to the study of pattern formation in symmetric physical systems. The purpose of this thesis is to discuss a possible model, namely the projection model, to explain the appearance and evolution of regular patterns in symmetric systems of equations.

Results found in Crystallography and Equivariant Bifurcation Theory are used extensively in our work. In particular, we provide a formalism of how the model of projection can be used and interpreted to understand experiments of reaction-diffusion systems.

We construct a scenario where systems of symmetric PDEs posed in different dimensions can be compared as projection. In particular, we show how we can overcome the boundary conditions imposed by the problems.

We prove a correspondence between irreducible representations and fixed points subspaces, given by the action of a $(n+1)$-dimensional crystallographic group, with the action of its projection on lower dimension. Such results are the first step to compare typical structures in dimension $(n+1)$, after projection, and the typical solutions of the posed problem in dimension n.

We show that complex structures, as the black-eye pattern, obtained both as projection and as an experimental observation in CIMA reactions are the same. In particular, we believe that the projection model provides extra information to the study of pattern forming system, since it allows us to embed the original problem into one with more symmetry.

Resumo

O trabalho desta tese está relacionado com o estudo de formação de padrões em sistemas físicos simétricos. O objetivo desta tese é discutir um possível modelo matemático, nomeadamente "modelo de projeção", para explicar a aparência e evolução de padrões regulares em sistemas de equações simétricas.

Nós usamos exaustivamente resultados encontrados na teoria de Cristalografia e na teoria dos estudos de bifurcações equivariantes. Em particular, nós obtemos um formalismo de como o modelo de projeção pode ser usado e interpretado para entender experimentos de reação-difusão.

Construímos um cenário que nos permite comparar, através do modelo de projeção, sistemas descritos por equações diferenciais parciais simétricas postos em diferentes dimensões. Em particular, nós mostramos como podemos lidar com as condições de fronteira impostas pelo problema.

Nós provamos uma correspondência entre representações e subespaços de pontos fixos, dada pela ação de um grupo cristalográfico de dimensão $(n+1)$, com aquelas obtidas pela ação do grupo projetado em dimensão n. Tais resultados nos permitem comparar soluções típicas de problemas postos em dimensões diferentes.

Mostramos que estruturas complexas, como padrões "black-eye", obtidos como projeção de padrões, representam a mesma estrutura obtida em experimentos. Em particular, nós acreditamos que o modelo de projeção pode prever informações adicionais no estudo de formação de padrões, pois tal modelo nos permite trabalhar com sistemas que contém mais simetrias.

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List of Symbols

Roman Symbols

$(-\delta^{-1}v, \delta^{-1})$	Inverse element of $(v, \delta) \in E(n+1)$, page 10
(v, δ)	Denotes an element in $E(n+1)$, page 9
α	Element of $O(n)$, page 18
δ	Denotes an element in $O(n+1)$, page 9
Γ	Crystallographic group , page 10
$\langle l_1, \ldots, l_k \rangle_{\mathbb{Z}}$	Lattice $\mathcal{L}$ in terms of its tenerators
$\langle \cdot, \cdot \rangle$	Inner product in $\mathbb{R}^{n+1}$, page 14
$\{y = 0\}$	Space given by $\{(x, y) \in \mathbb{R}^{n+1};\ y = 0\}$, page 47
$\mathbb{T}^{n+1} = \mathbb{R}^{n+1}/\mathcal{L}$	n-torus of translations modulo $\mathcal{L}$, page 23
$\mathcal{L}$	Lattice of a $(n+1)$-crystallographic group , page 10
$\mathcal{L}^*$	Dual lattice of $\mathcal{L}$, page 10
$\mathcal{P}$	Operator between function spaces , page 22
$\mathcal{X}$, $\mathcal{Y}$	Function Spaces
$\mathcal{X}_\Gamma = Fix_{\mathcal{X}_\mathcal{L}}(\Gamma)$	Space of Γ-invariant functions , page 16
$\mathcal{X}_\mathcal{L}$	Space of $\mathcal{L}$-periodic functions in $\mathcal{X}$, page 24
$\mathcal{X}_{\widetilde{\Gamma}}$	Space of $\widetilde{\Gamma}$-invariant functions , page 46
ω_k	Wave function given by $\exp(2\pi i \langle k, (x, y) \rangle)$, page 23
$\| \cdot \|$	Norm in $\mathbb{R}^{n+1}$, page 14
$\Pi_{y_0}(\mathcal{L}) = \widetilde{\mathcal{L}}$	Projection of $\mathcal{L}$, page 18
$\Pi_{y_0}(J) = \widetilde{J}$	Point group of $\widetilde{\Gamma}$, page 46
Π_{y_0}	Projection Operator , page 16

$\Pi_{y_0}(\Gamma) = \widetilde{\Gamma}$	Projection of Γ , page 46
σ, α_+, α_-	Side-reversing and side preserving elements of $O(n+1)$, page 18
Σ_z	Isotropy group of v , page 21
$\widehat{\Gamma}$	Subgroup of Γ given by $\{((v,y),\alpha_\pm);\ \alpha \in O(n),\ (v,y) \in \mathbb{R}^n \times \mathbb{R}\}$, page 18
$\widetilde{\mathcal{L}}$	n-Dimensional lattice. Notation also used for the projection of $\mathcal{L}$, page 13
$\widetilde{\mathcal{L}}_s$	Suspension of a n-dimensional lattice , page 13
$\widetilde{\mathcal{L}}^*$	Dual lattice of $\widetilde{\mathcal{L}}$, page 46
$\widetilde{f}$	Function in $\mathcal{X}_{\widetilde{\Gamma}}$, page 46
$\widetilde{J}^{\uparrow}$	Subgroup of J given by $\{\alpha_+ \in J;\ \alpha \in \widetilde{J}\} \cup \{\alpha_- \in J;\ \alpha \in \widetilde{J}\}$, page 49
$\widetilde{k}$	Element of $\widetilde{\mathcal{L}}^*$, page 49
A^t	Transpose of matrix A , page 14
A_{y_0}	Affine space given by $\{(x,y_0) \in \mathbb{R}^{n+1}\}$, page 48
$E(n+1)$	Euclidean group , page 9
E_{p_i}	Subset of $\mathcal{L}^*$ given by $\{x \in \widetilde{\mathcal{L}}^*;\ \|x\| = p_i\}$, page 64
f	Function in $\mathcal{X}_\Gamma$, page 46
$Fix(\Sigma)$	Fixed-point subspace associated with Σ , page 21
$GL(n+1,\mathbb{R})$	General Linear group , page 9
$H_{\mathcal{L}}$	Holohedy of $\mathcal{L}$, page 11
I_k	Generators for the space $\mathcal{X}_\Gamma$ given by $\sum_{\delta \epsilon J} \omega_{\delta k}\omega_{\delta k}(-v_\delta)$, page 45
Id_{n+1}	Identity mapping of $\mathbb{R}^{n+1}$, page 10
J	Point group of a $(n+1)$-crystallographic group , page 10
$J(\widetilde{k},z)$	Orbit of $(\widetilde{k},z)$ by J, page 49
$J^{\alpha}_{\widetilde{k}}$	Subset of J given by $\{\delta \in J;\ \delta(\widetilde{k},z)\|_1 = \alpha\widetilde{k}\}$, page 50
$Ker(\mathcal{DP})_q$	Kernel of the linearisation of $\mathcal{P}$ at a point $q \in \mathbb{R}^s$, page 23
$l_1,\ldots,l_{n+1}$	Base vectors of a $(n+1)$-dimensional lattice $\mathcal{L}$
$O(n+1)$	Orthogonal group , page 9
P	Primitive or Fundamental Cell of $\mathcal{L}$, page 11

P	Standard projection , page 45
S_2, C_{2h}, D_{2h}, D_{4h}, O_h, D_{3d}, D_{6h}	Holohedries of the Bravais lattice , page 15
$S_{\widetilde{k}}$	Union of the sets $J^{\alpha}_{\widetilde{k}}$, page 50
$SO(3)$	Special Orthogonal Group in three space , page 14
V	Vector Space
v	Denotes an element in $\mathbb{R}^{n+1}$, page 9
v_δ	Translation vector associated to δ , page 45
V_{k_c}	Subspace of $\mathcal{X}_{\mathcal{L}}$ given by $\bigoplus_{\lvert k\rvert=k_c} V_k$, page 24
V_k	Space generated by the real and imaginary part or the wave ω_k , page 24
$X0Y$	Plane given by $\{(x, y, 0);\ x,\ y \in \mathbb{R}\}$, page 27
CIMA	Chlorite-Iodide-Malonic-Acid
DBC	Dirichlet Boundary Condition , page 40
ITC	International Tables for Crystallography
NBC	Neumann Boundary Condition , page 40
PBC	Periodic Boundary Condition , page 39

Chapter 1

Introduction

> "In this section a mathematical model of the growing embryo will be described. This model will be a simplification and an idealization, and consequently a falsification. It is to be hoped that the features retained for discussion are those of greatest importance in the present state of knowledge."
>
> Alan Turing, *The Chemical Basis of Morphogenesis*

This work is related to the study of pattern formation in symmetric physical systems. The purpose of this thesis is to discuss a possible model, namely the projection model, to explain the appearance and evolution of regular patterns in symmetric systems of equations.

Regular patterns are usually seen directly in nature and experiments. Convection, reaction-diffusion systems and the Faraday waves experiment comprise three commonly studied pattern-forming systems, see for instance Busse [3], Crawford et al. [11], Turing [41]. The most regular patterns observed, both in nature and in experiments, are stripes, squares and hexagons.

Here the key word to study patterns and physical systems is *symmetry*. When symmetry is present in a physical phenomenon, a rich branch of theories can be used to study the problem. We are concerned with two main theories and a particular application:

- *Crystallography Theory:* which gives a primary source for the mathematical description of patterns;
- *Equivariant Bifurcation Theory:* which provides tools to relate our model with symmetric physical systems.
- *Reaction-Diffusion Experiments:* where we find motivation and wish to apply our results.

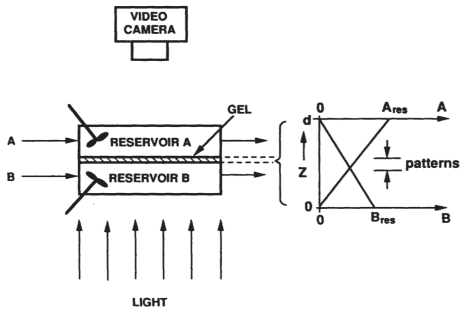


Fig. 1.1 Schematic diagram of the two-side-fed reactor for the reaction-diffusion experiments. The name "two-side-fed" relates to the fact that the experiment uses two reservoirs. (Reprinted figure by permission of Gunaratne et al. [24])

1.1 Reaction-Diffusion Experiments

Pattern formation in reaction-diffusion systems was first predicted by Alan Turing in his manuscript "The Chemical Basis of Morphogenesis", [41]. He suggests the profound idea that reaction and diffusion of chemicals in an initially uniform state could explain morphogenesis, that is, how biological patterns arise during growth.

The discussion about the process of morphogenesis is still a challenge in embryology. A broad discussion can be found in Murray [31]. Even facing this difficulty, Turing proposed a plausible mechanism towards the understanding of such a complex phenomenon.

The simplest model where a Turing structure is visible is an interaction of two chemicals. The chemicals will react, acting as an activator and an inhibitor for each other. A Turing instability occurs when a spatially homogeneous solution of the reaction-diffusion equation is unstable and yet it is stable under spatially homogeneous perturbations.

Intuitively, diffusion will smooth out spatial variation of the concentration of substances. Turing showed the opposite, diffusion in a reacting chemical system can actually be a destabilizing influence. That instability caused by diffusion will provide patterns, giving rise to a Turing instability and a corresponding Turing pattern.

The first Turing structure was observed in experiments in 1989, by Castets et al. [5]. The author considered a variant of the chlorite-iodide-malonic-acid (CIMA) reaction. Typically, reaction-diffusion experiments take place in a so called *open spatial reactor*, see Figure 1.1. The reaction occurs in a thin layer of gel, fed by diffusion from one or two faces with chemicals contained in stirred tanks, the contents of which are continuously renewed by pumps. This reaction generates a concentration gradient, also called chemical gradient. It corresponds to a gradual change in molecular density, and is observed in the z direction, see for instance Gomes [22], Kepper et al. [27]. Thus, variation on the chemical gradient is related to the thickness of the stratum where patterns are observed.

Although reaction-diffusion systems are ordinarily studied in mathematics, their quantitative comparison with experiments remains difficult, see Borckmans et al. [2], Szalai et al. [40]. Still, reaction-diffusion systems are such a broad and important class of non-equilibrium

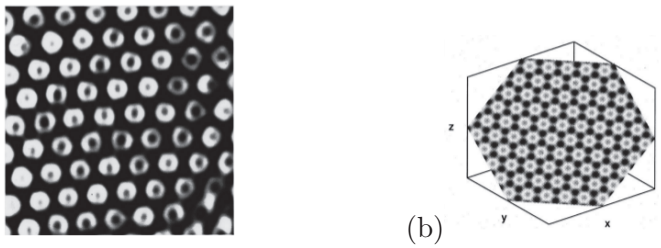


Fig. 1.2 Black-eye pattern (a) experimental observation; (b) as a projection into the plane $z = x + y$. (Reprinted figure with permission of [24] and [22])

systems - prevalent in biology, chemistry, ecology and engineering - that discussion about mathematical tools attempting to explain experiments is worthwhile.

In experiments as described before, the pattern itself and its observed state can occur in different dimensions. This happens for instance when an experiment is done in a 3-dimensional medium but the patterns are only observed on a surface, a 2-dimensional object, see [Kepper et al.], Section 4. This is the case for reaction-diffusion systems in the Turing instability regime, [41], which have often been described using a 2-dimensional representation, see Ouyang and Swinney [33]. The interpretation of this 2-dimensional outcome is subject to discussion: the black-eye pattern observed by [33] has been explained both as a mode interaction, Gunaratne et al. [24], and as a suitable projection of a 3-dimensional into a 2-dimensional lattice Gomes [22], see Figure 1.2. In her article, Gomes [22] shows how a 2-dimensional hexagonal pattern can be produced by a specific projection of a Body Centre Cubic (bcc) lattice.

Some approaches have been made to show that, in a two-side-fed reactor, the black-eye corresponds to a quasi-two-dimensional pattern instead of a 3-dimensional one, Yang et al. [45], Zhou et al. [46]. However, the 3-dimensional characteristics of patterns are difficult to analyse in experiments.

Two-side-fed reactors show a rich variety of patterns. However, the thickness of the stratum where patterns are observed cannot be controlled. This is a disadvantage since it has been shown that the thickness acts both as a bifurcation parameter and on the pattern dimensionality, see Kepper et al. [27].

To avoid difficulties present in a two-side-fed reaction, recent experiments are performed in a one-side-fed reactor, see [27]. Whereas such models have produced good agreement between theory and experiment, both one- or two- side-fed reactor experiments are based on observations on the projection of a 3-dimensional structures, see [40], from which we quote:

> "In this work we obtain results related to the projection introduced by [22]. We obtain theoretical results to interpret the effect of projection on the evolution of patterns in thin domains. We insist that the projection model can give high contribution to the understand of the transition of quasi-two- to 3-dimensional patterns in reaction-diffusion experiments. In particular we attempt to clarify the issue regarding to black-eye patterns."

1.2 Projection Model

There is no doubt that patterns observed in experiments, performed in open spatial reactors, are based on the top surface observations.

What is believed to be the first evidence of projection on CIMA reaction can be found in Winfree [44], chapter 13. The author gives details about the geometry of the formation of wave patterns in malonic-acid reaction performed on sufficiently thin layers. Through mechanical and chemical arguments, the author distinguishes reaction-diffusion systems from systems like the Faraday wave experiment. The distinction lies in the fact that, of these systems, only reaction-diffusion results can be interpreted through projection.

Later in 1991, a group in Bordeaux [Kepper et al.] conducted experiments on CIMA reaction and aimed at describing experimental observations of spontaneous symmetry breaking phenomena associated with steady-state instabilities. In [Kepper et al.], Section 4, the authors highlight the natural environment we must consider when we carry out CIMA reactions, in particular they state that all of their observations were based on projection of 3-dimensional structures and that the regions where Turing patterns are observed are associated by projection to a body-centred cubic pattern. More discussion on this can be found in Borckmans et al. [2], Dulos et al. [17], Kepper et al. [27], Szalai et al. [40].

Given all these evidence, does projection uncover hidden information that is not clear in experimental observations? If yes, how do we formalize projection in order to understand Turing patterns? How can projection give contributions to the study of reaction-diffusion systems?

The Euclidean group comprises the symmetries of the physical systems we are going to consider in this work. The symmetries of patterns are measured by a group, in our case, an appropriate subgroup of the Euclidean group: a crystallographic group.

To define a symmetric pattern, we consider the action of a crystallographic group on the space of functions $f : \mathbb{R}^n \to \mathbb{R}$. A n-dimensional pattern is the set of all level curves of a symmetric function f.

To obtain a n-dimensional pattern by projection, for $y_0 > 0$, we consider the restriction of a symmetric function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ to the region between the hyperplanes $y = 0$ and $y = y_0$. The projection of f is given by:

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x, y)dy.$$

In Figure 1.3, we see an illustration of the projection of functions $f : \mathbb{R}^2 \to \mathbb{R}$, defined as zero in the white region and one in the black. Projections of the particular functions yield functions whose value for each $x \in \mathbb{R}$ is the width of the black region above it.

The previous definition is motivated by the fact that we want to obtain information concerning the pattern, considering the depth of the stratum corresponding to its formation, as we see in experiments in open spatial reactors. However, fundamental questions arise:

- what are the symmetries of $\Pi_{y_0}(f)$, once f is symmetric?

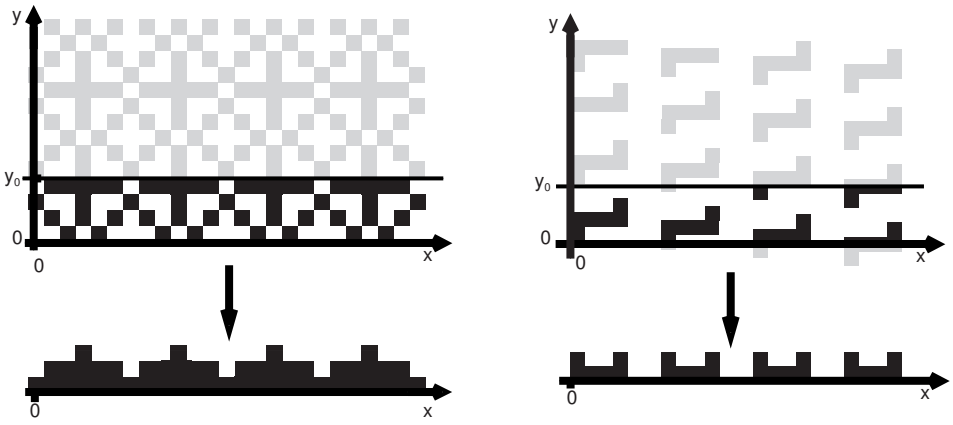


Fig. 1.3 The projection of periodic patterns in $\mathbb{R}^2$ restricted to the stripe of width y_0 defines functions with domain $\mathbb{R}$. (Reprinted figure by permission of [38])

- how can we relate a $(n+1)$-dimensional pattern to a specific n-dimensional one by projection?

- how can we study physical systems by interpreting patterns as projection?

Studies related to the projection defined before are recent. The only study performed to give a formalism of what projection means, in the context we are in, can be found in [38].

Pinho and Labouriau [38] study projections in order to understand how these affect symmetry. Their necessary and sufficient conditions for identifying projected symmetries are used extensively in our results.

In this thesis we intend to answer the questions posed in this section. We divided the problem in chapters. Each chapter makes use of different mathematical tools. As we mentioned before, Crystallography and Equivariant Bifurcation theories will play an important role in our work.

1.3 Crystallography Theory

The interdisciplinary nature of crystallography plays an important role in sciences as physics, chemistry and molecular biology. Despite this, there are few available introductory texts concerned with the foundation of the subject. An historical review of crystallographic research can be found in [25], chapter 1.1.

Crystallography aims at studying the structure of matter on an atomic scale. In particular, crystallography is used to determine, classify and interpret the geometrical structure of *crystals* and quasi-crystals. In this thesis we concentrate our ideas on crystals.

The most important characteristic properties in crystals are symmetry and periodicity. The foundation of studies in crystallography are found in modern algebra. Symmetry of crystals are expressed by *point groups*, the periodicity is described by translations and *lattices*, and the full structural symmetry is governed by a crystallographic group, usually named by

crystallographers as *space groups*. Consequently, crystallography is derived to a large extent from Euclidean Geometry.

In the study of crystals and quasicrystals, projection is a mathematical tool for lowering dimension, [39], [28]. A well developed study in crystallographic groups, their subgroups and the notion of projection used in crystallography can be found in the International Tables for Crystallography (ITC) volume A [25] and ITC volume E [29]. Tables therein provide information on projections of elements of crystallographic groups.

However, we intend to use crystallographic groups for a different purpose. The symmetries of solutions of partial differential equations, under certain boundary conditions, form a crystallographic group — see, for instance Golubitsky and Stewart [[20], chapter 5].

There are almost no references to give a relation between tools developed in crystallography and real periodic functions. Projections in [10] are seen as restriction of 3-dimensional periodic functions on a plane perpendicular to the z direction. Cochran [10] highlights that the full symmetry properties of periodic functions require symmetry elements which are not taken into account in the formulation of the usual full symmetry of crystals.

In order to study 3-dimensional patterns observed in a 2-dimensional environment, we use the projection of symmetric functions as defined in section 1.2. The symmetry group of the projected functions does not necessarily coincide with that of projections used in crystallography. The information contained in the ITC [25, 29], has to be organised in a different way before it can be used for this purpose.

1.4 Equivariant Bifurcation Theory

Group theory turns out to be very useful for analysing systems with symmetry. The application of group theory to the study of bifurcation problems with symmetry is known as equivariant bifurcation theory.

Equivariant bifurcation theory has been used extensively to study pattern formation via symmetry-breaking steady-state bifurcation in various physical systems modelled by $E(n)$-equivariant partial differential equations. In Golubitsky and Stewart [[20], chapter 5] there is a complete description of this method used in Dionne and Golubitsky [15], Dionne [14], Bosch Vivancos et al. [43], Callahan and Knobloch [4] and Dionne et al. [16], where the spatially periodic patterns are sometimes called planforms .

Using group theoretic methods, it is possible to understand how the symmetry of the observed patterns is affected or determined by the symmetry of the experimental set up or governing equations that produce them. Of central importance are the irreducible representations of a group. It will be by using representations, given by the action of a crystallographic group on different dimensional spaces, that we will find a way to relate features of symmetric systems and projection.

1.5 Description of the Chapters

We organize the thesis in the following way.

Chapter 2

The model we present involves multidisciplinary knowledge. In order to make the reading of this thesis easy, in Chapter 2, we restrict our attention to the statement of the necessary definitions and results that will be used in this work. In Section 2.2 we present many elementary, but nonetheless important ideas from crystallographic groups: of central importance are the characterization of a crystallographic group by its translation subgroup. In Section 2.3 we pay attention to 3-dimensional crystallographic groups, since we attempt to apply results to the 2- and 3-dimensional reaction-diffusion systems.

After dealing with the symmetries to describe symmetric patterns, in Section 2.4 we introduce the definition of projection. In Theorem 2.10, contained also in [38], we state the main result concerning the projection model, which will be extensively used throughout this thesis. Mainly, it establishes which are the symmetries of $\Pi_{y_0}(f)$, when f is symmetric.

The relation between projection and symmetric systems will be based on the language of groups too. The use of irreducible representations given by the group of symmetry of the problem is the main tool in Equivariant Bifurcation Theory. The remaining of Chapter 2 is related to how we use that tool to find solutions of symmetric systems.

Chapter 3

In Chapter 3, we establish which 3-dimensional patterns can be projected so as to obtain a 2-dimensional hexagonal pattern. We consider then the problem of how we can relate a $(n+1)$-dimensional pattern to a specific n-dimensional one by projection. We illustrate our results using the Primitive Cubic lattice.

Chapter 4

Chapter 4 presents our main results. These give us the first concrete evidence of how projection can be used to study symmetric physical systems, in particular the reaction-diffusion model. We establish how a 2-dimensional problem can be embedded in a 3-dimensional one by projection. This chapter involves a high level of group theory techniques. The results obtained allow us to use projection to compare the solutions involved in problems formulated in different dimensions. Then, we solve the question of how we can study physical systems by interpreting patterns as projection.

Chapter 5

The way solutions behave on the problem requires additional concepts from Equivariant Bifurcation Theory. We present the idea of mode interaction. In particular we conclude that, both the experimental results obtained by [24] and the theoretical results in [22], regarding black-eye patterns, are the same.

In the same chapter we interpret how forced symmetry breaking may be used to understand how the projection of a solution varies in terms of the band of projection. For this purpose,

we refer to studies made by Parker [34], where the author also studies the model of projection in this vein.

In the end of the chapter we address future work.

Chapter 2

Preliminaries

In this chapter we are interested in describing the symmetry of objects in the Euclidean space and in presenting the most relevant results to describe its projection. We will introduce the overall properties of the Euclidean Group and its crystallographic subgroups. Particular attention will be paid to 3-dimensional spaces. After that we introduce the model of projection followed by its main properties that are going to be used in our work. We conclude the chapter with concepts from Equivariant Bifurcation Theory. Further reading for this chapter can be found in Miller [30], Armstrong [1] chapter 24 to 26, Senechal [39], the International Tables for Crystallography (ITC) volume A [25], Pinho [37], Golubitsky et al. [21].

2.1 The Euclidean Group

Consider an object X that is a subset of $\mathbb{R}^{n+1}$. A permutation of a non-empty set X is a 1-1 mapping of X onto itself. The set S_X of all permutations of X forms a group, designated as the full symmetric group on X. We say that the elements of S_X act or operate on elements of X and a subgroup G of S_X is a transformation (permutation) group on X.

Given a transformation group G, we can look for all G-invariant subsets Y of X, where we say that Y is invariant by the action of G, or G-invariant, if for all $g \in G,\ g(Y) \subseteq Y$. On the other hand, given an arbitrary subset Y of X we can find a transformation subgroup that fixes Y, $K = \{k \in S_X;\ k(Y) = Y\}$.

In this work $X = \mathbb{R}^{n+1}$ and its symmetry group is the *Euclidean Group*, $E(n+1)$, of all linear isometries on $\mathbb{R}^{n+1}$. We are going to describe objects in $\mathbb{R}^{n+1}$ whose symmetries form a crystallographic group. In particular, we will use those objects to describe symmetric patterns.

Denote by T^{n+1} the group of all translations in dimension $n+1$, ($T^{n+1} \cong \mathbb{R}^{n+1}$). Let $O(n+1)$ be the subgroup of the general linear group $GL(n+1,\mathbb{R})$ of all linear length-preserving transformations. It is well-known that the Euclidean Group is the semi-direct product of T^{n+1} and $O(n+1)$. Hence, we may think of each isometry of $E(n+1) = \mathbb{R}^{n+1} \dot{+} O(n+1)$ as an ordered pair (v, δ), in which $v \in \mathbb{R}^{n+1}$ and $\delta \in O(n+1)$, with multiplication given by:

$$(v_1, \delta_1) \cdot (v_2, \delta_2) = (v_1 + \delta_1 v_2, \delta_1 \delta_2)$$

with identity element $(\mathbf{0}, Id_{n+1})$, where Id_{n+1} the identity mapping of $\mathbb{R}^{n+1}$, and the inverse element given by $(v,\delta)^{-1} = (-\delta^{-1}v, \delta^{-1})$.

Consider also the group *action* of $E(n+1)$ on $\mathbb{R}^{n+1}$ given by the function:

$$\begin{array}{rcl} E(n+1)\times\mathbb{R}^{n+1} & \to & \mathbb{R}^{n+1} \\ ((v,\delta),(x,y)) & \mapsto & (v,\delta)\cdot(x,y) = v+\delta(x,y) \end{array} \tag{2.1}$$

The symmetries of a subset Y in $\mathbb{R}^{n+1}$ are given by the elements of $E(n+1)$ that fix Y under the previous action.

2.2 Crystallographic Groups

Let Γ be a subgroup of $E(n+1)$. The homomorphism

$$\begin{array}{rccc} \phi: & \Gamma & \to & O(n+1) \\ & (v,\delta) & \mapsto & \delta \end{array}$$

has as image a group $\mathbf{J}$, called the *point group* of Γ, and its kernel forms the *translation subgroup* of Γ.

The definition of a crystallographic group is given in terms of its translation subgroup. We have the following definitions:

Definition 2.1. We say that the translation subgroup of Γ is a *k-dimensional lattice, $\mathcal{L}$,* if it is generated over the integers by k linearly independent elements $l_1,\ldots,l_k \in \mathbb{R}^{n+1}$, where $k \in \{1,\ldots,n+1\}$. We write:

$$\mathcal{L} = \langle l_1,\ldots,l_k\rangle_{\mathbb{Z}}.$$

Given a lattice $\mathcal{L}$, the definition of *dual lattice* is of great importance:

Definition 2.2. The *dual lattice* of a k-dimensional lattice $\mathcal{L}$ is defined as the set given by:

$$\mathcal{L}^* = \{k \in \mathbb{R}^{n+1};\ \langle k,l\rangle \in \mathbb{Z},\ \textit{for all } l \in \mathcal{L}\}.$$

Definition 2.3. A *crystallographic group* is a discrete subgroup of $E(n+1)$, such that its translation subgroup is an $(n+1)$-dimensional lattice.

A description of these concepts can be found in the ITC volume A [25], chapter 8.1, pp. 720-725 and in their suggested bibliography for the chapter; see also Miller [30].

Each isometry in the point group J of a crystallographic group Γ of $E(n+1)$ maps the lattice of Γ onto itself, and it is clear that $v\cdot\mathcal{L} = \mathcal{L}$ if and only if $v \in \mathcal{L}$. Therefore, Γ is a group of symmetry of $\mathcal{L}$. The complete symmetries of $\mathcal{L}$ can be obtained by finding all elements $\delta \in O(n+1)$ that leave $\mathcal{L}$ invariant. Such elements form a group called the *holohedry* of $\mathcal{L}$,

and it is denoted by $H_{\mathcal{L}}$. The holohedry is always a finite group, see Senechal [39], subsection 2.4.2.

Note that the term *holohedry* used here, as well as in Dionne and Golubitsky [15] and Golubitsky and Stewart [20], corresponds in [25], chapter 8.2, to the definition of point symmetry of the lattice.

With this information we can construct any crystallographic group in $\mathbb{R}^{n+1}$ (up to isomorphism), because it is just necessary to find a lattice $\mathcal{L}$ and a subgroup J of its holohedry.

In the next sections, we are going to review some important results regarding crystallographic group. In particular, for the 3-dimensional case we are going to describe representatives for crystallographic group in dimension 3. The next result shows why it is only necessary to consider these representatives.

Lemma 2.4. *Let $\xi = (u, \rho) \in E(n+1)$ and Γ be a crystallographic group with lattice $\mathcal{L}$ and point group J. Then, $\xi\Gamma\xi^{-1}$ is a crystallographic group with lattice $\mathcal{L}_\xi = \rho\mathcal{L}$ and point group $J_\xi = \rho J\rho^{-1}$. Moreover, the map $\Phi_\xi : \Gamma \longrightarrow \xi\Gamma\xi^{-1}$, given by $\phi_\xi(\gamma) = \xi\gamma\xi^{-1}$ is a group isomorphism.*

Proof. For every $\gamma = (v, \delta) \in \Gamma$, we have

$$\begin{aligned}(u,\rho)(v,\delta)(u,\rho)^{-1} &= (u,\rho)(v,\delta)(-\rho^{-1}u,\rho^{-1})\\ &= (u,\rho)(v-\delta\rho^{-1}u,\delta\rho^{-1})\\ &= (u+\rho v-\rho\delta\rho^{-1}u,\rho\delta\rho^{-1})\\ &= (\rho v+(Id_{n+1}-\rho\delta\rho^{-1})u,\rho\delta\rho^{-1})\end{aligned}$$

Therefore, the point group of $\xi\Gamma\xi^{-1}$ is $J_\xi = \rho J\rho^{-1}$. Its translation subgroup is given when the orthogonal coordinate of $\xi\gamma\xi^{-1}$, $\rho\delta\rho^{-1}$, is the identity Id_{n+1}, which yields $\mathcal{L}_\xi = \rho\mathcal{L}$.

The rest of the proof follows by group theory. □

2.2.1 Lattices

We saw in the previous section that the study of lattices is fundamental to describe crystallographic groups.

A lattice $\mathcal{L}$ is a non-trivial discrete subgroup of $\mathbb{R}^{n+1}$. By Definition 2.1, every $a \in \mathcal{L}$ can be written uniquely in the form

$$a = m_1 l_1 + \ldots + m_k l_k$$

where m_i, $i = 1, \ldots, k$ are integers. We call $l_1, \ldots, l_k$ *basic vectors*.

Let $x \in \mathbb{R}^{n+1}$ and $P = \{m_1 l_1 + \cdots + m_k l_k\} \subseteq \mathcal{L}$, for $m_i \in \{0,\ 1\}$. Applying to x those translations in P, we get a cell in $\mathbb{R}^k$ called a *primitive or fundamental cell*. By applying all elements of $\mathcal{L}$ to x, that is, by constructing the $\mathcal{L}$-orbit containing x, we form a geometrical lattice of points in $\mathbb{R}^k$. Usually it is most convenient to discuss lattices based on $x = \mathbf{0}$.

Given s linearly independent vectors $b_1, \ldots,\ b_s \in \mathcal{L}$, the set $\mathcal{L}' = \langle b_1, \ldots, b_s\rangle_{\mathbb{Z}}$, such that $0 \leqslant s \leqslant k$, is a sublattice of $\mathcal{L}$. If $\mathcal{L}'$ is k-dimensional, it will coincide with $\mathcal{L}$ if and only if

the volume of the primitive cell, all formed by the b_i, is equal to the volume of the primitive cell generated by the l_i, otherwise, $\mathcal{L}'$ is a proper sublattice of $\mathcal{L}$. The proof of this claim is analogous to the one given in Theorem 2.7 in [30]. Observe that if the vectors b_i in $\mathcal{L}$ satisfy:

$$b_i = \sum_{j=1}^{k} c_{ji} l_j, \quad i = 1, \ldots, k$$

then the b_i are basic vectors if and only if $\det C = \pm 1$, where $C = (c_{ji})$.

To finish our overview at lattices, in Corollary 2.2 in [30] we can see that a primitive cell is not unique. In fact, for any s-dimensional sublattice of $\mathcal{L}$, we can construct a primitive cell of $\mathcal{L}$. The next result is a generalization of the Theorem 2.6 in [30].

Proposition 2.5. *Let $\mathcal{L}$ be an $(n+1)$-dimensional lattice and suppose that $\mathcal{L} \cap \mathbb{R}^n$ is an n-dimensional sublattice of $\mathcal{L}$. Then there exists a primitive cell of $\mathcal{L}$ generated by $n+1$ linearly independent vectors, such that n of those vectors are in $\mathcal{L} \cap \mathbb{R}^n$.*

Proof. Let $a_1, \ldots, a_{n+1}$ be linearly independent vectors in $\mathcal{L}$ such that $a_1, \ldots, a_n \in \mathcal{L} \cap \mathbb{R}^n$, and let P be the cell in $\mathbb{R}^{n+1}$ determined by these vectors. There are only a finite number of elements of the discrete group $\mathcal{L}$ in P. Let l_1 be the shortest non-zero vector in $\mathcal{L} \cap P$ that is parallel to a_1. That is, on the edge of P with the endpoints zero and a_1 we choose the element $l_1 \neq 0$ of $\mathcal{L}$ closest to zero.

Now, let l_2 be an element of $\mathcal{L}$ in the parallelogram generated by $a_1,\ a_2$ such that the parallelogram generated by $l_1,\ l_2$ has the smallest possible non-zero area. Continuing with this method we can choose a $l_n \in \mathcal{L} \cap P$ such that the cell generated by $l_1, \ldots, l_n$ has the smallest possible non-zero volume in $\mathcal{L} \cap \mathbb{R}^n$. Again, choose l_{n+1} in the cell given by $a_1, \ldots, a_{n+1}$, such that the cell Q generated by $l_1, \ldots, l_{n+1}$ has the smallest volume in $\mathcal{L} \cap \mathbb{R}^{n+1}$.

We show that for all $l \in \mathcal{L}$

$$l = m_1 l_1 + \cdots + m_{n+1} l_{n+1}$$

where $m_i \in \mathbb{Z}, \quad \forall\ i = 1, \ldots, n+1$.

Clearly, the vectors $\{l_i\}_{i=1}^{n+1}$ are linearly independent and given any $l \in \mathcal{L}$ there exist unique real numbers α_i such that

$$l = \alpha_1 l_1 + \cdots + \alpha_{n+1} l_{n+1}$$

Let n_i be the largest integer less than α_i. Then

$$l - \sum_{i=1}^{n+1} n_i l_i = \sum_{i=1}^{n+1} \beta_i l_i = b \tag{2.2}$$

with $0 \leq \beta_i < 1$. The vector b defined in (2.2) is clearly an element $\mathcal{L} \cap Q$. We will show that $b = 0$.

Suppose $0 < \beta_{n+1} < 1$. Then the volume $V(Q')$ of the cell Q' generated by $l_1, \ldots, l_n,\ b$ is strictly less than the volume $V(Q)$. If $b \in P$ this is impossible since it contradicts our choice of l_{n+1}. If $b \notin P$ we can find integers $m_1, \ldots, m_n$ such that

$$b' = b + m_1 a_1 + \cdots + m_n a_n \in P$$

and the cell Q'' generated by $l_1, \ldots, l_n,\ b'$ has volume $V(Q'') < V(Q)$. This is impossible. Thus $\beta_{n+1} = 0$ and b lies in the space spanned by $a_1, \ldots, a_n$.

In a similar way we can prove that $\beta_i = 0$, for all $i = 1, \ldots, n$. □

In our work it will be useful to consider a primitive cell of $\mathcal{L}$, when the n-dimensional sublattice of $\mathcal{L}$ is a suspension of a n-dimensional lattice. This idea motivates the following definition:

Definition 2.6. We say that a lattice $\mathcal{L}_1$ is *rationally compatible* with a lattice $\mathcal{L}_2$ if there exists $r \in \mathbb{Z} \setminus \{0\}$ such that $r\mathcal{L}_1 \subset \mathcal{L}_2$. A vector $v \in \mathbb{R}^k$ is *rational with respect to a lattice* $\mathcal{L} \subset \mathbb{R}^k$ if $\langle v, \ell \rangle \in \mathbb{Q}$ for all $\ell \in \mathcal{L}$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^k$. Given a lattice $\widetilde{\mathcal{L}} \subset \mathbb{R}^k$, we define its *suspension* $\widetilde{\mathcal{L}}_s \subset \mathbb{R}^{k+1}$ as $\widetilde{\mathcal{L}}_s = \{(v, 0);\ v \in \widetilde{\mathcal{L}}\}$.

We conclude with the following result:

Lemma 2.7. *Let $\mathcal{L}$ be an $(n+1)$-dimensional lattice with dual lattice $\mathcal{L}^*$. Suppose that there are suspensions of n-dimensional lattices $\widetilde{\mathcal{L}}_1$ and $\widetilde{\mathcal{L}}_2$ that are rationally compatible with $\mathcal{L}$ and $\mathcal{L}^*$, respectively. Then the following conditions hold:*

1. *there exist bases $\{(l_1, 0), \ldots, (l_n, 0), (l_{n+1}, b)\}$ and $\{(k_1, 0), \ldots, (k_n, 0), (k_{n+1}, a)\}$ for $\mathcal{L}$ and $\mathcal{L}^*$, respectively.*

2. *if $m \in \mathbb{N} \setminus \{0\}$ is the minimum integer such that $(0, mb) \in \mathcal{L}$, then $(0, ma) \in \mathcal{L}^*$ and m is the minimum such that it happens.*

Proof. The condition 1. follows from Proposition 2.5.

To see that condition 2. holds, observe that if there is a minimum $m \neq 0$, such that the elements $(0, mb) \in \mathcal{L}$, then $\langle (0, mb), (k_{n+1}, a) \rangle \in \mathbb{Z} \setminus \{0\}$. Then $mba \in \mathbb{Z} \setminus \{0\}$. Therefore $(0, ma) \in \mathcal{L}^*$, due to the system of generators of $\mathcal{L}$.

Suppose that $(0, ra) \in \mathcal{L}^*$, for $r \in \mathbb{Z} \setminus \{0\}$ and $r < m$. Then,

$$\langle (l_{n+1}, b), (0, ra) \rangle \in \mathbb{Z} \setminus \{0\}.$$

Thus, $rba \in \mathbb{Z} \setminus \{0\}$, which implies that $(0, rb) \in \mathcal{L}$. Therefore, $m = r$. □

2.3 Crystallographic groups in 3-dimension

The special cases of 2-dimensional (wallpaper groups, see [1]) and 3-dimensional crystallographic groups are most heavily used in applications. As we will see in the next chapters, the knowledge of lattices and their holohedries will be of great importance to obtain plane sub-lattices in a three-dimensional one.

For our purpose it is worth paying more attention to the orthogonal group $O(3)$.

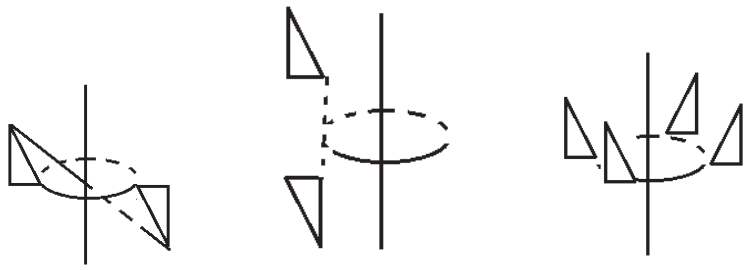


Fig. 2.1 Action of a side-reversing element.

2.3.1 The Orthogonal Group $O(3)$

The group $O(3)$ is the group of all linear transformations $\delta : \mathbb{R}^3 \to \mathbb{R}^3$ which preserve length.

Since all elements in $O(3)$ are linear, we may identify $O(3)$ with the group of orthogonal matrices $\{3\times3$ matrices A; $A^tA = Id_3\}$ (A^t = transpose of A). Note that $\det A = \pm 1$, $\forall A \in O(3)$ and the element $-Id_3$, with determinant -1, is in $O(3)$. We call $-Id_3$ the *inversion operator*.

The subset $SO(3) = \{A \in O(3); \det A = 1\}$ forms a normal subgroup of $O(3)$, called the *special orthogonal group in three space* or just the *group of rotations*, and $-Id_3 \notin SO(3)$. So, $O(3) = SO(3) \cup -Id_3SO(3)$. The elements of $SO(3)$ and $-Id_3SO(3)$ are all *side-preserving* (proper rotations) and *side-reversing* (rotations followed by inversion) elements, respectively.

The elements of $SO(3)$ are exactly the possible geometrical rotations about axes in $\mathbb{R}^3$ passing through the origin. This means that for any rotation A, there exists an unitary vector $w \in \mathbb{R}^3$, such that $Aw = w$ and the rotation axis of A is determined by $\pm w$, by Theorem 2.1, [30]. It follows that, if H is a subgroup of $SO(3)$ of order $n \geq 2$ consisting of rotations about a fixed axis then H is isomorphic to a cyclic group of order n (Lemma 2.2, [30]). This subgroup is generated by a rotation through the angle $2\pi/n$ and the fixed axis L is an *n-fold axis*.

A rotation-inversion of $-Id_3SO(3)$ is the combination of a rotation about an axis and a reflection in the plane perpendicular to this axis. We call this plane a *mirror plane*. It is equivalent to the combination of a rotation and an inversion about point on the axis.

2.3.2 Bravais Lattices and Their Holohedry

It is easy to find in the literature that, for dimension 3, we have only seven possible holohedries (up to isomorphism) and the possible lattices of a 3-dimensional crystallographic group form the well-known 14 *Bravais lattices*. For this subsection, we refer to [25], chapter 9.1.

Since $H_{\mathcal{L}} \cdot \mathcal{L} = \mathcal{L}$, an important result about crystallographic groups is the crystallographic restriction (Theorem 2.8, [30]), which says that not all finite groups are compatible with a discrete lattice. If $g \in \Gamma$ is of the form $g = (-Id_3)^i\delta$, $i = 0,\ 1$, with δ a non-trivial rotation, then δ is of order two, three, four, or six. Another observation is that the inversion $-Id_3$ is in $H_{\mathcal{L}}$ and, consequently, if $H_{\mathcal{L}}$ contains a cyclic subgroup C_m then it contains the subgroup $C_{2m} = \langle C_m, -I_3\rangle$.

The holohedries of the Bravais lattice, due to the crystallographic restriction theorem, are:

$$\begin{array}{ccccccccc} S_2 & \subset & C_{2h} & \subset & D_{2h} & \subset & D_{4h} & \subset & O_h \\ & & \cap & & \cap & & & & \\ & & D_{3d} & \subset & D_{6h} & & & & \end{array}$$

Here we are using the Schoenflies (or Schönflies) notation, named after the German mathematician Arthur Moritz Schoenflies. It is one of the two conventions commonly used to describe point groups.

Symmetry elements are denoted by C for proper rotation axes, σ for mirror planes, and S for improper rotation axes (rotation-inversion axes). The symbols C and S are usually followed by a subscript number (abstractly denoted n) denoting the order of rotation possible.

By convention, the axis of proper rotation of greatest order is defined as the principal axis. All other symmetry elements are described in relation to it. A vertical mirror plane (containing the principal axis) is denoted σ_v; a horizontal mirror plane (perpendicular to the principal axis) is denoted σ_h.

The general notation for the seven holohedries are:

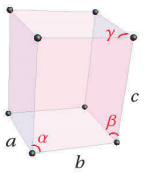
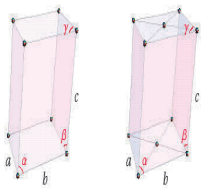
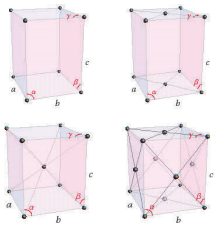
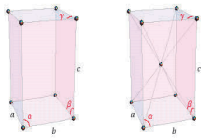
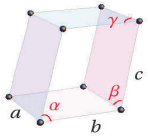
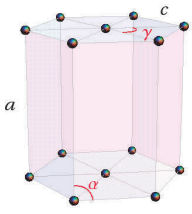
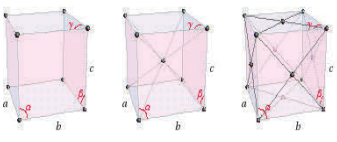
- S_{2n} (for Spiegel, German for mirror) that contains only a $2n$-fold rotation-inversion axis.
- C_n (for cyclic) has an n-fold rotation axis. C_{nh} is C_n with the addition of a mirror (reflection) plane perpendicular to the axis of rotation (horizontal plane). C_{nv} is C_n with the addition of n mirror planes containing the axis of rotation (vertical planes).
- D_n (for dihedral, or two-sided) has an n-fold rotation axis plus n twofold axes perpendicular to that axis. D_{nh} has, in addition, a horizontal mirror plane and, as a consequence, also n vertical mirror planes each containing the n-fold axis and one of the twofold axes. D_{nd} has, in addition to the elements of D_n, n vertical mirror planes which pass between twofold axes (diagonal planes).
- O (the chiral octahedral group) has the rotation axes of an octahedron or cube (three 4-fold axes, four 3-fold axes, and 6 diagonal 2-fold axes). O_h includes horizontal mirror planes and, as a consequence, vertical mirror planes. It contains also inversion center and improper rotation operations.

Table 2.1 summarizes the main information about the Bravais lattices and their respective holohedries.

2.4 Projected Patterns

The study of projections is related to patterns. Patterns are level curves of functions $f : \mathbb{R}^{n+1} \to \mathbb{R}$. In our work we suppose that these functions are invariant under the action of a crystallographic group.

Table 2.1 Bravais lattices and its Holohedry. The following figures were taken from the internet.

Lattice Family	Relations and Holohedry	
Triclinic	• **Angle relations:** $\alpha \neq \beta \neq \gamma$ • **Edge relation:** $a \neq b \neq c$ • **Holohedry:** S_2.	
Monoclinic: *primitive monoclinic* and *centred monoclinic* lattices, respectively.	• **Angle relations:** $\alpha \neq 90°$, $\beta = \gamma = 90°$ • **Edge relation:** $a \neq b \neq c$ • **Holohedry:** C_{2h}	
Orthorhombic: *simple orthorhombic*, *base-centred orthorhombic*, *body-centred orthorhombic* and *face-centred orthorhombic*, respectively.	• **Angle relations:** $\alpha = \beta = \gamma = 90°$ • **Edge relation:** $a \neq b \neq c$ • **Holohedry:** D_{2h}	
Tetragonal: *simple tetragonal* and *centred tetragonal*, respectively.	• **Angle relations:** $\alpha = \beta = \gamma = 90°$ • **Edge relation** $a = b \neq c$ • **Holohedry** D_{4h}	
Rhombohedral	• **Angle relations:** $\alpha = \beta = \gamma \neq 90°$ • **Edge relation** $a = b = c$ • **Holohedry** D_{3d}	
Hexagonal	• **Angle relations:** $\alpha = \beta = 90°$, $\gamma = 120°$ • **Edge relation** $a \neq c = b$ • **Holohedry** D_{6h}	
Cubic: *simple cubic*, *body-centred cubic (bcc)* and the *face-centered cubic (fcc)*, respectively	• **Angle relations:** $\alpha = \beta = \gamma = 90°$ • **Edge relation** $a = b = c$ • **Holohedry** O_h	

The set of all level curves of a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is defined by:

$$\mathcal{C}(f) = \bigcup_{c\in\mathbb{R}} \{(x, y) \in \mathbb{R}^{n+1};\ f(x, y) = c\}$$

Crystallographic groups are related to symmetries of pattern formation by the action of the group of symmetries on a space of functions, see Golubitsky and Stewart [20] chapter 5.

To see this, observe that the action (2.1) of the Euclidean group on $\mathbb{R}^{n+1}$ induces an action of a crystallographic group Γ on the space of functions $f : \mathbb{R}^{n+1} \to \mathbb{R}$ by:

$$(\gamma \cdot f)(x, y) = f(\gamma^{-1}(x, y)) \text{ for } \gamma \in \Gamma \text{ and } (x, y) \in \mathbb{R}^{n+1}.$$

Thus, we can construct a space $\mathcal{X}_\Gamma$ of Γ-invariant functions, that is

$$\mathcal{X}_\Gamma = \{f : \mathbb{R}^{n+1} \to \mathbb{R};\ \gamma \cdot f = f,\ \forall \gamma \in \Gamma\}$$

In particular a Γ-invariant function is $\mathcal{L}$ -invariant.

We have the following result:

Lemma 2.8. *let Γ be a crystallographic group. A function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is Γ-inavriant if and only if the subset $\mathcal{C}(f) \subseteq \mathbb{R}^{n+1}$ is invariant by the action of Γ on $\mathbb{R}^{n+1}$.*

Definition 2.9. A Γ-*symmetric pattern* consists of the set of all level curves of a Γ-invariant function $f : \mathbb{R}^{n+1} \to \mathbb{R}$.

In Gomes [22] the black-eye pattern is obtained as a projection of a function, whose set of all level curves form a bcc-pattern in $\mathbb{R}^3$. In terms of symmetries, the black-eye is a hexagonal pattern, as we can see in [22]. It is the set of all level curves of a function with periods in a hexagonal plane lattice, that is, a lattice that admits as its holohedry a group isomorphic to the dihedral group of symmetries of the regular hexagon, D_6. Moreover, we expect the point group of symmetries of the black-eye to be isomorphic to D_6.

For $y_0 > 0$, consider the restriction of $f \in \mathcal{X}_\Gamma$ to the region between the hyperplanes $y = 0$ and $y = y_0$. The projection operator Π_{y_0} integrates this restriction of f along the width y_0, yielding a new function with domain $\mathbb{R}^n$.

Definition 2.10. *(Pinho and Labouriau [38])* For $f \in \mathcal{X}_\Gamma$ and $y_0 > 0$, the *projection operator* Π_{y_0} is given by:

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x, y)dy$$

The region between $y = 0$ and $y = y_0$ is called the *projection band* and $\Pi_{y_0}(f) : \mathbb{R}^n \to \mathbb{R}$ is the *projected function.*

The previous definition states that the projections are made in a fixed direction. The next definition outlines that point.

Let $\mathcal{L} \subset \mathbb{R}^{n+1}$ be a lattice and $P \subset \mathbb{R}^{n+1}$ be an n-dimensional hyperspace such that $P \cap \mathcal{L} \neq \varnothing$. Given $v \in P \cap \mathcal{L}$ there is a rotation $\gamma \in O(n + 1)$ such that $\gamma(P - v)$ is the hyperspace $X0Y = \{y = 0\} = \{(x, y) \in \mathbb{R}^{n+1};\ y = 0\}$.

Definition 2.11. We define the y_0-*projection* of $\mathcal{L}$ into P as the lattice $\gamma^{-1}(\widetilde{\mathcal{L}}) \subset E(n)$ where $\widetilde{\mathcal{L}}$ is the symmetry group of $\Pi_{y_0}(\mathcal{X}_{\gamma(\mathcal{L}-v)})$.

The functions $\Pi_{y_0}(f)$ may be invariant under the action of some elements of the group $E(n) \cong \mathbb{R}^n \dot{+} O(n)$. The relation between the symmetries of f and those of $\Pi_{y_0}(f)$ was provided by [38].

To find the group of symmetries of the space of projected functions, $\Pi_{y_0}(\mathcal{X}_\Gamma) = \{\Pi_{y_0}(f) : \mathbb{R}^n \to \mathbb{R};\ f \in \mathcal{X}_\Gamma\}$, the authors consider the following data:

- for $\alpha \in O(n)$, the elements of $O(n+1)$:

$$\sigma := \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix},\ \alpha_+ := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \alpha_- := \sigma\alpha_+; \tag{2.3}$$

- the subgroup $\widehat{\Gamma}$ of $\Gamma \subset E(n+1)$, whose elements are of the form

$$((v,y),\alpha_\pm);\ \alpha \in O(n),\ (v,y) \in \mathbb{R}^n \times \mathbb{R}.$$

  The translation subgroups of $\widehat{\Gamma}$ and Γ are the same, while the point group of $\widehat{\Gamma}$ consists of those elements of Γ that fix the space $\mathbb{R}^n \times \{0\}$.

  For the 3-dimensional case $\widehat{\Gamma}$ coincides with the *scanning group* defined in [29], chapter 5.2.

- the projection $h : \widehat{\Gamma} \to E(n) \cong \mathbb{R}^n \dot{+} O(n)$ given by:

$$h\left((v,y),\alpha_\pm\right) = (v,\alpha)$$

The group of symmetries of $\Pi_{y_0}(\mathcal{X}_\Gamma)$ is the image by the projection h of the group Γ_{y_0} defined as:

- If $(0,y_0) \in \mathcal{L}$ then $\Gamma_{y_0} = \widehat{\Gamma}$.
- If $(0,y_0) \notin \mathcal{L}$ then Γ_{y_0} contains only those elements of $\widehat{\Gamma}$ that are either side preserving $((v,0),\alpha_+)$ or side reversing $((v,y_0),\alpha_-)$

The group $\widehat{\Gamma}$ consists of those elements of Γ that will contribute to the symmetries of the set of projected functions. Depending on whether the hypotheses above hold, the group Γ_{y_0} will be either the whole group $\widehat{\Gamma}$ or a subperiodic group of $\widehat{\Gamma}$, that is a subgroup whose lattice of translations has lower dimension than the space on which the group acts; see [25] chapter 8.1 and [29] chapter 1.2.

The group Γ_{y_0} depends on how the elements of Γ are transformed by the projection Π_{y_0} of Γ-invariant functions. The criterion that clarifies the connection between the symmetries of $\mathcal{X}_\Gamma$ and $\Pi_{y_0}(\mathcal{X}_\Gamma)$ is provided by the following result:

Theorem 2.12 (Theorem 1.2 in [38])**.** *All functions in* $\Pi_{y_0}(\mathcal{X}_\Gamma)$ *are invariant under the action of* $(v,\alpha) \in E(n)$ *if and only if one of the following conditions holds:*

I $((v,0),\alpha_+) \in \Gamma$*;*

II $((v,y_0),\alpha_-) \in \Gamma$*;*

III $(0,y_0) \in \mathcal{L}$ *and either* $((v,y_1),\alpha_+) \in \Gamma$ *or* $((v,y_1),\alpha_-) \in \Gamma$*, for some* $y_1 \in \mathbb{R}$*.*

Remark 2.13. We note that there is a non-trivial relation between the lattice $\widetilde{\mathcal{L}}$ of periods of the projected functions and that of the original one. In fact, consider an $(n+1)$-dimensional lattice $\mathcal{L}$ and $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. If $v \in \widetilde{\mathcal{L}}$ then (v, I_n) is a symmetry of $\Pi_{y_0}(\mathcal{X}_\Gamma)$. Applying theorem 2.12 with $\alpha = I_n$, one of the following holds for each $v \in \widetilde{\mathcal{L}}$:

I $((v,0),\alpha_+) = ((v,0),I_{n+1}) \in \Gamma$, or equivalently $(v,0) \in \mathcal{L}$;

II $((v,y_0),\alpha_-) = ((v,y_0),\sigma) \in \Gamma$ then $((v,y_0),\sigma)^2 \in \Gamma$ implying that $(2v,0) \in \mathcal{L}$;

III $(0,y_0) \in \mathcal{L}$ and either (v,y_1) or $(2v,0)$ is in $\mathcal{L}$, for some $y_1 \in \mathbb{R}$.

While condition I implies that $\mathcal{L} \cap \{(x,0) \in \mathbb{R}^{n+1}\} \subseteq \widetilde{\mathcal{L}}$, the other conditions show that this inclusion is often strict. Furthermore, conditions II and III show that we may have no element of the form (v,y_1) in $\mathcal{L}$ and yet $v \in \widetilde{\mathcal{L}}$. This is due to a possible non-zero translation vector associated to $\sigma \in J$.

2.5 Lie Groups and Their Representation

So far we have used crystallographic groups to measure the symmetries of functions, but they are nothing more than the special and well known groups used in the study of equivariant PDEs: Lie groups. A *Lie group* is a differentiable manifold, where the group operation is a differentiable map, as is the inversion operation that gives the inverse of a group element. For this section we refer to [21].

Let Γ be a Lie group and let V be a vector space (over either $\mathbb{R}$ or $\mathbb{C}$). We say that Γ acts on V if there is a continuous mapping

$$\begin{array}{rccc} \rho: & \Gamma \times V & \longrightarrow & V \\ & (\gamma, v) & \longmapsto & \gamma v \end{array}$$

such that (a) for each $\gamma \in \Gamma$ the mapping $\rho_\gamma : V \longrightarrow V$, defined by $\rho_\gamma(v) = \gamma v$ is linear and (b) if $\gamma_1,\ \gamma_2 \in \Gamma$ then $\gamma_1(\gamma_2 v) = (\gamma_1\gamma_2)v$. The map ρ that sends γ to $\rho_\gamma \in GL(V)$ is called a *representation* of Γ on V.

Within the collection of all representations or actions of a group, there are two classes of special ones: irreducible and absolutely irreducible representations/actions. Absolutely irreducible real representations/actions are particularly important in the theory of steady-state bifurcations in symmetric systems.

Consider a Lie group Γ acting on a vector space V. We say that a subspace $W \subseteq V$ is Γ-*invariant* if $\gamma w \in W$, for all $\gamma \in \Gamma$ and $w \in W$. A representation or action is *irreducible* if the only Γ-invariant subspaces are $\{0\}$ and V. An action or representation of Γ is said to be *absolutely irreducible* if the only linear mappings that commute with the corresponding

action of Γ on V are scalar multiples of the identity. A linear map $A : V \longrightarrow V$ commutes with the action of Γ if

$$\gamma(Av) = A(\gamma v), \quad \forall v \in V, \quad \forall \gamma \in \Gamma.$$

We say that the map A is Γ-symmetric.

2.6 Equivariant Bifurcation Problems

In Bifurcation Theory we consider maps $g : V \subseteq \mathbb{C}^m \times \mathbb{R} \longrightarrow V$ that are equivariant under the action of the Euclidean group. The map g is Euclidean equivariant if

$$\gamma(g(v,t)) = g(\gamma v, t), \quad \forall (v,t) \in V \times \mathbb{R}, \quad \forall \gamma \in E(n+1).$$

We say that g has Euclidean symmetry.

Definition 2.14. *A bifurcation problem with symmetry Γ is represented by a map Γ-equivariant map $g : V \times \mathbb{R} \longrightarrow V$, defined in a neighbourhood of the origin $(0,0)$ of the finite dimensional space V, that satisfies $g(0,0) = 0$ and $(dg)_{0,0} = 0$.*

In this vein, we work with the following system of ODEs:

$$\frac{dz}{dt} = g(z, \lambda) \tag{2.4}$$

where $z \in V$ and $\lambda \in \mathbb{R}$ is the bifurcation parameter.

In classical bifurcation theory, we seek to understand when a solution of a system of ODEs can lose stability as a parameter is varied. There are two main behaviours, considered in the literature, used to describe that loss of stability: steady-state and Hopf bifurcation. Some assumptions are necessary to define these two terms.

We assume that $z = 0$ is a trivial solution for all $\lambda \in \mathbb{R}$. We assume also that, this equilibrium $z = 0$ is asymptotically stable for $\lambda < 0$, that is, all eigenvalues of $(dg)_{0,\lambda}$ have negative real part when $\lambda < 0$; and that $z = 0$ loses stability at $\lambda = 0$, that is, some eigenvalue of $(dg)_{0,0}$ lies on the imaginary axis.

Definition 2.15. We refer to:

- *Steady-state bifurcation* when $(dg)_{0,0}$ has only zero eigenvalue and no other imaginary eigenvalue;
- *Hopf bifurcation* when $(dg)_{0,0}$ has simple eigenvalues $\pm\alpha i$ ($\alpha \neq 0$) and no other imaginary eigenvalue.

A useful result of equivariant bifurcation theory, that classifies a class of solutions to generic equations of the form (2.4), is the Equivariant Branching Lemma, proved by [42], [9]. It makes predictions about the symmetry of solutions at steady-state bifurcations, based on the symmetry of the bifurcation problem. Before stating the theorem, let us see some more definitions regarding symmetries.

If z is a solution of equation (2.4) then so is γz for all symmetries, $\gamma \in \Gamma$, because

$$\frac{d(\gamma z)}{dt} = \gamma g(z,\lambda) = g(\gamma z,\lambda).$$

The point γz is said to lie in the *orbit* Γz of $z \in V$ under the action of Γ, defined by

$$\Gamma z = \{\gamma x;\ \gamma \in \Gamma\}.$$

The point z need not be a stationary solution of equation (2.4) in order for its orbit to be defined. However all stationary solutions on the same orbit have the same existence properties.

Differentiating the equivariance condition $g(\gamma z,\lambda) = \gamma g(z,\lambda)$ gives

$$\gamma (dg)_{z,\lambda} = (dg)_{\gamma z,\lambda} \cdot \gamma, \quad \forall \gamma \in \Gamma. \tag{2.5}$$

If v is an eigenvector of $(dg)_{z,\lambda}$ with eigenvalue μ, then using the linearity of the action of Γ on V we have that γv is an eigenvector of $(dg)_{\gamma z,\lambda}$ with eigenvalue μ. Thus the eigenvalues of $(dg)_{z,\lambda}$ and $(dg)_{\gamma z,\lambda}$ are the same. If z and γz are fixed points on the same orbit of Γ they will have the same stability properties as well as the same existence properties, so they are the same type of solutions.

Now, as far the map g is concerned, we cannot distinguish between points which lie in the same orbit. The *isotropy group* of $z \in V$ is

$$\Sigma_z = \{\gamma \in \Gamma;\ \gamma z = z\}.$$

It provides a measure of how much symmetry a solution $z \in V$ has. The calculation of isotropy subgroups is greatly simplified by observing that points on the same orbit have conjugate isotropy subgroups, that is

$$\Sigma_{\gamma z} = \gamma \Sigma_z \gamma^{-1},$$

for all $\gamma \in \Gamma$.

The *fixed-point subspace*, $Fix(\Sigma)$, associated with a subgroup $\Sigma \subseteq \Gamma$ is defined by

$$Fix_V(\Sigma) = Fix(\Sigma) = \{z \in V;\ \sigma z = z \forall \sigma \in \Sigma\}.$$

Fixed-point subspaces are flow-invariant since

$$g(z,\lambda) = g(\sigma z,\lambda) = \sigma g(z,\lambda), \quad \forall z \in Fix(\Sigma).$$

It is easy to see that $Fix(\Gamma)$ is a Γ-invariant subspace of V. Thus, if Γ acts irreducibly then either $Fix(\Gamma) = \{0\}$ or $Fix(\Gamma) = V$. But if $Fix(\Gamma) = V$ then γ acts trivially, so for non-trivial actions, we are left with $Fix(\Gamma) = \{0\}$. Moreover, if the action of Γ on V is absolutely

irreducible, then using (2.5), we have

$$\gamma(dg)_{0,\lambda} = (dg)_{0,\lambda}\gamma.$$

Hence $(dg)_{0,\lambda} = c(\lambda)I_V$, here I_V is the identity matrix on V. Since $(dg)_{0,0} = 0$ it follows that $c(0) = 0$. Assuming generically that $c'(0) \neq 0$, we may now state the fundamental existence and uniqueness result for steady-state bifurcation problems with symmetry.

Theorem 2.16 (Equivariant Branching Lemma: Lemma 1.31 in [20])**.** *Let Γ be a compact Lie group acting on V. Assume*

(a) Γ acts absolutely irreducibly on V;

(b) $\Sigma \subset \Gamma$ is an isotropy subgroup satisfying $dim(Fix(\Sigma)) = 1$,

(c) $g : V \times \mathbb{R} \longrightarrow V$ is a Γ-equivariant bifurcation problem satisfying

$$(dg_\lambda)_{0,0}(v_0) \neq 0$$

where $v_0 \in Fix(\Sigma)$ is non-zero.

Then there exists a unique branch of stady-state solutions $(tv_0, \lambda(t))$, emanating from $(0,0)$ and with symmetry Σ, to the equation $g(v,\lambda) = 0$.

Using the terminology of [21], we call *axial subgroups* the isotropy subgroups, Σ, which satisfy the condition $dim(Fix(\Sigma)) = 1$.

2.7 Equivariant Partial Differential Equations (PDEs)

We study PDEs by reducing the system to a bifurcation problem described by a system of Ordinary Differential Equations (ODEs) which are equivariant under the action of a compact Lie group. When the reduction is complete, the bifurcation problem may be studied using some standard techniques. We discuss this point further.

Consider a parametrized family of PDEs, which has the form

$$\frac{\partial u}{\partial t}((x,y),t) = \mathcal{P}(u((x,y),t),\lambda) \tag{2.6}$$

where $\mathcal{P} : \mathcal{X} \times \mathbb{R} \longrightarrow \mathcal{Y}$ is an operator between suitable function spaces $\mathcal{X}$ and $\mathcal{Y}$, and $\lambda \in \mathbb{R}$ is a bifurcation parameter. The function $u : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}$ in $\mathcal{X}$ is a function of a spatial variable $(x,y) \in \mathbb{R}^n \times \mathbb{R}$ and time t.

We are interested in time-independent solutions of (2.6) given by

$$\mathcal{P}(u(x,y),\lambda) = 0. \tag{2.7}$$

The symmetry of an operator is given by the action of the Euclidean group on the function spaces. More precisely, we say that an operator $\mathcal{P}$ is *Euclidean equivariant* if for all $\gamma \in E(n+1)$

$$\gamma \cdot \mathcal{P}(u,\lambda) = \mathcal{P}(\gamma \cdot u,\lambda),$$

where the action of $E(n+1)$ on the functions u is given by

$$(\gamma \cdot u)((x,y),t) = u(\gamma^{-1}(x,y),t).$$

We assume that there is an Euclidean invariant time-independent solution for (2.6) for all values of λ. Without loss of generality we assume that this spatially uniform solution corresponds to $u = 0$, that is, $\mathcal{P}(0,\lambda) = 0$, for all $\lambda \in \mathbb{R}$.

Furthermore, we assume that this solution is stable for $\lambda < 0$, unstable for $\lambda > 0$ and that $\lambda = 0$ corresponds to a symmetry-breaking bifurcation point. The critical problem when considering symmetry-breaking bifurcations with $E(n+1)$ symmetry is that this group is not compact. This causes great difficulties when attempting to apply the Liapunov-Schmidt reduction, chapter VII in [19], to the PDE due to the presence of infinite dimensional representations. The standard method for overcoming this difficulty is to seek spatially periodic, time-independent solutions to (2.7).

When restricting the solutions of (2.7) to the subspace $\mathcal{X}_{\mathcal{L}}$ of $\mathcal{L}$-periodic functions in $\mathcal{X}$, the group of symmetries, Γ, of the problem we have to consider is a compact group. Specifically, the group Γ is the largest group constructed from $E(n+1)$ that leaves the spaces $\mathcal{X}_{\mathcal{L}}$ invariant. The group Γ can be written as a semi-direct product

$$\Gamma = \mathbb{T}^{n+1} \dot{+} H_{\mathcal{L}},$$

where $H_{\mathcal{L}}$ is the holohedry of the lattice $\mathcal{L}$ and $\mathbb{T}^{n+1} = \mathbb{R}^{n+1}/\mathcal{L}$ is the n-torus of translations modulo the lattice.

Remark 2.17. Observe that we have been using the same notation for the crystallographic group, which describes the symmetries of the patterns we are working with, and for the compact Lie group, $\Gamma = H_{\mathcal{L}} \dot{+} \mathbb{T}^{n+1}$, that is taken into account to describe the symmetries of the bifurcation problem we are going to consider.

Since (2.6) is assumed to undergo a time-independent bifurcation at $\lambda = 0$, we demand that

$$V = Ker(\mathcal{DP})_{(0,0)} \neq \{0\}$$

Moreover, since V is Γ-invariant and Γ is compact, the space V is a finite dimensional subspace of $\mathcal{X}_{\mathcal{L}}$. A discussion about this point can be found in [15].

We assume that all the functions $u : \mathbb{R}^{n+1} \times \mathbb{R} \longrightarrow \mathbb{R}$ in $\mathcal{X}_{\mathcal{L}}$ admit a unique formal Fourier expansion in terms of the *waves*

$$\omega_k(x,y) = \exp(2\pi i \langle k, (x,y) \rangle),$$

where k is a *wave vector* in the dual lattice, $\mathcal{L}^* = \{k \in \mathbb{R}^{n+1};\ \langle k, l_i \rangle \in \mathbb{Z},\ i = 1, \cdots,\ n+1\}$, of $\mathcal{L} = \langle l_1, \cdots, l_{n+1} \rangle_{\mathbb{Z}}$, with *wave number* $|k|$ and $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^{n+1}$. Thus,

$$u((x,y),t) = \sum_{k \in \mathcal{L}^*} C(k,t)\omega_k(x,y).$$

Here $C(k,t)$ is the (time-dependent) amplitude of the k^{th} Fourier mode, for each $k \in \mathcal{L}^*$, and with the constraint $C(-k,t) = \bar{C}(k,t)$, where the bar denotes the complex conjugate as usual.

Thus, the function u can be written in the form:

$$\begin{aligned} u((x,y),t) &= \sum_{k\in\mathcal{L}^*} 2Re(C(k,t)\omega_k(x,y)) \\ &= \sum_{k\in\mathcal{L}^*} 2[a_k(t)Re(\omega_k(x,y)) - b_k(t)Im(\omega_k(x,y))] \end{aligned} \tag{2.8}$$

with $C(k,t) = a_k(t) + ib_k(t)$, for each $k \in \mathcal{L}^*$. Here $a_k,\ b_k \in \mathbb{R}$.

Therefore, we can write

$$\mathcal{X}_\mathcal{L} = \bigoplus_{k\in\mathcal{L}^*} V_k \tag{2.9}$$

for

$$V_k = \langle Re(\omega_k(x,y))\rangle \oplus \langle Im(\omega_k(x,y))\rangle \cong \mathbb{C}.$$

In fact, the spaces V_k are isotypic components of the action of translations on $\mathcal{X}$, see [20] chapter 5.

The kernel of the linearisation of (2.6) at $\lambda = 0$ is a sum of subspaces generated by the waves $\omega_k(x,y)$, where $|k| = k_c$, which are called *critical* or *neutral* modes, see [14]. The dimension of the bifurcation problem depends on the number of vectors $k \in \mathcal{L}^*$ with $|k| = k_c$.

Therefore, we may identify V with

$$V = V_{k_c} = \bigoplus_{|k|=k_c} V_k = \bigoplus_{i=1}^{s} V_{k_s} \cong \mathbb{C}^s,$$

where $2s$ is the number of linearly independent wave vectors. The isomorphism between V and $\mathbb{C}^s$ is defined by

$$v = \sum_{i=1}^{s} z_i w_{k_i} + cc \longmapsto z = (z_1, \ldots, z_s)$$

Where cc denotes the complex conjugate. As a real vector space $dim(V) = 2s$.

An important result regarding $V = Ker(DP)_{(0,0)}$ is the following:

Proposition 2.18. *(Proposition 2.1 in [15]) The space V is Γ-irreducible if and only if the set of 2s dual vectors $\{\pm k_1, \ldots, \pm k_s\}$ is an orbit in $\mathcal{L}^*$ under the action of the holohedry $H_\mathcal{L}$.*

The PDE (2.6), by a Liapunov-Schmidt reduction gives a problem

$$\dot{z} = g(z,\lambda), \quad g : \mathbb{C}^s \times \mathbb{R} \longrightarrow \mathbb{C}^s \tag{2.10}$$

where $g(0,0) = 0$ and the Jacobian matrix at the bifurcation point $(dg)_{0,0}$ is the zero matrix. In the last section we show the standard method used to find solutions for the system (2.10).

Chapter 3

Hexagonal Projected Symmetries

This chapter has appeared as [32].

In the study of pattern formation in symmetric physical systems a 3-dimensional structure in thin domains is often modelled as a 2-dimensional one. We are concerned with functions in $\mathbb{R}^3$ that are invariant under the action of a crystallographic group and the symmetries of their projections into a function defined on a plane. We obtain a list of the crystallographic groups for which the projected functions have a hexagonal lattice of periods. The proof is constructive and the result may be used in the study of observed patterns in thin domains, whose symmetries are not expected in 2-dimensional models, like the black-eye pattern.

3.1 Hexagonal Projected Symmeties

As we saw in the last section, there is a connection between a crystallographic group Γ in dimension $n+1$ and the group of symmetries of the set of projected functions $\Pi_{y_0}(\mathcal{X}_\Gamma)$. In this section we aim to know which crystallographic groups in dimension 3 can yield hexagonal symmetries after projection. In other words, we want to describe how to obtain hexagonal plane patterns by projection.

Given a crystallographic group Γ, with an $(n+1)$-dimensional lattice $\mathcal{L}$, whose holohedry is $H_\mathcal{L}$, we denote by $\Pi_{y_0}(\mathcal{L}) = \widetilde{\mathcal{L}}$ the translation subgroup of the crystallographic group $\Pi_{y_0}(\Gamma) = \widetilde{\Gamma}$ of symmetries of $\Pi_{y_0}(\mathcal{X}_\Gamma)$, whose point group, $\widetilde{J}$, is a subset of the holohedry of $\Pi_{y_0}(\mathcal{L})$. From theorem 2.12 we obtain

Corollary 3.1. *Let $\widetilde{\Gamma}$ be a crystallographic group with lattice $\widetilde{\mathcal{L}} \subset \mathbb{R}^n$. Let $H_{\widetilde{\mathcal{L}}}$ and $H_\mathcal{L}$ be the holohedries of $\widetilde{\mathcal{L}}$ and $\mathcal{L} \subset \mathbb{R}^{n+1}$, respectively. If $\alpha \in H_{\widetilde{\mathcal{L}}}$ lies in the point group of $\widetilde{\Gamma}$ then either $\alpha_+ \in H_\mathcal{L}$ or $\alpha_- \in H_\mathcal{L}$.*

Proof. Since $\alpha \in H_{\widetilde{\mathcal{L}}}$ implies α lies in the point group of $\widetilde{\Gamma}$, then there exists $v \in \mathbb{R}^n$ such that f is (v, α)-invariant for all $f \in \Pi_{y_0}(\mathcal{X}_\Gamma)$. Hence, one of the three conditions of theorem 2.12 holds. Then, depending on whether (I), (II) or (III) is verified, either (w, α_+) or (w, α_-) is in Γ, where $w \in \{(v,0),\ (v,y_0),\ (v,y_1)\}$. By definition of holohedry, we have either $\alpha_+ \in H_\mathcal{L}$ or $\alpha_- \in H_\mathcal{L}$. □

As a converse to corollary 3.1 we have:

Corollary 3.2. *Let Γ be a crystallographic group with an $(n+1)$-dimensional lattice $\mathcal{L}$ and let $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$, with $H_{\widetilde{\mathcal{L}}}$ and $H_{\mathcal{L}}$ the holohedries of $\widetilde{\mathcal{L}} \subset \mathbb{R}^n$ and of $\mathcal{L} \subset \mathbb{R}^{n+1}$, respectively. Suppose either α_+ or α_- is in $H_{\mathcal{L}}$. If one of the following conditions holds*

1. *$\sigma \notin H_{\mathcal{L}}$;*
2. *either $(0, \alpha_+)$ or $(0, \alpha_-)$ is in Γ;*

then, for any $y_0 \in \mathbb{R}$, $\alpha \in H_{\widetilde{\mathcal{L}}}$.

Proof. Consider $v \in \widetilde{\mathcal{L}}$ and suppose condition 1 holds. By theorem 2.12, either $(v, 0) \in \mathcal{L}$ or $(0, y_0)$ and $(v, y_1) \in \mathcal{L}$.

If $(v, 0) \in \mathcal{L}$ then $(\alpha v, 0) \in \mathcal{L}$. Otherwise, $(0, y_0)$ and $(\alpha v, y_2) \in \mathcal{L}$ for $y_2 \in \{-y_1,\ y_1\}$. Applying theorem 2.12 we have $\alpha v \in \widetilde{\mathcal{L}}$ in both cases. Therefore, α is a symmetry of $\widetilde{\mathcal{L}}$.

Suppose now that condition 2 holds. If $(0, \alpha_+) \in \Gamma$ then $(0, \alpha)$ belongs to $\widetilde{\Gamma}$, for all $y_0 \in \mathbb{R}$, by condition I of theorem 2.12. The other possibility is that $(0, \alpha_-) \in \Gamma$. If for $v \in \widetilde{\mathcal{L}}$ either condition I or condition III of theorem 2.12 holds, the proof follows as in the case of condition 1. Suppose then that $((v, y_0), \sigma) \in \Gamma$, then $((v, y_0), \sigma) \cdot (0, \alpha_-) = ((v, y_0), \alpha_+) \in \Gamma$. Therefore, $(v, \alpha) \in \widetilde{\Gamma}$, by theorem 2.12, completing the proof. □

The analysis in [29] chapter 5.1 aims to find *sectional layer groups* and *penetration rod groups*, by a method of scanning a given crystallographic group.

Sectional layer and penetration rod groups are subgroups of a crystallographic group that leave a *crystallographic plane*, defined by two lattice points, and a *crystallographic straight line* invariant, respectively,

When a pattern is projected, it is not immediate that the plane of projection is crystallographic, as it may not contain two lattice points.

The next proposition provides conditions for a suspension of the projected lattice to be rationally compatible with the original lattice, see Definition 2.6.

Proposition 3.3. *Consider a crystallographic group Γ with a lattice $\mathcal{L} \subset \mathbb{R}^{n+1}$ and let $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L}) \subset \mathbb{R}^n$ be the translation subgroup of $\Pi_{y_0}(\Gamma)$ and denote its suspension by $\widetilde{\mathcal{L}}_s \subset \mathbb{R}^{n+1}$.*
If $(0, y_0) \notin \mathcal{L}$, then $\widetilde{\mathcal{L}}_s$ is always rationally compatible with $\mathcal{L}$.
If $(0, y_0) \in \mathcal{L}$, then $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$ if and only if the normal vector $(0, y_0)$ to the projection hyperplane is rational with respect to $\mathcal{L}$.

Note that if $(0, y_0) \in \mathcal{L}$, we are projecting the values of functions on a band of the width of one (or more) cell along a crystallographic direction. Otherwise the projected group is smaller. So, we must use different results from the ITC according to the specific case.

Proof. If $(0, y_0) \notin \mathcal{L}$, then only conditions I or II of remark 2.13 are applicable. Therefore, if $v \in \widetilde{\mathcal{L}}$ then $(2v, 0) \in \mathcal{L}$. Hence, $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$.

If $(0, y_0) \in \mathcal{L}$, then, using remark 2.13, it follows that $v \in \widetilde{\mathcal{L}}$ for all v such that $(v, y_1) \in \mathcal{L}$ for some $y_1 \in \mathbb{R}$.

Suppose first that $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$ and let $(v, y_1) \in \mathcal{L}$. Then $r(v, 0) \in \mathcal{L}$ and hence $(0, ry_1) \in \mathcal{L}$ for some $r \in \mathbb{Z}$. It follows that $y_1 = \frac{p}{q} y_0$ for some $p,\ q$ non-zero integers. Therefore $(0, y_0)$ is rational with respect to $\mathcal{L}$.

Now suppose $(0, y_0)$ is rational with respect to $\mathcal{L}$. We claim that for each $\widetilde{l_j}$ one of the following conditions holds:

(i) $((\widetilde{l_j}, 0), I_{n+1}) \in \Gamma$;

(ii) $((\widetilde{l_j}, y_1), \sigma) \in \Gamma$, for some $y_1 \in \mathbb{R}$;

(iii) $(0, y_0)$, and $(\widetilde{l_j}, \frac{p}{q} y_0) \in \mathcal{L}$, for some $p,\ q$ non-zero integers.

Condition (iii) is a stronger version of condition III in remark 2.13. The other conditions follow from remark 2.13. Any generator $\widetilde{l_j}$ of $\widetilde{\mathcal{L}}$ such that $(\widetilde{l_j}, y) \in \mathcal{L}$ must satisfy either (i) or (iii) above, because $(0, y_0)$ is rational with respect to $\mathcal{L}$. Any other generator of $\widetilde{\mathcal{L}}$ must satisfy (ii), proving our claim.

Conversely, if one of the conditions (i) or (ii) is true, then $(2v, 0) \in \mathcal{L}$, using remark 2.13. If condition (iii) holds for some $j \in \{1, \cdots, n\}$, then $(0, y_0)$, $(\widetilde{l_j}, \frac{p_j}{q_j} y_0) \in \mathcal{L}$, where $p_j,\ q_j$ are non-zero integers. Since $\mathcal{L}$ is a lattice $(q_j \widetilde{l_j}, 0) \in \mathcal{L}$. Therefore $\widetilde{\mathcal{L}_s}$ is rationally compatible with $\mathcal{L}$. □

As an illustration, take $\Gamma = \mathcal{L} = \langle (0,1), (\sqrt{2}, 1/2) \rangle_{\mathbb{Z}}$, for which $\widetilde{\mathcal{L}_s} = \langle (\sqrt{2}, 0) \rangle_{\mathbb{Z}}$ is always rationally compatible with $\mathcal{L}$, independently of y_0. Another example is given by $\Gamma = \mathcal{L} = \langle (0,1), (1, \sqrt{2}) \rangle_{\mathbb{Z}}$. For $y_0 = 1$ we have that $\widetilde{\mathcal{L}_s} = \langle (1,0) \rangle_{\mathbb{Z}}$ is not rationally compatible with $\mathcal{L}$, whereas for $y_0 \notin \mathbb{Z}$, we get $\widetilde{\mathcal{L}_s} = \{(0,0)\}$.

For 3-dimensional lattices, if the generators of $\widetilde{\mathcal{L}}$ are related by an orthogonal transformation, we can remove the condition on $(0, y_0)$ from the statement of proposition 3.3, at the price of having some more complicated conditions. This provides an alternative means of obtaining rational compatibility.

Our starting point is a specific 2-dimensional lattice $\widetilde{\mathcal{L}}$ and we want to characterize the 3-dimensional lattices $\mathcal{L}$ that project onto this. The first step is to establish that $\mathcal{L}$ must have a non-trivial intersection with the plane $X0Y = \{(x, y, 0);\ x,\ y \in \mathbb{R}\}$.

Theorem 3.4. *Let Γ be a crystallographic group with a lattice $\mathcal{L} \subset \mathbb{R}^3$ such that its projection $\widetilde{\Gamma} = \Pi_{y_0}(\Gamma)$ has a plane lattice $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$ generated by two linearly independent vectors*

$$\widetilde{l_1} \text{ and } \widetilde{l_2} = \rho \widetilde{l_1}$$

for ρ in the point group, $\widetilde{J}$, of $\widetilde{\Gamma}$.

Then the suspension $\widetilde{\mathcal{L}_s} \subset \mathbb{R}^3$ is rationally compatible with $\mathcal{L}$ if for each $v \in \{\widetilde{l_1},\ \widetilde{l_2}\}$ one of the following conditions holds:

a. $((v, 0), I_3) \in \Gamma$;

b. $((v, y_1), \sigma) \in \Gamma$, *for some* $y_1 \in \mathbb{R}$;

c. $(v, y_1) \in \mathcal{L}$, *for some* $y_1 \in \mathbb{R}$.

That the conditions are also necessary is immediate from remark 2.13. Note that the statement of theorem 3.4 excludes oblique and primitive rectangular lattices.

Proof. Since $\widetilde{l_2} = \rho\widetilde{l_1}$, it is sufficient to show that one of $r(\widetilde{l_1}, 0)$ or $r(\widetilde{l_2}, 0)$ is in $\mathcal{L}$. To see this, suppose, without loss of generality, that $r(\widetilde{l_1}, 0) \in \mathcal{L}$. Then since $\rho \in \widetilde{J}$, by corollary 3.1, either $\rho_+ \in H_{\mathcal{L}}$ or $\rho_- \in H_{\mathcal{L}}$. As $\rho_+(r\widetilde{l_1}, 0) = \rho_-(r\widetilde{l_1}, 0) = (r\widetilde{l_2}, 0)$, it implies that $r(\widetilde{l_2}, 0) \in \mathcal{L}$ and therefore, $\mathcal{L}$ has a sublattice $\mathcal{L}_r$.

If for some $v \in \{\widetilde{l_1},\ \widetilde{l_2}\}$ one of the conditions a or b is true then, by remark 2.13, $(rv, 0) \in \mathcal{L},$ for $r = 1$ or $r = 2$. Hence, all that remains to prove is the case when $\widetilde{l_1}$ and $\widetilde{l_2}$ only satisfy condition c.

By hypothesis,

$$(\widetilde{l_1}, y_1) \text{ and } (\widetilde{l_2}, y_2) \text{ are in } \mathcal{L}, \text{ for some } y_1,\ y_2\ \in \mathbb{R}. \tag{3.1}$$

This implies that

$$(\widetilde{l_1} + \widetilde{l_2}, y_1 + y_2) \in \mathcal{L}. \tag{3.2}$$

Using (3.1) and corollary 3.1

$$\text{either } \rho_+(\widetilde{l_1}, y_1) = (\widetilde{l_2}, y_1) \in \mathcal{L} \text{ or } \rho_-(\widetilde{l_1}, y_1) = (\widetilde{l_2}, -y_1) \in \mathcal{L}.$$

If $(\widetilde{l_2}, y_1) \in \mathcal{L}$ then

$$(\widetilde{l_2}, y_1) + (\widetilde{l_2}, y_2) = (2\widetilde{l_2}, y_1 + y_2) \in \mathcal{L}.$$

Thus, using (3.2)

$$(\widetilde{l_1} + \widetilde{l_2}, y_1 + y_2) - (2\widetilde{l_2}, y_1 + y_2) = (\widetilde{l_1} - \widetilde{l_2}, 0) \in \mathcal{L}.$$

Since $\{\widetilde{l_1},\ \widetilde{l_2}\}$ is a basis of $\widetilde{\mathcal{L}}$ and $\rho \in \widetilde{J}$ then

$$\rho(\widetilde{l_1} - \widetilde{l_2}) = m\widetilde{l_1} + n\widetilde{l_2},\ m,\ n \in \mathbb{Z}$$

where $m,\ n$ are not both equal to zero. Suppose that $n \neq 0$, then

$$n(\widetilde{l_1} - \widetilde{l_2}, 0),\ (m\widetilde{l_1} + n\widetilde{l_2}, 0) \in \mathcal{L} \tag{3.3}$$

implying that the sum of these last two vectors $((n + m)\widetilde{l_1}, 0) \in \mathcal{L}$. Therefore, if $n \neq -m$, $\mathcal{L}_r$ is a sublattice of $\mathcal{L}$, where $r = m + n \in \mathbb{Z}$. If $n = -m$, we subtract the two expressions in (3.3) to get $(2n\widetilde{l_1}, 0) \in \mathcal{L}$.

If $(\widetilde{l_2}, -y_1) \in \mathcal{L}$ then

$$-(\widetilde{l_2}, -y_1) + (\widetilde{l_2}, y_2) = (0, y_1 + y_2) \in \mathcal{L}.$$

Thus, using (3.2)

$$(\widetilde{l_1} + \widetilde{l_2}, y_1 + y_2) - (0, y_1 + y_2) = (\widetilde{l_1} + \widetilde{l_2}, 0) \in \mathcal{L}.$$

An analogous argument applied to $\rho(\widetilde{l_1} + \widetilde{l_2}, 0)$ finishes the proof. □

By Definition 2.11, we have:

Definition 3.5. We say that the y_0-*projection* of $\mathcal{L}$ into the plane P *is a hexagonal plane lattice* if and only if the lattice $\widetilde{\mathcal{L}}$ admits as its holohedry a group isomorphic to D_6.

Our main result is the following theorem. Note that the hypothesis of having a 3-fold rotation is not restrictive when one is looking for projections yielding a pattern with D_6 symmetry.

Theorem 3.6. *Let $\mathcal{L} \subset \mathbb{R}^3$ be a lattice of a crystallographic group Γ. Suppose that for some y_* the group $\Pi_{y_*}(\Gamma)$ contains a 3-fold rotation. Then for any y_0, the y_0-projection of $\mathcal{L}$ into the plane P is a hexagonal plane lattice if and only if:*

(1) $P \cap \mathcal{L}$ contains at least two elements;

(2) there exists $\beta \in H_{\mathcal{L}}$ such that:

- *β is a 3-fold rotation;*
- *P is β-invariant.*

Proof. Suppose first that $(0,0,0) \in P \cap \mathcal{L}$. To show that conditions (1) and (2) are necessary let us consider, without loss of generality, that $P = X0Y$. Therefore, the conditions hold by theorem 3.4.

To prove that condition (1) and (2) are sufficient consider the 3-fold rotation $\beta \in H_{\mathcal{L}}$. By [30], theorem 2.1 and the proof of the crystallographic restriction theorem, in the same reference, there exists only one subspace of dimension 2 invariant by β. Such a plane is the plane perpendicular to its rotation axis. So, let P be this plane.

Since $P \cap \mathcal{L} \neq \{(0,0,0)\}$, let v be a nonzero element of minimum length in $P \cap \mathcal{L}$ and consider the sublattice $\mathcal{L}' = \langle v, \beta v \rangle_{\mathbb{Z}}$. As β has order 3, the sublatice $\mathcal{L}'$ is a hexagonal plane lattice.

To finish the proof, consider $y_0 \in \mathbb{R}$; we prove that the y_0-projection of $\mathcal{L}$ into the plane P is a hexagonal plane lattice. Let $(0,\alpha) \in \Pi_{y_*}(\Gamma)$, where α is a 3-fold rotation. Then by theorem 2.12, one of the conditions holds:

1 $(0,\alpha_+) \in \Gamma$;

2 $((0,y_*),\alpha_-) \in \Gamma$;

3 $(0,y_*) \in \mathcal{L}$ and either $((0,y_1),\alpha_+)$ or $((0,y_1),\alpha_-)$ is in Γ.

Since the order of α is finite , we have that either $(0,\alpha_+) \in \Gamma$ or $(0,\alpha_-) \in \Gamma$. Then, the result follows by condition 2 of corollary 3.2.

If $(0,0,0) \notin P \cap \mathcal{L}$, note that the proof can be reduced to the previous case by a translation. □

Remark 3.7. Theorem 3.6 shows that a possible way to obtain patterns with hexagonal symmetry, by y_0-projection, is to project the functions $f \in \mathcal{X}_{\mathcal{L}}$ in a plane invariant by the action of some element $\beta \in H_{\mathcal{L}}$ with order three. After finding one of those planes, in order to obtain projections as in Definition 2.10, we only need to change coordinates. The reader can see an example with the bcc lattice in [22].

For certain specific widths of the projection this can be obtained by other means. However, in these cases the symmetry group of the projected functions, for most y_0, has a very small

point group and this is not interesting for the study of bifurcating patterns. More specifically, we are interested in relating hexagonal patterns of different complexity in solutions of the same differential equation with symmetry. As the projection width y_0 varies, one may obtain hexagonal patterns with different symmetry groups, corresponding to different patterns, as illustrated by the figures at the end of this chapter. Bifurcation occurs via symmetry-breaking and hence, more symmetry (a bigger point group) makes the bifurcation problem more interesting.

As a consequence of Theorem 3.6, we are able to list all the Bravais lattices that may be projected to produce a 2-dimensional hexagonal pattern.

Theorem 3.8. *The Bravais lattices that project to a hexagonal plane lattice, under the conditions of theorem 3.6 are:*

1. *Primitive Cubic lattice;*
2. *Body-centred Cubic lattice;*
3. *Face-centred Cubic lattice;*
4. *Hexagonal lattice; and*
5. *Rhombohedral lattice.*

Moreover, up to change of coordinates, for the first three lattices the plane of projection must be parallel to one of the planes in Table **??**. *For the Hexagonal and Rhombohedral lattices the plane of projection must be parallel to the plane* $X0Y$.

Proof. It is immediate from theorem 3.6 that we can exclude the following Bravais lattices: triclinic, monoclinic, orthorhombic and tetragonal, since the holohedries of these lattices do not have elements of order three.

To see if the other Bravais lattices have hexagonal projected symmetries, we need to examine the rotations of order three in their holohedries and see if the plane perpendicular to their rotation axes intersects the lattice.

The group of rotational symmetries of the Cubic lattice (as well as the Body Centred Cubic lattice and the Face Centred Cubic lattice) is isomorphic to S_4, the group of permutation of four elements. So, in the holohedry of the Cubic lattice we only have rotations of order one, two or three. Consider a systems of generators for a representative for the Cubic lattice $\mathcal{L}$, in the standard basis of $\mathbb{R}^3$, given by:

$$(1,0,0),\ (0,1,0),\ (0,0,1)$$

Then, the matrix representations of the rotations of order 3 in $H_{\mathcal{L}}$ are:

$$\gamma_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix},$$

Table 3.1 Two-dimensional spaces perpendicular to the rotation axis of each one of the rotations γ_i. Here we denote by $\langle v \rangle, v \in \mathbb{R}^3$ the subspace generated by v.

Rotation	**Rotation Axis**	**Perpendicular Plane**
γ_1	$\langle (1,1,-1) \rangle$	$P_1 = \{(x,y,z); z = x+y\}$
γ_2	$\langle (1,-1,-1) \rangle$	$P_2 = \{(x,y,z); z = x-y\}$
γ_3	$\langle (1,-1,1) \rangle$	$P_3 = \{(x,y,z); z = -x+y\}$
γ_4	$\langle (1,1,1) \rangle$	$P_4 = \{(x,y,z); z = -(x+y)\}$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Two-dimensional spaces perpendicular to the rotation axis of each one of these rotations are given in Table 3.1.

This means that for the first three lattices in the list, the projection of functions $f \in \mathcal{X}_{\mathcal{L}}$ into a plane have hexagonal symmetries only if the plane is parallel to one of the plane subspaces given in Table 3.1.

Consider now a 3-dimensional Hexagonal lattice. Its group of rotational symmetries has order twelve and it has a subgroup of order six consisting of the rotational symmetries of the Rhombohedral lattice.

Let the representatives for the Hexagonal and Rhombohedral lattices be generated by:

$$(1,0,0), (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0), (0,0,c) \quad c \neq 0, \ \pm 1.$$

$$(1,0,1), (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 1), (-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1)$$

respectively. Then, the twelve rotations in the holohedry of the Hexagonal lattice are generated by:

$$\rho_z = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \gamma_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The generators of the group of rotational symmetries of the Rhombohedral lattice are then ρ_z^2 and γ_x.

We conclude that the only rotations of order 6 in the holohedry of the Hexagonal lattice are ρ_z and ρ_z^5, and of order 3 ρ_z^2 and ρ_z^4.

Therefore, the y_0-projection of the Hexagonal and Rhombohedral lattices is a hexagonal plane sublattice if and only if the y_0-projection is made into a plane parallel to the plane $X0Y$. □

3.2 Hexagonal projected symmetries of the Primitive Cubic lattice

We conclude the chapter with an example to illustrate the hexagonal symmetries obtained by z_0-projection of functions with periods in the Primitive Cubic lattice, for all $z_0 \in \mathbb{R}$.

Consider a 3-dimensional crystallographic group, $\Gamma = \mathcal{L} \dot{+} H_{\mathcal{L}}$, where $\mathcal{L}$ is the Primitive Cubic Lattice generated by the vectors $(1,0,0),(0,1,0)$ and $(0,0,1)$ over $\mathbb{Z}$, and $H_{\mathcal{L}}$ its holohedry.

Without loss of generality, consider the projection of Γ on P_1 (see Table 3.1).

From Theorem 3.8, the Cubic lattice has a hexagonal plane sublattice that intersects P_1. This sublattice is generated by:

$$(0,1,1),\ (1,0,1) \tag{3.4}$$

To make our calculations easier and to set up the hexagonal symmetries in the standard way consider the new basis $\{(0,1,1),(1,0,1),(0,0,1)\}$ for the lattice $\mathcal{L}$. Now multiply $\mathcal{L}$ by the scalar $\frac{1}{\sqrt{2}}$ in order to normalise the vectors of (3.4). With these changes the crystallographic group Γ has the new translational subgroup generated by the vectors:

$$v_1 = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),\quad v_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}),\quad v_3 = (0, 0, \frac{1}{\sqrt{2}})$$

Projection of Γ on P_1, as in definition 2.10, can be done after a change of coordinates that transforms P_1 into $X0Y$. Consider that change given by the orthonormal matrix

$$A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

Then, in the new system of coordinates $X = Ax$, we obtain the base for the Primitive Cubic lattice given by:

$$l_1 = (1,0,0), l_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0), l_3 = (\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{-\sqrt{6}}{6}) \tag{3.5}$$

Observe that we changed the position of $\mathcal{L}$ as prescribed by theorem 3.6.

We proceed to describe the symmetries of the space $\Pi_{z_0}(\mathcal{X}_\Gamma)$, for each $z_0 \in \mathbb{R}$. For this, we need to obtain the subgroups $\widehat{\Gamma}$ and Γ_{z_0} of Γ. Denote by $\Sigma_{z_0} = \mathcal{L}_{z_0} \dot{+} J_{z_0}$ the subgroup of $E(2)$ of all symmetries of $\Pi_{z_0}(\mathcal{X}_\Gamma)$.

It is straightforward to see that the elements of Γ with orthogonal part $\alpha_\pm$, as in (2.3), are in the group

$$\widehat{\Gamma} = \{((v,z),\rho); (v,z) \in \mathcal{L},\quad \rho \in \widehat{J}\}$$

Table 3.2 Projection of $\Gamma = \mathcal{L} \dot{+} H_{\mathcal{L}}$, for each $z_0 \in \mathbb{R}$.

$z_0 \in \mathbb{R}$	Γ_{z_0}	$\Sigma_{z_0} = \mathcal{L}_{z_0} \dot{+} J_{z_0}$
$z_0 = \frac{3n}{\sqrt{6}},\ n \in \mathbb{Z}\backslash\{0\}$		
		$\mathcal{L}_{z_0} = \langle (\frac{1}{2}, \frac{\sqrt{3}}{6}), (\frac{1}{2}, \frac{-\sqrt{3}}{6}) \rangle_{\mathbb{Z}}$
then $(0, 0, \frac{3n}{\sqrt{6}}) \in \mathcal{L}$	$\Gamma_{z_0} = \widehat{\Gamma}$	$J_{z_0} = D_6 = \langle \gamma',\ \kappa' \rangle$
$z_0 = \frac{3n-1}{\sqrt{6}},\ n \in \mathbb{Z}\backslash\{0\}$		
	Γ_{z_0} contains	$\mathcal{L}_{z_0} = \langle (1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2}) \rangle_{\mathbb{Z}}$
then $(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{3n-1}{\sqrt{6}}) \in \mathcal{L}$	$((\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{3n-1}{\sqrt{6}}), \gamma)$	$J_{z_0} = \langle ((\frac{1}{2}, \frac{\sqrt{3}}{6}), \gamma'),\ \kappa' \rangle$
$z_0 = \frac{3n+1}{\sqrt{6}},\ n \in \mathbb{Z}\backslash\{0\}$		
	Γ_{z_0} contains	$\mathcal{L}_{z_0} = \langle (1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2}) \rangle_{\mathbb{Z}}$
then $(\frac{1}{2}, \frac{-\sqrt{3}}{6}, \frac{3n+1}{\sqrt{6}}) \in \mathcal{L}$	$((\frac{1}{2}, \frac{-\sqrt{3}}{6}, \frac{3n+1}{\sqrt{6}}), \gamma)$	$J_{z_0} = \langle ((\frac{1}{2}, \frac{-\sqrt{3}}{6}), \gamma'),\ \kappa' \rangle$
For z_0 different		
		$\mathcal{L}_{z_0} = \langle (1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2}) \rangle_{\mathbb{Z}}$
from the cases before	$\Gamma_{z_0} = H$	$J_{z_0} = \langle -\gamma',\ \kappa' \rangle$

where $\widehat{J}$ is the group generated by

$$\gamma = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \kappa = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the group Γ_{z_0} has a subgroup $H = \overline{\mathcal{L}} \dot{+} \overline{J}$, for all $z_0 \in \mathbb{R}$, where $\overline{\mathcal{L}}$ is the translation subgroup $\overline{\mathcal{L}} = \langle l_1,\ l_2 \rangle_{\mathbb{Z}}$ and $\overline{J}$ is the subgroup generated by $((0,0,0), \kappa)$ and $((0,0,0), -\gamma)$. Using statement I of Theorem 2.12, for all $z_0 \in \mathbb{R}$ all the functions $f \in \Pi_{z_0}(\mathcal{X}_\Gamma)$ are $(1,0)$, and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ periodic and invariant under the action of

$$\kappa' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } -\gamma' = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

In Table 3.2 we list the group Γ_{z_0}, for each $z_0 \in \mathbb{R}$, and describe the respective projected symmetries.

To illustrate our results, we choose planforms obtained in [14] and show its projections in different bands, in the hexagonal symmetry direction.

In [14] it is shown that there exists an isotropy subgroup with a one dimensional fixed-point subspace in a 6-dimensional representation of the action of Γ on the space $\mathcal{X}_{\mathcal{L}}$. As in Section

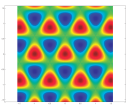
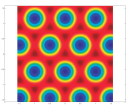
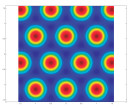
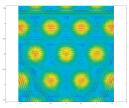
(a)  (b)  (c)  (d) 

Fig. 3.1 Projection of pattern u in a 6-dimensional representation with Primitive Cubic lattice periodicity. Contour plots of the integral of u over different depths z_0. (a) $z_0 = \frac{1}{2\sqrt{6}}$. (b) $z_0 = \frac{1}{\sqrt{6}}$. (c) $z_0 = \frac{2}{\sqrt{6}}$. (d) $z_0 = \frac{3}{\sqrt{6}}$.

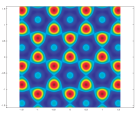
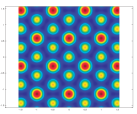
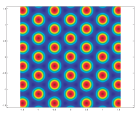
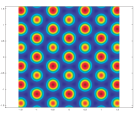
(a)  (b)  (c)  (d) 

Fig. 3.2 Projection of pattern u in a 12-dimensional representation with Body Centred Cubic lattice periodicity. Contour plots of the integral of u over different depths z_0. (a) $z_0 = \frac{1}{2\sqrt{6}}$. (b) $z_0 = \frac{1}{\sqrt{6}}$. (c) $z_0 = \frac{3}{2\sqrt{6}}$. (d) $z_0 = \frac{2}{\sqrt{6}}$.

2.7, the 6-dimensional representation is given by:

$$V = \bigoplus_{|k|=\sqrt{2}} V_k$$

A straightforward calculation shows that the function

$$u(x,y,z) = \sum_{|k|=\sqrt{2}} \exp(2\pi i k \cdot (x,y,z)) \tag{3.6}$$

is Γ-invariant function in V .

The contour plots of the projections of u are shown in Figure 3.1, with the symmetries given in Table 3.2. The same pictures occur for the projection of a strip of half this height of a pattern in an 8-dimensional representation with face centred cubic periodicity.

The Body Centred Cubic lattice shows a different configuration, illustrated in Figure 3.2.

As an illustration, consider a systems of generators

$$l_1 = (1,0,0),\ l_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0),\ l_3 = (0,0,c) \quad c \neq 0,\ \pm 1$$

$$r_1 = (1,0,0),\ r_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0),\ r_3 = (\frac{-1}{2}, \frac{\sqrt{3}}{6}, \frac{a}{3}), \qquad a \neq 0$$

for the Hexagonal and Rhombohedral lattices, respectively. A construction similar to that used for the Primitive Cubic lattice may be applied to these two cases, but here the parameters a and c will change the pattern of the projected functions. Examples are shown in Figures 3.3 and 3.4.

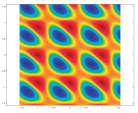
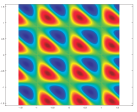
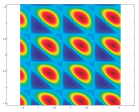
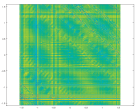
(a)  (b)  (c)  (d) 

Fig. 3.3 Projection of pattern u in a 6-dimensional representation with Rhombohedral lattice periodicity. Contour plots of the integral of u over different depths z_0 with parameter $a = 2$. (a) $z_0 = \frac{2}{6}$. (b) $z_0 = \frac{2}{3}$. (c) $z_0 = \frac{4}{3}$. (d) $z_0 = 2$.

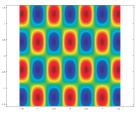
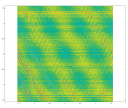
(a)  (b) 

Fig. 3.4 Projection of pattern u in a 12-dimensional representation with Hexagonal lattice periodicity. Contour plots of the integral of u over different depths z_0 with parameter $c = 2$. (a) $z_0 = 1$. (b) $z_0 = 2$.

Chapter 4

Projected Functions and Functions on Projections

When we study physical models in equivariant bifurcation theory, we are interested in results that are based on generic arguments, that is, on typical behaviour that is not destroyed by small perturbations of the equation that governs the system. The reason for considering genericity is simple: physical systems are often subject to perturbations. To proceed with the study of the problem, a first step is to describe the appropriate symmetries of the system. Then, by applying some mathematical tools, we can analyse what generic behaviour the system has.

Many authors have studied how to uncover the appropriate symmetries. In Crawford et al. [12], Golubitsky and Stewart [20], Gomes et al. [23] and references therein, we can see the description of how to obtain the desired symmetries taking into account some more features of the problem, as boundary conditions. Experimental and numerical results are also shown to illustrate the theory. The keyword in these works is *Hidden Symmetries*.

The theory of hidden symmetries is a method to identify all the symmetries in the Euclidean group that leave the PDE and the boundary condition, posed in a certain domain, invariant. It can be found in [23] and Crawford et al. [13] for the case of multidimensional rectangular domains and Field et al. [18] for spherical domains. The general idea of the method is to obtain information by doing an extension of the domain to the whole Euclidean space $\mathbb{R}^n$, by reflections on the boundaries and periodic replication. The result is that the original problem is embedded in a larger abstract one. The extended problem will typically have a larger class of solutions. The way back to the original problem will be by selecting, from this large class, the solutions that satisfy the original boundary conditions.

In Gomes [22] we can see a perspective of hidden symmetries and projections. The author proposes that projection can be developed into a variant of standard hidden symmetries methods as cited before. In this chapter we give a formalism of how we can do such an extension and how we can relate the solution of the extended problem to the original one. By the use of Equivariant Bifurcation Theory, we examine the effect of projection on bifurcation problems.

To formalise these ideas, consider an Equivariant PDE

$$\frac{\partial u}{\partial t}(x,t) = \mathcal{P}(u(x,t),\lambda) \tag{4.1}$$

where $\mathcal{P}:\mathcal{X}\times\mathbb{R}\longrightarrow\mathcal{Y}$ is an operator between suitable function spaces $\mathcal{X}$ and $\mathcal{Y}$, and $\lambda\in\mathbb{R}$ is a bifurcation parameter. The function $u:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ in $\mathcal{X}$ is a function of a spatial variable $x\in\mathbb{R}^n$ and time t. We are interested in time-independent solutions of (4.1) given by $\mathcal{P}(u(x,t),\lambda)=0$.

The standard method used in Equivariant Bifurcation Theory reduces the problem posed before to a steady-states of Γ-equivariant bifurcation problem

$$\dot{z} = g(z,\lambda), \quad g:\mathbb{C}^s\times\mathbb{R}\longrightarrow\mathbb{C}^s \tag{4.2}$$

where Γ is a a Lie group. The first approach to find solutions for $g(z,\lambda)=0$, and then for the problem (4.1), is by applying the Equivariant Branching Lemma.

As we see in Theorem 2.16, we can start looking for irreducible representations given by the action of the group Γ on $\mathcal{X}_{\mathcal{L}}$. After establishing these irreducible representations, we describe Γ-absolutely representations on V, the kernel of the linearisation of (4.1), which we suppose to be a Γ-irreducible subspace of $\mathcal{X}_{\mathcal{L}}$. Then, we reduce the symmetry of the problem by looking for fixed point subspaces of a subgroup Σ of Γ. Fixed point subspaces are flow-invariant, so we can restrict the bifurcation problem to $Fix(\Sigma)$ and look for solutions there.

Now, we have a symmetric n-dimensional model and a tool to find typical solutions for that problem. Suppose then that we can extend that model to a symmetric $(n+1)$-dimensional problem, where in the same way we can apply the previous tool to obtain its typical solutions, then the question must be asked:

> "Is there a relation between the typical solutions of the n-dimensional problem and the ones that come from the projection of the extended one?"

In this chapter we give the first necessary condition to make such a relation. We aim to establish, by using the projection method:

- a relation between irreducible representations posed in different dimensions;
- and a relation between the fixed point subspaces contained there in.

In the next sections we introduce a possible method to understand bifurcation problems from the point of view of projections. Although most of the technical results we prove in this chapter are independent of the dimension, we will be paying particular attention to the extension of 2- to 3-dimension problems.

We organise the chapter as follows. In section 4.1 we formulate how we extend 2- to 3-dimensional bifurcation problems in the point of view of hidden symmetries. We cover questions regarding the boundary conditions imposed by the problem. In particular, this section gives motivation for the application of our theory.

Once we have an idea of how to embed a n- in a $(n+1)$-dimensional problem, we wish to study how we can relate their solutions. To achieve this aim in Sections 4.2 and 4.3 we prove technical results that allow us to compare irreducible representations of the action of crystallographic groups, on the space of periodic functions, posed in different dimensions. We state the final results in Sections 4.4 and 4.5. We show how the projection of irreducible representations changes as the band of projection varies.

4.1 Hidden Symmetry

In this section we approach the theory of projection developed until now from the point of view of hidden symmetries. References for this section are [23] and [20], chapter 7.

To better understand the connection of this section and the remaining content of the chapter, let us come back to the motivation of our work: the black-eye pattern. We see in [22] that the 2-dimensional pattern, observed in the CIMA (chlorite-iodite-malonic-acid) experiment, is described as a slice of a fully 3-dimensional one by projection. The main challenge is to find out if the experimental black-eye pattern and its representation as projection are the same. To interpret this problem we must start looking for the effect of projection in the system we are considering, in this case a reaction-diffusion system.

When the problem is posed in the whole plane, we want to understand how to extend it to a 3-dimensional environment, so that we can obtain information concerning the 2-dimensional problem by the use of the projection method. It turns out that this extension has several points to be clarified.

The problem is posed as follows: we want to extend a hexagonal periodic pattern $\widetilde{u}$, to a 3-dimensional pattern u. The first intuitive question is how the boundary conditions of $\widetilde{u}$ are related to the 3-dimensional pattern. In other words, which boundary conditions should we impose on u such that its projection will satisfy periodic boundary condition (PBC) on the fundamental cell of $\mathcal{L}$? The answers to those questions are in chapter 3, where we relate periodic boundary conditions to symmetries by solving the question: suppose that there exist $u : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $y_0 \in \mathbb{R}$ such that $\Pi_{y_0}(u) = \widetilde{u}$, which are the symmetries of u?

The boundary condition problem is clarified in Theorem 3.8, stating that if $\widetilde{u}$ is invariant under the action of a crystallographic group with a hexagonal lattice and a point group containing a 3-fold rotation, then the possible symmetries of u are given by a 3-dimensional crystallographic group with one of the following lattices: simple cubic, ffc, bcc, hexagonal or rhombohedral.

Our next step is to understand what we can say about u, $\Pi_{y_0}(u)$ and y_0 if $\widetilde{u}$ is a solution for the PBC problem in the plane. In the previous section we saw the first approach to find typical solutions for a bifurcation problem is by finding fixed-point subspaces with dimension one in irreducible representations of the action of the group. The idea is then to look at what happens to the projection of functions in invariant subspaces given by the action of the imposed crystallographic group in higher dimension. We will deal with this second issue in the next section.

4.1.1 Reaction-Diffusion Systems and Boundary Conditions

Reaction-diffusion equations are usually defined on a bounded domain with Neumann or Dirichlet Boundary condition (NBC or DBC, respectively). Such equations have hidden symmetries that affect the form of the bifurcation equation. See [13], [18] and [23].

We can see in [13] and [23] that if u' is a solution, defined in a cell of a multi-rectangular lattice, of the NBC/DBC problem, then we can extend it to a periodic function, u, so that it will be a solution for the PBC problem. In the other way around, if we start with a solution, u, of the PBC problem, then we have to impose some symmetries on u such that its restriction in the initial bounded domain is a solution for the NBC/DBC problem. We explain here how the transition from NBC to PBC goes when the selected solution is defined in a fundamental cell of a plane hexagonal lattice.

4.1.2 Neumann Boundary conditions

Let us start with a hexagonal lattice $\widetilde{\mathcal{L}}$ generated by:

$$\widetilde{l_1} = (1,0), \quad \widetilde{l_2} = (\frac{-1}{2}, \frac{\sqrt{3}}{2})$$

Consider the domain

$$\mathcal{D}' = \{(x,y) \in \mathbb{R}^2;\ (x,y) = m\widetilde{l_1} + n\widetilde{l_2},\ \ m,\ n \in [0,1]\} \tag{4.3}$$

and an operator $\mathcal{P}' : \widetilde{\mathcal{X}} \times \mathbb{R} \to \widetilde{\mathcal{Y}}$ between function spaces $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$. Define a PDE

$$\frac{\partial u}{\partial t} = \mathcal{P}'(u, \lambda) \tag{4.4}$$

where $u : \mathcal{D}' \to \mathbb{R}$ in $\widetilde{\mathcal{X}}$, $\lambda \in \mathbb{R}$ is a bifurcation parameter.

The NBC problem is given by:

$$\begin{cases} \mathcal{P}'(u,\lambda) = 0 \\ \frac{\partial u}{\partial n}(x,y) = \nabla u(x,y) \cdot n = 0, \quad \forall (x,y) \in \partial\mathcal{D}' \ \text{ and } \ \forall n \perp \partial\mathcal{D}' \end{cases} \tag{4.5}$$

where here $\partial\mathcal{D}'$ is the boundary of the domain $\mathcal{D}'$.

In [23] solutions of a PDE with Newman boundary conditions on a rectangle are shown to be in one-to-one correspondence with solutions defined on the whole plane with some symmetries. We want to do a similar construction for the domain D'.

By virtue of the NBC, solutions for the problem (4.5) can be smoothly extended by reflection across the side walls to functions on a larger domain, see [13]. In our case, consider the following reflections

$$\tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

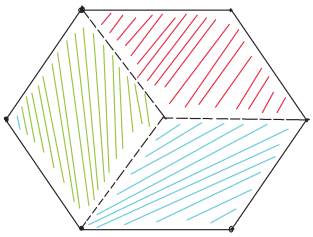


Fig. 4.1 The extended cell D. The red highlighted cell represents the set $\mathcal{D}'$. The blue and green one represent its reflections given by τ_1 and τ_2 respectively.

across the x-axis and the line $y = -\sqrt{3}x$, respectively. We extend a solution u', of the problem (4.5), to the hexagon

$$\mathcal{D} = \{(x,y) \in \mathbb{R}^2;\ (x,y) = m\widetilde{l}_1 + n\widetilde{l}_2,\quad m,\ n \in [-1,1]\}$$

by defining $u(\tau_i(x,y)) = u'(x,y),\ i = 1,2$. That is, by forcing u to be invariant under the two reflections. We proceed by extending the problem to the whole plane by doing periodic replication of the domain $\mathcal{D}$.

Now consider an operator $\widetilde{\mathcal{P}} : \widetilde{\mathcal{X}}_{\widetilde{\mathcal{L}}} \times \mathbb{R} \to \widetilde{\mathcal{Y}}_{\widetilde{\mathcal{L}}}$, where a vector valued function $u \in \widetilde{\mathcal{X}}_{\widetilde{\mathcal{L}}}$ is defined on the whole plane. Here we denote by $\widetilde{\mathcal{X}}_{\widetilde{\mathcal{L}}}$ and $\widetilde{\mathcal{Y}}_{\widetilde{\mathcal{L}}}$ the spaces of $\widetilde{\mathcal{L}}$–periodic functions in $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$, respectively.

Given the periodic boundary value problem:

$$\begin{cases} \widetilde{\mathcal{P}}(u,\lambda) = 0 \\ u((x,y) + \widetilde{l}) = u(x,y) \quad \forall \widetilde{l} \in \widetilde{\mathcal{L}}, \end{cases} \tag{4.6}$$

we have the following:

Lemma 4.1. *Solutions satisfying the Neumann boundary condition problem* (4.5) *on the domain* $\mathcal{D}'$ *are in* 1–1 *correspondence with solutions satisfying the periodic boundary value problem on* $\mathcal{D}$ *having* D_3 *symmetry.*

Proof. Let $u' : \mathcal{D}' \longrightarrow \mathbb{R}$ be a solution for the NBC problem (4.5). Define the extension u of u' to the whole plane by:

$$\begin{cases} u(x,y) \equiv u'(x,y) & \text{if } \ (x,y) \in \mathcal{D}' \\ u(\tau_i(x,y)) = u'(x,y) & \text{for } \ i = 1,2 \ \text{ and } \ \forall (x,y) \in \mathcal{D}' \\ u((x,y) + \widetilde{l}) = u(x,y) & \forall (x,y) \in \mathcal{D} \ \text{ and } \ \widetilde{l} \in \widetilde{\mathcal{L}} \end{cases} \tag{4.7}$$

By construction, u is differentiable along the segments

$$\begin{array}{rccc} \phi_i: & (0,1)\subset\mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ & t & \longmapsto & t\widetilde{l_i} \end{array}, \quad i=1,2$$

Also by construction, u satisfies the PBC problem (4.6) and is invariant under the group D_3 generated by τ_1 and τ_2.

Now let u be a solution to the PBC problem (4.6), we show that if u is D_3 invariant then $u|_{\mathcal{D}'}$ satisfies NBC on $\partial\mathcal{D}'$.

The boundary $\partial\mathcal{D}'$ is the set

$$\partial\mathcal{D}' = d_1 \cup d_2 \cup d_3 \cup d_4$$

for

$$\begin{aligned} d_1 &= \{(x,0)\in\mathbb{R}^2;\ x\in[0,1]\} \\ d_2 &= \{x(\frac{-1}{2},\frac{\sqrt{3}}{2})\in\mathbb{R}^2;\ x\in[0,1]\} \\ d_3 &= \{(x,\frac{\sqrt{3}}{2})\in\mathbb{R}^2;\ x\in[\frac{-1}{2},\frac{1}{2}]\} \\ d_4 &= \{(\frac{-x}{2}+1,\frac{x\sqrt{3}}{2})\in\mathbb{R}^2;\ x\in[0,1]\} \end{aligned}$$

Thus, u satisfies NBC on $\partial\mathcal{D}$ if

$$\nabla u(x,y)\cdot n = 0, \quad \forall (x,y)\in\partial\mathcal{D} \text{ and } \forall n\perp\partial\mathcal{D}.$$

This is equivalent to prove that

$$\begin{cases} \nabla u(x,y)\cdot(0,1) = 0, & \text{for} (x,y)\in d_1\cup d_3; \\ \nabla u(x,y)\cdot(1,\frac{1}{\sqrt{3}}) = 0, & \text{for} (x,y)\in d_2\cup d_4 \end{cases}$$

that is,

$$\begin{cases} \frac{\partial u}{\partial y}(x,y) = 0 & \text{for} (x,y)\in d_1\cup d_3; \\ \frac{\partial u}{\partial y}(x,y) = \sqrt{3}\frac{\partial u}{\partial x}(x,y) & \text{for} (x,y)\in d_2\cup d_4 \end{cases} \tag{4.8}$$

By τ_1-invariance,

$$u(x,y) = u(x,-y) \quad\Rightarrow\quad \frac{\partial u}{\partial y}(x,0) = -\frac{\partial u}{\partial y}(x,0) = 0$$

Moreover, $(x,y)\in d_3 \ \Leftrightarrow\ (x,y) = (\widehat{x},\widehat{y}) + \widetilde{l_2}$, for some $(\widehat{x},\widehat{y})\in d_1$, hence, using the PBC, $u(x,y) = u(\widehat{x},\widehat{y})$ and

$$\frac{\partial u}{\partial y}(x,y) = \frac{\partial u}{\partial y}(\widehat{x},\widehat{y}) = 0$$

proving the first equality in (4.8).

Now, τ_2-invariance gives $u(x,y) = u(\phi(x,y),\psi(x,y))$, where $\phi, \quad \psi : \mathbb{R}^2 \to \mathbb{R}$ are given by

$$\phi(x,y) = -\frac{x}{2} - \frac{y\sqrt{3}}{2} \quad \text{and} \quad \psi(x,y) = -\frac{x\sqrt{3}}{2} + \frac{y}{2}.$$

Note that, the functions ϕ and ψ are related by $\psi = \sqrt{3}\phi$, for $(x,y) \in d_2 \cup d_4$. Then

$$u(\phi,\psi) = u(\phi,\sqrt{3}\phi) \quad \forall (x,y) \in d_2 \cup d_4.$$

Differentiating this last equality with respect to x and y, we obtain

$$\frac{\partial u}{\partial x}(\phi,\psi) = \frac{1}{2}\frac{\partial u}{\partial \phi}(\phi,\psi) - \frac{\sqrt{3}}{2}\frac{\partial u}{\partial \psi}(\phi,\psi)$$
$$\frac{\partial u}{\partial y}(\phi,\psi) = \frac{\sqrt{3}}{2}\frac{\partial u}{\partial \phi}(\phi,\psi) - \frac{3}{2}\frac{\partial u}{\partial \psi}(\phi,\psi).$$

By multiplying the first equation by $\sqrt{3}$ we obtain $\frac{\partial u}{\partial y}(x,y) = \sqrt{3}\frac{\partial u}{\partial x}(x,y)$ for $(x,y) \in d_2 \cup d_4$, finishing the proof. □

4.1.3 Dirichlet Boundary conditions

Consider now the DBC problem given by:

$$\begin{cases} \mathcal{P}'(u,\lambda) = 0 & \\ u \equiv c & \text{on } \partial\mathcal{D}' \end{cases} \tag{4.9}$$

where c is a constant.

We extend a solution u', of the problem (4.9), to the hexagon $\mathcal{D}$, by defining $u(\tau_i(x,y)) = -u'(x,y)$, $i = 1,2$. We proceed by extending the problem to the whole plane by doing periodic replication of the domain $\mathcal{D}$.

Lemma 4.2. *Suppose that the operator in* (4.6) *satisfies* $\widetilde{\mathcal{P}}(-u,\lambda) = -\widetilde{\mathcal{P}}(u,\lambda)$. *Then, solutions satisfying the Dirichlet boundary condition problem* (4.9) *on the domain* $\mathcal{D}'$ *are in* 1–1 *correspondence with solutions satisfying the periodic boundary value problem on* $\mathcal{D}$ *having* D_6 *symmetry.*

Proof. As a contrast of the NBC problem, a solution of the DBC problem (4.9) satisfies the PBC if we require $\widetilde{\mathcal{P}}(-u,\lambda) = -\widetilde{\mathcal{P}}(u,\lambda)$.

The remaining of the prove is similar to the NBC problem. □

Both in the NBC and DBC problem, we have been assuming that all the components of u' have the same boundary condition. As in [18] and [13], similar ideas hold if we have different boundary conditions for diferent components of u'.

4.2 Generators for the space of invariant functions

An interpretation of how to identify a projected pattern as a 2- or 3-dimensional object can be given by understanding when the space $\Pi_{y_0}(\mathcal{X}_\Gamma)$ coincides with $\mathcal{X}_{\Pi_{y_0}(\Gamma)}$.

Consider an $(n+1)$-dimensional crystallographic group Γ with point group J and lattice $\mathcal{L}$ with holohedry $H_\mathcal{L}$. The point group J can be any subgroup of $H_\mathcal{L}$. Let $\mathcal{X}_\Gamma$ be the space of Γ-invariant functions.

Let $P:\mathbb{R}^{n+1}\to\mathbb{R}^n$ be the usual projection given by $P(x,y)=x$. The projection of $f\in\mathcal{X}_\Gamma$ is given by

$$\Pi_{y_0}(f)(x)=\sum_{k_1\in P(\mathcal{L}^*)} D(k_1)\omega_{k_1}(x) \tag{4.10}$$

where

$$D(k_1)=\sum_{k_2;\ (k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy.$$

For each $y_0\in\mathbb{R}$, we write the space of projected functions

$$\Pi_{y_0}(\mathcal{X}_\Gamma)=\{\Pi_{y_0}(f):\mathbb{R}^n\to\mathbb{R};\ f\in\mathcal{X}_\Gamma\}.$$

In contrast, for each $y_0\in\mathbb{R}$, consider the projection of the crystallographic group Γ, $\Pi_{y_0}(\Gamma)=\widetilde{\Gamma}$, with the respective projections $\Pi_{y_0}(\mathcal{L})=\widetilde{\mathcal{L}}$ and $\Pi_{y_0}(J)=\widetilde{J}$, as in chapter 3, section 3.1. If it is important to display the value y_0 explicitly we write $\widetilde{\Gamma}_{y_0}$, $\widetilde{\mathcal{L}}_{y_0}$ and $\widetilde{J}_{y_0}$, respectively. Then, we have the space of $\widetilde{\Gamma}$-invariant functions

$$\mathcal{X}_{\Pi_{y_0}(\Gamma)}=\{\widetilde{f}:\mathbb{R}^n\to\mathbb{R};\ \widetilde{f}\text{ is }\widetilde{\Gamma}-\text{invariant}\}=\mathcal{X}_{\widetilde{\Gamma}}.$$

In chapter 3, section 3.2, we saw an example of the projection, for different bands y_0, of the crystallographic group with simple cubic lattice. In general the projected group changes as the projection height varies. Here we can ask what happens with the space of functions. A natural question is: does the space of projected functions coincide with the space of functions that are invariant under the action of the projected group? One inclusion is clear, that is $\Pi_{y_0}(\mathcal{X}_\Gamma)\subseteq\mathcal{X}_{\Pi_{y_0}(\Gamma)}$. In this section we aim at establishing conditions under which the other one is true.

The results of this chapter are technical but with a deep consequence. We establish, by using the projection model, a relation between representations of the action of the group Γ on the space $\mathcal{X}_\mathcal{L}$ and the action of $\Pi_{y_0}(\Gamma)$ on $\mathcal{X}_{\widetilde{\mathcal{L}}}$. That is, we are looking to explain, via the projection model, three main concepts:

- Γ- and $\widetilde{\Gamma}$-invariance of subspaces in $\mathcal{X}_\mathcal{L}$ and $\mathcal{X}_{\widetilde{\mathcal{L}}}$, respectively;
- Γ- and $\widetilde{\Gamma}$-fixed point subspaces in $\mathcal{X}_\mathcal{L}$ and $\mathcal{X}_{\widetilde{\mathcal{L}}}$, respectively;
- Γ- and $\widetilde{\Gamma}$-irreducible representations in $\mathcal{X}_\mathcal{L}$ and $\mathcal{X}_{\widetilde{\mathcal{L}}}$, respectively.

For Γ- and $\widetilde{\Gamma}$-invariance of subspaces in $\mathcal{X}_\mathcal{L}$ and $\mathcal{X}_{\widetilde{\mathcal{L}}}$, respectively, we have the following:

Lemma 4.3. *Let Γ be a $(n+1)$-dimensional crystallographic group, with projection $\widetilde{\Gamma} = \Pi_{y_0}(\Gamma)$. If V is a Γ-invariant subspace of $\mathcal{X}_{\mathcal{L}}$ then $\widetilde{V} = \Pi_{y_0}(V)$ is a $\widetilde{\Gamma}$-invariant subspace of $\mathcal{X}_{\widetilde{\mathcal{L}}}$.*

Proof. The arguments in the proof are analogous to those in the proof of Lemma 3.1 of [38].

Let V be a Γ-invariant subspace of $\mathcal{X}_{\mathcal{L}}$ and $\widetilde{f} \in \widetilde{V}$. Then, there exists $f \in V$ such that $\Pi_{y_0}(f) = \widetilde{f}$.

For all $(v, \alpha) \in \widetilde{\Gamma}$, there exists an element $((v, y_1), \alpha_{\pm})$, depending on whether conditions $(i) - (iii)$ in Theorem 1.2 in [38] is verified. It follows that $\Pi_{y_0}(((v, y_1), \alpha_{\pm})f) = (v, \alpha)\widetilde{f}$. □

In order to study how invariant, irreducible and fixed point subspaces are transformed by projection, we must pay attention to the system of generators for such spaces. In our work we use the basis I_k, $k \in \mathcal{L}^*$, for the space of functions that are invariant under the action of a crystallographic group Γ. The functions I_k are obtained in the following way:

As in section 2.7, we assume that all the functions $f : \mathbb{R}^{n+1} \to \mathbb{R}$ in $\mathcal{X}_\Gamma$ admit a unique formal Fourier expansion in terms of the waves $\omega_{(k_1,k_2)}(x, y) = exp(2\pi i\langle(k_1, k_2) \cdot (x, y)\rangle)$

$$f(x, y) = \sum_{(k_1,k_2)\in\mathcal{L}^*} C(k_1, k_2)\omega_{(k_1,k_2)}(x, y),$$

where $\mathcal{L}^*$ is the $(n + 1)$-dimensional dual lattice of $\mathcal{L}$. To simplify our notation we write here the coefficients $C(k_1, k_2) := C((k_1, k_2), t)$, since the projection of f does not interfere with the temporal variable t. We write,

$$\mathcal{X}_{\mathcal{L}} = \bigoplus_{k\in\mathcal{L}^*} V_k = \langle Re(\omega_k(x, y))\rangle \oplus \langle Im(\omega_k(x, y))\rangle.$$

Now, given a crystallographic group Γ with point group J and lattice $\mathcal{L}$, we know by definition that for each $\delta \in J$ there exists $v \in \mathbb{R}^{n+1}$, such that the pair $(v, \delta) \in \Gamma$. Since for each $l \in \mathcal{L}$, the element $(l, Id_{n+1})(v, \delta) = (v + l, \delta) \in \Gamma$ and Γ is a discrete group, we can choose the smallest $v_\delta \in \mathbb{R}^{n+1}$ such that $(v_\delta, \delta) \in \Gamma$. We call v_δ the translation vector associated to δ. For more details see [37] chapter 2, section 2.1, or [25].

Now, the action of the element (v_δ, δ) on the wave function $\omega_{(k_1,k_2)}$ yields:

$$(v_\delta, \delta)\omega_{(k_1,k_2)}(x, y) = \omega_{(k_1,k_2)}((v_\delta, \delta)^{-1}(x, y)) = \omega_{\delta(k_1,k_2)}(-v_\delta)\omega_{\delta(k_1,k_2)}(x, y).$$

Therefore, if $f \in \mathcal{X}_\Gamma$, then we can associate the sum of all elements in the orbit of $\omega_{(k_1,k_2)}$ under the action of Γ in its Fourier expansion, that is, we can write $f \in \mathcal{X}_\Gamma$ in terms of the real and imaginary components of the function

$$I_{(k_1,k_2)}(x, y) = \sum_{\delta\in J} \omega_{\delta(k_1,k_2)}(x, y)\omega_{\delta(k_1,k_2)}(-v_\delta).$$

Observe that, the choice of $I_{(k_1,k_2)}$ does not depend on the choice of v_δ, since $\omega_{\delta(k_1,k_2)}(-v_\delta + l) = \omega_{\delta(k_1,k_2)}(-v_\delta)$, for all $l \in \mathcal{L}$.

As we saw in Section 2.7, given a wave $\omega_k \in \mathcal{X}_{\mathcal{L}}$, we can construct the subspace V_{k_c} of $\mathcal{X}_{\mathcal{L}}$ given by:

$$V_{k_c} = \bigoplus_{|k|=k_c} V_k = \bigoplus_{i=1}^{s} V_{k_s}.$$

An adptaion of Proposition 2.18 is in the following:

Proposition 4.4. *The space V_{k_c} is Γ-invariant if and only if the set of $2s$ dual vectors $\{\pm k_1, \ldots, \pm k_s\} \subset \mathcal{L}^*$ is invariant under the action of the point group J of Γ in $\mathcal{L}^*$. Moreover, this holds in particular if $\{\pm k_1, \ldots, \pm k_s\} \subset \mathcal{L}^*$ is a single J-orbit.*

Remark 4.5. As a contrast of Proposition 2.1 in [15], see Remark 2.17, the space V_{k_c} is Γ-invariant, but it is not Γ-irreducible. To see this, consider the subspace of V_{k_c}, generated by the by a function I_k. This subspace is a complex 1-dimensional Γ-invariant subspace of V_{k_c}.

Note that, the space generated by the real or the imaginary part of the function I_k is Γ-irreducible. Moreover, when $\{\pm k_1, \ldots, \pm k_s\} \in \mathcal{L}^*$ is a single J-orbit, we can decompose $V_{k_c} = \langle Re(I_k(x,y))\rangle \oplus \langle Im(I_k(x,y))\rangle \bigoplus W$, where $W \cap \mathcal{X}_\Gamma = \{0\}$.

We proceed now by analysing Γ- and $\widetilde{\Gamma}$-fixed point spaces. As a reminder, we want to show if for all function $\widetilde{f}$ in $\mathcal{X}_{\widetilde{\Gamma}} = Fix_{\mathcal{X}_{\widetilde{\mathcal{L}}}}(\widetilde{\Gamma})$, there is a Γ-invariant function in $f \in \mathcal{X}_\Gamma$, whose projection yields $\widetilde{f}$.

In order to analyse if $\Pi_{y_0}(\mathcal{X}_\Gamma) \supseteq \mathcal{X}_{\Pi_{y_0}(\Gamma)}$, we will work with the functions $I_{(k_1,k_2)}$. As a consequence, it is necessary to obtain more information about:

- the the dual lattice $\widetilde{\mathcal{L}}^*$ and $\mathcal{L}^*$;
- and the orbit of a dual vector $(k_1, k_2) \in \mathcal{L}^*$ under the action of J.

In particular, by Remark 4.5, an approach to analyze the projection of Γ-irreducible representations in $\mathcal{X}_{\mathcal{L}}$ can be done by looking for the functions I_k.

4.2.1 Projection of Dual Lattices

As long as we work with Γ-invariant function, dual lattices play an important rule in determining its Fourier terms. The results presented before and in chapter 3 only give direct information about a crystallographic group Γ, its translation subgroup $\mathcal{L}$, and its projection. In this subsection we will show the relation between $\mathcal{L}^*$ and the dual of the projected lattice, $\widetilde{\mathcal{L}}^*$ for each y_0.

Consider a lattice $\mathcal{L} \subset \mathbb{R}^{n+1}$, with dual lattice $\mathcal{L}^*$, and let $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L}) \subset \mathbb{R}^n$ be its projection with suspension $\widetilde{\mathcal{L}}_s = \{(v,0);\ v \in \widetilde{\mathcal{L}}\} \subset \mathbb{R}^{n+1}$. Denote by $\widetilde{\mathcal{L}}^*$ the dual lattice of $\widetilde{\mathcal{L}}$ with respective suspension $\widetilde{\mathcal{L}}^*_s$.

As mentioned in chapter 3, the projection of a crystallographic group does not always yield a crystallographic group. Proposition 3.6 gives conditions to analyse the effect of projection in the lattice $\mathcal{L}$ of a crystallographic group Γ. The conditions in that proposition are stated in terms of rational compatibility. Here, the importance of that definition is to establish how compatible the dual lattice $\mathcal{L}^*$ and the suspension $\widetilde{\mathcal{L}}^*_s$ are.

Remark 2.13 says that for each $\widetilde{v} \in \widetilde{\mathcal{L}}$, that is, for each symmetry $(\widetilde{v}, Id_n)$ of $\Pi_{y_0}(\mathcal{X}_\Gamma)$, we have, by Theorem 2.12 with $\alpha = Id_n$, that one of the following conditions holds:

I $(\widetilde{v}, 0) \in \mathcal{L}$;

II $((\widetilde{v}, y_0), \sigma) \in \Gamma$;

III $(0, y_0) \in \mathcal{L}$ and either $(\widetilde{v}, y_1) \in \mathcal{L}$ or $((\widetilde{v}, y_1), \sigma) \in \Gamma$, for some $y_1 \in \mathbb{R}$.

A useful and immediate observation from this remark is that, if $\sigma \notin H_{\mathcal{L}}$, then one of the conditions holds:

- $(0, y_0) \in \mathcal{L}$ and $\widetilde{\mathcal{L}} = P(\mathcal{L})$;
- $\widetilde{\mathcal{L}} = P(\mathcal{L} \cap \{y = 0\})$

where, from now on, we denote $\{y = 0\} = \{(x, y) \in \mathbb{R}^{n+1};\ y = 0\}$.

An analogous description, under certain conditions, will also hold when $(0, \sigma) \in \Gamma$, as we will see in the Lemma 4.12.

The first relation between the lattices $\mathcal{L}^*$ and $\widetilde{\mathcal{L}}^*$ is given in the next proposition.

Proposition 4.6. *Let Γ be a crystallographic group with lattice $\mathcal{L}$ and $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. If $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$, then for all $\widetilde{k} \in \widetilde{\mathcal{L}}^*$, there exists $z \in \mathbb{R}$ such that $(\widetilde{k}, z) \in \mathcal{L}^*$, that is, $\widetilde{\mathcal{L}}^* \subseteq P(\mathcal{L}^*)$.*

Proof. By Proposition 2.5, we can choose a system of generators of $\mathcal{L}$ given by:

$$(l_1, 0), \ldots, (l_n, 0), (l, y)$$

Denote by $\bar{\mathcal{L}}$ the n-dimensional lattice generated by $\{l_i\}$, for $i = 1, \ldots, n$, and let $(\bar{\mathcal{L}})^* = \langle \widetilde{k}_i \rangle_{\mathbb{Z}}$ be its dual. Then, the vectors

$$k_i = (\bar{k}_{1i}, \frac{\langle -\bar{k}_{1i}, l \rangle}{y}), \quad i = 1, \ldots, n, \quad \text{and} \quad k_{n+1} = (0, \frac{1}{y})$$

generate $\mathcal{L}^*$.

Since, by Theorem 2.12, if $(v, 0) \in \mathcal{L}$ then $v \in \widetilde{\mathcal{L}}$. This implies $\bar{\mathcal{L}}_s \subseteq \widetilde{\mathcal{L}}_s$, for all y_0. By properties of duality $(\widetilde{\mathcal{L}}_s)^* \subseteq (\bar{\mathcal{L}}_s)^* = \langle (\widetilde{k}_i, 0) \rangle_{\mathbb{Z}} + \langle (0, y) \rangle_{\mathbb{R}}$. Therefore, for all $\widetilde{k} \in \widetilde{\mathcal{L}}^*$, there exists $z \in \mathbb{R}$ such that $(\widetilde{k}, z) \in \mathcal{L}^*$. □

Lemma 4.7. *Let Γ be a crystallographic group with lattice $\mathcal{L}$ and $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. If $(0, y_0) \in \mathcal{L}$ then $\widetilde{\mathcal{L}}^* \subseteq P(\mathcal{L}^* \cap \{y = 0\})$.*

Proof. If $(0, y_0) \in \mathcal{L}$, then $P(\mathcal{L}) \subseteq \widetilde{\mathcal{L}}$. By duality, $\widetilde{\mathcal{L}}^* \subseteq P(\mathcal{L})^* = P(\mathcal{L}^* \cap \{y = 0\})$. □

Lemma 4.8. *Let Γ be a crystallographic group with lattice $\mathcal{L}$ and $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. If $\widetilde{\mathcal{L}} = P(\mathcal{L} \cap \{y = 0\})$ then $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^*)$.*

Proof. By Proposition 4.6, we only need to prove that $P(\mathcal{L}^*) \subseteq \widetilde{\mathcal{L}}^*$.

Wherever $v \in \widetilde{\mathcal{L}}$, then $(v,0) \in \mathcal{L}$. Thus we have that $< k_1, v >=< (k_1,k_2),(v,0) > \in \mathbb{Z}$, for all $(k_1,k_2) \in \mathcal{L}^*$. Hence, $k_1 \in \widetilde{\mathcal{L}}^*$. □

Remark 4.9. Observe that, if $\widetilde{\mathcal{L}} = P(\mathcal{L})$, then for all $(k,0) \in \mathcal{L}^*$ and for each $v \in \widetilde{\mathcal{L}}$ we have $\langle v,k\rangle \in \mathbb{Z}$, and hence $\widetilde{\mathcal{L}}^* \supseteq P(\mathcal{L}^* \cap \{y=0\})$.

When $(0,y_0) \in \mathcal{L}$, then by Remark 2.13, the vector $\widetilde{v} \in \widetilde{\mathcal{L}}$ if and only if either $(v,y_1) \in \mathcal{L}$ or $((v,y_1),\sigma) \in \Gamma$. It is immediate that $\widetilde{\mathcal{L}} \supseteq P(\mathcal{L})$. If the equality holds, that is $\widetilde{\mathcal{L}} = P(\mathcal{L})$, then by Proposition 4.6, we have $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^* \cap \{y=0\})$.

Moreover, as we saw before, we can conclude, by Proposition 4.6 and Lemma 4.8, that for any band of projection and any crystallographic group with lattice $\mathcal{L}$ and point group J, if $\sigma \notin J$ then one of the conditions holds:

- $(0,y_0) \in \mathcal{L}$, $\widetilde{\mathcal{L}} = P(\mathcal{L})$ and $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^* \cap \{y=0\})$;
- $\widetilde{\mathcal{L}} = P(\mathcal{L} \cap \{y=0\})$ and $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^*)$.

Define A_{y_0} to be the affine space $A_{y_0} = \{(x,y_0) \in \mathbb{R}^{n+1}\}$. In the next Lemma we work with the case when the crystallographic group Γ contains the element $(0,\sigma)$.

Lemma 4.10. *Let Γ be a crystallographic group with lattice $\mathcal{L}$ and $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. Suppose that $(0,\sigma) \in \Gamma$, the lattice $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$, and the affine A_{y_0} contains a point of the lattice. Then $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^* \cap \{y=0\})$.*

Proof. Since $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$, there exists a basis for $\mathcal{L}$ given by:

$$(l_1,0),\ldots,(l_n,0),(l,y)$$

where (l,y) is the vector with smallest norm in this direction.

If $l=0$, then $\widetilde{\mathcal{L}}$ does not depend on y_0. In this case A_{y_0} contains a point of the lattice if and only if $y_0 = my$, $m \in \mathbb{Z}$. Otherwise, if $l \neq 0$, then $(0,2y) = (l,y) - \sigma(l,y)$ is the smallest vector in the direction of projection.

Take $(x,y_0) \in A_{y_0} \cap \mathcal{L}$, then $(x,y_0) = m_1(l_1,0) + \cdots + m_n(l_n,0) + m_{n+1}(l,y)$, $m_j \in \mathbb{Z}$. Consequently $y_0 = m_{n+1}y$.

If m_{n+1} is even then $(0,y_0) = (0,m_{n+1}y) \in A_{y_0} \cap \mathcal{L}$. Thus, by Remark 4.9, $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^* \cap \{y=0\})$.

If m_{n+1} is odd, then $(l,m_{n+1}y) = (l,y) + (0,(m_{n+1}-1)y) \in A_{y_0} \cap \mathcal{L}$. Since $(0,\sigma) \in \Gamma$, we have $((l,m_{n+1}y),\sigma) \in \Gamma$, implying that $l \in \widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. Therefore, $\Pi_{y_0}(\mathcal{L}) = \Pi_b(\mathcal{L}) = P(\mathcal{L})$, and the result follows as in Remark 4.9. □

The next theorem summarizes the previous results.

Theorem 4.11. *Let Γ be a crystallographic group with point group J, lattice $\mathcal{L}$ and $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. Suppose that the lattice $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$ and if $\sigma \in J$ then $(0,\sigma) \in \Gamma$. Then we have the following relation between the lattices $\mathcal{L}$, $\mathcal{L}^*$, $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}^*$:*

1. *If $\sigma \notin J$, then one and only one of the conditions holds*

- $(0, y_0) \in \mathcal{L}$, $\widetilde{\mathcal{L}} = P(\mathcal{L})$ *and* $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^* \cap \{y = 0\})$;
- $\widetilde{\mathcal{L}} = P(\mathcal{L} \cap \{y = 0\})$ *and* $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^*)$

2. If $(0, \sigma) \in \Gamma$ *then one and only one of the conditions holds*

- $A_{y_0} \cap \mathcal{L} \neq \emptyset$, $\widetilde{\mathcal{L}} = P(\mathcal{L})$ *and* $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^* \cap \{y = 0\})$;
- $\widetilde{\mathcal{L}} = P(\mathcal{L} \cap \{y = 0\})$ *and* $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^*)$

Proof. The proof follows from the previous results. □

In general, if $(0, y_0) \in \mathcal{L}$, then $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^* \cap \{y = 0\})$. The criterion to find this equality is given in the following lemma. It aims to analyse what happens when the projected lattice contains an element that comes from the translation vector associated to σ.

Lemma 4.12. *Let* Γ *be a crystallographic group with lattice* $\mathcal{L}$ *and point group* J*, with* $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$*. Suppose that* $\widetilde{\mathcal{L}}_s$ *is rationally compatible with* $\mathcal{L}$*. Suppose also that* $(0, y_0) \in \mathcal{L}$ *and* $(v_\sigma, \sigma) \in \Gamma$*. Then,* $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^* \cap \{y = 0\})$ *if and only if* $\langle (k, 0), v_\sigma \rangle \in \mathbb{Z}$*, for each* $(k, 0) \in \mathcal{L}^*$*.*

Proof. By Lemma 4.7, we only need to show $\widetilde{\mathcal{L}}^* \supseteq P(\mathcal{L}^* \cap \{y = 0\})$, that is, $(k, 0) \in \mathcal{L}^*$ implies $k \in \widetilde{\mathcal{L}}^*$.

Since $(0, y_0) \in \mathcal{L}$, then by Theorem 2.12, for each $v \in \widetilde{\mathcal{L}}$ one of the following conditions holds for some $y_1 \in \mathbb{R}$:

(1) $(v, y_1) \in \mathcal{L}$;

(2) $((v, y_1), \sigma) \in \Gamma$.

In case (1), if $(k, 0) \in \mathcal{L}^*$ then $\langle k, v \rangle \in \mathbb{Z}$.

If (2) holds, then

$$(v_\sigma, \sigma) \cdot ((v, y_1), \sigma) = (v_\sigma + (v, -y_1), Id_{n+1}) \in \Gamma.$$

Thus, $v_\sigma + (v, -y_1) \in \mathcal{L}$. Therefore, $\langle (k, 0), (v, -y_1) \rangle \in \mathbb{Z}$ if and only if $\langle (k, 0), v_\sigma \rangle \in \mathbb{Z}$.

Hence, $k \in \widetilde{\mathcal{L}}^*$ if and only if $\langle (k, 0), v_\sigma \rangle \in \mathbb{Z}$. □

Note that it is always true that $2\langle (k, 0), v_\sigma \rangle \in \mathbb{Z}$, because $v_\sigma + \sigma v_\sigma \in \mathcal{L}$ and $\langle (k, 0), v_\sigma + \sigma v_\sigma \rangle = 2\langle (k, 0), v_\sigma \rangle$.

4.2.2 The orbit $J(\widetilde{k}, z)$

In this subsection we show how the orbit $J(\widetilde{k}, z)$ is related to the orbit of $\widetilde{J} \cdot \widetilde{k}$.

We want to decompose $P(J(\widetilde{k}, z))$ into $\widetilde{J}$-orbits. To do this, we decompose J in subsets such that the orbit of $(\widetilde{k}, z)$ under each of these subsets is projected into exactly one $\widetilde{J}$-orbit.

Consider the subset $\widetilde{J}^{\uparrow}$ of J given by

$$\widetilde{J}^{\uparrow} = \{\alpha_+ \in J;\ \alpha \in \widetilde{J}\} \cup \{\alpha_- \in J;\ \alpha \in \widetilde{J}\}. \tag{4.11}$$

Then for every $(\widetilde{k}, z) \in \mathcal{L}^*$ the projection $P(\widetilde{J}^{\uparrow}(\widetilde{k}, z)) \subset \widetilde{J}\widetilde{k}$.

By checking the group axioms, it is easy to see that the set $\widetilde{J}^{\uparrow}$ is a subgroup, but not necessarily a normal subgroup of J. To see this, recall from chapter 3 the subgroup $\widehat{\Gamma}$ of Γ, whose elements are of the form

$$((v,y),\alpha_{\pm})\,;\ \alpha\in O(n),\ (v,y)\in\mathbb{R}^n\times\mathbb{R}. \tag{4.12}$$

As we can see, the set $\widetilde{J}^{\uparrow}$ is a subgroup of the point group $\widehat{J}$ of $\widehat{\Gamma}$. In particular, when the band of projection satisfies $(0,y_0)\in\mathcal{L}$, then $\widetilde{J}^{\uparrow}=\widehat{J}$.

Observe that, for a given $(\widetilde{k},z)\in\mathcal{L}^*$, the set of $\delta\in J$ such that $P(\delta(\widetilde{k},z))\in\widetilde{Jk}$ is in general larger than $\widetilde{J}^{\uparrow}$. It is necessary to introduce some more notation in order to describe the complete set where its projection coincides with $\widetilde{Jk}$.

Given $(\widetilde{k},z)\in\mathcal{L}^*$ and $\alpha\in\widetilde{J}$, denote by $J^{\alpha}_{\widetilde{k}}$ the subset of J given by:

$$J^{\alpha}_{\widetilde{k}}=\{\delta\in J;\ \delta(\widetilde{k},z)|_1=\alpha\widetilde{k}\}.$$

Denote the union of the sets $J^{\alpha}_{\widetilde{k}}$ by

$$S_{\widetilde{k}}=\bigcup_{\alpha\in\widetilde{J}}J^{\alpha}_{\widetilde{k}}.$$

Then the projection of $P(S_{\widetilde{k}}(\widetilde{k},z))\subseteq\widetilde{Jk}$. We will show that $J(\widetilde{k},z)$ is decomposed into orbits given by the $S_{\widetilde{k}}$ sets and that $P(S_{\widetilde{k}}(\widetilde{k},z))=\widetilde{Jk}$.

Let us give a close look to the sets $J^{\alpha}_{\widetilde{k}}$.

Lemma 4.13. *Given $(\widetilde{k},z)\in\mathcal{L}^*$, if $\delta\in J^{Id_n}_{\widetilde{k}}$, then $\delta(\widetilde{k},z)|_2=\pm z$.*

Proof. For $\delta\in J^{Id_n}_{\widetilde{k}}$, $\delta(\widetilde{k},z)=(\widetilde{k},\delta(\widetilde{k},z)|_2)$. Then by orthogonality of J:

$$\|(\widetilde{k},z)\|=\|\delta(\widetilde{k},z)\|=\|(\widetilde{k},\delta(\widetilde{k},z)|_2\|$$

Therefore $|\delta(\widetilde{k},z)|_2|=|\pm z|$. □

Corollary 4.14. *$J^{Id_n}_{\widetilde{k}}$ does not depend of the choice of z.*

Corollary 4.15. *Let $\Sigma_{(\widetilde{k},z)}$ be the isotropy subgroup of $(\widetilde{k},z)$ in J. Then either $J^{Id_n}_{\widetilde{k}}=\Sigma_{(\widetilde{k},z)}$ or it is the disjoint union $\Sigma_{(\widetilde{k},z)}\cup\beta_-\Sigma_{(\widetilde{k},z)}$, for some $\beta_-\in J$, with $\beta\widetilde{k}=\widetilde{k}$.*

Denote the order of $J^{\alpha}_{\widetilde{k}}$ by $|J^{\alpha}_{\widetilde{k}}|$.

Lemma 4.16. *The order $|J^{\alpha}_{\widetilde{k}}|=|J^{Id_n}_{\widetilde{k}}|$, for all α such that α_+ or $\alpha_-\in\widehat{J}$. Moreover, $J^{\alpha}_{\widetilde{k}}=\gamma\cdot J^{Id_n}_{\widetilde{k}}$, for $\gamma=\alpha_+$ or $\gamma=\alpha_-$, when α_+ or $\alpha_-\in\widehat{J}$.*

Proof. To prove our claim, consider the map

$$\begin{aligned}\phi:&J^{Id_n}_{\widetilde{k}}\to J^{\alpha}_{\widetilde{k}}\\ &\delta\mapsto\phi(\delta)=\alpha_{\pm}\delta\end{aligned}$$

depending whether either α_+ or α_- is in J.

We show that ϕ is injective and onto.

If $\phi(\delta_1) = \phi(\delta_2)$, for some $\delta_1,\ \delta_2 \in J^{Id_n}$, then $\alpha_\pm\delta_1 = \alpha_\pm\delta_2$, implying that $\delta_1 = \delta_2$. Thus ϕ is injective.

Now consider $\rho \in J^\alpha_{\widetilde{k}}$, then there exist $\alpha_\pm^{-1}\rho \in J$ and $\widetilde{k}_2$, such that

$$\alpha_\pm^{-1}\rho(\widetilde{k}, z) = \alpha_\pm^{-1}(\alpha\widetilde{k}, \widetilde{k}_2) = (\widetilde{k}, \pm\widetilde{k}_2)$$

that is, $\alpha_\pm^{-1}\rho \in J^{Id_n}_{\widetilde{k}}$ and $\phi(\alpha_\pm^{-1}\rho) = \rho$. □

Corollary 4.17. *The set* $S_{\widetilde{k}} = \bigcup_{\alpha\in\widetilde{J}} J^\alpha_{\widetilde{k}} = \widetilde{J}^\uparrow J^{Id_n}_{\widetilde{k}}$. *Moreover,* $S_{\widetilde{k}}(\widetilde{k}, z) = \widetilde{J}^\uparrow(\widetilde{k}, z)$.

Observe that $S_{\widetilde{k}}$ is a subgroup of J if and only if $\widetilde{J}^\uparrow J^{Id_n}_{\widetilde{k}} = J^{Id_n}_{\widetilde{k}}\widetilde{J}^\uparrow$.

Example 4.18. Consider a 3-dimensional crystallographic group $\Gamma_1 = \mathcal{L}_1 \dot{+} H_{\mathcal{L}_1}$, where $\mathcal{L}_1$ is the simple cubic lattice generated by $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, with dual lattice $\mathcal{L}_1^*$ generated by the same vectors.

It is well known that the holohedry $H_{\mathcal{L}_1}$ has 24 rotations, they are: the identity Id_3, and

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\quad R_x^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\quad R_x^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$R_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\quad R_y^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\quad R_y^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$R_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\quad R_z^2 = -\sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\quad R_z^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_{(1,1,1)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\quad R^2_{(1,1,1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$R_{(1,0,-1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix},\quad R^2_{(1,0,-1)} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$R_{(1,-1,1)} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix},\quad R^2_{(1,-1,1)} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$R_{(1,-1,-1)} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\quad R^2_{(1,-1,-1)} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix},$$

$$R_{(1,1,0)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad R_{(1,-1,0)} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$R_{(1,0,1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad R_{(1,0,-1)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$R_{(0,1,1)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_{(0,1,-1)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

the remaining elements of $H_{\mathcal{L}_1}$ can be obtained by multiplying these matrices by $-Id_3$.

The projection of Γ yields a plane crystallographic group, $\widetilde{\Gamma} = \Pi_{z_0}(\Gamma) = \widetilde{\mathcal{L}} \dot{+} D_4$, for all $z_0 \in \mathbb{R}$, where the lattice $\widetilde{\mathcal{L}}$ is generated by the vectors $(1,0),\ (0,1)$, its dual, $\widetilde{\mathcal{L}}^*$ is generated by the same vectors.

The orbit of an element $(a,b,0) \in \mathcal{L}$ by $J = H_{\mathcal{L}_1}$ is given by:

$$J(a,b,0) = \{(\pm a, \pm b, 0), (\pm b, \pm a, 0)\} \cup \{(\pm a, 0, \pm b), (0, \pm a, \pm b)\} \cup \{(\pm b, 0, \pm a), (0, \pm b, \pm a)\}$$

Then, the elements given by the two first coordinates of the elements in $J(a,b,0)$ are

$$P(J(a,b,0)) = \{(\pm a, \pm b), (\pm b, \pm a)\} \cup \{(\pm a, 0),\ (0, \pm a)\} \cup \{(0, \pm b),\ (\pm b, 0)\}$$

On the other hand, we have that the projection of Γ has point group $\widetilde{J} = D_4$, for all $z_0 \in \mathbb{R}$, where D_4 is the dihedral group of symmetries of the square generated by:

$$\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The orbit $\widetilde{J}(a,b)$ is $\{(\pm a, \pm b),\ (\pm b, \pm a)\}$.

The subgroup $\widehat{J}$ is $\widehat{J} = \{\pm Id_3,\ \pm R_x^2,\ \pm R_y^2,\ \pm R_z,\ \pm R_z^2,\ \pm R_z^3,\ \pm R_{(1,1,0)},\ \pm R_{(1,-1,0)}\}$. As we can see, $\widehat{J} = \widetilde{J}^{\uparrow}$.

Note that, for all $(a,b,0) \in \mathcal{L}^*$, we have $J^{Id_2}_{(a,b)} = \Sigma_{(a,b,0)}$. Moreover, by Lemma 4.16, for each $\alpha \in \widetilde{J}$, the set $J^{\alpha}_{(a,b)} = \alpha J^{I_2}_{(a,b)}$.

In Table 4.1 we describe the subgroups $J^{Id_2}_{(a,b)}$ and the relation between the set $S_{(a,b)}$ and the group J, for all $(a,b) \in \mathcal{L}^*$.

Table 4.1 Relation between the set $S_{(a,b)}$ and the group J for Example 4.18

Relations between the values a **and** b	**Isotropy subgroup of** $(a,b,0)$	**The set** $S_{(a,b)}$
$a \neq 0$ **and** $b = 0$	$J^{Id_2}_{(a,0)} = \{Id_3, \sigma, R_x, R_x^2, R_x^3, -R_x^2, -R_{(0,1,1)}, -R_{(0,1,-1)}\}$	$\lvert S_{(a,b)}\rvert = 32$, then $S_{(a,b)} \not\leq J$
$a = b \neq 0$	$J^{Id_2}_{(a,a)} = \{Id_3, \sigma, R_{(1,1,0)}, -R_{(1,-1,0)}\}$	$S_{(a,a)} = \widehat{J}$
$a \neq 0 \neq b$ **and** $a \neq b$	$J^{I_2}_{(a,b)} = \{I_3, \sigma\}$	$S_{(a,b)} = \widehat{J}$

Example 4.19. Consider now a crystallographic group Γ_2 obtained from Γ_1, given in the previous example, by a change of coordinate given by the matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

After the change, the crystallographic group has a lattice $\mathcal{L}_2 = A\mathcal{L}_1$ given by:

$$\mathcal{L}_2 = \left\langle (1,0,0), (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0), (\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{1}{\sqrt{6}}) \right\rangle_{\mathbb{Z}},$$

with holohedry $H_{\mathcal{L}_2} = AH_{\mathcal{L}_1}A^t$. Here t denotes the transpose of the matrix.

If $(0, y_0) \in \mathcal{L}_2$, then $\widetilde{J}^{\uparrow} = \widehat{J}$, where $\widehat{J}$ is isomorphic to D_6, the dihedral group of symmetries of the hexagon.

Consider the vector $(\widetilde{k}, z) = (2,0,0) \in \mathcal{L}_2^*$. By doing some calculations, we can see that the product $S_{(2,0)} = \widetilde{J}^{\uparrow} J^{Id_n}_{(2,0)}$ is not a subgroup of J, since the element $AR_xA^t \in S_{(2,0)}$ does not have an inverse in $S_{(2,0)}$.

Now that we have established the notation $S_{\widetilde{k}}$, we are ready to state the main result of this subsection:

Theorem 4.20. *Given $\delta \in J$, the $\widetilde{J}$-orbit of $P(\delta(\widetilde{k}, z))$ is $P((S_u\delta)(\widetilde{k}, z))$, where $u = P(\delta(\widetilde{k}, z))$. It follows that $P(J(\widetilde{k}, z))$ is a disjoint union of $\widetilde{J}$-orbits, all of which have the same number of elements, counted with multiplicity.*

Proof. The first statement follows by definition.

To prove the second statement, consider $J = S_{\widetilde{k}} \cup S^c_{\widetilde{k}}$ and $\delta_1 \in S^c_{\widetilde{k}}$, where $S^c_{\widetilde{k}}$ is the complement of $S_{\widetilde{k}}$. Thus, for all $\alpha \in \widetilde{J}$, either $\alpha_+\delta_1$ or $\alpha_-\delta_1$ is in J. We show that $\alpha_\pm\delta_1 \in S^c_{\widetilde{k}}$.

In fact, if $\alpha_\pm\delta_1 \in S_{\widetilde{k}}$, then $\alpha_\pm\delta_1(\widetilde{k}, z) = \beta_\pm(\widetilde{k}, z)$, with $\beta \in \widetilde{J}$, implying that $\delta_1(\widetilde{k}, z) = \alpha_\pm^{-1}\beta_\pm(\widetilde{k}, z)$, which is a contradiction, since $\delta_1 \notin S_{\widetilde{k}}$.

Therefore, by Corollary (4.17), the set $\widetilde{J}^{\uparrow}(\delta(\widetilde{k},z)) = S_{u_1}(u_1,u_2) \subset J(\widetilde{k},z)$, where $(u_1,u_2) = \delta_1(\widetilde{k},z)$. Thus, we can write

$$J(\widetilde{k},z) = S_{\widetilde{k}}(\widetilde{k},z) \cup S_{u_1}\delta_1(\widetilde{k},z) \cup (S_{\widetilde{k}} \cup S_{u_1}\delta_1)^c(\widetilde{k},z).$$

Since J is a finite group, we can repeat the process till we get the expression:

$$J(\widetilde{k},z) = S_{k_1}(\widetilde{k},z) \cup S_{u_1}\delta_1(\widetilde{k},z) \cup \ldots \cup S_{u_r}\delta_r(\widetilde{k},z). \tag{4.13}$$

Then, by corollary (4.17)

$$P(J(\widetilde{k},z)) = \widetilde{J}k_1 \cup \widetilde{J}u_1 \cup \ldots \cup \widetilde{J}u_r. \tag{4.14}$$

By construction the union is disjoint. □

The number of different $\widetilde{J}$-orbits in $P(J(\widetilde{k},z))$ is given in the next lemma. The lemma gives a relation between the number r in the equation(4.14) and the order of the groups we use in this section.

Lemma 4.21. *Let $(\widetilde{k},z) \in \mathcal{L}^*$. Then, the number r in the equation 4.14 is*

$$r = \frac{|J| \cdot |\widetilde{J}^{\uparrow} \cap \Sigma_{(\widetilde{k},z)}|}{|\widetilde{J}^{\uparrow}||\Sigma_{(\widetilde{k},z)}|} - 1.$$

where $\Sigma_{(\widetilde{k},z)} = \{\delta \in J;\ \delta(\widetilde{k},z) = (\widetilde{k},z)\}$ is the isotropy subgroup of $(\widetilde{k},z)$.

Proof. We know that the cardinal number of the orbit of $(\widetilde{k},z)$ by J is the index

$$|J(\widetilde{k},z)| = \frac{|J|}{|\Sigma_{(\widetilde{k},z)}|}. \tag{4.15}$$

By Theorem (4.20),

$$|J(\widetilde{k},z)| = (r+1)|S_{\widetilde{k}}(\widetilde{k},z)|. \tag{4.16}$$

Then combining (4.15) and (4.16), we have

$$\begin{aligned} r+1 &= \frac{|J|}{|S_{\widetilde{k}}(\widetilde{k},z)||\Sigma_{(\widetilde{k},z)}|} \\ &= \frac{|J| \cdot |\widetilde{J}^{\uparrow} \cap \Sigma_{(\widetilde{k},z)}|}{|\widetilde{J}^{\uparrow}||\Sigma_{(\widetilde{k},z)}|} \end{aligned}$$

where we are using the fact that $S_{\widetilde{k}}(\widetilde{k},z) = \widetilde{J}^{\uparrow}(\widetilde{k},z)$ and that the isotropy subgroup of $(\widetilde{k},z)$ in $\widetilde{J}^{\uparrow}$ is $\widetilde{J}^{\uparrow} \cap \Sigma_{(\widetilde{k},z)}$. □

4.3 Projected Functions and Functions of the Projected Group

The aim of this section is to describe conditions under which the two spaces $\Pi_{y_0}(\mathcal{X}_\Gamma)$ and $\mathcal{X}_{\widetilde{\Gamma}}$ coincide.

For this purpose, we intend to obtain conditions under which the projections of the functions $I_{(\widetilde{k},z)}$ generate the space $\mathcal{X}_{\widetilde{\Gamma}}$, when $(\widetilde{k},z)$ varies in $\mathcal{L}^*$.

As denoted in the beginning of section 4.2,

$$I_{(\widetilde{k},z)}(x,y) = \sum_{\delta\in J} \omega_{\delta(\widetilde{k},z)}(x,y)\omega_{\delta(\widetilde{k},z)}(-v_\delta)$$

Thus, the projection of $I_{(\widetilde{k},z)}$ is given by:

$$\Pi_{y_0}(I_{(\widetilde{k},z)})(x) = \sum_{\delta\in J} \omega_{\delta(\widetilde{k},z)|_1}(x)\omega_{\delta(\widetilde{k},z)}(-v_\delta)\int_0^{y_0} \omega_{\delta(\widetilde{k},z)|_2}(y)dy \tag{4.17}$$

A first issue is: is the projection $\Pi_{y_0}(I_{(\widetilde{k},z)}) = \widetilde{I}_{\widetilde{k}}$? To answer this question we have two main steps:

1. rewrite 4.17 as a linear combination of the functions $\widetilde{I}_{\widetilde{k}}$, for $\widetilde{k} \in \widetilde{\mathcal{L}}^*$;
2. understand how the previous linear combination changes when the height y_0 varies.

To analyse 1. before, consider the orbit of $(\widetilde{k},z)$ by J given by $J(\widetilde{k},z) = \{\delta(\widetilde{k},z) = (\widetilde{k}_1,\widetilde{k}_2);\ \delta\in J\}$. Then we can rewrite (4.17) as:

$$\Pi_{y_0}(I_{(\widetilde{k},z)})(x) = \sum_{\widetilde{k}_1\in P(J(\widetilde{k},z))} \omega_{\widetilde{k}_1}(x) \sum_{\substack{\delta\in J^{Id_n}_{\widetilde{k}_1}:\\ \widetilde{k}_2=\delta(\widetilde{k},z)|_2}} D'(\delta,\widetilde{k}_2) \tag{4.18}$$

where

$$D'(\delta,\widetilde{k}_2) = \omega_{\delta(\widetilde{k},z)}(-v_\delta)\int_0^{y_0}\omega_{\widetilde{k}_2}(y)dy.$$

In the next examples we can see when the method of projecting the functions $I_{(\widetilde{k},z)}$ can contribute to get information about the equality of the spaces $\Pi_{y_0}(\mathcal{X}_\Gamma)$ and $\mathcal{X}_{\Pi_{y_0}(\Gamma)}$. The examples will also illustrate the use of the notations considered in this section.

Example 4.22. For the lattice $\mathcal{L}_1$ of Example 4.18, let us consider first the functions I_k, for $k = (\widetilde{k},0) \in \mathcal{L}^*$ and $\widetilde{k} \in \{(a,0),\ (0,a)\}$, $a \neq 0$. Observe that $I_{(a,0,0)} = I_{(0,a,0)}$, since the orbit $J(a,0,0) = J(0,a,0) = \{\delta(a,0,0);\ \delta\in J\} = \{(\pm a,0,0),\ (0,\pm a,0),\ (0,0,\pm a)\}$. The set of elements with 2 first coordinates of the elements in $J(a,0,0)$ is given by $J(a,0,0)|_1 = \{\delta(a,0,0)|_1;\ \delta\in J\} = \{(\pm a,0),\ (0,\pm a),\ (0,0)\}$. Therefore,

$$\begin{aligned} I_{(a,0,0)}(x,y,z) &= \sum_{\delta\in J} w_{\delta(a,0,0)}(x,y,z)w_{\delta(a,0,0)}(-v_\delta) \\ &= \sum_{k\in J(a,0,0)} w_k(x,y,z)\cdot 1 \\ &= 4\left(w_{(a,0,0)} + w_{(0,a,0)} + w_{(0,0,a)} + w_{(-a,0,0)} + w_{(0,-a,0)} + w_{(0,0,-a)}\right)(x,y,z) \end{aligned}$$

where, in the second summation we are using the fact that for all $\delta \in J$, $((0,0,0),\delta) \in \Gamma$. Thus,

$$\begin{aligned}\Pi_{z_0}(I_{(a,0,0)})(x,y) &= 4z_0\left(w_{(a,0)} + w_{(-a,0)} + w_{(0,a)} + w_{(0,-a)}\right)(x,y) + 4\left(\int_0^{z_0} w_a(z)dz + \int_0^{z_0} w_{-a}(z)dz\right)\\ &= a_1\widetilde{I}_{(a,0)}(x,y) + a_2 + \overline{a_2}\end{aligned}$$

where, $a_1 = 4z_0$, $a_2 = 4(\int_0^{z_0} w_a(z)dz + \int_0^{z_0} w_{-a}(z)dz)$ and $\overline{a_2}$ is the complex conjugate of a_2. Therefore, $\widetilde{I}_{(a,0)}(x,y) = \Pi_{z_0}(I_{(a,0,0)} - a_2 - \overline{a_2})(x,y)/a_1$.

Another example is, for $a \neq b$, $a \neq 0$ and $b \neq 0$:

$$\begin{aligned}I_{(a,b,0)} &= w_{(a,0,b)} + w_{(a,0,-b)} + w_{(0,a,b)} + w_{(0,a,-b)}\\ &\quad + w_{(0,-a,-b)} + w_{(0,-a,b)} + w_{(-a,0,b)} + w_{(-a,0,-b)}\\ &\quad + w_{(0,b,a)} + w_{(0,b,-a)} + w_{(b,0,a)} + w_{(b,0,-a)}\\ &\quad + w_{(-b,0,a)} + w_{(-b,0,-a)} + w_{(0,-b,a)} + w_{(0,-b,-a)}\\ &\quad + w_{(a,-b,0)} + w_{(-a,b,0)} + w_{(-b,a,0)} + w_{(-a,-b,0)}\\ &\quad + w_{(b,-a,0)} + w_{(b,a,0)} + w_{(-b,-a,0)} + w_{(a,b,0)} + cc\end{aligned}$$

then the projection is

$$\begin{aligned}\Pi_{z_0}(I_{(a,b,0)}) &= z_0\left(w_{(a,-b)} + w_{(-a,b)} + w_{(-b,a)} + w_{(-a,-b)} + w_{(b,-a)} + w_{(b,a)} + w_{(-b,-a)} + w_{(a,b)}\right)\\ &\quad + \left(w_{(0,b)} + w_{(b,0)} + w_{(-b,0)} + w_{(0,-b)}\right)\left(\int_0^{z_0}\exp(2\pi az)dz + \int_0^{z_0}\exp(-2\pi az)dz\right)\\ &\quad + \left(w_{(a,0)} + w_{(0,a)} + w_{(0,-a)} + w_{(-a,0)}\right)\left(\int_0^{z_0}\exp(2\pi bz)dz + \int_0^{z_0}\exp(-2\pi bz)dz\right)\\ &= c_1\widetilde{I}_{(a,0)} + c_2\widetilde{I}_{(0,b)} + z_0\widetilde{I}_{(a,b)}\end{aligned}$$

where

$$c_1 = \int_0^{z_0}\exp(2\pi bz)dz + \int_0^{z_0}\exp(-2\pi bz)dz$$

and

$$c_2 = \int_0^{z_0}\exp(2\pi az)dz + \int_0^{z_0}\exp(-2\pi az)dz$$

Therefore,

$$\begin{aligned}\widetilde{I}_{(a,b)} &= \Pi_{z_0}(I_{(a,b,0)}) - c_1\widetilde{I}_{(a,0)} - c_2\widetilde{I}_{(0,b)}\\ &= \Pi_{z_0}(I_{(a,b,0)}) - \frac{c_1}{a_1}\Pi_{z_0}(I_{(a,0,0)} + a_2 + \overline{a_2}) - \frac{c_2}{b_1}\Pi_{z_0}(I_{(0,b,0)} + b_2 + \overline{b_2})\\ &= \Pi_{z_0}[I_{(a,b,0)} - \frac{c_1}{a_1}(I_{(a,0,0)} + a_2 + \overline{a_2}) - \frac{c_2}{b_1}(I_{(0,b,0)} + b_2 + \overline{b_2}).]\end{aligned}$$

This completes the proof that for a 3-dimensional crystallographic group $\Gamma = \mathcal{L}\dot{+}H_{\mathcal{L}}$, where $\mathcal{L}$ is the simple cubic lattice, we have $\Pi_{y_0}(\mathcal{X}_\Gamma) = \mathcal{X}_{\widetilde{\Gamma}}$.

The next example shows that we cannot always obtain a direct correspondence between the functions $I_{(\widetilde{k},z)}$ and $\widetilde{I}_{\widetilde{k}}$.

Example 4.23. Consider the plane crystallographic group $\Gamma = \mathcal{L}\dot{+}H_{\mathcal{L}}$, with elements (v,δ), for $v \in \mathcal{L}$ and $\delta \in H_{\mathcal{L}}$, where

$$\mathcal{L} = \langle (\frac{1}{2}, \frac{\sqrt{3}}{6}),\ (\frac{1}{2}, \frac{-\sqrt{3}}{6}) \rangle_{\mathbb{Z}}, \quad \text{with dual} \quad \mathcal{L}^* = \langle (1,\sqrt{3}),\ (1,-\sqrt{3}) \rangle_{\mathbb{Z}}$$

and

$$H_{\mathcal{L}} = D_6 = \langle \gamma = \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},\ \ \kappa = -\sigma \rangle = \{\pm Id_2,\ \pm\gamma,\ \pm\gamma^2,\ \pm\sigma,\ \pm\gamma\sigma,\ \pm\gamma^2\sigma\}$$

In contrast with the previous example, projections of the group Γ yield different groups for different values of y_0. In fact, for $y_0 = \frac{m\sqrt{3}}{6}$, $m \in \mathbb{Z}$, the projected group is $\widetilde{\Gamma}_1 = \Pi_{y_0}(\Gamma) = \langle \frac{1}{2} \rangle_{\mathbb{Z}} \dot{+} \mathbb{Z}_2$, otherwise it is $\widetilde{\Gamma}_2 = \Pi_{y_0}(\Gamma) = \langle 1 \rangle_{\mathbb{Z}} \dot{+} \mathbb{Z}_2$. The dual lattices of the lattices of the groups $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ are $\langle 2 \rangle_{\mathbb{Z}}$ and $\langle 1 \rangle_{\mathbb{Z}}$, respectively.

Consider the vector $(\widetilde{k}, z) = (1,\sqrt{3}) \in \mathcal{L}^*$. We will calculate the projection of $I_{(1,\sqrt{3})}$.

By definition of the functions I_k, we have that $I_k = I_{\delta k}$, for all $\delta \in J$, the point group of Γ.

The orbit of $(1,\sqrt{3})$ by J is given by:

$$J(1,\sqrt{3}) = \{(\pm 1, \pm\sqrt{3}),\ (\pm 2, 0)\}$$

Thus,

$$\begin{aligned} I_{(1,\sqrt{3})}(x,y) &= \sum_{\delta\in J} w_{\delta(1,\sqrt{3})}(x,y) \\ &= 2\left(w_{(1,\sqrt{3})} + w_{(1,-\sqrt{3})} + w_{(-1,\sqrt{3})} + w_{(-1,-\sqrt{3})}\right)(x,y) \\ &+ 2\left(w_{(2,0)} + w_{(-2,0)}\right)(x,y) \end{aligned}$$

with projection

$$\begin{aligned} \Pi_{y_0}(I_{(1,\sqrt{3})})(x) = 2\left(w_{(1)}(x) + w_{(-1)}(x)\right)\left(\int_0^{y_0} w_{(\sqrt{3})}(y)dy + \int_0^{y_0} w_{(-\sqrt{3})}(y)dy\right) \\ + 2y_0\left(w_{(2)}(x) + w_{(-2)}(x)\right). \end{aligned}$$

If either $y_0 = \frac{m\sqrt{3}}{3}$, that is $(0,y_0) \in \mathcal{L}$, or $y_0 = \frac{m\sqrt{3}}{6}$, for $m \in \mathbb{Z}$, then

$$\int_0^{y_0} w_{(\sqrt{3})}(y)dy + \int_0^{y_0} w_{(-\sqrt{3})}(y)dy) = 0$$

and therefore, $\Pi_{y_0}(I_{(1,\sqrt{3})})(x) = \Pi_{y_0}(I_{(2,0)})(x) = 2y_0\widetilde{I}_{(2)}(x)$. Therefore, for $y_0 = \frac{m\sqrt{3}}{6}$, $m \in \mathbb{Z}$, we have $\Pi_{y_0}(\mathcal{X}_\Gamma) = \mathcal{X}_{\widetilde{\Gamma}}$.

If $y_0 \neq \frac{m\sqrt{3}}{6}$, for $m \in \mathbb{Z}$, then the projection of $I_{(1,\sqrt{3})}$ corresponds to the superposition of the functions $\widetilde{I}_{(1)}$ and $\widetilde{I}_{(2)}$. However, we can prove the equality of the spaces $\Pi_{y_0}(\mathcal{X}_\Gamma)$ and $\mathcal{X}_{\widetilde{\Gamma}}$ by constructing a Hilbert basis for $\mathcal{X}_{\widetilde{\Gamma}}$ by the projection of the functions I_k. To see this, observe that, like in the previous case, we have the orbit of the vectors $n(1,\sqrt{3})$, for $n \in \mathbb{Z}$,

by J given by:

$$J(n, n\sqrt{3}) = \{(\pm n, \pm n\sqrt{3}), (\pm 2n, 0)\}.$$

Thus,

$$\begin{aligned} I_{(n,n\sqrt{3})}(x,y) &= \sum_{\delta \in J} w_{\delta(n,n\sqrt{3})}(x,y) \\ &= 2\left(w_{(n,n\sqrt{3})} + w_{(n,-n\sqrt{3})} + w_{(-n,n\sqrt{3})} + w_{(-n,-n\sqrt{3})}\right)(x,y) \cdot \\ &+ 2\left(w_{(2n,0)} + w_{(-2n,0)}\right)(x,y) \end{aligned}$$

Therefore,

$$\begin{aligned} \Pi_{y_0}(I_{(n,n\sqrt{3})})(x) &= 2\left(w_{(n)}(x) + w_{(-n)}(x)\right)\left(\int_0^{y_0} w_{(n\sqrt{3})}(y) + w_{(-n\sqrt{3})}(y)dy\right) \\ &+ 2y_0\left(w_{(2n)}(x) + w_{(-2n)}(x)\right) \\ &= c_1\widetilde{I}_n + c_2\widetilde{I}_{2n} \end{aligned}$$

where $c_1 = \int_0^{y_0} w_{(n\sqrt{3})}(y) + w_{(-n\sqrt{3})}(y)dy \neq 0$ and $c_2 = 2y_0$. Then the set

$$\{c_1\widetilde{I}_n + c_2\widetilde{I}_{2n}\}_{n\in\mathbb{Z}}$$

forms a Hilbert basis for $\mathcal{X}_{\widetilde{\Gamma}}$. Thus we have shown that for $y_0 \neq \frac{m\sqrt{3}}{6}$, $m \in \mathbb{Z}$, we have $\Pi_{y_0}(\mathcal{X}_\Gamma) = \mathcal{X}_{\widetilde{\Gamma}}$.

4.3.1 Projection of the functions $I_{(k_1,k_2)}$

We are now able to prove the main result of this chapter to establish a correspondence between the functions $I_{(\widetilde{k},z)}$ and its projection.

From (4.18), we have

$$\Pi_{y_0}(I_{(\widetilde{k},z)})(x) = \sum_{\widetilde{k}_1 \in P(J(\widetilde{k},z))} \omega_{\widetilde{k}_1}(x) \sum_{\substack{\delta \in J_{\widetilde{k}_1}^{Id_n}: \\ \widetilde{k}_2 = \delta(\widetilde{k},z)|_2}} D'(\delta, \widetilde{k}_2)$$

where

$$D'(\delta, \widetilde{k}_2) = \omega_{\delta(\widetilde{k},z)}(-v_\delta)\int_0^{y_0} \omega_{\widetilde{k}_2}(y)dy.$$

Using Theorem 4.20, equation (4.14), we can rewrite (4.18) as:

$$\begin{aligned}\Pi_{y_0}(I_{(\widetilde{k},z)})(x) = &\sum_{\widetilde{k}_1\in\widetilde{J}\widetilde{k}} \omega_{\widetilde{k}_1}(x) \sum_{\substack{\delta\in J^{Id_n}_{\widetilde{k}_1}:\\ \widetilde{k}_2=\delta(\widetilde{k},z)|_2}} D'(\delta,\widetilde{k}_2)\\ &+\sum_{\widetilde{k}_1\in\widetilde{J}u_1} \omega_{\widetilde{k}_1}(x) \sum_{\substack{\delta\in J^{Id_n}_{\widetilde{k}_1}:\\ \widetilde{k}_2=\delta(\widetilde{k},z)|_2}} D'(\delta,\widetilde{k}_2)\\ &\vdots\\ &+\sum_{\widetilde{k}_1\in\widetilde{J}u_r} \omega_{\widetilde{k}_1}(x) \sum_{\substack{\delta\in J^{Id_n}_{\widetilde{k}_1}:\\ \widetilde{k}_2=\delta(\widetilde{k},z)|_2}} D'(\delta,\widetilde{k}_2)\end{aligned} \tag{4.19}$$

Now observe that, for $u_0 = \widetilde{k}$ and for $i = 0,\dots,r$, given $\widetilde{k}_1 \in \widetilde{J}u_i$, there exists $\alpha \in \widetilde{J}$ such that:

$$\sum_{\substack{\delta\in J^{Id_n}_{\widetilde{k}_1}:\\ \widetilde{k}_2=\delta(\widetilde{k},z)|_2}} D'(\delta,\widetilde{k}_2) = \sum_{\delta\in J^{\alpha}_{u_i}} D'(\delta,\widetilde{k}_2).$$

Lemma 4.24. *Let $\widetilde{\Gamma} = \Pi_{y_0}(\Gamma)$. Then, for $i = 0,\dots,r$ and $\beta \in \widetilde{J}$*

$$\sum_{\delta\in J^{\beta}_{u_i}} D'(\delta,\widetilde{k}_2) = \omega_{\beta u_i}(-v_\beta) \sum_{\delta\in J^{Id_n}_{u_i}} D'(\delta,\widetilde{k}_2),$$

for all $(v_\beta,\beta) \in \widetilde{\Gamma}$.

Proof. Observe that in (4.19), each term

$$\sum_{\widetilde{k}_1\in\widetilde{J}u_i} \omega_{\widetilde{k}_1}(x) \sum_{\substack{\delta\in J^{Id_n}_{\widetilde{k}_1}:\\ \widetilde{k}_2=\delta(\widetilde{k},z)|_2}} D'(\delta,\widetilde{k}_2)$$

is $\widetilde{\Gamma}$-invariant. Thus, we have

$$\begin{aligned}\sum_{\alpha u_i\in\widetilde{J}u_i} \omega_{\alpha u_i}(x) \sum_{\delta\in J^{\alpha}_{u_i}} D'(\delta,\widetilde{k}_2) &= (v_\beta,\beta) \sum_{\alpha u_i\in\widetilde{J}u_i} \omega_{\alpha u_i}(x) \sum_{\delta\in J^{\alpha}_{u_i}} D'(\delta,\widetilde{k}_2)\\ &= \sum_{\alpha u_i\in\widetilde{J}u_i} \omega_{\alpha u_i}(\beta^{-1}x-\beta^{-1}v_\beta) \sum_{\delta\in J^{\alpha}_{u_i}} D'(\delta,\widetilde{k}_2)\\ &= \sum_{\alpha u_i\in\widetilde{J}u_i} \omega_{\beta\alpha u_i}(x)\omega_{\beta\alpha u_i}(-v_\beta) \sum_{\delta\in J^{\alpha}_{u_i}} D'(\delta,\widetilde{k}_2).\end{aligned}$$

Comparing the right and the left side in the sum before, the term containing β in the left hand side is

$$\omega_{\beta u_i}(x) \sum_{\delta\in J^{\beta}_{u_i}} D'(\delta,\widetilde{k}_2).$$

On the last row of the right hand side terms containing β appear when $\alpha = Id_n$. Hence,

$$\sum_{\delta \in J^{\alpha}_{u_i}} D'(\delta, \widetilde{k}_2) = \omega_{\beta u_i}(-v_\beta) \sum_{\delta \in J^{Id_n}_{u_i}} D'(\delta, \widetilde{k}_2).$$

□

Denote the terms in (4.19) by:

$$s_i = \sum_{\alpha u_i \in \widetilde{J} u_i} w_{\alpha u_i}(x) \sum_{\delta \in J^{\alpha}_{u_i}} D'(\delta, \widetilde{k}_2)$$

for $i = 0, \ldots, r$, where $u_0 = \widetilde{k}$.

Lemma 4.25. *Let $\widetilde{\Gamma} = \Pi_{y_0}(\Gamma)$ and $(\widetilde{k}, z) \in \mathcal{L}^*$, such that $\widetilde{k} \in \widetilde{\mathcal{L}}^*$. The projection $\Pi_{y_0}(I_{(\widetilde{k},z)})$ is a linear combination of functions $\widetilde{I}_{\widetilde{k}} \in \mathcal{X}_{\widetilde{\Gamma}}$. Specifically, $\Pi_{y_0}(I_{(\widetilde{k},z)})$ is the sum of the terms $s_0 = c_0 \widetilde{I}_{\widetilde{k}}$ and $s_j = c_j \widetilde{I}_{u_j}$. Moreover, if $z = 0$ then the constant $c_0 \neq 0$.*

Proof. By Lemma 4.24, we have:

$$\begin{aligned} s_i &= \sum_{\alpha u_i \in \widetilde{J} u_i} w_{\alpha u_i}(x) \sum_{\delta \in J^{\alpha}_{u_i}} D'(\delta, \widetilde{k}_2) \\ &= \sum_{\alpha u_i \in \widetilde{J} u_i} w_{\alpha u_i}(x) w_{\alpha u_i}(-v_\alpha) \sum_{\delta \in J^{Id_n}_{u_i}} D'(\delta, \widetilde{k}_2). \end{aligned}$$

Now, recall from (4.13) that the u_i's in equation 4.19 are given by the projection $P(\delta_i(\widetilde{k}, z))$. Let us write $\delta_i(\widetilde{k}, z) = (u_i, u_{i2})$. Then $\delta(u_i, u_{i2})|_2 = \pm u_{i2}$, for all $\delta \in J^{Id_n}_{u_i}$. Thus,

$$c_i = \sum_{\delta \in J^{Id_n}_{u_i}} D'(\delta, \widetilde{k}_2) \quad = \sum_{\delta \in J^{Id_n}_{u_i}} D'(\delta, \delta(u_i, u_{i2})|_2)$$

is a constant. Therefore, $s_0 = c_0 \widetilde{I}_{\widetilde{k}}(x)$ and $s_j = c_j \widetilde{I}_{u_j}(x)$.

Now, observe that

$$c_0 = \begin{cases} |\, J^{Id_n}_{\widetilde{k}} \,| \int_0^{y_0} w_z(y) dy & \text{if } J^{Id_n}_{\widetilde{k}} = \Sigma_{(\widetilde{k},z)}; \\ \frac{|J^{Id_n}_{\widetilde{k}}|}{2} [\int_0^{y_0} w_z(y) dy + \int_0^{y_0} w_{-z}(y) dy] & \text{otherwise.} \end{cases} \tag{4.20}$$

□

Note that if $(\widetilde{k}, 0) \in \mathcal{L}^*$, then $c_0 \neq 0$ by Lemma 4.25. This holds in particular if $(0, y_0) \in \mathcal{L}$ (Lemma 4.7), or if $(0, \sigma) \in \Gamma$ and $(v, y_0) \in \mathcal{L}$ (Lemma 4.10).

It remains to find out if $c_0 \neq 0$ when this fails either because there does not exist an element of the form $(v, y_0) \in \mathcal{L}$, or because both $(0, y_0) \notin \mathcal{L}$ and $(0, \sigma) \notin \Gamma$.

Lemma 4.26. *Let $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$, suppose that $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$ and that a n-dimensional lattice in $\mathcal{L}^* \cap \{y = 0\}$ is rationally compatible with $\mathcal{L}^*$. Suppose there does not exist $m \in \mathbb{Z} \setminus \{0, \pm 1\}$, such that $(0, my_0) \in \mathcal{L}$. Then, given $\widetilde{k} \in \widetilde{\mathcal{L}}^*$ there exists $z \in \mathbb{R}$ such that $(\widetilde{k}, z) \in \mathcal{L}^*$ and the constant c_0 in the projection $\Pi_{y_0}(I_{(\widetilde{k},z)})$ is non-zero.*

Proof. Using the hypotheses of rational compatibility and Proposition 2.5, there exists a basis for $\mathcal{L}^*$ given by the vectors:

$$(k_1, 0), \ldots, (k_n, 0), (k_{n+1}, a) \tag{4.21}$$

Consider also the systems of generators for $\mathcal{L}$ given by:

$$(l_1, 0), \ldots, (l_n, 0), (l_{n+1}, b). \tag{4.22}$$

Now, given $(\widetilde{k}, z) \in \mathcal{L}^*$, we have $c_0 = 0$ if and only if $2zy_0 \in \mathbb{Z} \setminus \{0\}$, by the formula (4.20).

Next we show that if $c_0 = 0$, then there exists $m \in \mathbb{Z} \setminus \{0, \pm 1\}$, such that $(0, my_0) \in \mathcal{L}$.

To see this, since $\mathcal{L}^*$ is generated by (4.21), there exist an $m \in \mathbb{N} \setminus \{0\}$ where $z = ma$. Now, $c_0 = 0$, if and only if, $2zy_0 \in \mathbb{Z} \setminus \{0\}$. Or equivalently,

$$2zy_0 = 2may_0 \in \mathbb{Z} \setminus \{0\}.$$

Then, $(0, 2my_0) \in \mathcal{L}$, because $\langle (0, 2my_0), k \rangle \in \mathbb{Z}$, for all $k \in \mathcal{L}^*$. □

If $\mathcal{L}$ has generators of the form (4.22) and if $(0, d) \in \mathcal{L}$, $d > 0$, then $d = nb$ for some $n \in \mathbb{Z} \setminus \{0\}$. We may take $r > 0$ in $\mathbb{Z}$ minimal such that $(0, rb) \in \mathcal{L}$. If $\mathcal{L}^*$ has generators of the form (4.21), then by Lemma 2.7, the positive integer r is also minimal such that $(0, ra) \in \mathcal{L}^*$, and it is also the minimal such that $rab \in \mathbb{Z} \setminus \{0\}$.

Lemma 4.27. *Let $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$, suppose that $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$ and that a n-dimensional lattice in $\mathcal{L}^* \cap \{y = 0\}$ is rationally compatible with $\mathcal{L}^*$. Suppose also that there exists $(0, d) \in \mathcal{L}$, $d > 0$. If for some $\widetilde{k} \in \widetilde{\mathcal{L}}^*$, with $(\widetilde{k}, z) \in \mathcal{L}^*$ we have the constant $c_0 = 0$, then there exists $m \in \mathbb{Z} \setminus \{0\}$, such that $2zy_0 = mrab \in \mathbb{Z} \setminus \{0\}$.*

Proof. Since r is the least positive integer such that $(0, rb)$ be in $\mathcal{L}$, then for each $(\widetilde{k}, z) \in \mathcal{L}^*$ we can write

$$z = sa + rna \tag{4.23}$$

where s, $n \in \mathbb{Z}$ and $0 \leqslant s < r$.

If $z = rna$, then $(0, z) \in \mathcal{L}^*$, then $(\widetilde{k}, 0) \in \mathcal{L}^*$. Thus, the projection $\Pi_{y_0}(I_{(\widetilde{k},0)}) = c_0 \widetilde{I}_{\widetilde{k}} + \sum_{i=1}^{r} c_i \widetilde{I}_{u_i}$ yields $c_0 \neq 0$.

If $z = sa + rna$, with $s > 0$, then $(\widetilde{k}, z) - (0, rna) = (\widetilde{k}, sa) \in \mathcal{L}^*$.

Thus, without loss of generality we consider $(\widetilde{k}, z) \in \mathcal{L}^*$ with $z = sa$, for some $s \in \mathbb{Z}$, $0 < s < r$.

If $c_0 = 0$, then by formula (4.20) we have $2zy_0 \in \mathbb{Z} \setminus \{0\}$, and hence for $z = sa$ we get $2say_0 \in \mathbb{Z} \setminus \{0\}$. Then $(0, 2sy_0) \in \mathcal{L}$. Therefore $2sy_0 = mrb$, for some $m \in \mathbb{Z} \setminus \{0\}$, or $2y_0 = \frac{mbr}{s}$. Then $2zy_0 = \frac{2sambr}{s} = mrab \in \mathbb{Z} \setminus \{0\}$. □

Lemma 4.28. *Let $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$, suppose that $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$ and that a n-dimensional lattice in $\mathcal{L}^* \cap \{y = 0\}$ is rationally compatible with $\mathcal{L}^*$ and that there exists $(0, d) \in \mathcal{L}$, $d > 0$. Suppose also that $y_0 = mb$ and for some $\widetilde{k} \in \widetilde{\mathcal{L}}$ with $(\widetilde{k}, z) \in \mathcal{L}^*$, we have the constant $c_0 = 0$ and that $z = sa$, for some $s \in \mathbb{Z}$, $0 < s < r$. Then $2ms$ is divisible by r.*

Proof. By Lemma 4.7, we can suppose m is not divisible by r, otherwise $(0, y_0) \in \mathcal{L}$.

Writing $2ms = n_1 r + n_2$, $0 \leqslant n_2 < r$ we obtain

$$2zy_0 = n_1 rab + n_2 ab \in \mathbb{Z},$$

hence $n_2 ab \in \mathbb{Z}$. Since $n_2 ab = \langle (0, n_2 b), (k_{n+1}, a) \rangle \in \mathbb{Z}$. This contradicts the minimality of $(0, rb)$ unless $n_2 = 0$. □

Lemma 4.29. *Let $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$, suppose that $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$ and that a n-dimensional lattice in $\mathcal{L}^* \cap \{y = 0\}$ is rationally compatible with $\mathcal{L}^*$ and that there exists $(0, d) \in \mathcal{L}$, $d > 0$. Suppose also that $y_0 = \frac{mb}{s}$ and for some $\widetilde{k} \in \widetilde{\mathcal{L}}$ with $(\widetilde{k}, z) \in \mathcal{L}^*$, we have the constant $c_0 = 0$ and that $z = sa$, for some $s \in \mathbb{Z}$, $0 < s < r$. Then $2m$ is divisible by r. Moreover, if $\frac{2mrab}{s} \notin \mathbb{Z}$, then there exists $z' \in \mathbb{R}$ such that $(\widetilde{k}, z') \in \mathcal{L}^*$ and for this element of $\mathcal{L}^*$ the constant c_0 is not zero.*

Proof. Let $y_0 = \frac{mb}{s}$, with $m/s \notin \mathbb{Z}$, $m \in \mathbb{Z}$ and $1 < s < r$. Writing $2m = s_1 r + s_2$, $0 \leqslant s_2 < r$, we obtain $2zy_0 = s_1 rab + s_2 ab \in \mathbb{Z}$, hence $s_2 ab \in \mathbb{Z}$. It is a contradiction of the minimality of $rab \in \mathbb{Z}$, unless $s_2 = 0$.

If $s_2 = 0$, then $y_0 = \frac{s_1 rb}{2s}$. Write,

$$(\widetilde{k}, z') = (\widetilde{k}, z) + (0, ra) \in \mathcal{L}^*,$$

Then $2y_0 z' = 2y_0 z + \frac{s_1 rrab}{s} = 2y_0 z + \frac{2mrab}{s}$. Thus z' is as desired. □

We finish this section with some examples to illustrate the previous results.

Example 4.30. Consider the crystallographic group Γ_2, given in Example 4.19, whose lattice forms the simple cubic lattice positioned as

$$\mathcal{L}_2 = \left\langle (1,0,0), (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0), (\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{1}{\sqrt{6}}) \right\rangle_{\mathbb{Z}}.$$

Let $\widetilde{k} \in \widetilde{\mathcal{L}}^*$. We will show that there exists $z \in \mathbb{R}$, such that $(\widetilde{k}, z) \in \mathcal{L}^*$ and the constant c_0 in the projection $\Pi_{z_0}(I_{(\widetilde{k},z)})$ is non-zero, for all $z_0 \in \mathbb{R}$.

We have that, the values $a = \frac{2}{\sqrt{6}}$, $b = \frac{1}{\sqrt{6}}$ and the minimum positive integer r, such that $rab \in \mathbb{Z} \setminus \{0\}$ is $r = 3$. Let $\widetilde{k} \in \widetilde{\mathcal{L}}^*$, where $z = sa$, for some $0 < s < 3$.

Note that σ is not an element of the holohedry of $\mathcal{L}_2$ and the elements $(0, 0, z_0) = (0, 0, \frac{3m}{\sqrt{6}}) \in \mathcal{L}_2$. We need to show that the constant c_0 is non-zero, for $z_0 \neq \frac{3m}{\sqrt{6}}$. For this purpose, we just need to check the condition when $z_0 = mb$, where m is not divisible by 3, and when $z_0 = \frac{mb}{s}$, for $s \in \{1, 2\}$, by Lemma 4.28 and 4.29 respectively.

Let $(\widetilde{k}, z) \in \mathcal{L}^*$. If $z = a$ we can obtain a contradiction, since $(0, z_0) \notin \mathcal{L}$. If $z = 2a$, then since $(\widetilde{k}, -a) = (\widetilde{k}, 2a) - (0, 3a)$, we can choose $z' = -a$, so that the projection of $I_{(\widetilde{k},-a)}$ yields a non-zero constant c_0.

4.4 Equality of the spaces $\Pi_{y_0}(\mathcal{X}_\mathcal{L})$ and $\mathcal{X}_{\widetilde{\mathcal{L}}}$

The results presented before are intended to give tools to work with crystallographic groups with arbitrary point group. A particular case is when the point group is just the identity. In this case, the functions $I_{(\widetilde{k},z)}$ are just the wavefunctions $\omega_{(\widetilde{k},z)}$, for all $(\widetilde{k},z) \in \mathcal{L}^*$.

Theorem 4.31. *Let Γ be a crystallographic group with point group J, lattice $\mathcal{L}$ and $\widetilde{\mathcal{L}} = \Pi_{y_0}(\mathcal{L})$. Suppose that the lattice $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$ and if $\sigma \in J$ then $(0,\sigma) \in \Gamma$. Suppose also that given $\widetilde{k} \in \widetilde{\mathcal{L}}^*$ there exists $z \in \mathbb{R}$ such that $(\widetilde{k},z) \in \mathcal{L}^*$ and $zy_0 \notin \mathbb{Z} \setminus \{0\}$. Then $\Pi_{y_0}(\mathcal{X}_\mathcal{L}) = \mathcal{X}_{\widetilde{\mathcal{L}}}$.*

Proof. Any function in $\mathcal{X}_{\widetilde{\mathcal{L}}}$ is generated by the wavefunctions $\omega_{\widetilde{k}}$.

By Proposition 4.6, for all $\widetilde{k} \in \widetilde{\mathcal{L}}^*$ there exist $z \in \mathbb{R}$, such that $(\widetilde{k},z) \in \mathcal{L}^*$. Then,

$$\Pi_{y_0}(\omega_{(\widetilde{k},z)}) = \int_0^{y_0} w_z(y)dy\omega_{\widetilde{k}} = c_0\omega_{\widetilde{k}}.$$

By hypothesis $zy_0 \notin \mathbb{Z} \setminus \{0\}$. Thus we can chose $z \in \mathbb{R}$ such that the constant c_0 is non-zero. □

The decomposition of the projection of Γ-invariant subspace of $\mathcal{X}_\mathcal{L}$ is given in the following:

Lemma 4.32. *Let Γ be a $(n+1)$-dimensional crystallographic group, with projection $\widetilde{\Gamma} = \Pi_{y_0}(\Gamma)$. If V is a Γ-invariant subspace of $\mathcal{X}_\mathcal{L}$, then $\Pi_{y_0}(V)$ is a sum of $\widetilde{\Gamma}$-invariant subspaces of $\mathcal{X}_{\widetilde{\mathcal{L}}}$, that is, $\Pi_{y_0}(V)$ is decomposed in terms of*

$$\bigoplus_{i=0}^{r}\Big(\bigoplus_{k\in\widetilde{J}k_i} V_k\Big).$$

Moreover, the number r is given by:

$$r \;= \frac{|J|\cdot|\widetilde{J}^{\uparrow}\cap\Sigma_{(\widetilde{k},z)}|}{|\widetilde{J}^{\uparrow}||\Sigma_{(\widetilde{k},z)}|} - 1.$$

where $(\widetilde{k},z)$ defines the orbit of the wave vectors of the Γ-irreducible subspace V and $\Sigma_{(\widetilde{k},z)} = \{\delta \in J;\ \delta(\widetilde{k},z) = (\widetilde{k},z)\}$ is its isotropy subgroup.

Proof. Let V be a Γ-invariant subspace of $\mathcal{X}_\mathcal{L}$. By Proposition 4.4, we can decompose the space V in terms of

$$\bigoplus_{k\in J(\widetilde{k},z)} V_k \tag{4.24}$$

where J is the point group of Γ and $(\widetilde{k},z) \in \mathcal{L}^*$. Note that here a priori we do not require $\widetilde{k} \in \widetilde{\mathcal{L}}^*$.

Note that, since V is Γ-invariant, by Lemma 4.3 the projection $\widetilde{V} = \Pi_{y_0}(V)$ is a $\widetilde{\Gamma}$-invariant set of $\mathcal{X}_\mathcal{L}$. We will show how the subspace given in (4.24) is decomposed after projection.

We have two cases:

1. either for all $(k_1,k_2) \in J(\widetilde{k},z)$, the product $2k_2y_0 \in \mathbb{Z} \setminus \{0\}$;

2. or there exists at least one element $(k_1, k_2) \in J(\widetilde{k}, z)$, the product $2k_2y_0 \notin \mathbb{Z} \setminus \{0\}$. Without loss of generality, suppose that $(k_1, k_2) = (\widetilde{k}, z)$.

If 1. holds, this means that for all wavefunctions $\omega_{(k_1,k_2)} \in V$, the projection $\Pi_{y_0}(\omega_{(k_1,k_2)}) = 0$. Therefore, $\widetilde{V} = \Pi_{y_0}(V) = \{0\}$.

Suppose that 2. holds. Then, for each $f \in V$, written as $f(x,y) = \sum_{k \in J(k,z)} C(k)\omega_k(x,y)$, the projection of f has terms given by wavefunctions ω_{k_1}, where $k_1 \in P(J(k,z)) = \bigcup_{i=0}^{r} \widetilde{J}u_i$ and $u_0 = k$. Therefore, $\widetilde{V}$ decomposes in terms of

$$\Pi_{y_0}(V) = \bigoplus_{i=0}^{r} \Big(\bigoplus_{k \in \widetilde{J}k_i} V_k \Big).$$

The remaining of the proof follows by Lemma 4.21. □

4.5 Equality of the spaces $\Pi_{y_0}(\mathcal{X}_\Gamma)$ and $\mathcal{X}_{\widetilde{\Gamma}}$

The idea to prove the equality between the spaces $\Pi_{y_0}(\mathcal{X}_\Gamma)$ and $\mathcal{X}_{\widetilde{\Gamma}}$, is to show that there exists a Hilbert basis

$$\mathcal{B} = \Pi_{y_0}(\{\phi_i\}_{i \in \mathbb{N}}) \tag{4.25}$$

for $\mathcal{X}_{\widetilde{\Gamma}}$, for $\{\phi_i\}_{i \in \mathbb{N}} \subseteq \mathcal{X}_\Gamma$.

To find the set $\{\phi_i\}_{i \in \mathbb{N}}$, we will still use the functions I_k. To construct $\mathcal{B}$ we have to consider an "order" in $\widetilde{\mathcal{L}}^*$ and $\mathcal{L}^*$.

The possible lengths of dual wave vectors in $\widetilde{\mathcal{L}}^*$ form a sequence

$$0 < p_0 < p_1 < \cdots, \tag{4.26}$$

as we can see in the Theorem 5.6 in [20]. Then, define

$$\widetilde{E}_{p_i} = \{x \in \mathcal{L}^*;\ \|x\| = p_i\},$$

see Figure 4.2.

Now, every $\widetilde{k}$ defines a number p_i. It can be useful to use the following notation $\widetilde{E}_{\widetilde{k}}$ instead of $\widetilde{E}_{p_i}$.

The set $\widetilde{E}_{p_i}$ is finite and $\widetilde{J}$- invariant. Thus we can write:

$$\widetilde{E}_{p_i} = \widetilde{J}\widetilde{k}_{i0} \cup \widetilde{J}\widetilde{k}_{i1} \cup \ldots \cup \widetilde{J}\widetilde{k}_{is}$$

where the $\widetilde{k}_{ij}$'s are vectors in different orbits given by the action of $\widetilde{J}$ on $\widetilde{E}_{p_i}$. Analogously, we can define E_{p_i} in $\mathcal{L}^*$.

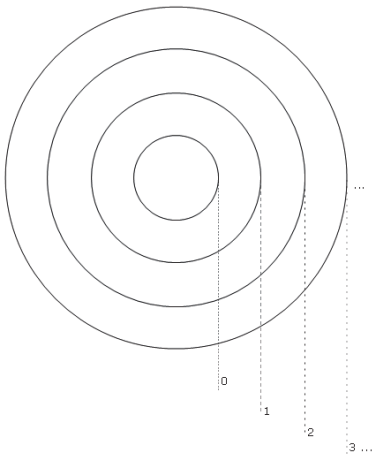


Fig. 4.2 Possible lengths of dual wave vectors in $\widetilde{\mathcal{L}}^*$. The circles represents the set of points E_{p_i}.

Comparing $\widetilde{E}_{\widetilde{k}}$ and orbits of $(\widetilde{k}, z)$ by J

Let $(\widetilde{k}, z) \in \mathcal{L}^*$, such that $\widetilde{k} \in \widetilde{\mathcal{L}}^*$. By the formulas (4.13) and (4.14), we have:

$$J(\widetilde{k}, z) = S_{k_1}(\widetilde{k}, z) \cup S_{u_1}\delta_1(\widetilde{k}, z) \cup \ldots \cup S_{u_r}\delta_r(\widetilde{k}, z)$$

and

$$P(J(\widetilde{k}, z)) = \widetilde{J}k_1 \cup \widetilde{J}u_1 \cup \ldots \cup \widetilde{J}u_r.$$

Denote by $\delta_i(\widetilde{k}, z) = (u_i, z_i)$, for $i = 1, \ldots, r$. Without loss of generality, using (4.26), we can assume

$$\|\widetilde{k}\| \leqslant \|u_1\| \leqslant \ldots \leqslant \|u_r\|. \tag{4.27}$$

Hence,

$$|z| \geqslant |z_1| \geqslant \ldots \geqslant |z_r|. \tag{4.28}$$

As we can see in (4.27), the projection of $P(J(\widetilde{k}, z))$ may intersect more then one set $\widetilde{E}_{p_i}$. Moreover, its also may have elements in different orbits given by $\widetilde{J}$ in $\widetilde{E}_{\widetilde{k}}$.

We will see that, it is useful to know when the inequalities in (4.27) are strict.

Remark 4.33. Let Γ be a crystallographic group with point group J and projection $\widetilde{\Gamma} = \Pi_{y_0}(\Gamma)$. The set $P(J(\widetilde{k}, z)) \cap \widetilde{E}_{\widetilde{k}} = \widetilde{J}\widetilde{k}$ if and only if all the inequalities in (4.27) and (4.28) are strict.

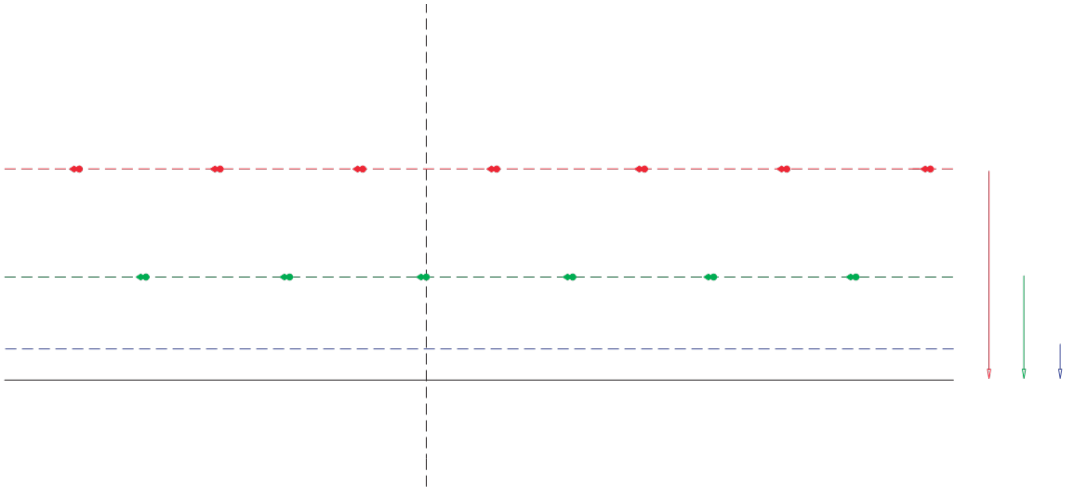


Fig. 4.3 Cases to analyse the projection of the spaces: (Red dashed line) $(v, y_0) \in \mathcal{L}$; (Green dashed line) $(0, y_0) \in \mathcal{L}$; (Blue dashed line) there does not exist $(v, y_0) \in \mathcal{L}$.

Now, we aim to understand how the previous linear combination changes when the height y_0 varies. The results that follows are separated in the following cases:

I Proposition 4.34, if $(0, y_0) \in \mathcal{L}$;

II Proposition 4.35, if $(0, \sigma) \in \Gamma$ and there exists $(v, y_0) \in \mathcal{L}$;

III Proposition 4.36, if either there does not exist $(v, y_0) \in \mathcal{L}$ or there exists $(v, y_0) \in \mathcal{L}$, but $(0, \sigma) \notin \Gamma$.

4.5.1 Case 1. The band of projection intersects a point $(0, y_0) \in \mathcal{L}$

Proposition 4.34. *Suppose that $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$. If $(0, y_0) \in \mathcal{L}$ then $\Pi_{y_0}(\mathcal{X}_\Gamma) = \mathcal{X}_{\widetilde{\Gamma}}$ if and only if for each $\widetilde{k} \in \widetilde{\mathcal{L}}^*$ we have $P(J(\widetilde{k}, z)) \cap \widetilde{E}_{\widetilde{k}} = \widetilde{Jk}$.*

Proof. To prove that $\Pi_{y_0}(\mathcal{X}_\Gamma) = \mathcal{X}_{\widetilde{\Gamma}}$, we show that $\Pi_{y_0}(I_{(\widetilde{k},0)}) = c\widetilde{I}_{\widetilde{k}}$, where c is a non-zero constant, for all $\widetilde{k} \in \widetilde{\mathcal{L}}^*$.

By Lemma 4.7, $\widetilde{\mathcal{L}}^* \subseteq P(\mathcal{L}^* \cap \{y = 0\})$. Consider $(\widetilde{k}, 0) \in \mathcal{L}^*$, for $\widetilde{k} \in \widetilde{\mathcal{L}}^*$. By Lemma 4.13, if $\delta \in J^\alpha_{\widetilde{k}}$ then $\delta(\widetilde{k}, 0) = (\alpha\widetilde{k}, 0)$.

By Lemma (4.25),

$$\Pi_{y_0}(I_{(\widetilde{k},0)})(x) = c_0\widetilde{I}_{\widetilde{k}} + \sum_{i=1}^{r} c_i\widetilde{I}_{u_i}.$$

Since $(0, y_0) \in \mathcal{L}$ the constant $c_0 \neq 0$. Moreover, we have that $\langle (0, y_0), (\widetilde{k}_1, \widetilde{k}_2) \rangle \in \mathbb{Z}$, that is, $y_0\widetilde{k}_2 \in \mathbb{Z}$, for all $(\widetilde{k}_1, \widetilde{k}_2) \in \mathcal{L}^*$. Then, if $\widetilde{k}_2 \neq 0$

$$\begin{aligned} D'(\delta, \widetilde{k}_2) &= w_{\delta(\widetilde{k},0)}(-v_\delta) \int_0^{y_0} w_{\widetilde{k}_2}(y)dy \\ &= w_{\delta(\widetilde{k},0)}(-v_\delta)(e^{2\pi\widetilde{k}_2 y_0} - 1) \\ &= 0 \end{aligned}$$

If $P(J(\widetilde{k},z)) \cap \widetilde{E}_{\widetilde{k}} = \widetilde{Jk}$, then reordering the sequence given in (4.28), we have $c_i = 0$, for all $i = 1,\dots,r$. Thus, the projection $\Pi_{y_0}(I_{(\widetilde{k},0)})(x) = c_0\widetilde{I}_{\widetilde{k}}$.

Now, if the $P(J(\widetilde{k},z)) \cap \widetilde{E}_{\widetilde{k}}$ contains more then one single orbit given by $\widetilde{J}$, there are $\widetilde{k} \in \widetilde{\mathcal{L}}^*$ and $\delta \in J$ such that $\delta(\widetilde{k},0) = (\widetilde{k}_1,0)$ and $\widetilde{k}_1 \notin \widetilde{Jk}$. This means that $\widetilde{I}_{\widetilde{k}}(x) \neq \widetilde{I}_{\widetilde{k}_1}(x)$, but that $I_{(\widetilde{k},0)}(x,y) = I_{(\widetilde{k}_1,0)}(x,y)$. Hence, in $\Pi_{y_0}(\mathcal{X}_\Gamma)$ it is not possible to obtain the two functions $\widetilde{I}_{\widetilde{k}}$ and $\widetilde{I}_{\widetilde{k}_1}(x)$. □

4.5.2 Case 2. $(0,\sigma) \in \Gamma$ and there exists $(v,y_0) \in \mathcal{L}$

As we saw in the previous cases, the equality of spaces can be obtained when the projection of a function $I_{(k,z)}$ corresponds exactly with the function $\widetilde{I}_k$. Thus, for all $\widetilde{k} \in \widetilde{\mathcal{L}}^*$, we can chose $z = 0$, so that the projection of $I_{(\widetilde{k},z)}$ yields a non-zero coefficient for the term $\widetilde{I}_{\widetilde{k}}$, but $c_i = 0$, for $i = 1,\dots,r$, as Lemma 4.25. Here we have a similar case.

Proposition 4.35. *Suppose that $(0,\sigma) \in \Gamma$, $\widetilde{\mathcal{L}}_s$ is rationally compatible with $\mathcal{L}$ and the affine space A_{y_0} intersects a point of the lattice. Then $\Pi_{y_0}(\mathcal{X}_\Gamma) = \mathcal{X}_{\widetilde{\Gamma}}$ if and only if for each $\widetilde{k} \in \widetilde{\mathcal{L}}^*$ we have $P(J(\widetilde{k},z)) \cap \widetilde{E}_{\widetilde{k}} = \widetilde{Jk}$.*

Proof. As in Proposition 4.34, we show that $\Pi_{y_0}(I_{(\widetilde{k},0)}) = c\widetilde{I}_{\widetilde{k}}$, where c is a non-zero constant, for all $\widetilde{k} \in \widetilde{\mathcal{L}}^*$.

By Lemma 4.10 $\widetilde{\mathcal{L}}^* = P(\mathcal{L}^* \cap \{y = 0\})$.

Observe that, since $\sigma \in J$, for all $\delta \in J$ we have $\sigma\delta \in J$. Thus, if $\delta(\widetilde{k},0) = (\widetilde{k}_1,\widetilde{k}_2)$, for $\widetilde{k}_2 \neq 0$, then $(\widetilde{k}_1,-\widetilde{k}_2) \in J(\widetilde{k},0)$.

Therefore, by Lemma (4.25),

$$c_i = \frac{|\, J_{\widetilde{k}}^{Id_n}\,|}{2}\Big[\int_0^{y_0} w_z(y)dy + \int_0^{y_0} w_{-z}(y)dy\Big]$$

for all $i = 1,\dots,r$.

Since the affine space A_{y_0} intersect a point of the lattice (v,y_0) and $\sigma \in H_{\mathcal{L}}$, then $(0,2y_0) \in \mathcal{L}$. Therefore, $2y_0k_2 \in \mathbb{Z}$, for all $(\widetilde{k}_1,\widetilde{k}_2) \in \mathcal{L}^*$. Then, for $\widetilde{k}_2 \neq 0$, the sum

$$\int_0^{y_0} w_{(\widetilde{k}_2)}(y)dy + \int_0^{y_0} w_{(-\widetilde{k}_2)}(y)dy = 0.$$

thus, $c_i = 0$, for all $i = 1,\dots,r$. □

4.5.3 Case 3. Either there does not exists $(v,y_0) \in \mathcal{L}$ or there exists $(v,y_0) \in \mathcal{L}$, but $\sigma \notin J$.

Let Γ be a crystallographic group with point group J, lattice $\mathcal{L}$ and let $\widetilde{\Gamma} = \Pi_{y_0}(\Gamma)$, for some $y_0 \in \mathbb{R}$, be its projection, with lattice $\widetilde{\mathcal{L}}$ and point group $\widetilde{J}$. In this section we deal with the case where one of the conditions holds:

1. either there does not exists $(v,y_0) \in \mathcal{L}$;
2. there exists $(v,y_0) \in \mathcal{L}$, but $\sigma \notin J$.

The question is: how do we construct a correspondence between bases of $\Pi_{y_0}(\mathcal{X}_\Gamma)$ and $\mathcal{X}_{\widetilde{\Gamma}}$?

As we discussed in the beginning of this section, the idea is to find $\phi_i \in \mathcal{X}_\Gamma$, $i \in \mathbb{N}$, such that $\Pi_{y_0}(\{\phi_i\}_{i\in\mathbb{N}})$ is a Hilbert basis for $\mathcal{X}_{\widetilde{\Gamma}}$. Using the notation already introduced, we can rewrite the usual basis for $\mathcal{X}_{\widetilde{\Gamma}}$ as the set of real and imaginary parts of the functions in

$$\widetilde{\mathcal{B}} = \bigcup_{i\in\mathbb{N}} \{\widetilde{I}_{\widetilde{k}_{ij}};\ j = 1, \ldots, s\} = \{\widetilde{I}_{\widetilde{k}_l}\}_{l\in\mathbb{N}}.$$

For each $k \in \mathcal{L}$ and each $a \geqslant 0$ consider the set

$$E(a, k) = P(Jk \cap \{y = a\}).$$

We have the following result:

Proposition 4.36. *Suppose that the lattices $\mathcal{L}$ and $\mathcal{L}^*$ are rationally compatible with $\widetilde{\mathcal{L}}_s$ and an n-dimensional lattice in $\mathcal{L}^* \cap \{y = 0\}$, respectively. Suppose also that given $\widetilde{k} \in \widetilde{\mathcal{L}}^*$ there exists $z \in \mathbb{R}$ such that $(\widetilde{k}, z) \in \mathcal{L}^*$ and the constant c_0 in the projection $\Pi_{y_0}(I_{(\widetilde{k},z)})$ is non-zero and for each $a \geqslant 0$ and $k \in \mathcal{L}^*$ one of the following conditions holds:*

a) $E(a, k)$ is empty;

b) $E(a, k) \cap \widetilde{E}_{p_i}$ is a single $\widetilde{J}$-orbit;

c) $E(a, k) \cap \widetilde{E}_{p_i}$ consists of exactly two $\widetilde{J}$-orbit and that there exists $\beta_- \in J$ such that $\widehat{J} = \widehat{J}^{\uparrow} \uplus \beta_- \widehat{J}^{\uparrow}$.

Then, $\Pi_{y_0}(\mathcal{X}_\Gamma) = \mathcal{X}_{\widetilde{\Gamma}}$.

Proof. Let us construct the set $\{\phi_i\}_{i\in\mathbb{N}}$ to define $\widetilde{\mathcal{B}}$.

Consider,

$$\widetilde{E}_{p_0} = \widetilde{J}\widetilde{k}_{00} \cup \widetilde{J}\widetilde{k}_{01} \cup \ldots \cup \widetilde{J}\widetilde{k}_{0s_0}.$$

By hypothesis, consider $z_{0j} \in \mathbb{R}$, for $j = 0, \ldots, s_0$, such that $(k_{0j}, z_{0j}) \in \mathcal{L}^*$ and the constant c_0, in the projection $\Pi_{y_0}(I_{(k_{0j},z_{0j})})$ is non-zero, that is $z_{0j}y_0 \notin \mathbb{Z} \setminus \{0\}$. Assume also that $z_{0j} \geqslant 0$ is the minimum value for which this happens.

Thus, define

$$\phi_{0j} = I_{(k_{0i},z_{0i})},$$

for $j = 0, \ldots, s_0$.

Next, we show that the function ϕ_{0j} is well defined and $\phi_{0j} \neq \phi_{0q}$, for $j \neq q$ and $j,\ q \in \{0, \ldots, s_0\}$. In fact, each function ϕ_{0j} is associated with the orbit of $J((k_{0i}, z_{0i}))$.

We have,

$$\widetilde{E}_{p_0} = P(S_{k_{00}}(k_{00}, z_{00})) \cup P(S_{k_{01}}(k_{01}, z_{01})) \cup \cdots \cup P(S_{k_{0s_0}}(k_{0s_0}, z_{0s_0}))$$

and $\|k_{0j}\| = \|k_{0q}\| = p_0$, for $j,\ q \in \{0, \ldots, s\}$. By hypothesis, one of the conditions a)-b) holds. If a) holds, then there is nothing to do.

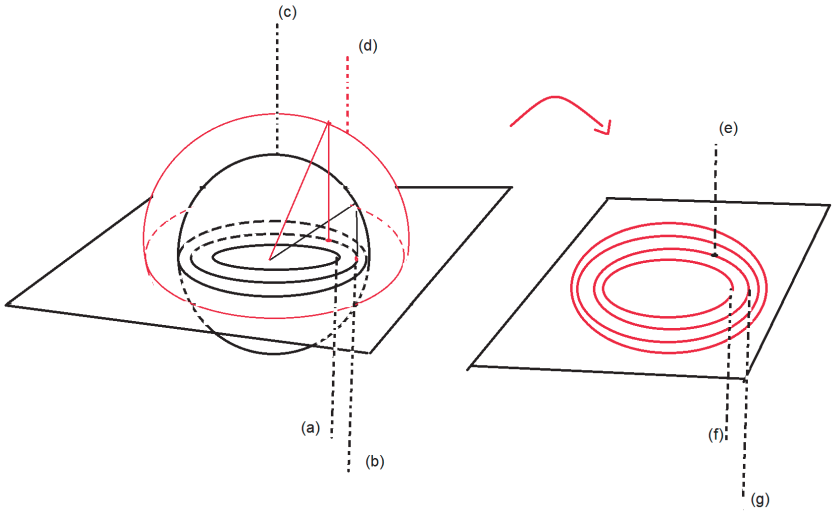


Fig. 4.4 Projection of orbits given by the group J. Each orbit $J(k_{0j}, z_{0j})$ is associated to a function $I_{(k_{0j},z_{0j})}$. (a) and (f) Circle containing $\widetilde{E}_{p_0}$; (b) and (g) Circle containing $\widetilde{E}_{p_1}$; (c) Circle containing the orbit of (k_{00}, z_{00}) by J; (d) Circle containing the orbit of (k_{0j}, z_{0j}) by J; (e) The point $k_{01} \in \widetilde{E}_{p_1}$;

Suppose that b) holds, that is $E(a, (k_{00}, z_{00}))$. Thus if $j \neq q$ then $\|(k_{0j}, z_{0j})\| \neq \|(k_{0q}, z_{0q})\|$. Therefore, $J((k_{0j}, z_{0j})) \neq J((k_{0q}, z_{0q}))$ and $P(J(k_{0j}, z_{0j})) \cap \widetilde{E}_{p_0} = P(S_{k_{0j}}(k_{0j}, z_{0j}))$.

Now, if condition c) holds then $\widehat{J} = \widehat{J}^{\uparrow} \cup \beta_- \widehat{J}^{\uparrow}$. Hence, the projection of $I_{(k_{0i},z_{0i})}$ is given by

$$\Pi_{y_0}(I_{(k_{0i},z_{0i})}) = a\widetilde{I}_{k_{0i}} + a'\widetilde{I}_{\beta k_{0i}} + ot$$

where ot means other terms which contains the function $\widetilde{I}_u$, where $\|u\| \neq \|k_{0i}\|$.

Since $\mathcal{L}^*$ is rationally compatible with an n-dimensional lattice in $\mathcal{L}^* \cap \{y = 0\}$, there exists $(0, q) \in \mathcal{L}^*$, for $q \in \mathbb{R}$ non-zero. Let $m \in \mathbb{Z}$ such that $z' = -z + mq > 0$.

Now, observe that $(\beta_-)^{-1} = (\beta^{-1})_- = \beta_-^{-1} \in \beta_- \widehat{J}$ and $\beta_-^{-1}(\beta k_{0i}, z') = (k_{0i}, -z')$. Thus, the projection of $I_{(\beta k_{0i},z')}$ yields:

$$\Pi_{y_0}(I_{(\beta k_{0i},z')}) = b\widetilde{I}_{k_{0i}} + b'\widetilde{I}_{\beta k_{0i}} + ot.$$

Define $\phi'_{0j} = b' I_{(k_{0i},z_{0i})} - a' I_{(\beta k_{0i},z')}$. Then, $\Pi_{y_0}(\phi'_{0j}) = c\widetilde{I}_{k_{0i}}$. We need to see that the constant $c \neq 0$. In fact, by hypothesis $zy_0 \notin \mathbb{Z} \setminus \{0\}$. Note that, $c = 0$ if and only if $zy_0 \in \mathbb{Z} \setminus \{0\}$ and $z'y_0 \in \mathbb{Z} \setminus \{0\}$.

Now, consider

$$\widetilde{E}_{p_1} = \widetilde{J}\widetilde{k}_{10} \cup \widetilde{J}\widetilde{k}_{11} \cup \ldots \cup \widetilde{J}\widetilde{k}_{1s_1}.$$

Then $\|k_{1j}\| > \|\widetilde{k}\|$, for all $\widetilde{k} \in \widetilde{E}_{p_0}$.

Consider $\widetilde{k}_{1j} \in \widetilde{E}_{p_1}$. We have two cases:

(i) $\widetilde{k}_{1j} \in P(J((k_{0i}, z_{0i})))$, for some $j \in \{0, \cdots, s_0\}$;

(ii) $\widetilde{k}_{1j} \notin P(J(k_{0i}, z_{0i}))$, for all $j \in \{0, \cdots, s_0\}$;

If (ii) holds, then define

$$\phi_{1j} = I_{(k_{1j}, z_{1j})},$$

for $j = 0, \ldots, s_1$, as before.

Suppose that (i) holds. Consider (k_{0j}, z_{0j}) with maximum norm, such that $k_{1j} \in P(J(k_{0i}, z_{0i}))$. Let $n \in \mathbb{N}$, such that

$$\|u\| < \|k_{1j} + nk_{n+1}\|,$$

for all $u \in P(J(k_{0i}, z_{0i}))$.

Let $z_{1j} \in \mathbb{R}$, such that the constant c_0, in the projection $\Pi_{y_0}(I_{(k_{1j}+nk_{n+1}, z_{1j})})$, is non-zero. Then, by construction, $\Pi_{y_0}(I_{(k_{1j}+nk_{n+1}, z_{1j})})$ has a term

$$c_0 \widetilde{I}_{(k_{1j}+nk_{n+1})}$$

that does not appear in the projection of the functions ϕ_{0j} defined before.

Therefore, consider

$$\phi_{1j} = I_{(k_{1j}+nk_{n+1}, z_{1j})}.$$

To define ϕ_{1i}, for all $i \in \{0, \ldots, s_1\}$, observe that for all $k_{1i} \in \widetilde{E}_{p_1}$,

$$k_{1i} + nk_{n+1} \in \widetilde{E}_{\|k_{1j}+nk_{n+1}\|}.$$

We can continue with this process to define ϕ_{ij}, for all $i,\ j \in \mathbb{N}$. Each ϕ_{ij} has a term after projection that ϕ_{lj} does not have, for all $l < i$. Moreover, ϕ_{ij} has a term that ϕ_{il} does not have for all $j \neq l$, and all $i \in \mathbb{N}$. □

After we know the effect of projection on irreducible representations given by the action of a $(n+1)$-dimensional crystallographic group Γ, we look at what happens in fixed point subspaces contained in those representations.

Given a finite dimensional Γ-invariant subspace, V, of $\mathcal{X}_{\mathcal{L}}$, consider an isotropy subgroup Σ of Γ. Then, after decomposing V into Γ-irreducible components, let us say $V_1 \oplus \ldots \oplus V_p$, we can write

$$Fix_V(\Sigma) = Fix_{V_1}(\Sigma) \oplus \ldots \oplus Fix_{V_p}(\Sigma).$$

Each $Fix_{V_i}(\Sigma)$, for $i = 1, \ldots, p$, is generated by functions

$$I_k(x, y) = \sum_{\sigma \in \Sigma_P} \omega_{\sigma k}(x, y)\omega_{\sigma k}(-v_\sigma),$$

where here we denote Σ_P by the point group of Σ, for some $k \in \mathcal{L}^*$.

The previous results applied for Σ, allow us to conclude that the projection $\Pi_{y_0}(W)$ of $W = Fix_V(\Sigma)$ corresponds to the sum of fixed point subspaces on $\Pi_{y_0}(V)$.

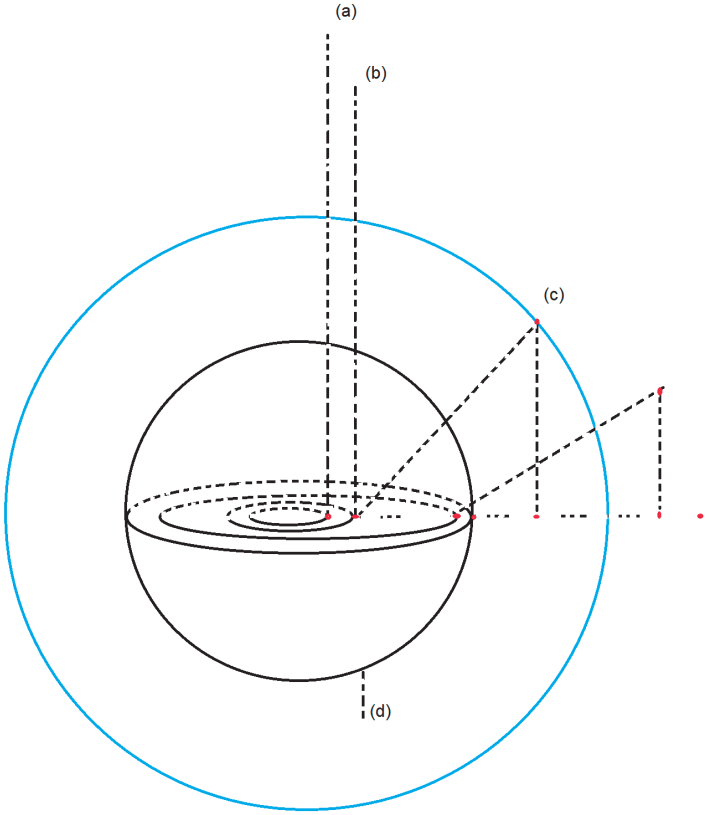


Fig. 4.5 Geometrical idea to define the sequence of functions ϕ_{ij}. (a) Circle containing $\widetilde{E}_{p_0}$; (b)Circle containing $\widetilde{E}_{p_1}$; (c) Circle containing the orbit of $(k_{1j} + nk_{n+1}, z_{1j})$ by J; (d) Circle containing the orbit of (k_{00}, z_{00}) by J.

We return again to our initial problem:

> "Is there a relation between the typical solutions of the n-dimensional problem and the ones that comes from the projection of the extended one?"

Typical solutions for equivariant PDE's can be found in irreducible representations given by the action of a compact Lie group. In this chapter, our results show how irreducible representations are transformed by projection.

The splitting of an irreducible representation after projection suggests that projection of typical solutions can be transformed into mode interactions in lower dimension. We postpone that problem to the next chapter, since it will be necessary to introduce some more tool used in Equivariant Bifurcation Theory, that are not in the scope of the current chapter.

Chapter 5

Projection, Mode Interaction and Forced Symmetry Breaking

After having used projection to compare $(n+1)$- and n-dimensional patterns in Chapters 3 and 4, we aim at discussing the dynamics involved in the problem.

In Chapter 2, we see how to use Equivariant Bifurcation Theory to study time-independent solutions with the variation of an external parameter in symmetric partial differential equations. Section 2.6 gives us a method to find typical solutions for those systems, as we see in Theorem 2.16. One of the hypothesis in this theorem is that the space $V = Ker(\mathcal{DP})_{(0,0)}$ is a Γ-irreducible subspace of $\mathcal{X}_{\mathcal{L}}$. Typical solutions are found in 1-dimensional fixed-point subspaces contained in V. The proof of the equality of spaces, in Chapter 4, gives us freedom to compare those irreducible representations in different dimensions and fixed point subspaces contained therein.

A question naturally arises: are the results in chapter 4 necessary to state that branches of solutions in fixed point subspaces are projected into solutions of a possible bifurcation problem in lower dimension? How does the projection of a solution change when the band of projection varies? To answer the first question, in Section 5.1, we introduce the idea of mode interaction. We show how the analysis of projection of irreducible representations can contribute to this issue. In Section 5.2, we apply our results to interpret how the study of forced symmetry breaking can explain the transition of projected patterns when the band of projection varies.

We review some concepts and analyse their relation with our model. We systematically compare our conclusions with experimental results related to Reaction-Diffusion systems.

5.1 Projection and Mode Interaction

The results in Chapter 4 suggest that typically projection may lead to bifurcations with higher corank.

As we see in Remark 4.5, for each $k \in \mathcal{L}^*$, we can consider the Γ-invariant subspace $V = V_{k_c}$ of $\mathcal{X}_{\mathcal{L}}$ given by:

$$V = \bigoplus_{|k|=k_c} V_k = \bigoplus_{i=1}^{s} V_{k_s}.$$

Without loss of generality, suppose that $\{\pm k_1, \ldots, \pm k_s\} \in \mathcal{L}^*$ is a single J-orbit. Then, we can decompose

$$V = \langle Re(I_k(x,y))\rangle \oplus \langle Im(I_k(x,y))\rangle \bigoplus W, \tag{5.1}$$

where $W \cap \mathcal{X}_\Gamma = \{0\}$.

The spaces $V_1 = \langle Re(I_k(x,y))\rangle$ and $V_2 = \langle Im(I_k(x,y))\rangle$ are 1-dimensional irreducible subspaces of $\mathcal{X}_{\mathcal{L}}$. Moreover, each element of V_i is fixed by the action of Γ.

Now, consider a $\Gamma = H_{\mathcal{L}} \dot{+} \mathbb{T}^{n+1}$-equivariant system described by

$$\frac{dz}{dt} = g(z,\lambda), \quad g : V \times \mathbb{R} \longrightarrow V. \tag{5.2}$$

Here $g(0,0) = 0$, the Jacobian matrix at the bifurcation point $(dg)_{0,0}$ is the zero matrix, and we assume that any bifurcation occurs at the origin, for $\lambda = 0$, as in Chapter 2.

By Theorem 2.16, generically, there exists a unique branch of solutions on V_i, $i = 1, 2$, with symmetry Γ, to the equation $g(z,\lambda) = 0$.

The projection of such branches of solutions is associated to the projection of the functions I_k. In Lemma 4.32, we see that for a dense set of points $y_0 \in \mathbb{R}$, the projection of irreducible representation, given by the subspaces V_i, yields at least a corank two bifurcation. This means that, after projection, solutions corresponding to wavenumbers $\|\widetilde{k}\|$ and $\|u_i\|$, for $i = 1, \ldots, r$, interact and bifurcate simultaneously.

5.1.1 Mode Interaction

We review here the concept of mode interaction. For what follows we refer to Castro [6–8] and Golubitsky et al. [21], chapter XIX and XX. At the end of this section we provide a new description of the black-eye pattern.

Definition 5.1. *([21], Chapter XIX, section 0 and Chapter XX, section 0.)* We have a *mode* when the kernel of $(dg)_{0,0}$ is Γ-irreducible.

We say that the eigenvector of the linearised equation $\dot{z} = (dg)_{0,0}z$ is a *steady-state mode* if it has a simple zero eigenvalue.

We say that the eigenvector of the linearised equation $\dot{z} = (dg)_{0,0}z$ is a *critical mode* if it has an eigenvalue that lies on the imaginary axis.

Observe that a steady-state mode is a critical mode. This happens when the associated eigenvalue is zero.

The idea of interaction, of two or more modes, corresponds to the decomposition of the kernel of $(dg)_{0,0}$ into two or more Γ-irreducible components, respectively.

As in [6], Chapter 2, let us consider the system of equations

$$\frac{dz_i}{dt} = g_i(z_1, \ldots, z_i, \ldots, z_{r+1}, \lambda) \tag{5.3}$$

for $i = 1, \ldots, r+1$ and some integer $r \leqslant n$, where z_i is in an s_i-dimensional subspace V_i in $\mathbb{C}^n$.

Definition 5.2. We say that the bifurcation defined by g is a $(r+1)$-*mode interaction* if the kernel of $(dg)_{0,0}$ decomposes as the direct sum of $(r+1)$ Γ-irreducible components.

We assume that the $(r+1)$ Γ-irreducible components are the V_i's. The mode interaction can be of steady-state/steady-state (or simply steady-state), Hopf/steady-state, Hopf/Hopf type, depending on the criticality of the modes.

Once we establish the context where we want to see the interactions, we become concerned with the genericity of those interactions. Consequently, it is necessary to perturb the system in order to see which features persist.

Perturbations in bifurcations cause interesting changes on the interaction of the modes. After a small perturbation, modes that were bifurcating together may split or bifurcate one from the other generating a secondary mode.

Perturbation will lead to multi-parameter systems. We call *non-linear interactions*, the interactions that we observe in this perturbed system. We can define mode interaction as before, where $\lambda \in \mathbb{R}^m$. Typically, these non-linear interactions are types of secondary bifurcations for parameter values at which there are multiple critical modes, see [21], page 413.

In the present work, we pay attention to linear and non-linear steady-state mode interactions.

5.1.2 Projection of Steady-State Linear Mode Interactions

Let $\Gamma = H_{\mathcal{L}} \dot{+} \mathbb{T}^{n+1}$ acting on the space of periodic functions, $\mathcal{X}_{\mathcal{L}}$. Consider a steady-state brunch arising in a 1-dimensional fixed point subspace, contained in the invariant subspace V given in (5.1). Consider also the projection of V. We show that the projection of the steady-state mode is a steady-state mode interaction in the projected space. The proof relies on results of Chapter 4.

Proposition 5.3. *Let $\Gamma = H_{\mathcal{L}} \dot{+} \mathbb{T}^{n+1}$ be a compact Lie group group acting on V. Assume that V is a Γ-irreducible representation and $\Sigma \subset \Gamma$ is an isotropy subgroup satisfying $dim(Fix_V(\Sigma)) = 1$. Then, generically, projection of a steady-state mode in $Fix_V(\Sigma)$ corresponds to a steady-state mode interaction on $\Pi_{y_0}(V)$.*

Proof. Let Σ be an isotropy subgroup such that $dim(Fix_V(\Sigma)) = 1$. Then, $Fix_V(\Sigma)$ is generated by the combination of real and imaginary part of

$$I_k(x, y) = \sum_{\sigma \in \Sigma_P} \omega_{\sigma k}(x, y)\omega_{\sigma k}(-v_\sigma),$$

as in Section 4.5.

By results of Section 4.6, the projection of $Fix_V(\Sigma)$ is either a fixed point subspace generated by $\Pi_{y_0}(I_k)$ or it is the sum of fixed point subspaces generated by the components resulting of the projection of I_k.

Suppose the set $P(Jk)$ is contained in a single $\widetilde{J}$-orbit, hence there is no mode interaction. For the latter to occur, that is, to have mode interaction, we want to be in the conditions of Section 4.6.3 and show that either $(v, y_0) \notin \mathcal{L}$ or that there exists $(v, y_0) \in \mathcal{L}$ but $\sigma \notin J$.

To achieve this, we are going to consider perturbations of the axis of projection (or equivalently perturbations of the position of the (n+1)-dimensional lattice) and show that, with respect to these the conditions of Propositions 4.34 and 4.35 fail.

Perturbation of an(n+1)-dimensional lattice is made by considering only changes of coordinates $\gamma \in SO(n+1)$. Consider a $(n+1)$-dimensional crystallographic group Γ with lattice $\mathcal{L}$ generated by

$$l_1, \ldots, l_{n+1}$$

with holohedry $H_{\mathcal{L}}$. Let $P = \{m_1 l_1 + \ldots + m_{n+1} l_{n+1};\ \ m_i \in \{-1, 0, 1\}\}$ be the primitive cell of $\mathcal{L}$ and $Q = \{m_1 l_1 + \ldots + m_{n+1} l_{n+1};\ \ m_i \in [-1, 1]\}$ the hyper-rectangle defined by P.

Let $y_0 \in \mathbb{R}$,be the value which defines the band of projection. Then there is $l \in \mathcal{L}$ such that the cell $P_l = l + P$ contains $(0, y_0)$.

Choose $(v, y_*) \in P_l$ such that the lengths $\|v\|$ and $\|(v, y_*) - (0, y_0)\|$ are minimal.

Define

$$D_v((0, y_*)) = \{(u, y_*) \in \mathbb{R}^{n+1};\ \|u\| < \frac{1}{q}\|v\|\}$$

where $q \in \mathbb{R}$. Note that $D_v((0, y_*))$ is the open disc of radius $r = \frac{\|v\|}{q} > 0$ centred at $(0, y_*)$ in $\mathbb{R}^{n+1}$.

For each point $(u, y_*) \in D_v((0, y_*))$ define the angles $\theta = \measuredangle((0, y_*), (u, y_*))$, $\psi = 90° - \measuredangle((0, \ldots, 0, 1, 0), (u, y_*))$. Consider the rotation ρ in $SO(n+1)$ given by:

$$\rho_{(\theta,\psi)} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & R_\theta R_\phi \end{pmatrix}$$

where

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \qquad R_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that the set of all such rotations $D = \{\rho_{(\theta,\psi)}\}$, defines an open set in $SO(n+1)$ containing the identity.

Consider a perturbation of the projection height y_0 of amplitude 2δ, $\delta \in \mathbb{R}$. Let y' be a point in the band $(y_0 - \delta, y_0 + \delta)$. We show that for any projection defined by y' either there is no $(v, y_0) \in \rho_{(\theta,\psi)}\mathcal{L}$ or there exists $(v, y_0) \in \rho_{(\theta,\psi)}\mathcal{L}$ but $\sigma \notin \rho_{(\theta,\psi)}J$, for all $\rho_{(\theta,\psi)} \in D$.

In fact, there is no point $(0, y') \in \rho_{(\theta,\psi)}\mathcal{L} \cap \langle(0, \ldots, 0, 1)\rangle$, because:

- either $(0, y_0) \in \mathcal{L}$ and thus $\rho_{(\theta,\psi)}(0, y_0) \in \langle (u, y_*) \rangle$, implying that $\rho_{(\theta,\psi)}(0, y_0) \notin \rho_\theta \mathcal{L} \cap \langle (0, \ldots, 0, 1) \rangle$;
- or $(0, y_0) \notin \mathcal{L}$, and then we use $P_l \cap \rho_{(\theta,\psi)}\mathcal{L} = \varnothing$. Thus the axis generated by $(0, \ldots, 0, 1)$ does not contain a point $(0, y')$.

Now, let $H_{\rho_{(\theta,\psi)}}$ be the holohedry of the perturbed lattice. We show that $\sigma \notin H_{\rho_{(\theta,\psi)}}$, for all $\rho_{(\theta,\psi)} \in S$, where

$$\sigma = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

We have two cases:

Case 1: $\sigma \in H_{\mathcal{L}}$

In this case $\sigma \notin H_{\rho_{(\theta,\psi)}}$ if and only if $\rho_{(\theta,\psi)} \in H_{\mathcal{L}}$. In fact,

$$\begin{aligned} \sigma \notin H_{\rho_{(\theta,\psi)}} &\Leftrightarrow \sigma = \rho_{(\theta,\psi)}\gamma,\ \gamma \in H_{\mathcal{L}} \\ &\Leftrightarrow \rho_{(\theta,\psi)} = \sigma\gamma^{-1} \in H_{\mathcal{L}}. \end{aligned}$$

But if $\rho_{(\theta,\psi)} \in H_{\mathcal{L}}$ then $\rho_{(\theta,\psi)}P = P$, which contradicts our construction.

Case 2: $\sigma \notin H_{\mathcal{L}}$

Suppose that $\sigma \in H_{\rho_{(\theta,\psi)}\mathcal{L}}$, then there is a minimal $y \in \mathbb{R}$ such that $(0, y) \in \rho_{(\theta,\psi)}\mathcal{L}$. Let

$$l' = \rho^{-1}_{(\theta,\psi)}(0, y) = (l_1, l_2) \tag{5.4}$$

Now, observe that $\rho_\theta P$ is a primitive cell for $\rho_{(\theta,\psi)}\mathcal{L}$. From (2.2), we have $(v, y_*) = v' + l$, for $v' \in P$. Take $(a, b) = \rho_\theta v' \in \rho_{(\theta,\psi)}P$, then $(a, b) - (0, y) \in \rho_{(\theta,\psi)}P$, by minimality. Thus

$$\begin{aligned} \|(a, b) - (0, y) + \rho_{(\theta,\psi)}l\| &= \|\rho_{(\theta,\psi)}v' - \rho_{(\theta,\psi)}l' + \rho_{(\theta,\psi)}l\| \\ &= \|v' + l - l'\| \leq \|(v, y_*)\| \end{aligned}$$

where in the last equality we are using the fact that $v' + l - l' \in P_l$ and (2.2). Again, by (2.2), we must have $l' = (0, y'')$, implying that $\theta = 0$, which is a contradiction, from (5.4).

Therefore, $\sigma \notin H_{\rho_{(\theta,\psi)}\mathcal{L}}$. Concluding the proof. □

5.1.3 Black-eye Patterns

In the light of our results, we are ready to give an interpretation of the black-eye pattern observed in experiments performed in a two-fed-open spatial reactor.

In Gunaratne et al. [24], Section VI, the authors describe properly black-eye patterns as:

> "a resonant interaction between the basic modes of the hexagonal array. These patterns consist of two hexagonal lattices: one of white spots and the other of black spots at the center of each white spot and at the center of the dark region in each equilateral triangle with three neighbouring white spots at its vertices. The hexagonal lattice of white spots has a wavelength of 0.15 mm while the lattice of

> black dots has a wavelength of 0.086 mm. The ration of the two wavelength is $\sqrt{3}$, suggesting that the lattice of black spots is a harmonic structure of white spots."

As we see in Gomes [22], the same pattern can be obtained as a projection of a body centred cubic structure.

The first main issue is: are they the same? An approach to answer this question can be found in Zhou et al. [46]. In this work, experiments are performed in order to clarify whether a black-eye pattern is a quasi-two-dimensional structure or a slice of a bcc planform as described by [22]. They suggested that the projection method can be excluded.

What defines a quasi-two-dimensional pattern?

When experiments are performed there are transitions of: 2-, quasi-two-, and 3-dimensional patterns. In Dulos et al. [17], experiments are performed to explain this transition. In particular, transition from quasi-two- to 3-dimensional Turing patterns can be considered as projection of the patterns happening in different layers, see [17], Section 4.

In [46], quasi-two-dimensional patterns are considered as patterns that appear at the onset of Turing instabilities. In [24], those patterns at the onset are described as non-linear interactions.

How are these interactions explained in theory?

In [24] the authors study the Swift-Hohenberg equation in order to show the generation of secondary modes as a result of interaction of hexagonal patterns. Despite this analysis, structures such as the black-eye are neglected in their work, as we see in Section VI.

Equivariant Bifurcation theory gives a formalism of how to study mode interaction. The results of Subsection 5.1.2 show how to relate mode interaction and projection. Generically, projection of steady-state modes correspond to a steady-state mode interaction.

Therefore we can conclude that both the experimental results obtained by [24] and the theoretical results in [22], regarding black-eye patterns, are the same.

We believe that studying 3-dimensional structures and interpreting the results as projection is still the best way of describing the transition of quasi-two- to 3-dimensional patterns.

However, there are still many issues regarding the theory of projection. As mentioned in Chapter 1, in reaction diffusion experiments, the reactants are fed through either one or two reservoirs, producing a chemical gradient in the z direction, see [22]. Thus, variation of the chemical gradient corresponds to variation in the band of projection.

We know that changes in the band of projection induces symmetry breaking. For instance, the projections corresponding to the band of black-eye pattern, break the symmetry from the group $\Gamma = \mathcal{L}\dot{+}H_{\mathcal{L}}$ to the subgroup $\mathcal{L}\dot{+}D_3$. How can that symmetry be transformed so that this chemical gradient can be observed in the z direction? The answer for this question is nothing more than a change of coordinates so that the plane $\{z = 0\}$ inherits the underlying symmetry. This is exactly the conclusion found in Parker [34]. Therefore, the chemical gradient can be studied via the techniques used to study forced symmetry breaking.

5.2 Future Work

To finish this chapter, we give suggestions for future work.

5.2.1 Projection and Forced Symmetry Breaking

It is well known that physical systems are subjected to perturbations. Any mathematical tool that attempts to study these systems must be sensitive to such disturbances.

A first approach for a physical system is by a mathematical model subject to reasonable hypotheses. These hypotheses lead to a model with more symmetry and that is easier to deal with. For instance, we consider the black-eye pattern obtained as projection in [22] as a first approximation for the black-eye pattern observed in experiments. Results found in [22] illustrate the black-eye pattern in an ideal circumstance, since the author does not report the effect of perturbations in the system. It is naive to perform experiments and compare the results with an illustration of a method that was not formalised to interpret a complex system.

In order to capture additional information of the experimental system, we slightly perturb a symmetric model to a less symmetric one. This gives an idea of a second order approximation given by forced symmetry breaking, see Parker [34], Parker et al. [35, 36].

As highlighted in [34], Chapter 2, there are many ways in which we can break the symmetry of a model in order to give approximate results to a real physical system. In this section we discuss the projection model in the light of forced symmetry breaking. In particular, we discuss the ideas given by Parker [34] and address future work.

Here, we look at the study of forced symmetry breaking found in [34]. A broad idea of the problem is as follows: consider the Γ-equivariant system described by (5.2) and a steady-state solution $x \in V$, with orbit $X = \Gamma x$.

The challenge in [34] is to answer to following question:

> "What happens to the orbit X when the vector field g is perturbed to the new system
>
> $$G(z, \lambda, \epsilon) = g(z, \lambda) + \epsilon h(z) \tag{5.5}$$
>
> where ϵ is real and small and h is equivariant with respect to a subgroup isomorphic to a subgroup of the holohedry of the underlying lattice?"

The new system given by G represents a forced system to break the symmetries of g. In [34], the author formalises and proves conditions of existence and persistence of equilibrium solutions for the new system. This problem is transformed to the study of the action of the symmetry group of the map G on the group orbit X. In particular, a partial classification for the behaviour of the group orbits is also obtained.

We can visualise symmetry breaking through projection when the band y_0 is varied. Depending on how the band of projection intersects the underlying lattice of the problem, we see a transition of more symmetric to less symmetric patterns after projection. How can we study this breaking of symmetry?

The relation of the work in [34] and projection can be found in the examples given in chapter 7, 8 and 9 of [34]. The author studies forced symmetry breaking in dimension 3.

In [34], projection of 3-dimensional patterns, under different direction of projection, are compared. Specifically, the author considers a planform, f, for (5.2) with symmetry

$\Sigma \subseteq \Gamma$. After breaking the symmetry of g, as in (5.5), by adding a term h with symmetry Δ, the examples in [34] illustrate how the structure of the planform f is altered after the forced symmetry breaking. Since Δ is subgroup isomorphic to a subgroup of the holohedry, perturbation of the system corresponds to perturbations of the lattice.

To describe planforms in dimension 3 as projection, Parker [34] considers the density plots of the planforms, the original one f and its correspondent perturbed pattern obtained from the system (5.5), in a cross section of the lattice. The planforms are compared after a projection in a fixed band. In [34], Example 9.2.8, Chapter 9, the visualization of planforms for the cubic lattices are made in a hexagonal cross section, in particular the black-eye pattern is compared with its corresponding perturbed pattern. A more detailed analysis of black-eye patterns is then made in [34], Section 10.2.

The difference between our result and the one presented in [34], is that projections of those steady-state, through different bands, can be directly compared with patterns obtained in the 2-dimensional problem, after performing a forced symmetry breaking.

Now, it is still early to tell more about the stability of the projected solutions. The model of projection should be compared with the bifurcation diagrams associated with the perturbed problem. The results proposed in this thesis should also be closely compared with experiments on the CIMA reaction.

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Appendix A

Matlab Programs to Plot Projections of Planforms

In this appendix we present the Matlab file used to plot 3-dimensional planforms as slices of the cube lattice.

Let $\mathcal{L}$ be a three-dimensional lattice and $\mathcal{L}^*$ its dual. Suppose that the y_0-projection of $\mathcal{L}$ in the plane P is a hexagonal plane lattice.

Consider $u : \mathbb{R}^3 \to \mathbb{R}$ in $V = \bigoplus_{\substack{m \in \mathcal{L}^*, \\ |m|=m_c}} V_m = \bigoplus_{i=1}^{n} V_{m_i}$, where $V_{m_i} = \{Re(zw_m(X));\ z \in \mathbb{C}\}$.

Thus, we can write:

$$u(x,y,z) = \sum_{|m|=m_c} \exp(2\pi i m \cdot (x,y,z))$$

$$= \sum_{|m|=m_c} (\cos 2\pi m \cdot (x,y,z) + \sin 2\pi m \cdot (x,y,z))$$

$$= \sum_{i=1}^{n} 2\cos 2\pi m_i \cdot (x,y,z)$$

To make our calculations easier, we can rotate the lattice in such a way that the plane P will be parallel to the plane $\{(x,y,z) \in \mathbb{R}^3;\ z = 0\}$. Therefore, in the new coordinates,

$$u(x,y,z) = \sum_{|k|=k_c} \exp(2\pi i k \cdot (x,y,z))$$

$$= \sum_{k=k_1}^{k_3} 2\cos 2\pi k \cdot (x,y,z)$$

where, A is the rotation matrix and $k_i = A \cdot m_i = (k_{i1}, k_{i2}, k_{i3})$

The projection $\Pi(u)(x,y) : \mathbb{R}^2 \to \mathbb{R}$ of u is given by:

$$\Pi(u)(x,y) = \int_{z_0}^{z_1} u(x,y,z)dz$$

$$= \int_{z_0}^{z_1} 2 \sum_{i=1}^{3} \cos 2\pi (k_{i1}x + k_{i2}y + k_{i3}z)$$

Now observe that, if $k_{i3} = 0$ for same $i \in \{1, 2, 3\}$ then:

$$\int_{z_0}^{z_1} \cos 2\pi k \cdot (x, y, z) dz = (z_1 - z_2) \cos 2\pi k \cdot (x, y, z)$$

otherwise

$$\int_{z_0}^{z_1} \cos 2\pi k \cdot (x, y, z) dz = \frac{1}{2\pi k_{i3}} \sin 2\pi k \cdot (x, y, z) \mid_{z_0}^{z_1}$$

We used Matlab program to plot the contour of the projection $\Pi(u)$, for some u.

Next we write the program used to plot patterns in the simple cubic lattice, where V is a 6-dimensional subspace of $\mathcal{X}_{\mathcal{L}}$.

```
function[]= plancubo6()

%Representation of dimension 6

%Setting the domain of x and y
d1 = input('periodicity in the x direction ');
d2 = input('periodicity in the y direction');
%Integration limit
z0 = input('write z0');
z1 = input('write z1');

[X,Y] = meshgrid(-d1:0.01:d1,-d2:0.01:d2);

l = size(X);

M = [sqrt(2) 0 0;0 sqrt(2) 0;0 0 sqrt(2)]; % matrix formed by the vectors m_{j}

A = [1/sqrt(2) 1/sqrt(2) 0;1/sqrt(6) -1/sqrt(6)  2/sqrt(6);
1/sqrt(3) -1/sqrt(3) -1/sqrt(3)]; % rotation matrix

K1 = (A*M(1,:)')';

i_K1 = (2*pi)*(K1(1,1)*X + K1(1,2)*Y + (K1(1,3)*z1)*ones(l));

s_K1 = (2*pi)*(K1(1,1)*X + K1(1,2)*Y + (K1(1,3)*z0)*ones(l));

if K1(1,3) == 0
            p_K1 = 2*(z1 - z0)*cos(2*pi*(K1(1,1)*X + K1(1,2)*Y));
    else
            p_K1 = (1/(pi*K1(1,3)))*(sin(i_K1) - sin(s_K1));
```

```
end

K2 = (A*M(2,:)')';

i_K2 = (2*pi)*(K2(1,1)*X + K2(1,2)*Y + (K2(1,3)*z1)*ones(l));

s_K2 = (2*pi)*(K2(1,1)*X + K2(1,2)*Y + (K2(1,3)*z0)*ones(l));

if K2(1,3) == 0
           p_K2 = 2*(z1 - z0)*cos(2*pi*(K2(1,1)*X + K2(1,2)*Y));
       else
            p_K2 = (1/(pi*K2(1,3)))*(sin(i_K2) - sin(s_K2));
end

K3 = (A*M(3,:)')';

i_K3 = (2*pi)*(K3(1,1)*X + K3(1,2)*Y + (K3(1,3)*z1)*ones(l));

s_K3 = (2*pi)*(K3(1,1)*X + K3(1,2)*Y + (K3(1,3)*z0)*ones(l));

if K3(1,3) == 0
            p_K3 = 2*(z1 - z0)*cos(2*pi*(K3(1,1)*X + K3(1,2)*Y));
       else
             p_K3 = (1/(pi*K3(1,3)))*(sin(i_K3) - sin(s_K3));

end

 Z = p_K1 + p_K2  + p_K3;

    figure
    colormap(flipud(gray))
    contour(X,Y,Z,20)
    contour3(X,Y,Z,20)
    pcolor(X,Y,Z)

    shading interp
    axis equal
```

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