Forest Algebras, ω-Algebras and A Canonical Form for Certain Relatively Free ω-Algebras

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Abstract

Forest algebras are defined for investigating languages of forests [ordered sequences] of unranked trees, where a node may have more than two [ordered] successors [9]. We show that several parameters on forests can be realized as forest algebra homomorphisms from the free forest algebra into algebras which retain the equational axioms of forest algebras. This includes the number of nodes, the number of connected parts, the set of labels of nodes, the depth, and the set of labels of roots of an element in the free forest algebra. We show that the horizontal monoid of a forest algebra is finite if and only if its vertical monoid is finite. By an example we show that the image of a forest algebra homomorphism may not be a forest algebra and also the pre-image of a forest subalgebra by a forest algebra homomorphism may not be a forest algebra.

Bojańczyk and Walukiewicz in [9] defined the syntactic forest algebra over a forest language. We define a new version of syntactic congruence of a subset of the free forest algebra, not just a forest language, which is used in the proof of an analog of Hunter's Lemma [23] in the third chapter. The new version of syntactic congruence is the natural extension of the syntactic congruence for monoids in case of forest algebras. We show that for an inverse zero action subset and a forest language which is the intersection of the inverse zero action subset with the horizontal monoid, the two versions of syntactic congruences coincide.

Almeida in [2] established some results on metric semigroups. We adapted some of his results to the context of forest algebras. We define on the free forest algebra a pseudo-ultrametric associated with a pseudovariety of forest algebras. We show that the basic operations on the free forest algebra are uniformly continuous, this pseudo-ultrametric space is totally bounded, and its completion is a forest algebra. The difficult part is how to handle the faithfulness property of forest algebras. We show that in a metric forest algebra with uniformly continuous basic operations, its horizontal monoid is compact if and only if its vertical monoid is compact. We show that every forest algebra homomorphism from the free forest algebra into a finite forest algebra is uniformly continuous. We show that the analog of Hunter's Lemma [23] holds for metric forest algebras, which leads to the result that zero-dimensional compact metric forest algebras are residually finite. We

establish an analog of Reiterman's Theorem [26], which is based on a study of the structure profinite forest algebras.

We define ω-algebras, which retain the equational axioms of forest algebras and are endowed with additional unary operations. We establish some useful properties of the free ω-algebra which entail that it is a forest algebra. A profinite algebra is defined to be a projective limit of a projective system of finite algebras [2]. For **BSS**[1], the pseudovariety of forest algebras generated by all syntactic forest algebras of piecewise-testable forest languages, we say that a profinite algebra S is pro-**BSS** if it is a projective limit of members of **BSS**. It is natural to study the free pro-**BSS** algebra as an ω-algebra. We show that the set of multiplicatively irreducible factors of the product of two elements is the union of the sets of multiplicatively irreducible factors of each one. We distinguish several kinds of non-trivial additively irreducible and non-trivial multiplicatively irreducible elements of the free ω-algebra. Then an algorithm to compute a canonical form for each element of the free ω-algebra in a certain variety is described and proved to be correct. If the relationship between the free ω-algebra in a certain variety and the free pro-**BSS** algebra is as in the word analog [1, Section 8.2], then the algorithm allows us to identify the structure of the latter.

[1] Stands for Bojańczyk, Segoufin, and Straubing as it was first introduced in [6].

Contents

Introduction

Several algebraic and combinatorial tools have played an important role in the development of Computer Science and its applications. The theory of formal languages, motivated both by linguistic studies and by development of computer languages in the 1950's, led to fruitful connections with Mathematics, in which algebraic tools such as semigroups, formal series, wreath products, and the combinatorics of words found an ideal ground for applications.

Eilenberg's treatise [12, 13] reflects already in the mid 1970's a significant development in the area, which both systematized earlier results and fostered further research. On the Computer Science side, finite automata proved to be a simple, yet very powerful model for efficient automatic processes. Their computing power is described by regular languages and thus it became important to determine whether given problems could be handled by restricted types of finite automata, which became a classification problem for regular languages. Eilenberg proposed as a framework for this classification the so-called varieties of languages and showed how they are in natural one-to-one correspondence with pseudovarieties of semigroups. The original Computer Science problem concerning formal languages thus became translated as a question about semigroups: to decide whether a given finite semigroup belongs to a given pseudovariety. Starting in the mid 1980's, Almeida showed how one could use profinite semigroups to handle some such decidability questions [1, 2].

But formal languages of words, in the sense of sequences of letters, are not always the most suitable tools to deal with Computer Science problems. Words correspond to sequential processes in which an action must be completed before the next one starts and only one action is executed at any given time. In many computing models, actions lead to branching and to the execution of other actions in parallel. Thus, trees are often subjacent to computer processes. Depending on the concrete Computer Science question, various algebraic tools have been introduced to deal with trees [6, 11, 14, 15, 17, 18, 28, 29]. For instance, (finite) trees may be regarded as terms in a suitable algebraic signature, which suggests as a possible approach to replace semigroups by more general (universal) algebraic structures in the classical algebraic theory of varieties of regular languages. The profinite method can be easily adapted to this setting, but it remains to be explored how this can be used to solve concrete problems. The limitations of such a model for trees to handle Computer Science problems [4, 8] have also led to the consideration of alternative models [6, 14, 15].

One of the major open problems in this area is the algebraic characterization of the class of trees that may be defined by first-order sentences.

In this dissertation we are planning to identify the structure of the free pro-**BSS** algebra.

First, we recall the forest algebra structures defined in [6], and verify some of their properties. We construct some examples provide useful parameters for the free forest algebra. We show the finiteness conditions for a forest algebras, the pre-image of a forest subalgebra over a forest algebra homomorphism may not be a forest subalgebra, and the image of a forest algebra homomorphism may not be a forest subalgebra.

After that, for a subset K of a forest algebra S, we define a binary relation $\backsim_K$ of K and we show that the relation $\backsim_K$ define a congruence relation of elements of S. Then we define a syntactic forest algebra which is the quotient of a forest algebra by $\backsim_K$ for some subset K of the forest algebra the so called syntactic congruence of K. Then we show that for an inverse zero action subset K of a forest algebra the quotient of the forest algebra by $\backsim_K$ is a forest algebra. Denote by A^Δ the free forest algebra on a finite alphabet A. A forest language is a subset of the horizontal monoid of A^Δ. For a forest language L, denote by V_L the set of all elements in the vertical monoid of A^Δ which map the identity element of the horizontal monoid of A^Δ into L. For a forest language L the set $K = (L, V_L)$ is called the inverse zero action subset of A^Δ determined by L; we show that the congruence relation $\sim_L$, which is defined in [6], coincides with the syntactic congruence $\backsim_K$, of the inverse zero action subset K. For a subset K of a forest algebra S, the syntactic congruence $\backsim_K$ is a more natural extension of the well-known syntactic congruence for monoids. Then we recall the theorem by Walukiewicz et.al. in [7], which gives a one to one correspondence between a pseudovariety of forest algebras and a variety of forest languages.

In Chapter 3, we define a metric on the free forest algebra A^Δ with respect to a pseudovariety of finite forest algebras $\mathbf{V}$ and we show that the basic operations with respect to this metric are contractive. We show that the completion of the free forest algebra with respect to the defined metric exists and is a forest algebra. We establish in this context an analog of Hunter's Lemma [23]. We also establish in this context an analog of Reiterman's Theorem [26].

In Chapter 4, we define ω-algebras: an ω-algebra is a set with two types of elements endowed with five binary operations and two unary operations, such that the equational axioms of forest algebras and three more conditions concerning the unary operations are satisfied. The class of ω-algebras is equational, so all free ω-algebras exist. Denote by $\mathcal{A}$ the free ω-algebra on a finite alphabet A. We give some examples of ω-algebras which are the key facts to show some useful properties of the free ω-algebra $\mathcal{A}$. In particular, we show that $\mathcal{A}$ is a forest algebra. We consider several subsets of the free ω-algebra $\mathcal{A}$ which are defined in terms of multiplicatively or additively irreducible factors or summands. These sets play a key role in the next chapter.

Finally in Chapter 5, we consider the variety $\mathcal{V}$ of ω-algebras, defined by the set Σ consisting of certain suitable identities motivated by the study of the pseudovariety **BSS**. We establish some consequences of the set of identities Σ and we describe an algorithm to compute the so-called canonical form of an element in the free ω-algebra $\mathcal{A}$ modulo Σ.

Chapter 1

Forest Algebra

Our main problem is to identify the structure of the free pro-**BSS** algebra. In order to tackle it, first we need to explore the structure of forest algebras.

Over a finite alphabet A, finite unranked ordered trees and forests are expressions defined inductively. If s is a forest and $a \in A$, then as is a *tree* where a is the root of the tree and it is the direct ancestor of the root of each tree in the forest s. Suppose that $t_1, \ldots, t_n$ is a finite sequence of trees, if we put each tree t_i on the right side of the tree t_{i-1} for $i = 2, 3, \ldots, n$ denoted by $t_1 + \cdots + t_n$ then the result is a *forest*. This applies as well to the empty sequence of trees, which thus gives rise to the *empty forest*, denoted by 0. The set of all forests is called the *horizontal* set.

A set L of forests over A is called a *forest language*.

If we take a forest and replace one of the leaves by a special symbol *hole*, which is denoted by $\square$, we obtain a *context*. A forest s can be substituted in place of the hole of a context p; the resulting forest is denoted by ps. There is a natural composition operation on contexts, the context qp is formed by replacing the hole of q with p. The set of all contexts is called the *vertical* set [9, 8].

In this chapter we explore the concept of forest algebra. We state several results which are used in the following chapters.

1.1 Preliminaries

Definition 1.1.1. A *forest algebra* S consists of a pair (H, V) of distinct monoids, subject to some additional requirements, which we describe below.

We write the operation in V, the vertical monoid, multiplicatively and the operation in H, the horizontal monoid, additively, although H is not assumed to be commutative. We accordingly denote the identity of V by $\square$ and that of H by 0.

We require that V acts on the left of H. That is, there is a map

$$(v, h) \in V \times H \mapsto vh \in H$$

such that $w(vh) = (wv)h$, for every $h \in H$ and every $v, w \in V$. We also require that this action be *monoidal*, that is, $\Box.h = h$, for every $h \in H$.

We further require that for every $h \in H$ and $v \in V$, V contains elements $h + v$ and $v + h$ such that for every $x \in S$,

$$(v + h)x = vx + h \quad \text{and} \quad (h + v)x = h + vx,$$

where vx is given by the action of v on x if x is a forest and by composition (multiplication) if x is a context.

We call the *equational axioms* of forest algebras, the preceding axioms on the elements of the forest algebras.

Finally in the definition of forest algebra we also require that the action be *faithful*, that is, if $vh = wh$, for every $h \in H$, then $v = w$.

Let (H_1, V_1) and (H_2, V_2) be algebras that satisfy the equational axioms of forest algebras. A forest algebra homomorphism

$$\alpha : (H_1, V_1) \to (H_2, V_2)$$

is a pair (γ, δ) of monoid homomorphisms

$$\begin{array}{ll} \gamma : & H_1 \to H_2, \\ \delta : & V_1 \to V_2 \end{array}$$

such that, for every $h \in H$ and every $v \in V$,

$$\gamma(vh) = \delta(v)\gamma(h) \qquad \text{and} \qquad \begin{cases} \delta(h + v) = \gamma(h) + \delta(v) \\ \delta(v + h) = \delta(v) + \gamma(h) \end{cases}$$

However, we will abuse notation slightly and denote both component maps by α.

Remark 1.1.2. Let (H_1, V_1) and (H_2, V_2) be algebras that satisfy the equational axioms of forest algebras. A mapping

$$\alpha = (\gamma, \delta) : (H_1, V_1) \to (H_2, V_2)$$

is called *forest algebra isomorphism*, if the mappings γ and δ are monoid isomorphisms and α is a forest algebra homomorphism.

Lemma 1.1.3. *In a forest algebra S the following equality holds:*

$$0 + \Box = \Box + 0 = \Box.$$

Proof. Let $v = \Box + 0$ and $v' = 0 + \Box$, since for all $x \in S$ we have

$$\begin{array}{lll} vx & = (\Box + 0)x & = x + 0, \\ v'x & = (0 + \Box)x & = 0 + x \end{array}$$

if $x \in H$, then both are equal to x. So for all $h \in H$, $vh = \Box h$ and also $v'h = \Box h$. Since the action is faithful, we conclude that $v = \Box$ and $v' = \Box$. $\Box$

The following lemma allows us to use the associativity of addition without reference to the type of elements.

Lemma 1.1.4. *In a forest algebra S the following equalities hold for all $x, y \in H$ and every $s \in S$:*

$$(x+y)+s = x+(y+s),$$
$$(x+s)+y = x+(s+y),$$
$$(s+x)+y = s+(x+y).$$

Proof. If $s \in H$, then the results hold from the associativity of H. Let $s \in V$. The terms $\Box + x$, $x + \Box$, $\Box + y$, and $y + \Box$ are in V and, for all $h \in H$, the following equalities hold:

$$((x+y)+\Box)h = (x+y)+h = x+(y+h) = x+(y+\Box)h = (x+(y+\Box))h,$$

which implies

$$(x+y)+\Box = x+(y+\Box),$$

and we have

$$((x+\Box)+y)h = (x+\Box)h+y = (x+h)+y$$

$$= x+(h+y) = x+(\Box+y)h = (x+(\Box+y))h,$$

which implies

$$(x+\Box)+y = x+(\Box+y).$$

Also we have

$$((\Box+(x+y))h = h+(x+y) = (h+x)+y = (\Box+x)h+y = ((\Box+x)+y)h$$

which implies

$$\Box+(x+y) = (\Box+x)+y.$$

So, for all $s \in V$, we have the following equalities

$$(x+y)+s = ((x+y)+\Box)s = (x+(y+\Box))s = x+(y+\Box)s = x+(y+s).$$

This shows the first equality and

$$(x+s)+y = (x+\Box)s+y = ((x+\Box)+y)s = (x+(\Box+y))s$$

$$= x+(\Box+y)s = x+(s+y),$$

which yields the second equality, while

$$s+(x+y) = (\Box+(x+y))s = ((\Box+x)+y)s = (\Box+x)s+y = (s+x)+y$$

yields the third equality. $\Box$

Let A be a finite alphabet, and let us denote by H^A the set of forests over A, and by V^A the set of contexts over A. Clearly H^A forms a monoid under $+$, see Figure 1.1, V^A forms a monoid under composition of contexts, see Figure 1.2, (the identity element is the empty context $\Box$), and substitution of a forest into a context defines a left action of V^A on H^A, see Figure 1.3. It is straightforward to verify that this action makes (H^A, V^A) into a forest algebra, which we denote by A^Δ.

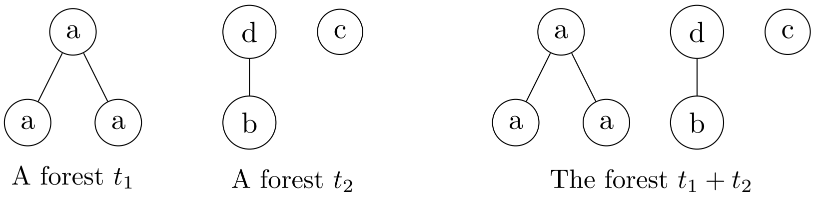

Figure 1.1: Forest addition

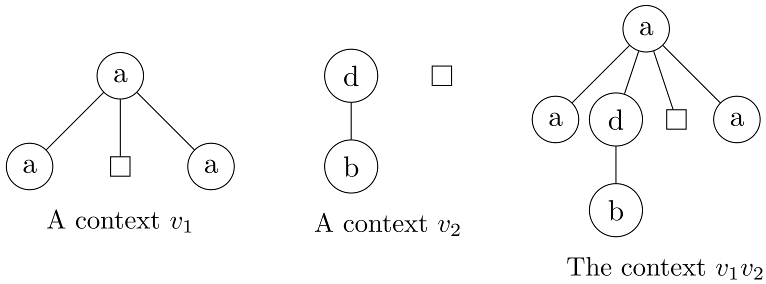

Figure 1.2: Context multiplication

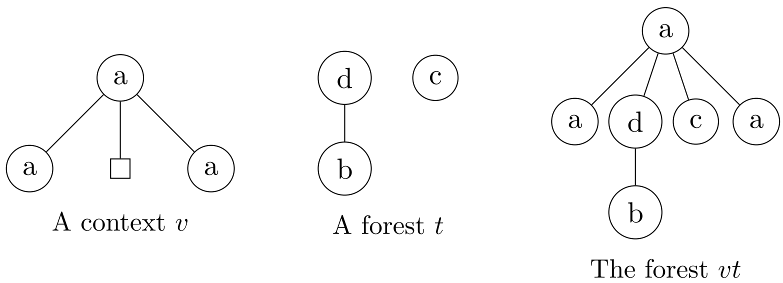

Figure 1.3: Action

Bojańczyk and Walukiewicz in [8, Lemma 3.6] showed that A^Δ is free in the sense of universal algebra: if (H, V) is a forest algebra, then every map $f : A \to V$ has a unique extension to a forest algebra homomorphism $\varrho : A^\Delta \to (H, V)$ such that $\varrho(a\Box) = f(a)$ for all $a \in A$. In view of this universal property we call A^Δ the free forest algebra on A. Since in the proof of [8, Lemma 3.6], the faithfulness does not play any role, we can state the following universal property:

Lemma 1.1.5. *For every algebra (H,V) that satisfies the equational axioms of forest algebras, every map $f : A \to V$ can be uniquely extended to a forest algebra homomorphism $(\alpha, \beta) : A^{\Delta} \to (H,V)$ such that $\beta(a\square) = f(a)$ for every $a \in A$.* $\square$

Bojańczyk and Walukiewicz in [8] denoted the set of forests over A by H_A and the set of contexts over A by V_A. To avoid confusion with syntactic forest algebras, which we consider later, we prefer instead the notation H^A and V^A.

For an algebra $S = (H,V)$ which satisfies the equational axioms of forest algebras, the relation $\sim_{\text{faith}}$ is defined as follows: for elements s and h in H, and for elements v and w in V,

$$\begin{aligned} h \sim_{\text{faith}} s &\quad \text{if and only if} \quad h = s \\ v \sim_{\text{faith}} w &\quad \text{if and only if} \quad \forall t \in H, vt = wt. \end{aligned} \tag{1.1}$$

Definition 1.1.6. A *congruence* relation is an equivalence relation $\equiv$ on an algebraic structure that satisfies

$$\mu(a_1, a_2, \ldots, a_n) \equiv \mu(a'_1, a'_2, \ldots, a'_n)$$

for every n-ary operation μ that defines the algebra structure, and all elements $a_1, \ldots, a_n, a'_1, \ldots, a'_n$ satisfying $a_i \equiv a'_i$ for each i.

Definition 1.1.7. Let u and v be elements of a forest algebra $S = (H,V)$. We mean by *basic operation*, which is denoted by $O(u,v)$, one of uv when $u \in V$, or $u + v$ when $u \in H$ or $v \in H$.

Note that the value of the operation $O(u,v)$ depends on the types of elements u and v in S.

Lemma 1.1.8. *The relation $\sim_{\text{faith}}$ as defined in* (1.1) *is a congruence relation.*

Proof. It is routine to check that, the relation $\sim_{\text{faith}}$ is an equivalence relation.

To show that $\sim_{\text{faith}}$ is a congruence, assume that

$$x \sim_{\text{faith}} y \quad \text{and} \quad p \sim_{\text{faith}} q.$$

Then we need to show that for the basic operations we have

$$O(x,p) \sim_{\text{faith}} O(y,q).$$

For the relation $x \sim_{\text{faith}} y$ and an element $p \in S$, we show that

$$O(x, p) \sim_{\text{faith}} O(y, p)$$

(similarly, $O(p, x) \sim_{\text{faith}} O(p, y)$). Indeed, if $x \sim_{\text{faith}} y$ and $p \sim_{\text{faith}} q$, then by what we have shown we have

$$O(x, p) \sim_{\text{faith}} O(y, p) \sim_{\text{faith}} O(y, q).$$

With respect to the type of x and y (they should have the same type) and a basic operation O, with respect to the type of p, we have the following:

1. Assume that x and y are in H. The relation $x \sim_{\text{faith}} y$ implies $x = y$ and since $p = p$ is always true, the equality $O(x, p) = O(y, p)$ and also the equality $O(p, x) = O(p, y)$ hold, which imply the relations

$$O(x, p) \sim_{\text{faith}} O(y, p) \quad \text{and} \quad O(p, x) \sim_{\text{faith}} O(p, y);$$

2. Assume that x and y are in V.

   (a) We show that for $p \in H$, and every $h \in H$, the equality $(x+p)h = (y+p)h$ holds. We have $x \sim_{\text{faith}} y$, so, for every $h \in H$, $xh = yh$. Therefore, for $p \in H$ we have $xh + p = yh + p$, for every $h \in H$. Since S satisfies the equational axioms of forest algebras, we have $(x+p)h = (y+p)h$, for every $h \in H$, which means $x+p \sim_{\text{faith}} y+p$. Similarly, we obtain $p + x \sim_{\text{faith}} p + y$.

   (b) We show that for $p \in V$, and every $h \in H$, the equality $(xp)h = (yp)h$ holds. We have $x \sim_{\text{faith}} y$, so, for every $h \in H$, $xh = yh$. Since $p \in V$, $ph \in H$ for every $h \in H$. Therefore, for every $h \in H$ we have $x(ph) = y(ph)$. Hence, for every $h \in H$, we have $(xp)h = (yp)h$, which means $xp \sim_{\text{faith}} yp$. Similarly, we obtain $px \sim_{\text{faith}} py$.

   (c) For $p \in H$, $xp \sim_{\text{faith}} yp$ is immediate by definition of the relation $\sim_{\text{faith}}$.

We have thus shown that the relation $\sim_{\text{faith}}$ is a congruence on S. □

Definition 1.1.9. A *subalgebra* of a forest algebra is a subset of a forest algebra, carrying the induced operations, that satisfies the equational axioms of forest algebras.

Definition 1.1.10. A *quotient* of a forest algebra is a forest algebra morphic image of a forest algebra.

We show in Remark 1.2.23 that a subalgebra and a quotient may not be a forest algebra (because of faithfulness). The solution is to take the faithful quotient of the result, which means for an algebra $S = (H, V)$ which satisfies the equational axioms of forest algebras, the *faithful quotient* of S which is denoted by $S/{\sim_{\text{faith}}}$. This is a forest algebra with the induced operations.

Let $S = (H, V)$ be a forest algebra and K be the faithful quotient of a subalgebra of S. Then we say that K is a *forest subalgebra* of S and we write $K \trianglelefteq S$.

Let $S_1 = (H_1, V_1)$ and $S_2 = (H_2, V_2)$ be forest algebras. Their *direct product* $S_1 \times S_2$ is $(H_1 \times H_2, V_1 \times V_2)$. The set

$$H_1 \times H_2 = \{(h_1, h_2) \mid h_1 \in H_1 \ \text{ and } \ h_2 \in H_2\}$$

is an additive monoid with identity $(0, 0)$ and the set

$$V_1 \times V_2 = \{(v_1, v_2) \mid v_1 \in V_1 \ \text{ and } \ v_2 \in V_2\}$$

is a multiplicative monoid with identity $(\Box, \Box)$. Operations are defined componentwise. The action is faithful. Indeed, for (v_1, v_2), $(w_1, w_2) \in V_1 \times V_2$, if for every $(h_1, h_2) \in H_1 \times H_2$,

$$(v_1, v_2)(h_1, h_2) = (w_1, w_2)(h_1, h_2),$$

then for every $h_1 \in H_1$, $v_1 h_1 = w_1 h_1$, which implies $v_1 = w_1$, and for every $h_2 \in H_2$, $v_2 h_2 = w_2 h_2$, which implies $v_2 = w_2$, thus $(v_1, v_2) = (w_1, w_2)$. So $S_1 \times S_2$ is a forest algebra.

1.2 Some More Examples

The following examples are useful in the rest of this thesis.

Example 1.2.1. The natural and evident example of forest algebras is the one with two elements

$$\mathcal{TFA} = (\ (H = \{0\}, +)\ ,\ (V = \{\Box\}, .)\ )$$

called *trivial forest algebra*.

Lemma 1.2.2. *Let $(S, +, 0)$ and $(T, \times, 1)$ be monoids where S is commutative. Assume that the monoid T acts on the monoid S by a monoid homomorphism*

$$\varphi : T \to \mathrm{End}(S)$$

where $\mathrm{End}(S)$ is assumed to be the monoid of semigroup homomorphisms from S to S. Let $S *_\varphi T$ be the semidirect product of S and T under the multiplication

$$(s, t) \cdot (s', t') = (s + \varphi(t)(s'), t \times t').$$

Denote $\varphi(t)(s)$ by ${}^t s$. Let

$$V = \{(s, t) \in S *_\varphi T \mid s + {}^t 0 = s\}.$$

Then (S, V) with respect to the following operations satisfies the equational axioms of forest algebras. For elements (s_1, t_1) and (s_2, t_2) in V and elements s and t in S define:

$$(s_1, t_1).(s_2, t_2) = (s_1 + {}^{t_1}s_2, t_1 \times t_2),$$

$$s +' (s_1, t_1) = (s + s_1, t_1) \quad \textit{and} \quad (s_1, t_1) +' s = (s_1 + s, t_1),$$

and

$$(s_1, t_1) * s = s_1 + {}^{t_1}p.$$

Proof. First, we show that V is a monoid. Let (s_1, t_1) and (s_2, t_2) be elements of V. Then we have

$$(s_1, t_1).(s_2, t_2) = (s_1 + {}^{t_1}s_2, t_1 \times t_2)$$

where we have

$$\begin{aligned} s_1 + {}^{t_1}s_2 + {}^{t_1 \times t_2}0 &= s_1 + {}^{t_1}(s_2 + {}^{t_2}0) \\ &= s_1 + {}^{t_1}(s_2), \end{aligned}$$

which implies that V is closed under the operation of $S *_\varphi T$. As we have $0 + {}^1 0 = 0$, $(0, 1)$ is in V. For $(s, t) \in V$ we have $(s, t).(0, 1) = (s + {}^t 0, t \times 1) = (s, t)$ and $(0, 1).(s, t) = (0 + {}^1 s, 1 \times t) = (s, t)$.

Since S is a commutative monoid, it is clear that for elements (s_1, t_1) and s respectively in V and S, the elements $s +' (s_1, t_1)$ and $(s_1, t_1) +' s$ are in V.

The following properties hold: for elements (s_1, t_1) and (s_2, t_2) in V and elements p and s in S we have

$$\begin{aligned} s +' ((s_1, t_1).(s_2, t_2)) &= s +' (s_1 + {}^{t_1}s_2, t_1 \times t_2) \\ &= (s + s_1 + {}^{t_1}s_2, t_1 \times t_2) \end{aligned}$$

and also

$$\begin{aligned} (s +' (s_1, t_1)).(s_2, t_2) &= (s + s_1, t_1).(s_2, t_2) \\ &= (s + s_1 + {}^{t_1}s_2, t_1 \times t_2), \end{aligned}$$

which imply the equality

$$(s +' (s_1, t_1)).(s_2, t_2) = s +' ((s_1, t_1).(s_2, t_2)).$$

We have

$$\begin{aligned} ((s_1, t_1).(s_2, t_2)) +' s &= (s_1 + {}^{t_1}s_2, t_1 \times t_2) +' s \\ &= (s_1 + {}^{t_1}s_2 + s, t_1 \times t_2) \end{aligned}$$

and also

$$\begin{aligned}((s_1,t_1)+' s).(s_2,t_2) &= (s_1+s,t_1).(s_2,t_2)\\ &= (s_1+s+{}^{t_1}s_2, t_1\times t_2),\end{aligned}$$

in which, as S is commutative, we have the equality

$$((s_1,t_1)+' s).(s_2,t_2) = ((s_1,t_1).(s_2,t_2))+' s.$$

We also have

$$s+((s_1,t_1)*p) = s+(s_1+{}^{t_1}p)$$

and also

$$\begin{aligned}(s+'(s_1,t_1))*p &= (s+s_1,t_1)*p\\ &= (s+s_1)+{}^{t_1}p,\end{aligned}$$

which imply the equality

$$(s+'(s_1,t_1))*p = s+((s_1,t_1)*p).$$

We have

$$((s_1,t_1)*p)+s = (s_1+{}^{t_1}p)+s$$

and also

$$\begin{aligned}((s_1,t_1)+' s)*p &= (s_1+s,t_1)*p\\ &= (s_1+s)+{}^{t_1}p,\end{aligned}$$

in which, as S is commutative, we have the equality

$$((s_1,t_1)+' s)*p = ((s_1,t_1)*p)+s.$$

We have the following equality

$$\begin{aligned}((s_1,t_1).(s_2,t_2))*p &= (s_1+{}^{t_1}s_2, t_1\times t_2)*p\\ &= s_1+{}^{t_1}s_2+{}^{(t_1\times t_2)}p\end{aligned}$$

and we also have

$$\begin{aligned}(s_1,t_1)*((s_2,t_2)*p) &= (s_1,t_1)*(s_2+{}^{t_2}p)\\ &= (s_1+{}^{t_1}(s_2+{}^{t_2}p)).\end{aligned}$$

As we assumed the monoid T acts on S, so we have

$${}^{(t_1\times t_2)}p = {}^{t_1}({}^{t_2}p)$$

and

$${}^{t_1}(s_2+{}^{t_2}p) = {}^{t_1}s_2+{}^{t_1}({}^{t_2}p).$$

We thus obtain the equality

$$((s_1,t_1).(s_2,t_2)) * p = (s_1,t_1) * ((s_2,t_2) * p).$$

The action $*$ is monoidal, as for an element p in S we have

$$(0,1) * p = 0 + {}^{1}p = id(p) = p. \qquad \square$$

Proposition 1.2.3. *Under the assumptions of Lemma 1.2.2, if one of the following holds, then (S, V) is a forest algebra.*

- *S is cancellative and the action of the monoid T on the monoid S is injective;*
- *T is a trivial monoid.*

Proof. Since by Lemma 1.2.2, (S, V) satisfies the equational axioms of forest algebras, in both cases we just need to show the faithfulness property. Let (s_1, t_1) and (s_2, t_2) be elements of V such that for all p in S the following equality holds:

$$(s_1,t_1) * p = (s_2,t_2) * p$$

which is

$$s_1 + {}^{t_1}p = s_2 + {}^{t_2}p.$$

So, for $p = 0$ we have $s_1 + {}^{t_1}0 = s_2 + {}^{t_2}0$ which implies the equality $s_1 = s_2$. We just need to check that for all s in S if for all p in S the equality $s + {}^{t_1}p = s + {}^{t_2}p$ holds, then the equality $t_1 = t_2$ holds.

In the first case S is cancellative implies that for all p in S the equality ${}^{t_1}p = {}^{t_2}p$ holds in which as the action of the monoid T on the monoid S is injective it implies that the equality $t_1 = t_2$ holds.

In the second case T is trivial implies the equalities $t_1 = t_2 = 1$. $\square$

Example 1.2.4. By Proposition 1.2.3 let T be the trivial monoid and S be the monoid of natural numbers $\mathbb{N}$ under usual addition. Then $S_N = (S, S *_\varphi T)$ is a forest algebra.

By the universal property of the free forest algebra A^Δ, there is a unique forest algebra homomorphism

$$\#_{\text{Nodes}} : A^\Delta \to S_N$$

such that

$$\#_{\text{Nodes}}(a\Box) = (1,1).$$

Definition 1.2.5. Let s be an element of the free forest algebra A^Δ, then the *number of nodes* of s is its image by the forest algebra homomorphism $\#_{\text{Nodes}}$ in Example 1.2.4.

Remark 1.2.6. We will abuse notation slightly and denote both forest algebra homomorphism from A^{Δ} to S_N and the number of nodes by $\#_{\text{Nodes}}$.

Example 1.2.7. In Lemma 1.2.2 let $T = \{1, c\}$ be the free idempotent monoid and S be the monoid of natural numbers $\mathbb{N}$ under usual addition. Let the monoid T acts on the monoid S by a monoid homomorphism

$$\begin{aligned} \varphi : T &\to \mathrm{End}(S) \\ 1 &\mapsto id_S \\ c &\mapsto 0_S \end{aligned}$$

Since for every element $(s,t) \in S *_{\varphi} T$ the equality $s + {}^{t}0 = s$ holds, as in both cases ${}^{t}0 = 0$, we have $S *_{\varphi} T$ with operation . as in Lemma 1.2.2 is a monoid.

Let operation $+'$ on elements of $S_C = (S, S *_{\varphi} T)$ be as in Lemma 1.2.2. We define the action of $S *_{\varphi} T$ on the left of S as follows: for an element (s,t) in $S *_{\varphi} T$ and an element p in S define

$$(s,t) * p = \begin{cases} s + {}^{t}p & \text{, if } \; t = 1 \\ s + {}^{t}p + 1 & \text{, if } \; t = c. \end{cases}$$

In order to show that S_C satisfies the equational axioms of forest algebras, in view of Lemma 1.2.2, we just need to check the following for elements s and p in S and elements $(s_1, t_1), (s_2, t_2) \in S *_{\varphi} T$:

- the equality $s + ((s_1, t_1) * p) = (s +' (s_1, t_1)) * p$ holds, since we have the following:

  $$s + (s_1, t_1) * p = \begin{cases} s + (s_1 + {}^{t_1}p) & \text{, if } \; t_1 = 1 \\ s + (s_1 + {}^{t_1}p + 1) & \text{, if } \; t_1 = c \end{cases}$$

  and also

  $$(s +' (s_1, t_1)) * p = (s + s_1, t_1) * p = \begin{cases} (s + s_1) + {}^{t_1}p) & \text{, if } \; t_1 = 1 \\ (s + s_1) + {}^{t_1}p + 1 & \text{, if } \; t_1 = c \end{cases}$$

  which imply the equality

  $$(s +' (s_1, t_1)) * p = s + ((s_1, t_1) * p);$$

- since S is commutative, the equality

  $$(s +' (s_1, t_1)) * p = s + ((s_1, t_1) * p)$$

  implies the equality:

  $$((s_1, t_1) * p) + s = ((s_1, t_1) +' s) * p;$$

- the equality $(s_1, t_1).(s_2, t_2)) * p = (s_1, t_1) * ((s_2, t_2) * p)$ holds, since we have the following:

$$((s_1, t_1).(s_2, t_2)) * p$$

$$= (s_1 + {}^{t_1}s_2, t_1 \times t_2) * p$$

$$= \begin{cases} s_1 + {}^{t_1}s_2 + {}^{(t_1 \times t_2)}p & \text{, if } \ t_1 \times t_2 = 1 \\ s_1 + {}^{t_1}s_2 + {}^{(t_1 \times t_2)}p + 1 & \text{, if } \ t_1 \times t_2 = c \end{cases}$$

$$= \begin{cases} s_1 + s_2 + p & \text{, if } \ t_1 = 1, t_2 = 1 \\ s_1 + s_2 + 1 & \text{, if } \ t_1 = 1, t_2 = c \\ s_1 + 1 & \text{, if } \ t_1 = c, t_2 = 1 \\ s_1 + 1 & \text{, if } \ t_1 = c, t_2 = c \end{cases}$$

and we also have

$$\begin{aligned} (s_1, t_1) * ((s_2, t_2) * p) &= (s_1, t_1) * \begin{cases} s_2 + {}^{t_2}p & \text{, if } \ t_2 = 1 \\ s_2 + {}^{t_2}p + 1 & \text{, if } \ t_2 = c \end{cases} \\ &= \begin{cases} s_1 + {}^{t_1}(s_2 + {}^{t_2}p) & \text{, if } \ t_1 = 1, t_2 = 1 \\ s_1 + {}^{t_1}(s_2 + {}^{t_2}p) + 1 & \text{, if } \ t_1 = c, t_2 = 1 \\ s_1 + {}^{t_1}(s_2 + {}^{t_2}p + 1) & \text{, if } \ t_1 = 1, t_2 = c \\ s_1 + {}^{t_1}(s_2 + {}^{t_2}p + 1) + 1 & \text{, if } \ t_1 = c, t_2 = c \end{cases} \\ &= \begin{cases} s_1 + s_2 + p & \text{, if } \ t_1 = 1, t_2 = 1 \\ s_1 + 1 & \text{, if } \ t_1 = c, t_2 = 1 \\ s_1 + s_2 + 1 & \text{, if } \ t_1 = 1, t_2 = c \\ s_1 + 1 & \text{, if } \ t_1 = c, t_2 = c \end{cases} \end{aligned}$$

We thus obtain the equality

$$((s_1, t_1).(s_2, t_2)) * p = (s_1, t_1) * ((s_2, t_2) * p);$$

- the action $*$ is monoidal, as for an element p in S we have

$$(0, 1) * p = 0 + {}^{1}p = id(p) = p.$$

This shows that S_C satisfies the equational axioms of forest algebras.

In view of Lemma 1.1.5, by the universal property of the free forest algebra A^Δ, there is a unique forest algebra homomorphism

$$\#_{\text{ConnectedParts}} : A^\Delta \to S_C$$

such that

$$\#_{\text{ConnectedParts}}(a\Box) = (1, c).$$

Definition 1.2.8. In view of the forest algebra homomorphism in Example 1.2.7, $\#_{\text{ConnectedParts}}$, we define the *number of connected parts* of a forest and a context as follows: for a forest h in the free forest algebra A^{Δ}, number of connected parts of the forest h is

$$\#_{\text{ConnectedParts}}(h),$$

and for a context v in the free forest algebra A^{Δ}, where

$$\#_{\text{ConnectedParts}}(v) = (n, t),$$

number of connected parts of the context v is

$$\begin{cases} n+1 & \text{, if } \ t = 1 \\ n & \text{, if } \ t = c. \end{cases}$$

Remark 1.2.9. For an element x in A^{Δ}, we denote the number of connected parts by CP.

Remark 1.2.10. By Definition 1.2.8 and Definition 1.2.5, if we have $s = h + t$ and h and t are non-trivial, then $\text{CP}(h) < \text{CP}(s)$ and $\text{CP}(t) < \text{CP}(s)$ and also $\#_{\text{Nodes}}(h) < \#_{\text{Nodes}}(s)$ and $\#_{\text{Nodes}}(t) < \#_{\text{Nodes}}(s)$.

Example 1.2.11. In Proposition 1.2.3 let T be the trivial monoid and S the powerset monoid of A, $P(A)$, under union. Then $S_L = (S, S *_{\varphi} T)$ is a forest algebra.

By the universal property of the free forest algebra A^{Δ}, there is a unique forest algebra homomorphism

$$\text{labels} : A^{\Delta} \to S_L$$

such that

$$\text{labels}(a\square) = (\{a\}, 1).$$

Definition 1.2.12. Let s be an element of the free forest algebra A^{Δ}, then the *set of labels of nodes* of s is its image by the forest algebra homomorphism labels in Example 1.2.11.

Remark 1.2.13. We will abuse notation slightly and denote both forest algebra homomorphism from A^{Δ} to S_L and the set of labels of nodes by labels.

Example 1.2.14. In Lemma 1.2.2 let T be the monoid of natural numbers $\mathbb{N}$ under usual addition and S the monoid of natural numbers $\mathbb{N}$ under operation max. Let the monoid T acts on the monoid S by a monoid homomorphism

$$\varphi : T \to \text{End}(S) \ \text{ via } \ t \mapsto (s \mapsto t + s).$$

As the set V contains every element $(s,t) \in S *_\varphi T$ such that the equality $\max\{s, t+0\} = s$ holds, it implies that the inequality $t \leq s$ holds. So we have

$$V = \{(s,t) \in S *_\varphi T \mid t \leq s\}.$$

Then Lemma 1.2.2 implies that $S_D = (S, V)$ satisfies the equational axioms of forest algebras.

By the universal property of the free forest algebra A^Δ, there is a unique forest algebra homomorphism

$$\mathrm{depth} : A^\Delta \to S_D$$

such that

$$\mathrm{depth}(a\Box) = (1,1).$$

Definition 1.2.15. In view of the forest algebra homomorphism depth in Example 1.2.14, we define the *depth* of a forest and a context as follows: for a forest h in the free forest algebra A^Δ, the depth of the forest h is $\mathrm{depth}(h)$, and for a context v in the free forest algebra A^Δ, where $\mathrm{depth}(v) = (n_1, n_2)$, the depth of the context v is n_1.

Remark 1.2.16. In view of the forest algebra homomorphism depth, by the way that we defined the action, for a context v in the free forest algebra A^Δ we have

$$\mathrm{depth}(v * 0) = (n_1, n_2) * 0 = \max\{n_1, n_2\},$$

and as we assumed $n_1 \geq n_2$, the depth of the context v is the depth of the forest $v * 0$.

Remark 1.2.17. We will abuse notation slightly and denote both forest algebra homomorphism from A^Δ to S_D and the depth by depth.

For instance, the elements 0 and $\Box$ of the free forest algebra A^Δ have depth 0.

Remark 1.2.18. If h_1 and u_1 are elements of A^Δ, such that $h_1 + u_1$ is an element of A^Δ, then

$$\mathrm{depth}(u_1 + h_1) = \max\{\mathrm{depth}(h_1), \mathrm{depth}(u_1)\}.$$

Moreover, if x is an element of A^Δ and $a \in A$, then $\mathrm{depth}(ax) = 1+\mathrm{depth}(x)$.

Example 1.2.19. In Lemma 1.2.2 let $T = \{1, c\}$ be the free idempotent monoid and S the powerset monoid of A, $P(A)$, under union. Let the monoid T acts on the monoid S by a monoid homomorphism

$$\begin{aligned} \varphi : T &\to \mathrm{End}(S) \\ 1 &\mapsto id_S \\ c &\mapsto 0_S \end{aligned}$$

Since the set V contains every element $(s,t) \in S *_{\varphi} T$ such that the equality $s \cup {}^{t}\emptyset = s$ holds, and for both cases ${}^{t}\emptyset = \emptyset$, we have $V = S *_{\varphi} T$.

Lemma 1.2.2 implies that (S, V) satisfies the equational axioms of forest algebras. Then $S_R = (S, S *_{\varphi} T)/\sim_{\text{faith}}$ is a forest algebra.

By the universal property of the free forest algebra A^{Δ}, there is a unique forest algebra homomorphism

$$\text{roots} : A^{\Delta} \to S_R$$

such that

$$\text{roots}(a\Box) = (\{a\}, c).$$

Definition 1.2.20. In view of the forest algebra homomorphism roots in Example 1.2.19, we define the *set of labels of roots* of a forest and a context as follows: for a forest h in the free forest algebra A^{Δ}, the set of labels of roots of the forest h is $\text{roots}(h)$, and for a context v in the free forest algebra A^{Δ}, where $\text{roots}(v) = (X, t)$, the set of labels of roots of the context v is X.

Remark 1.2.21. We will abuse notation slightly and denote both forest algebra homomorphism from A^{Δ} to S_R and the set of labels of roots by roots.

Example 1.2.22. Over a finite alphabet A, let $\sum_{i=1}^{n} a_i$ be a formal non-commuting sum of elements of A with $n \in \mathbb{N}$. Define sets

$$H = \{\sum_{i=1}^{n} a_i \mid n \in \mathbb{N}, a_i \in A\}$$

and

$$V = \{\sum_{i=1}^{n} a_i + \Box + \sum_{j=1}^{m} b_j \mid n, m \in \mathbb{N}, a_i, b_j \in A\}.$$

The set $S = (H, V)$ is a subset of the free forest algebra A^{Δ}, and it is closed under the basic operations in the free forest algebra.

Assume that $\{a, b\} \subseteq A$. Then, we show that $S = (H, V)$ is a forest algebra.

Since S is closed under the basic operations in the free forest algebra A^{Δ}, then $(V, ., \Box)$ and $(H, +, 0)$ are monoids, and S satisfies the equational axioms of forest algebras. It remains to check that the action is faithful. Let

$$v = \sum_{i_1=1}^{n_1} a_{i_1} + \Box + \sum_{j_1=1}^{m_1} b_{j_1} \quad \text{and} \quad w = \sum_{i_2=1}^{n_2} a'_{i_2} + \Box + \sum_{j_2=1}^{m_2} b'_{j_2}$$

be elements of V. Assume that v and w are such that for every $h = \sum_{k=1}^{n} c_k$ in H, the equality $v.h = w.h$ holds. From the definition of action, we have

$$v.h = \sum_{k=1}^{n_1+n+m_1} P_k \quad \text{and} \quad w.h = \sum_{k=1}^{n_2+n+m_2} Q_k,$$

together with the equality $v.h = w.h$, it implies that the equality $n_1 + m_1 = n_2 + m_2$ holds.

If the equality $n_1 = n_2$ holds, then $m_1 = m_2$. And from the equality $v.h = w.h$ we get that, for every i in $\{1, \ldots, n_1\}$ and j in $\{1, \ldots, m_1\}$, the equalities $a_i = a'_i$ and $b_j = b'_j$ hold. So, we have the equality $v = w$.

Now assume that the inequality $n_1 < n_2$ holds. Again, since for every h in H the equality $v.h = w.h$ holds, so we have, for every i in $\{1, \ldots, n_1\}$, the equality $a_i = a'_i$ holds. And since for every h, the equality

$$h + \sum_{j_1=1}^{m_2} b_{j_1} = \sum_{i_2=n_1+1}^{n_2} a'_{i_2} + h + \sum_{j_2=1}^{m_2} b_{j_2}$$

holds, so we can choose $h = a$ or $h = b$. Therefore, we have $a'_{n_1+1} = a$ and also $a'_{n_1+1} = b$, which is a contradiction.

For the case $n_1 > n_2$, we just need to exchange the roles of v and w and again we get to a contradiction. So, we have the equality $v = w$.

Now, assume that $A = \{a\}$, then $S = (H, V)$ is not a forest algebra. Because $v = a + \Box$ and $w = \Box + a$ are two different contexts, but for every $h = \sum_{i=1}^{n} a$, we have the equality $v.h = w.h$, since both are equal to $\sum_{k=1}^{n+1} a$.

For the case $|A| \geq 2$, by the universal property of the free forest algebra A^Δ, there is a unique forest algebra homomorphism

$$\psi : A^\Delta \to S,$$

such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A^\Delta \arrow[d, dotted, "\psi"] \\ & S \end{tikzcd}$$

Remark 1.2.23. Let $A = \{a, b\}$ and $S = (H, V)$ be the forest algebra and ψ be the forest algebra homomorphism in Example 1.2.22. In view of the universal property of the free forest algebra the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\iota_1"] \arrow[dr, "\iota_2"'] & A^\Delta \arrow[d, dotted, "f"] \\ & (A \setminus \{b\})^\Delta \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[dr] & A^\Delta \arrow[d, dotted, "\psi"] \\ & S \end{tikzcd}$$

where $\iota_2 : A \to (A \setminus \{b\})^\Delta$ is $\iota_2(a) = \iota_1(a)$ and $\iota_2(b) = \Box$. Let $i_1 : S \to A^\Delta$ and $i_2 : (A \setminus \{b\})^\Delta \to A^\Delta$ be the natural injections. Then the mapping

$\varphi : S \to S$ with $\varphi = \psi \circ i_2 \circ f \circ i_1$ is a forest algebra homomorphism, since φ is a composition of forest algebra homomorphism:

$$\begin{tikzcd} S \arrow[r, "i_1"] \arrow[rrrr, bend right, "\varphi"'] & A^\Delta \arrow[r, "f"] & (A \setminus \{b\})^\Delta \arrow[r, "i_2"] & A^\Delta \arrow[r, "\psi"] & S \end{tikzcd}$$

The set $T = \varphi(S)$ is a non-empty subset of S, in which by the forest algebra homomorphism ψ, T is not a forest subalgebra of S.

The trivial forest algebra $K' = \{0, \square\}$ is a forest subalgebra of S, but in view of the forest algebra homomorphism ψ, the set $K = \varphi^{-1}(K')$ is not a forest subalgebra of S.

1.3 Elementary Properties

Lemma 1.3.1. *In a forest algebra $S = (H, V)$, if $h_1, h_2 \in H$ then the following hold*

$$h_1 = h_2 \;\Leftrightarrow\; h_1 + \square = h_2 + \square \;\Leftrightarrow\; \square + h_1 = \square + h_2.$$

Proof. If $h_1 = h_2$ then the equality $h_1 + h = h_2 + h$ holds for all $h \in H$. By properties of the forest algebra S, this is equivalent to $(h_1 + \square)h = (h_2 + \square)h$ for every $h \in H$, which yields $h_1 + \square = h_2 + \square$.

On the other hand, if $h_1 + \square = h_2 + \square$, then to get the result it is enough to drop the $\square$ from contexts, which follows from $(h_1 + \square)0 = (h_2 + \square)0$, that is $h_1 = h_2$.

In a similar way

$$h_1 = h_2 \;\Leftrightarrow\; \square + h_1 = \square + h_2.$$

□

Lemma 1.3.2. *In a forest algebra $S = (H, V)$, the horizontal monoid H is finite if and only if the vertical monoid V is finite.*

Proof. First we will show that V finite implies H finite. By definition of forest algebra for every $h \in H$, $h + \square \in V$ and also for $h + \square \in V$, $h = (h + \square)0 \in H$, define

$$V_1 = \{h + \square \mid h \in H\} \subseteq V.$$

By Lemma 1.3.1, we showed that elements of V_1 are in one to one correspondence with elements of H. Indeed, the mapping $h \mapsto h + \square$ is injective on H. Since V is finite so is V_1 as subset of V. Therefore H is finite.

For the converse,[1] assume that H is finite. Since S is a forest algebra, because of faithfulness property, every context completely determined by its actions on all forests. So, the number of elements of V is bounded by the number of functions from H to H, which is finite. □

[1] We thank Prof. Igor Walukiewicz for this observation. Our original argument was much more involved.

Corollary 1.3.3. *A forest algebra $S = (H, V)$ is finite if and only if H is finite.* $\square$

For a forest algebra $S = (H, V)$, the following mapping is the action of the contexts on the left of the forest 0.

$$\begin{aligned} \_0 : V &\to H \\ v &\mapsto v0 \end{aligned}$$

Definition 1.3.4. A subset K of a forest algebra $S = (H, V)$ is called an *inverse zero action subset* if, for every context v, $v \in K$ if and only if $v0 \in K$.

Definition 1.3.5. Let h be a forest and v be a context in the free forest algebra A^{Δ}. We say that h is a *connected forest* or a *tree* if we cannot write h as a summation of two non-trivial forests. Also we say that v is a *connected context* if we cannot write v as a summation of a context and a non-trivial forest and vice versa.

Note that 0 and $\square$ are respectively a connected forest and a connected context.

Lemma 1.3.6. *Let $h_1 = t_1 + \cdots + t_n$ and $h_2 = s_1 + \cdots + s_m$ be sums of non-trivial trees in the free forest algebra A^{Δ}. Then $h_1 = h_2$ if and only if $m = n$ and for every $i = 1, \ldots, n$ the equality $s_i = t_i$ holds.*

Proof. The reverse implication is easy, we just need to sum equal trees.

Now, we show the direct implication. Recall that elements of the horizontal set are finite unranked ordered trees and forests and respectively $t_1 + \cdots + t_n$ and $s_1 + \cdots + s_m$ are the formal expressions for putting each tree t_i on the right side of the tree t_{i-1} for $i = 2, 3, \ldots, n$ and respectively putting each tree s_j on the right side of the tree s_{j-1} for $j = 2, 3, \ldots, m$. The equality $h_1 = h_2$, since they are ordered forests, implies the componentwise equality which is the equality $t_i = s_i$ for all i and the equality $n = m$. $\square$

Recall that a context v in the free forest algebra A^{Δ} is a forest in which exactly one leaf is the $\square$. Hence v can be written uniquely as $v = t_1 + \cdots + t_n$ as a sum of non-trivial trees over the alphabet $A \cup \{\square\}$, in which exactly one t_i has the leaf $\square$; we denote by $C(v)$ this tree t_i. So, every context v of the free forest algebra is uniquely of the form $H_1 + C(v) + H_2$ where H_1 and H_2 are forests and $C(v)$ is a tree over $A \cup \{\square\}$.

Lemma 1.3.7. *Let $v_1 = H_1 + C(v_1) + H_2$ and $v_2 = S_1 + C(v_2) + S_2$ be contexts in the free forest algebra A^{Δ}. Then $v_1 = v_2$ if and only if the equalities $H_1 = S_1$, $H_2 = S_2$, and $C(v_1) = C(v_2)$ hold.* $\square$

Lemma 1.3.8. *In the free forest algebra A^{Δ}, let a_1 and a_2 be elements of A. Then the contexts $a_1\square$ and $a_2\square$ are equal if and only if the equality $a_1 = a_2$ holds.*

Proof. The direct implication is obtained by acting on the forest 0. The reverse implication comes from the fact that, for every forest h, the equality $a_1 h = a_2 h$ holds. $\square$

For the free forest algebra A^Δ, every context v has a *factorization* of the form:

$$v = \prod_{i \in \mathbb{N}} v_i$$

where, for all $k \in \mathbb{N}$,

$$\begin{array}{ll} v_i = a_i \square & \text{for } \; i = 2k \\ v_i = h_{i,1} + \square + h_{i,2} & \text{for } \; i = 2k+1 \end{array}$$

with forests $h_{i,1}$ and $h_{i,2}$ in H^A and $a_i \in A \cup \{0\}$, for $a_i = 0$ let $0\square = \square$, such that if, for a positive integer k, $a_{2k} = 0$, then for all $j \geq 2k$, $v_j = \square$. We show that this factorization is unique by iteration on the number of nodes of $C(v)$. There are forests h_1 and h_2 and a context $C(v)$ such that $v = h_1 + C(v) + h_2$. If $C(v) = \square$, then result is immediate by Lemma 1.3.7 since we have $(h_1 + \square + h_2)$ as the factorization of v. Now, consider the case $C(v) \neq \square$, then $C(v) = a\square v_1$ and the product of $(h_1 + \square + h_2)a\square$ by the factorization of v_1 will give the result, uniqueness of h_1 and h_2 is from Lemma 1.3.7 and the uniqueness of $a\square$ is from Lemma 1.3.7 together with the forest algebra homomorphism roots.

Definition 1.3.9. Let s and t be elements of a forest algebra S. We say that t is a *scattered divisor* of s and denote it by $t \mid_s s$, when t has a decomposition of the form $t_1 \dots t_n$ and s has a decomposition of one of the following forms:

$$u_1 t_1 u_2 t_2 \dots u_n t_n \quad \text{or} \quad u_1 t_1 u_2 t_2 \dots u_n t_n u_{n+1}.$$

Note that, for some i's we may have $u_i = \square$.

We say that t is a *divisor* of s, if s has one of the following decompositions:

$$u_1 t u_2 \quad \text{or} \quad u_1 t.$$

In this case, we write $t \mid s$.

Remark 1.3.10. Let h be a forest and v be a context of the free forest algebra A^Δ. Let n be the maximum integer for which $a_1 \dots a_n \mid_s h$ with $a_i \in A$ for all $i \in \{1, \dots, n\}$, i.e.:

$$n = \max\{m \mid a_1 \dots a_m \mid_s h, a_i \in A \ (i = 1, \dots, m)\}.$$

In a similar way, let d be the maximum integer for which $a_1 \dots a_d \mid_s v0$ with $a_i \in A$ for all $i \in \{1, \dots, d\}$.

We claim that $\mathrm{depth}(h) = n$ and $\mathrm{depth}(v) = d$.

First, for a forest h in the free forest algebra A^{Δ}, we have $h = h_1 + \cdots + h_m$ a sum of non-trivial connected forests. By Remark 1.2.18, there is a positive integer i, such that $\mathrm{depth}(h) = \mathrm{depth}(h_i)$. As h_i is a non-trivial connected forest, then there is an element $a_1 \in A$ and a forest h^1, such that $h_i = a_1\Box(h^1)$, again by Remark 1.2.18, we have

$$\mathrm{depth}(h) = \mathrm{depth}(h_i) = \mathrm{depth}(a_1\Box(h^1)) = 1 + \mathrm{depth}(h^1).$$

Iterate the same argument for h^1. Since $\#_{\mathrm{Nodes}}(h)$ is finite and

$$\#_{\mathrm{Nodes}}(h) > \#_{\mathrm{Nodes}}(h^1),$$

there is a sequence of elements $a_1, \ldots, a_k$ in A such that $\mathrm{depth}(h) = k$ and $a_1 \ldots a_k \mid_s h$. We show that k is the maximum integer for which $a_1 \ldots a_k \mid_s h$ with $a_i \in A$ for all $i \in \{1, \ldots, k\}$. Assume that $n > k$ and $b_1 \ldots b_n \mid_s h$ where, for all $i \in \{1, \ldots, n\}$, $b_i \in A$. By applying the forest algebra homomorphism depth, we have $\mathrm{depth}(h) \geq n$, which is a contradiction. This establishes the claim $\mathrm{depth}(h) = n$. By Remark 1.2.16, we have $\mathrm{depth}(v) = \mathrm{depth}(v * 0)$, then the claim $\mathrm{depth}(v) = d$, is a consequence of $\mathrm{depth}(v * 0) = d$.

Definition 1.3.11. Let S be a forest algebra. We say that an element x of S is a *subterm* of an element y of S, if there exists an n-ary operation f, which is a composition of basic operations, and there are elements $t_1, \ldots, t_{n-1}$ in S such that $f(x, t_1, \ldots, t_{n-1}) = y$.

Lemma 1.3.12. *Let A be a finite alphabet. For elements x_1 and x in the free forest algebra A^{Δ}, if x_1 is a subterm of x then $\#_{\mathrm{Nodes}}(x_1) \leq \#_{\mathrm{Nodes}}(x)$.*

Proof. By definition of a subterm, there is an n-ary operation f, and there are elements $t_1, \ldots, t_{n-1}$ such that $f(x_1, t_1, \ldots, t_{n-1}) = x$. So, we have

$$\#_{\mathrm{Nodes}}(f(x_1, t_1, \ldots, t_{n-1})) = \#_{\mathrm{Nodes}}(x).$$

Since $\#_{\mathrm{Nodes}}$ is a forest algebra homomorphism then we have

$$\#_{\mathrm{Nodes}}(x) = \#_{\mathrm{Nodes}}(x_1) + \#_{\mathrm{Nodes}}(t_1) + \cdots + \#_{\mathrm{Nodes}}(t_{n-1}),$$

which implies the result. $\Box$

Lemma 1.3.13. *Let A be a finite alphabet. For an element x in the free forest algebra A^{Δ}, $\#_{\mathrm{Nodes}}(x) = 0$ if and only if x is a trivial element.*

Proof. If x is a trivial element, then $\#_{\mathrm{Nodes}}(x) = 0$. Now, assume that $\#_{\mathrm{Nodes}}(x) = 0$ but x is a non-trivial element, then there is an element $d\Box$, which is a subterm of x. By Lemma 4.1.7, we have $1 = \#_{\mathrm{Nodes}}(d\Box) \leq 0$, which is a contradiction. $\Box$

Chapter 2

Syntactic Congruence and Pseudovarieties

An important ingredient in our profinite approach to the study of the pseudovariety **BSS** is the syntactic congruence of a certain subset of a forest algebra, which we introduce in this chapter. We also consider the notion of a pseudovariety of finite forest algebras.

2.1 The Relation $\backsim_K$

Let $S = (H, V)$ be a forest algebra and K a subset of S. We take $H' = K \cap H$ and $V' = K \cap V$. We may define on S a relation $\backsim_K = (\sigma_K, \sigma'_K)$, the so-called *syntactic congruence* of K, as follows:

- for $h_1, h_2 \in H$, $h_1 \; \sigma_K \; h_2$ if for all $t, w, r \in V$:

  I. $th_1 \in K \iff th_2 \in K$;

  II. $t(rh_1 + w) \in K \iff t(rh_2 + w) \in K$;

  III. $t(w + rh_1) \in K \iff t(w + rh_2) \in K$.

- for $u, v \in V$, $u \; \sigma'_K \; v$ if for all $t, w \in V$ and $h \in H$:

  I. $tuh \; \sigma_K \; tvh$;

  II. $tuw \in K \iff tvw \in K$.

The relation $\backsim_K$ is defined only over elements of the same type, so for $u \in H$ and $v \in V$ and vice versa, they are not related, which we indicate by writing $u \not\backsim_K v$.

It is easy to check that σ_K and σ'_K are equivalence relations.

Lemma 2.1.1. *For a forest algebra S and a subset K of S, the equivalence relations σ_K and σ'_K are congruences with respect to the basic operations of S.*

Proof. See Appendix A, Section A.1. $\square$

Lemma 2.1.1, guarantees that the quotient of the forest algebra S with respect to equivalence $\backsim_K$ is well defined. Note that the equational axioms of forest algebras are preserved by taking quotients. If the quotient satisfies the faithfulness property, then it is a forest algebra.

Proposition 2.1.2. *Let $S = (H_S, V_S)$ be a forest algebra and let K be either a subset of H_S or an inverse zero action subset of S, see Definition 1.3.4. Then the quotient $S/\backsim_K$ is a forest algebra.*

Proof. We show that, if $uh\ \sigma_K\ vh$ for every $h \in H$ then $u\ \sigma'_K\ v$. Since $uh\ \sigma_K\ vh$ then by definition we have $tuh\ \sigma_K\ tvh$ for every $t \in V$. So, in order to show that $u\ \sigma'_K\ v$ we just need to show that for every t and w in V the following holds:

$$tuw \in K \iff tvw \in K.$$

Assume that $tuw \in K$ and K is an inverse zero action subset of S. Then we have $tuw0 \in K$. As $tuw0\ \sigma_K\ tvw0$, it follows that $tvw0 \in K$. Again since K is an inverse zero action subset of S, we deduce that $tvw \in K$. This shows that

$$tuw \in K \Rightarrow tvw \in K,$$

and the converse is obtained by interchanging the roles of u and v.

Now, assume that K is a subset of H_S. Then the following holds:

$$tuw \in K \iff tvw \in K,$$

due to the fact that $K \cap V_S = \emptyset$. $\square$

In the following definition we assume that K is a subset of a forest algebra S such that the quotient $S/\backsim_K$ is a forest algebra.

Definition 2.1.3. The *syntactic forest algebra* for K is the quotient of S with respect to the equivalence $\backsim_K$, where the horizontal semigroup H_K consists of equivalence classes σ_K of forests in S, while the vertical semigroup V_K consists of equivalence classes σ'_K of contexts in S.

The *syntactic homomorphism*

$$\alpha_K = (\gamma_K, \delta_K) : S \longrightarrow S/\backsim_K$$

assigns to every element of S its equivalence class in (H_K, V_K).

Let K be a subset of S such that the quotient $S/\backsim_K$ is a forest algebra, then the set K is saturated by the congruence $\backsim_K$, i.e. $u \backsim_K v$ and $u \in K$ implies $v \in K$. This means that $\alpha_K^{-1}\alpha_K(K) = K$.

Proposition 2.1.4. *The syntactic congruence of K is the largest one that saturates K.*

Proof. We show that, if $\propto$ is a congruence over S and K is a union of classes of $\propto$, then $s \propto t$ implies $s \backsim_K t$. Let a and b be elements of H_S, and let $t, r, w \in V_S$; since $a \propto b$ and $\propto$ is a congruence, then the relations $ta \propto tb$, $t(ra + w) \propto t(rb + w)$ and $t(w + ra) \propto t(w + rb)$ hold. However, K is a union of classes of $\propto$, therefore the elements in each of the pairs ta and tb, $t(ra + w)$ and $t(rb + w)$, and $t(w + ra)$ and $t(w + rb)$ are either both in K or both outside K. This is true for all $t, r, w \in V_S$, thus $a \ \sigma_K \ b$.

Now, let a and b be elements of V_S, and let $t, r, w \in V_S$ and $h \in H_S$; since $a \propto b$ and $\propto$ is a congruence, then the relations $tah \propto tbh$, $t(rah + w) \propto t(rbh + w)$, $t(w + rah) \propto t(w + rbh)$, and $taw \propto tbw$ hold. However, K is a union of classes of $\propto$, therefore each pair tah and tbh, $t(rah + w)$ and $t(rbh + w)$, $t(w + rah)$ and $t(w + rbh)$, and taw and tbw has either both elements in K or both outside K. This is true for all $t, r, w \in V_S$ and $h \in H_S$, thus $a \ \sigma'_K \ b$. $\square$

For $L \subset H^A$, Bojańczyk and Walukiewicz in [9], defined an equivalence relation $\sim_L$ over the free forest algebra as follows:

$$h_1 \sim_L h_2 \ ; \quad \forall v \in V^A, vh_1 \in L \Leftrightarrow vh_2 \in L$$

$$v_1 \sim_L v_2 \ ; \quad \forall h \in H^A, v_1 h \in L \Leftrightarrow v_2 h \in L$$

Then they showed that $\sim_L$ is a congruence relation. They defined the syntactic forest algebra over a forest language L, which they denote it by $A^\Delta/\sim_L$.

Lemma 2.1.5. *For a forest language $L \subset H^A$ let K be the inverse zero action subset of A^Δ where $K \cap H^A = L$. Then, the congruence relation $\sim_L$ coincides with the congruence relation $\backsim_K$.*

Proof. First, we show that $v_1 \sim_L v_2$ implies $v_1 \backsim_K v_2$ for contexts $v_1, v_2 \in V^A$. By the way the equivalence relation $\sim_L$ is defined and since $\sim_L$ is a congruence relation, then for every $h \in H^A$ and every $u \in V^A$, we have

$$uv_1 h \in L \Leftrightarrow uv_2 h \in L.$$

Now, assume that there exist $u, w \in V^A$ such that $uv_1 w \notin V_L$ but $uv_2 w \in V_L$, then by definition of inverse zero action subset $uv_1 w0 \notin L$ and $uv_2 w0 \in L$. Let $h = w0$, then $uv_1 h \notin L$ and $uv_2 h \in L$. Which is in contradiction with the assumption.

Conversely, that $v_1 \backsim_K v_2$ implies $v_1 \sim_L v_2$ is immediate from the definition of $\backsim_K$.

Similarly, for forests $h_1, h_2 \in H^A$, $h_1 \backsim_K h_2$ implies $h_1 \sim_L h_2$.

Finally, we show that $h_1 \sim_L h_2$ implies $h_1 \backsim_K h_2$. By definition of the equivalence relation $\sim_L$ and since $\sim_L$ is a congruence relation, then for every $v \in V^A$, we have

$$vh_1 \in L \Longleftrightarrow vh_2 \in L.$$

Now, assume that there exist u, v and w in V^A such that $u(vh_1 + w) \notin K$ but $u(vh_2 + w) \in K$, then by definition of inverse zero action subset $u(vh_1 + w)0 \notin L$ and $u(vh_2 + w)0 \in L$. Let $v' = u(v + w0)$, then $v'h_1 \notin L$ and $v'h_2 \in L$, which is in contradiction with the assumption. Similarly, we can show that for all contexts u, v and w in V^A, the following holds:

$$u(w + vh_1) \in K \iff u(w + vh_2) \in K. \qquad \Box$$

The following examples show that there are subsets K of A^Δ such that $A^\Delta/\backsim_K$ is a forest algebra even though K is neither a subset of H^A nor an inverse zero action subset of A^Δ.

Example 2.1.6. Over a finite alphabet A, let S be the free forest algebra $A^\Delta = (H^A, V^A)$ and K be the set of non-trivial elements of the form

$$\sum_{i=1}^{n} a_i + \Box + \sum_{j=1}^{m} b_j \quad \text{with} \quad n, m \in \mathbb{N}, a_i, b_j \in A.$$

It is easy to see that the quotient $S/\backsim_K = (H_K, V_K)$ is a forest algebra and

$$\begin{aligned} H_K &= \{\{0\}, K * 0, H^A \setminus (K * 0)\}, \\ V_K &= \{\{\Box\}, K, V^A \setminus K\}. \end{aligned}$$

Example 2.1.7. Over a finite alphabet A, let S be the free forest algebra $A^\Delta = (H^A, V^A)$, and $K = (H, V)$ with $H = V * 0$ and V is the set of all elements of the form

$$\prod_{i=1}^{n} a_i \Box \quad \text{with} \quad n \geq 1 \quad \text{and} \quad a_i \in A.$$

Let W be the set of elements of one of the forms

$$v.(\Box + h) \quad \text{or} \quad v.(h + \Box) \qquad \text{with} \quad v \in V \cup \{\Box\} \quad \text{and} \quad h \in H.$$

Then by easy calculations we can show that the quotient $S/\backsim_K$ is a forest algebra and

$$\begin{aligned} H_K &= \{\{0\}, H, H^A \setminus H\}, \\ V_K &= \{\{\Box\}, V, W, V^A \setminus (V \cup W)\}. \end{aligned}$$

Lemma 2.1.5 imply that we can adapt the results concerning the congruence relation $\sim_L$ defined in [9], to the results with congruence relation $\backsim_K$. The congruence relation $\backsim_K$ is defined specially for proof of the analog of Hunter's Lemma 3.1.27 which is shown in the next chapter.

2.2 On Pseudovarieties

Definition 2.2.1. A nonempty class $\mathbf{V}$ of finite forest algebras is called a *pseudovariety* if the following conditions hold:

(i) if $S \in \mathbf{V}$ and B is a forest subalgebra of S, then $B \in \mathbf{V}$;

(ii) if $S \in \mathbf{V}$ and $S \to B$ is an onto forest algebra homomorphism, then $B \in \mathbf{V}$;

(iii) $\mathbf{V}$ is closed under finite direct products.

We denote by $\mathbf{F}$ the pseudovariety of all finite forest algebras. Pseudovarieties are used in the next chapter, specially when we defined a metric on the free forest algebra.

Let (H_L, V_L) be the syntactic forest algebra of a forest language L.

Definition 2.2.2. We say that a forest language $L \subseteq H^A$ is recognized by a forest algebra homomorphism $\varphi : A^\Delta \to S$ into a forest algebra $S = (H_S, V_S)$ if there exists a subset $P \subseteq H_S$ such that $L = \varphi^{-1}P$ or, equivalently, if $L = \varphi^{-1}\varphi L$.

For a forest algebra S, we say that a subset K of S is $\mathbf{V}$*-recognizable* if

$$\exists S' \in \mathbf{V}, \quad \exists \varphi : S \to S' : \quad K = \varphi^{-1}\varphi(K).$$

By Proposition 2.1.4, the syntactic forest algebra of the forest language L is the smallest forest algebra which recognizes L. Indeed for a subset K of the free forest algebra A^Δ such that the quotient $A^\Delta/\backsim_K$ is a forest algebra the syntactic homomorphism α_K recognizes K, and if $\alpha : A^\Delta \to (H, V)$ is any other forest algebra homomorphism recognizing K, then α_K factors through α; that is, there is a forest algebra homomorphism $\beta : (H, V) \to (H/\sigma_K, V/\sigma'_K)$ such that $\beta\alpha = \alpha_K$.

For a forest algebra S, a subset K of S is called *recognizable* if it is $\mathbf{F}$-recognizable.

Bojańczyk, Straubing and Walukiewicz established in 2007 a version of Eilenberg's correspondence theorem for forest algebras[1].

Definition 2.2.3. Let $\mathbf{V}$ be a pseudovariety of forest algebras. For every finite alphabet A define

$$\mathcal{V}(A) = \{L \subseteq H^A \mid (H_L, V_L) \in \mathbf{V}\}.$$

We call $\mathcal{V}$ the *variety of forest languages* associated to $\mathbf{V}$, and write

$$\mathbf{V} \to \mathcal{V}.$$

[1]Personal communication by Mikolaj Bojańczyk.

Walukiewicz et al. in [7], showed that the mapping $\mathbf{V} \to \mathcal{V}$ is one to one. And also Walukiewicz et.al. in [7] showed that the following theorem holds.

Theorem 2.2.4. *Let $\mathcal{V}$ be an operator assigning to each finite alphabet A a family $\mathcal{V}(A)$ of A-languages. Then $\mathcal{V}$ is a variety of languages if and only if the following conditions hold:*

i.1 if $L \in \mathcal{V}(A)$, then $H^A \setminus L \in \mathcal{V}(A)$;

i.2 if $L_1, L_2 \in \mathcal{V}(A)$, then $L_1 \cap L_2 \in \mathcal{V}(A)$;

i.3 if $L \in \mathcal{V}(A)$ and $v \in V^A$, then the set

$$v^{-1}L = \{w \in H^A \mid vw \in L\}$$

is in $\mathcal{V}(A)$;

i.4 if $f : A^\Delta \to B^\Delta$ is a forest algebra homomorphism and if $L \in \mathcal{V}(B)$, then $Lf^{-1} \in \mathcal{V}(A)$.

We note that conditions i.1 and i.2 jointly assert that $\mathcal{V}(A)$ is closed under boolean operations. □

Let t be an element of the free forest algebra A^Δ. A *piece* of t is obtained by removing nodes from t. A forest language L over A is called *piecewise testable* if there exists $n \geq 0$ such that membership of t in L is determined by the set of pieces of t of size n or less. The size of a piece is the size of the forest, i.e. the number of nodes [6].

The pseudovariety **BSS** of finite forest algebras is generated by all syntactic forest algebras of piecewise testable forest languages.

2.2.1 Connection to a Pseudovariety of Finite Monoids

Let **W** be a pseudovariety of finite monoids. One can define a pseudovariety of finite forest algebras **HW** consisting of all finite forest algebras whose horizontal monoids are in **W** called the pseudovariety of *horizontally*-**W** forest algebras. Also we can define a pseudovariety of finite forest algebras **VW** which consisting of all finite forest algebras whose vertical monoids are in **W** we call it the pseudovariety of *vertically*-**W** forest algebras. The pseudovariety of all finite forest algebras whose horizontal and vertical monoids are in **W** is called the pseudovariety of *fully*-**W** forest algebras and denoted **FW**.

Recall that, for every forest algebra $S = (H, V)$, Lemma 1.3.1 shows that the mapping $h \mapsto h + \Box$ is injective on H, that is an embedding of the additive monoid H in the multiplicative monoid V. This implies the following result.

Lemma 2.2.5. *We have* $\mathbf{VW} \subset \mathbf{HW}$. □

Corollary 2.2.6. *The pseudovariety of vertically-***W** *forest algebras and of fully-***W** *forest algebras coincide* ($\mathbf{VW} = \mathbf{FW}$). □

2.3 Conclusions

We introduced syntactic forest algebras of a subset of a forest algebra which is a more natural extension of the well-known syntactic congruence for monoids. By Lemma 2.1.5, one can easily translate results concerning $\sim_L$ into results concerning $\backsim_K$. Our aim in considering the new congruence is to prove the analog of Hunter's Lemma 3.1.27 which is shown in the next chapter. Pseudovarieties are introduced as a class of finite forest algebras that is closed under taking subalgebras, onto homomorphic images and finite direct products. Pseudovarieties are important when we define a metric on the free forest algebra. We recall the theorem by Walukiewicz et.al. in [7], which gives a one to one correspondence between a pseudovariety of forest algebras and a variety of forest languages. As Salehi [27] puts it, most of the interesting classes of algebraic structures are varieties, and similarly as Walukiewicz et al. [7] put it, most of the interesting families of tree or string languages studied in the literature turn out to be varieties of some kind. The aforementioned variety theorem connects as a one-to-one correspondence these interesting families to each other. For a variety of languages there exists a characterization in terms of the structure of the syntactic forest algebra. Theorem 2.2.4 says that such a characterization exists, but it will not give any information about the algebraic structure.

Also many classes of languages fail to be a variety. But we may still have a characterization in terms of the syntactic homomorphism. As an example of such a case, for each finite alphabet A consider the family $\mathcal{V}(A)$ of A-languages consisting only of the forest language H of Example 2.1.7. Then $\mathcal{V}$ is not a variety because it does not satisfy i.4, but we can still characterize it in terms of the syntactic homomorphisms of these languages, see [7].[2]

By Lemma 2.2.5, for a pseudovariety $\mathbf{W}$ of finite monoids, we have $\mathbf{VW} \subset \mathbf{HW}$.

[2]We did this individually but as it states in [7] is more convenient.

Chapter 3

Metric Forest Algebras

A *profinite* forest algebra is a projective limit of finite forest algebras (which are viewed as discrete topological forest algebras). Alternatively, a profinite forest algebra may be defined as a compact forest algebra which, as a topological forest algebra, is residually finite. In particular, observe that a closed forest subalgebra of a profinite forest algebra is necessarily profinite.

For a pseudovariety of finite forest algebras $\mathbf{V}$, a profinite forest algebra is said to be *pro*-$\mathbf{V}$ if it is residually in $\mathbf{V}$. For each finite set A, there exists a free pro-$\mathbf{V}$ forest algebra on A, which is denoted $\overline{\Omega}_A\mathbf{V}$. Up to forest algebra isomorphism, it depends only on the cardinality $|A|$ and not on the set A itself so that we may sometimes write $\overline{\Omega}_{|A|}\mathbf{V}$ instead of $\overline{\Omega}_A\mathbf{V}$. The forest algebra $\overline{\Omega}_A\mathbf{V}$ may be constructed by completion of the free forest algebra A^Δ with respect to a pseudo-ultrametric naturally associated with $\mathbf{V}$.

For a pseudovariety of finite forest algebras $\mathbf{V}$, we show that the completion of the free forest algebra A^Δ with respect to the pseudo-ultrametric associated with $\mathbf{V}$ exists and is a forest algebra.

In this chapter we adapt some of the results on metric semigroups in [2] to the context of forest algebras.

3.1 Metrics Associated with a Pseudovariety of Forest Algebras

For two elements $u, v \in A^\Delta$ and a forest algebra B if for every forest algebra homomorphism

$$\varphi : A^\Delta \to B$$

the equality $\varphi(u) = \varphi(v)$ holds, then we say that B *satisfies the identity* $u = v$ and we write $B \models u = v$. For a pseudovariety of finite forest algebras $\mathbf{V}$, define:

$$r(u, v) = \min\{|B| \mid \ B \in \mathbf{V} \ \ and \ \ B \not\models u = v\}$$

and

$$d(u,v) = 2^{-r(u,v)}$$

where we take $\min \emptyset = \infty$ and $2^{-\infty} = 0$.

Since a finite forest algebra B has at least the identity elements $\square$ and 0, for all $u, v \in A^{\Delta}$ we have $r(u,v) \geq 2$ or, equivalently, $d(u,v) \leq 2^{-2}$.

Note that in the above definition every forest algebra homomorphism $\varphi : A^{\Delta} \to B$ can be assumed to be onto.

Example 3.1.1. For every $u \in H^A$ and $v \in V^A$, $d(u,v) = 2^{-2}$. Indeed, for every forest algebra homomorphism $\varphi : A^{\Delta} \to \mathcal{TFA}$, we have $\varphi(u) \neq \varphi(v)$. This means that $r(u,v) = 2$, whence $d(u,v) = 2^{-2}$.

Definition 3.1.2. A function

$$d : X \times X \to \mathbb{R}^{\geq 0}$$

is said to be a *pseudo-ultrametric* on the set X if the following properties hold for all $u, v, w \in X$:

1. $d(u,u) = 0$;
2. $d(u,v) = d(v,u)$;
3. $d(u,w) \leq \max\{d(u,v), d(v,w)\}$.

We then also say that X is a *pseudo-ultrametric space.*

If instead of Condition 3, the following weaker condition holds

4. $d(u,w) \leq d(u,v) + d(v,w)$ (triangle inequality),

then d is said to be a *pseudo-metric* on X, and X is said to be a *pseudo-metric space.*

If the following condition holds:

5. $d(u,v) = 0$ implies $u = v$,

then we drop the prefix "pseudo".

Proposition 3.1.3. *Let* $\mathbf{V}$ *be a pseudovariety of finite forest algebras. The function* d *is a pseudo-ultrametric on* A^{Δ}*.*

Proof. Let $u, v, w \in A^{\Delta}$. For every forest algebra homomorphism $\varphi : A^{\Delta} \longrightarrow B$ and $B \in \mathbf{V}$, $u = v$ implies $\varphi(u) = \varphi(v)$ so

$$\begin{aligned} r(u,u) &= \min\left\{|B| \mid B \in \mathbf{V} \text{ and } \exists \psi : A^{\Delta} \longrightarrow B \ : \ \psi(u) \neq \psi(u)\right\} \\ &= \min \emptyset = \infty, \end{aligned}$$

$$\Rightarrow d(u,u) = 2^{-\infty} = 0.$$

For every forest algebra homomorphism $\varphi : A^\Delta \longrightarrow B$ and $B \in \mathbf{V}$, the inequalities $\varphi(u) \neq \varphi(v)$ and $\varphi(v) \neq \varphi(u)$ are equivalent, so

$$\begin{aligned} r(u,v) &= \min\big\{|B| \mid \ B \in \mathbf{V} \ \text{ and } \ \exists \psi : A^\Delta \longrightarrow B \ : \ \psi(u) \neq \psi(v)\big\} \\ &= \min\big\{|B| \mid \ B \in \mathbf{V} \ \text{ and } \ \exists \psi : A^\Delta \longrightarrow B \ : \ \psi(v) \neq \psi(u)\big\} \\ &= r(v,u) \end{aligned}$$

$\Rightarrow d(u,v) = 2^{-r(u,v)} = 2^{-r(v,u)} = d(v,u)$.

For showing $d(u,w) \leq \max\{d(u,v),\ d(v,w)\}$, it is enough to show that the following inequality holds.

$$r(u,w) \geq \min\{r(u,v),\ r(v,w)\} \tag{3.1}$$

If $r(u,w) = \infty$ then the inequality (3.1), is clear. Now suppose that $n = r(u,w) < \infty$ and $r(u,w) \ngeq \min\{r(u,v), r(v,w)\}$. Thus, $r(u,v) > n$ and $r(v,w) > n$. This means that for every $B \in \mathbf{V}$ with $|B| \leq n$ and for every forest algebra homomorphism $\varphi : A^\Delta \longrightarrow B$, $\varphi(u) = \varphi(v)$ by the first of the preceding inequalities and $\varphi(v) = \varphi(w)$ by the second one. Hence $\varphi(u) = \varphi(w)$, which contradicts the equality $r(u,w) = n$.

Properties $(1)-(4)$ hold, so d is a pseudo-ultrametric and A^Δ is a pseudo-ultrametric space. □

Property (5) may not hold for d. Let $h_1, h_2 \in H^A$ be two distinct forests and let $\mathbf{V}$ be the pseudovariety of trivial forest algebras. Then, for every forest algebra homomorphism $\varphi : A^\Delta \longrightarrow \mathcal{TFA}$ we have $\varphi(h_1) = \varphi(h_2)$; so $d(h_1, h_2) = 0$ but $h_1 \neq h_2$.

Definition 3.1.4. A function $f : (X, d_X) \longrightarrow (Y, d_Y)$ between two pseudo-metric spaces is said to be *uniformly continuous* if the following condition holds:

$$\forall \epsilon > 0 \ \ \exists \delta > 0 \ \ \forall x_1, x_2 \in X \ , \quad (d_X(x_1,x_2) < \delta \Longrightarrow d_Y(f(x_1), f(x_2)) < \epsilon).$$

Proposition 3.1.5. *The basic operations are contractive:*

$$d(O(u,w), O(v,z)) \leq \max\{d(u,v), d(w,z)\}.$$

In particular, the basic operations on A^Δ are uniformly continuous.

Proof. By definition of the metric, the claim is equivalent to showing that

$$r(O(u,w), O(v,z)) \geq \min\{r(u,v), r(w,z)\}.$$

If either u and v, or w and z do not have the same type, then $r(u,v) = 2$ or $r(w,z) = 2$ and in both cases $\min\{r(u,v), r(w,z)\} = 2$ and the above inequality becomes $2 \leq r(O(u,w), O(v,z))$, which is always true. We can therefore assume that both pairs of elements u, v and w, z have the same type. Let $n_1 = r(u,v)$ and $n_2 = r(w,z)$ and let $n = \min\{n_1, n_2\}$. Then we have:

$$\forall B \in \mathbf{V} \ , \quad |B| < n \quad \forall \, \varphi : \ A^\Delta \longrightarrow B \ , \quad \begin{cases} \varphi(u) = \varphi(v) \\ \varphi(w) = \varphi(z) \end{cases} \tag{3.2}$$

From Equation (3.2), it follows directly that for every forest algebra homomorphism $\varphi : A^\Delta \longrightarrow B$,

$$O(\varphi(u), \varphi(w)) = O(\varphi(v), \varphi(z))$$

which implies the equality $\varphi(O(u,w)) = \varphi(O(v,z))$. We have thus shown that the basic operations are contractive. Hence, they are uniformly continuous. $\square$

By Lemma 1.3.1 and the fact that, for $v_1, v_2 \in V$, we have

$$v_1 = v_2 \ \Rightarrow \ v_1 0 = v_2 0,$$

we can easily see that:

Lemma 3.1.6. *For $h_1, h_2 \in H^A$ and $v_1, v_2 \in V^A$, we have the following facts:*

1. $r(h_1, h_2) = r(h_1 + \square, h_2 + \square)$;
2. $r(h_1, h_2) = r(\square + h_1, \square + h_2)$;
3. $r(v_1, v_2) \leq r(v_1 0, v_2 0)$. $\square$

Example 3.1.7. Let A be a finite alphabet and $v_1 = a + \square$ and $v_2 = a\square$ in V^A. In view of the forest algebra homomorphism roots in Example 1.2.19 and since the forest algebra S_R is finite, we have

$$\mathrm{roots}(a\square) = (\{a\}, 1) \ \text{ and } \ \mathrm{roots}(a + \square) = (\{a\}, 0),$$

which implies that there exists $n \in \mathbb{N}$ such that $r(v_1, v_2) = n < \infty$.

On the other hand, $v_1 \neq v_2$ but $v_1 0 = v_2 0$, so $r(v_1 0, v_2 0) = \infty > n$. So we may have strict inequality in Lemma 3.1.6.

Definition 3.1.8. For a (pseudo-ultra)metric d on a set X, $u \in X$, and a positive real number ϵ, consider the *open ball*

$$B_\epsilon(u) = \{v \in X \mid d(u,v) < \epsilon\}.$$

The point u is the *center* and ϵ is the *radius* of the ball.

A metric space that can be covered by a finite number of balls of any given positive radius is said to be *totally bounded*.

Proposition 3.1.9. *Let* $\mathbf{V}$ *be a pseudovariety of finite forest algebras. The pseudo-ultrametric space* (A^Δ, d) *is totally bounded.*

Proof. For a given $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $2^{-n} < \epsilon$. Up to forest algebra isomorphism, there are only finitely many forest algebras of cardinality at most n. For such a forest algebra $T_i = (H_i, V_i)$ in $\mathbf{V}$, consider all possible (there are only finitely many) forest algebra homomorphisms $\varphi_{i,j} : A^\Delta \longrightarrow T_i$ and let $T = \prod_{i,j} T_i$ and

$$\begin{aligned} \varphi : \ & A^\Delta \longrightarrow T \\ & w \mapsto \left(\varphi_{i,j}(w)\right)_{i,j} \end{aligned}$$

where $T \in \mathbf{V}$.

For all $t \in T$, choose $u_t \in A^\Delta$, such that $\varphi(u_t) = t$. For $k \in A^\Delta$, if $t = \varphi(k)$ then $\varphi(k) = \varphi(u_t)$ which implies $k \in B_\epsilon(u_t)$. Thus

$$A^\Delta \subseteq \bigcup_{t \in T} B_\epsilon(u_t).$$

Hence, A^Δ is totally bounded, since $T = (H, V)$ is finite. □

Definition 3.1.10. A sequence $\{u_n\}_n$ in a (pseudo-ultra)metric space X is said to be a *Cauchy sequence*, if

$$\forall \epsilon > 0 \ \ \exists N \quad (m, n \geq N \implies d(u_m, u_n) < \epsilon)\,.$$

Note that every convergent sequence is a Cauchy sequence. The space X is *complete* if every Cauchy sequence in X converges in X.

Recall that, if $u, u \in A^\Delta$ have different types, then $d(u, w) = 2^{-2}$. This yields immediately the following result;

Lemma 3.1.11. *A Cauchy sequence of elements of* A^Δ*, cannot have an infinite number of elements of both* H^A *and* V^A*.* □

Definition 3.1.12. A *(pseudo-)metric forest algebra* is a forest algebra endowed with a pseudo-metric d and that the basic operations are uniformly continuous.

A metric forest algebra B is called *complete* if every Cauchy sequence in B converges in B.

Remark 3.1.13. Note that, by [22, Theorem 1.15], every metric space has a completion.

By Lemma 3.1.11, it is natural to consider the completion of A^Δ, denoted $\overline{\mathtt{C}}_A\mathbf{V}$, as the union of the completions of H^A and V^A which denoted respectively $\overline{\mathtt{C}}_{\mathbf{V}}H^A$ and $\overline{\mathtt{C}}_{\mathbf{V}}V^A$.

Since operations on A^Δ are uniformly continuous, they do extend to uniformly continuous operations on $\overline{\mathtt{C}}_A\mathbf{V}$. Hence, $\overline{\mathtt{C}}_A\mathbf{V}$ satisfies naturally the equational axioms of forest algebras.

Proposition 3.1.14. *There exists a complete metric forest algebra* $\overline{\mathsf{C}}_A\mathbf{V}$ *and a uniformly continuous forest algebra homomorphism* $\iota : A^{\Delta} \longrightarrow \overline{\mathsf{C}}_A\mathbf{V}$ *with the following* universal property*: for every uniformly continuous forest algebra homomorphism* $f : A^{\Delta} \longrightarrow B$ *into a complete metric forest algebra* B*, there exists a unique uniformly continuous forest algebra homomorphism* $\hat{f} : \overline{\mathsf{C}}_A\mathbf{V} \longrightarrow B$ *such that* $\hat{f} \circ \iota = f$.

$$\begin{tikzcd} A^{\Delta} \arrow[r, "\iota"] \arrow[dr, "f"'] & \overline{\mathsf{C}}_A\mathbf{V} \arrow[d, dotted, "\hat{f}"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} & A^{\Delta} \arrow[dl, "\iota"'] \arrow[dr, "\eta"] & \\ \overline{\mathsf{C}}_A\mathbf{V} \arrow[rr, dotted, bend right, "\hat{\iota}"'] & & D \arrow[ll, dotted, bend right, "\hat{\eta}"'] \end{tikzcd}$$

Moreover, if $\eta : A^{\Delta} \longrightarrow D$ *is another uniformly continuous forest algebra homomorphism into a complete metric forest algebra with the above universal property, then the induced unique uniformly continuous forest algebra homomorphisms* $\hat{\iota} : \overline{\mathsf{C}}_A\mathbf{V} \longrightarrow D$ *and* $\hat{\eta} : D \longrightarrow \overline{\mathsf{C}}_A\mathbf{V}$ *are mutually inverse.*

Proof. By [32, Theorem 24.4], the completion exists and $\iota(A^{\Delta})$ is dense in $\overline{\mathsf{C}}_A\mathbf{V}$. And we have the following universal property of the completion $\overline{\mathsf{C}}_A\mathbf{V}$ of A^{Δ} as a metric space.

For every uniformly continuous forest algebra homomorphism $f : A^{\Delta} \to B$ of A^{Δ} into a complete metric forest algebra B, there exists a unique lifting of f to a uniformly continuous map $\hat{f} : \overline{\mathsf{C}}_A\mathbf{V} \to B$ making the diagram

$$\begin{tikzcd} A^{\Delta} \arrow[r, "\iota"] \arrow[dr, "f"'] & \overline{\mathsf{C}}_A\mathbf{V} \arrow[d, dotted, "\hat{f}"] \\ & B \end{tikzcd} \tag{3.3}$$

commute. Up to forest algebra isomorphism, the completion of A^{Δ} is the unique metric forest algebra satisfying this property. Therefore we just need to check that ι and $\hat{f}$ are forest algebra homomorphisms.

Note that for every element x in the completion $\overline{\mathsf{C}}_A\mathbf{V}$ there is a sequence $\{x_n\}_n$ of elements of A^{Δ} such that $\lim \iota(x_n) = x$.

We claim that the mapping ι respects basic operations of forest algebra. For every $x, y \in \overline{\mathsf{C}}_A\mathbf{V}$, since $\iota(A^{\Delta})$ is dense in $\overline{\mathsf{C}}_A\mathbf{V}$, there are sequences $\{x_n\}_n$ and $\{y_n\}_n$ of elements of A^{Δ} such that $\lim \iota(x_n) = x$ and $\lim \iota(y_n) = y$. By using the fact that ι and basic operations are uniformly continuous, we have

$$\lim O(\iota(x_n), \iota(y_n)) = O(\lim \iota(x_n), \lim \iota(y_n)) = O(x, y)$$

and

$$\lim \iota(O(x_n, y_n)) = \iota(\lim O(x_n, y_n)) = O(x, y).$$

Therefore the equality $\lim \iota(O(x_n, y_n)) = O(\lim \iota(x_n), \lim \iota(y_n))$ holds. So ι respects basic operations of forest algebra.

We show that $\overline{\mathsf{C}}_A\mathbf{V}$ is a forest algebra. Consider elements $u, v \in \overline{\mathsf{C}}_\mathbf{V}V^A$ be such that for every element $h \in \overline{\mathsf{C}}_\mathbf{V}H^A$ the equality $uh = vh$ holds. As $u, v \in \overline{\mathsf{C}}_\mathbf{V}V^A$ and for all h with $h \in \overline{\mathsf{C}}_\mathbf{V}H^A$, consider sequences $\{u_n\}_n$, $\{v_n\}_n$ and $\{h_n\}_n$, sequences of elements of A^Δ, such that $u = \lim \iota(u_n)$, $v = \lim \iota(v_n)$ and $h = \lim \iota(h_n)$. Since the equality $uh = vh$ holds, we have $\lim \iota(u_n)\iota(h_n) = \lim \iota(v_n)\iota(h_n)$ which implies the equality $\lim \iota(u_n h_n) = \lim \iota(v_n h_n)$. We show that for every $\varepsilon > 0$, there is a positive integer N such that for all $n \geq N$ the inequality $d(u_n, v_n) < \varepsilon$ holds.

The equality $us = vs$ holds for all s in H^A, which implies the equality $\lim \iota(u_n s) = \lim \iota(v_n s)$, that is for every positive integer m there is a positive integer M such that for all $n \geq M$ the inequality $d(u_n s, v_n s) < 2^{-m}$ holds. By definition of d, for every forest algebra B in $\mathbf{V}$ such that $|B| \leq m$ and every forest algebra homomorphism

$$\varphi : A^\Delta \to B$$

the equality $\varphi(u_n s) = \varphi(v_n s)$ holds. Since B is a forest algebra and

$$\varphi(u_n)\varphi(s) = \varphi(v_n)\varphi(s),$$

we have $\varphi(u_n) = \varphi(v_n)$ which implies that $d(u_n, v_n) < 2^{-m}$. Hence, the equality $u = v$ holds.

By assumption, since f is a forest algebra homomorphism, for all elements $x, y \in A^\Delta$ and basic operation $O(x, y)$, the following equality holds

$$f(O(x, y)) = O(f(x), f(y)).$$

Now, we will show that $\hat{f}$ respects basic operations of forest algebras. For $x, y \in \overline{\mathsf{C}}_A\mathbf{V}$, if $O(x, y)$ be a basic operation in the forest algebra, then by using the fact that $\hat{f}$ and basic operations are uniformly continuous, f is a forest algebra homomorphism and Diagram (3.3), commutes, we have the following:

$$\begin{aligned}
\hat{f}(O(x, y)) &= \hat{f}(\lim \iota(O(x_n, y_n))) \\
&= \lim \hat{f}(\iota(O(x_n, y_n))) \\
&= \lim f(O(x_n, y_n)) \\
&= \lim O(f(x_n), f(y_n)) \\
&= O(\lim f(x_n), \lim f(y_n)) \\
&= O(\lim \hat{f}(\iota(x_n)), \lim \hat{f}(\iota(y_n))) \\
&= O(\hat{f}(\lim \iota(x_n)), \hat{f}(\lim \iota(y_n))) \\
&= O(\hat{f}(x), \hat{f}(y)).
\end{aligned}$$

□

Recall that a *Hausdorff space* is a topological space in which distinct points have disjoint neighbourhoods. The *Hausdorff completion* of a pseudo-metric space X is a complete metric space X' together with a uniformly continuous map $\iota_X : X \to X'$ such that $\iota_X(X)$ is dense in X', and for every uniformly continuous map f from X to a complete metric space X'' there is a unique uniformly continuous map $g : X' \to X''$ such that $f = g \circ \iota_X$. In view of [31, Theorem 19.36], such a completion always exists. The Hausdorff completion of the ultrametric space (A^{Δ}, d), is denoted by $\overline{\Omega}_A\mathbf{V}$. In view of the proof of Proposition 3.1.14, $\overline{\Omega}_A\mathbf{V}$ is a forest algebra.

Definition 3.1.15. A subset of a metric space is *clopen* if it is both closed and open.

Definition 3.1.16. A metric space is said to be *zero-dimensional* if every open set is a union of clopen subsets.

Definition 3.1.17. A *topological forest algebra* is a forest algebra which is also a topological space, and whose operations are uniformly continuous.

Recall that a metric space is *compact* if every sequence admits some convergent subsequence. Equivalently, every covering by open subsets contains a finite covering.

Definition 3.1.18. A *compact forest algebra* is a topological forest algebra whose topology is compact and Hausdorff. Finite forest algebras are viewed as topological forest algebras under the discrete topology.

Note that a metric forest algebra $S = (H_S, V_S)$ is compact if and only if H_S and V_S are compact.

Lemma 3.1.19. *Let $S = (H, V)$ be an arbitrary metric forest algebra with uniformly continuous basic operations, then H is compact if and only if V is compact.*

Proof. ($\Rightarrow$) Assume that H is compact. Since the basic operations are uniformly continuous, the following mapping is onto uniformly continuous:

$$\begin{aligned}(\_0) : V &\to H\\ v &\mapsto v0\end{aligned}$$

Hence, $V = (\_0)^{-1}(H)$ is compact, as it is the inverse image of the compact set H.

($\Leftarrow$) Assume that V is compact. Let $H \subseteq \bigcup_{\alpha \in \Gamma} B_\alpha$ be an open covering for H. Since the mapping $(\_0)$ is onto uniformly continuous,

$$V \subseteq \bigcup_{\alpha \in \Gamma} (\_0)^{-1}(B_\alpha)$$

is an open covering for V. Since, by assumption, V is compact, it admits a finite open covering

$$V \subseteq \bigcup_{i=1}^{n} (\_0)^{-1}(B_i).$$

so, we have

$$(\_0)(V) \subseteq (\_0)(\bigcup_{i=1}^{n} (\_0)^{-1} B_i).$$

Hence, as $\_0$ is onto we have $H \subseteq \bigcup_{i=1}^{n} B_i$. □

Lemma 3.1.20. *Let* $\mathbf{V}$ *be a pseudovariety of finite forest algebras. If* $B \in \mathbf{V}$*, then every forest algebra homomorphism* $f : A^{\Delta} \longrightarrow B$ *is uniformly continuous.*

Proof. Suppose that $u, v \in A^{\Delta}$ are such that both have the same type with $d(u, v) < 2^{-|B|}$. By definition, $d(u, v) = 2^{-r(u,v)}$, where

$$r(u, v) = \min\{|C| \mid C \in \mathbf{V} \text{ and } \exists g : A^{\Delta} \longrightarrow C \ : \ g(u) \neq g(v)\}.$$

So for $d(u, v) < 2^{-|B|}$ we have

$$\min\{|C| \mid C \in \mathbf{V} \text{ and } \exists g : A^{\Delta} \longrightarrow C \ : \ g(u) \neq g(v)\} > |B|.$$

Hence, for every $g : A^{\Delta} \longrightarrow B$ and every $\epsilon > 0$, we have $d(g(u), g(v)) = 0 < \epsilon$, since $g(u) = g(v)$. Thus, every forest algebra homomorphism $f : A^{\Delta} \longrightarrow B$ into $B \in \mathbf{V}$ is uniformly continuous. □

For every uniformly continuous forest algebra homomorphism $f : A^{\Delta} \longrightarrow S$, by Proposition 3.1.14, there exists a unique uniformly continuous forest algebra homomorphism $\hat{f} : \overline{\Omega}_A\mathbf{V} \longrightarrow S$ such that the following diagram commutes:

$$\begin{tikzcd}
A^{\Delta} \arrow[r, "\iota"] \arrow[dr, "f"'] & \overline{\mathtt{C}}_A\mathbf{V} = \overline{\Omega}_A\mathbf{V} \arrow[d, dotted, "\hat{f}"] \\
& S
\end{tikzcd}$$

Let $u, v \in \overline{\Omega}_A\mathbf{V}$ and $S \in \mathbf{V}$. We write $S \models u = v$ if, for every uniformly continuous forest algebra homomorphism $f : A^{\Delta} \longrightarrow S$, the equality $\hat{f}(u) = \hat{f}(v)$ holds, and we then also say that S satisfies $u = v$.

For elements $u, v \in \overline{\Omega}_A\mathbf{V}$, the formal equality $u = v$ in $\overline{\Omega}_A\mathbf{V}$ is called a $\mathbf{V}$*-pseudoidentity*.

Proposition 3.1.21. *Let $u, v \in \overline{\Omega}_A\mathbf{V}$ and $S \in \mathbf{V}$. If $u = \lim u_n$ and $v = \lim v_n$, then*

$$(S \models u = v) \iff (\exists N > 0 \;\; \forall n \geq N \;, \;\; S \models u_n = v_n)\,. \tag{3.4}$$

Proof. ($\Rightarrow$) Suppose that $S \models u = v$. For every uniformly continuous forest algebra homomorphism $f : A^{\Delta} \longrightarrow S$, the equality $\hat{f}(u) = \hat{f}(v)$ holds. As we assumed that $u = \lim \iota(u_n)$ and $v = \lim \iota(v_n)$, we have $\hat{f}(\lim \iota(u_n)) = \hat{f}(\lim \iota(v_n))$, whence $\lim \hat{f}(\iota(u_n)) = \lim \hat{f}(\iota(v_n))$ since $\hat{f}$ is uniformly continuous.

Since the equality $\lim \hat{f}(\iota(u_n)) = \lim \hat{f}(\iota(v_n))$ holds, for every positive integer m there is a positive integer M such that for all $n \geq M$ the inequality $d(\hat{f}(\iota(u_n)), \hat{f}(\iota(v_n))) < 2^{-m}$ holds. By definition of d, for every forest algebra B in $\mathbf{V}$ such that $|B| \leq m$ and every forest algebra homomorphism

$$\varphi : A^{\Delta} \to B$$

the equality $\varphi(u_n) = \varphi(v_n)$ holds. Let $m > |S|$. Indeed, there exists $N > 0$ such that, for all $n \geq N$, $\hat{f}(\iota(u_n)) = \hat{f}(\iota(v_n))$ which is equivalent to $f(u_n) = f(v_n)$. Hence, we have $S \models u_n = v_n$.

($\Leftarrow$) Suppose that there exists $N > 0$ such that, for all $n \geq N$, we have $S \models u_n = v_n$. So for every uniformly continuous forest algebra homomorphism

$$f : A^{\Delta} \longrightarrow S$$

and for all $n \geq N$, the equality $f(u_n) = f(v_n)$ holds, which yields that the equality $\hat{f}(u_n) = \hat{f}(v_n)$ holds; hence, so does the equality $\lim \hat{f}(u_n) = \lim \hat{f}(v_n)$. We assumed that $u = \lim u_n$ and $v = \lim v_n$. Since $\hat{f}$ is uniformly continuous, the equality $\hat{f}(\lim u_n) = \hat{f}(\lim v_n)$ holds, and so does the equality $\hat{f}(u) = \hat{f}(v)$. Therefore, we have $S \models u = v$. □

Remark 3.1.22. For elements x and y in $\overline{\Omega}_A\mathbf{V}$ consider sequences $\{x_n\}_n$ and $\{y_n\}_n$ of elements of A^{Δ} such that $x = \lim \iota(x_n)$ and $y = \lim \iota(y_n)$, then $d(x, y) = \lim d(x_n, y_n)$. This implies that for elements x' and y' in A^{Δ}, as we have $\iota(x')$ and $\iota(y')$ are elements of $\overline{\Omega}_A\mathbf{V}$ and constant sequences $\{x'\}_n$ and $\{y'\}_n$ of elements of A^{Δ} are such that $x' = \lim \iota(x')$ and $y' = \lim \iota(y')$, we have $d(\iota(x'), \iota(y')) = \lim d(x', y') = d(x', y')$.

Proposition 3.1.23. *For $u, v \in \overline{\Omega}_A\mathbf{V}$, we have $d(u, v) = 2^{-r(u,v)}$, where*

$$r(u, v) = \min\{|B| : B \in \mathbf{V} \;\; \textit{and} \;\; B \not\models u = v\}.$$

Proof. For given $u, v \in \overline{\Omega}_A\mathbf{V}$, with $u \neq v$, there exists $m \in \mathbb{N}$ such that $d(u, v) \geq 2^{-m}$.

Consider $\{u_n\}_n$ and $\{v_n\}_n$, sequences of elements of A^{Δ}, such that $u = \lim \iota(u_n)$ and $v = \lim \iota(v_n)$. There is a positive integer N such that, for every $n \geq N$,

$$d(u, \iota(u_n)) < 2^{-m} \quad \text{and} \quad d(v, \iota(v_n)) < 2^{-m}.$$

Since $d(u,v) \geq 2^{-m}$, by Remark 3.1.22, we have for all $n \geq N$, $d(u_n, v_n) = d(\iota(u_n), \iota(v_n)) \geq 2^{-m}$.

This shows that, for every $B \in \mathbf{V}$ with $|B| < m$, the identity $u_n = v_n$ fails for all sufficiently large n, and therefore by Proposition 3.1.21, we have $B \not\models u = v$.

We have thus shown that $d(u,v) \geq 2^{-m}$ implies $r(u,v) \leq m$.

To complete the proof, we should show that $r(u,v) \leq m$ implies $d(u,v) \geq 2^{-m}$. By contraposition, we will show that $d(u,v) < 2^{-m}$ implies $r(u,v) > m$.

For given $u, v \in \overline{\Omega}_A\mathbf{V}$, with $u \neq v$, let $m \in \mathbb{N}$ be such that $d(u,v) < 2^{-m}$. Such an m exists because we have at least $m = 2$.

As before, consider sequences $\{u_n\}_n$ and $\{v_n\}_n$ of elements of A^Δ such that $u = \lim \iota(u_n)$ and $v = \lim \iota(v_n)$. Then there is a positive integer N such that for $n \geq N$,

$$d(u, \iota(u_n)) < 2^{-m} \quad \text{and} \quad d(v, \iota(v_n)) < 2^{-m}.$$

Since $d(u,v) < 2^{-m}$, by Remark 3.1.22, for all $n \geq N$,

$$d(u_n, v_n) = d(\iota(u_n), \iota(v_n)) < 2^{-m}.$$

Hence, for every $S \in \mathbf{V}$ with $|S| \leq m$, there is $M \in \mathbb{N}$ large enough such that for all $n \geq M$, $S \models u_n = v_n$. So for every $S \in \mathbf{V}$ with $|S| \leq m$, by Proposition 3.1.21 and Remark 3.1.20, $S \models u = v$. Therefore $r(u,v) > m$.

We have thus established the claim that $d(u,v) < 2^{-m}$ implies $r(u,v) > m$. □

Proposition 3.1.24. *In a metric space X we have the following:*

1. *If X is a totally bounded pseudo-ultrametric space, then its completion is also totally bounded.*

2. *If X is a totally bounded complete metric space, then X is compact.*

Proof. See [16, Corollary 15.3.6 and Theorem 15.4.1]. □

Lemma 3.1.25. *Let $K = (H', V')$ be an inverse zero action subset of a compact metric forest algebra $S = (H, V)$. Then H' is a clopen subset of H if and only if V' is a clopen subset of V.*

Proof. Assume that H' is a clopen subset of H. Since the action of the contexts on the left of the forest 0 is uniformly continuous the set $V' = \{v \in V \mid v0 \in H'\}$ is clopen.

Now assume that V' is a clopen subset of V. Since V is a compact metric space, V' is compact. The mapping $\_0 : V \to H$ is uniformly continuous so $H' = (\_0)(V')$ is compact, whence H' is closed. We just need to show that H' is open. Since V is compact and V' is open, the complement V'^c of V'

in V is closed, whence it is compact. So $H'' = (\_0)(V'^c)$ is compact thus it is closed. By the way that we define H'' and H' and since $\_0$ is an onto uniformly continuous mapping, it follows that H'' is the complement of H' in H. As H'' is closed, we conclude that H' is open. $\square$

Corollary 3.1.26. *An inverse zero action subset $K = (H', V')$ of a compact metric forest algebra $S = (H, V)$ is clopen if and only if H' is clopen.* $\square$

Lemma 3.1.27. (Similar to Hunter's Lemma) *Let K be a clopen inverse zero action subset of a compact and zero-dimensional metric forest algebra S. Then there is a continuous forest algebra homomorphism $\psi : S \to T$ into a finite forest algebra T such that $K = \psi^{-1} \circ \psi(K)$.*

Proof. It suffices to show that the classes of the syntactic congruence of K are open. Then there are only finitely many of them since S is a compact forest algebra. So that $S/\backsim_K = (H/\sigma_K, V/\sigma'_K)$ is a finite forest algebra and the natural mapping $S \to S/\backsim_K$ is a continuous forest algebra homomorphism.

We want to show that, for a sequence $\{u_n\}_n$ of elements of S which converge to an element u, all but finitely many terms in the sequence are $\backsim_K$-equivalent to u. By Lemma 3.1.11, it suffices to consider the cases $\{u_n\} \subset H$ and $\{u_n\} \subset V$.

If $\{u_n\} \subset H$ then we will show that:

$$\exists N \text{ such that } \forall n > N \text{ we have } u_n \ \sigma_K \ u.$$

And if $\{u_n\} \subset V$, then we will show that:

$$\exists N \text{ such that } \forall n > N \text{ we have } u_n \ \sigma'_K \ u.$$

In both cases, we argue by contradiction, assuming that there is a subsequence consisting of terms which fails the above property. We may as well assume that so does the original sequence. In other words, we can assume that $\{u_n\} \not\backsim_K u$.

Since $\{u_n\}_n \to u$, by Lemma 3.1.11, we may assume that u_n and u have the same type.

If $\{u_n\} \subset H$, then for each n there are $t_n, r_n, w_n \in V$ such that at least one of the following conditions does not hold:

I. $t_n u_n \in K \iff t_n u \in K$;

II.
   1. $t_n(r_n u_n + w_n) \in K \iff t_n(r_n u + w_n) \in K$;
   2. $t_n(w_n + r_n u_n) \in K \iff t_n(w_n + r_n u) \in K$.

So there is a subsequence for which the same condition among these three does not hold. For each n, we denote by $P(u_n, t_n, r_n, w_n)$ the term in the left side of the above statement which fails and $P(u, t_n, r_n, w_n)$ the related one from the right side.

Thus, there exists $\{n_k\}$ such that

$$P(u_{n_k}, t_{n_k}, r_{n_k}, w_{n_k}) \in K \not\Leftrightarrow P(u, t_{n_k}, r_{n_k}, w_{n_k}) \in K.$$

Since K is clopen, we may as well assume that $P(u_{n_k}, t_{n_k}, r_{n_k}, w_{n_k}) \in K$ and $P(u, t_{n_k}, r_{n_k}, w_{n_k}) \notin K$. Since S is compact, we may assume that the following limits exist in S:

$$\lim t_{n_k} = t \ , \quad \lim w_{n_k} = w \ , \quad \lim r_{n_k} = r.$$

Then we have

$$P(u, t, w, r) = \lim P(u_{n_k}, t_{n_k}, w_{n_k}, r_{n_k}) = \lim P(u, t_{n_k}, w_{n_k}, r_{n_k}).$$

Since K is open and we assumed that the sequence

$$\{P(u, t_{n_k}, w_{n_k}, r_{n_k})\}_{n_k}$$

takes it values in the complement of K, we have $p(u, t, w, r) \notin K$. And since the sequence

$$\{P(u_{n_k}, t_{n_k}, w_{n_k}, r_{n_k})\}_{n_k}$$

takes its values in K, we have $p(u, t, w, r) \in K$. So, $P(u, t, w, r)$ must belong to both K and its complement, which is a contradiction.

If the sequence $\{u_n\}$ is contained in V, then, in a similar way to the preceding case, the is a positive integer N, large enough, such that for every $n \geq N$ we have $u_n \ \sigma'_K \ u$.

Hence, $\backsim_K$-classes are open. □

In view of proof of Lemma 3.1.27, we get the following result.

Corollary 3.1.28. *Let K be a clopen inverse zero action subset of a compact zero-dimensional metric forest algebra S. The following statements hold:*

- *The classes of the syntactic congruence of K are open;*
- *$S/\backsim_K$ is a forest algebra, then since S is compact we can conclude that $S/\backsim_K$ is finite;*
- *The natural mapping $S \to S/\backsim_K$ is a continuous forest algebra homomorphism.* □

Lemma 3.1.29. *Let s and t be two distinct forests in a compact and zero-dimensional forest algebra $S = (H, V)$. Then, there is a clopen inverse zero action subset K which separates s and t.*

Furthermore, the quotient forest algebra homomorphism $i : S \to S/\backsim_K$ sends s and t to two distinct points.

Proof. By zero-dimensionality, s and t may be separated by a clopen subset $H' \subseteq H$ in the sense that s lies in H' and t does not. Let $V' = \bigcup_{h \in H'} V_h$. By Lemma 3.1.25, the inverse zero action subset $K = (H', V')$ is clopen and it is such that s lies in K and t does not. Since the syntactic congruence $\backsim_K$ saturates K, the congruence classes of s and t are distinct, that is the quotient forest algebra homomorphism $i : S \to S/\backsim_K$ sends s and t to two distinct points. $\square$

Theorem 3.1.30. *A zero-dimensional and compact metric forest algebra is residually finite.*

Proof. We show that for any given two distinct points $s, t \in S$, there is a continuous forest algebra homomorphism $\rho : S \to T$ into a finite forest algebra T such that $\rho(s) \neq \rho(t)$.

For any given distinct points $s, t \in S$ exactly one of the following conditions holds:

1. s and t have different types;
2. $s, t \in H$;
3. $s, t \in V$ with $s0 \neq t0$;
4. $s, t \in V$ with $s0 = t0$.

If s and t have different types then there is a continuous forest algebra homomorphism $\eta : S \to \mathcal{TFA}$ into the trivial forest algebra $\mathcal{TFA}$ which maps forests to 0 and contexts to $\Box$. So $\eta(s) \neq \eta(t)$.

For $s, t \in H$ by Lemma 3.1.29, there is a clopen inverse zero action subset K which separates them and $i : S \to S/\backsim_K$ sends s and t to two distinct points. Hence, to prove that S is residually finite, it suffices to show that $S/\backsim_K$ is finite and i is continuous, which is the result of Corollary 3.1.28.

Now assume that $s, t \in V$ with $s0 \neq t0$. By Lemma 3.1.29, there is a clopen inverse zero action subset K' that separates $s0$ and $t0$ and $i : S \to S/\sim_{K'}$ sends $s0$ and $t0$ to two distinct points and, therefore, so does with s and t. Now, Corollary 3.1.28, shows that $S/\sim_{K'}$ is finite and i is continuous.

Finally for $s, t \in V$ with $s0 = t0$. Since $s \neq t$ and S is a forest algebra, there is a forest $h \in H$ such that $sh \neq th$. For every $w \in V_h$ we have $sw \neq tw$, because otherwise, $swh' = twh'$ for every $h' \in H$; in particular, for $h' = 0$ we have $sw0 = tw0$, which is in contradiction with $sh \neq th$. So, for distinct contexts s and t, there is a context w such that $sw \neq tw$ with $sw0 \neq tw0$. Again, by Lemma 3.1.29, there is a clopen inverse zero action subset K'' such that $sw0 \in K''$ and $tw0 \notin K''$. As K'' is an inverse zero action subset of S, we have $sw \in K''$ and $tw \notin K''$. Since the syntactic congruence $\sim_{K''}$ saturates K'', the congruence classes of sw and tw are distinct, that is the quotient forest algebra homomorphism $\varphi : S \to S/\sim_{K''}$ sends sw and tw to two distinct points. We have $\varphi(s) \neq \varphi(t)$ because, since $\varphi(w) \in V_{S/\sim_{K''}}$ and φ is a forest algebra homomorphism; therefore if $\varphi(s) = \varphi(t)$ then

$$\varphi(sw) = \varphi(s)\varphi(w) = \varphi(t)\varphi(w) = \varphi(tw)$$

which is a contradiction. The result of Corollary 3.1.28, gives that $S/\sim_{K''}$ is finite and φ is continuous. $\square$

Definition 3.1.31. Fix a set A, and consider the category of A-generated topological forest algebras whose objects are the mappings $A \to S$ into topological forest algebras whose images generate dense subalgebras, and whose morphisms $\theta : \varphi \to \psi$, from $\varphi : A \to S$ to $\psi : A \to T$, are given by continuous forest algebra homomorphisms $\theta : S \to T$ such that $\theta \circ \varphi = \psi$. Now, consider a *projective system* in this category, given by a directed set I of indices, for each $i \in I$ an object $\varphi_i : A \to S_i$ in our category of A-generated topological forest algebras and, for each pair $i, j \in I$ with $i \geq j$ a *connecting morphism* $\psi_{i,j} : \varphi_i \to \varphi_j$ such that the following conditions hold for all $i, j, k \in I$:

- $\psi_{i,i}$ is the identity morphism on φ_i;
- if $i \geq j \geq k$ then $\psi_{j,k} \circ \psi_{i,j} = \psi_{i,k}$.

The *projective limit* of this projective system is an A-generated topological forest algebra $\Phi : A \to S$ together with morphisms $\Phi_i : \Phi \to \varphi_i$ such that for all $i, j \in I$ with $i \geq j$, $\psi_{i,j} \circ \Phi_i = \Phi_j$ and, moreover, the following universal property holds:

> For any A-generated topological forest algebra $\Psi : A \to T$ and morphisms $\Psi_i : \Psi \to \varphi_i$ such that for all $i, j \in I$ with $i \geq j$, $\psi_{i,j} \circ \Psi_i = \Psi_j$ there exists a morphism $\theta : \Psi \to \Phi$ such that $\Phi_i \circ \theta = \Psi_i$ for every $i \in I$.

Fix a set A. Let $\mathbf{V}$ be a pseudovariety of finite forest algebras. Assume that a directed set I of indices, a projective system $(S_i)_{i \in I}$ of A-generated forest algebras in $\mathbf{V}$, and onto forest algebra homomorphisms $\varphi_{i,j} : S_i \to S_j$ for each pair $i, j \in I$ with $i \geq j$ are given. Consider the subset of the direct product $\prod_{i \in I} S_i$ consisting of all those $(s_i)_{i \in I}$ such that $s_i \in S_i$ and $\varphi_{i,j}(s_i) = s_j$ whenever $i \geq j$. Let $(s_i)_{i \in I}$ and $(s'_i)_{i \in I}$ be elements of S and O is a basic operation such that $O(s_j, s'_j)$ is defined for some $j \in I$, see Definition 1.1.7. Note that, since the $\varphi_{i,j}$'s are forest algebra homomorphisms, $(s_i)_{i \in I} \in \prod_{i \in I} H_{S_i}$ if and only if there is a $j \in I$ such that $s_j \in H_{S_j}$. So, O is a basic operation such that $O(s_i, s'_i)$ is defined for every $i \in I$. We claim that $O((s_i)_{i \in I}, (s'_i)_{i \in I}) = (O(s_i, s'_i))_{i \in I}$ is also an element of S. Since $\varphi_{i,j}(s_i) = s_j$ and $\varphi_{i,j}(s'_i) = s'_j$ whenever $i \geq j$ and $\varphi_{i,j}$ is a forest algebra homomorphism, the following equalities hold

$$\varphi_{i,j}(O(s_i, s'_i)) = O(\varphi_{i,j}(s_i), \varphi_{i,j}(s'_i)) = O(s_j, s'_j).$$

Hence, S is a subalgebra of the direct product $\prod_{i\in I} S_i$.

Remark 3.1.32. Let s_j be an element of an S_j with $j \in I$. Since S_j is finite and A-generated, there is an element w in the free forest algebra A^Δ which maps to s_j under the homomorphism induced by the generating mapping. Then the image of w in S maps to s_j under the j-component projection $S \to S_j$. Hence, there is an element $(s_i)_{i\in I}$ of S with the j-component equal to s_j. Therefore, the natural projection $S \to S_j$ is onto.

We claim that S is a forest algebra. We only need to check the faithfulness property of S. Let $v = (v_i)_{i\in I}$ and $w = (w_i)_{i\in I}$ be elements of V_S such that for all $h = (h_i)_{i\in I}$ in H_S the equality $vh = wh$ holds. Since the restriction of the natural projection $S \to S_j$ to the horizontal part is onto, the equality $vh = wh$ implies that the equality $v_ih_i = w_ih_i$ holds for all $h_i \in H_{S_i}$ and every $i \in I$. Since S_i is a forest algebra, then the equality $v_i = w_i$ holds for every $i \in I$. Hence, $(v_i)_{i\in I} = (w_i)_{i\in I}$. Therefore, S is a forest algebra.

We claim that the mapping $\Phi : A \to S$ given by $\Phi(a) = (\varphi_i(a))_{i\in I}$ is such that $\Phi(A)$ generates a dense subalgebra T of S. We want to find an approximation $(t_i)_{i\in I} \in T$ to the element $(s_i)_{i\in I}$ of S such that for every j, $t_{i_j} = s_{i_j}$. Since the system is projective, to find $(t_i)_{i\in I} \in T$, take $k \in I$ such that $k \geq i_1, \ldots, i_n$. Then by Remark 3.1.32, there is an element $w \in A^\Delta$ which represents the element s_k. This element w then represents an element $(t_i)_{i\in I}$ of T which is an approximation as required.

Now, assume that T is an A-generated topological forest algebra and that the forest algebra homomorphisms $\pi_i : T \to S_i$ are such that for all $i, j \in I$ with $i \geq j$, $\varphi_{i,j} \circ \pi_i = \pi_j$. Define a mapping $\varphi : T \to S$ with $\varphi(t) = (\pi_i(t))_{i\in I}$. We show that φ is a forest algebra homomorphism. For elements x and y in T the following equalities hold:

$$\begin{aligned}
\varphi(O(x,y)) &= (\pi_i(O(x,y)))_{i\in I} \\
&= (O(\pi_i(x), \pi_i(y)))_{i\in I} \\
&= O((\pi_i(x))_{i\in I}, (\pi_i(y))_{i\in I}) \\
&= O(\varphi(x), \varphi(y)).
\end{aligned}$$

Hence, S has the required universal property and therefore it is the projective limit of the projective system $(S_i)_{i\in I}$.

Definition 3.1.33. Fix a set A. A *profinite forest algebra* is defined to be a projective limit of a projective system of A-generated finite forest algebras. And for a pseudovariety $\mathbf{V}$ of finite forest algebras a *pro-$\mathbf{V}$ forest algebra* is defined to be a projective limit of a projective system of A-generated finite forest algebras in $\mathbf{V}$.

Hence, a profinite forest algebra is a pro-$\mathbf{F}$ forest algebra.

Theorem 3.1.34. *Let* $\mathbf{V}$ *be a pseudovariety of finite forest algebras and* A *be a finite set. An* A*-generated compact forest algebra* S *is a pro-*$\mathbf{V}$ *forest algebra if and only if* S *is residually in* $\mathbf{V}$ *as a topological forest algebra.*

Proof. Assuming that S is an A-generated compact pro-$\mathbf{V}$ forest algebra, there are a directed set I of indices, a projective system $(S_i)_{i\in I}$ of A-generated finite forest algebras in $\mathbf{V}$, and onto continuous forest algebra homomorphisms $\varphi_{i,j} : S_i \to S_j$ for each pair $i, j \in I$ with $i \geq j$, such that S is its projective limit. By the above construction, the projective limit is the forest subalgebra of the direct product $\prod_{i\in I} S_i$ consisting of all those $(s_i)_{i\in I}$ such that $s_i \in S_i$ and $\varphi_{i,j}(s_i) = s_j$ whenever $i \geq j$. Note that, if $(s_i)_{i\in I}$ is in $H_{\prod_{i\in I} S_i}$, then for every $i \in I$, $s_i \in H_{S_i}$ and similarly for the vertical part. By construction of the projective limit, we conclude that S is residually in $\mathbf{V}$.

Conversely, suppose that an A-generated compact forest algebra S is residually in $\mathbf{V}$. Take a set D that contains all the A-generated elements of $\mathbf{V}$ up to forest algebra isomorphism and consider the set I of all onto continuous forest algebra homomorphisms $\varphi : S \to T$ with $T \in D$ and order them by letting $\varphi \geq \psi$ for another continuous forest algebra homomorphism $\psi : S \to U$ if there is a forest algebra homomorphism $\theta : T \to U$ such that $\theta \circ \varphi = \psi$. Note that I is a directed set: two onto continuous forest algebra homomorphisms $\varphi : S \to T$ and $\psi : S \to U$ induce a continuous forest algebra homomorphism $\lambda : S \to T \times U$ in which if we replace the direct product $T \times U$ by a member of D isomorphic to the image of λ we obtain a member of I which is above both φ and ψ. We thus obtain a projective system of forest algebra homomorphisms between A-generated members of $\mathbf{V}$. Let S' be its A-generated projective limit. We claim that S' and S are isomorphic as topological forest algebras. The forest algebra homomorphisms $\varphi : S \to T$ in I induce a continuous forest algebra homomorphism Φ from S into the direct product of all the T's which by construction takes its values in S'. Since S is compact, Φ is a closed mapping. Since S is residually in $\mathbf{V}$, Φ is injective.

It remains to show that Φ is onto. Given $s' = (t_\varphi)_{\varphi\in I}$ in S', for each $\varphi \in I$ the closed set $\varphi^{-1}(t_\varphi)$ is nonempty. Note that, if $s' \in H_{S'}$, then for every $\varphi \in I$, $t_\varphi \in H_{\varphi(S)}$ and similarly for the vertical part. The fact that I is directed and the given family belongs to S' implies that any finite intersection of such closed subsets is still nonempty. By compactness of S, we deduce that there is some $s \in \bigcap_{\varphi\in I} \varphi^{-1}(t_\varphi)$ and for such s we have $\Phi(s) = s'$. Hence, Φ is indeed onto. □

Corollary 3.1.35. *The forest algebra* $\overline{\Omega}_A\mathbf{V}$ *is a pro-*$\mathbf{V}$ *forest algebra.*

Proof. Theorem 3.1.30 together with Theorem 3.1.34, imply that the zero-dimensional and compact metric forest algebras are profinite. Since by

Proposition 3.1.14, $\overline{\Omega}_A\mathbf{V}$ is an A-generated free forest algebra as a topological forest algebra.

We just need to show that $\overline{\Omega}_A\mathbf{V}$ is zero-dimensional. Let $x \in H_{\overline{\Omega}_A\mathbf{V}}$. It suffices to show that the open ball $B_\varepsilon(x)$ with $\varepsilon < 2^{-2}$ contains some clopen subset which contains x. Let $y \in H_{\overline{\Omega}_A\mathbf{V}} \setminus B_\varepsilon(x)$. There is a positive integer n such that $\varepsilon > 2^{-n}$. Since $d(x, y) \geq \varepsilon$, there is a forest algebra S_y with $|S_y| \geq n$ and a continuous forest algebra homomorphism $\varphi_y : \overline{\Omega}_A\mathbf{V} \to S_y$ such that $\varphi_y(x) \neq \varphi_y(y)$. Then $K_y = \varphi^{-1} \circ \varphi(y)$ is a clopen set which contains y but not x. In particular K_y form a clopen covering of the closed set $H_{\overline{\Omega}_A\mathbf{V}} \setminus B_\varepsilon(x)$, from which the finite covering $\mathcal{K}$ can be extracted. The union of the clopen sets in $\mathcal{K}$ is itself a clopen set K. Note that $H_{\overline{\Omega}_A\mathbf{V}} \setminus K$ is also clopen, contain x, and also is contained in $B_\varepsilon(x)$. And similarly for $x \in V_{\overline{\Omega}_A\mathbf{V}}$.

Hence, $\overline{\Omega}_A\mathbf{V}$ is a compact and zero-dimensional forest algebra. So, it is a pro-$\mathbf{V}$ forest algebra. $\square$

3.2 Reiterman's Theorem

In this section we establish an analog of Reiterman's Theorem [26].

Recall that a $\mathbf{V}$-pseudoidentity is a formal equality $u = v$ with $u, v \in \overline{\Omega}_A\mathbf{V}$ for some finite set A. And for $S \in \mathbf{V}$, we write $S \models u = v$ if, for every continuous forest algebra homomorphism $\varphi : \overline{\Omega}_A\mathbf{V} \to S$, the equality $\varphi(u) = \varphi(v)$ holds.

For a set Σ of $\mathbf{V}$-pseudoidentities, let $[\![\Sigma]\!]_{\mathbf{V}}$ denote the class of all $S \in \mathbf{V}$ such that $S \models u = v$ for every pseudoidentity $u = v$ from Σ.

If $S_1, S_2 \in [\![\Sigma]\!]_{\mathbf{V}}$, then clearly $S_1 \times S_2 \in [\![\Sigma]\!]_{\mathbf{V}}$. If T is a forest subalgebra of $S \in [\![\Sigma]\!]_{\mathbf{V}}$, then there is an embedding $\psi : T \to S$. Let $\varphi : \overline{\Omega}_A\mathbf{V} \to T$ be any forest algebra homomorphism. Then the composite $\alpha = \psi \circ \varphi$ is a continuous forest algebra homomorphism $\overline{\Omega}_A\mathbf{V} \to S$ and so, for every pseudoidentity $u = v \in \Sigma$, we obtain the equality $\alpha(u) = \alpha(v)$. Consequently $\varphi(u) = \varphi(v)$ and so $T \in [\![\Sigma]\!]_{\mathbf{V}}$. Now, let T be a forest algebra such that there is an onto forest algebra homomorphism $\psi : S \to T$. Then it is equally easy to obtain $T \in [\![\Sigma]\!]_{\mathbf{V}}$. So $[\![\Sigma]\!]_{\mathbf{V}}$ is a pseudovariety of finite forest algebras.

For a subpseudovariety $\mathbf{W}$ of $\mathbf{V}$, let $\pi_{\mathbf{W}} : \overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_A\mathbf{W}$ be the natural continuous forest algebra homomorphism:

$$\begin{tikzcd} A \arrow[r, "\iota_{\mathbf{V}}"] \arrow[dr, "\iota_{\mathbf{W}}"'] & \overline{\Omega}_A\mathbf{V} \arrow[d, dotted, "\pi_{\mathbf{W}}=\widehat{\iota_{\mathbf{W}}}"] \\ & \overline{\Omega}_A\mathbf{W} \end{tikzcd}$$

where the two mappings $\iota_{\mathbf{V}}$ and $\iota_{\mathbf{W}}$ giving $\overline{\Omega}_A\mathbf{V}$ and $\overline{\Omega}_A\mathbf{W}$ as free respectively pro-$\mathbf{V}$ and pro-$\mathbf{W}$ forest algebras over the set A.

Lemma 3.2.1. *A pseudoidentity $u = v$, with $u, v \in \overline{\Omega}_A\mathbf{V}$, holds in every member of a subpseudovariety $\mathbf{W}$ of $\mathbf{V}$ if and only if $\pi_{\mathbf{W}}(u) = \pi_{\mathbf{W}}(v)$.*

Proof. ($\Rightarrow$) Let $S \in \mathbf{W}$. Let $\varphi : \overline{\Omega}_A\mathbf{V} \to S$ be any continuous forest algebra homomorphism. Note that, the following diagram commutes:

$$\begin{tikzcd}
 & & \overline{\Omega}_A\mathbf{V} \arrow[dl, "\varphi"] \arrow[dd, "\pi_{\mathbf{W}}=\widehat{\iota_{\mathbf{W}}}"] \\
A \arrow[urr, "\iota_{\mathbf{V}}"] \arrow[r, "\iota"] \arrow[drr, "\iota_{\mathbf{W}}"'] & S & \\
 & & \overline{\Omega}_A\mathbf{W} \arrow[ul, "\psi=\widehat{\iota}"']
\end{tikzcd}$$

where the image of the mapping ι is dense in S. There exists then a forest algebra homomorphism $\psi : \overline{\Omega}_A\mathbf{W} \to S$ such that $\psi \circ \pi_{\mathbf{W}} = \varphi$. If $\pi_{\mathbf{W}}(u) = \pi_{\mathbf{W}}(v)$, then $\psi(\pi_{\mathbf{W}}(u)) = \psi(\pi_{\mathbf{W}}(v))$ and the equality $\varphi(u) = \varphi(v)$ holds.

($\Leftarrow$) Let, for all $S \in \mathbf{W}$, $S \models u = v$ and suppose that $\pi_{\mathbf{W}}(u) \neq \pi_{\mathbf{W}}(v)$. Since $\pi_{\mathbf{W}}(u)$ and $\pi_{\mathbf{W}}(v)$ are distinct elements of $\overline{\Omega}_A\mathbf{W}$, there is a forest algebra $T \in \mathbf{W}$ and a forest algebra homomorphism $\alpha : \overline{\Omega}_A\mathbf{W} \to T$ such that $\alpha(\pi_{\mathbf{W}}(u)) \neq \alpha(\pi_{\mathbf{W}}(v))$, which contradicts the assumption that for all $S \in \mathbf{W}$, $S \models u = v$. $\square$

Theorem 3.2.2. *Let A be a finite set. Let $\mathbf{W}$ be a pseudovariety of finite forest algebras and S an A-generated finite forest algebra. If there is an onto continuous forest algebra homomorphism $\varphi : \overline{\Omega}_A\mathbf{W} \to S$, then $S \in \mathbf{W}$.*

Proof. Since, by Corollary 3.1.35, $\overline{\Omega}_A\mathbf{W}$ is a pro-$\mathbf{W}$ forest algebra, by Theorem 3.1.34, there are a directed set I of indices, a projective system $(S_i)_{i\in I}$ of A-generated finite forest algebras in $\mathbf{W}$, and onto continuous forest algebra homomorphisms $\psi_{i,j} : S_i \to S_j$ for each pair $i, j \in I$ with $i \geq j$ such that $\overline{\Omega}_A\mathbf{W}$ is a projective limit of $(S_i)_{i\in I}$. Let $\pi_i : \overline{\Omega}_A\mathbf{W} \to S_i$ be a continuous forest algebra morphism such that $\pi_i \circ \iota_{\mathbf{W}} = \iota_{S_i}$, where image of the mapping $\iota_{S_i} : A \to S_i$ is dense in S and the mapping $\iota_{\mathbf{W}} : A \to \overline{\Omega}_A\mathbf{W}$ giving $\overline{\Omega}_A\mathbf{W}$ as free pro-$\mathbf{W}$ forest algebra over the set A. The $\pi_i^{-1}(y)$ ($i \in I$, $y \in S_i$) constitute a subbasis of open sets for the topology of $\overline{\Omega}_A\mathbf{W}$. Since φ is continuous, for every $x \in \overline{\Omega}_A\mathbf{W}$, there exist finitely many elements $i_{x,1}, \ldots, i_{x,m}$ of I such that, putting $y_{x,j} = \pi_{i_{x,j}}(x)$ for $j = 1, \ldots, m$, we have $\bigcap_{j=1}^{m} \pi_{i_{x,j}}^{-1}(y_{x,j}) \subseteq \varphi^{-1}\varphi(x)$. Since for each pair $i, j \in I$ with $i \geq j$ the following diagram commutes

$$\begin{tikzcd}
\overline{\Omega}_A\mathbf{W} \arrow[d, "\pi_i"'] \arrow[dr, "\pi_j"] & \\
S_i \arrow[r, "\psi_{i,j}"'] & S_j
\end{tikzcd}$$

for every $x \in \overline{\Omega}_A \mathbf{W}$ we have $\pi_i^{-1}(\pi_i(x)) \subseteq \pi_j^{-1}(\pi_j(x))$. Hence, for an element $k_x \geq i_{x,1}, \ldots, i_{x,m}$ of I and $y_x = \pi_{k_x}(x)$, $\pi_{k_x}^{-1}(y_x) \subseteq \varphi^{-1}\varphi(x)$.

Note that $\overline{\Omega}_A \mathbf{W}$ is compact and $\overline{\Omega}_A \mathbf{W} = \bigcup_x \pi_{k_x}^{-1}(y_x)$. So there exists an integer n, elements $x_1, \ldots, x_n \in \overline{\Omega}_A \mathbf{W}$, elements $k_{x_1}, \ldots, k_{x_n} \in I$, and elements $y_{x_1}, \ldots, y_{x_n}$ respectively of $S_{k_{x_1}}, \ldots, S_{k_{x_n}}$ such that

$$\overline{\Omega}_A \mathbf{W} = \bigcup_{j=1}^{n} \pi_{k_{x_j}}^{-1}(y_{x_j})$$

with $x_j \in \pi_{k_{x_j}}^{-1}(y_{x_j})$ and $\pi_{k_{x_j}}^{-1}(y_{x_j}) \subseteq \varphi^{-1}(\varphi(x_j))$ for all j.

Consider an element i of I, with $i \geq k_{x_1}, \ldots, k_{x_n}$. For every $x \in \overline{\Omega}_A \mathbf{W}$, there exists $1 \leq j \leq n$ such that $x \in \pi_{k_{x_j}}^{-1}(y_{x_j})$. Therefore, $\pi_i^{-1}(\pi_i(x)) \subseteq \pi_{k_{x_j}}^{-1}(\pi_{k_{x_j}}(x)) \subseteq \varphi^{-1}(\varphi(x_j))$. Hence, for each $x \in \overline{\Omega}_A \mathbf{W}$, also $\pi_i^{-1}(\pi_i(x)) \subseteq \varphi^{-1}(\varphi(x))$. Thus, $\ker(\pi_i) \subseteq \ker(\varphi)$ and so φ factors through π_i. So there is a forest algebra homomorphism φ_i such that the following diagram commutes.

$$\begin{tikzcd} \overline{\Omega}_A \mathbf{W} \arrow[r, "\varphi"] \arrow[d, "\pi_i"'] & S \\ S_i \arrow[ur, "\varphi_i"'] & \end{tikzcd}$$

In the preceding diagram, φ and π_i are onto, whence so is φ_i. As S_i is in $\mathbf{W}$, so is S. $\square$

Theorem 3.2.3. (Analog of Reiterman's Theorem) *A subclass* $\mathbf{W}$ *of* $\mathbf{V}$ *is a subpseudovariety if and only if it is of the form* $[\![\Sigma]\!]$ *for some set* Σ *of* $\mathbf{V}$*-pseudoidentities.*

Proof. The reverse direction has already been verified. To prove the direct implication, let $\mathbf{W} \subseteq \mathbf{V}$ be a pseudovariety and Σ be the set of all pseudoidentities which are satisfied by every member of $\mathbf{W}$, and let $\mathbf{W}' = [\![\Sigma]\!]_{\mathbf{V}}$. Clearly $\mathbf{W} \subseteq \mathbf{W}'$. To complete the proof we show that $S \in \mathbf{W}'$ implies $S \in \mathbf{W}$. Since $S \in \mathbf{W}'$ is finite, there is a finite set A and there is an onto continuous forest algebra homomorphism $\varphi : \overline{\Omega}_A \mathbf{W}' \to S$. Let $\pi : \overline{\Omega}_A \mathbf{W}' \to \overline{\Omega}_A \mathbf{W}$ be the natural projection. By Lemma 3.2.1, $\ker(\pi) \subseteq \ker(\varphi)$ and so φ factors through π. So the following diagram commutes.

$$\begin{tikzcd} \overline{\Omega}_A \mathbf{W}' \arrow[r, "\varphi"] \arrow[d, "\pi"'] & S \\ \overline{\Omega}_A \mathbf{W} \arrow[ur, dotted, "\psi"'] & \end{tikzcd}$$

In the preceding diagram, $\overline{\Omega}_A \mathbf{W}'$ is compact and φ and π are onto and continuous, whence so is ψ. Now Theorem 3.2.2 implies that $S \in \mathbf{W}$. $\square$

3.3 Conclusions

We defined a metric on the free forest algebra with respect to a pseudovariety of finite forest algebras and we showed that the basic operations with respect to this metric are contractive. We showed that the completion of the free forest algebra with respect to the defined metric exists and is a forest algebra. We established in this context an analog of Hunter's Lemma [23]. We showed that compact and zero-dimensional metric forest algebras are residually finite, whence profinite. We also established an analog of Reiterman's Theorem (3.2.3). For a pseudovariety **V** of finite forest algebras, by Theorem 3.2.3, a simple basis may be seen as a formalization of a simple algebraic criterion for membership in **V**. For **BSS** such a basis was obtained by Bojańczyk, Segoufin, and Straubing in [6]. In the same paper, they also did it for the pseudovarieties of finite forest algebras generated by all syntactic forest algebras of cca[1]-piecewise testable[2] forest languages and commutative piecewise testable forest languages. For a pseudovariety **W** of finite monoids, if a basis is known, then by Lemma 2.2.5, we can find easily a basis of pseudoidentities for the pseudovarieties **VW** and **HW**. For other pseudovarieties finding a basis of pseudoidentities may be a very difficult task.

There are several results on metric semigroups (for examples, see [2, 1] for results and references) that we still do not know if they have a natural analog in the context of forest algebras.

[1] Stands for closest common ancestor: given a forest s and three nodes x, y, and z of s we say that z is the *closest common ancestor* of x and y if z is an ancestor of both x and y and all other nodes of s with this property are ancestors of z.

[2] A forest s is a *cca-piece* of a forest t, if there is an injective mapping from nodes of s to nodes of t that preserves the label of the node together with the forest-order and the closest common ancestor relationship. A forest language L is called *cca-piecewise testable* if there exists $n > 0$ such that membership of t in L depends only on the set of cca-pieces of t of size n.

Chapter 4

Forest Algebras as ω-Algebras

The natural analog for forest algebras of the structural identification of the relatively free pro-**J** semigroup $\overline{\Omega}_n\mathbf{J}$ as an algebra of type (2,1), see [1, Section 8.2], is to study the relatively free pro-**BSS** forest algebra $\overline{\Omega}_A\mathbf{BSS}$, as an ω-algebra.

In this chapter we introduce ω-algebras which satisfy the equational axioms of forest algebras with some extra assumptions. Forest algebras can be viewed as special cases of ω-algebras. We show several results on free ω-algebras.

4.1 ω-Algebra

An *ω-algebra* $\mathsf{B} = (\mathsf{H}, \mathsf{V})$ is a set with two *types of elements* endowed with five binary operations $+, +_1, +_2, .,$ and $*$ and two unary operations $\omega()$ on H and $()^\omega$ on V, such that the following conditions are satisfied:

1. $\langle \mathsf{H}; + \rangle$ is a monoid with identity 0;
2. $\langle \mathsf{V}; . \rangle$ is a monoid with identity $\square$;
3. for every $h \in \mathsf{H}$ and $v \in \mathsf{V}$, $v * h$ is in H;
4. for every $h \in \mathsf{H}$ and $v, w \in \mathsf{V}$, $v * (w * h) = (v.w) * h$;
5. for every $h \in \mathsf{H}$, $\square * h = h$;
6. for every $h \in \mathsf{H}$ and $v \in \mathsf{V}$, $h +_1 v$ and $v +_2 h$ are in V;
7. for every $h, s \in \mathsf{H}$ and $v \in \mathsf{V}$, $h +_1 (v +_2 s) = (h +_1 v) +_2 s$;
8. for every $h \in \mathsf{H}$ and $v, w \in \mathsf{V}$, $(h +_1 v).w = h + v.w$ and $(v +_2 h).w = v.w + h$;

9. for every $h, s \in \mathsf{H}$ and $v \in \mathsf{V}$, $(h +_1 v) * s = h + v * s$ and $(v +_2 h) * s = v * s + h$;

10. $\Box +_2 0 = \Box = 0 +_1 \Box$;

11. $\omega(0) = 0$;

12. $(\Box)^\omega = \Box$;

13. for every $h, s \in \mathsf{H}$, $(h +_1 \Box +_2 s)^\omega = \omega(h) +_1 \Box +_2 \omega(s)$.

The *class of* ω*-algebras* of type $\tau = (2, 2, 2, 2, 2, 1, 1)$ satisfying the above conditions is denoted by $\mathfrak{B}$.

If for instance we take the unary operations ω as identities, then every forest algebra satisfies the above conditions. Hence, every forest algebra may be thus viewed as an element of the class of ω-algebras $\mathfrak{B}$.

Lemma 4.1.1. *Let* $S = (H_S, V_S)$ *be a zero-dimensional, see Definition 3.1.16, and compact metric forest algebra, see Definition 3.1.18. Let* v *be an element of* V_S *and* h *be an element of* H_S*, and let* $k \in \mathbb{Z}$*. Then sequences of products* $\{v^{n!+k}\}_n \geq |k|$ *and additions* $\{(n! + k)(h)\}_n \geq |k|$ *converge. For* $k = 0$ *the limit is an idempotent.*

Proof. For proof see [2, page 20]. $\Box$

The limit $\lim v^{n!}$ is denoted by v^{ω_1}, and the limit $\lim(n!)(h)$ is denoted by $\omega_2(h)$.

Every forest algebra $S = (H, V)$ endowed with unary operations ω_1 and ω_2 satisfies the properties of ω-algebras. The axioms $(1)-(10)$ are immediate by the equational axioms of forest algebras, and we have

$$(\Box)^{\omega_1} = \lim \Box^{n!} = \lim \Box = \Box$$

and

$$\omega_2(0) = \lim(n!)(0) = \lim 0 = 0,$$

also for a context $v = h_1 + \Box + h_2$ we have the following

$$\begin{aligned}(h_1 + \Box + h_2)^{\omega_1} &= \lim(h_1 + \Box + h_2)^{n!} \\ &= \lim(n!)(h_1) + \lim \Box + \lim(n!)(h_2) \\ &= \omega_2(h_1) + \Box + \omega_2(h_2).\end{aligned}$$

Hence, the forest algebra S becomes an ω-algebra.

An *ω-algebra homomorphism*

$$\eta : (H_1, V_1) \to (H_2, V_2)$$

of ω-algebras is a pair (λ, μ) of monoid homomorphisms

$$\begin{aligned} \lambda : &\quad H_1 \to H_2, \\ \mu : &\quad V_1 \to V_2 \end{aligned}$$

such that, for every $h \in H$ and every $v \in V$,

$$\lambda(v * h) = \mu(v) * \lambda(h) \qquad \text{and} \qquad \begin{cases} \mu(h +_1 v) = \lambda(h) +_1 \mu(v) \\ \mu(v +_2 h) = \mu(v) +_2 \lambda(h), \end{cases}$$

and

$$\mu(v^\omega) = (\mu(v))^\omega \qquad \text{and} \qquad \lambda(\omega(h)) = \omega(\lambda(h)).$$

However, we will abuse notation slightly and denote both component maps by η.

An *ω-subalgebra* is a subset of an ω-algebra, closed under all its operations, and carrying the induced operations.

Let $S_1 = (H_1, V_1)$ and $S_2 = (H_2, V_2)$ be ω-algebras. Their *direct product* $S_1 \times S_2$ is $(H_1 \times H_2, V_1 \times V_2)$ where

$$H_1 \times H_2 = \{(h_1, h_2) \mid h_1 \in H_1 \text{ and } h_2 \in H_2\}$$

and

$$V_1 \times V_2 = \{(v_1, v_2) \mid v_1 \in V_1 \text{ and } v_2 \in V_2\}.$$

Operations are defined componentwise.

The class of ω-algebras $\mathfrak{B}$ is closed under direct products and subalgebras. So, by [5, 19] and also since it is defined by equational axioms, all the free ω-algebras exist.

Over a finite alphabet $A = \{a_i \mid i = 1, \dots, n\}$ an ω-algebra $\mathcal{A} = (\mathsf{H}, \mathsf{V})$ is said to be *A-free ω-algebra* over $\mathfrak{B}$ with the free generating set A via the mapping $\eta : A \to \mathcal{A}$ such that $\eta(A)$ generates $\mathcal{A}$, if we have the following universal property: for every ω-algebra $S = (H, V) \in \mathfrak{B}$ with any subset $\{y_i \mid i = 1, \dots, n\}$ of V, there is a unique ω-algebra homomorphism $\phi : \mathcal{A} \to S$ such that $\phi(\eta(a_i)) = y_i$.

Consider the map $\eta : A \to \mathcal{A}$ such that $\eta(a) = a\square$. Define a set $A' = \{\eta(a) \mid a \in A\}$, then $\mathcal{A}$ can be viewed as an A'-free ω-algebra via the natural injection ι. Let $S = (H, V)$ be an ω-algebra in $\mathfrak{B}$ such that there is a mapping $f : A' \to V$. By the universal property of free ω-algebras, the following diagram commutes:

$$\begin{tikzcd} A' \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathcal{A} \arrow[d, dotted, "\exists ! \hat{f}"] \\ & S \end{tikzcd}$$

Later in this chapter we will show that the faithfulness axiom holds for $\mathcal{A}$. Hence, $\mathcal{A}$ is a forest algebra. Under the claim that $\mathcal{A}$ is a forest algebra, for every forest algebra $S = (H, V)$ in $\mathfrak{B}$ if there is a mapping $f : A' \to V$, then there exists a unique ω-algebra homomorphism φ such that the following diagram commutes.

$$\begin{tikzcd} A' \arrow[r] \arrow[dr, "f"'] & \mathcal{A} \arrow[d, dotted, "\exists!\varphi"] \\ & S \end{tikzcd}$$

In view of the universal property of the free ω-algebras, $\mathcal{A}$ is a free forest algebra.

Remark 4.1.2. The axioms $(1) - (10)$ imply that over a finite alphabet $A = \{a_i \mid i = 1, \dots, n\}$ the A'-free ω-algebra $\mathcal{A}$ satisfies the equational axioms of forest algebras. Hence, in view of the universal property, Lemma 1.1.5, of the free forest algebra A^{Δ}, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "\eta"'] & A^{\Delta} \arrow[d, dotted, "\exists!\varphi"] \\ & \mathcal{A} \end{tikzcd}$$

For a finite alphabet A, by *term algebra* we mean the freely generated algebraic structure generated by A over a signature τ, which we denote by $\mathbf{A}$, for more details see [5, 19].

In an ω-algebra, we will denote all operations $+$, $+_1$ and $+_2$ by $+$, except in some places to avoid confusions.

Remark 4.1.3. Let $S = (H, V)$ be a forest algebra. Hence, the axioms $(1)-(10)$ of ω-algebras hold in S. Endow S with additional unary operations $\omega(\_) : H \to H$ and $\_^{\omega} : V \to V$ defined as identity mapping. Hence, the axioms $(11) - (13)$ of ω-algebras also hold in S. Therefore, S is an ω-algebra in $\mathfrak{B}$.

Example 4.1.4. For an alphabet A let $\mathcal{A}$ be the A'-free ω-algebra in $\mathfrak{B}$. Let S_N be the forest algebra in Example 1.2.4. In view of Remark 4.1.3, S_N is an ω-algebra.

By the universal property of the free ω-algebra $\mathcal{A}$, there is a unique ω-algebra homomorphism

$$\#_{\text{Nodes}} : \mathcal{A} \to S_N$$

such that

$$\#_{\text{Nodes}}(a\Box) = 1'.$$

Definition 4.1.5. We say that the *number of nodes* of an element $x \in \mathcal{A}$ is n, if x is an element of H and $\#_{\text{Nodes}}(x) = n$ or x is an element of V and $\#_{\text{Nodes}}(x) = n'$.

Definition 4.1.6. Let S be an ω-algebra in $\mathfrak{B}$. We say that an element x of S is a *subterm* of an element y of S, if there exists an n-ary operation f, which is a composition of operations from $\{+, +_1, +_2, ., *, \omega(), ()^\omega\}$, and there are elements $t_1, \ldots, t_{n-1}$ in S such that $f(x, t_1, \ldots, t_{n-1}) = y$.

Lemma 4.1.7. *For elements x_1 and x in the free ω-algebra $\mathcal{A}$, if x_1 is a subterm of x then $\#_{\text{Nodes}}(x_1) \leq \#_{\text{Nodes}}(x)$.*

Proof. By definition of a subterm, there is an n-ary operation f, and there are elements $t_1, \ldots, t_{n-1}$ such that $f(x_1, t_1, \ldots, t_{n-1}) = x$. So, we have

$$\#_{\text{Nodes}}(f(x_1, t_1, \ldots, t_{n-1})) = \#_{\text{Nodes}}(x).$$

Since $\#_{\text{Nodes}}$ is an ω-algebra homomorphism then we have

$$\#_{\text{Nodes}}(x) = \#_{\text{Nodes}}(x_1) + \#_{\text{Nodes}}(t_1) + \cdots + \#_{\text{Nodes}}(t_{n-1}),$$

which implies the result. $\square$

Lemma 4.1.8. *For an element x in the free ω-algebra $\mathcal{A}$, $\#_{\text{Nodes}}(x) = 0$ if and only if x is a trivial element.*

Proof. If x is a trivial element, then $\#_{\text{Nodes}}(x) = 0$. Now, assume that $\#_{\text{Nodes}}(x) = 0$ but x is a non-trivial element, then there is an element $d\square$, which is a subterm of x. By Lemma 4.1.7, we have $1 = \#_{\text{Nodes}}(d\square) \leq 0$, which is a contradiction. $\square$

Lemma 4.1.8 is used in the following section for distinguishing all kinds of non-trivial additively irreducible and non-trivial multiplicatively irreducible elements of the free ω-algebras.

4.2 About Free ω-Algebra $\mathcal{A}$

In this section, we use the universal property of free ω-algebras to show the following key theorem:

Theorem 4.2.1. *For the A'-free ω-algebra $\mathcal{A} = (\mathsf{H}, \mathsf{V})$ we have the following:*

- $\omega(\mathsf{H} \setminus \{0\}) \cap +(\mathsf{H} \setminus \{0\}, \mathsf{H} \setminus \{0\}) = \emptyset$*;*
- $(\mathsf{V} \setminus (\mathsf{H} + \square + \mathsf{H}))^\omega \cap .(\mathsf{V} \setminus \{\square\}, \mathsf{V} \setminus \{\square\}) = \emptyset$*;*
- *for every $a\square \in A'$, $a\square * \mathsf{H} \cap +(\mathsf{H} \setminus \{0\}, \mathsf{H} \setminus \{0\}) = \emptyset$;*
- *for every $a\square \in A'$, $a\square.\mathsf{V} \cap (+(\mathsf{H} \setminus \{0\}, \mathsf{V}) \cup +(\mathsf{V}, \mathsf{H} \setminus \{0\})) = \emptyset$.*

Where

$$\begin{aligned}
\omega(\mathsf{H}\setminus\{0\}) &= \{\omega(h) \mid h \in \mathsf{H}\setminus\{0\}\},\\
\mathsf{H}+\Box+\mathsf{H} &= \{h+\Box+s \mid h,s \in \mathsf{H}\},\\
\mathsf{V}\setminus(\mathsf{H}+\Box+\mathsf{H}) &= \{v \in \mathsf{V} \mid v \notin (\mathsf{H}+\Box+\mathsf{H})\},\\
(\mathsf{V}\setminus(\mathsf{H}+\Box+\mathsf{H}))^{\omega} &= \{v^{\omega} \mid v \in \mathsf{V}\setminus(\mathsf{H}+\Box+\mathsf{H})\},\\
.(\mathsf{V}\setminus\{\Box\},\mathsf{V}\setminus\{\Box\}) &= \{v.w \mid v,w \in \mathsf{V}\setminus\{\Box\}\},\\
a\Box * \mathsf{H} &= \{a\Box * h \mid h \in \mathsf{H}\},\\
a\Box.\mathsf{V} &= \{a\Box.v \mid v \in \mathsf{V}\},\\
+(\mathsf{H}\setminus\{0\},\mathsf{H}\setminus\{0\}) &= \{h+s \mid h,s \in \mathsf{H}\setminus\{0\}\},\\
+(\mathsf{H}\setminus\{0\},\mathsf{V}) &= \{h+v \mid h \in \mathsf{H}\setminus\{0\}, v \in \mathsf{V}\},\\
+(\mathsf{V},\mathsf{H}\setminus\{0\}) &= \{v+h \mid h \in \mathsf{H}\setminus\{0\}, v \in \mathsf{V}\}.
\end{aligned}$$

An element of $\mathcal{A}$ is said to be a *p-forest* or a *p-context*, if it belongs, respectively, to H or V. We call an element of $\mathcal{A}$ a *finite p-forest* or a *finite p-context* if it does not involve the unary operations; otherwise, we call it an *infinite p-forest* or an *infinite p-context*. We call an infinite p-forest and an infinite p-context respectively an *ω-forest* and an *ω-context*, if it is of the form $\omega(h)$ for some p-forest h or it is of the form v^{ω} for some p-context v (respectively).

We say that a p-context v is a *factor* of a p-context t, if there exist p-contexts u and w such that $t = uvw$. And we say that a p-context v is a factor of a p-forest t, if there exist a p-context u and a p-forest h such that $t = uvh$.

The p-forest 0 and the p-context $\Box$ are called respectively the *trivial* p-forest and p-context.

Let v be a p-context. We say that a p-context w is a *prefix* of v if there exists a p-context u such that $v = w.u$. The set of all prefixes of v is denoted by $\mathrm{Pref}(v)$. Let h be a p-forest. We say that a p-context w is a prefix of h if there exists a p-forest s such that $h = w * s$. The set of all prefixes of h is denoted by $\mathrm{Pref}(h)$. Note that every prefix of an element t in $\mathcal{A}$ is a factor of t.

We call a non-trivial p-context t in V a *$\Box$-pure* p-context if, whenever $h_1, h_2 \in \mathsf{H}$ and $u \in \mathsf{V}$ are such that $t = h_1+u+h_2$, the equalities $h_1 = h_2 = 0$ hold.

For a p-context v in V we define $C(v)$ as follows. If v is a $\Box$-pure p-context or the trivial p-context, then let $C(v) = v$. Otherwise, there are p-forests h_1 and h_2 in H and a p-context v_1 in V such that at least one of the p-forests h_1 and h_2 is non-trivial and the equality $v = h_1 + v_1 + h_2$ holds. Iterate the same procedure on v_1. By Lemma 4.1.7, since

$$\#_{\text{Nodes}}(v_1) < \#_{\text{Nodes}}(v) < \infty,$$

after finitely many steps we will find a p-context v_n such that $C(v_n) = v_n$. We then let $C(v) = v_n$.

The following results are immediate:

Lemma 4.2.2. *A p-context v in V is $\Box$-pure if and only if $C(v) = v$.* $\Box$

Lemma 4.2.3. *For every p-context v in V there are p-forests h_1 and h_2 in H, and a p-context u in V such that u is $\Box$-pure and $v = h_1 + u + h_2$.* $\Box$

For a p-context v with $C(v) \neq \Box$, we define a *factorization* of v by

$$v = \prod_{i \in \mathbb{N}} v_{2i+1} v_{2i+2}$$

where the p-contexts v_{2i+1}'s are $\Box$-pure, $v_{2i+2} = h_{i,1} + \Box + h_{i,2}$ with p-forests $h_{i,1}$ and $h_{i,2}$ in H, and the v_i's are such that, if $k > 0$ and $v_{2k+1} = \Box$, then $v_j = \Box$ for all $j \geq 2k+1$. For a p-context v with $C(v) \neq \Box$, such a factorization exists: by the way that we defined $C(v)$, since there are finitely many p-forests h_i and h'_i such that $v = h_1 + \cdots + h_n + C(v) + h'_n + \cdots + h'_1$, if $n = 0$, then v is a $\Box$-pure, $v_1 = v$ and for every $i \geq 2$, $v_i = \Box$; otherwise, $v_1 = \Box$, $v_2 = h_1 + \cdots + h_n + \Box + h'_n + \cdots + h'_1$, $v_3 = C(v)$, and for every $i \geq 4$, $v_i = \Box$.

For a p-context v with $C(v) = \Box$ the factorization of v is v itself.

We say that a p-context v in V is *multiplicatively irreducible* if there do not exist non-trivial p-contexts u_1 and u_2 such that $v = u_1 u_2$.

Lemma 4.2.4. *Every p-context can be written as product of its non-trivial multiplicatively irreducible factors.*

Proof. Let v be a non-trivial p-context. If v is multiplicatively irreducible, then v can be written as a product of itself. Otherwise, there exist non-trivial p-contexts v_1 and v_2 such that $v = v_1 v_2$. Iterate the same procedure on v_1 and v_2. Since

$$\#_{\text{Nodes}}(v_1) < \#_{\text{Nodes}}(v) \ , \quad \#_{\text{Nodes}}(v_2) < \#_{\text{Nodes}}(v)$$

and $\#_{\text{Nodes}}(v)$ is finite, we will get the result after finitely many steps. $\Box$

We say that an element P in $\mathcal{A}$ is *additively irreducible* if, for the case that P is a p-forest, there do not exist non-trivial p-forests s_1 and s_2 such that $P = s_1 + s_2$ and, for the case that P is a p-context, there does not a exist non-trivial p-forest s and a p-context v such that $P = s + v$ or $P = v + s$. In view of the definition of $\square$-pure, every $\square$-pure is an additively irreducible p-context.

Let t be a p-forest which is not additively irreducible, then we show that t can be written as a sum of non-trivial additively irreducible p-forests, which we call *summands* of t.

Lemma 4.2.5. *Every p-forest can be written as sum of its non-trivial additively irreducible summands.*

Proof. Let t be a non-trivial p-forest in H. If t is additively irreducible, then t can be written as a sum of itself. Otherwise, there exist non-trivial p-forests s_1 and s_2 such that $t = s_1 + s_2$. Iterate the same procedure on s_1 and s_2. Since

$$\#_{\mathrm{Nodes}}(s_1) < \#_{\mathrm{Nodes}}(t) < \infty \quad \text{and} \quad \#_{\mathrm{Nodes}}(s_2) < \#_{\mathrm{Nodes}}(t) < \infty,$$

after finitely many steps we will find non-trivial additively irreducible p-forests h_i's and then $t = h_1 + \cdots + h_n$. $\square$

In the case where h is a finite p-forest, in view of Remark 4.1.2, h is the sum of its connected forest summands. And in the case where h is an infinite p-forest, we will show that h is the sum of its summands which are ω-forests or p-forests which are of the form $v * s$ for some non-trivial $\square$-pure p-context v and p-forest s.

Let s and t be elements of the free ω-algebra $\mathcal{A}$. We say that t is a *scattered divisor* of s when t has a decomposition of the form $t_1 \dots t_n$ and s has one of the following decompositions:

$$u_1 t_1 u_2 t_2 \dots u_n t_n \quad \text{or} \quad u_1 t_1 u_2 t_2 \dots u_n t_n u_{n+1}.$$

Note that, for some i's we may have $u_i = \square$.

We say that an element t of $\mathcal{A}$ is an *divisor* of an element P in $\mathcal{A}$ if the following conditions hold:

- in case t and P are p-contexts, there exist p-contexts u and v such that $P = utv$;
- in case t is a p-context and P is a p-forest, there exist a p-context u and a p-forest h such that $P = uth$;
- in case t and P are p-forests, there exists a p-context u such that $P = ut$.

Note that a divisor of a p-context cannot be a p-forest.

Remark 4.2.6. The difference between factor and divisor is that: a divisor may be a p-forest but a factor is always a p-context.

The following result gives some conditions for being a subterm, see Definition 4.1.6, of an element of the free ω-algebra $\mathcal{A}$.

Lemma 4.2.7. *In the free ω-algebra $\mathcal{A}$ the following conditions hold:*

- *a divisor of an element y is a subterm of y;*
- *for a p-forest h, h is a subterm of the p-forest $\omega(h)$;*
- *for a p-context v, v is a subterm of the p-context v^ω;*
- *for a p-forest h, h is a subterm of the p-contexts $h + \Box$ and $\Box + h$;*
- *if an element P is a subterm of an element Q and the element Q is a subterm of an element t, then the element P is a subterm of the element t.*

Proof. We just show the first one the next three conditions are handled similarly. Assume that, x and y are p-forests and x is a divisor of y. Then there is a p-context v such that $v * x = y$. For this case, let $f(\_,\_) = \_ * \_$.

Now assume that, x and y are p-contexts and x is a divisor of y. Then there are p-contexts v_1 and v_2 such that $v_1.x.v_2 = y$. For this case, let $f(\_,\_,\_) = \_.(\_.\_)$ or $f(\_,\_,\_) = (\_.\_).\_$.

Assume that, x is a p-context and y is a p-forest and x is a divisor of y. Then there is a p-context v and there is a p-forest h such that $v * (x * h) = (v.x) * h = y$. For this case, let $f_1(\_,\_,\_) = \_ * (\_ * \_)$ or $f_2(\_,\_,\_) = (\_.\_) * \_$.

Now we show the last one. Assume that, an element P is a subterm of an element Q, and the element Q is a subterm of an element t then, by definition of subterm there are an n-ary operation f_1, an m-ary operation f_2, elements $t_1, \ldots, t_{n-1}$, and also elements $t'_1, \ldots, t'_{m-1}$ such that the equalities $f_1(Q, t_1, \ldots, t_{n-1}) = t$ and $f_2(P, t'_1, \ldots, t'_{m-1}) = Q$ hold. Let

$$f(x_1, x_2, \ldots, x_{m+n-1}) = f_1(f_2(x_1, x_2, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1})$$

be the $(n + m - 1)$-ary operation, then we have

$$\begin{aligned} f(P, t'_1, \ldots, t'_{m-1}, t_1, \ldots, t_{n-1}) &= f_1(f_2(P, t'_1, \ldots, t'_{m-1}), t_1, \ldots, t_{n-1}) \\ &= f_1(Q, t_1, \ldots, t_{n-1}) \\ &= t, \end{aligned}$$

which means P is a subterm of t. $\Box$

Since a factor of x is a divisor of x, then a factor of x is a subterm of x. Let us give another example of ω-algebras.

Example 4.2.8. Let m be a non-negative integer and $T_m = A^\Delta$ be the free forest algebra with additional operations $\omega : H^A \to H^A$, which sends the forest h to mh, which is the sum of m copies of h, and $\omega : V^A \to V^A$ which sends the context v to v^m, which is the product of m copies of v.

The ω-algebra T_m satisfies the identities in $\mathfrak{B}$:

1. $\omega(0) = 0$;
2. $(\Box)^\omega = \Box$;
3. for every $h, s \in H^A$, $(h + \Box + s)^\omega = \omega(h) + \Box + \omega(s)$.

By the universal property of the free ω-algebra $\mathcal{A}$, for ω-algebra T_m, there is a unique ω-algebra homomorphism

$$\_^{m,m} : \mathcal{A} \to T_m$$

such that

$$\_^{m,m}(a\Box) = a\Box.$$

Let t be an element of $\mathcal{A}$. We denote $\_^{m,m}(t)$ by $t^{m,m}$.

By Remark 4.1.2, we have $A^\Delta = \{t^{m,m} \mid t \in \mathcal{A}\ ,\ \ m \in \mathbb{N}\}$.

4.2.1 Some More Examples of ω-Algebras in $\mathfrak{B}$

Now we construct more examples of ω-algebras which are used to show that every free ω-algebra is a forest algebra. The following example is the most important example in the rest of this chapter.

Example 4.2.9. Let $\mathcal{S} = (H_\mathcal{S}, V_\mathcal{S})$ be the free forest algebra

$$(A \uplus \{a\Box, b\Box, c\Box\})^\Delta,$$

with additional operations $\omega : H_\mathcal{S} \to H_\mathcal{S}$ given by

$$\omega(h) = \begin{cases} a\Box * h & , \text{ if } \ h \ \text{ is non-trivial} \\ 0 & , \text{ if } \ h = 0 \end{cases}$$

and $\omega : V_\mathcal{S} \to V_\mathcal{S}$ given by

$$v^\omega = \begin{cases} b\Box.v.c\Box & , \text{ if } \ C(v) \neq \Box \\ \omega(h_1) + \Box + \omega(h_2) & , \text{ if } \ v = h_1 + \Box + h_2. \end{cases}$$

Then $\mathcal{S}$ is an ω-algebra:

1. $\omega(0) = 0$;
2. $(\Box)^\omega = \Box$;

3. for every $h, s \in H_{\mathcal{S}}$,

$$(h + \Box + s)^{\omega} = \omega(h) + \Box + \omega(s).$$

By the universal property of the free ω-algebra $\mathcal{A}$, there is a unique ω-algebra homomorphism

$$\Phi : \mathcal{A} \to \mathcal{S}$$

such that for $d\Box \in A'$

$$\Phi(d\Box) = d\Box.$$

Example 4.2.10. Let, for every $n \in \mathbb{N}$, $S_n = A^{\Delta}$ be the free forest algebra with additional operations $\omega : H^A \to H^A$ which sends every forest h to the forest $n(h)$, which is n-times addition of h by itself, for the case $n = 0$ we assume that $0(h) = 0$, and $\omega : V^A \to V^A$ which sends every context v to a context v^n, which is n-times product of v by itself, for the case $n = 0$ we assume that $v^0 = \Box$.

For every $n \in \mathbb{N}$ the ω-algebra S_n satisfies the identities in $\mathfrak{B}$:

1. $\omega(0) = n(0) = 0$;
2. $(\Box)^{\omega} = \Box^n = \Box$;
3. for every $h, s \in H^A$,

$$\begin{aligned}(h + \Box + s)^{\omega} &= (h + \Box + s)^n \\ &= n(h) + \Box + n(s) \\ &= \omega(h) + \Box + \omega(s).\end{aligned}$$

By the universal property of the free ω-algebra $\mathcal{A}$, there is a unique ω-algebra homomorphism

$$f_n : \mathcal{A} \to S_n$$

such that

$$f_n(x) = x^{n,n}.$$

Lemma 4.2.11. *For an alphabet A, let $\mathcal{A} = (\mathsf{H}, \mathsf{V})$ be the A'-free ω-algebra in $\mathfrak{B}$. Let $h \in \mathsf{H}$ and $v \in \mathsf{V}$. Then the following conditions hold:*

- $\omega(h) = 0$ *if and only if* $h^{1,1} = 0$;
- $v^{\omega} = \Box$ *if and only if* $v^{1,1} = \Box$.

Proof. We show that $h^{1,1} = 0$ if and only if $h = 0$ and that $v^{1,1} = \Box$ if and only if $v = \Box$.

The equalities $0^{1,1} = 0$ and $\Box^{1,1} = \Box$ are immediate from the definitions. Now, assume that $h^{1,1} = 0$ and $v^{1,1} = \Box$. We have the following facts:

- if P^ω is a subterm of h with P is a finite p-context, that is $P^{1,1} = P$, then P is a subterm of $h^{1,1}$, so $P = \Box$;
- if $\omega(Q)$ is a subterm of h with Q is a finite p-forest, that is $Q^{1,1} = Q$, then Q is a subterm of $h^{1,1}$, so $Q = 0$;
- for a finite p-context P, that is $P^{1,1} = P$, if P is a subterm of h, then P is a subterm of $h^{1,1}$, so $P = \Box$;
- for a finite p-forest Q, that is $Q^{1,1} = Q$, if Q is a subterm of h which is a p-forest, then Q is a subterm of $h^{1,1}$, so $Q = 0$;
- every p-forest in $\mathcal{A}$, is made by combinations of some of its finite subterms, where by combination we mean addition, multiplication, action, and applying the operations ω.

We have thus shown that $h = 0$.

In a similar way we can show that $v = \Box$.

By using the identity $\omega(0) = 0$ if $h = 0$, then $\omega(h) = 0$. Now, assume that $\omega(h) = 0$, then $h^{1,1} = 0$ and so we have $h = 0$.

In a similar way, by using the identity $(\Box)^\omega = \Box$, if $v = \Box$, then $v^\omega = \Box$. Now, assume that $v^\omega = \Box$, then $v^{1,1} = \Box$ and so we have $v = \Box$. $\Box$

Corollary 4.2.12. *Since for every p-context $v \in \mathsf{V}$, $C(v)$ is also a p-context, we have $C(v) = \Box$ if and only if $C(v)^{1,1} = \Box$.* $\Box$

Remark 4.2.13. If there is a forest algebra homomorphism $\delta : A^\Delta \to S$ into a forest algebra S, then we can be viewed as an ω-algebra homomorphism $f : \mathcal{A} \to S$ such that $f = \delta \circ f_1$. Note that, in the forest algebra S, the unary operations ω's are assumed to be identities. That is the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f_1"] \arrow[dr, "f"'] & A^\Delta \arrow[d, "\delta"] \\ & S \end{tikzcd}$$

Remark 4.2.14. According to Remark 4.2.13 and by using the forest algebra homomorphism labels : $A^\Delta \to S_L$ in Example 1.2.11, there is a unique ω-algebra homomorphism

$$\text{labels} : \mathcal{A} \to S_L$$

such that

$$\text{labels}(a\Box) = \{a, \Box\}.$$

Definition 4.2.15. We say that the *set of labels* of an element $x \in \mathcal{A}$ is X if x is an element of H and $\text{labels}(x) = X$ or x is an element of V and $\text{labels}(x) = X \cup \{\Box\}$.

Example 4.2.16. Let $S_1' = A^\Delta$ be the free forest algebra with additional operations $\omega : H^A \to H^A$ and $\omega : V^A \to V^A$ which, for $a \in A$ fixed, are defined as follows:

$$\omega(h) = \begin{cases} a & \text{, if } \ h \neq 0 \\ 0 & \text{, otherwise} \end{cases}$$

and

$$v^\omega = \omega(h_1) + C(v) + \omega(h_2) \qquad \text{where } \ v = h_1 + C(v) + h_2.$$

The ω-algebra S_1' satisfies the identities in $\mathfrak{B}$:

1. $\omega(0) = 0$;
2. $(\Box)^\omega = \Box$;
3. for every $h, s \in H^A$, $(h + \Box + s)^\omega = \omega(h) + \Box + \omega(s)$.

Example 4.2.17. Let $S_2' = A^\Delta$ be the free forest algebra with additional operations $\omega : H^A \to H^A$, which sends forest h to 0, and $\omega : V^A \to V^A$ which sends context v to $a\Box$ if $C(v) = a\Box u$ and sends to $\Box$ if $C(v) = \Box$.

The ω-algebra S_2' satisfies the identities in $\mathfrak{B}$:

1. $\omega(0) = 0$;
2. $(\Box)^\omega = \Box$;
3. for every $h, s \in H^A$, $(h + \Box + s)^\omega = \omega(h) + \Box + \omega(s) = \Box$.

Example 4.2.18. Let $S_3' = A^\Delta$ be the free forest algebra with additional operations $\omega : H^A \to H^A$, which sends the forest $h = a_1\Box s_1 + \cdots + a_n\Box s_n$ to $a_1 + \cdots + a_n$ and sends 0 to 0 and $\omega : V^A \to V^A$ which sends the context $v = h_1 + C(v) + h_2$ with $C(v) = c\Box u$, $h_1 = a_1\Box s_1 + \cdots + a_n\Box s_n$ and $h_2 = b_1\Box t_1 + \cdots + b_m\Box t_m$ to $a_1 + \cdots + a_n + c\Box + b_1 + \cdots + b_m$ and sends $\Box$ to $\Box$.

The ω-algebra S_3' satisfies the identities in $\mathfrak{B}$:

1. $\omega(0) = 0$;
2. $(\Box)^\omega = \Box$;
3. for every $h, s \in H^A$, $(h + \Box + s)^\omega = \omega(h) + \Box + \omega(s)$.

By the universal property of the free ω-algebra $\mathcal{A}$, for ω-algebras S_1', S_2' and S_3', there is a unique ω-algebra homomorphism

$$f_i' : \mathcal{A} \to S_i'$$

such that

$$f_i'(a\Box) = a\Box,$$

respectively for $i = 1, 2$ and 3.

Lemma 4.2.19. *For an element x of $\mathcal{A}$, x is trivial if and only if $f_3'(x)$ is trivial.*

Proof. The direct implication is trivial. We show the reverse implication. If x is a non-trivial finite p-forest or p-context, then $f_3'(x) = x$ is also non-trivial. If $x = a_1\Box s_1 + \cdots + a_n\Box s_n$ is sum of non-trivial finite p-forest, then, since a_i's are non-trivial, $f_3'(\omega(x)) = a_1 + \cdots + a_n$ is non-trivial. And if $x = h_1 + C(x) + h_2$ is a non-trivial finite p-context with $C(x) = c\Box v$, $h_1 = a_1\Box s_1 + \cdots + a_n\Box s_n$ and $h_2 = b_1\Box t_1 + \cdots + b_m\Box t_m$, then, since at least one of a_i's, b_j's or c is non-trivial, $f_3'(x^\omega) = a_1 + \cdots + a_n + c\Box + b_1 + \cdots + b_m$ is non-trivial. Every p-forest and similarly p-context in $\mathcal{A}$, is made by combinations of some of its finite subterms, where by combination we mean addition, multiplication, action, and applying the operations ω. Every non-trivial element of $\mathcal{A}$ has a non-trivial finite subterm. Hence, x is non-trivial implies $f_3'(x)$ is non-trivial. $\Box$

Lemma 4.2.19 is used later on in the proof of the fact that the free ω-algebra is a forest algebra.

We proceed with another example of ω-algebras which is constructed from a monoid together with an action on itself. The next couple of examples of ω-algebras are obtained as particular cases.

Let M be a monoid. Let $\varphi : M \to \operatorname{End} M$ be a mapping into the monoid of monoid endomorphisms of M, acting on the left. Denote $\varphi(v)(u)$ by ${}^v u$. Define on M a *skew multiplication* $\odot$ as follows:

$$u \odot v = u\,{}^u v,$$

and denote the resulting structure by M^φ. We say φ is a *skew mapping* if $\varphi : M^\varphi \to \operatorname{End} M$ is a semigroup homomorphism, that is, if the following condition holds:

$$\varphi(u \odot v) = \varphi(u)\varphi(v) \quad \text{for all } u, v \in M. \tag{4.1}$$

Proposition 4.2.20. *If φ is a skew mapping then M^φ is a monoid and $\varphi : M^\varphi \to \operatorname{End} M$ is a monoid homomorphism.*

Proof. Let $u, v, w \in M$ be arbitrary elements. Condition (4.1) yields the equality ${}^u(v\,{}^v w) = {}^u v\,{}^{u\odot v} w$. Hence, the following equalities also hold:

$$u \odot (v \odot w) = u\,{}^u(v\,{}^v w) = u\,{}^u v\,{}^{u\odot v} w = (u \odot v)\,{}^{u\odot v} w = (u \odot v) \odot w$$
$$1 \odot u = 1\,{}^1 u = 1\,u = u = u\,{}^u 1 = u \odot 1.$$

The last statement is now obvious. $\Box$

Assuming that φ is a skew mapping, we call the monoid M^φ the *skew monoid determined by φ*.

For a semigroup S, let

$$S^I = S \uplus \{I\} \tag{4.2}$$

be the *monoid which is obtained from S by adding a (new) identity element* I, even if S is already a monoid.

The following result yields an example of application of Proposition 4.2.20.

Lemma 4.2.21. *Let S be a commutative semigroup, denoted additively. Let $T = S^I \times S^I \times S^I$ be the direct product of three copies of S^I and consider the mapping $\varphi : T \to \operatorname{End} T$ defined by*

$$\varphi(s_1, s_2, s_3) = \begin{cases} \mathrm{Id}_T & \text{if } s_2 = I, \\ \sigma_2 & \text{otherwise,} \end{cases}$$

where Id_T is the identity mapping on T and

$$\sigma_2(u_1, u_2, u_3) = (I, u_1 + u_2 + u_3, I).$$

Then φ is a skew mapping.

Proof. Since the monoid S^I is commutative, φ does take its values in the monoid $\operatorname{End} T$. It is a simple calculation to verify that φ is a skew mapping. Indeed, since $s_2 u_2 = I$ if and only if $s_2 = u_2 = I$, we have

$$\varphi\big((s_1, s_2, s_3) \odot (u_1, u_2, u_3)\big) = \mathrm{Id}_T = \varphi(s_1, s_2, s_3)\, \varphi(u_1, u_2, u_3)$$

if and only if $s_2 = u_2 = I$. The case $s_2 u_2 \neq I$ is then immediate since σ_2 is an idempotent. $\square$

Combining with Proposition 4.2.20, we obtain the following result.

Corollary 4.2.22. *If S is a commutative semigroup and T and φ are as in Lemma 4.2.21, then T^φ is a monoid.* $\square$

In order to define a structure of ω-algebra on (S^I, T), we consider the following operations, where we already call the elements of S^I p-forests and those of T p-contexts:

- p-forest addition is the addition in S^I;
- for $s \in S^I$ and $(u_1, u_2, u_3) \in T$, we take

$$\begin{aligned} s + (u_1, u_2, u_3) &= (s + u_1, u_2, u_3) \\ (u_1, u_2, u_3) + s &= (u_1, u_2, u_3 + s); \end{aligned}$$

- p-context multiplication is the skew multiplication in T;

- the action of $(u_1, u_2, u_3) \in T$ on $s \in S^I$ is given by

$$(u_1, u_2, u_3) * s = u_1 + u_2 + u_3 + s;$$

- for $s \in S^I$, we let

$$\omega(s) = \begin{cases} s & \text{if } s = I, \\ s + s_0 & \text{otherwise,} \end{cases}$$

  where $s_0 \in S^I$ is a fixed element;

- for $(u_1, u_2, u_3) \in T$, we let

$$(u_1, u_2, u_3)^\omega = \begin{cases} (\omega(u_1), u_2, \omega(u_3)) & \text{if } u_2 = I, \\ (u_1, u_2 + s_0, u_3) & \text{otherwise.} \end{cases}$$

Proposition 4.2.23. *For the above operations, (S^I, T) is an ω-algebra.*

Proof. It takes just a very few simple calculations to check the axioms of ω-algebras, the only ones that require any additional verification being 4, 8, and 9. $\square$

Example 4.2.24. For an alphabet A let $\mathcal{A} = (\mathsf{H}, \mathsf{V})$ be the A'-free ω-algebra in $\mathfrak{B}$. Let S be the monoid of natural numbers $\mathbb{N}$ under operation $+$, and let $I = -\infty$. Then Proposition 4.2.23 implies that $\mathcal{ID} = (S^I, T)$ is an ω-algebra. Note that, the element I is the identity element of S^I.

By the universal property of the free ω-algebra $\mathcal{A}$, there is a unique ω-algebra homomorphism

$$\text{Idem} : \mathcal{A} \to \mathcal{ID}$$

such that

$$\#_{\text{Idem}}(a\square) = (-\infty, 0, -\infty).$$

Definition 4.2.25. For a non-trivial element $x \in \mathcal{A}$, the *number of idempotents* of x with multiplicities, we denote by $\#_{\text{IDEM}}(x)$, is $\#_{\text{Idem}}(x)$ if $x \in \mathsf{H}$ and $\#_{\text{Idem}}(x * 0)$ otherwise. In addition we assumed that the number of idempotents of the trivial elements of $\mathcal{A}$ are also 0.

Consider the following monoids:

- the monoid $P(\mathcal{A})$ under union;
- the direct product $E_1' = \mathsf{H} \times P(\mathcal{A})$, where H is the additive monoid of p-forests;

- the skew monoid T^φ given by Corollary 4.2.22, where S is the semigroup $P(\mathcal{A}) \setminus \{\emptyset\}$ under union and $I = \emptyset$; we denote the skew multiplication by $\oplus$, which is given by

$$(U_1, U_2, U_3) \oplus (V_1, V_2, V_3) = \begin{cases} (U_1 \cup V_1, V_2, U_3 \cup V_3) & \text{if } U_2 = \emptyset \\ (U_1, U_2 \cup V_1 \cup V_2 \cup V_3, U_3) & \text{otherwise;} \end{cases}$$

- the product monoid $E_2' = \mathsf{V} \times T^\varphi$ of the multiplicative monoid V of p-contexts with the skew monoid T^φ; to simplify the notation, we may sometimes write (u, U_1, U_2, U_3) instead of $(u, (U_1, U_2, U_3))$.

The operation of E_2', denoted $\odot$, is then given by the following formula:

$$\begin{aligned} &(u, U_1, U_2, U_3) \odot (v, V_1, V_2, V_3) \\ &= \begin{cases} (uv, U_1 \cup V_1, V_2, U_3 \cup V_3) & \text{if } U_2 = \emptyset \\ (uv, U_1, U_2 \cup V_1 \cup V_2 \cup V_3, U_3) & \text{otherwise.} \end{cases} \end{aligned}$$

Consider the subset E_2'' of E_2' consisting of the elements (u, U_1, U_2, U_3) such that $C(u) = \Box$ if and only if $U_2 = \emptyset$. Note that it is a submonoid of E_2' for which the operation is given by the following formula:

$$\begin{aligned} &(u, U_1, U_2, U_3) \odot (v, V_1, V_2, V_3) \\ &= \begin{cases} (uv, U_1 \cup V_1, V_2, U_3 \cup V_3) & \text{if } C(u) = \Box \\ (uv, U_1, U_2 \cup V_1 \cup V_2 \cup V_3, U_3) & \text{otherwise.} \end{cases} \end{aligned}$$

Next, to obtain an ω-algebra structure (E_1', E_2'), define the mixed operations as follows:

$$\begin{aligned} (h, U) + (v, V_1, V_2, V_3) &= (h + v, U \cup V_1, V_2, V_3) \\ (v, V_1, V_2, V_3) + (h, U) &= (v + h, V_1, V_2, V_3 \cup U) \\ (v, V_1, V_2, V_3) * (h, U) &= (v * h, V_1 \cup V_2 \cup V_3 \cup U). \end{aligned}$$

Note that, if the p-contexts are restricted to E_2'' then the two mixed sums do take their values in E_2''.

Finally, define the ω-operations as follows:

$$\omega(h, U) = \begin{cases} (0, \emptyset) & \text{if } h = 0 \\ (\omega(h), U \cup \{\omega(h)\}) & \text{otherwise;} \end{cases}$$

$$(u, U_1, U_2, U_3)^\omega = \begin{cases} (u^\omega, \emptyset, U_1 \cup U_2 \cup U_3 \cup \{u^\omega\}, \emptyset) & \text{if } C(u) \neq \Box \\ \omega(h, U_1) + (\Box, \emptyset, \emptyset, \emptyset) + \omega(k, U_3) & \text{if } u = h + \Box + k. \end{cases}$$

Note that the ω-power of an element of E_2'' remains in E_2''.

Proposition 4.2.26. *For the above operations, (E'_1, E'_2) is an ω-algebra and (E'_1, E''_2) is an ω-subalgebra.*

Proof. For the axioms not involving the ω-operations, it suffices to observe that the reduced structure is just the direct product of $\mathcal{A}$ with the ω-algebra (S^I, T) given by Proposition 4.2.23. Thus, it only remains to check the axioms involving the ω-operations, which amounts to an easy verification. $\square$

Example 4.2.27. For an alphabet A, let $\mathcal{A} = (\mathsf{H}, \mathsf{V})$ be the A'-free ω-algebra in $\mathfrak{B}$. Let $\mathcal{IS} = (E'_1, E''_2)$ be the ω-algebra as in Proposition 4.2.26.

By the universal property of the free ω-algebra $\mathcal{A}$, there is a unique ω-algebra homomorphism

$$\mathrm{Ist} : \mathcal{A} \to \mathcal{IS},$$

such that

$$\mathrm{Ist}(a\Box) = (a\Box, \emptyset, \emptyset, \emptyset).$$

In addition, the mapping

$$f : \mathcal{IS} \to \mathcal{A}$$

which is the first component projection, is an ω-algebra homomorphism and the composite $f \circ \mathrm{Ist}$ is the identity on free ω-algebra.

Definition 4.2.28. For an element $x \in \mathcal{A}$, the *set of idempotent subterms* of x, we denote by $\mathrm{IST}(x)$, is second component of $\mathrm{Ist}(x)$ if $x \in \mathsf{H}$ and second component of $\mathrm{Ist}(x * 0)$ otherwise.

Example 4.2.29. For an alphabet A, let $\mathcal{A} = (\mathsf{H}, \mathsf{V})$ be the A'-free ω-algebra in $\mathfrak{B}$. Let $\mathrm{NERVE} = (H, V)$ where H is the trivial monoid and $V = A^*$ is the free monoid on A.

We consider the only action of the monoid V on the left of the monoid H which we denote by $*$ as follows: let v be an element of V, define $v * 0 = 0$. And let v be an element of the monoid V. We define operation $+'$ as follows:

$$v +' 0 = v \quad , \quad 0 +' v = v$$

which are clearly elements of the monoid V. And define the unary operations as identity. It is immediate to verify that NERVE is an ω-algebra.

By the universal property of the free ω-algebra $\mathcal{A}$, there is a unique ω-algebra homomorphism

$$\mathrm{nerve} : \mathcal{A} \to \mathrm{NERVE}$$

such that

$$\mathrm{nerve}(a\Box) = a\Box.$$

Definition 4.2.30. Let x be an element of the free ω-algebra $\mathcal{A}$, then the *nerve* of x is its image by the ω-algebra homomorphism, the unique ω-algebra homomorphism nerve in Example 4.2.29.

In view of Remark 4.1.2, for an element y in the free forest algebra A^{Δ}, we define nerve of y by the image of the restriction of the ω-algebra homomorphism nerve, to the free forest algebra A^{Δ}, that is $\text{nerve}|_{A^{\Delta}}$.

4.2.1.1 Rank in Free ω-Algebra $\mathcal{A}$

For an alphabet A, let $\mathcal{A} = (\mathsf{H}, \mathsf{V})$ be the A'-free ω-algebra in $\mathfrak{B}$. Let $I = -\infty$, and let M be the monoid of $\mathbb{N}^I$ under operation max, assuming that $-\infty < 0$. Let $M' = \{n' \mid n \in M\}$ be a copy of M. There is a monoid isomorphism $\varphi : M \to M'$ via $n \mapsto n'$. In view of Proposition 1.2.3, let T be the trivial monoid and $S = M$. As $T * S$ and M are isomorphic, Proposition 1.2.3 implies that $\mathcal{R} = (M, M')$ is a forest algebra. Hence, the axioms $(1) - (10)$ of ω-algebras hold in $\mathcal{R}$. Endow $\mathcal{R}$ with additional operations $\omega(\_) : M \to M$ and $\_^{\omega} : M' \to M'$ defined as follows:

$$\omega(n) = n + 1 \qquad ,\text{for } \; n \geq 0$$

$$(m')^{\omega} = (m+1)' \quad ,\text{for } \; m \geq 0$$

$$\omega(-\infty) = -\infty$$

$$(-\infty')^{\omega} = -\infty'$$

It is immediate that the axioms (11) and (12) of ω-algebras hold in $\mathcal{R}$. In order to show that $\mathcal{R}$ is an ω-algebra we just need to show that it satisfies the axiom (13) of ω-algebras: for every $m, n \in N$ we have

$$(m + -\infty' + n)^{\omega} = \omega(m) + -\infty' + \omega(n),$$

since both are equal to $(\max\{m, n, -\infty\})'^{\omega}$.

By the universal property of the free ω-algebra $\mathcal{A}$, there is a unique ω-algebra homomorphism

$$\text{Rank} : \mathcal{A} \to \mathcal{R}$$

such that

$$\text{Rank}(a\Box) = 0'.$$

We say that an element $x \in \mathcal{A}$ has *rank* n, if x is an element of H and $\text{Rank}(x) = n$, or x is an element of V and $\text{Rank}(x) = n'$.

Corollary 4.2.31. *All non-trivial finite p-forests and non-trivial finite p-contexts have rank* 0. *By Remark 4.1.2, we have* $\text{Rank}^{-1}(0) = H^A \setminus \{0\}$ *and* $\text{Rank}^{-1}(0') = V^A \setminus \{\Box\}$.

4.2.2 Some Properties of the Free ω-Algebra $\mathcal{A}$

Recall that, in Example 4.2.9, $\mathcal{S} = (A \uplus \{a, b, c\})^{\Delta}$ is an ω-algebra and Φ is the ω-algebra homomorphism from $\mathcal{A}$ to $\mathcal{S}$. These notation will apply for the reminder of current subsection.

Lemma 4.2.32. *Let x and y be elements of the free ω-algebra $\mathcal{A}$, if x is a subterm of y in $\mathcal{A}$ then $\Phi(x)$ is a subterm of $\Phi(y)$ in the free forest algebra $(A \uplus \{a, b, c\})^{\Delta}$.*

Proof. There is an n-ary operation f, and there are elements $t_1, \ldots, t_{n-1}$ such that $f(x, t_1, \ldots, t_{n-1}) = y$. So, we have

$$\Phi(f(x, t_1, \ldots, t_{n-1})) = \Phi(y).$$

Since Φ is an ω-algebra homomorphism and f is an n-ary operation we have

$$\Phi(y) = f(\Phi(x), \Phi(t_1), \ldots, \Phi(t_{n-1})),$$

which implies the result. $\square$

Lemma 4.2.33. *Let x be an element of the free ω-algebra $\mathcal{A}$. Then we have* $\mathrm{Rank}(x) \leq 0$ *if and only if* $\mathrm{labels}(\Phi(x)) \subseteq A$.

Proof. If $\mathrm{Rank}(x) \leq 0$, then $\Phi(x) = x$ and therefore,

$$\mathrm{labels}(\Phi(x)) = \mathrm{labels}(x) \subseteq A.$$

If $\mathrm{Rank}(x) > 0$, then for some p-forest h there is a subterm $\omega(h)$ of x or for some p-context v there is a subterm v^{ω} of x. By Lemma 4.2.32, respectively, $\Phi(\omega(h))$ or $\Phi(v^{\omega})$ is a subterm of $\Phi(x)$. So, respectively, $a\square * \Phi(h)$ or $b\square.\Phi(v).c\square$ is a subterm of $\Phi(x)$, which implies that, respectively, $a\square$ or $b\square$ belongs to $\mathrm{labels}(\Phi(x))$. Therefore, $\mathrm{labels}(\Phi(x))$ is not a subset of A. $\square$

Lemma 4.2.34. *Let x be an element of the free ω-algebra $\mathcal{A}$. Then $\Phi(x)$ is trivial if and only if x is trivial.*

Proof. If x is trivial, then $\Phi(x)$ is also trivial.

Assume that, $\Phi(x)$ is trivial, then since $\mathrm{labels}(\Phi(x)) \subseteq A$ by Lemma 4.2.33, we have $\mathrm{Rank}(x) \leq 0$. So, we have $\Phi(x) = x$ which yields x is trivial. $\square$

Lemma 4.2.35. *For elements x_1 and x in the free ω-algebra $\mathcal{A}$, if x_1 is a subterm of x then* $\mathrm{labels}(x_1) \subseteq \mathrm{labels}(x)$.

Proof. By definition of a subterm, there is an n-ary operation f, and there are elements $t_1, \ldots, t_{n-1}$ such that $f(x_1, t_1, \ldots, t_{n-1}) = x$. So, we have the following equality:

$$\text{labels}(f(x_1, t_1, \ldots, t_{n-1})) = \text{labels}(x).$$

Since labels is an ω-algebra homomorphism, the unique ω-algebra homomorphism labels in Remark 4.2.14, then the equality

$$\text{labels}(x) = \text{labels}(x_1) \cup \text{labels}(t_1) \cup \ldots \cup \text{labels}(t_{n-1})$$

holds which implies the result. $\square$

Lemma 4.2.36. *For an alphabet A, let $\mathcal{A} = (\mathsf{H}, \mathsf{V})$ be the A'-free ω-algebra in $\mathfrak{B}$. Let $a\square \in A'$ and $h \in \mathsf{H}$, then there do not exist non-trivial p-forests h_1 and h_2 in H such that $a\square * h = h_1 + h_2$.*

In addition, for $v \in \mathsf{V}$, there do not exist a non-trivial p-forest h_1 and a p-context v_1 in $\mathcal{A}$ such that $a\square.v = h_1 + v_1$ or $a\square.v = v_1 + h_1$.

Proof. Assume that, there exist p-forests h_1 and h_2 such that $a\square * h = h_1 + h_2$. By applying the ω-algebra homomorphism f_1 from Example 4.2.10, we have

$$f_1(a\square * h) = f_1(h_1 + h_2),$$

which implies $h_1^{1,1} = 0$ or $h_2^{1,1} = 0$. By Lemma 4.2.11, we have $h_1 = 0$ or $h_2 = 0$ that is h_1 or h_2 is the trivial p-forest.

In a similar way we can show that $a\square.v$ is additively irreducible. $\square$

We showed that:

Corollary 4.2.37. *Let $\mathcal{A} = (\mathsf{H}, \mathsf{V})$ be the A'-free ω-algebra in $\mathfrak{B}$, we have*

$$a\square * \mathsf{H} \cap +(\mathsf{H} \setminus \{0\}, \mathsf{H} \setminus \{0\}) = \emptyset,$$

and also

$$a\square.\mathsf{V} \cap (+(\mathsf{H} \setminus \{0\}, \mathsf{V}) \cup +(\mathsf{V}, \mathsf{H} \setminus \{0\})) = \emptyset.$$

Lemma 4.2.38. *Let $\mathcal{A} = (\mathsf{H}, \mathsf{V})$ be the A'-free ω-algebra in $\mathfrak{B}$. We have*

$$\omega(\mathsf{H} \setminus \{0\}) \cap +(\mathsf{H} \setminus \{0\}, \mathsf{H} \setminus \{0\}) = \emptyset,$$

and also we have

$$(\mathsf{V} \setminus (\mathsf{H} + \square + \mathsf{H}))^\omega \cap .(\mathsf{V} \setminus \{\square\}, \mathsf{V} \setminus \{\square\}) = \emptyset.$$

Proof. Assume that, h is a non-trivial p-forest and there exist p-forests h_1 and h_2 such that $\omega(h) = h_1 + h_2$. By the ω-algebra homomorphism Φ, since by assumption the p-forest h is non-trivial, the following equalities hold in $\mathcal{S}$:

$$a\square * h = \Phi(\omega(h)) = \Phi(h_1 + h_2) = \Phi(h_1) + \Phi(h_2).$$

Since $a\square * \Phi(h)$ is a connected forest, Lemma 1.3.6 yields $\Phi(h_1) = 0$ or $\Phi(h_2) = 0$. By Lemma 4.2.34, it follows that $h_1 = 0$ or $h_2 = 0$, which establishes the desired disjointness relation:

$$\omega(\mathsf{H} \setminus \{0\}) \cap +(\mathsf{H} \setminus \{0\}, \mathsf{H} \setminus \{0\}) = \emptyset.$$

Now, assume that, for a p-context v with $v \notin (\mathsf{H} + \square + \mathsf{H})$ there exist p-contexts v_1 and v_2 such that $v^\omega = v_1.v_2$. By the ω-algebra homomorphism Φ, we have

$$\Phi(v^\omega) = \Phi(v_1).\Phi(v_2).$$

Since v_1 is a p-context, there are p-forests H_1 and H_2, and additively irreducible p-context u such that $v_1 = H_1 + u + H_2$. So, we have the equality

$$\Phi(v^\omega) = \Phi(H_1) + \Phi(u).\Phi(v_2) + \Phi(H_2)$$

in the free forest algebra. Since $\Phi(v^\omega)$ is connected, we have the equalities $\Phi(H_1) = \Phi(H_2) = 0$. Hence, Lemma 4.2.34 implies the equalities $H_1 = H_2 = 0$. If we assume that $u = \square$, then we get the result.

Suppose that $u \neq \square$, then $u \notin \mathsf{H} + \square + \mathsf{H}$. Now, by the ω-algebra homomorphism f_3' in Example 4.2.18, we have

$$f_3'(v^\omega) = f_3'(u).f_3'(v_2).$$

So, there are some $a_1, \ldots, a_n, b_1, \ldots, b_m, c \in A$ such that

$$f_3'(v^\omega) = a_1 + \cdots + a_n + c\square + b_1 + \cdots + b_m.$$

Since u is a non-trivial additively irreducible p-context and $u \notin \mathsf{H} + \square + \mathsf{H}$, we have $C(f_3'(u)) \neq \square$. So, there are forests S_1 and S_2, and a non-trivial context w such that $f_3'(u) = S_1 + w + S_2$. Hence, we obtain the following equality in the free forest algebra

$$a_1 + \cdots + a_n + c\square + b_1 + \cdots + b_m = S_1 + w.f_3'(v_2) + S_2,$$

which implies the equalities

$$a_1 + \cdots + a_n = S_1,$$

$$b_1 + \cdots + b_m = S_2$$

and

$$c\square = w.f_3'(v_2).$$

Hence, we obtain the equality $f_3'(v_2) = \square$, and Lemma 4.2.19 yields the equality $v_2 = \square$. $\square$

Lemma 4.2.39. *Let h be a p-forest in $\mathcal{A}$. Every factor of $h + \square$ is of the form $s + \square$ for some p-forest s. Dually, every factor of $\square + h$ is of the form $\square + s$ for some p-forest s.*

Proof. If $h + \square$ is multiplicatively irreducible then the only factors of $h + \square$ are $\square$ and $h + \square$ itself.

Assume that, $h + \square$ is not multiplicatively irreducible. So, there are non-trivial p-contexts $v_1 = H_1 + C(v_1) + H_2$ and $v_2 = H_1' + C(v_2) + H_2'$ such that $h + \square = v_1.v_2$ which implies that

$$\square = C(h + \square) = C(H_1 + C(C(v_1).v_2) + H_2) = C(C(v_1).v_2).$$

We have

$$C(C(v_1).v_2) = \begin{cases} C(v_1).v_2 & , C(v_1) \neq \square \\ C(v_2) & , C(v_1) = \square. \end{cases}$$

Since the equality $\square = C(C(v_1).v_2)$ holds, we have the equality

$$0 = \#_{\text{Nodes}}(\square) = \#_{\text{Nodes}}(C(C(v_1).v_2)).$$

If the inequality $C(v_1) \neq \square$ holds then, by Lemma 4.1.8, we have the inequality $\#_{\text{Nodes}}(C(v_1)) \neq 0$, which implies

$$\#_{\text{Nodes}}(C(C(v_1).v_2)) = \#_{\text{Nodes}}(C(v_1).v_2) = \#_{\text{Nodes}}(C(v_1)) + \#_{\text{Nodes}}(v_2) \neq 0$$

yielding a contradiction. So, we have $C(v_1) = \square$, which implies

$$0 = \#_{\text{Nodes}}(\square) = \#_{\text{Nodes}}(C(C(v_1).v_2)) = \#_{\text{Nodes}}(C(v_2))$$

whence $C(v_2) = \square$. So, we have $v_1 = H_1 + \square + H_2$ and $v_2 = H_1' + \square + H_2'$, together with the equality $h + \square = v_1.v_2$ yielding the equality

$$h + \square = H_1 + H_1' + \square + H_2' + H_2.$$

Applying the ω-algebra homomorphism f_1 of Example 4.2.10 on both sides we obtain the following equality:

$$(h + \square)^{1,1} = (H_1 + H_1' + \square + H_2' + H_2)^{1,1}$$

which is

$$h^{1,1} + \square = (H_1 + H_1')^{1,1} + \square + (H_2' + H_2)^{1,1}$$

that is, by Remark 4.1.2, an equality in the free forest algebra A^{Δ}. Hence, the equality $(H_2' + H_2)^{1,1} = 0$ holds in A^{Δ}. By Lemma 4.2.11 it follows that the equality $H_2' + H_2 = 0$. Since the equalities

$$0 = \#_{\text{Nodes}}(0) = \#_{\text{Nodes}}(H_2' + H_2) = \#_{\text{Nodes}}(H_2') + \#_{\text{Nodes}}(H_2)$$

hold by Lemma 4.1.8, we deduce that $\#_{\text{Nodes}}(H_2) = \#_{\text{Nodes}}(H_2') = 0$ and, therefore, again by Lemma 4.1.8, we have the equality $H_2 = H_2' = 0$. So, we have $v_1 = H_1 + \square$ and $v_2 = H_1' + \square$.

Dually considerations yield the dual case. $\square$

Corollary 4.2.40. *Let s be a non-trivial p-forest in $\mathcal{A}$. The p-forest s is additively irreducible if and only if the p-context $s + \Box$ is multiplicatively irreducible. Dually, the p-forest s is additively irreducible if and only if the p-context $\Box + s$ is multiplicatively irreducible.*

Proof. ($\Rightarrow$) Assume that, the p-context $s + \Box$ is multiplicatively irreducible and s is not additively irreducible. Then there exist non-trivial p-forests s_1 and s_2 such that $s = s_1 + s_2$. Since we have the equality $s_1 + s_2 + \Box = (s_1 + \Box).(s_2 + \Box)$ and we have $\#_{\text{Nodes}}(s_i + \Box) = \#_{\text{Nodes}}(s_i) \geq 1$ for $i = 1, 2$, both p-contexts $s_1 + \Box$ and $s_2 + \Box$ are non-trivial. It follows that $s + \Box$ is not multiplicatively irreducible which is a contradiction.

($\Leftarrow$) Assume that, the p-forest s is additively irreducible. We show that the p-context $s + \Box$ is multiplicatively irreducible. Indeed, otherwise, there are non-trivial p-contexts v_1 and v_2 such that $s + \Box = v_1.v_2$. By Lemma 4.2.39, we have $v_1 = H_1 + \Box$ and $v_2 = H_1' + \Box$. And since v_1 and v_2 are non-trivial p-contexts, H_1 and H_1' must be non-trivial p-forests. Now, the equality $s+\Box = H_1+H_2+\Box$ implies that the equality $s = H_1+H_1'$ holds, by action of both sides on the forest 0. Hence, s is not an additively irreducible p-forest which yields a contradiction. $\Box$

By Lemma 4.2.38 and Corollary 4.2.40, the following is immediate.

Corollary 4.2.41. *Let w be a p-context in $\mathcal{A}$ with $C(w) \neq \Box$. Then w^ω is additively irreducible.* $\Box$

Lemma 4.2.42. *Let w and v be p-contexts in $\mathcal{A}$ with $C(w) \neq \Box$. Then $w^\omega.v$ is additively irreducible.*

Proof. The case $v = \Box$ is given by Corollary 4.2.41.

Suppose then that $v \neq \Box$. We show that $w^\omega.v$ is additively irreducible. If not, then for some non-trivial p-forest h and some p-context u it is of the form $h+u$ (or $u+h$). Now, by applying Φ to both sides of the equality $w^\omega.v = h+u$ we have the following equality in the free forest algebra $(A \uplus \{a\Box, b\Box, c\Box\})^\Delta$:

$$b\Box.\Phi(w).c\Box.v = \Phi(h) + \Phi(u),$$

in which the left side is a connected context while the right side is not since the p-forest h is assumed to be non-trivial which is a contradiction. $\Box$

Lemma 4.2.43. *The only multiplicatively irreducible p-contexts of $\mathcal{A}$ are v^ω, $s + \Box$, $\Box + h$, and $a\Box$, where v is a p-context with $C(v) \neq \Box$, s and h are additively irreducible p-forests, and $a\Box \in A'$.*

Proof. Assume that, u is a non-trivial p-context which is multiplicatively irreducible. Since u is a p-context it is of the form $H_1 + C(u) + H_2$ for some p-forests H_1 and H_2 in the free ω-algebra $\mathcal{A}$. By properties of the free ω-algebra $\mathcal{A}$ we have $u = (H_1 + \Box).(\Box + H_2).C(u)$. As u is multiplicatively irreducible we have $u = H_1 + \Box$, $u = \Box + H_2$ or $u = C(u)$ and the other factors of u are trivial.

If $u = H_1 + \Box$ then, by Corollary 4.2.40, since u is multiplicatively irreducible, H_1 is additively irreducible. Similarly, way if $u = \Box + H_2$ then H_2 is additively irreducible.

Now, assume that, $u = C(u)$. Then u has one of the forms $u = z^\omega.w$, $u = a\Box.x$ or $u = \Box$ where z, w and x are p-contexts and $C(z) \neq \Box$. Since u is a non-trivial p-context, we have $u \neq \Box$. If $u = z^\omega.w$ with $C(z) \neq \Box$, then $\#_{\text{Nodes}}(z^\omega) \neq 0$. So, we have $z^\omega \neq \Box$ whence u is multiplicatively irreducible if and only if $w = \Box$, that is, $u = z^\omega$. And if $u = a\Box.x$, since $a\Box \neq \Box$, then u is multiplicatively irreducible if and only if $x = \Box$, that is, $u = a\Box$. $\Box$

Definition 4.2.44. We distinguish the following *kinds* of non-trivial additively irreducible p-forests:

- *kind* 1: $\omega(h)$, for some non-trivial p-forest h;
- *kind* 2: $d\Box * h$, for some $d\Box \in A'$ and p-forest h;
- *kind* 3: $u^\omega * h$, for some p-forest h and some p-context u with $C(u) \neq \Box$.

Lemma 4.2.45. *If x is a non-trivial additively irreducible p-forest in $\mathcal{A}$, then $\Phi(x)$ is connected.*

Proof. As x is a non-trivial additively irreducible p-forest we may have one of the following conditions:

- If x has kind 1, then $\Phi(x) = a\Box * \Phi(h)$ which is connected.
- If x has kind 2, then $\Phi(x) = d\Box * \Phi(h)$ which is connected.
- And if x has kind 3, then $\Phi(x) = (b\Box.\Phi(u).c\Box) * \Phi(h)$ which is also connected. $\Box$

Lemma 4.2.46. *Let x be a non-trivial additively irreducible p-forest and y be a p-forest in $\mathcal{A}$. If $\Phi(x) = \Phi(y)$, then y is also a non-trivial additively irreducible p-forest and x and y have the same kind.*

Proof. First, we observe that y is non-trivial. Indeed, if y is trivial then so is x by Lemma 4.2.34.

Next, we show that y is additively irreducible. If not, then $y = y_1 + \cdots + y_n$ where for every i, y_i is a non-trivial p-forest in $\mathcal{A}$. Now, Lemma 4.2.45 together with the equality $\Phi(x) = \Phi(y)$, which is an equality in the free forest algebra, imply that $\Phi(y_1) + \cdots + \Phi(y_n)$ is connected which, in view of Lemma 4.2.34, is a contradiction.

Finally, we show that x and y have the same kind. If x has kind 1, then $\Phi(x) = a\Box * \Phi(h)$. Since y is non-trivial additively irreducible if y has kind 2 or 3, then we have $\Phi(y) = d\Box * \Phi(h')$ or $\Phi(y) = (b\Box.\Phi(u).c\Box) * \Phi(h')$ where

d and b are different from a. So, in both cases we have $\Phi(x) \neq \Phi(y)$ which is a contradiction. Thus, y has kind 1. The cases where x has kinds 2 or 3 are handled similarly. $\square$

Lemma 4.2.47. *Let x be a non-trivial p-context in $\mathcal{A}$. If we have $x = C(x)$, then $\Phi(x)$ is connected.*

Proof. If x is a non-trivial p-context in $\mathcal{A}$ and $x = C(x)$, then x has one of the forms: $d\square.v$ for some $d\square \in A$ and p-context v, or $u^\omega.v$ for some p-contexts u and v with $C(u) \neq \square$. Applying the ω-algebra homomorphism Φ, we obtain, respectively, $\Phi(x) = d\square.\Phi(v)$ or $\Phi(x) = (b\square.\Phi(u).c\square).\Phi(v)$. In either case $\Phi(x)$ is a connected context. $\square$

Lemma 4.2.48. *For every context v in the free forest algebra $\mathcal{S}$ we have $a\square.v \notin \Phi(\mathcal{A})$.*

Proof. We have $\text{roots}(a\square.v) = (\{a\}, 1)$. Since Φ is an ω-algebra homomorphism, the image of a p-context is a context and the image of a p-forest is a forest.

Assume that, there is a p-context y such that the equality $\Phi(y) = a\square.v$ holds. Since y is a p-context, for some p-forests H_1 and H_2 we have $y = H_1 + C(y) + H_2$. If at least one of H_1 or H_2 is a non-trivial p-forest, then, by Lemma 4.2.34, we have $\#\text{roots}(\Phi(y)) \geq 2$ which is a contradiction. So, we have the equalities $H_1 = H_2 = 0$ which means $y = C(y)$, that is, y is a $\square$-pure. Since $a\square.v$ is non-trivial, by Lemma 4.2.34 we deduce that y is non-trivial. And since y is a $\square$-pure, there are $d\square \in A'$ and $x \in \mathsf{V}$, or $z, x \in \mathsf{V}$ with $C(z) \neq \square$, such that $y = d\square.x$ or $y = z^\omega.x$. In both cases $\Phi(y) \neq a\square.v$. $\square$

Let x be an element of $\Phi(\mathcal{A})$, *depth-first pre-order traversal* is one way to go through the nodes of a tree. The Figure 4.1 is an example of a tree which is traversed with depth-first pre-order traversal algorithm; for more details see [20].

Remark 4.2.49. Let x be an element of $\Phi(\mathcal{A})$. By depth-first pre-order traversal of a tree, we can define a unique mapping from the set of natural numbers, actually the set

$$\{n \mid n \in \mathbb{N} \setminus \{0\}, n \leq \#_{\text{Nodes}}(x)\},$$

to the nodes of x. This is one of the ways that we can determine the position of a node in an element of $\Phi(\mathcal{A})$.

We denote by $\text{Pos}(x, i)$ the value of the above mapping for the positive integer i which gives the label of the node in the position i.

Remark 4.2.50. Let x be an element of $\Phi(\mathcal{A})$. By definition of the ω-algebra homomorphism Φ, the number of nodes with label b and the number of nodes with label c in x are equal.

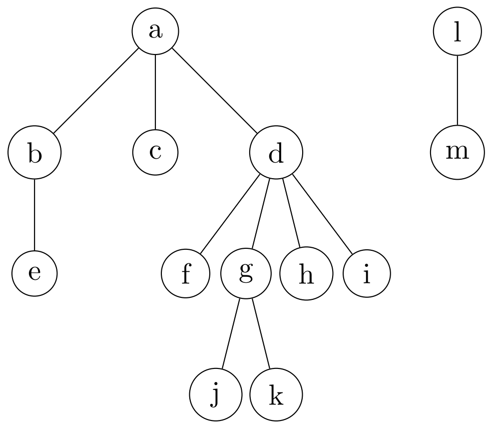

Figure 4.1: depth-first pre-order traversal: a be c d f gjk h i lm

Let x be an element of $\mathcal{S}$. By $x|_{b,c}$ we mean substitute every subterm $d\Box$ of x where $d\Box \in A' \cup \{a\Box\}$, by $\Box$.

We denote by $D(b,c)$ the *Dyck language* over $\{b,c\}$, see [21, 24, 30] for more details on Dyck Languages. The following conditions are some properties of the Dyck language $D(b,c)$:

- if $x \in D(b,c)$, then there is a non-negative integer n such that $|x| = 2n$;
- if $x, y \in D(b,c)$ and $x = z_1 y z_2$, then $z_1 z_2 \in D(b,c)$;
- if $x, y \in D(b,c)$ and $x = z_1 z_2$, then $z_1 y z_2 \in D(b,c)$;
- if $x \in D(b,c)$ with $x = x_1 \ldots x_{2n} \neq \varepsilon$ then, for every $i \geq 1$ with $x_i = b$, there is a unique integer $j \geq i$ such that $x_j = c$ and $x_{i+1} \ldots x_{j-1} \in D(b,c)$.

Lemma 4.2.51. *For every p-context u in $\mathcal{A}$, the following statement holds:*

$$\mathrm{nerve}(\Phi(u))|_{b,c} \in D(b,c).$$

Proof. We argue by induction on the number of nodes of $\mathrm{nerve}(\Phi(u))$.

If $\#_{\mathrm{Nodes}}(\mathrm{nerve}(\Phi(u))) = 0$ then, by Lemma 4.1.8 and since $\Phi(u)$ is a context, we have $\mathrm{nerve}(\Phi(u)) = \Box$ and, therefore, we have $\mathrm{nerve}(\Phi(u))|_{b,c} = \varepsilon \in D(b,c)$.

Assume that, for every p-context u with $\#_{\mathrm{Nodes}}(\mathrm{nerve}(\Phi(u))) \leq n$, we have $\mathrm{nerve}(\Phi(u))|_{b,c} \in D(b,c)$.

Let u be a p-context with $\#_{\mathrm{Nodes}}(\mathrm{nerve}(\Phi(u))) = n+1$. In view of the definition of the ω-algebra homomorphism nerve, we have $\mathrm{nerve}(\Phi(u)) = \mathrm{nerve}(\Phi(C(u))$. For $C(u)$ we have: there exists a p-context u_2 and $d\Box \in A'$ such that $C(u) = d\Box.u_2$ or there are p-contexts u_1 and u_2 with $C(u_1) \neq \Box$

such that $C(u) = u_1^\omega.u_2$. If $C(u) = d\Box.u_2$, then we have $\mathrm{nerve}(\Phi(u)) = d\Box.\mathrm{nerve}(\Phi(u_2))$. Since we have

$$\begin{aligned}
\#_{\mathrm{Nodes}}(\mathrm{nerve}(\Phi(u))) &= \#_{\mathrm{Nodes}}(d\Box.\mathrm{nerve}(\Phi(u_2))) \\
&= 1 + \#_{\mathrm{Nodes}}(\mathrm{nerve}(\Phi(u_2))) \\
&= n + 1,
\end{aligned}$$

we also have $\#_{\mathrm{Nodes}}(\mathrm{nerve}(\Phi(u_2))) = n$. By induction hypothesis, it follows that $\mathrm{nerve}(\Phi(u_2))|_{b,c} \in D(b,c)$. Combining with the equalities

$$\mathrm{nerve}(\Phi(u))|_{b,c} = d\Box.\mathrm{nerve}(\Phi(u_2))|_{b,c} = \mathrm{nerve}(\Phi(u_2))|_{b,c}$$

we deduce that $\mathrm{nerve}(\Phi(u))|_{b,c} \in D(b,c)$.

Now assume that $C(u) = u_1^\omega.u_2$. Then the following equality holds:

$$\mathrm{nerve}(\Phi(u)) = b\Box.\mathrm{nerve}(\Phi(u_1)).c\Box.\mathrm{nerve}(\Phi(u_2)).$$

Since we have

$$\begin{aligned}
&\#_{\mathrm{Nodes}}(\mathrm{nerve}(\Phi(u))) \\
&= \#_{\mathrm{Nodes}}(b\Box.\mathrm{nerve}(\Phi(u_1)).c\Box.\mathrm{nerve}(\Phi(u_2))) \\
&= 1 + \#_{\mathrm{Nodes}}(\mathrm{nerve}(\Phi(u_1))) + 1 + \#_{\mathrm{Nodes}}(\mathrm{nerve}(\Phi(u_2))) \\
&= n + 1,
\end{aligned}$$

it follows that $\#_{\mathrm{Nodes}}(\mathrm{nerve}(\Phi(u_i))) < n$ for $i = 1, 2$. By induction hypothesis, we have $\mathrm{nerve}(\Phi(u_i))|_{b,c} \in D(b,c)$ for $i = 1, 2$. In view of the equalities

$$\begin{aligned}
\mathrm{nerve}(\Phi(u))|_{b,c} &= (b\Box.\mathrm{nerve}(\Phi(u_1)).c\Box.\mathrm{nerve}(\Phi(u_2)))|_{b,c} \\
&= b\Box.(\mathrm{nerve}(\Phi(u_1))|_{b,c}).c\Box.(\mathrm{nerve}(\Phi(u_2))|_{b,c}),
\end{aligned}$$

we conclude that $\mathrm{nerve}(\Phi(u))|_{b,c} \in D(b,c)$. $\Box$

Remark 4.2.52. Over a finite alphabet A let Σ be the set $A \cup \{\Box\}$ and Σ^* be the free monoid generated by Σ. Let ε be the empty word in Σ^*. Let x be a word of Σ^*. We define $\#_{hole}(x)$ the *number of occurrences of the letter* $\Box$ in the word x. Define sets

$$\Sigma_H^* = \{x \in \Sigma^* \mid \#_{hole} = 0\} \quad \text{and} \quad \Sigma_V^* = \{x \in \Sigma^* \mid \#_{hole} = 1\}.$$

Then Σ_H^* is a monoid under concatenation of words and Σ_V^* becomes a monoid under insertion, denoted by $\triangleleft$, which is defined as follows: for words x and y in Σ_V^*, where $x = x_1 \cdots x_n$ assume that $x_i = \Box$, define

$$x \triangleleft y = x_1 \cdots x_{i-1} y x_{i+1} \cdots x_n.$$

It is immediate to see that $S = (\Sigma_H^*, \Sigma_V^*)$ endowed with the following additional operations satisfies the equational axioms of forest algebras: for elements h and v respectively in Σ_H^* and Σ_V^* define:

$$\begin{aligned} &h + v = hv; \\ &v + h = vh; \\ &v * h = v \lhd h, \end{aligned}$$

where the operations on the right are concatenation and insertion.

By the universal property of the free forest algebra $(A \uplus \{a, b, c\})^{\Delta}$, there is a unique forest algebra homomorphism

$$\text{traversal} : A^{\Delta} \to S$$

such that

$$\text{traversal}(a\Box) = a\Box.$$

Let x be an element of $\mathcal{A}$. Then we have

$$\Phi(x) \in \text{Im}(\Phi) \subseteq (A \uplus \{a, b, c\})^{\Delta}$$

and so traversal$(\Phi(x))$ is a word in Σ^* where $\Sigma = A \uplus \{a, b, c, \Box\}$.

Example 4.2.53. Consider the following element of $\mathcal{A}$:

$$x = d\Box.((f\Box + g))^{\omega}.$$

Then we have

$$\Phi(x) = d\Box.b\Box.((f\Box + g)).c\Box).$$

And so we have

$$\begin{aligned} \text{traversal}(\Phi(x)) &= \text{traversal}(d\Box) \lhd \text{traversal}(b\Box.(f\Box + g).c\Box) \\ &= d\,\text{traversal}(b\Box.(f\Box + g).c\Box) \\ &= d(\text{traversal}(b\Box) \lhd \text{traversal}((f\Box + g).c\Box)) \\ &= db(\text{traversal}(f\Box + g) \lhd \text{traversal}(c\Box)) \\ &= db((\text{traversal}(f\Box)\,\text{traversal}(g)) \lhd (c\Box)) \\ &= db((f\Box g) \lhd (c\Box)) \\ &= dbfc\Box g \end{aligned}$$

Lemma 4.2.54. *The following property holds for every element x of $\mathcal{A}$:*

$$\text{traversal}(\Phi(x))|_{b,c} \in D(b, c).$$

Proof. Note that for every p-context x, the p-forest $x * 0$ and the p-context x have the same number of nodes and the same rank.

We argue by simultaneous induction on the number of nodes and the rank of x.

Assume that, for every element x in $\mathcal{A}$ with

$$\#_{\mathrm{Nodes}}(x) \leq n \quad \text{and} \quad \mathrm{Rank}(x) \leq k$$

and at least one of the inequalities strict, we have $\mathrm{traversal}(\Phi(x))|_{b,c} \in D(b,c)$.

Let x be a p-forest with $\#_{\mathrm{Nodes}}(x) = n$ and $\mathrm{Rank}(x) = k$. We show that $\mathrm{traversal}(\Phi(x))|_{b,c} \in D(b,c)$. Without loss of generality we may assume that x is an additively irreducible element. Indeed, otherwise $x = x_1 + x_2$ for some non-trivial elements x_1 and x_2 so that, by Lemma 4.1.7, we have for $i = 1, 2$, $\#_{\mathrm{Nodes}}(x_i) < \#_{\mathrm{Nodes}}(x)$ while $\mathrm{Rank}(x_i) \leq \mathrm{Rank}(x)$. By the induction hypothesis we have

$$\mathrm{traversal}(\Phi(x_i))|_{b,c} \in D(b,c).$$

Since the following equality holds:

$$\mathrm{traversal}(\Phi(x))|_{b,c} = \mathrm{traversal}(\Phi(x_1))|_{b,c}\, \mathrm{traversal}(\Phi(x_2))|_{b,c},$$

we have

$$\mathrm{traversal}(\Phi(x))|_{b,c} \in D(b,c).$$

Hence, x is additively irreducible and one of the following must hold:

- Assume that, for some non-trivial p-forest h, we have $x = \omega(h)$. Then we have $\mathrm{traversal}(\Phi(x))|_{b,c} = \mathrm{traversal}(\Phi(h))|_{b,c}$ where $\#_{\mathrm{Nodes}}(h) = n$ and $\mathrm{Rank}(h) = k - 1$. And by induction hypothesis we get the result.

- Assume that, for some p-forest h and $d \in A$, we have $x = d\Box * h$. Then we have

  $$\mathrm{traversal}(\Phi(x))|_{b,c} = \mathrm{traversal}(\Phi(h))|_{b,c}$$

  where $\#_{\mathrm{Nodes}}(h) = n - 1$. Since $\mathrm{Rank}(h) = k$, by induction hypothesis we have

  $$\mathrm{traversal}(\Phi(h))|_{b,c} \in D(b,c).$$

- Finally, assume that, for some p-forest h and some p-context v with $C(v) \neq \Box$, we have $x = v^{\omega} * h$. It follows that

  $$\mathrm{traversal}(\Phi(x))|_{b,c} = b\, \mathrm{traversal}(\Phi(v))|_{b,c} c\, \mathrm{traversal}(\Phi(h))|_{b,c}.$$

  By induction hypothesis, since

  $$\#_{\mathrm{Nodes}}(h) < n \quad \text{and} \quad \mathrm{Rank}(h) < k$$

we have

$$\text{traversal}(\Phi(h))|_{b,c} \in D(b,c).$$

And since $\text{Rank}(v) < k$, while $\#_{\text{Nodes}}(h) \leq n$, by the induction hypothesis we have

$$\text{traversal}(\Phi(v))|_{b,c} \in D(b,c).$$

Since, for a p-context x, the equality

$$\text{traversal}(\Phi(x))|_{b,c} = \text{traversal}(\Phi(x * 0))|_{b,c}$$

holds, the induction step and proof are complete. □

Let $y = y_1 \cdots y_n \in D(b,c)$ and let i and j be positive integers such that $1 \leq i < j \leq n$ with $y_i = b$ and $y_j = c$. We say that i *is related with* j if $y_{i+1} \cdots y_{j-1} \in D(b,c)$.

Let x be an element of $\mathcal{A}$. And let

$$y = \text{traversal}(\Phi(x))|_{b,c} \quad \text{and} \quad t = \text{traversal}(\Phi(x)).$$

Assume that length of the words y and t are respectively n and m. In view of Remark 4.2.49, we say that i is related with j if the following conditions hold:

- $t_i = b$;
- $t_j = c$;
- $t_1 \cdots t_i|_{b,c} = y_1 \cdots y_{i_1}$;
- $t_j \cdots t_m|_{b,c} = y_{j_1} \cdots y_n$;
- i_1 is related with j_1.

Remark 4.2.55. Let $x \in \mathcal{A}$ and $y = \text{traversal}(\Phi(x))|_{b,c}$. Lemma 4.2.54 implies that, if $y = y_1 \cdots y_n$, where each y_k is a letter, and, for a certain $1 \leq i \leq n$, $y_i = b$, then there is a unique j with $i < j \leq n$ such that $y_j = c$ and i is related with j. And if $y_i = c$, then there is a unique j with $1 \leq j < i$ such that $y_j = b$ and j is related with i.

Lemma 4.2.56. *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\Phi"] \arrow[dr, "\#_{\text{Nodes}}"'] & (A \uplus \{a,b,c\})^{\Delta} \arrow[d, "\#_{\text{Nodes}_A}"] \\
 & (\mathbb{N}, \mathbb{N}')
\end{tikzcd}$$

where, for $x \in \{a, b, c\}$, we define $\#_{\text{Nodes}_A}(x\Box) = 0'$, and for $x \in A$ we define $\#_{\text{Nodes}_A}(x\Box) = 1'$.

Proof. All the mappings $\#_{\text{Nodes}}$, Φ and $\#_{\text{Nodes}_A}$ are ω-algebra homomorphisms.

We just need to show that, whenever x is a generator, the following equality holds:

$$\#_{\text{Nodes}}(x) = (\#_{\text{Nodes}_A} \circ \Phi)(x).$$

Let $d\Box$ be an element of A'. Then we have $\#_{\text{Nodes}}(d\Box) = 1'$ and $\Phi(d\Box) = d\Box$ which implies the following equality:

$$\#_{\text{Nodes}_A} \circ \Phi(d\Box) = \#_{\text{Nodes}_A}(d\Box) = 1'. \qquad \Box$$

Lemma 4.2.57. *For p-contexts x and y, we have the following:*

- *if for some forests h_1 and h_2 of $\mathcal{S}$ with the property that the number of occurrences of the label b is less than or equal to the number of occurrences of the label c in* traversal(h_1) *and* traversal(h_2)*, then the equality $\Phi(x) * (c\Box * h_1) = \Phi(y) * (c\Box * h_2)$ implies the equalities $\Phi(x) = \Phi(y)$ and $h_1 = h_2$;*

- *if for some contexts u_1 and u_2 of $\mathcal{S}$ with the property that the number of occurrences of the label b is less than or equal to the number of occurrences of the label c in* traversal(u_1) *and* traversal(u_2)*, then the equality $\Phi(x).(c\Box.u_1) = \Phi(y).(c\Box.u_2)$ implies the equalities $\Phi(x) = \Phi(y)$ and $u_1 = u_2$.*

Proof. We will show just the first one, the second one can be handled similarly.

We argue by simultaneous induction on the number of nodes and the rank of x.

Assume that, for every p-context x with

$$\#_{\text{Nodes}}(x) \leq n \quad \text{and} \quad \text{Rank}(x) \leq k$$

and at least one of the inequalities strict, the equality $\Phi(x) * (c\Box * h_1) = \Phi(y) * (c\Box * h_2)$ implies the equalities $\Phi(x) = \Phi(y)$ and $h_1 = h_2$.

Let x be a p-context with $\#_{\text{Nodes}}(x) = n$ and $\text{Rank}(x) = k$. There are p-forests H_1, H_2, S_1, and S_2 such that $x = H_1 + C(x) + H_2$ and also $y = S_1 + C(y) + S_2$. By Lemma 4.2.47, the image of a $\Box$-pure p-context is connected. As the equality $\Phi(x) * (c\Box * h_1) = \Phi(y) * (c\Box * h_2)$ holds in the free forest algebra $\mathcal{S}$, and the number of occurrences of the label b and the number of occurrences of the label c are equal in $\Phi(x)$. So, there is a unique tree in both sides which does not have equal number of occurrences of the label b and the label c. So, the equality

$$\Phi(H_1) + \Phi(C(x)) * (c\Box * h_1) + \Phi(H_2) = \Phi(S_1) + \Phi(C(y)) * (c\Box * h_2) + \Phi(S_2)$$

holds if and only if the following equalities hold:

$$\begin{aligned}\Phi(H_1) &= \Phi(S_1),\\ \Phi(H_2) &= \Phi(S_2),\\ \Phi(C(x))) * (c\Box * h_1) &= \Phi(C(y)) * (c\Box * h_2).\end{aligned}$$

If at least one of the H_1 and H_2 are non-trivial, then, since $\#_{\text{Nodes}}(C(x)) < n$ and $\text{Rank}(x) \leq k$, by induction hypothesis we obtain the following equalities:

$$\Phi(C(x)) = \Phi(C(y)) \quad \text{and} \quad h_1 = h_2,$$

which yield to the equalities $\Phi(x) = \Phi(y)$ and $h_1 = h_2$.

We may assume that $H_1 = H_2 = 0$. Then either $x = d\Box.v_1$ or $u_1^\omega.v_1$, and either $y = d'\Box.v_2$ or $y = u_2^\omega.u_2$, where u_1, u_2, v_1, and v_2 are p-contexts, with $C(u_1) \neq \Box$ and $C(u_2) \neq \Box$, and $d\Box, d'\Box \in A'$. Since the equality $\Phi(x) * (c\Box * h_1) = \Phi(y) * (c\Box * h_2)$ holds, by applying traversal, we obtain that $x = d\Box.v_1$ if and only if $y = d\Box.v_2$. Hence, one of the following must hold:

- Assume that, $x = d\Box.v_1$ and $y = d\Box.v_2$. Then by the equality $\Phi(x) * (c\Box * h_1) = \Phi(y) * (c\Box * h_2)$ and Lemma 1.3.7, we obtain the equality $\Phi(v_1) * (c\Box * h_1) = \Phi(v_2) * (c\Box * h_2)$. Since $\#_{\text{Nodes}}(v_1) = n - 1$ and $\text{Rank}(v_1) = k$, by induction hypothesis the equality $\Phi(v_1) * (c\Box * h_1) = \Phi(v_2) * (c\Box * h_2)$ implies the equalities $\Phi(v_1) = \Phi(v_2)$ and $h_1 = h_2$.

- Assume that, $x = u_1^\omega.v_1$ and $y = u_2^\omega.v_2$. Then by the equality $\Phi(x) * (c\Box * h_1) = \Phi(y) * (c\Box * h_2)$ and Lemma 1.3.7, we obtain the equality

  $$\Phi(u_1) * (c\Box * \Phi(v_1) * c\Box * h_1) = \Phi(u_2) * (c\Box * \Phi(v_2) * c\Box * h_2).$$

  Since $\#_{\text{Nodes}}(u_1) \leq n$ and $\text{Rank}(u_1) < k$, by induction hypothesis we obtain the following equalities:

  $$\Phi(u_1) = \Phi(u_2) \quad \text{and} \quad \Phi(v_1) * (c\Box * h_1) = \Phi(v_2) * (c\Box * h_2).$$

  And since $\#_{\text{Nodes}}(v_1) < n$ and $\text{Rank}(v_1) \leq k$, by induction hypothesis we have the equalities $\Phi(v_1) = \Phi(v_2)$ and $h_1 = h_2$. □

Corollary 4.2.58. *For elements x and y of the free ω-algebra $\mathcal{A}$, we have the following:*

- *if for some p-forests h and h' and some p-contexts v and v' with $C(v) \neq \Box$ and $C(v') \neq \Box$ we have $x = v^\omega * h$ and $y = v'^\omega * h'$, then the equality $\Phi(x) = \Phi(y)$ implies the equalities $\Phi(v) = \Phi(v')$ and $\Phi(h) = \Phi(h')$;*

- *if for some p-contexts w and w' and some p-contexts v and v' with $C(v) \neq \Box$ and $C(v') \neq \Box$ we have $x = v^\omega.w$ and $y = v'^\omega.w'$, then the equality $\Phi(x) = \Phi(y)$ implies the equalities $\Phi(v) = \Phi(v')$ and $\Phi(w) = \Phi(w')$.*

Proof. Assume that, for some p-forests h and h' and some p-contexts v and v' with $C(v) \neq \Box$ and $C(v') \neq \Box$ we have $x = v^\omega * h$ and $y = v'^\omega * h'$. And assume that the equality $\Phi(x) = \Phi(y)$ holds. The equality $\Phi(x) = \Phi(y)$ implies that the following equality holds

$$b\Box.\Phi(v).c\Box * \Phi(h) = b\Box.\Phi(v').c\Box * \Phi(h'),$$

which is the equality $\Phi(v) * (c\Box * \Phi(h)) = \Phi(v') * (c\Box * \Phi(h'))$. Since Remark 4.2.55 implies that in $\Phi(h)$ and $\Phi(h')$ the number of occurrences of the label b and the number of occurrences of the label c are equal, Lemma 4.2.57 gives the result.

We can apply similar arguments in the second one, and then Lemma 4.2.57 gives the result. $\Box$

The following result is immediate by Lemma 1.3.6 and Lemma 1.3.7.

Lemma 4.2.59. *For elements x and y of the free ω-algebra $\mathcal{A}$ we have the following:*

- *if for some p-forests h and h' we have $x = \omega(h)$ and $y = \omega(h')$, then the equality $\Phi(x) = \Phi(y)$ implies the equality $\Phi(h) = \Phi(h')$;*
- *if for some p-contexts $d\Box$ and $d'\Box$ and some p-contexts v and v' we have $x = d\Box.v$ and $y = d'\Box.v'$, then the equality $\Phi(x) = \Phi(y)$ implies the equalities $\Phi(v) = \Phi(v')$ and $d = d'$;*
- *if for some p-contexts $d\Box$ and $d'\Box$ and some p-forests h and h' we have $x = d\Box * h$ and $y = d'\Box * h'$, then the equality $\Phi(x) = \Phi(y)$ implies the equalities $\Phi(h) = \Phi(h')$ and $d = d'$.* $\Box$

The following is one of the main results in this chapter.

Theorem 4.2.60. *The ω-algebra homomorphism $\Phi : \mathcal{A} \to \mathcal{S}$ is injective.*

Proof. Let x and y be elements of $\mathcal{A}$ such that $\Phi(x) = \Phi(y)$. We show that $x = y$. We argue by simultaneous induction on the number of nodes and the rank of x.

Assume that, for every element x of $\mathcal{A}$ with

$$\#_{\text{Nodes}}(x) \leq n \quad \text{and} \quad \text{Rank}(x) \leq k$$

and at least one of the inequalities strict, the equality $\Phi(x) = \Phi(y)$ implies the equality $x = y$.

Let x be a p-forest with $\#_{\text{Nodes}}(x) = n$ and $\text{Rank}(x) = k$. We show that the equality $\Phi(x) = \Phi(y)$ implies the equality $x = y$.

Assume that x is a p-forest. Then $x = x_1 + \cdots + x_n$ and let y be a p-forest with $y = y_1 + \cdots + y_m$ where x_i and y_j are non-trivial additively irreducible. Hence, Lemma 4.2.45 implies that $\Phi(x_i)$ and $\Phi(y_j)$ are connected. The equality $\Phi(x) = \Phi(y)$ holds in the free forest algebra $\mathcal{S}$. By Lemma 1.3.6, we have $m = n$ and for all i the equalities $\Phi(x_i) = \Phi(y_i)$ hold. Since $\#_{\text{Nodes}}(x_i) < n$ and $\text{Rank}(x_i) \leq k$, by induction hypothesis we have for all i, $x_i = y_i$. So, we may assume that x and y are additively irreducible.

Hence, x is additively irreducible and one of the following must hold:

- Assume that, for some non-trivial p-forests h and h', we have $x = \omega(h)$ and $y = \omega(h')$. Then by the equality $\Phi(x) = \Phi(y)$ and Lemma 4.2.59, we have $\Phi(h) = \Phi(h')$, where $\#_{\text{Nodes}}(h) = n$ and $\text{Rank}(h) = k - 1$. And by induction hypothesis we get the result.

- Assume that, for some p-forests h and h' and $d, d' \in A$, we have $x = d\Box * h$ and $y = d'\Box * h'$. Then by the equality $\Phi(x) = \Phi(y)$ and Lemma 4.2.59, we have $\Phi(h) = \Phi(h')$ and $d = d'$, where $\#_{\text{Nodes}}(h) = n - 1$. Since $\text{Rank}(h) = k$, by induction hypothesis the equality $\Phi(h) = \Phi(h')$ implies the equality $h = h'$.

- Finally, assume that, for some p-forests h and h' and some p-contexts v and v' with $C(v) \neq \Box$ and $C(v') \neq \Box$, we have $x = v^\omega * h$ and $y = v'^\omega * h'$. Since the equality $\Phi(x) = \Phi(y)$ holds, Corollary 4.2.58 implies the equalities $\Phi(v) = \Phi(v')$ and $\Phi(h) = \Phi(h')$. Hence, by induction hypothesis, since

  $$\#_{\text{Nodes}}(h) < n \quad \text{and} \quad \text{Rank}(h) \leq k,$$

  we have the equality $h = h'$. And since $Rank(v) < k$, while

  $$\#_{\text{Nodes}}(h) \leq n,$$

  by the induction hypothesis we have the equality $v = v'$.

Now assume that, x is a p-context. So, there are p-forests H_1, H_2, S_1, and S_2 such that $x = H_1 + C(x) + H_2$ and also $y = S_1 + C(y) + S_2$. By Lemma 4.2.47, the image of a $\Box$-pure p-context is connected. As the equality $\Phi(x) = \Phi(y)$ holds in the free forest algebra $\mathcal{S}$, Lemma 1.3.7 implies the following equalities:

$$\Phi(H_1) = \Phi(S_1) \ , \quad \Phi(H_2) = \Phi(S_2) \quad \text{and} \quad \Phi(C(x)) = \Phi(C(y)).$$

Hence, the equalities $H_1 = S_1$ and $H_2 = S_2$ is from the preceding arguments on the case where x is a p-forest. So, we may assume that x is an additively irreducible p-context.

To complete the proof, we can do the similar arguments as in the preceding arguments for the p-forest case. $\Box$

Definition 4.2.61. For an element x in the free ω-algebra $\mathcal{A}$, we define the *number of summands* of x to be $\text{CP}(x) = \text{CP}(\Phi(x))$ where $\text{CP}(\Phi(x))$ is the number of connected parts of $\Phi(x)$ in the free forest algebra $\mathcal{S}$ (See Definition 1.2.8).

Lemma 4.2.62. *Let u and v be p-contexts in the free ω-algebra $\mathcal{A}$. If for all p-forests h in $\mathcal{A}$ the equality $u * h = v * h$ holds, then we have the following statements:*

- *there are some p-forests T_1 and T_2 such that $u = T_1 + C(u) + T_2$ and $v = T_1 + C(v) + T_2$, and for every p-forest h in $\mathcal{A}$ the equality $C(u) * h = C(v) * h$ holds;*
- $\mathrm{Rank}(u) = \mathrm{Rank}(v)$*;*
- $\#_{\mathrm{Nodes}}(u) = \#_{\mathrm{Nodes}}(v)$.

Proof. Since for the trivial p-forest the equality $u * 0 = v * 0$ holds, the last two statements are immediate.

Let $u = H_1 + C(u) + H_2$ and $v = S_1 + C(v) + S_2$. For the p-forest $h = d\Box*(H_1+H_2+S_1+S_2)$ we have $u*h = v*h$. As $C(v)*d\Box*(H_1+H_2+S_1+S_2)$ and $C(u)*d\Box*(H_1+H_2+S_1+S_2)$ are additively irreducible for any choice of $C(u)$ and $C(v)$, Lemma 4.2.45 implies that $\Phi(C(v)*d\Box*(H_1+H_2+S_1+S_2))$ and $\Phi(C(u)*d\Box*(H_1+H_2+S_1+S_2))$ are connected. Applying the ω-algebra homomorphism Φ, we obtain the following equality:

$$\begin{aligned}&\Phi(H_1 + C(u) * d\Box * (H_1 + H_2 + S_1 + S_2) + H_2)\\ &= \Phi(S_1 + C(v) * d\Box * (H_1 + H_2 + S_1 + S_2) + S_2)\end{aligned}$$

in the free forest algebra $(A \uplus \{a, b, c\})^{\Delta}$. Lemma 1.3.6 implies the componentwise equality of the forests. Since the second summands on both sides of the preceding equality are the only ones with maximum number of nodes, we can conclude that the following equalities hold:

$$\begin{aligned}&\Phi(C(v) * d\Box * (H_1 + H_2 + S_1 + S_2))\\ &= \Phi(C(u) * d\Box * (H_1 + H_2 + S_1 + S_2)),\end{aligned}$$

$$\Phi(H_1) = \Phi(S_1) \quad \text{and} \quad \Phi(H_2) = \Phi(S_2).$$

Theorem 4.2.60 implies the equalities $H_1 = S_1$ and $H_2 = S_2$. Since for every p-forest h in $\mathcal{A}$ the equality $H_1 + C(u) * h + H_2 = S_1 + C(v) * h + S_2$ holds, applying the ω-algebra homomorphism Φ, we obtain the following equality:

$$\Phi(H_1 + C(u) * h + H_2) = \Phi(S_1 + C(v) * h + S_2)$$

which is

$$\Phi(H_1) + \Phi(C(u) * h) + \Phi(H_2) = \Phi(S_1) + \Phi(C(v) * h) + \Phi(S_2)).$$

As the preceding equality holds for all h, it implies that either both $C(v)$ and $C(u)$ are trivial or both are non-trivial. In the former case, the proof is complete. In the latter case, as $C(v) * h$ and $C(u) * h$ are additively irreducible, Lemma 4.2.45 implies that $\Phi(C(v) * h)$ and $\Phi(C(u) * h)$ are connected. Now, by the equalities $\Phi(H_1) = \Phi(S_1)$ and $\Phi(H_2) = \Phi(S_2)$, Lemma 1.3.6 implies the equality $\Phi(C(u)*h) = \Phi(C(v)*h)$, and the equality $C(u) * h = C(v) * h$ for every p-forest h follows by Theorem 4.2.60. $\square$

The following is the main result in this chapter.

Theorem 4.2.63. *Every free ω-algebra $\mathcal{A}$ is a forest algebra.*

Proof. As $\mathcal{A}$ satisfies the equational axioms of forest algebras, all we need to show is that for given two p-contexts u and v in $\mathcal{A}$ such that, for every p-forest h in $\mathcal{A}$, the equality $u * h = v * h$ holds, then the equality $u = v$ also holds. In view of Lemma 4.2.62, we just need to consider the cases where $u = C(u)$ and $v = C(v)$. We proceed by induction on the number of nodes of u.

For p-contexts u and v with $\#_{\text{Nodes}}(u) = 0$, if for every p-forest h in $\mathcal{A}$ the equality $u * h = v * h$ holds, then Lemma 4.2.62 implies $\#_{\text{Nodes}}(v) = 0$, so that Lemma 4.1.8 implies the equalities $u = v = \square$.

Assume that for p-contexts u and v with $\#_{\text{Nodes}}(u) \leq n$ if, for every p-forest h in $\mathcal{A}$, the equality $u * h = v * h$ holds, then the equality $u = v$ also holds.

Now, consider p-contexts u and v with $\#_{\text{Nodes}}(u) = n + 1$ such that for every p-forest h in $\mathcal{A}$ the equality $u*h = v*h$ holds. As we assumed $u = C(u)$ and $v = C(v)$, then either $u = d\square.w$ or $x^\omega.w$, and either $v = d'\square.w'$ or $v = y^\omega.w'$, where w, w', x, and y are p-contexts, with $C(x) \neq \square$ and $C(y) \neq \square$, and $d\square, d'\square \in A'$.

Since for every p-forest h in $\mathcal{A}$ the equality $u * h = v * h$ holds, applying traversal $\circ\Phi$ on $u * h = v * h$, implies that $u * h$ and $v * h$ have the same kind. In the first case $d = d'$, which means $u = d\square.w$ if and only if $v = d'\square.w'$.

Since for every p-forest h in $\mathcal{A}$ we have $\Phi(u * h) = \Phi(v * h)$, we obtain

$$\Phi(d\square * (w * h)) = \Phi(d\square * (w' * h)) \quad \text{or} \quad \Phi(x^\omega * (w * h)) = \Phi(y^\omega * (w' * h)).$$

In the first case, we obtain the equality $\Phi(w * h) = \Phi(w' * h)$ and for the second case, by Corollary 4.2.58, we obtain the equalities:

$$\Phi(x^\omega) = \Phi(y^\omega) \quad \text{and} \quad \Phi(w * h) = \Phi(w' * h).$$

In both cases, Theorem 4.2.60 implies for every h the equality $w * h = w' * h$ holds, where in the second case Theorem 4.2.60 also implies that $x^\omega = y^\omega$, and by induction hypothesis, since $\#_{\text{Nodes}}(w) < \#_{\text{Nodes}}(u)$, we have the equality $w = w'$. So we have the equality $u = v$. $\square$

4.2.3 Some Notation in the Free ω-Algebra $\mathcal{A}$

From now on, we will work on elements of the free ω-algebra $\mathcal{A}$ over an alphabet A.

For a p-context v, we define the following sets:

$$\begin{aligned}
&\mathrm{IrrF}(v) = \{w \mid w \text{ is a non-trivial multiplicatively irreducible factor of } v\};\\
&\mathrm{LIrrF}(v) = \{h + \Box \mid h + \Box \in \mathrm{IrrF}(v)\};\\
&\mathrm{RIrrF}(v) = \{\Box + h \mid \Box + h \in \mathrm{IrrF}(v)\};\\
&\mathrm{PIrrF}(v) = \{u \mid u \in \mathrm{IrrF}(v)\ ,\quad u \text{ is } \Box\text{-pure}\};\\
&\mathrm{IdemF}(v) = \{u \mid u \in \mathrm{IrrF}(v)\ ,\quad u = w^{\omega} \text{ for a } p\text{-context } w\};\\
&\mathrm{IrrNIdemF}(v) = \mathrm{IrrF}(v) \setminus \mathrm{IdemF}(v);\\
&\mathrm{IrrNIdemF}^{*}(v) = \mathrm{IrrNIdemF}(v) \cup (\bigcup_{u^{\omega} \in \mathrm{IdemF}(v)} \mathrm{IrrNIdemF}^{*}(u)).
\end{aligned}$$

And for a p-forest h, we define the following sets:

$$\begin{aligned}
&\mathrm{IrrS}(h) = \{t \mid t \text{ is a non-trivial additively irreducible summand of } h\};\\
&\mathrm{IdemS}(h) = \{t \mid t \in \mathrm{IrrS}(h)\ ,\quad t = \omega(s) \text{ for a } p\text{-forest } s\};\\
&\mathrm{IrrNIdemS}(h) = \mathrm{IrrS}(h) \setminus \mathrm{IdemS}(h);\\
&\mathrm{IrrNIdemS}^{*}(h) = \mathrm{IrrNIdemS}(h) \cup (\bigcup_{\omega(t) \in \mathrm{IdemS}(h)} \mathrm{IrrNIdemS}^{*}(t)).
\end{aligned}$$

Lemma 4.2.64. *Let v be a non-trivial p-context in $\mathcal{A}$. If we have* $\mathrm{IrrF}(v) = \{x\}$*, then there exists a unique positive integer k such that the equality $v = x^k$ holds.*

Proof. If v is multiplicatively irreducible, then we have the equality $v = x$. Assume that, v is not a multiplicatively irreducible p-context. Then we can write $v = x_1 \dots x_k$, as a product of its non-trivial multiplicatively irreducible factors. By definition of $\mathrm{IrrF}(v)$ we must have $x_i = x$, whence $v = x^k$.

We show that k is unique, that is, if there are positive integers k_1 and k_2 such that $v = x^{k_1} = x^{k_2}$, then the equality $k_1 = k_2$ holds.

As x is a non-trivial multiplicatively irreducible p-context, by Lemma 4.2.43, x has one of the following forms:

1. $d\Box$, where $d\Box \in A$;
2. u^{ω}, where $C(u) \neq \Box$;
3. $s + \Box$, where s is a non-trivial additively irreducible p-forest;
4. $\Box + s$, where s is a non-trivial additively irreducible p-forest.

If $x = d\Box$ and $v = x^{k_1} = x^{k_2}$, then by applying the ω-algebra homomorphism $\#_{\mathrm{Nodes}}$ we have the following equalities:

$$\#_{\mathrm{Nodes}}(v) = \#_{\mathrm{Nodes}}(x^{k_1}) = \#_{\mathrm{Nodes}}(x^{k_2}),$$

which implies the equality $k_1 = k_2$.

If $x = u^\omega$ with $C(u) \neq \Box$ and $v = x^{k_1} = x^{k_2}$, then by using the ω-algebra homomorphism f_2', we have

$$f_2'(v) = f_2'(x^{k_1}) = f_2'(x^{k_2})$$

which is $f_2'(v) = (q\Box)^{k_1} = (q\Box)^{k_2}$, for some element $q\Box \in A'$. Now by using $\#_{\text{Nodes}}$ we have the following equalities:

$$\#_{\text{Nodes}}(f_2'(v)) = \#_{\text{Nodes}}((b\Box)^{k_1}) = \#_{\text{Nodes}}((b\Box)^{k_2}),$$

which implies the equality $k_1 = k_2$.

For element of the form $x = s+\Box$ in $\mathcal{A}$ where s is a non-trivial additively irreducible p-forest, if $s = d\Box * h$, then by using f_1 we have the following equalities:

$$f_1(v) = f_1(x^{k_1}) = f_1(x^{k_2}).$$

Now by the forest algebra homomorphism $\#_{\text{ConnectedParts}}$ which gives the number of connected parts we have the following equalities:

$$\begin{aligned}\#_{\text{ConnectedParts}}(f_1(v)) &= \#_{\text{ConnectedParts}}(f_1(x^{k_1}))\\ &= \#_{\text{ConnectedParts}}(f_1(x^{k_2})),\end{aligned}$$

which is

$$\#_{\text{ConnectedParts}}((d\Box * f_1(h) + \Box)^{k_1}) = \#_{\text{ConnectedParts}}((d\Box * f_1(h) + \Box)^{k_2})$$

and it implies the equality $k_1 = k_2$.

If $s = \omega(h)$ then by using f_1' we obtain

$$f_1'(v) = f_1'(x^{k_1}) = f_1'(x^{k_2})$$

which is

$$f_1'(v) = (a + \Box)^{k_1} = (a + \Box)^{k_2}.$$

Again by using $\#_{\text{ConnectedParts}}$ we have the following equalities:

$$\begin{aligned}\#_{\text{ConnectedParts}}(f_1'(v)) &= \#_{\text{ConnectedParts}}((a + \Box)^{k_1})\\ &= \#_{\text{ConnectedParts}}((a + \Box)^{k_2}),\end{aligned}$$

which implies the equality $k_1 = k_2$.

Finally, if $s = u^\omega * h$ with $C(u) \neq \Box$, then by using f_2' we have

$$f_2'(v) = f_2'(x^{k_1}) = f_2'(x^{k_2})$$

which implies the following equalities:

$$f_2'(v) = (a\Box * f_2'(h) + \Box)^{k_1} = (a\Box * f_2'(h) + \Box)^{k_2}.$$

Now by using $\#_{\text{ConnectedParts}}$ we have

$$\begin{aligned}\#_{\text{ConnectedParts}}(f_2'(v)) &= \#_{\text{ConnectedParts}}((a\Box * f_2'(h) + \Box)^{k_1})\\ &= \#_{\text{ConnectedParts}}((a\Box * f_2'(h) + \Box)^{k_2})\end{aligned}$$

which implies the equality $k_1 = k_2$.

In a similar way, we can get the uniqueness of k in case $\Box + s$. $\Box$

Lemma 4.2.65. *Let $w_1, \ldots, w_n$ be non-trivial multiplicatively irreducible p-contexts in $\mathcal{A}$. Then the following equality holds:*

$$\text{IrrF}(w_1. \cdots .w_n) = \{w_1\} \cup \text{IrrF}(w_2. \cdots .w_n).$$

Proof. See Appendix A, Section A.2. $\Box$

Corollary 4.2.66. *For p-contexts v_1 and v_2 in $\mathcal{A}$, we have the following equality:*

$$\text{IrrF}(v_1.v_2) = \text{IrrF}(v_1) \cup \text{IrrF}(v_2).$$

Proof. The result is trivial by considering the cases where at least one of the p-contexts v_1 and v_2 is trivial.

We can assume that v_1 and v_2 are non-trivial p-contexts, then there are non-trivial multiplicatively irreducible p-contexts $w_1, \ldots, w_n$ and $w_1', \ldots, w_n'$ such that $v_1 = w_1. \cdots .w_n$ and $v_2 = w_1'. \cdots .w_m'$. To show the result we apply Lemma 4.2.65 to $w_1. \cdots .w_n.w_1'. \cdots .w_m'$. $\Box$

By definition of $\text{IrrNIdemF}^*(v)$ and by Lemma 4.2.65, we have the following equality:

$$\text{IrrNIdemF}^*(v_1.v_2) = \text{IrrNIdemF}^*(v_1) \cup \text{IrrNIdemF}^*(v_2)$$

and also we have

$$\begin{aligned}\text{IrrNIdemF}^*(v^\omega) &= \text{IrrNIdemF}(v^\omega) \cup (\bigcup_{u^\omega \in \text{IdemF}(v^\omega)} \text{IrrNIdemF}^*(u)),\\ \text{IrrNIdemF}(v^\omega) &= \emptyset,\\ \text{IdemF}(v^\omega) &= \{v^\omega\}.\end{aligned}$$

In particular, we obtain equalities:

$$\text{IrrNIdemF}^*(v) = \text{IrrNIdemF}^*(v^2) = \text{IrrNIdemF}^*(v^\omega).$$

Lemma 4.2.67. *For p-forests h_1 and h_2 in $\mathcal{A}$, the following equality holds:*

$$\text{IrrS}(h_1 + h_2) = \text{IrrS}(h_1) \cup \text{IrrS}(h_2).$$

Proof. By Lemma 4.2.39, for a p-forest h in $\mathcal{A}$ we have the following equality:

$$\mathrm{IrrF}(h + \Box) * 0 = \mathrm{IrrS}(h).$$

Now, by Lemma 4.2.65, we have the equality

$$\mathrm{IrrF}(h_1 + h_2 + \Box) = \mathrm{IrrF}(h_1 + \Box) \cup \mathrm{IrrF}(h_2 + \Box),$$

which implies the following equality:

$$\mathrm{IrrS}(h_1 + h_2) = \mathrm{IrrS}(h_1) \cup \mathrm{IrrS}(h_2). \qquad \Box$$

By definition of $\mathrm{IrrNIdemS}^*(h)$ and by Lemma 4.2.67, we have the following equality:

$$\mathrm{IrrNIdemS}^*(h_1 + h_2) = \mathrm{IrrNIdemS}^*(h_1) \cup \mathrm{IrrNIdemS}^*(h_2),$$

and also we have

$$\mathrm{IrrNIdemS}^*(\omega(h)) = \mathrm{IrrNIdemS}(\omega(h)) \cup \Big(\bigcup_{\omega(t) \in \mathrm{IdemS}(\omega(h))} \mathrm{IrrNIdemS}^*(t)\Big),$$
$$\mathrm{IrrNIdemS}(\omega(h)) = \emptyset,$$
$$\mathrm{IdemS}(\omega(h)) = \{\omega(h)\}.$$

So, the following equalities hold:

$$\mathrm{IrrNIdemS}^*(h) = \mathrm{IrrNIdemS}^*(2h) = \mathrm{IrrNIdemS}^*(\omega(h)).$$

For a p-forest h, recall that we considered in Definition 1.3.4, the set

$$V_{\mathrm{IrrNIdemS}^*(h)} = (\_ * 0)^{-1}(\mathrm{IrrNIdemS}^*(h)),$$

and every p-context v in $V_{\mathrm{IrrNIdemS}^*(h)}$ can be written as a product of its non-trivial multiplicatively irreducible factors:

$$v = w_1. \cdots .w_m.$$

There is a positive integer n such that

$$v = u_1 v_1^{\omega} \cdots u_n v_n^{\omega} u_{n+1},$$

where each

$$u_i = \prod_{j=n_{i-1}}^{n_i} w_j$$

is a product of successive context factors of v which are not ω-context and v_i^{ω} is w_{n_i+1}, which is an ω-context factor of v. For every such p-context v, which is of the form

$$u_1 v_1^{\omega} \cdots u_n v_n^{\omega} u_{n+1}$$

define a set H_v, which contains all possible non-trivial forests of the form

$$P_1 Q_1 \cdots P_n Q_n P_{n+1} 0$$

such that each P_i is a scattered divisor of u_i and each Q_j is a product of some of the elements of $\mathrm{IrrNIdemF}^*(v_j)$ in some order.

For a p-forest h, we define the following set:

$$\mathrm{Special}_{\mathrm{H}}(h) = \mathrm{IrrNIdemS}^*(h) \cup \left(\bigcup_{v \in V_{\mathrm{IrrNIdemS}^*(h)}} H_v \right).$$

Lemma 4.2.68. *For p-forests h and s, the following equality holds:*

$$\mathrm{Special}_{\mathrm{H}}(h+s) = \mathrm{Special}_{\mathrm{H}}(h) \cup \mathrm{Special}_{\mathrm{H}}(s).$$

Proof. Because, we have the following equalities:

$$\begin{aligned}
&\mathrm{Special}_{\mathrm{H}}(h+s) \\
&= \mathrm{IrrNIdemS}^*(h+s) \cup \left(\bigcup_{v \in V_{\mathrm{IrrNIdemS}^*(h+s)}} H_v \right) \\
&= \mathrm{IrrNIdemS}^*(h) \cup \mathrm{IrrNIdemS}^*(s) \cup \left(\bigcup_{v \in V_{\mathrm{IrrNIdemS}^*(h) \cup \mathrm{IrrNIdemS}^*(s)}} H_v \right) \\
&= \mathrm{IrrNIdemS}^*(h) \cup \mathrm{IrrNIdemS}^*(s) \cup \left(\bigcup_{v \in V_{\mathrm{IrrNIdemS}^*(h)} \cup V_{\mathrm{IrrNIdemS}^*(s)}} H_v \right)
\end{aligned}$$

$$\begin{aligned}
&= \mathrm{IrrNIdemS}^*(h) \cup \mathrm{IrrNIdemS}^*(s) \cup \left(\bigcup_{v \in V_{\mathrm{IrrNIdemS}^*(h)}} H_v \right) \\
&\qquad \cup \left(\bigcup_{v \in V_{\mathrm{IrrNIdemS}^*(s)}} H_v \right) \\
&= \mathrm{IrrNIdemS}^*(h) \cup \left(\bigcup_{v \in V_{\mathrm{IrrNIdemS}^*(h)}} H_v \right) \cup \mathrm{IrrNIdemS}^*(s) \\
&\qquad \cup \left(\bigcup_{v \in V_{\mathrm{IrrNIdemS}^*(s)}} H_v \right) \\
&= \mathrm{Special}_{\mathrm{H}}(s) \cup \mathrm{Special}_{\mathrm{H}}(s).
\end{aligned}$$

□

Lemma 4.2.69. *For a p-context v and a p-context h, the sets* $\mathrm{IrrF}(v)$ *and* $\mathrm{IrrS}(h)$ *are finite.*

Proof. We argue by induction on the number of nodes of v. If $\#_{\mathrm{Nodes}}(v) = 0$, then $\mathrm{IrrF}(v) = \emptyset$ which has finite number of elements. Assume that for every p-context v with $\#_{\mathrm{Nodes}}(v) \leq k$, $\mathrm{IrrF}(v)$ has finite number of elements. Let v be a p-context with $\#_{\mathrm{Nodes}}(v) = k + 1$. If v is multiplicatively irreducible, then $\mathrm{IrrF}(v) = \{v\}$. We may assume that v is not multiplicatively irreducible, then, there are non-trivial p-contexts v_1 and v_2 such that $v = v_1.v_2$, and Corollary 4.2.66 implies that $\mathrm{IrrF}(v) = \mathrm{IrrF}(v_1) \cup \mathrm{IrrF}(v_2)$. Since $\#_{\mathrm{Nodes}}(v_1) \leq k$ and $\#_{\mathrm{Nodes}}(v_2) \leq k$, by induction hypothesis, $\mathrm{IrrF}(v_1)$ and $\mathrm{IrrF}(v_2)$ are finite, so does $\mathrm{IrrF}(v)$.

We can do similar arguments for a p-forest h. □

Corollary 4.2.70. *For a p-context v, the sets* $\mathrm{LIrrF}(v)$, $\mathrm{RIrrF}(v)$, $\mathrm{PIrrF}(v)$, $\mathrm{IdemF}(v)$, $\mathrm{IrrNIdemF}(v)$, *and* $\mathrm{IrrNIdemF}^*(v)$ *are finite.*

And similarly, for a p-forest h, the sets $\mathrm{IdemS}(h)$, $\mathrm{IrrNIdemS}(h)$ *and* $\mathrm{IrrNIdemS}^*(h)$ *are finite.*

Proof. The sets $\mathrm{LIrrF}(v)$, $\mathrm{RIrrF}(v)$, $\mathrm{PIrrF}(v)$, $\mathrm{IdemF}(v)$, and $\mathrm{IrrNIdemF}(v)$ are subsets of set $\mathrm{IrrF}(v)$, and Lemma 4.2.69 implies that all are finite.

Since finite union of finite sets is a finite set, by definition of $\mathrm{IrrNIdemF}^*$, $\mathrm{IrrNIdemF}^*(v)$ is finite.

For a p-forest h, we can do the similar argument. □

4.3 Conclusion

We introduced ω-algebras which satisfy the equational axioms of forest algebras with some extra assumptions. Since the class of ω-algebras is defined by equational axioms, all the free ω-algebras exist. By introducing additional partial operations on a forest algebra we make it into an ω-algebra. By using the universal property of the free ω-algebra we showed that the free ω-algebra is a forest algebra. We distinguished all kinds of non-trivial additively irreducible and non-trivial multiplicatively irreducible elements of the free ω-algebras. We showed that the set of non-trivial multiplicatively irreducible factor of a product of p-contexts is the union of the set of non-trivial multiplicatively irreducible factor of each one. By Lemma 4.1.1, it is natural to study the free profinite forest algebra as an ω-algebra. We still do not know if the free ω-algebra is the answer for the corresponding term algebra for the relatively free pro-**BSS** forest algebras.

Analog of Birkhoff theorem for partial algebras also holds as studies of Németi and Sain [25], Andréka and Németi [3], and briefly studied by Burmeister [10, p. 314]. The class of ω-algebras $\mathfrak{B}$ is a variety and it is defined by a set of equations on the free ω-algebra $\mathcal{A}$ [5]. The latter means:

there is a family E of equations $p = q$, where p and q are polynomial symbols, such that an algebra B of the type τ belongs to $\mathfrak{B}$ if and only if for each equation $p = q$ in E the induced operations p_B and q_B coincide. Every subvariety of $\mathfrak{B}$ satisfies the equational axioms of ω-algebras with more equational axioms [5]. To identify the free object in a subvariety of $\mathfrak{B}$ we just need to identify the quotient of the free ω-algebra by the new set of equations.

Chapter 5

Canonical Forms

In the study of the pseudovariety **BSS**, from [6, Theorem 2 and Proposition 19] and [1, Section 8.2], we obtained certain suitable identities denoted by Σ. We describe an algorithm to compute the so-called canonical form for an element of the free ω-algebra $\mathcal{A}$ modulo Σ and we prove it is correct.

In this chapter we use the same notation as in Chapters 1 and 4.

5.1 Identities

Given any finite monoid M, there is a number $\omega(M)$ [denoted by ω when M is understood from the context] such that for each element x of M, x^ω is an idempotent: $x^\omega = x^\omega x^\omega$. Therefore for any finite forest algebra (H, V) and any element u of V and g of H we will write u^ω and $\omega(g)$ for the corresponding idempotents [6].

Let **V** is a pseudovariety of finite forest algebras. We say that an algebra is *pro*-**V** if it is a projective limit of a projective system of forest algebras from **V**.

Let A be a finite alphabet and let **BSS** be the pseudovariety of finite forest algebras generated by all syntactic forest algebras of piecewise-testable forest languages. By Lemma 2.1.5 and in view of [6, Theorem 2 and Proposition 19], we get $\mathbf{BSS} \subset \mathbf{VJ}$, where **J** is the pseudovariety of $\mathcal{J}$-trivial monoids. And by Lemma 2.2.5, we have $\mathbf{BSS} \subset \mathbf{FJ}$. For a multiplicative finite monoid M and additive finite monoid S, and $m \in M$ and $s \in S$, there exists exactly one idempotent of the form m^n and ns with $n \geq 1$; these idempotents will be represented respectively by m^ω and $\omega(s)$. We thus define new unary operations $m \mapsto m^\omega$ and $s \mapsto \omega(s)$ on the pseudovariety of all finite forest algebras. In order to verify that the unary operations $m \mapsto m^\omega$ and $s \mapsto \omega(s)$ defined on the pseudovariety of all finite forest algebras commutes with all forest algebra homomorphisms let $\alpha : S_1 = (H_1, V_1) \to S_2 = (H_2, V_2)$ be a forest algebra homomorphism of finite forest algebras and $h \in H_1$ and $v \in V_1$, then $\alpha(\omega(h)) = \omega(\alpha(h))$

and $\alpha(v^\omega) = \alpha(v)^\omega$. For construction of all elements of the free pro-**BSS**, $\overline{\Omega}_A$**BSS**, from the projections $a_1, \ldots, a_n$, it is natural to consider the basic operations and two unary operations $x \mapsto \omega(x)$ and $y \mapsto y^\omega$. And so we can study the free pro-**BSS** as an ω-algebra.

Consider the variety $\mathcal{V}$ of ω-algebras of type τ, defined by the set Σ consisting of the following identities, for context terms u and v and forest term h,

$$(uv)^\omega = (vu)^\omega = (u^\omega v^\omega)^\omega \tag{5.1}$$

$$v^\omega v = v^\omega = vv^\omega \tag{5.2}$$

$$(v^\omega)^\omega = v^\omega \tag{5.3}$$

$$vh + \omega(vuh) = \omega(vuh) = \omega(vuh) + vh \tag{5.4}$$

Lemma 5.1.1. *For forest terms h and s, the following identities are consequences of Σ:*

$$\omega(h + s) = \omega(s + h) = \omega(\omega(h) + \omega(s)) \tag{5.5}$$

$$\omega(h) + h = \omega(h) = h + \omega(h) \tag{5.6}$$

$$\omega(\omega(h)) = \omega(h) \tag{5.7}$$

Proof. The identities 5.5, 5.6 and 5.7 are immediate respectively from the identities 5.1, 5.2 and 5.3 by letting $u = s + \Box$ and $v = h + \Box$ and then acting on the trivial forest term 0. $\Box$

Lemma 5.1.2. *The following identities are consequences of Σ:*

I.1 $v^\omega v^\omega = v^\omega$;

I.2 $(uv)^\omega u = (uv)^\omega = v(uv)^\omega$;

I.3 $u^\omega = v^\omega$ *where u with the factorization $\prod_{i \in \mathbb{N}} u_i$ is a p-context and v is the product, in any order, of the factors of u;*

I.4 $\omega(h) + \omega(h) = \omega(h)$;

I.5 $\omega(h + s) + h = \omega(h + s) = s + \omega(h + s)$;

I.6 $\omega(h) = \omega(s)$ *where h is a p-forest and s is the sum, in any order, of the elements of* $\mathrm{IrrS}(h)$;

I.7 $(vv)^\omega = v^\omega$;

I.8 $(uv^\omega)^\omega = (uv)^\omega$;

I.9 $u^\omega = (\prod_{v \in \mathrm{IrrNIdemF}^*(u)} v)^\omega$;

I.10 $\mathrm{IrrNIdemF}^*(u) = \mathrm{IrrNIdemF}^*(v)$ *if and only if* $u^\omega = v^\omega$;

I.11 *if* $\mathrm{IrrNIdemF}^*(u) \subseteq \mathrm{IrrNIdemF}^*(v)$, *then* $u^\omega v^\omega = v^\omega u^\omega = v^\omega$;

I.12 *if* $\mathrm{IrrNIdemF}^*(u) \subseteq \mathrm{IrrNIdemF}^*(v)$, *then* $uv^\omega = v^\omega u = v^\omega$;

I.13 $\omega(h + h) = \omega(h)$;

I.14 $\omega(uh + uwh) = \omega(uwh)$;

I.15 $\omega(\omega(h) + s) = \omega(h + s)$;

I.16 *for a p-context* $v = h_1 + C(v) + h_2$, *if* $C(v) = \square$ *then* $v^\omega = \omega(h_1) + \square + \omega(h_2)$. *And for* $C(v) \neq \square$, *if* $C(v) \neq v$ *then there is a p-context* u *with* $u = C(u)$ *such that* $v^\omega = u^\omega$;

I.17 *for every p-context* u *and* v *and every p-forest* h *we have* $\omega(uvs) = \omega(uvs) + \omega(us) = \omega(us) + \omega(uvs)$;

I.18 *for every p-context* u *and every p-forest* t *if p-contexts* w *and* v *are such that one of the identities* $wv = w$ *or* $w = vw$ *holds, then the identities* $uvt + \omega(uwt) = \omega(uwt) = \omega(uwt) + uvt$ *hold;*

I.19 *for p-contexts* $v_1, \ldots, v_n$ *and a p-forest* s, *if we have* $u_1, \ldots, u_n$ *are product of some of multiplicatively irreducible factors of respectively* $v_1, \ldots, v_n$ *or the trivial p-context* $\square$ *and* h *is a suffix of* s, *then*

$$\omega(v_1^\omega \cdots v_n^\omega s) + u_1 \cdots u_n h = \omega(v_1^\omega \cdots v_n^\omega s).$$

And similarly, we have the identity

$$\omega(v_1^\omega \cdots v_n^\omega s + u_1 \cdots u_n h) = \omega(v_1^\omega \cdots v_n^\omega s);$$

I.20 *for contexts* v *and* u *with* $u \mid_s v$ *we have the identity* $\omega(v0) + u0 = \omega(v0)$. *And similarly, we have the identity* $\omega(v0 + u0) = \omega(v0)$;

I.21 $\omega(h) = \omega(\sum_{t \in \mathrm{IrrNIdemS}^*(h)} t)$;

I.22 *for every forest* $s \in \mathrm{Special_H}(h)$, $\omega(h + s) = \omega(h)$;

I.23 $\omega(h) = \omega(\sum_{t \in \mathrm{Special_H}(h)} t)$;

I.24 *for a p-context* v *with factorization* $\prod_{i=1}^{n} v_i$ *and a p-forest* $h = t_1 + \cdots + t_{\mathrm{CP}(h)}$, *we have the following results:*

G.1 *if* $t_1 \in \mathrm{LIrrF}(v)0$, *then* $v^\omega h = v^\omega h'$ *where* $h' = t_2 + \cdots + t_{\mathrm{CP}(h)}$;

G.2 *if* $t_{\mathrm{CP}(h)} \in \mathrm{RIrrF}(v)0$, *then* $v^\omega h = v^\omega h'$ *where* $h' = t_1 + \cdots + t_{\mathrm{CP}(h)-1}$;

G.3 *if there is a p-context* $w \in \mathrm{Pref}(h)$ *such that* $\mathrm{IrrNIdemF}^*(w) \subseteq \mathrm{IrrNIdemF}^*(v)$, *then* $v^\omega h = v^\omega h'$ *where* $h = wh'$;

G.4 *if for a positive integer* j *with* $1 \le j \le \mathrm{CP}(h)$ *and a nonempty set* $L \subseteq \mathrm{LIrrF}(v)0$ *there is a p-forest* s *which is a sum of, in any order, of elements of* L *such that there are p-contexts* u *and* w *and a p-forest* r *with* $t_1 + \cdots + t_j = ur$ *and* $s = uwr$, *then* $v^\omega h = v^\omega h'$ *where* $h' = t_{j+1} + \cdots + t_{\mathrm{CP}(h)}$;

G.5 *if for a positive integer* j *with* $1 \le j \le \mathrm{CP}(h)$ *and a nonempty set* $R \subseteq \mathrm{RIrrF}(v)0$ *there is a p-forest* s *which is a sum of, in any order, of elements of* R *such that there are p-contexts* u *and* w *and a p-forest* r *with* $t_j + \cdots + t_{\mathrm{CP}(h)} = ur$ *and* $s = uwr$, *then* $v^\omega h = v^\omega h'$ *where* $h' = t_1 + \cdots + t_{j-1}$;

G.6 *for* $v_n = H_1 + \Box + H_2$ *with* $H_1 = s_1 + \cdots + s_{\mathrm{CP}(H_1)}$ *and* $H_2 = s'_1 + \cdots + s'_{\mathrm{CP}(H_2)}$ *if*

$$s_{\mathrm{CP}(H_1)} \in \mathrm{Special_H}(h) \quad \textit{or} \quad s'_1 \in \mathrm{Special_H}(h),$$

then we have the identities

$$v\omega(h) = v_1 \ldots v_{n-1}(s_1 + \cdots + s_{\mathrm{CP}(H_1)-1} + \omega(h) + H_2)$$

or

$$v\omega(h) = v_1 \ldots v_{n-1}(H_1 + \omega(h) + s'_2 + \cdots + s'_{\mathrm{CP}(H_2)});$$

G.7 *for* $v_n = H_1 + \Box + H_2$ *with* $H_1 = s_1 + \cdots + s_{\mathrm{CP}(H_1)}$ *and* $H_2 = s'_1 + \cdots + s'_{\mathrm{CP}(H_2)}$ *if for a positive integer* j *with*

$$1 \le j \le \mathrm{CP}(H_1) \qquad (1 \le j \le \mathrm{CP}(H_2))$$

and a nonempty set $D \subseteq \mathrm{Special_H}(h)$ *there is a p-forest* p *which is a sum of, in any order, of elements of* D *such that there are p-contexts* u *and* w *and a p-forest* r *with*

$$s_j + \cdots + s_{\mathrm{CP}(H_1)} = ur \quad \textit{or} \quad s'_1 + \cdots + s'_j = ur$$

and $p = uwr$, *then we have the identities*

$$v\omega(h) = v_1 \ldots v_{n-1}(s_1 + \cdots + s_{j-1} + \omega(h) + H_2)$$

or

$$v\omega(h) = v_1 \ldots v_{n-1}(H_1 + \omega(h) + s'_{j+1} + \cdots + s'_{\mathrm{CP}(H_2)});$$

G.8 *if* $\mathrm{Special_H}(h) \subseteq \mathrm{Special_H}(\sum_{x \in \mathrm{LIrrF}(v)0} x)$*, then* $v^\omega \omega(h) = v^\omega 0$*;*

G.9 *if* $\mathrm{Special_H}(h) \subseteq \mathrm{Special_H}(\sum_{x \in \mathrm{RIrrF}(v)0} x)$*, then* $v^\omega \omega(h) = v^\omega 0$*.*

G.10 *if* $\mathrm{Special_H}(h) \subseteq \mathrm{Special_H}(\sum_{x \in \mathrm{LIrrF}(v)0} x)$*, then* $v^\omega h = v^\omega 0$*;*

G.11 *if* $\mathrm{Special_H}(h) \subseteq \mathrm{Special_H}(\sum_{x \in \mathrm{RIrrF}(v)0} x)$*, then* $v^\omega h = v^\omega 0$*.*

Proof. See Appendix A, Section A.3. $\square$

Remark 5.1.3. For an ω-context v^ω, let

$$h_1 = \sum_{h \in \mathrm{LIrrF}(v)0} h \quad \text{and} \quad h_2 = \sum_{h \in \mathrm{RIrrF}(v)0} h.$$

Then we have the following identities:

$$\begin{aligned}
v^\omega &= (h_1 + \prod_{u \in \mathrm{PIrrF}(v)} u + h_2)^\omega \\
&\qquad\qquad\qquad\qquad \text{by I.3} \\
&= ((h_1 + \Box)(\Box + h_2)(\prod_{u \in \mathrm{PIrrF}(v)} u))^\omega \\
&= ((h_1 + \Box)^\omega(\Box + h_2)^\omega(\prod_{u \in \mathrm{PIrrF}(v)} u)^\omega)^\omega \\
&\qquad\qquad\qquad\qquad \text{by (5.1)} \\
&= (((h_1 + \Box)^\omega)^\omega((\Box + h_2)^\omega)^\omega(\prod_{u \in \mathrm{PIrrF}(v)} u)^\omega)^\omega \\
&\qquad\qquad\qquad\qquad \text{by (5.3)} \\
&= ((h_1 + \Box)^\omega(\Box + h_2)^\omega \prod_{u \in \mathrm{PIrrF}(v)} u)^\omega \\
&\qquad\qquad\qquad\qquad \text{by I.3 and I.8} \\
&= ((\omega(h_1) + \Box)(\Box + \omega(h_2)) \prod_{u \in \mathrm{PIrrF}(v)} u)^\omega \\
&= (\omega(h_1) + \prod_{u \in \mathrm{PIrrF}(v)} u + \omega(h_2))^\omega \\
&= \omega(h_1) + (\omega(h_1) + \prod_{u \in \mathrm{PIrrF}(v)} u + \omega(h_2))^\omega + \omega(h_2) \\
&\qquad\qquad\qquad\qquad \text{by I.2.}
\end{aligned}$$

Lemma 5.1.4. *For an* ω*-context* v^ω *with*

$$h_1 = \sum_{h \in \mathrm{LIrrF}(v)0} h \quad \textit{and} \quad h_2 = \sum_{h \in \mathrm{RIrrF}(v)0} h$$

we have the following identities:

1. $v^\omega + \omega(h_2) = \omega(h_1) + v^\omega = v^\omega$;

2. $(v + h_2)^\omega = v^\omega$ *and* $(h_1 + v)^\omega = v^\omega$.

Proof. By Remark 5.1.3, we have the following identity

$$v^\omega = (\omega(h_1) + \prod_{u \in \mathrm{PIrrF}(v)} u + \omega(h_2))^\omega \tag{5.8}$$

$$= \omega(h_1) + (\omega(h_1) + \prod_{u \in \mathrm{PIrrF}(v)} u + \omega(h_2))^\omega + \omega(h_2). \tag{5.9}$$

The identities in 1 follows from (5.9) and I.4, the identities in 2 follows from (5.8), (5.1), and 1. □

Lemma 5.1.5. *We have the following results:*

1. *for every forest* $s \in \mathrm{Special}_{\mathrm{H}}(h)$, $\omega(h) + s = \omega(h)$;

2. *if* $\mathrm{Special}_{\mathrm{H}}(s) \subseteq \mathrm{Special}_{\mathrm{H}}(h)$, *then* $\omega(h) + \omega(s) = \omega(s) + \omega(h) = \omega(h)$;

3. *if* $\mathrm{Special}_{\mathrm{H}}(s) \subseteq \mathrm{Special}_{\mathrm{H}}(h)$, *then* $\omega(h) + s = s + \omega(h) = \omega(h)$.

Proof. 1. By using I.23 and I.6 and I.5 we obtain:

$$\begin{aligned}
\omega(h) + s &= \omega(\sum_{t \in \mathrm{Special}_{\mathrm{H}}(h)} t) + s \\
&= \omega(s + \sum_{t \in \mathrm{Special}_{\mathrm{H}}(h) \setminus \{s\}} t) + s \\
&= \omega(s + \sum_{t \in \mathrm{Special}_{\mathrm{H}}(h) \setminus \{s\}} t) \\
&= \omega(\sum_{t \in \mathrm{Special}_{\mathrm{H}}(h)} t) \\
&= \omega(h).
\end{aligned}$$

2. Assume that the p-forests s and h are such that

$$\mathrm{Special}_{\mathrm{H}}(s) \subseteq \mathrm{Special}_{\mathrm{H}}(h).$$

Then, by I.6, I.23 and (5.5), there is a p-forest h_1 such that

$$\omega(h) = \omega(h_1 + s) = \omega(s + h_1).$$

We have

$$\begin{aligned}
\omega(h) + \omega(s) &= \omega(s + h_1) + \omega(s) && \text{by I.6, I.23 and (5.5)} \\
&= \omega(\omega(s) + \omega(h_1)) + \omega(s) && \text{by (5.5)} \\
&= \omega(\omega(s) + \omega(h_1)) && \text{by I.5} \\
&= \omega(s + h_1) && \text{by (5.5)} \\
&= \omega(h) && \text{by I.6, I.23 and (5.5).}
\end{aligned}$$

We can do the similar arguments for the symmetric case.

3. By using the preceding identities and (5.6) we have

$$\begin{aligned}
\omega(h) + \omega(s) &= \omega(h) + (\omega(s) + s) \\
&= (\omega(h) + \omega(s)) + s \\
&= \omega(h) + s.
\end{aligned}$$

□

The following is the immediate result of Lemma 5.1.5 and (5.5)

Corollary 5.1.6. *If for p-forests s and h we have*

$$\mathrm{Special}_{\mathrm{H}}(s) \subseteq \mathrm{Special}_{\mathrm{H}}(h),$$

then the identity $\omega(h + s) = \omega(h)$ holds. □

Lemma 5.1.7. *Let v be a p-context and consider p-context*

$$u = \prod_{w \in \mathrm{IrrNIdemF}^*(v)} w.$$

Define elements v_1, h_1, h_2, s_l, and s_r as follows:

$$v_1 = \prod_{w \in \mathrm{PIrrF}(u)} w, \qquad h_1 = \sum_{s \in \mathrm{RIrrF}(u)0} s,$$

$$h_2 = \sum_{s \in \mathrm{LIrrF}(u)0} s, \qquad s_r = \sum_{s \in \mathrm{Special}_{\mathrm{H}}(h_1)} s,$$

$$s_l = \sum_{s \in \mathrm{Special}_{\mathrm{H}}(h_2)} s.$$

Then we have the following identity:

$$v^\omega = (v_1(s_l + \Box + s_r))^\omega.$$

Proof. By I.9 and I.3 we have $u^\omega = (v_1.(h_1 + \Box + h_2))^\omega$, while (5.1) implies

$$(v_1.(h_1 + \Box + h_2))^\omega = (v_1^\omega.(h_1 + \Box + h_2)^\omega)^\omega.$$

By properties of ω-algebras we have $(h_1 + \Box + h_2)^\omega = \omega(h_1) + \Box + \omega(h_2)$. By I.21 we have $\omega(h_1) = \omega(s_r)$ and $\omega(h_2) = \omega(s_l)$, which imply that the following identity:

$$(v_1^\omega.(h_1 + \Box + h_2)^\omega)^\omega = (v_1^\omega.(s_r + \Box + s_l)^\omega)^\omega.$$

Hence, (5.1) implies the result. □

Remark 5.1.8. Under the assumptions of Lemma 5.1.7, we have the following results:

1. if $\mathrm{Special_H}(s) \subseteq \mathrm{Special_H}(h_1)$, then $\omega(s) + v^\omega = v^\omega$;
2. if $\mathrm{Special_H}(s) \subseteq \mathrm{Special_H}(h_2)$, then $v^\omega + \omega(s) = v^\omega$;
3. if $\mathrm{Special_H}(s) \subseteq \mathrm{Special_H}(h_1)$, then $s + v^\omega = v^\omega$;
4. if $\mathrm{Special_H}(s) \subseteq \mathrm{Special_H}(h_2)$, then $v^\omega + s = v^\omega$.

which can be easily proved by Lemmas 5.1.5 and 5.1.7.

5.2 Canonical Forms

We define relations $<_H$ and $<_V$ respectively on H^A and V^A over alphabet $A = \{a_1, \dots, a_n\}$, recursively, as follows:

- for every $i \le n$, $0 <_H a_i$, $\square <_V a_i\square$;
- for every $i, j \le n$, $a_i\square <_V a_j\square$ if $i < j$;
- for forests t and s expressed as sums of non-trivial trees $t_1 + \cdots + t_i$ and $s_1 + \cdots + s_j$:

$$s <_H t \quad \text{if} \quad \begin{cases} i < j \\ \quad \text{or} \\ i = j \quad \text{and} \quad \exists k \le i \;\; \forall l < k \;\; t_l = s_l \quad \text{and} \quad t_k <_H s_k; \end{cases}$$

- for trees a_ih and a_jr:

$$a_ih <_H a_jr \quad \text{if} \quad \begin{cases} a_i\square <_V a_j\square \\ \quad \text{or} \\ a_i = a_j \quad \text{and} \quad h <_H r; \end{cases}$$

- for connected contexts a_iv and a_ju:

$$a_iv <_V a_ju \quad \text{if} \quad \begin{cases} a_i\square <_V a_j\square \\ \quad \text{or} \\ a_i = a_j \quad \text{and} \quad v <_V u; \end{cases}$$

- for contexts $v = H_1 + C(v) + H_2$ and $u = S_1 + C(u) + S_2$:

$$v <_V u \quad \text{if} \quad \begin{cases} C(v) <_V C(u) \\ \quad \text{or} \\ C(v) = C(u) \quad \text{and} \quad H_1 <_H S_1 \\ \quad \text{or} \\ H_1 + C(v) = S_1 + C(u) \quad \text{and} \quad H_2 <_H S_2. \end{cases}$$

Lemma 5.2.1. *The relations $<_H$ and $<_V$ are strict total orders respectively on H^A and V^A.*

Proof. We show that, for given forests h_1 and h_2, one of the following holds:

- $h_1 = h_2$;
- $h_1 <_H h_2$;
- $h_2 <_H h_1$.

We argue by induction on the minimum of the number of nodes of h_1 and h_2. We assume that h_1 is the forest which has the minimum of the number of nodes between h_1 and h_2. Let h_1 be a forest with $\#_{\text{Nodes}}(h_1) = 0$, then we have $h_1 = 0$ which implies that $0 <_H h_2$ or $h_2 = 0$. Assume that for a forest h_1 with $\#_{\text{Nodes}}(h_1) \leq k$ the result holds. We show that a forest h_1 with $\#_{\text{Nodes}}(h_1) = k + 1$ the result holds. We may assume that $h_1 \neq h_2$. We have the following three cases:

1. If $\text{CP}(h_1) < \text{CP}(h_2)$, then by definition of $<_H$, $h_1 <_H h_2$.
2. If $\text{CP}(h_2) < \text{CP}(h_1)$, then by definition of $<_H$, $h_2 <_H h_1$.
3. If $\text{CP}(h_1) = \text{CP}(h_2)$, then we have the following two cases:
   (a) If $\text{CP}(h_1) = 1$, then we have $\text{roots}(h_1) <_H \text{roots}(h_2)$ which implies $h_1 <_H h_2$ and vice versa, or there are forests s_1 and s_2, and an element $d \in A$ such that $h_1 = d\Box * s_1$ and $h_2 = d\Box * s_2$, since
   $$\#_{\text{Nodes}}(s_1) = k,$$
   induction hypothesis and definition of $<_H$ imply that $s_1 <_H s_2$ yields to $h_1 <_H h_2$ and vice versa, while $s_1 = s_2$ yields $h_1 = h_2$;
   (b) If $\text{CP}(h_1) = n$, then there are forests
   $$s_1, \ldots, s_n, t_1, \ldots, t_n$$
   such that $h_1 = s_1 + \cdots + s_n$ and $h_2 = t_1 + \cdots + t_n$. Since
   $$\#_{\text{Nodes}}(s_1) \leq k,$$
   induction hypothesis and definition of $<_H$ imply that $s_1 <_H t_1$ yields to $h_1 <_H h_2$ and vice versa, while for $s_1 = t_1$, since
   $$\#_{\text{Nodes}}(s_2 + \cdots + s_n) \leq k,$$
   induction hypothesis and definition of $<_H$ imply that $s_2 + \cdots + s_n <_H t_2 + \cdots + t_n$ yields to $h_1 <_H h_2$ and vice versa, while $s_2 + \cdots + s_n = t_2 + \cdots + t_n$ yields $h_1 = h_2$.

We can do the similar arguments for contexts v_1 and v_2. $\square$

Let $\mathcal{B} = (\mathsf{H}, \mathsf{V})$ be the free ω-algebra over the alphabet $B = \{a_4, \ldots, a_n\}$ with $B \cap \{a, b, c\} = \emptyset$. We define $<'_H$ and $<'_V$ respectively on H and V as follows: for p-contexts u and v and p-forests h and s

$$\begin{aligned} u <'_V v \text{ if } \quad & \Phi(u) <_V \Phi(v) \\ h <'_H s \text{ if } \quad & \Phi(h) <_H \Phi(s), \end{aligned}$$

where $<_H$ and $<_V$ are strict total orders respectively on H^A and V^A, Φ is the ω-algebra homomorphism in Example 4.2.9, and

$$A = \{a_1 = a, a_2 = b, a_3 = c, a_4, \ldots, a_n\}.$$

From now on by order we mean $<'_H$ and $<'_V$.

Lemma 5.2.2. *By using identities in Σ, for an ω-context u^ω there is a p-context v such that $u^\omega = v^\omega$ where v admits a factorization $v_1 v_2$ such that v_1 is a product of $a_i\square$ with $a_i\square \in A'$ in increasing order and v_2 is of the form $H_1 + \square + H_2$, where H_1 and H_2 are sums of non-trivial additively irreducible forests in increasing order and no summand of H_1 and H_2 is an ω-forest.*

Proof. By Lemma 5.1.2, part I.9, for the p-context

$$w = \prod_{v \in \mathrm{IrrNIdemF}^*(u)} v,$$

we have $u^\omega = w^\omega$. By definition of $\mathrm{IrrNIdemF}^*(u)$, the p-context w does not have ω-context factors. Again by Lemma 5.1.2, part I.3, there is a p-context z, with $w^\omega = z^\omega$, of the form $v_1 v_2 v_3$ where

$$v_1 = \prod_{x \in \mathrm{PIrrF}(w)} x \ , \qquad \text{in increasing order,}$$

$$v_2 = \prod_{y \in \mathrm{LIrrF}(w)} y \ , \qquad \text{in increasing order}$$

and

$$v_3 = \prod_{y' \in \mathrm{RIrrF}(w)} y' \ , \qquad \text{in increasing order.}$$

Since v_1 is product of non-trivial $\square$-pure multiplicatively irreducible factors of w and w does not have ω-context factors, v_1 is the product of some $a_i\square$ with $a_i \in A$. By definition of RIrrF and LIrrF there are p-forests S_1 and S_2 such that

$$\prod_{y \in \mathrm{LIrrF}(w)} y = S_1 + \square \quad \text{and} \quad \prod_{y' \in \mathrm{RIrrF}(w)} y' = \square + S_2.$$

We claim that S_1 and S_2 do not have ω-forest summands. If $\omega(p)$ is an ω-forest summand of S_1, then $\omega(p) + \Box$ is a factor of $S_1 + \Box$ and also a factor of w, which contradicts with the assumption that factors of w are not ω-contexts, and similarly for S_2. $\Box$

The p-context v^ω in Lemma 5.2.2 is called the *ordered form context* of u^ω.

For a given ω-forest $\omega(h)$, by Lemma 5.1.2, part I.21, for the p-forest

$$s = \sum_{x \in \mathrm{IrrNIdemS}^*(h)} x,$$

we have $\omega(h) = \omega(s)$. Again by Lemma 5.1.2, part I.6, there is a p-forest r, with $\omega(s) = \omega(r)$, where

$$r = \sum_{x \in \mathrm{IrrS}(s)} x \quad , \qquad \text{in increasing order.}$$

The ω-forest $\omega(r)$ is called the *ordered form forest* of $\omega(h)$.

Definition 5.2.3. Assume that $h = h_1 + \cdots + h_n$ is a p-forest decomposed as the sum of its non-trivial additively irreducible summands. We denote by $h^{(i)}$ the p-forest which is obtained from h by elimination of its i-th summand. That is,

$$\begin{aligned} h^{(i)} &= h_1 + \cdots + \widehat{h_i} + \cdots + h_n \\ &= h_1 \cdots + h_{i-1} + h_{i+1} + \cdots + h_n. \end{aligned}$$

Definition 5.2.4. Let v be a p-context in the free ω-algebra $\mathcal{A}$. Then, by Lemma 4.2.4, we have $v = v_1 \cdots v_n$ where the v_i's are non-trivial multiplicatively irreducible factors of v. For a positive integer k, we say that $NLex_V(v_1, \dots, v_n)$ is k if there are positive integers

$$i_1, \dots, i_k, i_{k+1} \in \{1, \dots, n+1\},$$

such that the following conditions hold:

- $i_1 = 1$ and $i_{k+1} = n + 1$;
- for every $j \in \{1, \dots, k\}$ we have $i_j < i_{j+1}$;
- for every $j \in \{1, \dots, k\}$ and every $t \in \{i_j, \dots, i_{j+1} - 2\}$ we have $v_t <'_V v_{t+1}$;
- for every $j \in \{2, \dots, k-1\}$ we have $v_{i_j} <'_V v_{i_j - 1}$.

Note that, for non-trivial additively irreducible p-forests s and t, the equality $(s + \Box).(\Box + t) = (\Box + t).(s + \Box)$ holds. And also we have:

$$NLex_V((s + \Box), (\Box + t)) = 1 \quad \text{and} \quad NLex_V((\Box + t), (s + \Box)) = 2.$$

Definition 5.2.5. Let v be a p-context in $\mathcal{A}$ and let the v_i's be its non-trivial multiplicatively irreducible factors of v. Define

$$VLex(v) = \min\{k \in \mathbb{N} \mid v = \prod_{i=1}^{n} v_i, \quad k = NLex_V(v_1, \ldots, v_n)\}.$$

Definition 5.2.6. Let h be a p-forest in the free ω-algebra $\mathcal{A}$. Then, by Lemma 4.2.5, we have $h = h_1 + \cdots + h_m$ where the h_i's are non-trivial additively irreducible summands of h. For a positive integer k, we say that $HLex(h_1 + \cdots + h_m)$ is k if there are positive integers

$$i_1, \ldots, i_k, i_{k+1} \in \{1, \ldots, m+1\},$$

such that the following conditions hold:

- $i_1 = 1$ and $i_{k+1} = n+1$;
- for every $j \in \{1, \ldots, k\}$ we have $i_j < i_{j+1}$;
- for every $j \in \{1, \ldots, k\}$ and every $t \in \{i_j, \ldots, i_{j+1}-2\}$ we have $h_t <'_H h_{t+1}$;
- for every $j \in \{2, \ldots, k-1\}$ we have $h_{i_j} <'_H h_{i_j-1}$.

Note that, the number of idempotent subterms of a given element of $\mathcal{A}$ is finite.

Let $P = (H_P, V_P)$ be an ω-algebra and for every $u, v \in V_P$ and $h \in H_P$ the set of identities Σ, consisting of the following identities, hold in P.

$$\begin{aligned}
&(uv)^\omega = (vu)^\omega = (u^\omega v^\omega)^\omega \\
&v^\omega v = v^\omega = v v^\omega \\
&(v^\omega)^\omega = v^\omega \\
\\
&vh + \omega(vuh) = \omega(vuh) = \omega(vuh) + vh
\end{aligned}$$

Definition 5.2.7. Let t_1 and t_2 be two elements with the same type in $\mathcal{A}$. We say that t_1 and t_2 are connected and we denote it by $t_1 \sim_\Sigma t_2$, if there exists a finite sequence of elements called *connecting sequences* $S_0, \ldots, S_n$ in $\mathcal{A}$, all have the same type as the type that t_1 and t_2 have, such that $S_0 = t_1$, $S_n = t_2$ and for all $i \in \{1, \ldots, n\}$ there are subterms X and Y of respectively S_{i-1} and S_i, that is:

$$S_{i-1} = f_i(X; U_{i,1}, \ldots, U_{i,m}) \quad \text{and} \quad S_i = f_i(Y; U_{i,1}, \ldots, U_{i,m}),$$

for some elements $U_{i,1}, \ldots, U_{i,m}$ in $\mathcal{A}$ and f_i is an n-ary operation which is a composition of operations from $\{+, +_1, +_2, ., *, \omega(), ()^\omega\}$, and there exists an ω-algebra homomorphism:

$$\varphi : \mathcal{A} \to P,$$

such that $\varphi(X) = u$ and $\varphi(Y) = v$, or $\varphi(Y) = u$ and $\varphi(X) = v$ and the identity $u = v$ is in Σ.

The result [1, Proposition 1.3.6] justify the following definition:

Definition 5.2.8. Let $u = v$ be an identity of type τ on A' (cf. Section 4.1). The *congruence defined by the identity* $u = v$ is given by

$$\bigcap\{\, \ker(\varphi) \mid \varphi : \mathcal{A} \to S \text{ is an } \omega\text{-algebra homomorphism with } S \in [\![u = v]\!] \,\},$$

where $[\![u = v]\!]$ is the class of ω-algebras of type τ satisfying the identity $u = v$.

For more details about congruences see [1, pp. 24-31].

Since $[\![u = v]\!]$ is equational, by Birkhoff Theorem [1, Theorem 1.3.8], $[\![u = v]\!]$ is a variety of ω-algebras of type τ.

Lemma 5.2.9. *The relation $\sim_{u=v}$ is the congruence on $\mathcal{A} = (\mathsf{H}, \mathsf{V})$ defined by the identity $u = v$.*

Proof. The relation $\sim_{u=v}$ is an equivalence relation.

For a p-forest h in H, the relation $h \sim_{u=v} h$ holds. Because, let $n = 0$ and $S_0 = h$ then the result is immediate.

Assume that, the relation $h \sim_{u=v} t$ holds. So, the connecting sequence of p-forests $S_0, \ldots, S_n$ in H exists. We show that the relation $t \sim_{u=v} h$ holds. For a connecting sequence in H, we can choose $S'_0, \ldots, S'_n$ in H such that, for every $i \in \{0, \ldots, n\}$, $S'_i = S_{n-i}$. The required properties hold in view of the assumption $h \sim_{u=v} t$. So, the relation $t \sim_{u=v} h$ holds.

Now, assume that, $h \sim_{u=v} t$ and $t \sim_{u=v} r$ hold. So, the connecting sequences of p-forests $S_0, \ldots, S_n$ and $S'_0, \ldots, S'_m$ exist with $S_0 = h$, $S_n = t$, $S'_0 = t$, and $S'_m = r$. We just need to take the sequence

$$Q_0, \ldots, Q_n, Q_{n+1}, \ldots, Q_{m+n}$$

such that $Q_i = S_i$ for all $i \in \{0, \ldots, n\}$ and $Q_{n+i} = S'_i$ for all $i \in 0, \ldots, m$. The required properties hold in view of the assumptions $h \sim_{u=v} t$ and $t \sim_{u=v} r$. So, the relation $h \sim_{u=v} r$ holds.

This shows that $\sim_{u=v}$ is an equivalence relation on H. In a similar way the relation $\sim_{u=v}$ is an equivalence relation on V.

To show that $\sim_{u=v}$ is a congruence, assume that $x \sim_{u=v} y$ and $p \sim_{u=v} q$, then we need to show for the basic operations, we have $O(x, p) \sim_{u=v} O(y, q)$.

It is easy to see that for p-forests x and y if $x \sim_{u=v} y$, then $\omega(x) \sim_{u=v} \omega(y)$. And for p-contexts t and z if $t \sim_{u=v} z$, then $t^\omega \sim_{u=v} z^\omega$. It is because x is a subterm of $\omega(x)$ and t is a subterm of t^ω.

Since the relation $x \sim_{u=v} y$ holds, there is a connecting sequence

$$S_0, \ldots, S_n.$$

Since S_i's, have the same type, the sequence

$$O(S_0, p), \ldots, O(S_n, p),$$

is from the element $O(x, p)$ to the element $O(y, p)$. With respect to the type of x and y (they should have the same type) and basic operation O, with respect to the type of p, we have the following:

1. If the operation is addition, then let

   $$f_i'(q; U_{i,1}, \ldots, U_{i,m}, p) = f_i(q; U_{i,1}, \ldots, U_{i,m}) + p,$$

   and put the sequence of elements $S_0', \ldots, S_n'$ in $\mathcal{A}$ with $S_0' = x + p$, $S_n' = y + p$ and for every $i \in \{1, \ldots, n\}$,

   $$S_{i-1}' = f_i'(X; U_{i,1}, \ldots, U_{i,m}, p)$$

   and

   $$S_i' = f_i'(Y; U_{i,1}, \ldots, U_{i,m}, p)$$

   then the required properties hold in view of the assumption $x \sim_{u=v} y$. Similarly, we have $p + x \sim_{u=v} p + y$.

2. If the operation is multiplication, then let

   $$f_i'(q; U_{i,1}, \ldots, U_{i,m}, p) = f_i(q; U_{i,1}, \ldots, U_{i,m}).p,$$

   and put the sequence of elements $S_0', \ldots, S_n'$ in $\mathcal{A}$ with $S_0' = x.p$, $S_n' = y.p$ and for every $i \in \{1, \ldots, n\}$,

   $$S_{i-1}' = f_i'(X; U_{i,1}, \ldots, U_{i,m}, p)$$

   and

   $$S_i' = f_i'(Y; U_{i,1}, \ldots, U_{i,m}, p)$$

   then the required properties hold in view of the assumption $x \sim_{u=v} y$. Similarly, we have $p.x \sim_{u=v} p.y$.

3. Assume that, x and y are p-contexts, p is a p-forest and the operation is action. Then let

   $$f_i'(q; U_{i,1}, \ldots, U_{i,m}, p) = f_i(q; U_{i,1}, \ldots, U_{i,m}) * p,$$

   and put the sequence of elements $S_0', \ldots, S_n'$ in $\mathcal{A}$ with $S_0' = x * p$, $S_n' = y * p$ and for every $i \in \{1, \ldots, n\}$,

   $$S_{i-1}' = f_i'(X; U_{i,1}, \ldots, U_{i,m}, p)$$

   and

   $$S_i' = f_i'(Y; U_{i,1}, \ldots, U_{i,m}, p)$$

   then the required properties hold in view of the assumption $x \sim_{u=v} y$.

4. Assume that, x and y are p-forests, p is a p-context and the operation is action. Then let

$$f'_i(q; U_{i,1}, \ldots, U_{i,m}, p) = p * f_i(q; U_{i,1}, \ldots, U_{i,m}),$$

and put the sequence of elements $S'_0, \ldots, S'_n$ in $\mathcal{A}$ with $S'_0 = p * x$, $S'_n = p * y$ and for every $i \in \{1, \ldots, n\}$,

$$S'_{i-1} = f'_i(X; U_{i,1}, \ldots, U_{i,m}, p)$$

and

$$S'_i = f'_i(Y; U_{i,1}, \ldots, U_{i,m}, p)$$

then the required properties hold in view of the assumption $x \sim_{u=v} y$.

For the relation $x \sim_{u=v} y$ and an element $p \in \mathcal{A}$, we have shown that

$$O(x,p) \sim_{u=v} O(y,p)$$

(similarly, $O(p,x) \sim_{u=v} O(p,y)$). Indeed, if $x \sim_{u=v} y$ and $p \sim_{u=v} q$, then by what we have shown we have

$$O(x,p) \sim_{u=v} O(y,p) \sim_{u=v} O(y,q).$$

Thus, we have shown that the relation $\sim_{u=v}$ is a congruence on A^{Δ}.

The relation $\sim_{u=v}$ is the congruence defined by the identity $u = v$. Because, if θ is the congruence defined by the identity $u = v$, then $x\theta y$ implies $x \sim_{u=v} y$. To show this, take the sequence $S_0 = x$ and $S_1 = y$ the existence of ω-algebra homomorphism comes from the congruence $x\theta y$. $\square$

Let $\mathcal{A}$ and $\mathcal{B}$ be respectively the A' and $A' \uplus \{\mathfrak{d}\square\}$ free ω-algebras, where $\mathfrak{d} \notin \{a, b, c\}$ is fixed and

$$\{a\square, b\square, c\square\} \cap A' = \emptyset.$$

Let h be a fixed p-forest of $\mathcal{A}$ and let v be a fixed p-context of $\mathcal{A}$. Let $\sim_{\omega(h)=\mathfrak{d}}$ and $\sim_{v^\omega=\mathfrak{d}\square}$ be the congruence defined by the identity respectively $\omega(h) = \mathfrak{d}$ and $v^\omega = \mathfrak{d}\square$ (see Definition 5.2.8). Note that, there is the inclusion map

$$\iota : \mathcal{A} \to \mathcal{B}.$$

For an element $x \in \mathcal{A}$ with $\omega(h) \in \mathrm{IST}(x)$ (cf. Example 4.2.27), the *multiplicity of* $\omega(h)$ in x is defined as follows:

$$m_x(\omega(h)) = \max\{\mathrm{length}(\mathrm{traversal}(\Phi(z))|_{\mathfrak{d}}) \mid z \in \iota(x)/\sim_{\omega(h)=\mathfrak{d}}\},$$

where length of a word is the number of its letters.

Similarly, for an element $x \in \mathcal{A}$ with $v^\omega \in \mathrm{IST}(x)$, the *multiplicity of* v^ω in x is defined as follows:

$$m_x(v^\omega) = \max\{\text{length}(\text{traversal}(\Phi(z))|_{\mathfrak{d}}) \mid z \in \iota(x)/\sim_{v^\omega=\mathfrak{d}\Box}\}.$$

Let t be an element of $\mathcal{A}$, $\omega(t_1), \ldots, \omega(t_{n'_1})$ be the ω-forest subterms of t, and $t_1'^\omega, \ldots, t_{n'_2}'^\omega$ be the ω-context subterms of t with $C(t'_i) \neq \Box$, given by $\mathrm{IST}(t)$. Let $n = \#_{\mathrm{IDEM}}(t)$ (cf. Example 4.2.24). Note that, the equality $n = n_1 + n_2$ holds if and only if all the idempotent subterms have multiplicity 1. Consider a sequence of elements of $\mathcal{A}$, $\{Q_j\}_{j=1}^{n'_1+n'_2}$ giving an ordering of t_i and t'_k by decreasing order of rank. For $q \leq n'_1 + n'_2$, let $m_q = m_t(E_1) + \cdots + m_t(E_q)$ and $m_0 = 0$, where for every i, E_i is $\omega(Q_i)$ or Q_i^ω, if defined. Define the sequence of elements of $\mathcal{A}$, $\{P_c\}_{c=1}^n$ as follows:

$$P_{m_{q-1}+1} = \cdots = P_{m_q} = Q_q.$$

Let $M_t = (m'_1, \ldots, m'_n)$ be n-tuple of natural numbers whose entries are respectively $HLex(P_i)$ or $VLex(P_i)$ in case P_i is a p-forest or a p-context and, for every i, if $\mathrm{Rank}(P_i) = \mathrm{Rank}(P_{i+1})$ (cf. Section 4.2.1.1), then their respective entries m'_i and m'_{i+1} satisfy the inequality $m'_i \geq m'_{i+1}$. We denote M_t by $LLex_H(t)$ or $LLex_V(t)$ respectively, if t is a p-forest or a p-context.

Note that, for given p-forests h and s, and for p-contexts u and v we can compare the n-tuple $LLex_H(s)$ and the m-tuple $LLex_H(h)$, and also the n'-tuple $LLex_V(v)$ and the m'-tuple $LLex_V(u)$ as follows: $LLex_H(s) \leq LLex_H(h)$ if one of the following conditions holds:

- $n < m$;
- $n = m$ and for

  $$LLex_H(s) = (q_1, \ldots, q_n) \quad \text{and} \quad LLex_H(h) = (z_1, \ldots, z_n),$$

  there is a positive integer $1 \leq j \leq n$ such that the following conditions hold:

  $$\begin{cases} z_i = q_i & , \forall i > j \\ q_j < z_j. \end{cases}$$

and similarly, $LLex_V(v) \leq LLex_V(u)$ if one of the following conditions holds:

- $n' < m'$;
- $n' = m'$ and for

  $$LLex_V(v) = (q'_1, \ldots, q'_{n'}) \quad \text{and} \quad LLex_V(u) = (z'_1, \ldots, z'_{n'}),$$

  there is a positive integer $1 \leq j \leq n'$ such that the following conditions hold:

  $$\begin{cases} z'_i = q'_i & , \forall i > j \\ q'_j < z'_j. \end{cases}$$

That is, we can compare $LLex_H(s)$ and $LLex_H(h)$, and also $LLex_V(v)$ and $LLex_V(u)$ by *reverse-lexicographic order*, i.e., the right-most component is the most significant.

Note that for a p-forest t if the summands of t are in increasing order and for a p-context v if v is the product of its non-trivial multiplicatively irreducible factors in increasing order, then we have the following equalities:

$$VLex(v) = 1 \quad \text{and} \quad HLex(t) = 1.$$

Remark 5.2.10. By definition of ordered form forest and ordered form context, if we substitute an ω-context or ω-forest subterm, v^ω or $\omega(h)$, of an element t in $\mathcal{A}$ by respectively its ordered form ω-context or ω-forest, then we may reduce at least one of $\#_{\text{IDEM}}(t)$, $\#_{\text{Nodes}}(t)$, and $LLex_H(t)$ or $LLex_V(t)$ if t is respectively a p-context or a p-forest.

5.2.1 Algorithm of Canonical Form

Let t be an element of $\mathcal{A}$. For every $i = 0, \dots, \text{Rank}(t) - 1$ define $O(i)$ as follows:

O.1 substitute an ω-forest subterm $\omega(h)$ of t with $\text{Rank}(h) = i$ by its ordered form forest, if it is not in ordered form;

O.2 substitute an ω-context subterm v^ω of t with $\text{Rank}(v) = i$ by its ordered form context, if it is not in ordered form.

By applying the rules O.1 and O.2 of $O(i)$ on t we may reduce $\#_{\text{IDEM}}(t)$, $\#_{\text{Nodes}}(t)$, or one of $LLex_V(t)$ and $LLex_H(t)$ which the last two depends on the type of t.

Let $t_1 = t, t_2, \dots, t_n$ be the sequence of elements such that for each j, t_{j+1} is obtained from t_j by applying one of the rules O.1 or O.2. For every j in the step from t_j to t_{j+1} at least one of $\#_{\text{IDEM}}(t_j)$, $\#_{\text{Nodes}}(t_j)$, or $LLex_H(t_j)$ or $LLex_V(t_j)$ is reduced.

Assume that, from t_j to t_{j+1} the following equality holds:

$$\#_{\text{Nodes}}(t_j) = \#_{\text{Nodes}}(t_{j+1}).$$

Note that if we reduce $\#_{\text{IDEM}}(t_j)$, then, by definition of $LLex_H$ and $LLex_V$, it implies that $LLex_H(t_j)$ or $LLex_V(t_j)$ is reduced respectively when t_j is a p-forest or a p-context.

In this step one of the rules O.1 and O.2 is applied on an ω-context or an ω-forest subterm of t_j which is v^ω or $\omega(h)$. If v has an ω-context factor or h has an ω-forest summand, then by applying the rules O.1 and O.2 we reduce $\#_{\text{IDEM}}(t_j)$ and therefore we reduce $LLex_H(t_j)$ or $LLex_V(t_j)$ respectively when t_j is a p-forest or a p-context.

Assume that, v does not have an ω-context factor and h does not have an ω-forest summand. So, from t_j to t_{j+1} the following equalities hold:

$$\#_{\text{Nodes}}(t_j) = \#_{\text{Nodes}}(t_{j+1})$$
$$\#_{\text{IDEM}}(t_i) = \#_{\text{IDEM}}(t_{i+1}),$$

which implies that $VLex(v) > 1$ and $HLex(h) > 1$. So, by applying the rules O.1 and O.2 on the ω-context v^ω or on the ω-forest $\omega(h)$ subterm of t_j we may reduce $LLex_H(t_j)$ or $LLex_V(t_j)$ respectively when t_j is a p-forest or a p-context.

Now, assume that from t_j to t_{j+1} the following equality holds,

$$LLex_V(t_j) = LLex_V(t_{j+1}) \quad \text{or} \qquad LLex_H(t_j) = LLex_H(t_{j+1}).$$

This implies that $\#_{\text{IDEM}}(t_j) = \#_{\text{IDEM}}(t_{j+1})$. Then in this step we applied one of the rules O.1 and O.2 on a subterm of t_j which is of the form v^ω or $\omega(h)$, where $v = v_1.(H_1+\square+H_2)$ with $v_1 = \prod_{x\in\text{PIrrF}(v)} x$, $H_1 = h_1+\cdots+h_m$ and $H_2 = h'_1+\cdots+h'_{m'}$ in which all are in increasing order or $h = s_1+\cdots+s_{n'}$ and is in increasing order. Since we could apply the rules O.1 and O.2, this implies that we have repetitions of a factor or repetitions of a summand of a subterm of t_j and so we reduced $\#_{\text{Nodes}}(t_j)$.

Since for the element t in $\mathcal{A}$ we have $\#_{\text{IDEM}}(t)$, $\#_{\text{Nodes}}(t)$, and $LLex_H(t)$ or $LLex_V(t)$, respectively when t is a p-forest or a p-context, are finite, we just can apply this reduction rules finitely many times.

Assume that, an element t in $\mathcal{A}$ is given. Note that the rules O.1 and O.2 just will be applied on the ω-context and ω-forest subterms. Applying the rules O.1 and O.2 on two disjoint subterms will not collide with each other.

The rule O.1 can be applied on an ω-forest subterm $\omega(h)$ with $\text{Rank}(h) = i$. It implies that h does not have an ω-context subterm u^ω or an ω-forest subterm $\omega(s)$ with

$$\text{Rank}(u) = i \quad \text{and} \quad \text{Rank}(s) = i.$$

This means that we can not apply the rules O.1 and O.2 on the subterms of $\omega(h)$.

Also, the rule O.2 can be applied on an ω-context subterm v^ω with $\text{Rank}(v) = i$. It implies that v does not have an ω-context subterm u^ω or an ω-forest subterm $\omega(s)$ with

$$\text{Rank}(u) = i \quad \text{and} \quad \text{Rank}(s) = i.$$

This means that we can not apply the rules O.1 and O.2 on the subterms of v^ω.

Assume that, for every j with $j \leq i$ we applied $O(j)$ on t. By applying the rule O.2 of $O(i+1)$ on an ω-context subterm v^ω of t we do the following:

let $v = P_0 u_1^\omega P_1 \cdots u_n^\omega P_n$ where P_k's are p-contexts which does not have an ω-context factor. Let $w = P_0 u_1 P_1 \cdots u_n P_n$ then we have $\mathrm{Rank}(v) - 1 \leq \mathrm{Rank}(w) \leq \mathrm{Rank}(v)$. Let w' be the p-context which is the product of non-trivial multiplicatively irreducible factor of w in increasing order.

Note that, as for every j with $j \leq i$ we applied $O(j)$ on t, the u_k's do not have an ω-context factor.

Then w'^ω is the ordered form context of v^ω and we have

$$\mathrm{Rank}(v) - 1 \leq \mathrm{Rank}(w') \leq \mathrm{Rank}(v).$$

Similarly, for the rule O.1 on an ω-forest subterm $\omega(h)$ of t we have

$$\mathrm{Rank}(h) - 1 \leq \mathrm{Rank}(s') \leq \mathrm{Rank}(h),$$

where $\omega(s')$ is the ordered form forest of $\omega(h)$. This shows that by applying O.1 and O.2 on an ω-context and an ω-forest subterm of t the rank of that subterm will be reduced by 1 or the rank will be the same.

We may apply the rule O.1 and then the rule O.1 on the subterm $\omega(H_1 + \omega(S_1 + \omega(s) + S_2) + H_2)$ then we should first apply O.1 on $\omega(S_1 + \omega(s) + S_2)$ to get the result $\omega(s')$ and then exactly on some steps after that we can apply the rule O.1 on $\omega(H_1 + \omega(s') + H_2)$. The order of applying these rules can not be changed. Similarly, we may apply the rule O.2 and then the rule O.2 on the subterm $(P_1(Q_1 w^\omega Q_2)^\omega P_2)^\omega$ then we should first apply O.2 on $(Q_1 w^\omega Q_2)^\omega$ to get the result u^ω and then exactly on some steps after that we can apply the rule O.2 on $(P_1 u^\omega P_2)^\omega$. The order of applying these rules can not be changed.

This shows that if for $i = 0, \ldots, \mathrm{Rank}(t) - 1$ consecutive $O(i)$ is applied, then by applying the rules O.1 and O.2 of $O(i)$ in any order on an arbitrary element t the result will be unique.

Let t be an element of $\mathcal{A}$ where t is the result of applying consecutive $O(j)$. For every $i = 0, \ldots, \mathrm{Rank}(t) - 1$ define $S(i)$ as follows:

S.1 substitute an ω-forest subterm $\omega(s)$ of t with $\mathrm{Rank}(s) = i$ and where $s = s_1 + \cdots + s_n$ is in increasing order by $\omega(s^{(j)})$ where $s^{(j)} = s_1 + \cdots + \widehat{s_j} + \cdots + s_n$ if $s_j \in \mathrm{Special}_\mathrm{H}(s^{(j)})$;

S.2 substitute an ω-context subterm v^ω of t with $\mathrm{Rank}(v) = i$ and where $v = v_1.(H_1 + \Box + H_2)$ with $v_1 = \prod_{x \in \mathrm{PIrrF}(v)} x$, $H_1 = h_1 + \cdots + h_m$ and $H_2 = h'_1 + \cdots + h'_{m'}$ and all are in increasing order by $(v^{l,(j)})^\omega$ where $v^{l,(j)} = v_1.(H_1^{(j)} + \Box + H_2)$ if $h_j \in \mathrm{Special}_\mathrm{H}(H_1^{(j)})$;

S.3 substitute an ω-context subterm v^ω of t with $\mathrm{Rank}(v) = i$ and where $v = v_1.(H_1 + \Box + H_2)$ with $v_1 = \prod_{x \in \mathrm{PIrrF}(v)} x$, $H_1 = h_1 + \cdots + h_m$ and $H_2 = h'_1 + \cdots + h'_{m'}$ and all are in increasing order by $(v^{r,(j)})^\omega$ where $v^{r,(j)} = v_1.(H_1 + \Box + H_2^{(j)})$ if $h'_j \in \mathrm{Special}_\mathrm{H}(H_2^{(j)})$.

Note that, in $S(i)$ if the rules S.1, S.2 or S.3 is applied on an ω-context or an ω-forest subterm, then the result is still in order since we just remove a forest summand and this will not change the order.

By applying the rules S.1, S.2 and S.3 of $S(i)$ on a given element t in $\mathcal{A}$ where t is the result of applying consecutive $O(j)$ the number of nodes of t will be reduced. And since $\#_{\text{Nodes}}(t)$ and $\#_{\text{IDEM}}(t)$ are finite, we just can apply this reduction rules finitely many times.

Assume that, an element t in $\mathcal{A}$ is given, where t is the result of applying consecutive $O(j)$. The rules S.1, S.2 and S.3 of $S(i)$ just will be applied on ω-context and ω-forest subterms. Applying the rules S.1, S.2 and S.3 on two disjoint subterms will not collide with each other.

The rule S.1 can be applied on an ω-forest subterm $\omega(h)$ with $\text{Rank}(h) = i$. It implies that h does not have an ω-context subterm u^ω or an ω-forest subterm $\omega(s)$ with

$$\text{Rank}(u) = i \quad \text{and} \quad \text{Rank}(s) = i.$$

This means if the rule S.1 is applied on $\omega(h)$, then we can not apply the rules S.1, S.2 or S.3 on the subterms of h.

Also, the rules S.2 and S.3 can be applied on an ω-context subterm v^ω with $\text{Rank}(v) = i$. It implies that v does not have an ω-context subterm u^ω or an ω-forest subterm $\omega(s)$ with

$$\text{Rank}(u) = i \quad \text{and} \quad \text{Rank}(s) = i.$$

This means that if the rules S.2 or S.3 is applied on v^ω, then we can not apply the rules S.1, S.2 and S.3 on subterms of v.

Applying the rules S.2 and S.3 on an ω-context subterm v^ω will not collide with each other.

If we can apply the rule S.1 on an ω-forest subterm $\omega(h)$ and again applying the rule S.1 on it, then this mean that we first eliminate the summand s_j and then after that eliminate the summand s_k, and it does not matter we eliminate which one first. If we apply the rule S.2 on the ω-context subterm v^ω and again the rule S.2 on it or similarly if we apply the rule S.3 on an ω-context subterm v^ω and again the rule S.3 on it, then this mean that we first eliminate the summand h_j and then after that eliminate the summand h_k in H_1 or similarly we first eliminate the summand h'_j and then after that eliminate the summand h'_k in H_2 where from the result of applying consecutive $O(j)$ we have $v = v_1(H_1 + \Box + H_2)$, it does not matter we eliminate which one first.

So, for $i = 0, \ldots, \text{Rank}(t) - 1$ if we apply consecutive $S(i)$ then by applying the rules S.1, S.2 and S.2 of $S(i)$, in any order, on an arbitrary element t where t is the result of applying consecutive $O(j)$, the result will be unique.

Let t be an element of $\mathcal{A}$ where t is the result of applying consecutive $S(j)$. For every $i = 0, \ldots, \text{Rank}(t) - 1$ define $R(i)$ as follows:

R.1 substitute a forest subterm of t which is of the form $s + \omega(h)$ with $\mathrm{Rank}(h) = i$ by $\omega(h)$ if $s \in \mathrm{Special}_{\mathrm{H}}(h)$;

R.2 substitute a forest subterm of t which is of the form $\omega(h) + s$ with $\mathrm{Rank}(h) = i$ by $\omega(h)$ if $s \in \mathrm{Special}_{\mathrm{H}}(h)$;

R.3 substitute a forest subterm of t which is of the form $\omega(s) + \omega(h)$ with $\mathrm{Rank}(h) = i$ by $\omega(h)$ if $\mathrm{Special}_{\mathrm{H}}(s) \subseteq \mathrm{Special}_{\mathrm{H}}(h)$;

R.4 substitute a forest subterm of t which is of the form $\omega(h) + \omega(s)$ with $\mathrm{Rank}(h) = i$ by $\omega(h)$ if $\mathrm{Special}_{\mathrm{H}}(s) \subseteq \mathrm{Special}_{\mathrm{H}}(h)$;

R.5 substitute a context subterm of t which is of the form $a\Box.v^{\omega}$ with $\mathrm{Rank}(v) = i$ by v^{ω} if $a\Box \in \mathrm{PIrrF}(v)$;

R.6 substitute a context subterm of t which is of the form $v^{\omega}.a\Box$ with $\mathrm{Rank}(v) = i$ by v^{ω} if $a\Box \in \mathrm{PIrrF}(v)$;

R.7 substitute a context subterm of t which is of the form $h + v^{\omega}$ with $\mathrm{Rank}(v) = i$ by v^{ω} if $h \in \mathrm{Special}_{\mathrm{H}}(\sum_{y \in \mathrm{LIrrF}(v)0} y)$;

R.8 substitute a context subterm of t which is of the form $v^{\omega}.(h + \Box)$ with $\mathrm{Rank}(v) = i$ by v^{ω} if $h \in \mathrm{Special}_{\mathrm{H}}(\sum_{y \in \mathrm{LIrrF}(v)0} y)$;

R.9 substitute a context subterm of t which is of the form $v^{\omega} + h$ with $\mathrm{Rank}(v) = i$ by v^{ω} if $h \in \mathrm{Special}_{\mathrm{H}}(\sum_{y \in \mathrm{RIrrF}(v)0} y)$;

R.10 substitute a context subterm of t which is of the form $v^{\omega}.(\Box + h)$ with $\mathrm{Rank}(v) = i$ by v^{ω} if $h \in \mathrm{Special}_{\mathrm{H}}(\sum_{y \in \mathrm{RIrrF}(v)0} y)$;

R.11 substitute a context subterm of t which is of the form $u^{\omega}v^{\omega}$ with $\mathrm{Rank}(v) = i$ by v^{ω} if the following conditions satisfy:

(a) $\mathrm{PIrrF}(u) \subseteq \mathrm{PIrrF}(v)$;
(b) $\mathrm{Special}_{\mathrm{H}}(\sum_{x \in \mathrm{LIrrF}(u)0} x) \subseteq \mathrm{Special}_{\mathrm{H}}(\sum_{y \in \mathrm{LIrrF}(v)0} y)$;
(c) $\mathrm{Special}_{\mathrm{H}}(\sum_{x \in \mathrm{RIrrF}(u)0} x) \subseteq \mathrm{Special}_{\mathrm{H}}(\sum_{y \in \mathrm{RIrrF}(v)0} y)$.

R.12 substitute a context subterm of t which is of the form $v^{\omega}u^{\omega}$ with $\mathrm{Rank}(v) = i$ by v^{ω} if the following conditions satisfy:

(a) $\mathrm{PIrrF}(u) \subseteq \mathrm{PIrrF}(v)$;
(b) $\mathrm{Special}_{\mathrm{H}}(\sum_{x \in \mathrm{LIrrF}(u)0} x) \subseteq \mathrm{Special}_{\mathrm{H}}(\sum_{y \in \mathrm{LIrrF}(v)0} y)$;
(c) $\mathrm{Special}_{\mathrm{H}}(\sum_{x \in \mathrm{RIrrF}(u)0} x) \subseteq \mathrm{Special}_{\mathrm{H}}(\sum_{y \in \mathrm{RIrrF}(v)0} y)$.

By applying the rules R.1-R.12 of $R(i)$ on a given element t in $\mathcal{A}$ where t is the result of applying consecutive $S(j)$ the number of nodes of t will be reduced. And since $\#_{\mathrm{Nodes}}(t)$ and $\#_{\mathrm{IDEM}}(t)$ are finite, this reduction rules can be applied only finitely many times.

It is easy to check that for $i = 0, \ldots, \mathrm{Rank}(t) - 1$ if we apply consecutive $R(i)$, then by applying the rules R.1-R.12 of $R(i)$, in any order, on an arbitrary element t where t is the result of applying consecutive $S(j)$ the result will be unique.

5.2.1.1 Main Algorithm

Assume that, an element t in $\mathcal{A}$ where each ω-context subterm v^ω of t has the property $C(v) \neq \square$ is given. Over the element t we will do the following consecutive steps:

Step 1. make the element t in order: for $i = 0, \ldots, \mathrm{Rank}(t) - 1$ apply consecutive $O(i)$;

Step 2. reduce ω-context and ω-forest subterms: for $i = 0, \ldots, \mathrm{Rank}(t) - 1$ apply consecutive $S(i)$;

Step 3. reduce the element t: for $i = 0, \ldots, \mathrm{Rank}(t) - 1$ apply consecutive $R(i)$.

For a given element t in $\mathcal{A}$ the result of the above steps is called the *canonical form* of the element t and denote by t.

Lemma 5.2.11. *For p-contexts v and w in $\mathcal{A}$ with $C(v) \neq \square$, if v^ω is a subterm of w, then v^ω is a subterm of at least one of the non-trivial multiplicatively irreducible factor of w.*

Proof. Since v^ω is an ω-context subterm of w, then we have v^ω is in $\mathrm{IST}(w)$.

Let $w = w_1. \cdots .w_m$ where w_i's are non-trivial multiplicatively irreducible factors of w. We have

$$\mathrm{IST}(w) = \mathrm{IST}(w_1) \cup \cdots \cup \mathrm{IST}(w_m),$$

and therefore, there is a positive integer i such that $v^\omega \in \mathrm{IST}(w_i)$, which implies that: there is a non-trivial multiplicatively irreducible factor w_i of w such that v^ω is an ω-context subterm of w_i. $\square$

Since every non-trivial p-forest in $\mathcal{A}$ can be written as a sum of its non-trivial additively irreducible summands and every non-trivial p-context in $\mathcal{A}$ can be written as a product of its non-trivial multiplicatively irreducible factors, Lemma 5.2.11 implies the following facts:

Corollary 5.2.12. *Let h and t be p-forests in $\mathcal{A}$ and h non-trivial. If $\omega(h)$ is a subterm of t, then $\omega(h)$ is a subterm of at least one of the non-trivial additively irreducible summand of t.* $\square$

Corollary 5.2.13. *Let h be a non-trivial p-forest and w a p-context in $\mathcal{A}$. If $\omega(h)$ is a subterm of w, then $\omega(h)$ is a subterm of at least one of the non-trivial multiplicatively irreducible factor of w.* $\square$

Corollary 5.2.14. *Let t be a p-forest and v a p-context in $\mathcal{A}$ with $C(v) \neq \square$. If v^ω is a subterm of t, then v^ω is a subterm of at least one of the non-trivial additively irreducible summand of t.* $\square$

Remark 5.2.15. Since for every i, $S(i)$ and $O(i)$ are applied only on ω-context and ω-forest subterms, Lemma 5.2.11, Corollaries 5.2.12, 5.2.13 and 5.2.14 imply the following equalities:

$$\begin{cases} \text{if at least one of } P_1 \text{ and } P_2 \text{ is a } p\text{-forest:} \\ \qquad O(i)(P_1 + P_2) = O(i)(P_1) + O(i)(P_2); \\ \\ \text{if both of } P_1 \text{ and } P_2 \text{ are } p\text{-contexts:} \\ \qquad O(i)(P_1.P_2) = O(i)(P_1).O(i)(P_2); \\ \\ \text{if } P_1 \text{ is a } p\text{-context and } P_2 \text{ is a } p\text{-forest:} \\ \qquad O(i)(P_1 * P_2) = O(i)(P_1) * O(i)(P_2), \end{cases}$$

and also the following equalities:

$$\begin{cases} \text{if at least one of } P_1 \text{ and } P_2 \text{ is a } p\text{-forest:} \\ \qquad S(i)(P_1 + P_2) = S(i)(P_1) + S(i)(P_2); \\ \\ \text{if both of } P_1 \text{ and } P_2 \text{ are } p\text{-contexts:} \\ \qquad S(i)(P_1.P_2) = S(i)(P_1).S(i)(P_2); \\ \\ \text{if } P_1 \text{ is a } p\text{-context and } P_2 \text{ is a } p\text{-forest:} \\ \qquad S(i)(P_1 * P_2) = S(i)(P_1) * S(i)(P_2). \end{cases}$$

In addition, for an ω-context or an ω-forest P in $\mathcal{A}$ the following equalities hold.

$$O(i)(P) = \begin{cases} \begin{cases} \text{if } \ 0 \leq i < \text{Rank}(h): \\ \qquad \omega(O(i)(h)) \\ \\ \text{for } \ j = 0, \ldots, \text{Rank}(h) - 1: \\ \qquad O(\text{Rank}(h))\omega(O(j)(h)) \end{cases} & , \text{ if } \quad P = \omega(h) \\ \\ \begin{cases} \text{if } \ 0 \leq i < \text{Rank}(v): \\ \qquad (O(i)(v))^\omega \\ \\ \text{for } \ j = 0, \ldots, \text{Rank}(v) - 1: \\ \qquad O(\text{Rank}(v))((O(j)(v))^\omega) \end{cases} & , \text{ if } \quad P = v^\omega \end{cases}$$

and

$$S(i)(P) = \begin{cases} \begin{cases} \text{if } \ 0 \le i < \text{Rank}(h): \\ \qquad \omega(S(i)(h)) \\ \\ \text{for } \ j = 0, \dots, \text{Rank}(h) - 1: \\ \qquad S(\text{Rank}(h))\omega(S(j)(h)) \end{cases} & , \text{ if } \quad P = \omega(h) \\ \\ \begin{cases} \text{if } \ 0 \le i < \text{Rank}(v): \\ \qquad (S(i)(v))^\omega \\ \\ \text{for } \ j = 0, \dots, \text{Rank}(v) - 1: \\ \qquad S(\text{Rank}(v))((S(j)(v))^\omega) \end{cases} & , \text{ if } \quad P = v^\omega. \end{cases}$$

Let $P = (H_P, V_P)$ be an ω-algebra and for every $u, v \in V_P$ and $h, s \in H_P$ the set of identities Σ, consisting of the following identities, hold in P.

$$\begin{aligned} &(uv)^\omega = (vu)^\omega = (u^\omega v^\omega)^\omega \\ &v^\omega v = v^\omega = vv^\omega \\ &(v^\omega)^\omega = v^\omega \\ \\ &vh + \omega(vuh) = \omega(vuh) = \omega(vuh) + vh \end{aligned}$$

Assume that $t_1 \sim_\Sigma t_2$. We show that if we apply the reduction rules on S_{i-1} and S_i, witnesses for elementary steps of the congruence $\sim_\Sigma$, the results are the same.

Let

$$S_{i-1} = f_i(X; U_{i,1}, \dots, U_{i,m}) \quad \text{and} \quad S_i = f_i(Y; U_{i,1}, \dots, U_{i,m}),$$

where $U_{i,1}, \dots, U_{i,m}$ are elements of $\mathcal{A}$, f_i is an n-ary operation which is a composition of operations from $\{+, +_1, +_2, ., *, \omega(), ()^\omega\}$, and there exists an ω-algebra homomorphism:

$$\varphi : \mathcal{A} \to P,$$

such that $\varphi(X) = u'$ and $\varphi(Y) = v'$, or $\varphi(Y) = u'$ and $\varphi(X) = v'$ and the identity $u' = v'$ is in Σ. Without loss of generality we may assume that $\varphi(X) = u'$ and $\varphi(Y) = v'$. We argue on the choice of the identities in Σ:

1. for the identity $(uv)^\omega = (vu)^\omega$ in Σ we do the following: apply

$$O(\text{Rank}(uv)) \qquad \text{on} \qquad f_i(X; U_{i,1}, \dots, U_{i,m})$$

and

$$O(\text{Rank}(vu)) \qquad \text{on} \qquad f_i(Y; U_{i,1}, \dots, U_{i,m}).$$

And Remark 5.2.15 implies that the results of both are the same;

2. for the identity
$$(uv)^\omega = (u^\omega v^\omega)^\omega$$
in Σ we do the following: apply $O(\mathrm{Rank}(uv))$ on
$$f_i(X; U_{i,1}, \ldots, U_{i,m})$$
and $O(\mathrm{Rank}(uv)+1)$ which is $O(\mathrm{Rank}(u^\omega v^\omega))$ on
$$f_i(Y; U_{i,1}, \ldots, U_{i,m}).$$
And Remark 5.2.15 implies that the results of both are the same;

3. for the identity $v^\omega v = v^\omega$ in Σ one of the following conditions holds:

   - if there is an ω-context subterm w_1^ω of S_{i-1} such that $v^\omega v$ is a factor of w_1, so respectively there is an ω-context subterm w_2^ω of S_i such that v^ω is a factor of w_2, then we apply $O(\mathrm{Rank}(w_1))$ on
   $$f_i(X; U_{i,1}, \ldots, U_{i,m})$$
   and we apply $O(\mathrm{Rank}(w_2))$ on
   $$f_i(Y; U_{i,1}, \ldots, U_{i,m}).$$
   And Remark 5.2.15 implies that the results of both are the same;
   - if the previous case does not hold, then we apply $O(\mathrm{Rank}(S_{i-1}))$ and after that $S(\mathrm{Rank}(S_{i-1}))$ on
   $$f_i(X; U_{i,1}, \ldots, U_{i,m})$$
   and then we apply $R(\mathrm{Rank}(v))$ by the rules R.6, R.8 and R.10 only on the subterm X of
   $$f_i(X; U_{i,1}, \ldots, U_{i,m})$$
   and on the other side we apply the rules $O(\mathrm{Rank}(S_i))$ and after that we apply the rules $S(\mathrm{Rank}(S_i))$ on
   $$f_i(Y; U_{i,1}, \ldots, U_{i,m}),$$
   and then we apply $R(\mathrm{Rank}(v)-1)$ on it. And Remark 5.2.15 implies that the results of both are the same.

4. for the identity $vv^\omega = v^\omega$ in Σ one of the following conditions holds:

- if there is an ω-context subterm w_1^ω of S_{i-1} such that vv^ω is a factor of w_1, so respectively there is an ω-context subterm w_2^ω of S_i such that v^ω is a factor of w_2, then we apply $O(\mathrm{Rank}(w_1))$ on

$$f_i(X; U_{i,1}, \ldots, U_{i,m})$$

and we apply $O(\mathrm{Rank}(w_2))$ on

$$f_i(Y; U_{i,1}, \ldots, U_{i,m}).$$

And Remark 5.2.15 implies that the results of both are the same;

- if the previous case does not hold, then we apply $O(\mathrm{Rank}(S_{i-1}))$ and after that $S(\mathrm{Rank}(S_{i-1}))$ on

$$f_i(X; U_{i,1}, \ldots, U_{i,m})$$

and then we apply $R(\mathrm{Rank}(v))$ by the rules R.5, R.7 and R.9 only on the subterm X of

$$f_i(X; U_{i,1}, \ldots, U_{i,m})$$

and on the other side we apply the rules $O(\mathrm{Rank}(S_i))$ and after that we apply the rules $S(\mathrm{Rank}(S_i))$ on

$$f_i(Y; U_{i,1}, \ldots, U_{i,m}),$$

and then we apply $R(\mathrm{Rank}(v) - 1)$ on it. And Remark 5.2.15 implies that the results of both are the same.

5. for the identity $(v^\omega)^\omega = v^\omega$ in Σ we do the following: apply $O(\mathrm{Rank}(v))$ on

$$f_i(Y; U_{i,1}, \ldots, U_{i,m})$$

and $O(\mathrm{Rank}(v))$ on

$$f_i(X; U_{i,1}, \ldots, U_{i,m}),$$

after that apply $O(\mathrm{Rank}(v)+1)$ which is $O(\mathrm{Rank}(v^\omega))$ by the rule O.2 only on the subterm X of

$$f_i(X; U_{i,1}, \ldots, U_{i,m}).$$

And Remark 5.2.15 implies that the results of both are the same;

6. for the identity $\omega(uvh) + uh = \omega(uvh)$ in Σ one of the following conditions holds:

- if there is an ω-context subterm w_1^ω of S_{i-1} such that $\omega(uvh) + uh + \Box$ or $\Box + \omega(uvh) + uh$ is a factor of w_1, so respectively there is an ω-context subterm w_2^ω of S_i such that $\omega(uvh) + \Box$ or $\Box + \omega(uvh)$ is a factor of w_2, then we apply $O(\text{Rank}(S_{i-1}))$ and after that we apply $S(\text{Rank}(w_1))$ by the rules S.2 or S.3, depends on the factor, only on the subterm X of

$$f_i(X; U_{i,1}, \dots, U_{i,m})$$

and we apply $O(\text{Rank}(S_i))$ and after that $S(\text{Rank}(w_2) - 1)$ on

$$f_i(Y; U_{i,1}, \dots, U_{i,m}).$$

And Remark 5.2.15 implies that the results of both are the same;

- if there is an ω-forest subterm $\omega(h_1)$ of S_{i-1} such that $h_1 = P_1 + \omega(uvh) + uh + P_2$ for some p-forests P_1 and P_2, so respectively there is an ω-forest subterm $\omega(h_2)$ of S_i such that $h_2 = P_1 + \omega(uvh) + P_2$, then we apply $O(\text{Rank}(S_{i-1}))$ and after that we apply $S(\text{Rank}(h_1))$ by the rule S.1 only on the subterm X of

$$f_i(X; U_{i,1}, \dots, U_{i,m})$$

and we apply $O(\text{Rank}(S_i))$ and after that $S(\text{Rank}(h_2) - 1)$ on

$$f_i(Y; U_{i,1}, \dots, U_{i,m}).$$

And Remark 5.2.15 implies that the results of both are the same;

- if the previous cases do not hold, then we apply $O(\text{Rank}(S_{i-1}))$ and after that $S(\text{Rank}(S_{i-1}))$ on

$$f_i(X; U_{i,1}, \dots, U_{i,m})$$

and then we apply $R(\text{Rank}(uvh))$ by the rule R.2 only on the subterm X of

$$f_i(X; U_{i,1}, \dots, U_{i,m})$$

and on the other side we apply the rules $O(\text{Rank}(S_i))$ and after that we apply the rules $S(\text{Rank}(S_i))$ on

$$f_i(Y; U_{i,1}, \dots, U_{i,m})$$

and then we apply $R(\text{Rank}(uvh) - 1)$ on it. And Remark 5.2.15 implies that the results of both are the same.

7. for the identity $uh + \omega(uvh) = \omega(uvh)$ in Σ one of the following conditions holds:

- if there is an ω-context subterm w_1^ω of S_{i-1} such that $uh + \omega(uvh) + \Box$ or $\Box + uh + \omega(uvh)$ is a factor of w_1, so respectively there is an ω-context subterm w_2^ω of S_i such that $\omega(uvh) + \Box$ or $\Box + \omega(uvh)$ is a factor of w_2, then we apply $O(\text{Rank}(S_{i-1}))$ and after that we apply $S(\text{Rank}(w_1))$ by the rules S.2 or S.3, depends on the factor, only on the subterm X of

$$f_i(X; U_{i,1}, \dots, U_{i,m})$$

and we apply $O(\text{Rank}(S_i))$ and after that $S(\text{Rank}(w_2) - 1)$ on

$$f_i(Y; U_{i,1}, \dots, U_{i,m}).$$

And Remark 5.2.15 implies that the results of both are the same;

- if there is an ω-forest subterm $\omega(h_1)$ of S_{i-1} such that $h_1 = P_1 + uh + \omega(uvh) + P_2$ for some p-forests P_1 and P_2, so respectively there is an ω-forest subterm $\omega(h_2)$ of S_i such that $h_2 = P_1 + \omega(uvh) + P_2$, then we apply $O(\text{Rank}(S_{i-1}))$ and after that we apply $S(\text{Rank}(h_1))$ by the rule S.1 only on the subterm X of

$$f_i(X; U_{i,1}, \dots, U_{i,m})$$

and we apply $O(\text{Rank}(S_i))$ and after that $S(\text{Rank}(h_2) - 1)$ on

$$f_i(Y; U_{i,1}, \dots, U_{i,m}).$$

And Remark 5.2.15 implies that the results of both are the same;

- if the previous cases do not hold, then we apply $O(\text{Rank}(S_{i-1}))$ and after that $S(\text{Rank}(S_{i-1}))$ on

$$f_i(X; U_{i,1}, \dots, U_{i,m})$$

and then we apply $R(\text{Rank}(uvh))$ by the rule R.2 only on the subterm X of

$$f_i(X; U_{i,1}, \dots, U_{i,m})$$

and on the other side we apply the rules $O(\text{Rank}(S_i))$ and after that we apply the rules $S(\text{Rank}(S_i))$ on

$$f_i(Y; U_{i,1}, \dots, U_{i,m})$$

and then we apply $R(\text{Rank}(uvh) - 1)$ on it. And Remark 5.2.15 implies that the results of both are the same.

We then have a system of reduction rules which is noetherian and confluent. This implies that for elements t_1 and t_2 in $\mathcal{A}$ with $t_1 \sim_\Sigma t_2$ if we apply the reduction rules on t_1 and t_2, then the results are the same.

The variety $\mathcal{V}$ certainly contains **BSS**. Denoting by $F_A\mathcal{V}$ the $\mathcal{V}$-free algebra on A, we then have an ω-algebra homomorphism

$$\varphi : F_A\mathcal{V} = (H_1, V_1) \to \overline{\Omega}_A\mathbf{BSS} = (H_2, V_2)$$

such that $x_i \mapsto x_i$ $(i = 1, \dots, n)$.

If two p-contexts or p-forests have the same canonical form, then in $F_A\mathcal{V}$ they are equal and so they have the same image by φ. Therefore, their image by φ have the same scattered divisors.

Note that, φ is a pair (α, β) of monoid homomorphisms

$$\begin{aligned} \alpha :\ & H_1 \to H_2, \\ \beta :\ & V_1 \to V_2. \end{aligned}$$

In order to show that φ is injective, it suffices to show that β is injective.

Lemma 5.2.16. *If β is injective, then φ is injective.*

Proof. We just need to show that α is injective. Assume that for h_1 and h_2 in H_1, $\alpha(h_1) = \alpha(h_2)$. It implies that $\alpha(h_1) + \Box = \alpha(h_2) + \Box$ which is $\beta(h_1 + \Box) = \beta(h_2 + \Box)$. And since by assumption β is injective, we conclude that $h_1 + \Box$ and $h_2 + \Box$ have the same canonical form and so does h_1 and h_2. $\Box$

We solved the word problem for the free ω-algebra in the variety $\mathcal{V}$ of ω-algebras defined by the set Σ.

5.3 Open Problems

The following problems remain open.

Open problem 5.3.1. *The monoid homomorphism β is injective.*

Open problem 5.3.2. *The ω-algebra homomorphism φ is surjective.*

Open problem 5.3.3. *What about other pseudovarieties?*

If the first two open problems admit affirmative solutions, then together with Lemma 5.2.16, we get the following result:

Theorem 5.3.4. *The variety of type τ generated by* **BSS** *is defined by the identities*

$$\begin{aligned} &(uv)^\omega = (vu)^\omega = (u^\omega v^\omega)^\omega \\ &v^\omega v = v^\omega = vv^\omega \\ &(v^\omega)^\omega = v^\omega \end{aligned}$$

$$vh + \omega(vuh) = \omega(vuh) = \omega(vuh) + vh$$

and $\overline{\Omega}_A\mathbf{BSS}$ *is the free object on* A *in this variety. Two terms in the variables from* A *coincide in* $\overline{\Omega}_A\mathbf{BSS}$ *if and only if they have the same canonical form with respect to the reduction rules in the Algorithm 5.2.1.1. In particular, the word problem for* $\overline{\Omega}_A\mathbf{BSS}$ *is decidable.* $\square$

Appendix A

Long Proofs

A.1 Lemma 2.1.1

Lemma A.1.1. *For a forest algebra S and a subset K of S, the equivalence relations σ_K and σ'_K are congruences with respect to the basic operations of S.*

Proof. Let $h_1 \; \sigma_K \; h_2$ and $v_1 \; \sigma'_K \; v_2$. We should show that for every $s \in H_S$ and $u \in V_S$ we have the following relations:

I.1 $h_1 + s \; \sigma_K \; h_2 + s$ and $s + h_1 \; \sigma_K \; s + h_2$;

I.2 $h_1 + u \; \sigma'_K \; h_2 + u$ and $u + h_1 \; \sigma'_K \; u + h_2$;

I.3 $uh_1 \; \sigma_K \; uh_2$;

I.4 $uv_1 \; \sigma'_K \; uv_2$ and $v_1 u \; \sigma'_K \; v_2 u$;

I.5 $v_1 s \; \sigma_K \; v_2 s$;

I.6 $v_1 + s \; \sigma'_K \; v_2 + s$ and $s + v_1 \; \sigma'_K \; s + v_2$.

We claim that the following equivalence holds for every context t:

$$t(h_1 + s) \in K \iff t(h_2 + s) \in K.$$

Let $q_t = t(\Box + s)$. We get $q_t h_1 = t(\Box + s)h_1 = t(h_1 + s)$. Hence,

$$t(h_1 + s) = q_t h_1 \in K \iff t(h_2 + s) = q_t h_2 \in K,$$

where the middle equivalence follows from the relation $h_1 \; \sigma_K \; h_2$.

Now we show that for all contexts t, r and w we have:

1. $t(r(h_1 + s) + w) \in K \iff t(r(h_2 + s) + w) \in K$,

2. $t(w + r(h_1 + s)) \in K \iff t(w + r(h_2 + s)) \in K$.

By using the term q_r we get

$$t(r(h_1+s)+w) = t(q_r h_1 + w) \in K \Leftrightarrow t(r(h_2+s)+w) = t(q_r h_2 + w) \in K$$

and

$$t(w+r(h_1+s)) = t(w + q_r h_1) \in K \Leftrightarrow t(w+r(h_2+s)) = t(w + q_r h_2) \in K$$

where the equivalences follow from the hypothesis $h_1 \; \sigma_K \; h_2$. So $h_1 + s \; \sigma_K \; h_2 + s$ holds and, similarly, so does $s + h_1 \; \sigma_K \; s + h_2$.

We next claim that the following equivalence holds for every context t and every forest h:

$$t(h_1+u)h \; \sigma_K \; t(h_2+u)h.$$

From definition of forest algebra S we get $t(h_1+uh) = t(h_1+u)h$. So we have:

$$t(h_1+u)h = t(h_1+uh) \; \sigma_K \; t(h_2+uh) = t(h_2+u)h,$$

where the equivalence follows from I.1.

Now we show that, for all contexts t and w, we have:

$$t(h_1+u)w \in K \iff t(h_2+u)w \in K.$$

From definition of forest algebra S we get $t(h_1+uw) = t(h_1+u)w$. On the other hand the relation

$$t(h_1+uw) \in K \iff t(h_2+uw) \in K,$$

follows from the hypothesis $h_1 \; \sigma_K \; h_2$. Hence, $h_1 + u \; \sigma'_K \; h_2 + u$ holds and, similarly, so does $u + h_1 \; \sigma'_K \; u + h_2$.

We claim that the following equivalence holds for every context t:

$$t(uh_1) \in K \iff t(uh_2) \in K.$$

Let $p_t = tu$. We get $p_t h_1 = (tu)h_1 = t(uh_1)$. Hence,

$$t(uh_1) = p_t h_1 \in K \Leftrightarrow t(uh_2) = p_t h_2 \in K,$$

where the equivalence follows from $h_1 \; \sigma_K \; h_2$.

Now we show that for all contexts t, r and w we have:

1. $t(r(uh_1)+w) \in K \iff t(r(uh_2)+w) \in K$;
2. $t(w+r(uh_1)) \in K \iff t(w+r(uh_2)) \in K$.

By using the term p_r we get:

$$t(r(uh_1)+w) = t(p_r h_1 + w) \in K \Leftrightarrow t(r(uh_2)+w) = t(p_r h_2 + w) \in K$$

and

$$t(w + r(uh_1)) = t(w + p_r h_1) \in K \Leftrightarrow t(w + r(uh_2)) = t(w + p_r h_2) \in K$$

where the equivalences follow from $h_1 \; \sigma_K \; h_2$. So $uh_1 \; \sigma_K \; uh_2$.

We next claim that the following equivalence holds for every context t and every forest h:

$$t(uv_1)h \; \sigma_K \; t(uv_2)h.$$

By using the term p_t we have:

$$t(uv_1)h = p_t v_1 h \; \sigma_K \; p_t v_2 h = t(uv_2)h,$$

where the equivalence follows from $v_1 \; \sigma'_K \; v_2$.

Now we show that for all contexts t and w we have:

$$t(uv_1)w \in K \iff t(uv_2)w \in K.$$

By using the term p_t we get:

$$t(uv_1 w) = p_t v_1 w \in K \iff t(uv_2 w) = p_t v_2 w \in K,$$

follows from $v_1 \; \sigma'_K \; v_2$. So $uv_1 \; \sigma'_K \; uv_2$ holds and, similarly, so does $v_1 u \; \sigma'_K \; v_2 u$.

The equivalence $v_1 s \; \sigma_K \; v_2 s$ is clear by definition of the relation $v_1 \; \sigma'_K \; v_2$.

We next claim that the following equivalence holds for every context t and every forest h:

$$t(v_1 + s)h \; \sigma_K \; t(v_2 + s)h.$$

By using the term q_t we have:

$$t(v_1 + s)h = q_t v_1 h \; \sigma_K \; q_t v_2 h = t(v_2 + s)h,$$

where the equivalence follows from $v_1 \; \sigma'_K \; v_2$.

Now we show that for all contexts t and w we have:

$$t(v_1 + s)w \in K \iff t(v_2 + s)w \in K.$$

By using the term q_t we have:

$$t(v_1 + s)w = q_t v_1 w \in K \Leftrightarrow t(v_2 + s)w = q_t v_2 w \in K,$$

follows from $v_1 \; \sigma'_K \; v_2$. So $v_1 + s \; \sigma'_K \; v_2 + s$ holds and, similarly, so does $s + v_1 \; \sigma'_K \; s + v_2$.

So $\backsim_K$ is a congruence. $\square$

A.2 Lemma 4.2.65

Lemma A.2.1. *Let $w_1, \ldots, w_n$ be non-trivial multiplicatively irreducible p-contexts in $\mathcal{A}$. Then the following equality holds:*

$$\mathrm{IrrF}(w_1.\cdots.w_n) = \{w_1\} \cup \mathrm{IrrF}(w_2.\cdots.w_n). \tag{A.1}$$

Proof. By induction on n we will show that for $n \geq 1$ the equality (A.1) holds. For $n = 1$, the result follows directly from the $\mathrm{IrrF}(w_1)$, since w_1 is a non-trivial multiplicatively irreducible p-context.

Assume that, for $n = k$ the equality (A.1) holds.

We show that for $n = k+1$ the equality (A.1) also holds.

We show that, if v be a non-trivial multiplicatively irreducible p-context in $\mathcal{A}$ which is a factor of $w_1.\cdots.w_{k+1}$, then one of the following conditions holds:

$$v = w_1 \quad \text{or} \quad v \in \mathrm{IrrF}(w_2.\cdots.w_{k+1}).$$

Since w_1 is non-trivial multiplicatively irreducible, w_1 has one of the forms

$$\begin{cases} d\Box \\ s+\Box \\ \Box+s \\ u^\omega \end{cases}$$

where s is a non-trivial additively irreducible p-forest and u is a p-context with $C(u) \neq \Box$ and $d\Box \in A'$.

Assume that, $w_1 = d\Box$ and v is a non-trivial multiplicatively irreducible p-context which is a factor of $w_1.\cdots.w_{k+1}$. Then there are p-contexts P_1 and P_2 such that the equality

$$P_1.v.P_2 = w_1.\cdots.w_{k+1}$$

holds. Since P_1 is a p-context, there are p-forests H_1 and H_2 such that $P_1 = H_1 + C(P_1) + H_2$. Applying the ω-algebra homomorphism f_1, by Lemmas 4.2.11 and 1.3.7, we have $H_1 = H_2 = 0$. So, we must have one of the following equalities:

$$P_1 = \Box \text{ or } P_1 = p\Box.w \text{ or } P_1 = x^\omega.w \text{ with } C(x) \neq \Box.$$

If $P_1 = \Box$, then we have

$$v.P_2 = d\Box.w_2.\cdots.w_{k+1}.$$

Again, applying the ω-algebra homomorphism f_n, for $v = G_1 + C(v) + G_2$ we have $G_1 = G_2 = 0$ which means

$$v = e\Box \quad \text{or} \quad v = y^\omega.$$

If we have $v = y^\omega$, then applying the ω-algebra homomorphism Φ, we have the following equality:

$$b\Box.\Phi(y).c\Box.\Phi(P_2) = d\Box.\Phi(w_2.\cdots.w_{k+1}),$$

which contradicts the fact that $d \neq b$. So, we must have $v = e\Box$. And again applying the ω-algebra homomorphism Φ, we have the following equality:

$$e\Box.\Phi(P_2) = d\Box.\Phi(w_2.\cdots.w_{k+1}),$$

which implies the equality $e = d$ that is $v = d\Box$.

If we have $P_1 = p\Box.w$ with $p\Box \in A'$, then we have the following equality:

$$p\Box.w.v.P_2 = d\Box.w_2.\cdots.w_{k+1},$$

and applying the ω-algebra homomorphism Φ, it implies that the following equality holds:

$$p\Box.\Phi(w.v.P_2) = d\Box.\Phi(w_2.\cdots.w_{k+1}).$$

This yields the equalities $p = d$ and

$$\Phi(w.v.P_2) = \Phi(w_2.\cdots.w_{k+1}).$$

By Theorem 4.2.60, we conclude that

$$w.v.P_2 = w_2.\cdots.w_{k+1}$$

which means v is a factor of $w_2.\cdots.w_{k+1}$ that is $v \in \mathrm{IrrF}(w_2.\cdots.w_{k+1})$.

If $P_1 = x^\omega.w$ with $C(x) \neq \Box$, then we have

$$x^\omega.w.v.P_2 = d\Box.w_2.\cdots.w_{k+1}$$

and applying the ω-algebra homomorphism Φ, we have the equality

$$b\Box.\Phi(x).c\Box.\Phi(w.v.P_2) = d\Box.\Phi(w_2.\cdots.w_{k+1})$$

which contradicts the fact that $d \neq b$.

Assume that $w_1 = u^\omega$ with $C(u) \neq \Box$ and v a non-trivial multiplicatively irreducible p-context which is a factor of $w_1.\cdots.w_{k+1}$. There are p-contexts P_1 and P_2 such that the equality

$$P_1.v.P_2 = w_1.\cdots.w_{k+1}$$

holds. Since P_1 is a p-context, there are p-forests H_1 and H_2 such that $P_1 = H_1 + C(P_1) + H_2$. Applying the ω-algebra homomorphism Φ, by Lemma 1.3.7 and Theorem 4.2.60, we have $H_1 = H_2 = 0$. So, we must have one of the following equalities:

$$P_1 = \Box \quad \text{or} \quad P_1 = p\Box.w \quad \text{or} \quad P_1 = x^\omega.w \ \text{ with } \ C(x) \neq \Box.$$

If $P_1 = \Box$, then we have

$$v.P_2 = u^\omega.w_2.\cdots.w_{k+1}.$$

Again, applying the ω-algebra homomorphism f_n, for $v = G_1 + C(v) + G_2$ we have $G_1 = G_2 = 0$ which means

$$v = e\Box \quad \text{or} \quad v = y^\omega.$$

If $v = e\Box$, then applying the ω-algebra homomorphism Φ, we have the following equality holds:

$$e\Box.\Phi(P_2) = b\Box.\Phi(u).c\Box.\Phi(w_2.\cdots.w_{k+1})$$

which contradicts the fact that $e \neq b$. So, we may have $v = y^\omega$ so that, by applying the ω-algebra homomorphism Φ, we have the following equality:

$$b\Box.\Phi(y).c\Box.\Phi(P_2) = b\Box.\Phi(u).c\Box.\Phi(w_2.\cdots.w_{k+1}).$$

By Corollary 4.2.58, we have the equality $\Phi(y) = \Phi(u)$ and Theorem 4.2.60 yields the equality $y = u$ which implies that the equality $y^\omega = u^\omega$ holds, that is the equality $v = w_1$.

If $P_1 = p\Box.w$ with $p\Box \in A'$, then the following equality holds:

$$p\Box.w.v.P_2 = u^\omega.w_2.\cdots.w_{k+1}.$$

So that, by applying the ω-algebra homomorphism Φ, it implies the following equality:

$$p\Box.\Phi(w.v.P_2) = b\Box.\Phi(u).c\Box.\Phi(w_2.\cdots.w_{k+1})$$

which contradicts the fact that $p \neq b$.

Now, if $P_1 = x^\omega.w$ with $C(x) \neq \Box$, then the following equality holds:

$$x^\omega.w.v.P_2 = u^\omega.w_2.\cdots.w_{k+1}.$$

Applying the ω-algebra homomorphism Φ, we obtain the following equality:

$$b\Box.\Phi(x).c\Box.\Phi(w.v.P_2) = b\Box.\Phi(u).c\Box.\Phi(w_2.\cdots.w_{k+1}).$$

By Corollary 4.2.58, the equality

$$\Phi(w.v.P_2) = \Phi(w_2.\cdots.w_{k+1})$$

holds. And by Theorem 4.2.60, we have the equality

$$w.v.P_2 = w_2.\cdots.w_{k+1}$$

which means v is a factor of $w_2.\cdots.w_{k+1}$ that is

$$v \in \mathrm{IrrF}(w_2.\cdots.w_{k+1}).$$

Assume next that $w = s + \Box$, where s is a non-trivial additively irreducible p-forest and let v be a non-trivial multiplicatively irreducible p-context which is a factor of $w_1. \cdots .w_{k+1}$. There are p-contexts P_1 and P_2 such that the equality

$$P_1.v.P_2 = w_1. \cdots .w_{k+1}$$

holds. Since P_1 and P_2 are p-contexts, there are p-forests H_1, H_1', H_2, and H_2' such that $P_1 = H_1 + C(P_1) + H_2$ and $P_2 = H_1' + C(P_2) + H_2'$. We may have

$$H_1 \neq 0 \quad \text{or} \quad H_1 = 0.$$

If $H_1 \neq 0$, then $H_1 = h_1 + \cdots + h_m$ where $h_1, \ldots, h_m$ are non-trivial additively irreducible p-forests. Let

$$w = h_2 + \cdots + h_m + C(P_1) + H_2.$$

Applying the ω-algebra homomorphism Φ, we have the following equality:

$$\Phi(h_1) + \Phi(w.v.P_2) = \Phi(s) + \Phi(w_2. \cdots .w_{k+1}).$$

By Lemma 1.3.7, together with Lemma 1.3.6, we have $\Phi(h_1) = \Phi(s)$, as both are connected in the free forest algebra, and $\Phi(w.v.P_2) = \Phi(w_2. \cdots .w_{k+1})$. So that, by Theorem 4.2.60, we have $w.v.P_2 = w_2. \cdots .w_{k+1}$ that is v is a factor of $w_2. \cdots .w_{k+1}$ which means $v \in \mathrm{IrrF}(w_2. \cdots .w_{k+1})$.

Now, assume that $H_1 = 0$. Since $s \neq 0$, applying the ω-algebra homomorphism Φ, we have $C(P_1) = \Box$; otherwise, we have

$$\Phi(C(P_1).v.P_2) + \Phi(H_2) = \Phi(s) + \Phi(w_2. \cdots .w_{k+1})$$

so that, by Lemma 1.3.7, the equality $\Phi(s) = 0$ holds and by Lemma 4.2.34, we have $s = 0$ which contradicts the assumption that $s \neq 0$. So, we have $P_1 = \Box + H_2$ which yields the equality

$$v.P_2 + H_2 = s + w_2. \cdots .w_{k+1}.$$

Now, since v is a non-trivial multiplicatively irreducible p-context, v has one of the following forms:

$$\begin{cases} e\Box \\ t + \Box \\ \Box + t \\ y^\omega \end{cases}$$

where t is a non-trivial additively irreducible p-forest, y is a p-context with $C(y) \neq \Box$ and $e\Box \in A'$.

If we have $v = e\Box$, then applying the ω-algebra homomorphism Φ, we obtain the following equality:

$$e\Box.\Phi(P_2) + \Phi(H_2) = \Phi(s) + \Phi(w_2. \cdots .w_{k+1})$$

so that, by Lemma 1.3.7, we have $\Phi(s) = 0$ and Lemma 4.2.34 yields $s = 0$ which contradicts the assumption that $s \neq 0$.

If we have $v = y^\omega$, then applying the ω-algebra homomorphism Φ, we obtain the following equality:

$$b\square.\Phi(y).c\square.\Phi(P_2) + \Phi(H_2) = \Phi(s) + \Phi(w_2.\cdots.w_{k+1})$$

so that, by Lemma 1.3.7, we have $\Phi(s) = 0$ and again Lemma 4.2.34 implies that $s = 0$ which contradicts the assumption that $s \neq 0$.

If we have $v = t + \square$, then applying the ω-algebra homomorphism Φ, we obtain the following equality:

$$\Phi(t) + \Phi(P_2) + \Phi(H_2) = \Phi(s) + \Phi(w_2.\cdots.w_{k+1})$$

so that, by Lemma 1.3.7, we have $\Phi(t) = \Phi(s)$ and Theorem 4.2.60 leads to the equality $t = s$ which in turn yields $v = s + \square$.

And if we have $v = \square + t$, then applying the ω-algebra homomorphism Φ, the following equality holds:

$$\Phi(P_2) + \Phi(t) + \Phi(H_2) = \Phi(s) + \Phi(w_2.\cdots.w_{k+1}).$$

We may have

$$H'_1 \neq 0 \quad \text{or} \quad H'_1 = 0.$$

If we have $H'_1 \neq 0$, then there are non-trivial additively irreducible p-forests $h'_1, \ldots, h'_r$ such that $H'_1 = h'_1 + \cdots + h'_r$. Let

$$w = h'_2 + \cdots + h'_r + C(P_2) + H'_2.$$

Applying the ω-algebra homomorphism Φ, the following equality holds:

$$\Phi(h'_1) + \Phi(P_1.v.w) = \Phi(s) + \Phi(w_2.\cdots.w_{k+1}),$$

so that, by Lemma 1.3.7, we have the following equalities:

$$\Phi(h'_1) = \Phi(s) \quad \text{and} \quad \Phi(P_1.v.w) = \Phi(w_2.\cdots.w_{k+1})$$

and by Theorem 4.2.60, we have $P_1.v.w = w_2.\cdots.w_{k+1}$ which means v is a factor of $w_2.\cdots.w_{k+1}$ that is $v \in \mathrm{IrrF}(w_2.\cdots.w_{k+1})$.

If we have $H'_1 = 0$, then applying the ω-algebra homomorphism Φ, we have the following equality:

$$\Phi(C(P_2)) + \Phi(H'_2) + \Phi(t) + \Phi(H_2) = \Phi(s) + \Phi(w_2.\cdots.w_{k+1}),$$

so that, by Lemma 1.3.7, we have $\Phi(s) = 0$, and Lemma 4.2.34 which yields $s = 0$ which contradicts the assumption that $s \neq 0$.

In the dual case where $w_1 = \square + s$ with s a non-trivial additively irreducible p-forest and v a factor of $w_1.\cdots.w_{k+1}$, we can use similar arguments to show that $v = w_1$ or $v \in \mathrm{IrrF}(w_2.\cdots.w_{k+1})$. $\square$

A.3 Lemma 5.1.2

Recall that, the set Σ consisting of the following identities, for contexts u and v and forest h,

$$(uv)^\omega = (vu)^\omega = (u^\omega v^\omega)^\omega \tag{A.2}$$
$$v^\omega v = v^\omega = vv^\omega \tag{A.3}$$
$$(v^\omega)^\omega = v^\omega \tag{A.4}$$

$$vh + \omega(vuh) = \omega(vuh) = \omega(vuh) + vh. \tag{A.5}$$

And also the following identities, for forests h and s,

$$\omega(h+s) = \omega(s+h) = \omega(\omega(h)+\omega(s)) \tag{A.6}$$
$$\omega(h) + h = \omega(h) = h + \omega(h) \tag{A.7}$$
$$\omega(\omega(h)) = \omega(h) \tag{A.8}$$

are the result of Lemma 5.1.1.

We are going to establish a number of identities as consequences of Σ.

Since for an ω-algebra $S = (H, V)$, H and V are algebras of type $(2, 1)$ and the identities

$$\omega(h+s) = \omega(s+h) = \omega(\omega(h)+\omega(s))$$
$$\omega(h) + h = \omega(h) = h + \omega(h)$$
$$\omega(\omega(h)) = \omega(h)$$

and the identities

$$(uv)^\omega = (vu)^\omega = (u^\omega v^\omega)^\omega$$
$$v^\omega v = v^\omega = vv^\omega$$
$$(v^\omega)^\omega = v^\omega,$$

respectively, hold in H and V, by [1, Lemma 8.2.2], we obtain I.1-I.6.

I.7 . By using (A.2), (A.4), and I.1 we have $(vv)^\omega = (v^\omega v^\omega)^\omega = (v^\omega)^\omega = v^\omega$.

I.8 . By using the identities $(uv)^\omega = (u^\omega v^\omega)^\omega$ and $(v^\omega)^\omega = v^\omega$ we have

$$(uv^\omega)^\omega = (u^\omega (v^\omega)^\omega)^\omega = (u^\omega v^\omega)^\omega = (uv)^\omega.$$

I.9 . By using the identities (A.2), I.7, I.3, and I.8 we have

$$u^\omega = \Big(\prod_{v \in \mathrm{IrrNIdemF}^*(u)} v\Big)^\omega.$$

I.10 . If the identity $u^\omega = v^\omega$ holds, then we have the following equalities:

$$\begin{aligned}\mathrm{IrrNIdemF}^*(u) &= \mathrm{IrrNIdemF}^*(u^\omega)\\ &= \mathrm{IrrNIdemF}^*(v^\omega)\\ &= \mathrm{IrrNIdemF}^*(v),\end{aligned}$$

and if the equality $\mathrm{IrrNIdemF}^*(u) = \mathrm{IrrNIdemF}^*(v)$ holds, then by using 4 and 7 we have

$$u^\omega = (\prod_{v_1 \in \mathrm{IrrNIdemF}^*(u)} v_1)^\omega = (\prod_{v_1 \in \mathrm{IrrNIdemF}^*(v)} v_1)^\omega = v^\omega.$$

I.11 . Assume that the p-contexts u and v are such that

$$\mathrm{IrrNIdemF}^*(u) \subseteq \mathrm{IrrNIdemF}^*(v).$$

Then by using I.9 there is a p-context v_1 such that $v^\omega = (v_1 u)^\omega = (uv_1)^\omega$. By using the identities $ww^\omega = w^\omega = w^\omega w$, $(w_1 w_2)^\omega = (w_1^\omega w_2^\omega)^\omega$ and I.2 we have

$$\begin{aligned}v^\omega u^\omega &= (uv_1)^\omega u^\omega\\ &= (u^\omega v_1^\omega)^\omega u^\omega\\ &= (u^\omega v_1^\omega)^\omega\\ &= (uv_1)^\omega\\ &= v^\omega.\end{aligned}$$

We can do the similar arguments for the symmetric case.

I.12 . By using I.11 and (A.3) we have

$$\begin{aligned}v^\omega u^\omega &= v^\omega(u^\omega u)\\ &= (v^\omega u^\omega)u\\ &= v^\omega u.\end{aligned}$$

We can do the similar arguments for the symmetric case.

I.13 . By using (A.6), (A.8), and I.4 we have

$$\omega(h+h) = \omega(\omega(h)+\omega(h)) = \omega(\omega(h)) = \omega(h).$$

I.14 . By (A.6), (A.8), and (A.5) we have

$$\begin{aligned}\omega(uh+uwh) &= \omega(\omega(uh)+\omega(uwh))\\ &= \omega(\omega(uh)+\omega(\omega(uwh)))\\ &= \omega(uh+\omega(uwh))\\ &= \omega(\omega(uwh))\\ &= \omega(uwh).\end{aligned}$$

I.15 . By using (A.6) and (A.8) we have

$$\omega(\omega(h) + s) = \omega(\omega(\omega(h)) + \omega(s)) = \omega(\omega(h) + \omega(s)) = \omega(h + s).$$

I.16 . For a p-context $v = h_1 + C(v) + h_2$ if $C(v) = \Box$, then $v^\omega = \omega(h_1) + \Box + \omega(h_2)$, it is one of the identities defining ω-algebras. And for $C(v) \neq \Box$ if $C(v) \neq v$, then by using I.3 we have $v^\omega = (C(v)(h_1 + \Box + h_2))^\omega$ and since $C(v)$ is a $\Box$-pure p-context, it gives the result.

I.17 . By using (A.6), (A.8), (A.5), and I.5 we have:

$$\begin{aligned}
\omega(uvs) &= \omega(\omega(uvs)) \\
&= \omega(us + \omega(uvs)) \\
&= \omega(\omega(us) + \omega(\omega(uvs))) \\
&= \omega(\omega(us) + \omega(\omega(uvs))) + \omega(us) \\
&= \omega(us + \omega(uvs)) + \omega(us) \\
&= \omega(uvs) + \omega(us),
\end{aligned}$$

and in a similar way we have $\omega(uvs) = \omega(us) + \omega(uvs)$.

I.18 . If the identity $wv = w$ holds, then by (A.5) we have:

$$\begin{aligned}
uvt + \omega(uwt) &= uvt + \omega(uwvt) \\
&= \omega(uwvt) \\
&= \omega(uwt).
\end{aligned}$$

In a similar way we can conclude that $\omega(uwt) + uvt = \omega(uwt)$. And also in a similar way if the identity $vw = w$ holds.

I.19 . By using I.1 and I.17 we have the following identities:

$$\omega(v_1^\omega \cdots v_n^\omega s) = \omega(v_1^\omega \cdots v_n^\omega s) + \omega(v_1^\omega \cdots v_n^\omega h)$$

$$\omega(v_1^\omega \cdots v_n^\omega h) = \omega(v_1^\omega \cdots v_n^\omega h) + \omega(v_1^\omega \cdots v_{n-1}^\omega u_n h)$$

$$\omega(v_1^\omega \cdots v_{n-1}^\omega u_n h) = \omega(v_1^\omega \cdots v_{n-1}^\omega u_n h) + \omega(v_1^\omega \cdots v_{n-2}^\omega u_{n-1} u_n h)$$

$$\vdots$$

$$\omega(v_1^\omega u_2 \cdots u_{n-1} u_n h) = \omega(v_1^\omega u_2 \cdots u_{n-1} u_n h) + \omega(u_1 u_2 \cdots u_{n-1} u_n h)$$

$$\omega(u_1 u_2 \cdots u_{n-1} u_n h) = \omega(u_1 u_2 \cdots u_{n-1} u_n h) + u_1 \cdots u_n h.$$

By combining the above equations and I.15 we get the result.

I.20 . Since $u \mid_s v$, there exist p-contexts

$$P_1, \ldots, P_n, Q_1, \ldots, Q_{n+1}$$

such that

$$u = P_1 \cdots P_n \quad \text{and} \quad v = Q_1 P_1 \cdots Q_n P_n Q_{n+1}.$$

By using I.17 we have the following identities:

$$\omega(v0) = \omega(Q_1 P_1 \cdots Q_n P_n Q_{n+1} 0) + \omega(P_1 Q_2 \cdots Q_n P_n Q_{n+1} 0)$$

$$\begin{aligned} &\omega(P_1 Q_2 \cdots Q_n P_n Q_{n+1} 0) \\ &= \omega(P_1 Q_2 \cdots Q_n P_n Q_{n+1} 0) + \omega(P_1 P_2 Q_3 \cdots Q_n P_n Q_{n+1} 0) \end{aligned}$$

$$\begin{aligned} &\omega(P_1 P_2 Q_3 \cdots Q_n P_n Q_{n+1} 0) \\ &= \omega(P_1 P_2 Q_3 \cdots Q_n P_n Q_{n+1} 0) + \omega(P_1 P_2 P_3 Q_4 \cdots Q_n P_n Q_{n+1} 0) \end{aligned}$$

$$\begin{aligned} &\omega(P_1 P_2 \cdots P_{n-1} P_n Q_{n+1} 0) \\ &= \omega(P_1 P_2 \cdots P_{n-1} P_n Q_{n+1} 0) + \omega(P_1 P_2 \cdots P_{n-1} P_n 0) \end{aligned}$$

$$\vdots$$

$$\omega(P_1 P_2 \cdots P_{n-1} P_n 0) = \omega(P_1 P_2 \cdots P_{n-1} P_n 0) + P_1 P_2 \cdots P_{n-1} P_n 0$$

By combining the above equations and I.15 we get the result.

I.21 . By (A.4), I.13 and I.6 $\omega(h) = \omega(\sum\limits_{t \in \mathrm{IrrS}(h)} t)$ and by using I.15 we get the result.

I.22 . If for a p-forest s we have $s \in \mathrm{Special}_{\mathrm{H}}(h)$, then we have

$$s \in \mathrm{IrrNIdemS}^*(h) \quad \text{or} \quad s \in \mathrm{Special}_{\mathrm{H}}(h) \setminus \mathrm{IrrNIdemS}^*(h).$$

If $s \in \mathrm{IrrNIdemS}^*(h)$, then by I.5, I.6 and I.21 we get the result.

It remains to show the result for $s \in \mathrm{Special}_{\mathrm{H}}(h) \setminus \mathrm{IrrNIdemS}^*(h)$. From the definition of the set $\mathrm{Special}_{\mathrm{H}}(h)$ it follows that

$$s = P_1 Q_1 \cdots P_n Q_n P_{n+1} 0 \neq 0,$$

where

$$u_1 v_1^{\omega} \cdots u_n v_n^{\omega} u_{n+1} 0 \in \mathrm{IrrNIdemS}^*(h)$$

such that each p-context P_i is a scattered divisor of u_i and each p-context Q_j is a product of some of the elements of $\mathrm{IrrNIdemF}^*(v_j)$ in any order and some but not all of the p-contexts P_i and Q_j are $\Box$. Then the result is by I.19 and I.20.

I.23 . By I.21 and I.22 for every $s \in \mathrm{Special}_{\mathrm{H}}(h)$, we obtain

$$\omega(h) = \omega(\sum_{t \in \mathrm{IrrNIdemS}^*(h)} t + s).$$

So, we have

$$\omega(h) = \omega(\sum_{t \in \mathrm{IrrNIdemS}^*(h)} t + \sum_{s \in \mathrm{Special}_{\mathrm{H}}(h)} s).$$

By I.6 we have the following identity:

$$\begin{aligned}
&\omega(h) \\
&= \omega(\sum_{t \in \mathrm{IrrNIdemS}^*(h)} t \\
&\quad + \sum_{s' \in \mathrm{IrrNIdemS}^*(h)} s' \\
&\quad + \sum_{s'' \in \mathrm{Special}_{\mathrm{H}}(h) \setminus \mathrm{IrrNIdemS}^*(h)} s''),
\end{aligned}$$

where I.6 and I.15 implies that

$$\omega(h) = \omega(\sum_{t \in \mathrm{IrrNIdemS}^*(h)} t + \sum_{s'' \in \mathrm{Special}_{\mathrm{H}}(h) \setminus \mathrm{IrrNIdemS}^*(h)} s''),$$

that is

$$\omega(\sum_{t \in \mathrm{Special}_{\mathrm{H}}(h)} t) = \omega(h).$$

I.24 . For a p-context v with factorization $\prod_{i=1}^{n} v_i$ let

$$h_1 = \sum_{s \in \mathrm{LIrrF}(v)0} s \quad \text{and} \quad h_2 = \sum_{s \in \mathrm{RIrrF}(v)0} s,$$

and let $h = t_1 + \cdots + t_{\mathrm{CP}(h)}$ we have the following conditions:

G.1 . For a p-context v let

$$w = (\prod_{u \in \mathrm{PIrrF}(v)} u)(\prod_{u \in \mathrm{LIrrF}(v)} u)(\prod_{u \in \mathrm{RIrrF}(v)} u),$$

then by I.3 we obtain $v^\omega = w^\omega$. Since $t_1 \in \mathrm{LIrrF}(v)0$, we have

$$
\begin{aligned}
v^\omega h &= w^\omega h \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by I.3} \\
&= w^\omega\big((\prod_{u \in \mathrm{LIrrF}(v)} u)^\omega (\prod_{u \in \mathrm{RIrrF}(v)} u)^\omega\big)(t_1 + \cdots + t_{\mathrm{CP}(h)}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by (A.2) and I.2}
\end{aligned}
$$

$$
\begin{aligned}
&= w^\omega(\omega(h_1) + \Box + \omega(h_2))(t_1 + \cdots + t_{\mathrm{CP}(h)}) \\
&= w^\omega(\omega(h_1) + (t_1 + \cdots + t_{\mathrm{CP}(h)}) + \omega(h_2)) \\
&= w^\omega((\omega(h_1) + t_1) + (t_2 + \cdots + t_{\mathrm{CP}(h)}) + \omega(h_2)) \\
&= w^\omega(\omega(h_1) + (t_2 + \cdots + t_{\mathrm{CP}(h)}) + \omega(h_2)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by I.5 and I.6} \\
&= w^\omega(\omega(h_1) + \Box + \omega(h_2))(t_2 + \cdots + t_{\mathrm{CP}(h)}) \\
&= w^\omega(t_2 + \cdots + t_{\mathrm{CP}(h)}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by I.2} \\
&= v^\omega h' \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by I.3.}
\end{aligned}
$$

G.2 . We can do the similar arguments as in the proof of G.1.

G.3 . Since there is a p-context $w \in \mathrm{Pref}(h)$ such that

$$\mathrm{IrrNIdemF}^*(w) \subseteq \mathrm{IrrNIdemF}^*(v),$$

there is a p-context u and a p-forest h' such that by using I.3 we have $v^\omega = (wu)^\omega$ and $h = wh'$, while I.12 implies the result.

G.4 . We want to show that: if for a positive integer j with $1 \leq j \leq \mathrm{CP}(h)$ and a nonempty set $L \subseteq \mathrm{LIrrF}(v)0$ there is a p-forest s which is a sum of, in any order, of elements of L such that there are p-contexts u and w and a p-forest r with $t_1 + \cdots + t_j = ur$ and $s = uwr$, then $v^\omega h = v^\omega h'$ where $h' = t_{j+1} + \cdots + t_{\mathrm{CP}(h)}$.
Let

$$w = (\prod_{u \in \mathrm{PIrrF}(v)} u)(\prod_{u \in \mathrm{LIrrF}(v)} u)(\prod_{u \in \mathrm{RIrrF}(v)} u).$$

Since the positive integer j with $1 \leq j \leq \mathrm{CP}(h)$ and the nonempty set $L \subseteq \mathrm{LIrrF}(v)0$ are such that there is a p-forest s which is a sum, in any order, of elements of L such that there are p-contexts u and z and a p-forest r such that $t_1 + \cdots + t_j = ur$ and $s = uzr$.

Then by I.6 we obtain for some p-forest h'', $\omega(h_1) = \omega(s + h'')$. So, we have

$$
\begin{aligned}
& v^\omega h \\
&= w^\omega(\omega(s+h'') + \Box + \omega(h_2))h \\
&\hspace{15em} \text{by (A.2) and I.2} \\
&= w^\omega(\omega(s+h'') + h + \omega(h_2)) \\
&= w^\omega(\omega(\omega(s) + \omega(h'')) + \omega(s) + h + \omega(h_2)) \\
&\hspace{15em} \text{by (A.6) and I.5} \\
&= w^\omega(\omega(\omega(s) + \omega(h'')) \\
&\qquad + \omega(uzr) + (ur + t_{j+1} + \cdots + t_{\mathrm{CP}(h)}) + \omega(h_2)) \\
&= w^\omega(\omega(\omega(s) + \omega(h'')) \\
&\qquad + (\omega(uzr) + ur) + (t_{j+1} + \cdots + t_{\mathrm{CP}(h)}) + \omega(h_2)) \\
&= w^\omega(\omega(\omega(s) + \omega(h'')) \\
&\qquad + \omega(s) + (t_{j+1} + \cdots + t_{\mathrm{CP}(h)}) + \omega(h_2)) \\
&\hspace{15em} \text{by (A.5)} \\
&= w^\omega(\omega(\omega(s) + \omega(h'')) \\
&\qquad + \omega(s) + \Box + \omega(h_2))(t_{j+1} + \cdots + t_{\mathrm{CP}(h)}) \\
&= w^\omega(\omega(\omega(s) + \omega(h'')) \\
&\qquad + \Box + \omega(h_2))(t_{j+1} + \cdots + t_{\mathrm{CP}(h)}) \\
&\hspace{15em} \text{by I.5} \\
&= w^\omega(\omega(s+h'') + \Box + \omega(h_2))(t_{j+1} + \cdots + t_{\mathrm{CP}(h)}) \\
&= w^\omega(t_{j+1} + \cdots + t_{\mathrm{CP}(h)}) \\
&\hspace{15em} \text{by I.2} \\
&= v^\omega h' \\
&\hspace{15em} \text{by I.3.}
\end{aligned}
$$

G.5 . We can do the similar arguments as in the proof of G.4.

G.6 . Let $v_n = H_1 + \Box + H_2$ with $H_1 = s_1 + \cdots + s_{\mathrm{CP}(H_1)}$ and $H_2 = s'_1 + \cdots + s'_{\mathrm{CP}(H_2)}$. If $s_{\mathrm{CP}(H_1)} \in \mathrm{Special}_{\mathrm{H}}(h)$, then we have

$$
\begin{aligned}
v\omega(h) &= v_1 \dots v_n \omega(h) \\
&= v_1 \dots v_{n-1}(H_1 + \Box + H_2)\omega(h) \\
&= v_1 \dots v_{n-1}(H_1 + \omega(h) + H_2) \\
&= v_1 \dots v_{n-1}(s_1 + \cdots + s_{\mathrm{CP}(H_1)} + \omega(h) + H_2) \\
&= v_1 \dots v_{n-1}(s_1 + \cdots + (s_{\mathrm{CP}(H_1)} + \omega(h)) + H_2) \\
&= v_1 \dots v_{n-1}(s_1 + \cdots + s_{\mathrm{CP}(H_1)-1} + \omega(h) + H_2) \\
&\hspace{12em} \text{by I.5 and I.23.}
\end{aligned}
$$

We can do the similar arguments for the symmetric case.

G.7 . We want to show that: for $v_n = H_1 + \Box + H_2$ with $H_1 = s_1 + \cdots + s_{\mathrm{CP}(H_1)}$ and $H_2 = s'_1 + \cdots + s'_{\mathrm{CP}(H_2)}$ if

$$s_{\mathrm{CP}(H_1)} \in \mathrm{Special}_{\mathrm{H}}(h) \quad \text{or} \quad s'_1 \in \mathrm{Special}_{\mathrm{H}}(h),$$

then we have the identities

$$v\omega(h) = v_1 \ldots v_{n-1}(s_1 + \cdots + s_{\mathrm{CP}(H_1)-1} + \omega(h) + H_2)$$

or

$$v\omega(h) = v_1 \ldots v_{n-1}(H_1 + \omega(h) + s'_2 + \cdots + s'_{\mathrm{CP}(H_2)}).$$

Let for a positive integer j with $1 \leq j \leq \mathrm{CP}(H_1)$ and a nonempty set $D \subseteq \mathrm{Special}_{\mathrm{H}}(h)$ and there is a p-forest p which is a sum, in some order, of elements of D such that there are p-contexts u and w and a p-forest r such that $s_j + \cdots + s_{\mathrm{CP}(H_1)} = ur$ and $p = uwr$. Then by I.6 we obtain for some p-forest h'', $\omega(h) = \omega(p+q)$. And we have

$$\begin{aligned}
&v\omega(h)\\
&= v_1 \ldots v_n\omega(h)\\
&= v_1 \ldots v_{n-1}(H_1 + \Box + H_2)\omega(h)\\
&= v_1 \ldots v_{n-1}(H_1 + \omega(q+p) + H_2)\\
&= v_1 \ldots v_{n-1}(H_1 + \omega(\omega(q) + \omega(p)) + H_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by (A.6)}\\
&= v_1 \ldots v_{n-1}(H_1 + \omega(p) + \omega(\omega(q) + \omega(p)) + H_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by I.5}\\
&= v_1 \ldots v_{n-1}(s_1 + \cdots + s_{j-1} + ur + \omega(p)\\
&\quad + \omega(\omega(q) + \omega(p)) + H_2)\\
&= v_1 \ldots v_{n-1}(s_1 + \cdots + s_{j-1} + \omega(p) + \omega(\omega(q) + \omega(p)) + H_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by (A.5)}\\
&= v_1 \ldots v_{n-1}(s_1 + \cdots + s_{j-1} + \omega(\omega(q) + \omega(p)) + H_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by I.5}\\
&= v_1 \ldots v_{n-1}(s_1 + \cdots + s_{j-1} + \omega(h) + H_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by (A.6).}
\end{aligned}$$

We can do the similar arguments for the symmetric case.

G.8 . Assume that $\mathrm{Special}_{\mathrm{H}}(h) \subseteq \mathrm{Special}_{\mathrm{H}}(\sum_{x \in \mathrm{LIrrF}(v)0} x)$. Let

$$w = (\prod_{u \in \mathrm{PIrrF}(v)} u)(\prod_{u \in \mathrm{LIrrF}(v)} u)(\prod_{u \in \mathrm{RIrrF}(v)} u).$$

Then $v^\omega = w^\omega$. Therefore,

$$\begin{aligned}
v^\omega \omega(h) &= w^\omega(\omega(h+h_1) + \Box + \omega(h_2))\omega(h) \\
&\qquad\qquad \text{by (A.2), I.2, I.3, and I.22} \\
&= w^\omega(\omega(h+h_1) + \omega(h) + \omega(h_2)) \\
&= w^\omega(\omega(\omega(h)+\omega(h_1)) + \omega(h) + \omega(h_2)) \\
&\qquad\qquad \text{by (A.6)} \\
&= w^\omega((\omega(\omega(h)+\omega(h_1)) + \omega(h)) + \omega(h_2)) \\
&= w^\omega(\omega(\omega(h)+\omega(h_1)) + \omega(h_2)) \\
&\qquad\qquad \text{by I.5} \\
&= w^\omega(\omega(\omega(h)+\omega(h_1)) + \Box + \omega(h_2))0 \\
&= w^\omega(\omega(h+h_1) + \Box + \omega(h_2))0 \\
&\qquad\qquad \text{by (A.6)} \\
&= w^\omega 0 \\
&\qquad\qquad \text{by (A.2), I.2, I.3, and I.22} \\
&= v^\omega 0 \\
&\qquad\qquad \text{by I.3.}
\end{aligned}$$

G.9 . The identity is symmetric to G.8 and we can do the similar arguments for the symmetric case.

G.10 . Assume that $\mathrm{Special_H}(h) \subseteq \mathrm{Special_H}(\sum_{x \in \mathrm{LIrrF}(v)0} x)$. Let

$$w = (\prod_{u \in \mathrm{PIrrF}(v)} u)(\prod_{u \in \mathrm{LIrrF}(v)} u)(\prod_{u \in \mathrm{RIrrF}(v)} u).$$

Then $v^\omega = w^\omega$. Therefore,

$$\begin{aligned}
v^\omega h &= w^\omega(\omega(h+h_1) + \Box + \omega(h_2))h \\
&\qquad\qquad \text{by (A.2), I.2, I.3, and I.22} \\
&= w^\omega(\omega(h+h_1) + h + \omega(h_2)) \\
&= w^\omega(\omega(h+h_1) + \omega(h_2)) \\
&\qquad\qquad \text{by I.5} \\
&= w^\omega(\omega(h+h_1) + \Box + \omega(h_2))0 \\
&= w^\omega 0 \\
&\qquad\qquad \text{by (A.2), I.2, I.3, and I.22} \\
&= v^\omega 0 \\
&\qquad\qquad \text{by I.3.}
\end{aligned}$$

G.11 . The identity is symmetric to G.10 and we can do the similar arguments for the symmetric case.

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Notation