OPTIMALITY CONDITIONS FOR ASYMPTOTICALLY STABLE CONTROL PROCESSES

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Abstract: In this article, we present and discuss the infinite horizon optimal control problem subject to stability constraints. First, we consider optimality conditions of the Hamilton-Jacobi-Bellman type, and present a method to define a feedback control strategy. Then, we address necessary conditions of optimality in the form of a maximum principle. These are derived from an auxiliary optimal control problem with mixed constraints.

Keywords: Optimality Conditions, Optimal Control, Stability Constraints

1. INTRODUCTION

In this article, we address the infinite horizon problem of optimizing a performance criterion by choosing control strategies whose trajectories are asymptotically stable. In a first stage, we state and discuss sufficient conditions of optimality in the form of an Hamilton-Jacobi-Bellman equation, and, based on them, we also present a method to synthesize a feedback control strategy. Then, we present necessary conditions of optimality in the form of a maximum principle and show how it can be derived from an auxiliary optimal control problem with mixed constraints.

In many references, by optimal stabilization it is meant time-optimal stabilization, i.e. finding a control that steers the state of the system to the origin in minimum time. However, here, our goal is substantially different. Given a dynamic control system and a set in the state space containing at least one equilibrium point, we are interested in finding a feedback control strategy which stabilizes the system in the given set (in the sense that the corresponding trajectory converges asymptotically to an equilibrium point) and, simultaneously, minimizes the given cost functional. Notice that the value of the optimal cost depends not only on the equilibrium point, but also on the particular trajectory driving the system to this point.

There has been a significant demand for results for this problem. A small sample of optimal stabilization application problems include micro-electro-mechanical (MEMS) control systems (Chase and Bhashyam, 1999), economic systems under a variety of constraints and assumptions, (Benigno

and Woodford, 2004; Adam, 2002; Suescun, 1998), rigid body mechanical systems (El-Gohary, 2003), biological, medical, health care systems (Gomez and McLaughlin, 1991), and general chaotic systems (Basso *et al.*, 1998), to name just a few.

This contrasts with what appears to be a small body of results available for the general nonlinear dynamic optimization framework addressing the pertinent issues. See for example, (Ugrinovskii and Petersen, 1999) for results on the stabilization and minimax optimal control in the context of stochastic control systems, and (Prieur and Trelat, 2004) for a very specific problem and approach. The problem of stabilizing general dynamic nonlinear control systems has been receiving a considerable attention in the control literature, (Brockett, 1983; Clarke *et al.*, 1997; Sontag, 1998; Clarke *et al.*, 1998; Sontag, 1999; Prieur, 2000) and references cited therein. It has also emerged the important role of dynamic optimization and methods of nonsmooth analysis to derive stability results, see (Clarke *et al.*, 1997; Clarke *et al.*, 1998). However, to the best of our knowledge, no results have been derived for optimal control problems where control strategies are restricted to the subset of stabilizing ones.

This article is organized as follows. In the next section, we introduce a precise and detailed statement of our problem. Then, in section three, the Hamilton-Jacobi-Bellman equation corresponding to this problem is presented followed in the ensuing section by the definition and pertinent results concerning the verification function. In section five, a mechanism of feedback synthesis based on a dynamic programming approach is presented and a result on the convergence of a sequence of sampled feedback control processes is discussed. In section six, we present and outline the derivation of the necessary conditions of optimality. Finally, some brief conclusions are presented.

2. PROBLEM STATEMENT

Let us consider the following dynamic control system

$$
\begin{cases}
\dot{x}(t) = f(x(t), u(t)), & t \in [\tau, +\infty) \text{ a.e.,} \\
x(\tau) = z, \\
u \in \mathcal{U},
\end{cases} \quad (1)
$$

where $\mathcal{U} := \{u(\cdot) \in L^\infty[\tau, \infty): u(t) \in \Omega \subset \mathbb{R}^m \text{ a.e.}\}$ with Ω being a closed convex set. Assume that the closed set $S \subset \mathbb{R}^n$, called target set, contains at least an asymptotic equilibrium of the former, i.e., $\exists \xi \in S$ and $\bar{u}(\cdot) \in \mathcal{U}$ such that $x_{\bar{u}}(\cdot)$, the response of the system to the control $\bar{u}$ with $x_{\bar{u}}(\tau) = z$, satisfies $x_{\bar{u}}(t) \to \xi$ as $t \to \infty$. Let us consider the following optimal control problem:

$$
P_\infty(\tau, z) \text{ Minimize } g(\xi) + \int_\tau^\infty e^{-\delta t} f_0(x(t), u(t)) dt \quad (2)
$$

$$
\text{subject to (1), and}
$$

$$
x(t) \to \xi \text{ as } t \to \infty, \quad (3)
$$

$$
\xi \in S \subset \mathbb{R}^n \quad (4)
$$

Here, the constant $\delta > 0$ is the discount rate, $f_0 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are given functions, and $S \subset \mathbb{R}^n$ is a closed set.

This is an infinite time horizon problem in which the optimization is taken over arcs x such that $x(t)$ converges to an equilibrium point, ξ, of (1), where ξ is also a choice variable. The pair (x, ξ) satisfying the constraints (1), (3), and (4) is called an admissible process of $P_\infty(\tau, z)$. We sometimes refer to an admissible arc x leaving implicit the existence of $\xi \in S$ such that the pair (x, ξ) is an admissible process of $P_\infty(\tau, z)$. Now, we specify the sense of the convergence $x(t) \to \xi$ as $t \to \infty$. By this we mean that, $\exists \gamma > 0, \lim_{t \to \infty} \int_\tau^t e^{\gamma s} \|x(s) - \xi\| ds < \infty$.

Therefore, the optimal trajectory has to approach an equilibrium point in the given set S.

Our approach consists in considering a family of auxiliary optimal control problems where this asymptotic convergence constraint gives rise to a penalization term added to the cost function of the original problem, i.e., we consider the problem:

$$
(P_\infty^l(\tau, z)) \text{ Minimize } g(\xi) + \int_\tau^\infty e^{-\delta t} f_0(x(t), u(t)) dt
$$

$$
+ \int_{\tau+l}^\infty e^{\gamma s} \|x(s) - \xi\| ds
$$

$$
\text{subject to (1), and } \xi \in S \subset \mathbb{R}^n.
$$

Note that we should have $\int_{\tau+l}^\infty e^{\gamma s} \|x(s) - \xi\| ds \to 0$ as $l \to \infty$, thus recovering the original optimization problem without the explicit constraint. Then, we show how to construct an (almost) optimal feedback control for problem $P_\infty^l(\tau, \xi)$. This framework allows us to construct stabilizing optimal feedback controls.

In order to state the assumptions required by the data of our problem, let $F:[0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n+1}$ be defined by $F(t, x, u) = \begin{bmatrix} e^{-\delta t} f_0(x, u) \\ f(x, u) \end{bmatrix}$. They are as follows:

H1) F is continuous and locally Lipschitz in x.
H2) There exists $c > 0$ such that
$F(t, x, u) \in c(1 + \|x\|)B, \ \forall (t, x) \in [0, \infty) \times \mathbb{R}^n$.
H3) $\forall (t, x) \in [0, \infty) \times \mathbb{R}^n$ the set $F(t, x, \Omega)$ is convex-valued.
H4) The set Ω is compact.
H5) g is lower semicontinuous.

3. HAMILTON-JACOBI-BELLMAN EQUATION

In this section we present a number of preliminary concepts and results needed in order to construct an optimal solution of optimal stabilization problem by using the concept of verification function, which can be shown to be a solution to the so called Hamilton-Jacobi-Bellman (HJB) partial differential equation associated with the optimal control problem.

Let $\mathcal{H} : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the Hamiltonian function for this problem defined by

$$\mathcal{H}(t, x, \eta) := \sup_{v \in \Omega}\{\langle \eta, f(x, v)\rangle + e^{-\delta t}f_0(x, v)\}. \quad (5)$$

Then, the continuous function $\phi : [\tau, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is a viscosity solution to the HJB equation if

$$\phi_t(t, x) - \mathcal{H}(t, x, -\phi_x) = 0, \quad \forall (t, x) \in [\tau, \infty) \times \mathbb{R}^n,$$

wherever

$$\nabla_t w(t, x) - \mathcal{H}(t, x, -\nabla_x w(t, x)) \begin{cases} \leq 0 \ \forall (t,x) \in A^-_{\phi-w} \\ \geq 0 \ \forall (t,x) \in A^+_{\phi-w} \end{cases}$$

for any C^1 function $w : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. Here $A^+_{\phi-w}$ and $A^-_{\phi-w}$ denote, respectively, the *argmax* and the *argmin* of the function $(\phi - w)(\cdot, \cdot)$ in $[0, \infty) \times \mathbb{R}^n$. This solution concept satisfies the uniqueness and nonsmoothness requirements of the generalized solution to the HJB equation, but a characterization of an extended valued, lower semicontinuous solution is needed when endpoint state constraints are present. So, now we introduce the proximal sub-gradient.

The *proximal sub-gradient* of Φ at (t, x), denoted by $\partial_P \Phi(t, x)$, is the set of all vectors $(\alpha, \xi) \in \mathbb{R}^{n+1}$ such that $\exists \sigma > 0$ and a neighborhood U of (t, x) satisfying

$$\Phi(\tau, y) \geq \Phi(t, x) + \alpha(\tau - t) + \langle \xi, y - x \rangle \quad (6)$$
$$- \sigma(\|\tau - t\|^2 + \|y - x\|^2),$$

$\forall (\tau, y) \in U$. Analogously, the *proximal super-gradient* of Φ at (t, x), denoted by $\partial^P \Phi(t, x)$, is the set of all vectors $(\alpha, \xi) \in \mathbb{R}^{n+1}$, such that $\exists \sigma > 0$ and a neighborhood U of (t, x) satisfying

$$\Phi(\tau, y) \leq \Phi(t, x) + \alpha(\tau - t) + \langle \xi, y - x \rangle \quad (7)$$
$$- \sigma(\|\tau - t\|^2 + \|y - x\|^2),$$

$\forall (\tau, y) \in U$. The proximal super-gradient can also be defined by $\partial^P \Phi(t, x) = -\partial_P(-\Phi)(t, x)$.

A lower semicontinuous function $v : [\tau, \infty) \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proximal solution to the HJB equation if $\forall (t, x) \in [\tau, \infty) \times \mathbb{R}^n$, such that $\partial^P v(t, x) \neq \emptyset$,

$$\eta_0 - \mathcal{H}(t, x, -\eta) = 0, \quad \forall (\eta_0, \eta) \in \partial^P v(t, x). \quad (8)$$

There are well known results in the literature providing a characterization of the value function,

$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, for an optimal control problem (for our problem in this article $V(\tau, z) := \text{Inf}\{P_\infty(\tau, z)\}$) as a generalized lower semicontinuous solution to the HJB equation (see for example Theorem 12.3.7 in (Vinter, 2000)). Such a result was derived for the infinite time horizon in (Baumeister *et al.*, submitted in 2005).

Clearly, invariance type results provide more detailed information on optimal control processes than this characterization of the value function and thus we proceed with the definition and properties of verification functions in the next section.

4. LOCAL VERIFICATION FUNCTIONS

Next, we provide a result, standard in Dynamic Programming, for the conventional optimal control problem. In fact, we extend the concept of local verification function for this new problem formulation and provide conditions under which the existence of a verification function for a reference process $(\bar{x}, \bar{\xi}, \bar{u})$ is necessary and sufficient for its optimality.

Let $\bar{x}$ be an admissible arc of problem $P_\infty(\tau, z)$. Let $T(\bar{x}, \epsilon)$ be a tube centered at $\bar{x}$ defined by $T(\bar{x}, \epsilon) := \{(t, x) \in [\tau, \infty) \times \mathbb{R}^n : \|x - \bar{x}(t)\| \leq \epsilon\}$. A function $\phi : T(\bar{x}, \epsilon) \to \mathbb{R} \cup +\infty$ is a lower semicontinuous local verification function for $(\bar{x}, \bar{\xi}, \bar{u})$ if ϕ is lower semicontinuous and the following conditions are satisfied.

(1) $\forall (t, x) \in \text{int } T(\bar{x}, \epsilon)$ such that $\partial^P \phi(t, x) \neq \emptyset$,

$$\eta^0 + \min_{u \in \Omega}\{\langle \eta, f(x, u)\rangle + e^{-\delta t}f_0(x, u)\} \geq 0,$$
$$\forall (\eta^0, \eta) \in \partial^P \phi(t, x).$$

(2) $\liminf_{t \to \infty} \phi(t, \xi) \leq g(\xi) + \int_\tau^\infty e^{-\delta t}f_0(x(t), u(t))dt,$

$\forall \xi \in S$ and admissible control process (x, u).

(3) $\liminf_{t \uparrow \infty, \xi' \to \xi} \phi(t, \xi') = \liminf_{t \uparrow \infty} \phi(t, \xi)$ for all $\xi \in S \cap [\bar{\xi} + \epsilon B]$.

(4) $\phi(\tau, z) = g(\bar{\xi}) + \int_\tau^\infty e^{-\delta t}f_0(\bar{x}(t), \bar{u}(t))dt.$

We have the following necessary and sufficient conditions of optimality.

<u>Theorem</u>. Let $(\bar{x}, \bar{\xi}, \bar{u})$ be an admissible process of problem $P_\infty(\tau, z)$. Assume that the basic hypotheses $(H1) - (H5)$ hold. We have the following:

(1) If there exists a lower semicontinuous local verification function for $(\bar{x}, \bar{\xi}, \bar{u})$, then this control process is a strong local minimizer for $P_\infty(\tau, z)$.

(2) Conversely, if $|g(\xi)| + \left|\int_\tau^\infty f_0(x(t), u(t))dt\right|$ is bounded for all admissible processes (x, ξ, u), and $(\bar{x}, \bar{\xi}, \bar{u})$ is a strong local minimizer of

$P_\infty(\tau, z)$, then there exists a lower semi-continuous local verification function for $(\bar{x}, \bar{\xi}, \bar{u})$.

The proof is a slight modification of a similar result for finite time interval problems in (Vinter, 2000).

Take a large $T > 0$, with $T > \tau + l$, and consider the approximate problem $P_T(\tau, z)$ of $P_\infty(\tau, z)$.

$$\text{Min} \qquad J(x, u)$$
$$\text{subject to (1), and } x(T) \in S.$$

Here B is the unit ball of $\mathbb{R}^n$ and $J(x, u)$ is

$$g(x(T)) + \int_\tau^T e^{-\delta t} f_0(x(t), u(t)) dt + \int_{T-l}^T e^{\gamma s} \|x(s) - x(T)\| ds.$$

The rationale behind this approximating problem is that, when $T \to \infty$, the last term in the cost function goes to zero and we can show that there is subsequence $x(T_k)$ converging to some $\xi \in S$.

Now, we define verification function for a process $(\bar{x}, \bar{u})$ of problem $P_T(\tau, z)$. The function $\phi : T(\bar{x}, \delta) \to \mathbb{R} \cup \{\infty\}$ is a lower semicontinuous local verification function for $\bar{x}$ with parameter $\delta > 0$ if ϕ is lower semicontinuous and satisfies:

a) $\forall (t, x) \in int T(\bar{x}, \delta)$ such that $\partial^P \phi(t, x) \neq \emptyset$,
$$\eta^0 + \min_{u \in \Omega} \{\langle \eta, f(x, u) \rangle + f_0(x, u)\} \geq 0 \quad (9)$$
$$\forall (\eta^0, \eta) \in \partial^P \phi(t, x).$$

b) $\phi(T, \zeta) \leq g(z), \qquad \forall \zeta \in S.$

c) $\liminf_{t' \uparrow T, x' \to \zeta} \phi(t', x') = \phi(T, \zeta), \ \forall \zeta \in S \cap [\bar{x}(T) + \delta B].$

d) $\phi(\tau, z) = g(\bar{x}(T)) + \int_\tau^T e^{-\delta t} f_0(\bar{x}(t), \bar{u}(t)) dt$
$$+ \int_{T-l}^T e^{\gamma s} \|\bar{x}(s) - \bar{x}(T)\| ds.$$

We have the following result which can be found in (Vinter, 2000).

<u>Theorem.</u> Let $(\bar{x}, \bar{u})$ be an admissible process of problem $P_T(\tau, z)$. Assume that the basic hypotheses $(H1) - (H5)$ hold. We have the following:

(1) If there exists a lower semicontinuous local verification function for $(\bar{x}, \bar{u})$, then this control process is a strong local minimizer for $P_T(\tau, z)$.

(2) Conversely, if $(\bar{x}, \bar{u})$ is a strong local minimizer of $P_T(\tau, z)$ and $|\int_\tau^T f_0(\bar{x}(t), \bar{u}(t)) dt + \int_{T-l}^T e^{\gamma s} \|x(s) - x(T)\| ds| + |g(\bar{x}(T))|$ is bounded for all admissible processes $(\bar{x}, \bar{u})$, then there exists a a lower semicontinuous local verification function for $(\bar{x}, \bar{u})$.

5. FEEDBACK SYNTHESIS

Here, we present and discuss an algorithm for feedback control synthesis for problem $P_\infty(\tau, z)$

that, essentially, is a version of the procedure in (Vinter, 2000) modified in order to force the state to reach the target set S. A partition $\pi = \{t_k\}$ of $[\tau, \infty)$ is a countably, strictly increasing sequence t_k such that $t_i > t_j$, whenever $i > j$, $t_k \to \infty$ as $k \to \infty$. The diameter of π, denoted by h_π, is defined by $\sup_{k \geq 0} \{\Delta_k\}$, where $\Delta_k = t_{k+1} - t_k$. Let us assume that $\tau = 0$.

Let ϕ be a given local verification function as defined in the previous section and let $x \in \mathbb{R}^n$ be a given state. Define

$$U(x) := \{u \in \Omega, \langle N_S^P(p_S(x)), f(x, u) \rangle \leq 0\}$$

where $p_S(x)$ is the proximal point of x at S.

Let us start with $x(0) = x_0$. Then, an approximating optimal control process is constructed recursively by computing a piecewise constant control function given, for each $k = 0, 1, \ldots$ by

$$\bar{u}_k^\pi \in \arg \max_{u \in U(x^\pi(t_k^\pi))} \left\{ \phi\left(t_k^\pi, x^\pi(t_k^\pi) + \Delta_k f(x^\pi(t_k^\pi), u)\right)\right.$$
$$\left. + \Delta_k f^0(x^\pi(t_k^\pi), u)\right\}$$

and the corresponding trajectory is obtained by integrating the dynamics differential equation with the boundary condition given by the last value of the state variable in the previous time subinterval of the partition. Namely, $x^\pi(t)$ is defined on $[t_k^\pi, t_{k+1}^\pi)$ as the solution of

$$\dot{x}(t) = f(t, x(t), \bar{u}_k^\pi) \quad \text{a.e.} t \in (t_k^\pi, t_{k+1}^\pi],$$

with initial value $x(t_k^\pi)$ given by the value of the state variable in the previous interval.

We have the following main result of this work.

<u>Theorem.</u> Assume that $(H1) - (H5)$ hold. Let ϕ be a lower semicontinuous solution to the Hamilton-Jacobi-Bellman equation. Take (x^π, u^π), the control process obtained by the recursive procedure described above. Then, x^π has a cluster point [3] with respect to the topology of uniform convergence on compact intervals, and, associated with such a point $x(\cdot)$, there is a pair, control $u(\cdot)$ and limit point ξ, such that $(x(\cdot), \xi, u(\cdot))$ is an optimal process of $P_\infty(0, x_0)$.

Here, we just outline the proof. Given $N \in \mathbb{N}$, there exist a partition $\pi(N)$, and a corresponding process (x^π, u^π) constructed by the procedure described above, such that (x^π, u^π) restricted to the time interval $[0, T_N]$ is admissible for the slightly perturbed problem $P_{T_N}(0, x_0)$, defined by:

$$(P_N) \text{ Minimize } J(x, u)$$
$$\text{subject to (1), and } x(T^N) \in S + \frac{1}{N} B.$$

---

[3] A cluster point of a given sequence is a point to which there is a convergent subsequence.

Our aim is to extract the limit as $N \to \infty$ to recover the original problem. So, we assume that the sequence T_N satisfies $T_N \to \infty$ as $N \to \infty$. Following arguments in (Vinter, 2000), it is possible to show that, under assumptions $(H1)-(H5)$, $\{x^\pi\}(\cdot)$ has a cluster point, $\bar{x}^N(\cdot)$, on $[0, T_N]$. Moreover, by using Filippov's selection theorem, the existence of $\bar{u}^N$ such that $(\bar{x}^N(\cdot), \bar{u}^N(\cdot))$ is an optimal process for P_N is asserted. Since $\bar{x}^N(T_N) \in S + (1/N)B$ and S is compact, there exists a subsequence of $\bar{x}^N(T_N)$ converging to some point $\xi \in S$. We denote it by $\{\bar{x}_k^N\}$ and consider the corresponding processes $(\bar{x}_k^N(\cdot), \bar{u}_k^N(\cdot))$ on the respective intervals $[0, T_k]$, $k = 1, 2, \ldots$. T_k is a subsequence of T_N.

Now, we show how to obtain the optimal control process for the original problem. Restrict $(\bar{x}_k^N(\cdot), \bar{u}_k^N(\cdot))$ to $[0, T_1]$. Again it is possible to show that $\bar{x}_k^N(\cdot)$ has a cluster point, $x^1(\cdot)$, in the uniform convergence topology on $[0, T_k]$ and there exists a corresponding control function $u^1(\cdot)$. Now, by considering this subsequence $(\bar{x}_k^N(\cdot), \bar{u}_k^N(\cdot))$ restricted to $[0, T_2]$ for $k = 2, 3, \ldots$ and repeating the same argument as before, we can find a process $(x^2(\cdot), u^2(\cdot))$ satisfying (1) and (2) restricted to $[0, T_2]$, in which $(x^2(\cdot))$ is a cluster point of $(\bar{x}_k^N(\cdot))$. By continuing this process, for all $k \in N$, we can show $(x^k(\cdot), u^k(\cdot))$ satisfying (1) and (2) restricted to $[0, T_k]$, in which $(x^k(\cdot))$ is a cluster point of $(\bar{x}_k^N(\cdot))$. For each $T > 0$, there exists $k \in N$ such that $T \in [T_{k-l}, T_k]$.

Define $(\bar{x}(\cdot), \bar{u}(\cdot)) : [0, \infty) \to I\!\!R^n \times I\!\!R^m$ by

$$(\bar{x}(t), \bar{u}(t)) \equiv (x^k(t), u^k(t)) \quad t \in [0, T]. \quad (10)$$

<u>Lemma</u>. We have the following.

(1) The function given by (10) is well defined.
(2) $(\bar{x}(t), \bar{\xi}, \bar{u}(t))$ is an optimal process for $P_\infty(0, x_0)$.

By construction $(\bar{x}(t), \bar{u}(t))$ is well defined. The second assertion follows from the existence of ϕ and from the first theorem in the previous section.

6. NECESSARY CONDITIONS OF OPTIMALITY

Consider problem $(P_\infty(\tau, z))$ formulated in section 2. In order to derive the necessary conditions we specify further the constraint of asymptotic convergence, $x(t)$ to ξ as $t \to \infty$. We impose that the rate of asymptotic convergence is not smaller than some given positive number γ. It is not difficult to see that this condition can be expressed as the following inequality mixed constraint

$$h(x, u) := \frac{x^T f(x, u)}{\|x\|^2} + \gamma \leq 0.$$

Let us fix $\tau = 0$ and $z = x_0$, and consider the following optimal control problem

(P) Minimize $J(u)$ $\qquad\qquad$ (11)

subject to $\dot{x}(t) = f(x(t), u(t))$ $\mathcal{L}-a.e.$ (12)

$\qquad\qquad x(0) = x_0, \quad x(t) \to \xi \in S$ (13)

$\qquad\qquad h(x(t), u(t)) \leq 0 \quad \forall t \geq 0$ (14)

$\qquad\qquad u(t) \in \Omega \quad \forall t \geq 0$ $\qquad\qquad$ (15)

where $J(u) := g(\xi) + \int\limits_0^\infty f_0(x(t), u(t))dt$. Notice that it is enough to specify u^* since it will follow that $x^*(t) \to \xi^* \in S$ as $t \to \infty$.

Let us state now our optimality conditions. Let the Pontryagin function or pseudo-Hamiltonian be defined as

$$H(x, p, q, \lambda, u) := p^T f(x, u) + q h^+(x, u) + \lambda f_0(x, u))$$

where $h^+(x, u) = \max\{0, h(x, u)\}$.

<u>Theorem</u>. Let (x^*, u^*) be an optimal control process for problem (P).

Then, there exists an absolutely continuous function $p : [0, \infty) \to I\!\!R^n$, a monotonically increasing function $q : [0, \infty) \to I\!\!R$, and a number $\lambda \geq 0$ satisfying:

$$-\dot{p}(t) \in \partial_x H(x^*(t), p(t), q(t), \lambda, u^*(t)) \quad (16)$$

$$-\lim_{s \to \infty} p(s) \in \lambda \partial_x g(\xi^*) + N_S(\xi^*) \quad (17)$$

$$q(t) = -\int\limits_{[t, \infty)} d\nu(s) \quad (18)$$

$$\lim_{s \to \infty} q(s) = 0 \quad (19)$$

$$u^*(t) \quad \text{maximizes the mapping} \quad (20)$$

$$v \to H(x^*(t), p(t), q(t), \lambda, v) \text{ on } \Omega. \quad (21)$$

Here, $d\nu$ is an arbitrary positive measure supported on $[0, \infty)$. Note that, by denoting the generalized gradient (in the sense of Clarke, see (Clarke *et al.*, 1998)) by ∂f, we have

$$\partial_x h^+(x, u) = \begin{cases} \{0\} & \text{if } h(x, u) < 0 \\ co\{0, \partial_x^> h(x, u)\} & \text{if } h(x, u) = 0, \end{cases}$$

where $\partial_x^> h(x, u) = \lim\limits_{x' \to x} \sup\limits_{h(x', u) > 0} \partial_x h(x', u)$.

We will need a refinement of the previously stated assumptions on the data of the optimal control problem

H1) The functions g, f_0, f and h are Lipschitz continuous in x uniformly w.r.t. all other variables.

H2) The functions f_0, f and h are Borel measurable w.r.t. the control variable.

H3) The sets $S \in I\!\!R^n$ and $\Omega \in I\!\!R^m$ are closed and bounded.

H4) There is at least one equilibrium point in S.

Now, we outline the proof. We consider the following steps:

a) The result is proved for an auxiliary optimal control problem with mixed constraints and a certain finite time T, denoted by (P_T).

b) The infinite horizon is regarded as the limit of the conditions for finite time for the problem (P_T). Given an optimal control process for the infinite time horizon, its truncation to some finite interval $[0, T]$ for T sufficiently large is proved to be an almost minimizer of the auxiliary finite time optimal control problem. Then, Ekeland's variational principle is applied and the necessary conditions of optimality proved in $a)$ are applied. Finally, limits are extracted in order to get the desired result.

c) By extending the dynamics, we produce another auxiliary problem (Q_T) exhibiting only conventional state constraints and without mixed constraints. Well known necessary conditions of optimality in the form of a maximum principle, (Vinter, 2000), can be readily written down for (Q_T) and expressed in terms of the data of (P_T).

7. CONCLUSIONS

Here we presented and discussed an infinite time horizon control optimization problem in which a given objective functional is optimized by choosing control strategies which ensure the stabilization of the dynamic control system within a given target set with respect to which the system is invariant. Therefore, the trajectory associated to the optimal control process converges asymptotically to an optimal equilibrium within a given target set. We provided a dynamic programming based algorithm which yields a control process defined in a feedback form that approximates the optimal process. The method proposed here is modification of previous construct in (Rowland and Vinter, 1991) for a simpler problem. In this article, the model is finite time interval and there are no target set or set constraints. We also present necessary conditions of optimality in the form of a maximum principle for an optimal control process satisfying a prescribed minimum rate for the asymptotic convergence towards the optimal equilibrium point in a given target set.

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