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The word problem and some reducibility properties for pseudovarieties of the form DRH

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To my parents

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Abstract

The present work is focused on the study of some properties of pseudovarieties of the form DRH, which consists of all finite semigroups whose regular $\mathcal{R}$-classes lie in a certain pseudovariety of groups H.

Let κ denote the canonical implicit signature containing two implicit operations: multiplication and $(\omega-1)$-power. The latter is the unary implicit operation that assigns to each element s of a finite semigroup S the inverse of $s1_{G(s)}$ in the unique maximal subgroup $G(s)$ of the subsemigroup of S generated by s (we are using $1_{G(s)}$ to denote the identity of $G(s)$). Given an arbitrary implicit signature σ, we call σ-word any element of the free σ-semigroup, denoted $\Omega_A^\sigma \mathsf{S}$.

We start by considering the κ-word problem over DRH, that consists in deciding whether the natural interpretations of two given κ-words coincide in every semigroup of DRH. We show that this problem is decidable assuming so is the analogous problem for H. We also exploit the concept of DRH-automaton in order to define a canonical form for the elements of $\Omega_A^\kappa \mathsf{DRH}$, based on the knowledge of a canonical form for the elements of $\Omega_A^\kappa \mathsf{H}$. The approach conducted consists in a generalization of the tools used by Almeida and Zeitoun when they solved the analogous problems for the pseudovariety R, consisting of all $\mathcal{R}$-trivial finite semigroups.

Next, we study some reducibility properties of pseudovarieties of the form DRH, with respect to certain classes of systems of equations. Informally, given an implicit signature σ, an alphabet X with a constraint on each element, and a class $\mathcal{C}$ of finite sets of formal equalities between σ-words on X (known as systems of σ-equations), we say that a pseudovariety V is σ-reducible with respect to $\mathcal{C}$ if the existence of a solution modulo V of any system in $\mathcal{C}$ yields the existence of a solution modulo V given by σ-words (both satisfying the given constraints). Fix an implicit signature σ that contains a non-explicit operation. First, we prove that H being σ-reducible with respect to systems of the form $x_1 = \cdots = x_n$ implies that so is DRH. Let us include a definition before proceeding. Given a graph, we associate to each edge $x \xrightarrow{y} z$ the equation $xy = z$. Systems of equations obtained in this way from a finite graph are said to be systems of graph equations. We show that DRH is σ-reducible with respect to systems of graph equations if and only if the same happens for H. This is inspired by some of the results of Almeida, Costa, and Zeitoun which were established in order to derive κ-reducibility of some joins of the form $\mathsf{R} \vee \mathsf{W}$ with respect to systems of graph equations. Finally, we prove that, if we further require that a non-explicit operation with value 1 over H may be obtained from elements of σ by composition, then H being σ-reducible with respect to systems of graph equations suffices for DRH being σ-reducible with respect to systems of the form $x_1 = \cdots = x_n = x_n^2$. Both the first and last mentioned properties for DRH depend on a certain periodicity that may be found on the given constraints.

Our last result states that a pseudovariety DRH is κ-reducible with respect to any system of κ-equations if and only if H enjoys the same property. In fact, we show something slightly more general. However, the results known so far do not suggest any advantage in the general formulation we present. We choose to do so for two reasons: firstly, there is no significant extra effort in doing it; and secondly, we hope that some other results may appear in the near future. The solution relies on a generalization of the adaptation of Makanin's algorithm (originally used by Makanin to solve word equations in free semigroups) carried out by Almeida, Costa, and Zeitoun in order to show that R is κ-reducible with respect to any finite system of κ-equations.

Resumo

Neste trabalho concentramo-nos no estudo de algumas propriedades das pseudovariedades DRH, isto é, pseudovariedades formadas por todos os semigrupos finitos cujas $\mathcal{R}$-classes regulares pertencem a uma dada pseudovariedade de grupos H.

Denotemos por κ a assinatura implícita canónica, a qual contém duas operações implícitas: a multiplicação, e a potência $(\omega - 1)$. A última, é a operaçao unária que associa a cada elemento s num semigrupo finito S, o inverso de $s1_{G(s)}$ no único subgrupo maximal $G(s)$ do subsemigrupo de S gerado por s (estamos a denotar por $1_{G(s)}$ a identidade de $G(s)$). Se σ for uma assinatura implícita arbitrária, chamamos σ-palavras aos elementos do σ-semigrupo livre, denotado $\Omega_A^\sigma \mathsf{S}$.

A primeira questão abordada é o problema da κ-palavra em DRH, o qual consiste em decidir quando é que a interpretação natural de duas κ-palavras coincide em todos os semigrupos de DRH. Mostra-se que este problema é decidível, assumindo que o problema análogo para H também é. A definição dos chamados DRH-autómatos é aproveitada para exibir uma forma canónica para os elementos de $\Omega_A^\kappa \mathsf{DRH}$, partindo do conhecimento de uma forma canónica para os elementos de $\Omega_A^\kappa \mathsf{H}$. A abordagem que fazemos consiste numa generalização das ferramentas usadas por Almeida e Zeitoun quando resolveram os problemas análogos para a pseudovariedade R dos semigrupos finitos cujas $\mathcal{R}$-classes são triviais.

De seguida, é estudada a redutibilidade das pseudovariedades DRH para certas classes de sistemas de equações. Informalmente, dada uma assinatura implícita σ, um alfabeto X com certas restrições, e uma classe $\mathcal{C}$ de conjuntos finitos de igualdades formais entre σ-palavras obtidas a partir de X (ditos sistemas de σ-equações), dizemos que a pseudovariedade V é σ-redutível em relação a $\mathcal{C}$ se a existência de uma solução módulo V para qualquer sistema em $\mathcal{C}$ garante a existência de uma solução módulo V dada por σ-palavras (ambas respeitando as restrições dadas). Tomemos uma assinatura implícita σ que contém pelo menos uma operação não explícita. Primeiro mostramos que se H for σ-redutível para sistemas da forma $x_1 = \cdots = x_n$, então DRH também é. Incluímos uma definição antes de apresentar o resultado seguinte. Dado um certo grafo, associa-se a cada aresta $x \xrightarrow{y} z$ a equação $xy = z$. Todo o sistema de equações obtido desta forma a partir de um grafo finito diz-se um sistema de equações de grafos. É mostrado que a pseudovariedade DRH é σ-redutível em relação a esta classe de sistemas se e só se H também o for. A solução apresentada é inspirada em resultados que Almeida, Costa, e Zeitoun provaram aquando do estudo da κ-redutibilidade em relação a sistemas de equações de grafos de algumas pseudovariedades da forma $\mathsf{R} \vee \mathsf{W}$. Finalmente, são considerados sistemas da forma $x_1 = \cdots = x_n = x_n^2$ e supõe-se que existe uma operação não explícita com valor 1 em H que pode ser obtida de elementos de σ por composição. Mostra-se que para DRH ser σ-redutível em relação a este tipo de equações é suficiente que H seja redutível em relação a sistemas de equações de

grafos. Tanto o primeiro como o último problemas abordados dependem de uma certa periodicidade que pode ser encontrada nas restrições das variáveis dos sistemas em questão.

Este trabalho culmina com a prova de que a pseudovariedade DRH é κ-redutível em relação a qualquer sistema de κ-equações se e só se o mesmo acontece com H. Na realidade, mostra-se algo ligeiramente mais forte. Contudo, os resultados conhecidos até à data não evidenciam nenhuma vantagem na formulação mais geral que apresentamos. Uma vez que não acarreta esforço extra significativo, essa formulação é a escolhida na expectativa de que outros resultados possam aparecer no futuro. O método usado consiste numa generalização da adaptação do Algoritmo de Makanin (usado por Makanin para resolver equações de palavras em semigrupos livres) feita por Almeida, Costa, e Zeitoun para mostrarem que a pseudovariedade R é κ-redutível em relação a qualquer sistema finito de κ-equações.

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Chapter 1

Introduction

Although the notion of semigroup goes back at least to the early 1900s [36], the first remarkable result, a description of the structure of finite simple semigroups, was obtained by Suschkewitsch [68] in 1928. Later on, in 1940, Rees [57] improved that result by characterizing completely 0-simple semigroups. It was Kleene [46] in 1956 who introduced for the first time what is nowadays known as *rational languages*. It appeared in the wake of his study of models of biological neurons, which was an emergent subject by then. By the same time, Green [40] introduced the relations that later received his name. During all the fifties and sixties, the study of semigroups was essentially motivated by its application in Theoretical Computer Science, namely, by the study of automata and rational languages. This led to results relating the study of combinatorial properties of rational languages to the study of algebraic properties of their syntactic semigroups. The work of Brzozowski, McNaughton, Schützenberger, Simon and Zalcstein generated a lot of results in the sixties and seventies, revealing many of the connections between finite automata, recognizable languages and finite semigroups (see, for instance, [31, 33, 51, 52, 64, 66, 71]). In particular, the characterization of star-free languages as those that are recognized by finite aperiodic monoids due to Schützenberger [64] and Simon's Theorem [66], stating that piecewise testable languages are precisely those recognized by a finite $\mathcal{J}$-trivial monoid, were of great relevance.

Meanwhile, in 1965, Krohn and Rhodes [47] also came up with a significant result. They proved the so-called Krohn-Rhodes Prime Decomposition Theorem. In its original form, it states that every finite semigroup S divides a semidirect product whose factors are either simple groups dividing S or the monoid U_3 (that is, the three element monoid containing two right-zeros and an identity). This led to the notion of *Krohn-Rhodes complexity* of a finite semigroup S: it is the smallest non-negative integer n such that S divides the wreath product $A_n \wr G_n \wr A_{n-1} \wr G_{n-1} \wr \cdots \wr A_1 \wr G_1 \wr A_0$, where $A_0, A_1, \ldots, A_n$ are aperiodic semigroups and $G_1, \ldots, G_n$ are groups [48]. About 50 years later, it is still an open problem to prove or disprove the decidability of the complexity of a semigroup. Some partial results have been appearing since then, namely in the form of lower and upper bounds for complexity, the most recent in [43].

It was merely in 1976 that Eilenberg and Schützenberger [39] introduced the formal definition of *pseudovariety*. In Eileinberg's treatise [37, 38], the relationship between pseudovarieties of finite semigroups and varieties of rational languages is finally explicitly established [38, Chapter VII, Theorem 3.4s]. And then started a revolutionary growth of the theory of finite semigroups. Following

Eilenberg's work, it is worth mentioning the results obtained by Tilson, with special emphasis on the Derived Category Theorem [69], whose main purpose was to show how essential it is to study categories as generalizations of monoids.

Another line of development came from the fact that Eilenberg considered pseudovarieties ultimately defined by sequences of equations [39]. In an attempt to obtain a finite analogue of the Birkhoff Variety Theorem [30] known from Universal Algebra, in 1982, Reiterman [58] introduced *implicit operations* and decribed pseudovarieties by *equations of implicit operations*.

An additional adversity in the study of pseudovarieties was that, unlike in Universal Algebra, pseudovarieties have no free objects. However, they appear naturally when profinite constructions over pseudovarieties are taken. Indeed, the structure of metric space that Reiterman defined on the space of A-ary implicit operations in a pseudovariety V is precisely the projective limit of the A-generated members of V [2]. The latter is nowadays known as the *free A-generated pro-V semigroup* and denoted $\overline{\Omega}_A\mathsf{V}$ (a notation rather similar to the one already used in [58]). The decade of 1990 was crucial to the development of profinite semigroups as a tool for studying pseudovarieties of finite semigroups, having been carried out mainly by Almeida [3, 6, 7].

Quite often, one of the questions arising when facing a certain class consists in determining whether a given object belongs or not to that class, the so-called *membership problem*. In what concerns pseudovarieties, Eilenberg's correspondence justifies the interest in studying the decidability of the membership problem in that context. That means to prove either that there exists an algorithm deciding whether a given finite semigroup belongs to a certain pseudovariety, in which case the pseudovariety is said to be *decidable*; or to prove that such an algorithm does not exist, being thus in the presence of an *undecidable* pseudovariety. Since many relevant pseudovarieties are a result of the application of some natural operators on pseudovarieties V and W, such as the join $\mathsf{V} \vee \mathsf{W}$, the semidirect product $\mathsf{V} * \mathsf{W}$, the two-sided semidirect product $\mathsf{V} ** \mathsf{W}$, or the Mal'cev product $\mathsf{V} \mathbin{ⓜ} \mathsf{W}$, it is also relevant to decide the membership problem for the resulting pseudovariety. Witnessing this statement there is, for instance, Rhodes' paper [59] in which he shows interest in finding out whether some particular joins of pseudovarieties are decidable. By then, it was already known that the join operator does not preserve decidability [1].[1] Studying the decidability of pseudovarieties of the form $\mathsf{V} * \mathsf{W}$ seems natural in view of the Krohn-Rhodes problem. Also in this setting, the Mal'cev product plays a role, thanks to the Fundamental Lemma of Complexity [62]. In turn, the two-sided semidirect product shows up as a natural construction when studying the kernel category of a relational morphism. Many other results appeared highlighting the relevance of these operators. Again, none of $*$, ⓜ and $**$ preserves decidability [60].

Aiming to guarantee the decidability of pseudovarieties obtained through the application of $*$ from a stronger property for the involved pseudovarieties, the notion of *hyperdecidability* was introduced [4]. That notion seemed natural after Almeida and Weil [22, Theorem 5.3] proved that, given pseudovarieties of semigroups V and W, a basis for a pseudovariety $\mathsf{V} * \mathsf{W}$ could be obtained from a basis of the pseudovariety of semigroupoids $g\mathsf{V}$.[2] Shortly after, the notion of *(weak) reducibility* [16]

[1] The careful reader may realize that [59] was published in 1987 while [1] was just published in 1992. However, there was a preprinted version of [1] dating from 1986, under the name of "Undecidability of the identity problem for finite semigroups with applications" by Albert and Rhodes.

[2] It was later realized that the referred theorem has a gap in its proof. Although it is not known so far the full generality of the result, it remains valid if the pseudovariety of semigroupoids $g\mathsf{V}$ has finite vertex-rank (see also [61]).

emerged as a method of establishing hyperdecidability of pseudovarieties. Almeida and Steinberg [16, Theorem 4.7] proved that every recursively enumerable pseudovariety V for which there exists a highly computable implicit signature σ making V into a σ-recursive and σ-reducible pseudovariety is also hyperdecidable. That may be seen as a reason for the introduction of *tame* pseudovarieties [17]. It turns out that tameness is a stronger, but in general easier to prove, property. Yet, we point out that σ-recursiveness (under certain reasonable conditions) is equivalent to the decidability of the word problem in the free σ-semigroup over V, denoted $\Omega_A^\sigma \mathsf{V}$ [16, Theorem 3.1] (such word problem is also called *σ-word problem over* V). Therefore, solving the word problem in $\Omega_A^\sigma \mathsf{V}$ is not only a matter appearing transversely when dealing with any algebraic structure, but also something really useful in proving the decidability results already mentioned. Some other variants of these strong versions of decidability may be found in the literature (see [6] for an overview).

It is also worth mentioning that a particular instance of hyperdecidability, known as *strong decidability*, was already considered for several years under the name of *computable pointlike sets*. For instance, in 1988 Henckell [41] proved that finite aperiodic semigroups have computable pointlike sets or, in other words, that the pseudovariety A of finite aperiodic semigroups is strongly decidable. This study was conducted to produce progress in the question of decidability of the Krohn-Rhodes complexity for semigroups. Along the same line, Ash [28] introduced *inevitable sequences* in a finite monoid (for finite groups) in order to prove the Rhodes type II conjecture [45, Conjecture 1.3]. Deciding whether a sequence $(s_1,\dots,s_n)$ from a finite monoid is inevitable in Ash's sense translates to hyperdecidability of the pseudovariety G of finite groups with respect to the equation $x_1\cdots x_n = 1$. Also, Pin and Weil [55, Theorem 4.1] described a defining set of identities for a Mal'cev product, which in turn implies that the decidability of idempotent pointlike sets may be used as a sufficient condition for decidability of Mal'cev products of pseudovarieties [8, Theorem 4.2]. The diversity of motivations behind these works somehow indicates that hyperdecidability may lead the way to a better understanding of the structure of finite semigroups. Indeed, many researchers have shown interest in studying these properties for pseudovarieties. Just to name a few results, it follows from Ash's work that G is κ-tame [27]; Almeida and Zeitoun [23] proved that the pseudovariety J of all $\mathcal{J}$-trivial semigroups is hyperdecidable, and later Almeida [6] that it is completely κ-tame; although the pseudovariety G_p of p-groups is not κ-tame [6, 29], Steinberg [67] proved that it is hyperdecidable, and Almeida [5] that there exists an implicit signature σ that makes it σ-tame; Almeida and Trotter [18] proved hyperdecidability and κ-reducibility of the pseudovariety OCR of orthogroups; Almeida and Zeitoun [24] that the pseudovarieties N (nilpotent semigroups), K and D (semigroups whose idempotents are left and right zeros, respectively), and LI (the smallest pseudovariety containing both K and D) are κ-tame; Henckell [41] proved that A is hyperdecidable with respect to systems of the form $x_1 = \cdots = x_n = x_n^2$; complete κ-tameness of the pseudovariety Ab of Abelian groups was proved by Almeida and Delgado [13]; and Henckell, Rhodes and Steinberg [42] proved that the pseudovariety $\overline{\mathsf{G}}_p$ of semigroups whose subgroups belong to G_p is strongly decidable.

On the other hand, Brzozowski and Fich [32] conjectured that $\mathsf{Sl} * \mathsf{L} = \mathsf{GLT}$ and established the inclusion $\mathsf{Sl} * \mathsf{L} \subseteq \mathsf{GLT}$. Here, Sl is the pseudovariety of finite semilattices, L is the pseudovariety of finite $\mathcal{L}$-trivial semigroups, and GLT is the pseudovariety of semigroups S for which $eS_ee \in \mathsf{Sl}$, for every idempotent $e \in S$, where S_e is the subsemigroup generated by the elements lying $\mathcal{J}$-above e. Motivated by this problem, Almeida and Weil [21] considered the dual of the pseudovariety L, the pseudovariety R of $\mathcal{R}$-trivial finite semigroups, and described the structure of the free pro-R semigroup. Later on, it was proved by Almeida and Silva [15] that the pseudovariety R is SC-hyperdecidable for the canonical implicit signature κ, and by Almeida, Costa and Zeitoun that R is tame [9], completely κ-reducible [10], and strongly decidable [11]. Although tameness implies strong decidability, the improvement of [11] with respect to [9] lies in the fact that an algorithm is presented in the former.

A natural generalization of R is found in the pseudovarieties of the form DRH for a pseudovariety of groups H. This class contains all finite semigroups whose regular $\mathcal{R}$-classes are groups lying in H. Observe that, when H is the trivial pseudovariety $[\![x = y]\!]$, the pseudovariety DRH is nothing but R.

Also, the pseudovarieties DRH may be seen as a specialization of the pseudovariety DS, of all finite semigroups whose regular $\mathcal{D}$-classes are subsemigroups. The interest in the latter has been pointed out by Schützenberger in [65], where he characterizes the varieties of rational languages corresponding to some subpseudovarieties of DS under Eilenberg's correspondence, among which DRH.

These considerations motivated us to study the pseudovarieties of the form DRH. Our main concern in this work is to generalize for any pseudovariety DRH some of the already known properties of R (when imposing some reasonable conditions on H).

The thesis is organized in five chapters (besides the present Introduction) as follows.

Chapter 2 serves the purpose of standardizing most of the definitions and notation used in the rest of the thesis. In particular, definitions of $\mathcal{C}$-decidability (where $\mathcal{C}$ is a class of formal equalities between elements of $\overline{\Omega}_A\mathsf{S}$), hyperdecidability, strong decidability, (complete) σ-reducibility and (complete) σ-tameness are provided in Section 2.4, in the way they are used in the subsequent chapters. It also includes the statement of some results involving these notions. Section 2.6 contains a summarized exposition of the representation of the elements of $\overline{\Omega}_A\mathsf{DRH}$ in terms of certain *decorated reduced A-labeled ordinals* found in [21]; while in Section 2.7 we state and prove some results used later that, although they are not new, as far as we know, they are not formulated in the literature.

In Chapter 3 we solve the κ-word problem over DRH, assuming that there exists a solution for the κ-word problem over H. We also take the opportunity to present a canonical form for the elements of $\Omega_A^\kappa\mathsf{DRH}$, based on the knowledge of a canonical form for the elements of $\Omega_A^\kappa\mathsf{H}$. Here, the symbol κ represents the canonical implicit signature, that is, the implicit signature consisting of multiplication and $(\omega-1)$-power. The approach followed is analogous to that which has been carried out in [25] for the pseudovariety R. Let us describe the main steps of the solution of the referred word problem. First, we define the class of DRH-*automata* as well as an equivalence relation $\sim$ on it. Associating a value of $\overline{\Omega}_A\mathsf{DRH}$ to each such automaton it is proved the existence of a bijection $\overline{\pi}$ between the $\sim$-classes of DRH-automata and the $\mathcal{R}$-classes of $\overline{\Omega}_A\mathsf{DRH}$. Also, a certain language is associated to each DRH-automaton. That language has the property of supplying a complete characterization of the relation $\sim$, meaning that two DRH-automata are equivalent if and only if the associated languages are the same. Thus, solving the word problem in $\Omega_A^\kappa\mathsf{DRH}$ turns into solving the word problem in $\Omega_A^\kappa\mathsf{H}$ plus comparing languages associated to DRH-automata (since the pseudovariety DRH satisfies a pseudoidentity $u = v$ if and only if it satisfies $u \mathrel{\mathcal{R}} v$ and H satisfies $u = v$). After proving some technical results, we devote Section 3.4 of this chapter to prove the existence of an algorithm deciding whether two given κ-words over DRH lie in the same $\mathcal{R}$-class. We do that with the help of DRH-graphs, which are DRH-automata constructed from a given representation of a κ-word u and corresponding to the $\mathcal{R}$-class of u modulo DRH under the above announced bijection $\overline{\pi}$. Let u and v be κ-words and suppose that the word problem over H of a certain finite set of factors of u and v (to be precisely defined in the sequel) may be solved in $O(p(u,v))$-time. The main theorem of this chapter states that it takes at most $O((p(u,v)+m)m\,|A|)$-time to decide whether u and v are the same modulo DRH, where $m = \max\{|u|, |v|\}$. We further explain an algorithm doing so. As an illustration of that result, we prove that the κ-word problem for DRG may be solved in $O(m^3\,|A|)$-time.

We start the approach to some reducibility questions in Chapter 4, which contains three sections. We study the σ-reducibility of DRH for finite systems of pointlike equations in the first, for finite systems of graph equations in the second, and for finite systems of idempotent pointlike equations in the last. Let σ contain some non-explicit operation. The results in Sections 4.1 and 4.3 make use of a certain periodicity that may be found in DRH when iterating left basic factorizations of pseudowords infinitely many times to the right. In Section 4.1 we prove that DRH is σ-reducible for *finite systems of pointlike equations* if so is H, whereas in Section 4.3 we prove that, if a non-explicit operation with value 1 over H may be obtained from elements of σ by composition, then DRH is σ-reducible for *finite systems of idempotent pointlike equations* provided H is σ-reducible. Interchanging the roles of the pseudovarieties DRH and H in the last statement we still obtain a valid result, although that is only a simple observation. Until now, we do not know whether H being σ-reducible is also a necessary condition for σ-reducibility of DRH with respect to finite systems of idempotent pointlike equations. On the contrary, Section 4.2 depends deeply on [9, Lemma 5.14]. Inspired by the notion of *splitting points* [9] in the setting of the pseudovariety R, we generalize them for the pseudovariety DRH and show what is the relationship between the original notion of splitting points and the new one. By then, we have all the tools to transform any finite system of graph equations into a more treatable one. We prove that DRH is σ-reducible if and only if so is H. Some examples of applications are given.

Finally, in Chapter 5 we extend for pseudovarieties of the form DRH the techniques that Almeida, Costa and Zeitoun [10] used for proving that the pseudovariety R is completely κ-reducible. Firstly, the problem is reduced to the study of a "special" single word equation plus some extra conditions.[3] Intuitively, we turn our attention to the "special" word equation and look at the propagation of the factorizations present in each member. This may be seen as an analogue of propagation of splitting points. To deal with its possible infinite propagation, we generalize the adaptation of the Makanin's algorithm [50] made in [10]. More precisely, we prove that DRH is a completely κ-reducible pseudovariety if and only if the pseudovariety of groups H is completely κ-reducible as well. Of course, the latter condition holds for every locally finite pseudovariety H. However, so far, the unique known instance of a completely κ-reducible non-locally finite pseudovariety of groups is Ab, the pseudovariety of Abelian groups [13]. Hence, the pseudovariety DRAb is completely κ-reducible. On the contrary, since neither the pseudovarieties G and G_p (respectively, of all finite groups, and of all p-groups, for a prime p) nor proper non-locally finite subpseudovarieties of Ab are completely κ-reducible [29, 34, 35], we obtain a family of pseudovarieties of the form DRH that are not completely κ-reducible.

To conclude this thesis, we discuss in Chapter 6 some open problems that naturally appear following our work.

[3] "Special" means that the equation admits a solution δ modulo DRH such that, for each product of variables xy occurring as a factor of the product of the members of the equation, the product $\delta(x)\cdot\delta(y)$ is reduced.

Chapter 2

Preliminaries

For the basic concepts and results on pseudovarieties and (pro)finite semigroups the reader is referred to [3, 7]. Some knowledge of automata theory may be useful, although no use of deep results is made. For this topic, we refer to [63]. The required topological tools may be found in [70].

2.1 Semigroups

Let S be a semigroup. We denote by S^I the monoid whose underlying set is $S \uplus \{I\}$, where S is a subsemigroup and I plays the role of a neutral element. If S is a monoid, then we usually represent by 1 its identity (except in the case where the neutral element I is added). Given n elements $s_1, \ldots, s_n$ of S, we use the notation $\prod_{i=1}^{n} s_i$ for the product $s_1 s_2 \cdots s_n$. Given a sequence $(s_n)_{n \geq 1}$ in S we call *infinite product* the sequence $(\prod_{i=1}^{n} s_i)_{n \geq 1}$. An element $s \in S$ is said to be *regular* if there is $t \in S$ such that the equality $sts = s$ holds. A subset of S is *regular* if all its elements are regular. We let $\mathcal{R}$, $\leq_{\mathcal{R}}$, $\mathcal{L}$, $\mathcal{H}$, and $\mathcal{D}$ denote some of Green's relations on a semigroup. A straightforward computation shows the next result, which we use without further mention.

Proposition 2.1 ([54, Chapter 3, Proposition 1.1])**.** *The relations $\leq_{\mathcal{R}}$ and $\mathcal{R}$ are left compatible with multiplication.*

The free semigroup on the set A is denoted A^+, while the free monoid is denoted A^*. Since elements of A^* are often seen as words, we call *empty word* the identity of A^* and denote it by ε. We use FG_A to denote the free group on A, that is, the quotient $(A \cup A^{-1})^*/{\sim}$, where $A^{-1} = \{a^{-1} : a \in A\}$ is disjoint from A and $aa^{-1} \sim \varepsilon \sim a^{-1}a$ $(a \in A)$. Abusing the notation, the $\sim$-class of ε in FG_A is also denoted ε.

For $u \in A^*$, we write $|u| = n$ if $u = a_1 \cdots a_n$, and $|u| = 0$ if $u = \varepsilon$. We denote by 2^A the set of all subsets of A.

2.2 Automata

A *deterministic automaton* over a finite alphabet Δ is a tuple $\mathcal{A} = \langle V, \delta, \mathrm{q}, F \rangle$, where

- V is a set (not necessarily finite) whose elements are called *states*;

- $\delta : V \times \Delta \to V$ is a partial map, which is called the *transition function*;
- $\mathsf{q} \in V$ is the *initial state*;
- $F \subseteq V$ is the set of *final states*.

The transition function δ naturally extends to a partial map $\overline{\delta} : V \times \Delta^* \to V$ by letting $\overline{\delta}(\mathsf{v}, \varepsilon) = \mathsf{v}$ and $\overline{\delta}(\mathsf{v}, s_1 \cdots s_n) = \delta(\overline{\delta}(\mathsf{v}, s_1 \cdots s_{n-1}), s_n)$, where v is a state of V and $s_1, \ldots, s_n \in \Delta$. The function $\overline{\delta}$ is also denoted δ. We usually write $\delta(\mathsf{v}, s) = \mathsf{v}.s$, for $s \in \Delta^*$. Also, if $\mathsf{v}_1.s = \mathsf{v}_2$, then we may write $\mathsf{v}_1 \xrightarrow{s} \mathsf{v}_2$. We say that the automaton $\mathcal{A}$ is *trim* if for every $\mathsf{v} \in V$ there exist $s_1, s_2 \in \Delta^*$ such that $\mathsf{q}.s_1 = \mathsf{v}$ and $\mathsf{v}.s_2 \in F$. The *language accepted by* $\mathcal{A}$ is the set $\{s \in \Delta^* : \mathsf{q}.s \in F\}$.

Given a state $\mathsf{v} \in V$, we denote by $\mathcal{A}_\mathsf{v}$ the *sub-automaton of* $\mathcal{A}$ *rooted at* v, that is, the deterministic automaton $\mathcal{A}_\mathsf{v} = \langle \mathsf{v}.\Delta^*, \delta|_{\mathsf{v}.\Delta^* \times \Delta}, \mathsf{v}, F \cap (\mathsf{v}.\Delta^*)\rangle$.

2.3 Pseudovarieties and profinite semigroups

Unless otherwise stated, V and W stand for arbitrary pseudovarieties of semigroups. We list below some of the pseudovarieties mentioned in this work.

S consists of all finite semigroups;

Sl consists of all finite semilattices;

G consists of all finite groups;

Ab consists of all finite Abelian groups;

G_p consists of all finite p-groups (for a prime number p);

G_{sol} consists of all finite solvable groups.

We denote arbitrary subpseudovarieties of G by H. The class of finite semigroups whose subgroups belong to H is also a pseudovariety, denoted $\overline{\mathsf{H}}$. Our main focus are the pseudovarieties of the form DRH, that is, the class of all finite semigroups whose regular $\mathcal{R}$-classes are groups lying in H, and hence, are also $\mathcal{H}$-classes. Clearly, we have $\mathsf{DRH} = \mathsf{DRG} \cap \overline{\mathsf{H}}$. If H is the trivial pseudovariety of groups $\mathsf{I} = [\![x = y]\!]$, then $\mathsf{DRH} = \mathsf{DRI}$ is the pseudovariety R of all finite $\mathcal{R}$-trivial semigroups. Each pseudovariety DRH is contained in the pseudovariety DS of finite semigroups whose regular $\mathcal{D}$-classes are subsemigroups, and it contains the pseudovariety R.

We recall that a *profinite semigroup* is a compact residually finite topological semigroup. A *pro-*V *semigroup* S is a profinite semigroup that is *residually* V, meaning that given two distinct elements $s, t \in S$, there exists a continuous homomorphism $\psi : S \to T$ into a finite semigroup $T \in \mathsf{V}$ such that $\psi(s) \neq \psi(t)$. The *free A-generated pro-*V *semigroup* $\overline{\Omega}_A\mathsf{V}$ is characterized in the following proposition:

Proposition 2.2 ([7, Proposition 3.4]). *The profinite semigroup $\overline{\Omega}_A\mathsf{V}$ is the A-generated topological semigroup with the following universal property: the generating function $\iota : A \to \overline{\Omega}_A\mathsf{V}$ is such that,* *for every mapping $\psi : A \to S$ into a pro-V semigroup there exists a unique continuous homomorphism $\widehat{\psi} : \overline{\Omega}_A\mathsf{V} \to S$ such that the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "\psi"'] & \overline{\Omega}_A\mathsf{V} \arrow[d, dashed, "\widehat{\psi}"] \\ & S . \end{tikzcd}$$

The subsemigroup $\iota(A)^+$ is denoted $\Omega_A\mathsf{V}$. Since $\overline{\Omega}_A\mathsf{V}$ is an A-generated topological semigroup, we have $\overline{\Omega_A\mathsf{V}} = \overline{\Omega}_A\mathsf{V}$. If the pseudovariety V contains at least one non-trivial semigroup, then it is easily checked that the generating mapping $\iota : A \to \overline{\Omega}_A\mathsf{V}$ is injective. So, we often identify the elements of A with their images under ι. In particular, we sometimes call empty pseudoword/word the identity element $I \in (\overline{\Omega}_A\mathsf{V})^I$. Also, if $B \subseteq A$, then the inclusion mapping induces an injective continuous homomorphism $\overline{\Omega}_B\mathsf{V} \to \overline{\Omega}_A\mathsf{V}$. Hence, we look at $\overline{\Omega}_B\mathsf{V}$ as a subset of $\overline{\Omega}_A\mathsf{V}$. From now on, A denotes a finite set, also called an *alphabet*.

For a given $u \in \overline{\Omega}_A\mathsf{V}$ and a pro-V semigroup S, we denote by $u_S : S^A \to S$ the natural interpretation of u in S, that is, the mapping sending each element $\psi \in S^A$ (seen as a function from A to S) to the element $\widehat{\psi}(u)$, where $\widehat{\psi}$ stands for the unique continuous homomorphism given by Proposition 2.2. It may be proved that $(u_S)_{S\in\mathsf{V}}$ defines an A-ary implicit operation on V [7, Proposition 4.1]. Furthermore, the mapping assigning to each such u the A-ary implicit operation $(u_S)_{S\in\mathsf{V}}$ is a bijection onto the class of all A-ary implicit operations on V [7, Theorem 4.2]. The implicit operations corresponding to the elements of A^+ are called *explicit operations*. On the other hand, if W is another pseudovariety contained in V then, by Proposition 2.2, there is a unique (onto) continuous homomorphism $\rho_{\mathsf{V},\mathsf{W}} : \overline{\Omega}_A\mathsf{V} \to \overline{\Omega}_A\mathsf{W}$ such that $\rho_{\mathsf{V},\mathsf{W}}(a) = a$, for every $a \in A$ (see also [7, Proposition 4.4]). We call *natural projection of* $\overline{\Omega}_A\mathsf{V}$ *onto* $\overline{\Omega}_A\mathsf{W}$ the map $\rho_{\mathsf{V},\mathsf{W}}$. We shall write ρ_{W} when V is clear from the context. Whenever the pseudovariety Sl is contained in V, we denote the projection $(\rho_{\mathsf{Sl}} = \rho_{\mathsf{V},\mathsf{Sl}})$ by c and call it the *content function*.

The most natural example of an implicit operation is the multiplication $\_\cdot\_$, which is a binary operation. In the pro-V semigroup $\overline{\Omega}_{\{x_1,x_2\}}\mathsf{V}$ it corresponds to the element $x_1 \cdot x_2$. If $\mathsf{V} = \mathsf{G}$, then it is also natural to consider the unary implicit operation $\_^{-1}$ sending each element to its inverse in the group. This operation can be generalized for any pseudovariety of semigroups as follows. Let $S \in \mathsf{V}$. Since S is finite, given any $s \in S$, the subsemigroup of S generated by s contains a unique maximal subgroup G_s with identity 1_{G_s}. Hence, there is a power of s that belongs to G_s, say s^k, and satisfies $ss^k = 1_{G_s} = s^k s$. This power s^k is precisely the limit of the sequence $(s^{n!-1})_{n\geq 1}$, which becomes constant for n large enough. Thus, the sequence $(x_1^{n!-1})_{n\geq 1}$ converges in $\overline{\Omega}_{\{x_1\}}\mathsf{V}$. We denote its limit by $x_1^{\omega-1}$. More generally, we use the notation $x_1^{\omega+k}$ for $\lim_{n\geq 1} x_1^{n!+k}$, where $k \in \mathbb{Z}$.

An *implicit signature*, usually denoted σ, is a set of implicit operations on S containing the multiplication. Of course, every implicit signature σ endows $\overline{\Omega}_A\mathsf{V}$ with a structure of σ-algebra. We denote by $\Omega_A^\sigma\mathsf{V}$ the σ-subalgebra of $\overline{\Omega}_A\mathsf{V}$ generated by A (more precisely, by $\iota(A)$). The implicit signature $\kappa = \{\_\cdot\_, \_^{\omega-1}\}$ is the *canonical implicit signature*. Elements of $\overline{\Omega}_A\mathsf{V}$ are called *pseudowords over* V (or simply *pseudowords if* $\mathsf{V} = \mathsf{S}$), while elements of $\Omega_A^\sigma\mathsf{V}$ are σ*-words over* V (or simply σ*-words if* $\mathsf{V} = \mathsf{S}$). We let $\langle\sigma\rangle$ denote the implicit signature obtained from σ through composition of its elements (see [7, Proposition 4.7]). For instance, we have $\_^{\omega} \in \langle \kappa \rangle$, since $u^{\omega} = u^{\omega-1} \cdot u$ for every pseudoword u. Finally, we define *σ-terms* over an alphabet A inductively as follows:

- the empty word I and each letter $a \in A$ are σ-terms;
- if $u_1, \ldots, u_n$ are σ-terms and $\eta \in \sigma$ is an n-ary implicit operation, then $\eta(u_1, \ldots, u_n)$ is a σ-term.

Of course, each σ-term may naturally be seen as representing an element of $\Omega_A^{\sigma}\mathsf{S}$ and, on the other hand, for each element of $\Omega_A^{\sigma}\mathsf{S}$ there is a (usually non-unique) σ-term representing it. Further, note that σ-words are $\langle \sigma \rangle$-words and conversely, but a $\langle \sigma \rangle$-term may not be a σ-term.

By a *pseudoidentity* (respectively, *σ-identity*) we mean a formal equality $u = v$, for $u, v \in \overline{\Omega}_A\mathsf{S}$ (respectively, for $u, v \in \Omega_A^{\sigma}\mathsf{S}$). We say that a profinite semigroup S *satisfies* the pseudoidentity (respectively, σ-identity) $u = v$ if the interpretations of u and v coincide on S. Expressions like "V *satisfies* $u = v$", "$u = v$ *holds modulo* V", and "$u = v$ *holds in* V" mean that every semigroup $S \in \mathsf{V}$ satisfies $u = v$. If that is the case, then we may write $u =_{\mathsf{V}} v$. Note that $u =_{\mathsf{V}} v$ if and only if $\rho_{\mathsf{V}}(u) = \rho_{\mathsf{V}}(v)$. If $\mathcal{S}$ is a set of pseudoidentities, then we denote by $[\![\mathcal{S}]\!]$ the class of all finite semigroups that satisfy every pseudoidentity of $\mathcal{S}$. By Reiterman's Theorem [58], a class of finite semigroups is a pseudovariety if and only if it is of the form $[\![\mathcal{S}]\!]$. We say that V is *σ-equational* if there exists a set of σ-identities $\mathcal{S}$ such that $\mathsf{V} = [\![\mathcal{S}]\!]$.

2.4 Decidability

The *membership problem* for a pseudovariety V amounts to determining whether a given finite semigroup belongs to V. If there exists an algorithm to solve this problem, then the pseudovariety V is said to be *decidable*. Otherwise, it is called *undecidable*. As we already referred in the Introduction, other stronger notions of decidability have been set up over the years. They are related with so called systems of pseudoequations.

Let X be a finite set of *variables* and P a finite set of *parameters*, disjoint from X. A *pseudoequation* is a formal expression $u = v$ with $u, v \in \overline{\Omega}_{X \cup P}\mathsf{S}$, together with an *evaluation of the parameters* $\mathsf{ev} : P \to \overline{\Omega}_A\mathsf{S}$. If $u, v \in \Omega_{X \cup P}^{\sigma}\mathsf{S}$ and $\mathsf{ev}(P) \subseteq \Omega_A^{\sigma}\mathsf{S}$, then $u = v$ is said to be a *σ-equation*. If $u, v \in (X \cup P)^+$ and $\mathsf{ev}(P) \subseteq A^+$, then it is called a *word equation*. A *finite system of pseudoequations* (respectively, *σ-equations*, *word equations*) is a finite set

$$\{u_i = v_i \colon i = 1, \ldots, n\}, \tag{2.1}$$

where $u_i = v_i$ is a pseudoequation (respectively, σ-equation, word equation), for $i = 1, \ldots, n$. For each variable $x \in X$, we consider a *constraint* given by a clopen subset K_x of $\overline{\Omega}_A\mathsf{S}$. A *solution modulo* V of the system (2.1) *satisfying the given constraints and subject to the evaluation of the parameters* ev is a continuous homomorphism $\delta : \overline{\Omega}_{X \cup P}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ such that the following conditions are satisfied:

(S.1) $\delta(u) =_{\mathsf{V}} \delta(v)$;

(S.2) $\delta(p) = \mathsf{ev}(p)$, for every parameter $p \in P$;

(S.3) $\delta(x) \in K_x$, for every variable $x \in X$.

If $\delta(X \cup P) \subseteq \Omega_A^\sigma \mathsf{S}$, then we say that δ is a solution modulo V of (2.1) *in σ-words*.

Remark 2.3. It follows from Hunter's Lemma that, for each clopen set K_x, there exists a finite semigroup S_x and a continuous homomorphism $\varphi_x : \overline{\Omega}_A \mathsf{S} \to S_x$ such that K_x is the preimage of $\varphi_x(K_x)$ under φ_x (see [7, Proposition 3.5], for instance). It is sometimes more convenient to think of the constraints of the variables in terms of a fixed pair (φ, ν), where $\varphi : \overline{\Omega}_A \mathsf{S} \to S$ is a continuous homomorphism into a finite semigroup S and $\nu : X \to S$ is a map. In that way, the requirement (S.3) becomes a finite union of requirements of the form "$\varphi(\delta(x)) = \nu_j(x)$, for every variable $x \in X$", for a certain finite family $(\nu_j : X \to S)_j$ of mappings. We may also assume, without loss of generality that S has a content function (see [19, Proposition 2.1]), that is, that the homomorphism $c : \overline{\Omega}_A \mathsf{S} \to \overline{\Omega}_A \mathsf{Sl}$ factors through φ. Moreover, we occasionally wish to allow δ to take its values in $(\overline{\Omega}_A \mathsf{S})^I$. For that purpose, we naturally extend the function φ to a continuous homomorphism $\varphi^I : (\overline{\Omega}_A \mathsf{S})^I \to S^I$ by letting $\varphi^I(I) = I$. It is worth noticing that this assumption does not lead to trivial solutions since the constraints must be satisfied. We allow ourselves some flexibility in these points, adopting each approach according to which is the most suitable. In the case where we consider the homomorphism φ^I, we abuse notation and also denote it by φ.

Given a class $\mathcal{C}$ of finite systems of pseudoequations, we may pose the following problem:

> determine whether a given system from $\mathcal{C}$ (together with an evaluation of the parameters and constraints on variables) has a solution modulo V.

The pseudovariety V is $\mathcal{C}$-*decidable* if the above decision problem is decidable.

An important instance of a class of systems of equations arises from finite graphs. Let $\Gamma = V \uplus E$ be a directed graph, where V and E are finite sets, respectively, of *vertices* and *edges*. We consider Γ equipped with two maps $\alpha : E \to V$ and $\omega : E \to V$, such that an edge $e \in E$ goes from the vertex $v_1 \in V$ to the vertex $v_2 \in V$ if and only if $\alpha(e) = v_1$ and $\omega(e) = v_2$. We may associate to each edge $e \in E$ the equation $\alpha(e)e = \omega(e)$. We denote by $\mathcal{S}(\Gamma)$ the finite system of equations obtained in this way from Γ. Whenever $\mathcal{S}$ is a finite system of that form, we call $\mathcal{S}$ a *system of graph equations*. We notice that any system of graph equations is of the form $\{x_i y_i = z_i\}_{i=1}^N$, where $y_i \neq y_j$ for $i \neq j$ and $y_i \notin \{x_j, z_j\}$, for all i, j. If $\mathcal{C}$ is the class of all systems of graph equations arising from a graph with n vertices at most, then $\mathcal{C}$-decidability deserves the name of *n-hyperdecidability* in [4]. The pseudovariety V is *hyperdecidable* if it is n-hyperdecidable for all $n \geq 1$.

When the constraints on the variables $e \in E$ are all given by the clopen subset $K_e = \{I\}$, the system $\mathcal{S}(\Gamma)$ is called a *system of pointlike equations*. Observe that any system of pointlike equations may be seen as a system of the form $\{x_{i,1} = \cdots = x_{i,n_i}\}_{i=1}^N$. We say that V is *strongly decidable* if it is decidable for the class of all systems of pointlike equations.

Here are some remarkable results involving these notions.

Proposition 2.4 ([4, Corollary 4])**.** *Every strongly decidable pseudovariety is also decidable.*

Theorem 2.5 ([4, Theorem 14])**.** *Let n be a natural number, V a decidable pseudovariety of rank n containing the Brandt semigroup B_2,*[1] *and W a $(n+1)$-hyperdecidable pseudovariety. Then, $\mathsf{V} * \mathsf{W}$ is decidable.*

[1] The Brandt semigroup B_2 is the multiplicative semigroup of 2×2 real matrices generated by $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and by $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

Recall that a pseudovariety W is said to be *order-computable* if it is decidable and there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \geq |\overline{\Omega}_n \mathsf{W}|$, for all $n \in \mathbb{N}$.

Proposition 2.6 ([14, Corollary 5])**.** *If* V *is strongly decidable and* W *is order-computable, then* $\mathsf{V} * \mathsf{W}$ *is strongly decidable.*

Theorem 2.7 ([4, Theorem 15])**.** *Let* V *be a hyperdecidable (respectively, strongly decidable) pseudovariety and let* W *be an order-computable pseudovariety. Then,* $\mathsf{V} \vee \mathsf{W}$ *is hyperdecidable (respectively, strongly decidable).*

Theorem 2.8 ([55, Theorem 4.1; 8, Theorem 4.2])**.** *If* V *is decidable and* W *is* $\mathcal{C}$*-decidable for* $\mathcal{C}$ *consisting of systems of the form* $x_1 = \cdots = x_n = x_n^2$*, then* $\mathsf{V} ⓜ \mathsf{W}$ *is decidable.*

We call systems of equations of the form exhibited in Theorem 2.8 *systems of idempotent pointlike equations*.

Since the semigroups $\overline{\Omega}_A \mathsf{V}$ are very often uncountable, it is in general hard to say whether a pseudovariety V is $\mathcal{C}$-decidable, for a given class of systems $\mathcal{C}$. That was the motivation for the emergence of the next few concepts.

Given a class $\mathcal{C}$ of finite systems of σ-equations, we say that a pseudovariety V is *σ-reducible with respect to* $\mathcal{C}$ (or simply, σ-reducible for $\mathcal{C}$) provided a solution modulo V of a system of $\mathcal{C}$ guarantees the existence of a solution modulo V of that system given by σ-words. The pseudovariety V is said to be *σ-reducible* if it is σ-reducible for the class of finite systems of graph equations and it is *completely σ-reducible* if it is σ-reducible for the class of all finite systems of σ-equations. The following result involves the notion of reducibility.

Proposition 2.9 ([6, Proposition 10.2])**.** *If* V *is σ-reducible with respect to the equation* $x = y$*, then* V *is σ-equational.*

Since we are aiming to achieve decidability results for V, it is reasonable to require that V is recursively enumerable and that σ is *highly computable*, meaning that it is a recursively enumerable set and that all of its elements are computable operations. Henceforth, we make this assumption without further mention. Also, we should be able to decide whether two given σ-words have the same value over V, the so-called *σ-word problem*. Based on [16, Theorem 3.1], we say, for short, that the pseudovariety V is *σ-recursive* (with V recursively enumerable and σ highly computable) if the word problem is decidable in $\Omega_A^\sigma \mathsf{V}$, for every alphabet A. We say that V is *σ-tame with respect to* $\mathcal{C}$, for a highly computable implicit signature σ, if it is both σ-recursive, and σ-reducible with respect to $\mathcal{C}$. We say that V is *σ-tame* (respectively, *completely σ-tame*) when it is σ-tame with respect to the class of finite systems of graph equations (respectively, to the class of all finite systems of σ-equations).

Theorem 2.10 ([6, Theorem 10.3])**.** *Let* $\mathcal{C}$ *be a recursively enumerable class of finite systems of σ-equations, without parameters. If* V *is a pseudovariety which is σ-tame with respect to* $\mathcal{C}$*, then* V *is* $\mathcal{C}$*-decidable.*

Despite being a stronger requirement, it is sometimes easier to prove that a given pseudovariety is tame with respect to $\mathcal{C}$, rather than its $\mathcal{C}$-decidability.

We end this section with a list of decidability results concerning some pseudovarieties of groups, to which we refer later.

Theorem 2.11. *We have the following:*

- *the pseudovariety* Ab *is completely* κ*-tame ([13]);*
- *the pseudovariety* G *is* κ*-tame ([27] and [16]), but it is not completely* κ*-reducible ([34]);*
- *for every extension closed pseudovariety of groups* H*, there exists an implicit signature* $\sigma(\mathsf{H}) \supseteq \kappa$ *such that* H *is* $\sigma(\mathsf{H})$*-reducible ([5]);*
- *no proper subpseudovariety of* G *containing a pseudovariety* G_p *(for a certain prime* p*) is* κ*-reducible for the equation* $x = y$ *(Proposition 2.9 and [29]);*
- *no proper non locally finite subpseudovariety of* Ab *is* κ*-reducible ([35]).*

2.5 Some structural aspects of free pro-DRH semigroups

This section mostly follows [21].

Before describing how to represent pseudowords over DRH conveniently, we need to introduce a few concepts and results. We start by a well known result on factorization of pseudowords.

Proposition 2.12 ([25, Proposition 2.1])**.** *Let* $x,y,z,t \in \overline{\Omega}_A\mathsf{S}$ *and* $a,b \in A$ *be such that* $xay = zbt$*. Suppose that* $a \notin c(x)$ *and* $b \notin c(z)$*. If either* $c(x) = c(z)$ *or* $c(xa) = c(zb)$*, then* $x = z$*,* $a = b$*, and* $y = t$*.*

This motivates the definition of *left basic factorization* of a pseudoword $u \in \overline{\Omega}_A\mathsf{S}$: it is the unique triple $\mathsf{lbf}(u) = (u_\ell, a, u_r)$ of $(\overline{\Omega}_A\mathsf{S})^I \times A \times (\overline{\Omega}_A\mathsf{S})^I$ such that $u = u_\ell a u_r$, $a \notin c(u_\ell)$, and $c(u_\ell a) = c(u)$. This kind of factorization is also well defined over each pseudovariety DRH.

Proposition 2.13 ([21, Proposition 2.3.1])**.** *Every element* $u \in \overline{\Omega}_A\mathsf{DRH}$ *admits a unique factorization of the form* $u = u_\ell a u_r$ *such that* $a \notin c(u_\ell)$ *and* $c(u_\ell a) = c(u)$*.*

Notice that, for a pseudoword $u \in \overline{\Omega}_A\mathsf{S}$, the factorization of $\rho_{\mathsf{DRH}}(u)$ mentioned in Proposition 2.13 may be obtained from the left basic factorization of u, by projecting onto $\overline{\Omega}_A\mathsf{DRH}$. For that reason, we may also refer to the triple (u_ℓ, a, u_r) in Proposition 2.13 as the left basic factorization of u, with no possible ambiguity. Applying inductively Propositions 2.12 and 2.13 to the leftmost factor of the left basic factorization of a pseudoword, we obtain the following result.

Corollary 2.14. *Let* u *be a pseudoword.*

(a) *There exists a unique factorization* $u = a_1u_1a_2u_2\cdots a_nu_n$ *such that* $a_i \notin c(a_1u_1\cdots a_{i-1}u_{i-1})$*, for* $i = 2,\ldots,n$*, and* $c(u) = \{a_1,\ldots,a_n\}$*.*

(b) *Using the notation in* (a)*, suppose that the pseudovariety* DRH *satisfies the pseudoidentity*

$$a_1u_1a_2u_2\cdots a_nu_n = b_1v_1b_2v_2\cdots b_mv_m,$$

where $b_i \notin c(b_1v_1\cdots b_{i-1}v_{i-1})$*, for* $i = 2,\ldots,m$*, and* $c(u) = \{b_1,\ldots,b_m\}$*. Then,* $m = n$*,* $a_i = b_i$ *for* $i = 1,\ldots,m$ *and* DRH *satisfies* $u_i = v_i$*, for* $i = 1,\ldots,m$*.*

In view of Corollary 2.14, we refer to both the factorizations in (a) and in (b) as the *first-occurrences factorization* whenever $u \in \overline{\Omega}_A\mathsf{S}$ or $u \in \overline{\Omega}_A\mathsf{DRH}$.

For a pseudoword u, we may also iterate the left basic factorization of u to the right as follows. Set $u'_0 = u$. For each $k \geq 1$, whenever $u'_{k-1} \neq I$, we let $\mathsf{lbf}(u'_{k-1}) = (u_k, a_k, u'_k)$. Then, for every such k, the equality $u = u_1a_1 \cdot u_2a_2 \cdots u_ka_k \cdot u'_k$ holds. Moreover, the content of each factor u_ka_k decreases as k increases: $c(u_1a_1) \supseteq c(u_2a_2) \supseteq \cdots$. Since the alphabet A is finite, this sequence of contents either is finite or stabilizes. The *cumulative content* of u, denoted $\vec{c}(u)$, is the empty set in the former case, and is the ultimate value of the sequence otherwise. In particular, Proposition 2.13 yields that the cumulative content of a pseudoword is completely determined by its projection onto $\overline{\Omega}_A\mathsf{R}$, so that we may also refer to the cumulative content of an element of $\overline{\Omega}_A\mathsf{DRH}$. We further define the *irregular* and *regular* parts of u, respectively denoted $\mathsf{irr}(u)$ and $\mathsf{reg}(u)$: if $\vec{c}(u) = \emptyset$, then $\mathsf{irr}(u) = u$ and $\mathsf{reg}(u) = I$; if $\vec{c}(u) = c(u'_k)$ and k is minimal for this equality, then $\mathsf{irr}(u) = \mathsf{lbf}_1(u)\cdots\mathsf{lbf}_k(u)$ and $\mathsf{reg}(u) = u'_k$. Similarly, for a given $u \in \overline{\Omega}_A\mathsf{DRH}$ and $v \in \rho^{-1}_{\mathsf{DRH}}(u)$, we may refer to $\rho_{\mathsf{DRH}}(\mathsf{irr}(v))$ and to $\rho_{\mathsf{DRH}}(\mathsf{reg}(v))$ as the irregular and regular parts of u, respectively. Note that this does not depend on the choice of v. This terminology is justified by the following result, which characterizes regular elements of $\overline{\Omega}_A\mathsf{DRH}$ in terms of the relationship between its content and its cumulative content.

Proposition 2.15 ([21, Corollary 6.1.5])**.** *Let $u \in \overline{\Omega}_A\mathsf{DRH}$. Then, u is regular if and only if $c(u) = \vec{c}(u)$ (and, hence,* $\mathsf{reg}(u) = u$*) and it is idempotent if and only if it is regular and its projection onto $\overline{\Omega}_A\mathsf{H}$ is* 1.

If $\vec{c}(u) = \emptyset$, then we set $\lceil u \rceil = k$ if $u'_k = I$. Otherwise, we set $\lceil u \rceil = \infty$. We also write $\mathsf{lbf}_\infty(u)$ for the sequence $(u_1a_1, \ldots, u_{\lceil u \rceil}a_{\lceil u \rceil}, I, I, \ldots)$ if $\vec{c}(u) = \emptyset$, and for the sequence $(u_ka_k)_{k\geq 1}$ otherwise. We denote the k-th element of $\mathsf{lbf}_\infty(u)$ by $\mathsf{lbf}_k(u)$ and we say that a certain pseudovariety V satisfies $\mathsf{lbf}_\infty(u) = \mathsf{lbf}_\infty(v)$ for pseudowords u and v if it satisfies $\mathsf{lbf}_k(u) = \mathsf{lbf}_k(v)$ for all $k \geq 1$.

Remark 2.16. Let u and u_0 be pseudowords such that DRH satisfies $u = uu_0$. Then, by uniqueness of left basic factorization in $\overline{\Omega}_A\mathsf{DRH}$, the equality $\mathsf{lbf}_\infty(u) = \mathsf{lbf}_\infty(uu_0)$ holds modulo DRH. Therefore, using the above notation, u_0 is a suffix of each factor u'_k and so, $c(u_0) \subseteq \vec{c}(u)$. Conversely, by definition of left basic factorization and of cumulative content it is easy to check that if $c(u_0) \subseteq \vec{c}(u)$, then the equality $\mathsf{lbf}_\infty(u) = \mathsf{lbf}_\infty(uu_0)$ holds.

Suppose that the iteration of the left basic factorization of $u \in \overline{\Omega}_A\mathsf{S}$ to the right runs forever. Since $\overline{\Omega}_A\mathsf{S}$ is a compact monoid, the infinite sequence $(\mathsf{lbf}_1(u)\cdots\mathsf{lbf}_k(u))_{k\geq 1}$ has, at least, one accumulation point. Plus, any two accumulation points are $\mathcal{R}$-equivalent [21, Lemma 2.1.1]. If, in addition, $\rho_{\mathsf{DRH}}(u)$ is regular, then the projection onto DRH of the $\mathcal{R}$-class containing the accumulation points of the mentioned sequence is regular.

Proposition 2.17 ([21, Proposition 2.1.4])**.** *Let* V *be a pseudovariety such that* $\mathsf{R} \subseteq \mathsf{V} \subseteq \mathsf{DS}$ *and let $(s_1\cdots s_n)_{n\geq 1}$ be an infinite product in $\overline{\Omega}_A\mathsf{V}$. If every letter occurring in any s_n occurs in an infinite number of them, then the unique $\mathcal{R}$-class containing the accumulation points of $(s_1\cdots s_n)_{n\geq 1}$ is regular.*

Since the regular $\mathcal{R}$-classes of $\overline{\Omega}_A\mathsf{DRH}$ are groups, given an infinite product $(s_1\cdots s_n)_{n\geq 1}$ in $\overline{\Omega}_A\mathsf{DRH}$ satisfying the hypothesis of Proposition 2.17, we may define the *idempotent designated* by it as the identity of the group to which its accumulation points belong. It also happens that each regular $\mathcal{R}$-class of $\overline{\Omega}_A\mathsf{DRH}$ is homeomorphic to a free pro-H semigroup. This claim consists of a particular case of the following proposition, which is the key ingredient for proving in [21] the results on the representation of elements of $\overline{\Omega}_A\mathsf{DRH}$, some of which we state later.

Proposition 2.18 ([21, Proposition 5.1.2])**.** *Let* V *be a pseudovariety such that* $\mathsf{H} \subseteq \mathsf{V} \subseteq \mathsf{DO} \cap \overline{\mathsf{H}}$. *Then, the regular* $\mathcal{H}$*-classes of* $\overline{\Omega}_A\mathsf{V}$ *are free pro-*H *groups on their content. More precisely, if* e *is an idempotent of* $\overline{\Omega}_A\mathsf{V}$ *and if* H_e *is its* $\mathcal{H}$*-class, then letting* $\psi_e(a) = eae$ *for each* $a \in c(e)$ *defines a unique homeomorphism* $\psi_e : \overline{\Omega}_{c(e)}\mathsf{H} \to H_e$ *whose inverse is the restriction of* ρ_H *to* H_e.

Remark 2.19. We are denoting by DO the pseudovariety of semigroups in which regular $\mathcal{D}$-classes are orthodox subsemigroups. It is clear that $\mathsf{H} \subseteq \mathsf{DRH} \subseteq \overline{\mathsf{H}}$. Indeed, if H is a group of H, then H is its unique (regular) $\mathcal{R}$-class and if $S \in \mathsf{DRH}$, then any subgroup $H \leq S$ is a subgroup of the regular $\mathcal{R}$-class of its identity, which in turn is a group of H. On the other hand, the inclusion $\mathsf{DRH} \subseteq \mathsf{DO}$ also holds, since any regular $\mathcal{D}$-class of a semigroup $S \in \mathsf{DRH}$ is both a subsemigroup and an $\mathcal{L}$-class. Hence, Proposition 2.18 applies to the pseudovariety DRH.

Let v and w be pseudowords. We say that the product vw is *reduced* if w is nonempty and the first letter of w does not belong to $\vec{c}(v)$. The following is an important consequence of Proposition 2.18, which we use later on.

Corollary 2.20. *Let* u *be a pseudoword and* $v, w \in (\overline{\Omega}_A\mathsf{S})^I$ *be such that* $c(v) \cup c(w) \subseteq \vec{c}(u)$ *and* H *satisfies* $v = w$. *Then, the pseudovariety* DRH *satisfies* $uv = uw$.

Proof. Suppose that $c(u) \neq \vec{c}(u)$. Then, by definition of cumulative content, considering the iterations of the left basic factorizations of u to the right, we may write u as a reduced product $u_1 \cdot u_2$ such that $c(u_2) = \vec{c}(u_2) = \vec{c}(u)$. If we prove that DRH satisfies $u_2v = u_2w$, then it is immediate that it also satisfies the desired pseudoidentity $uv = uv$. So, we assume without loss of generality that $c(u) = \vec{c}(u)$. Let $e \in \overline{\Omega}_A\mathsf{DRH}$ be the idempotent designated by the infinite product $(\mathsf{lbf}_1(u) \cdots \mathsf{lbf}_k(u))_{k\geq 1}$ and ψ_e the homeomorphism described in Proposition 2.18. Since v and w represent the same element over H, it follows that $\psi_e(\rho_\mathsf{H}(v)) = \psi_e(\rho_\mathsf{H}(w))$, which in turn implies

$$\rho_{\mathsf{DRH}}(u)\psi_e(\rho_\mathsf{H}(v)) = \rho_{\mathsf{DRH}}(u)\psi_e(\rho_\mathsf{H}(w)). \tag{2.2}$$

Since $\rho_{\mathsf{DRH}}(u) \in H_e$, the $\mathcal{H}$-class of the images of ψ_e, the equality (2.2) holds inside H_e. Moreover, by Proposition 2.18, the inverse of the homeomorphism ψ_e is precisely the restriction of ρ_H to H_e so that we may deduce from (2.2) that $\rho_{\mathsf{DRH}}(u)\rho_{\mathsf{DRH}}(v) = \rho_{\mathsf{DRH}}(u)\rho_{\mathsf{DRH}}(w)$ which, in other words, means that DRH satisfies $uv = uw$, as required. □

We now have all the necessary ingredients to describe the elements of $\overline{\Omega}_A\mathsf{DRH}$ by means of the so-called "decorated reduced A-labeled ordinals", which we do along the next section. The construction is based on [21].

2.6 Decorated reduced A-labeled ordinals

A *decorated reduced A-labeled ordinal* is a triple (α, ℓ, g) where

- α is an ordinal.[2]

[2] For some facts about ordinal numbers see Appendix A.

- $\ell : \alpha \to A$ is a function. For a limit ordinal $\beta \leq \alpha$, we let the *cumulative content of β with respect to ℓ* be given by

$$\vec{c}(\beta, \ell) = \{a \in A \colon \exists (\beta_n)_{n\geq 1} \mid \cup_{n\geq 1}\beta_n = \beta,\ \beta_n < \beta \text{ and } \ell(\beta_n) = a\}.$$

  In Remark 2.22 below, we observe that the relationship between the cumulative content of an ordinal and the cumulative content of a pseudoword makes this terminology adequate. We further require for ℓ the following property:

  for every limit ordinal $\beta < \alpha$, the letter $\ell(\beta)$ does not belong to the set $\vec{c}(\beta, \ell)$.

- $g : \{\beta \leq \alpha \colon \beta \text{ is a limit ordinal}\} \to \overline{\Omega}_A\mathsf{H}$ is a function such that $g(\beta) \in \overline{\Omega}_{\vec{c}(\beta,\ell)}\mathsf{H}$.

We denote the set of all decorated reduced A-labeled ordinals by $\mathsf{rLO}_{\mathsf{H}}(A)$.

For a pseudoword v and a letter a, let us say that the product va is *end-marked* if $a \notin \vec{c}(v)$.

To each pseudoword u, we assign an element of $\mathsf{rLO}_{\mathsf{H}}(A)$ as follows.

Proposition 2.21 ([10, Proposition 4.8]). *The set of all end-marked pseudowords over a finite alphabet constitutes a well-founded forest under the partial order $\leq_{\mathcal{R}}$.*

Then, α_u is the unique ordinal such that there exists an isomorphism (also unique)

$$\theta_u : \alpha_u \to \{\text{end-marked prefixes of } u\} \tag{2.3}$$

such that $\theta_u(\beta) >_{\mathcal{R}} \theta_u(\gamma)$ whenever $\beta < \gamma$. We let $\ell_u : \alpha_u \to A$ be the function assigning to each ordinal $\beta \leq \alpha$ the letter a if $\theta_u(\beta) = va$.

Remark 2.22. Let u be a pseudoword. Then, the equality $\vec{c}(\alpha_u, \ell_u) = \vec{c}(u)$ holds. In fact, given $a \in \vec{c}(u)$, by definition of cumulative content, the letter a appears in all factors of u of the form $\mathsf{lbf}_k(u)$. By Corollary 2.14, the first occurrence of each letter in a pseudoword is well defined, so that, for each $k \geq 1$, there exists a factorization $\mathsf{lbf}_k(u) = u_k a w_k$, with $a \notin c(u_k)$. Therefore, each product $(\mathsf{lbf}_1(u) \cdots \mathsf{lbf}_{m-1}(u) u_{m-1}) \cdot a$ is an end-marked prefix of u. Hence, to conclude that $a \in \vec{c}(\alpha_u, \ell_u)$ it is enough to prove that $\alpha_u = \bigcup_{m\geq 1} \theta_u^{-1}(\mathsf{lbf}_1(u) \cdots \mathsf{lbf}_m(u))$. Write $v_m = \mathsf{lbf}_1(u) \cdots \mathsf{lbf}_m(u)$ and let vb be an end-marked prefix of u. By Proposition 2.21, we know that, for each m, either $vb <_{\mathcal{R}} v_m$ or $vb \geq_{\mathcal{R}} v_m$. Should the former occur for all m, then $\vec{c}(vb)$ would be equal to $\vec{c}(u) \neq \emptyset$, a contradiction with the assumption of vb being an end-marked pseudoword. Thus, each end-marked prefix of u is a prefix of some v_m, resulting that $\alpha_u = \bigcup_{m\geq 1} \theta_u^{-1}(v_m)$. Conversely, for every $a \in \vec{c}(\alpha_u, \ell_u)$ there exists an infinite increasing sequence $(\beta_n)_{n\geq 1}$ such that $\beta_n < \alpha_u$, $\alpha_u = \bigcup_{n\geq 1} \beta_n$ and $\ell_u(\beta_n) = a$. This implies that for every factorization $u = vbw$, with vb an end-marked prefix, the letter a belongs to $c(w)$. In particular, since the sequence $(c(\mathsf{lbf}_k(u)))_{k\geq 1}$ is ultimately constant, it follows that $a \in c(\mathsf{lbf}_k(u))$, for every k. Consequently, we have $a \in \vec{c}(u)$. More generally, for a limit ordinal $\beta \leq \alpha$ such that $\Theta_u(\beta) = va$, the equality $\vec{c}(v) = \vec{c}(\beta, \ell_u)$ holds.

It remains to define g_u. We first observe that the isomorphism (2.3) yields that we may iterate infinitely many times the left basic factorization of v to the right, whenever β is a limit ordinal and $\theta_u(\beta) = va$. Indeed, if $v = v_1 a_1 \cdots v_k a_k$ were the result of the k-th iteration of the left basic factorization of v to the right, then we would have $\theta_u(\theta_u^{-1}(v) + 1) = va = \theta_u(\beta)$ and β would not be a limit ordinal. The function g_u sends the limit ordinal $\beta \leq \alpha$ to the projection onto $\overline{\Omega}_A\mathsf{H}$ of the regular part of v, where $\Theta_u(\beta) = va$. We notice that, by Remark 2.22, the sets $\vec{c}(v)$ and $\vec{c}(\beta,\ell_u)$ coincide. Thus, the projection of the regular part of v belongs to $\overline{\Omega}_{\vec{c}(\beta,\ell_u)}\mathsf{H}$. We thus defined a function

$$\begin{aligned} F : \overline{\Omega}_A\mathsf{S} &\to \mathsf{rLO_H}(A) \\ u &\mapsto (\alpha_u, \ell_u, g_u) \end{aligned}$$

Although formulated differently, this is essentially the construction performed in [21]. In fact, the results stated in [21, Theorem 5.2.3] and [21, Proposition 5.3.2] together imply that two decorated reduced A-labeled ordinals $F(u)$ and $F(v)$ coincide if and only if the projections of u and of v onto $\overline{\Omega}_A\mathsf{DRH}$ represent the same element. Hence, we have a well defined bijection

$$\begin{aligned} \widehat{F} : \overline{\Omega}_A\mathsf{DRH} &\to \mathsf{rLO_H}(A) \\ u &\mapsto (\alpha_{u'}, \ell_{u'}, g_{u'}) \text{ for a pseudoword } u' \in \rho^{-1}_{\mathsf{DRH}}(u) \end{aligned}$$

Abusively, we write $\widehat{F}(u) = (\alpha_u, \ell_u, g_u)$.

Example 2.23. Let $u = xyz(xy)^\omega y$ and $v = yty^\omega$. Then, $\alpha_u = \omega = \alpha_v$ and the functions ℓ_u, g_u, ℓ_v and g_v are given by:

$$\begin{array}{ll} \ell_u(0) = x & \ell_v(0) = y \\ \ell_u(1) = y & \ell_v(1) = t \\ \ell_u(2) = z & \ell_v(k) = y, \text{ for all } k \geq 2 \\ \ell_u(2k+1) = x, \text{ for all } k \geq 1 & \\ \ell_u(2k) = y, \text{ for all } k \geq 2 & \\ g_u(\omega) = (xy)^\omega y = y & g_v(\omega) = y^\omega = 1 \end{array}$$

Consider also the product $uv = xyz(xy)^\omega yyty^\omega$. Then, we have $\alpha_{uv} = \omega \cdot 2$ and the functions ℓ_{uv} and g_{uv} are given by

$$\begin{array}{ll} \ell_{uv}(0) = x & g_{uv}(\omega) = (xy)^\omega y^2 = y^2 \\ \ell_{uv}(1) = y & g_{uv}(\omega \cdot 2) = y^\omega = 1 \\ \ell_{uv}(2) = z & \\ \ell_{uv}(2k+1) = x, \text{ for all } k \geq 1 & \\ \ell_{uv}(2k) = y, \text{ for all } k \geq 2 & \\ \ell_{uv}(\omega) = t & \\ \ell_{uv}(\omega + k) = y, \text{ for all } k \geq 1 & \end{array}$$

The *product of two decorated reduced A-labeled ordinals* (α,ℓ,g) and (α',ℓ',g') is defined as being the triple $(\alpha + \alpha'_2, p, h)$ where:

- the ordinal α' can be written as $\alpha' = \alpha'_1 + \alpha'_2$, with $\alpha'_1 = 0$ if α is not a limit ordinal; $\alpha'_1 = \alpha'$ if α is a limit ordinal and $\ell'(\alpha') \subseteq \vec{c}(\alpha, \ell)$; and α'_1 is the least ordinal such that $\alpha'_1 < \alpha'$ and $\ell'(\alpha'_1) \notin \vec{c}(\alpha, \ell)$ otherwise;

- the function p is given by

$$p(\gamma) = \ell(\gamma), \text{ if } \gamma < \alpha;$$
$$p(\alpha + \gamma) = \ell'(\alpha'_1 + \gamma), \text{ if } \gamma < \alpha'_2;$$

- for each limit ordinal $\gamma < \alpha$, we set $h(\gamma) = g'(\gamma)$ while, for a limit ordinal $\gamma \leq \alpha'_2$, we take $h(\alpha + \gamma) = g'(\alpha'_1 + \gamma)$. If α is also a limit ordinal, then we define

$$h(\alpha) = g(\alpha)\rho_{\mathsf{H}}(\widehat{F}^{-1}(\alpha'_1, \ell'|_{\alpha'_1}, g'|_{\{\gamma \leq \alpha'_1 \,:\, \gamma \text{ is a limit ordinal}\}})).$$

The following theorem is a consequence of [21, Theorem 6.1.1].

Theorem 2.24. *Using the notation above, the map* $\widehat{F} : \overline{\Omega}_A\mathsf{DRH} \to \mathsf{rLO}(A)$ *is an isomorphism and so, the map* $F : \overline{\Omega}_A\mathsf{S} \to \mathsf{rLO}(A)$ *is a homomorphism.*

Notation 2.25. *Let* $u \in \overline{\Omega}_A\mathsf{S}$ *and take ordinals* $\beta \leq \gamma \leq \alpha_u$. *Let* $\theta_u(\beta) = va$ *and* $\theta_u(\gamma) = wb$. *If* $\beta < \gamma$, *then* va *is a prefix of* w. *From Theorem 2.24 and taking into account the definition of the product in* $\mathsf{rLO}_{\mathsf{H}}(A)$, *it follows the existence of a unique* $z \in \overline{\Omega}_A\mathsf{S}$ *such that* $w = vaz$. *We write* $u[\beta, \gamma[= az$. *If* $\beta = \gamma$, *then we let* $u[\beta, \gamma[= I$. *It is worth noticing that this notation is well defined when we consider the projections onto* $\overline{\Omega}_A\mathsf{DRH}$, *meaning that if* $u =_{\mathsf{DRH}} v$, *then* $u[\beta, \gamma[=_{\mathsf{DRH}} v[\beta, \gamma[$.

If u is a κ-word, then the factors of u of the form $u[\beta, \gamma[$ are κ-words as well. This fact arises as a consequence of the following lemma when we iterate it inductively.

Lemma 2.26 ([25, Lemma 2.2])**.** *Let* $u \in \Omega_A^{\kappa}\mathsf{S}$ *and let* (u_ℓ, a, u_r) *be its left basic factorization. Then,* u_ℓ *and* u_r *are* κ*-words.*

The property of the implicit signature κ stated in this lemma becomes crucial if we intend to generalize part of the reducibility results of Chapter 5 for a larger implicit signature σ. It is then worth to explicitly formulate it:

for every σ-word $z = z_1z_2$, if the product $z_1 \cdot z_2$ is reduced, then the pseudowords z_1 and z_2 are also σ-words. (sig)

Of course, this holds for $\sigma = \kappa$.

2.7 More on the structure of free pro-DRH semigroups

We proceed with the statement of some structural results to handle pseudowords modulo DRH. They seem to be already used in the literature, however, since we could not find the exact statement that fits our purpose, we include the proofs for the sake of completeness.

We start with a characterization of the $\mathcal{R}$-classes of $\overline{\Omega}_A\mathsf{DRH}$ by means of iteration of left basic factorizations to the right.

Lemma 2.27. *Let u, v be pseudowords. Then, $\rho_{\mathsf{DRH}}(u)$ and $\rho_{\mathsf{DRH}}(v)$ lie in the same $\mathcal{R}$-class if and only if the pseudovariety* DRH *satisfies* $\mathsf{lbf}_\infty(u) = \mathsf{lbf}_\infty(v)$.

Proof. Suppose that $u \mathrel{\mathcal{R}} v$ modulo DRH and let u_0 and v_0 be possibly empty pseudowords such that DRH satisfies $uu_0 = v$ and $u = vv_0$. This implies that DRH also satisfies $u = uu_0v_0$, which in turn, by Remark 2.16, yields that $c(u_0) \subseteq \vec{c}(u)$. Hence, $\mathsf{lbf}_\infty(u) = \mathsf{lbf}_\infty(uu_0) =_{\mathsf{DRH}} \mathsf{lbf}_\infty(v)$, where the last equality follows from Proposition 2.13.

Conversely, suppose that $\mathsf{lbf}_\infty(u) =_{\mathsf{DRH}} \mathsf{lbf}_\infty(v)$. Then, we may choose accumulation points of $(\mathsf{lbf}_1(u) \cdots \mathsf{lbf}_k(u))_{k \geq 1}$ and of $(\mathsf{lbf}_1(v) \cdots \mathsf{lbf}_k(v))_{k \geq 1}$, say u' and v', respectively, having the same value in DRH. Since the accumulation points of these sequences are $\mathcal{R}$-above u and v, respectively, there exist possibly empty pseudowords u_0 and v_0 such that $u = u'u_0$ and $v = v'v_0$. Clearly, we have $\mathsf{lbf}_\infty(v) = \mathsf{lbf}_\infty(v')$ and so, again by Remark 2.16, it follows that $c(v_0) \subseteq \vec{c}(v')$. Therefore, the following equalities hold modulo DRH

$$u = u'u_0 = v'u_0 \overset{\text{Corollary 2.20}}{=} v'(v_0v_0^{\omega-1})u_0 = v(v_0^{\omega-1}u_0).$$

Hence, u is $\mathcal{R}$-below v modulo DRH. By symmetry, we also get that DRH satisfies $v \leq_{\mathcal{R}} u$. □

Corollary 2.28. *Let $u, v \in \overline{\Omega}_A\mathsf{S}$. Then, the pseudovariety* DRH *satisfies the relation $u \mathrel{\mathcal{R}} v$ if and only if $\alpha_u = \alpha_v$, $\ell_u = \ell_v$ and $g_u|_{\{\beta < \alpha_u \colon \beta \text{ is a limit ordinal}\}} = g_v|_{\{\beta < \alpha_v \colon \beta \text{ is a limit ordinal}\}}$.*

Proof. It is enough to observe that the end-marked prefixes of a pseudoword u suffice to completely characterize $\widehat{F}(u)$, except the element $g_u(\alpha_u)$ when α_u is a limit ordinal. Indeed, since for every end-marked prefix of u, say wa, there exists a big enough index k such that wa is a prefix of $\mathsf{lbf}_1(u) \cdots \mathsf{lbf}_k(u)$, the result follows from Lemma 2.27. □

Notation 2.29. *In what follows, given a pseudoword u, we denote by $F^-(u)$ the triple*

$$(\alpha_u, \ell_u, g_u|_{\{\beta < \alpha_u \colon \beta \text{ is a limit ordinal}\}}).$$

From the previous corollary, it follows that $F^-(u) = F^-(v)$ if and only if $u \mathrel{\mathcal{R}} v$ modulo DRH. *Also, if α_u is a successor ordinal, then $F(u) = F^-(u)$.*

Corollary 2.30. *Let u and v be pseudowords that are $\mathcal{R}$-equivalent modulo* DRH. *Suppose that they admit factorizations $u = u_1au_2$ and $v = v_1bv_2$ such that u_1a and v_1b are end-marked. If $\alpha_{u_1} = \alpha_{v_1}$, then $a = b$ and* DRH *satisfies $u_1 = v_1$ and $u_2 \mathrel{\mathcal{R}} v_2$. If, in addition, u and v are the same element over* DRH, *then* DRH *also satisfies $u_2 = v_2$.*

Proof. Since u_1a and v_1b are end-marked pseudowords, the ordinals α_{u_1a} and α_{v_1b}, which by hypothesis are the same, are necessarily successors. Then, Theorem 2.24 and Corollary 2.28 together imply that $F(u_1a)F^-(u_2) = F^-(u) = F^-(v) = F(v_1b)F^-(v_2)$. By definition of the product of decorated reduced A-labeled ordinals, it follows that $F(u_1) = F(v_1)$, $a = b$ and $F^-(u_2) = F^-(v_2)$. That means that in DRH we have $u_1 = v_1$ and $u_2 \mathrel{\mathcal{R}} v_2$. Moreover, if $u = v$ modulo DRH, then $F(u) = F(v)$. A similar argument yields that $u_2 = v_2$ modulo DRH. □

The following result is just a gathering of the observations made in Notation 2.25 and of Corollary 2.30 that we state for later reference.

Corollary 2.31. *Let $u, v \in \overline{\Omega}_A\mathsf{S}$ be such that* DRH *satisfies $u \mathrel{\mathcal{R}} v$. Let $\beta < \gamma < \alpha_u = \alpha_v$. Then, the pseudovariety* DRH *also satisfies $u[\beta,\gamma[= v[\beta,\gamma[$ and $u[\gamma,\alpha_u[\mathrel{\mathcal{R}} v[\gamma,\alpha_v[$. Moreover, if $u = v$ modulo* DRH*, then $u[\gamma,\alpha_u[= v[\gamma,\alpha_v[$ modulo* DRH. □

The next observation can be thought as the key ingredient when proving our main result. It becomes trivial when $\mathsf{DRH} = \mathsf{R}$.

Lemma 2.32. *Let $u, v \in \overline{\Omega}_A\mathsf{S}$ and $u_0, v_0 \in (\overline{\Omega}_A\mathsf{S})^I$ be such that $c(u_0) \subseteq \vec{c}(u)$ and $c(v_0) \subseteq \vec{c}(v)$. Then, the pseudovariety* DRH *satisfies $uu_0 = vv_0$ if and only if it satisfies $u \mathrel{\mathcal{R}} v$ and if, in addition, the pseudovariety* H *satisfies $uu_0 = vv_0$. In particular, by taking $u_0 = I = v_0$, we get that $u =_{\mathsf{DRH}} v$ if and only if $u \mathrel{\mathcal{R}} v$ modulo* DRH *and $u =_{\mathsf{H}} v$.*

Proof. Suppose that $uu_0 = vv_0$ modulo DRH. Since $c(u_0) \subseteq \vec{c}(u)$, it follows from Corollary 2.20 that DRH satisfies $u = u(u_0u_0^{\omega-1}) = vv_0u_0^{\omega-1}$ and so, the pseudoword v is $\mathcal{R}$-above u in $\overline{\Omega}_A\mathsf{DRH}$. By symmetry, we also get that DRH satisfies $v \leq_{\mathcal{R}} u$.

Conversely, suppose that u and v are in the same $\mathcal{R}$-class modulo DRH and that H satisfies $uu_0 = vv_0$. From the fact that $u \mathrel{\mathcal{R}} v$ modulo DRH it follows the existence of a possibly empty pseudoword v_0' such that DRH satisfies $u = vv_0' \mathrel{\mathcal{R}} v$. Thus, Remark 2.16 and Lemma 2.27 together yield the inclusion $c(v_0') \subseteq \vec{c}(v)$. On the other hand, since the pseudoidentities $\{u = vv_0', uu_0 = vv_0\}$ are valid in H, it follows that H satisfies $v_0'u_0 = v_0$. Therefore, Corollary 2.20 may be used to conclude that DRH satisfies $uu_0 = v(v_0'u_0) = vv_0$ as desired. □

Chapter 3

The κ-word problem over DRH

The aim of this chapter is to solve the word problem in the semigroup $\Omega_A^\kappa \mathsf{DRH}$, based on the knowledge of a solution of the word problem in $\Omega_A^\kappa \mathsf{H}$. We borrow the main idea from [25]. Supplementing this chapter, we present in Appendix B the computation in Python of the solution of the word problem in $\Omega_A^\kappa \mathsf{DRG}$.

Throughout this chapter, generic (finite) alphabets are denoted A, while $\Sigma = \{0,1\}$ is a fixed two-element alphabet. We also use $\mathbb{Z}$ for the set of integer numbers $\{\dots,-2,-1,0,1,2,\dots\}$ and $\mathbb{N}$ for the set of natural numbers $\{0,1,2,\dots\}$.

3.1 DRH-automata

We start by introducing the notion of a DRH-automaton.

Definition 3.1. *An A-labeled* DRH-automaton *is a tuple $\mathcal{A} = \langle V, \to, \mathsf{q}, F, \lambda_{\mathsf{H}}, \lambda \rangle$, where $\langle V, \to, \mathsf{q}, F \rangle$ is a nonempty deterministic trim automaton over Σ and $\lambda_{\mathsf{H}} : V \to (\overline{\Omega}_A \mathsf{H})^I$ and $\lambda : V \to A \uplus \{\varepsilon\}$ are functions. We further require that $\mathcal{A}$ satisfies the following conditions (A.1)–(A.6).*

(A.1) the set of final states is $F = \lambda^{-1}(\varepsilon)$ and $\lambda_{\mathsf{H}}(F) = \{I\}$;

(A.2) there is no outgoing transition from F;

(A.3) for every $\mathsf{v} \in V \setminus F$, both $\mathsf{v}.0$ and $\mathsf{v}.1$ are defined;

(A.4) for every $\mathsf{v} \in V \setminus F$, the following equality holds:

$$\lambda(\mathsf{v}.\Sigma^*) = \lambda(\mathsf{v}.0\Sigma^*) \uplus \{\lambda(\mathsf{v})\}.$$

We observe that if conditions (A.1)–(A.4) hold for $\mathcal{A}$, then the reduct *$\mathcal{A}_{\mathsf{R}} = \langle V, \to, \mathsf{q}, F, \lambda \rangle$ is an A-labeled* R*-automaton (see [25, Definition 3.11]). Since the cumulative content of a pseudoword over* DRH *depends only on its projection onto $\overline{\Omega}_A \mathsf{R}$, and hence, also its regularity, we may use the known results for the word problem in* R *(namely, [25, Theorem 3.21]) as intuition for defining the* length $\|\mathcal{A}\|$*, the* regularity index $\mathsf{r.ind}(\mathcal{A})$ *and the* cumulative content $\vec{c}(\mathcal{A})$ *of a* DRH*-automaton $\mathcal{A}$*

from the knowledge of its reduct $\mathcal{A}_{\mathsf{R}}$. We set:

$$\|\mathcal{A}\| = \sup\{k \geq 0 \colon \mathsf{q}.1^k \textit{ is defined}\};$$

$$\mathsf{r.ind}(\mathcal{A}) = \begin{cases} \infty, & \textit{if } \|\mathcal{A}\| < \infty; \\ \min\{m \geq 0 \colon \forall k \geq m \quad \lambda(\mathsf{q}.1^k\Sigma^*) = \lambda(\mathsf{q}.1^m\Sigma^*)\}, & \textit{otherwise}; \end{cases}$$

$$\vec{c}(\mathcal{A}) = \begin{cases} \emptyset, & \textit{if } \|\mathcal{A}\| < \infty; \\ \lambda(\mathsf{q}.1^{\mathsf{r.ind}(\mathcal{A})}\Sigma^*), & \textit{otherwise}. \end{cases}$$

We are now able to state the further required properties for $\mathcal{A}$:

(A.5) if $\mathsf{v} \in V \setminus F$, then $\lambda_{\mathsf{H}}(\mathsf{v}) = I$ if and only if $\|\mathcal{A}_{\mathsf{v}.0}\| < \infty$;

(A.6) if $\mathsf{v} \in V \setminus F$ and $\|\mathcal{A}_{\mathsf{v}.0}\| = \infty$, then $\lambda_{\mathsf{H}}(\mathsf{v}) \in \overline{\Omega}_{\vec{c}(\mathcal{A}_{\mathsf{v}.0})}\mathsf{H}$.

We say that $\mathcal{A}$ is a DRH-tree *if it is a* DRH-*automaton such that for every $\mathsf{v} \in V$ there exists a unique $\alpha \in \Sigma^*$ such that $\mathsf{q}.\alpha = \mathsf{v}$.*

Example 3.2. Here is an example of a DRH-automaton, call it $\mathcal{A}$. The first label in each state corresponds to its image under λ_{H} and the second to its image under λ.

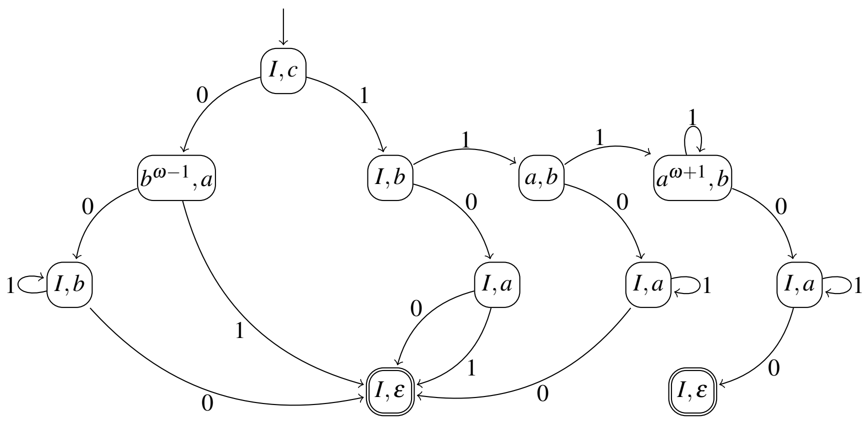

Fig. 3.1 A DRH-automaton.

Let q be the initial state. We may check that $\|\mathcal{A}\| = \infty = \|\mathcal{A}_{\mathsf{q}.\alpha}\|$, for all $\alpha \in \Sigma^* \setminus \{0, 10\}$ such that $\mathsf{q}.\alpha$ is defined, and $\|\mathcal{A}_{\mathsf{q}.0}\| = 1 = \|\mathcal{A}_{\mathsf{q}.10}\|$. This means that the only non trivial sub-automata with infinity regularity index, and hence, with empty cumulative content, are $\mathcal{A}_{\mathsf{q}.0}$ an d $\mathcal{A}_{\mathsf{q}.10}$. Also, the regularity index is 0 for all the sub-automata $\mathcal{A}_{\mathsf{q}.\alpha}$, where $\alpha \in \Sigma^+ \setminus \{0, 10\}$ is such that $\mathsf{q}.\alpha$ is defined. The regularity index of the automaton itself is $\mathsf{r.ind}(\mathcal{A}) = 1$. On the other hand, we have for instance, that $\vec{c}(\mathcal{A}) = \{a, b\}$, while $\vec{c}(\mathcal{A}_{\mathsf{q}.110}) = \{a\}$ and $\vec{c}(\mathcal{A}_{\mathsf{q}.00}) = \{b\}$.

Definition 3.3. *We say that two* DRH-*automata $\mathcal{A}_i = \langle V_i, \to_i, \mathsf{q}_i, F_i, \lambda_{i,\mathsf{H}}, \lambda_i \rangle$, $i = 1, 2$, are* isomorphic *if there exists a bijection $f : V_1 \to V_2$ such that*

(J.1) $f(\mathsf{q}_1) = \mathsf{q}_2$;

(J.2) for every $\mathsf{v} \in V_1$ *and* $\alpha \in \Sigma$, $f(\mathsf{v}) \cdot \alpha = f(\mathsf{v} \cdot \alpha)$;

(J.3) for every $\mathsf{v} \in V_1$, *the equalities* $\lambda_{1,\mathsf{H}}(\mathsf{v}) = \lambda_{2,\mathsf{H}}(f(\mathsf{v}))$ *and* $\lambda_1(\mathsf{v}) = \lambda_2(f(\mathsf{v}))$ *hold.*

Isomorphic DRH-automata are essentially the same, up to the name of the states. Therefore, we consider DRH-automata only up to isomorphism.

We denote the trivial DRH-automaton by **1** and the set of all A-labeled DRH-automata by $\mathbb{A}_A$.

Definition 3.4. *Let* $\mathcal{A}_i = \langle V_i, \to_i, \mathsf{q}_i, F_i, \lambda_{i,\mathsf{H}}, \lambda_i \rangle$, $i = 1,2$, *be two* DRH-*automata. We say that* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are* equivalent *if*

$$\forall \alpha \in \Sigma^*, \quad \lambda_1(\mathsf{q}_1.\alpha) = \lambda_2(\mathsf{q}_2.\alpha) \text{ and } \lambda_{1,\mathsf{H}}(\mathsf{q}_1.\alpha) = \lambda_{2,\mathsf{H}}(\mathsf{q}_2.\alpha). \tag{3.1}$$

We agree that (3.1) *means that either both equalities hold or both* $\mathsf{q}_1.\alpha$ *and* $\mathsf{q}_2.\alpha$ *are undefined. We write* $\mathcal{A}_1 \sim \mathcal{A}_2$, *when* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are equivalent.*

Example 3.5. An example of a DRH-automaton equivalent to the one in Example 3.2 is represented in Figure 3.2.

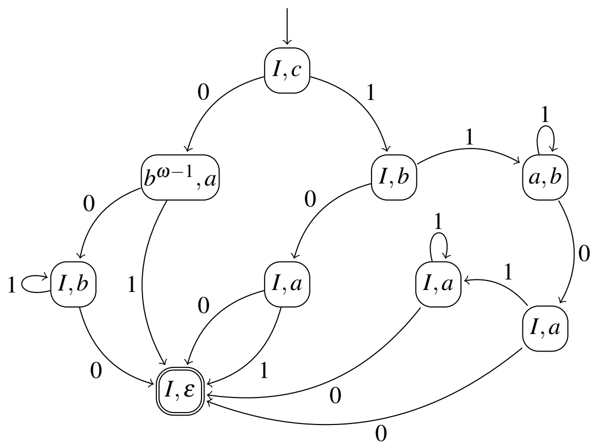

Fig. 3.2 A DRH-automaton equivalent to the automaton in Figure 3.1.

We observe that equivalent DRH-trees are necessarily isomorphic. Indeed, for $i = 1,2$, let $\mathcal{T}_i = \langle V_i, \to_i, \mathsf{q}_i, F_i, \lambda_{i,\mathsf{H}}, \lambda_i \rangle$ be two equivalent DRH-trees. Then, the mapping $f : V_1 \to V_2$ that sends each state $\mathsf{v} \in V_1$ to the state $\mathsf{q}_2.\alpha$, where $\alpha \in \Sigma^*$ is the unique element such that $\mathsf{q}_1.\alpha = \mathsf{v}$, is a bijection satisfying (J.1)–(J.3).

The following lemma is useful when defining a bijective correspondence between the equivalence classes of $\mathbb{A}_A$ and the $\mathcal{R}$-classes of $\overline{\Omega}_A\mathsf{DRH}$.

Lemma 3.6 (cf. [25, Lemma 3.16])**.** *Every* DRH-*automaton has a unique equivalent* DRH-*tree.*

Proof. Consider a DRH-automaton $\mathcal{A} = \langle V, \to, \mathsf{q}, F, \lambda_{\mathsf{H}}, \lambda \rangle$ and let $\mathcal{T} = \langle V', \to', \mathsf{q}', F', \lambda'_{\mathsf{H}}, \lambda' \rangle$ be the DRH-tree defined as follows. We set $V' = \{\alpha \in \Sigma^* : \mathsf{q}.\alpha \text{ is defined}\}$ and put $\mathsf{q}' = \varepsilon$. The labels of each state $\alpha \in V'$ are given by $\lambda'_{\mathsf{H}}(\alpha) = \lambda_{\mathsf{H}}(\mathsf{q}.\alpha)$ and by $\lambda'(\alpha) = \lambda(\mathsf{q}.\alpha)$. We also take $F' = \lambda'^{-1}(\varepsilon)$. Finally, the transitions in $\mathcal{T}$ are given by $\alpha.0 = \alpha 0$ and by $\alpha.1 = \alpha 1$, whenever $\lambda'(\alpha) \neq \varepsilon$. We claim that $\mathcal{T}$ is a DRH-tree equivalent to $\mathcal{A}$. We first prove that $\mathcal{T}$ is a DRH-tree. Properties (A.1)–(A.3) follow from construction. For Property (A.4) we use the same property for $\mathcal{A}$: taking $\alpha \in V' \setminus F'$ we may compute

$$\lambda'(\alpha.\Sigma^*) = \lambda(\mathsf{q}.\alpha\Sigma^*) = \lambda(\mathsf{q}.\alpha 0\Sigma^*) \uplus \{\lambda(\mathsf{q}.\alpha)\} = \lambda'(\alpha 0\Sigma^*) \uplus \{\lambda'(\alpha)\} = \lambda'(\alpha.0\Sigma^*) \uplus \{\lambda'(\alpha)\}.$$

At last, let $\alpha \in V' \setminus F'$. Then, since $\lambda'_{\mathsf{H}}(\alpha) = \lambda_{\mathsf{H}}(\mathsf{q}.\alpha)$, we have $\lambda'_{\mathsf{H}}(\alpha) = I$ if and only if $\|\mathcal{A}_{\mathsf{q}.\alpha 0}\| < \infty$, which in turn means that $\|\mathcal{T}_{\alpha.0}\| < \infty$ by definition of $\to'$, thereby proving (A.5). In the same way, Property (A.6) for $\mathcal{T}$ is inherited from Property (A.6) for $\mathcal{A}$. This proves that $\mathcal{T}$ is a DRH-automaton. Further, it is a tree by construction. The definition of λ'_{H}, λ' and $\to'$ guarantees that it is equivalent to $\mathcal{A}'$. On the other hand, as we already observed above, equivalent DRH-trees are necessarily isomorphic. Thus, up to isomorphism, a DRH-tree equivalent to $\mathcal{A}$ is unique. $\square$

Example 3.7. We represent in Figure 3.3 the unique DRH-tree equivalent to the DRH-automaton in Figure 3.1.

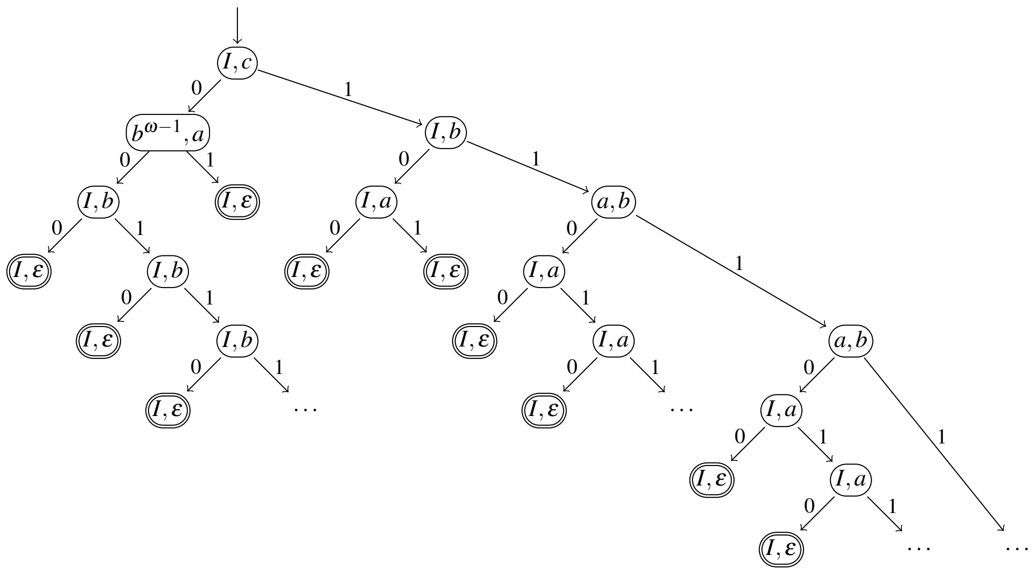

Fig. 3.3 The DRH-tree equivalent to the DRH-automaton in Figure 3.1.

Given a DRH-automaton $\mathcal{A}$, we denote by $\vec{\mathcal{A}} = \langle \vec{V}, \to, \vec{\mathsf{q}}, \vec{F}, \vec{\lambda}_{\mathsf{H}}, \vec{\lambda} \rangle$ the unique DRH-tree which is equivalent to $\mathcal{A}$. Denoting both transition functions of $\mathcal{A}$ and of $\vec{\mathcal{A}}$ by $\to$ is an abuse of notation justified by the construction made in the proof of Lemma 3.6. Given $0 \le i \le \|\mathcal{A}\| - 1$, we denote by $\mathcal{A}_{[i]}$ the DRH-subtree rooted at $\vec{\mathsf{q}}.1^i 0$ as illustrated in Figure 3.4.

Notation 3.8. *Let $u \in \overline{\Omega}_A\mathsf{DRH}$ and $v \in \overline{\Omega}_B\mathsf{H}$ be such that $B \subseteq \vec{c}(u)$. By Corollary 2.20, the set $u\rho^{-1}_{\mathsf{DRH},\mathsf{H}}(v)$ is a singleton. It is convenient to denote by uv the unique element of $u\rho^{-1}_{\mathsf{DRH},\mathsf{H}}(v)$. In this case, the notation $\rho_{\mathsf{H}}(uv)$ refers to the element $\rho_{\mathsf{H}}(uv) = \rho_{\mathsf{H}}(u)\,v$ of $\overline{\Omega}_A\mathsf{H}$.*

Definition 3.9. *Let $\mathcal{A} = \langle V, \to, \mathsf{q}, F, \lambda_{\mathsf{H}}, \lambda \rangle$ be an A-labeled* DRH*-automaton. The* value $\pi(\mathcal{A})$ *of $\mathcal{A}$ in $(\overline{\Omega}_A\mathsf{DRH})^I$ is inductively defined as follows:*

- *if $\mathcal{A} = \mathbf{1}$, then $\pi(\mathcal{A}) = I$;*

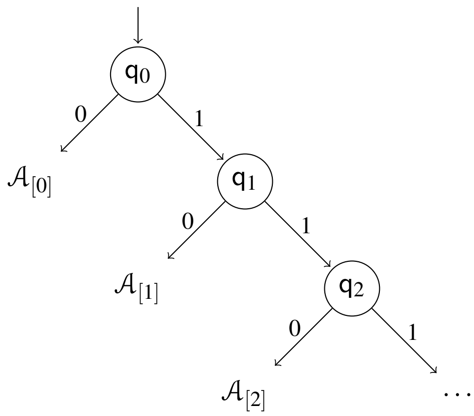

Fig. 3.4 Representation of the DRH-trees of the form $\mathcal{A}_{[i]}$.

- *otherwise, we consider two different cases according to whether or not* $\|\mathcal{A}\| < \infty$.

  - *If* $\|\mathcal{A}\| < \infty$, *then we set*

$$\pi(\mathcal{A}) = \prod_{i=0}^{\|\mathcal{A}\|-1} \pi(\mathcal{A}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i)\lambda(\mathsf{q}.1^i).$$

  - *If* $\|\mathcal{A}\| = \infty$, *then we first define the* idempotent associated to $\mathcal{A}$, $\mathsf{id}(\mathcal{A}) \in \overline{\Omega}_A\mathsf{DRH}$. *Noticing that, for* $k \geq \mathsf{r.ind}(\mathcal{A})$, *all the elements* $\pi(\mathcal{A}_{[k]})\lambda_{\mathsf{H}}(\mathsf{q}.1^k)\lambda(\mathsf{q}.1^k)$ *have the same content, we let* $\mathsf{id}(\mathcal{A})$ *be the idempotent designated by the infinite product*

$$(\pi(\mathcal{A}_{[\mathsf{r.ind}(\mathcal{A})]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{\mathsf{r.ind}(\mathcal{A})})\lambda(\mathsf{q}.1^{\mathsf{r.ind}(\mathcal{A})})\cdots\pi(\mathcal{A}_{[k]})\lambda_{\mathsf{H}}(\mathsf{q}.1^k)\lambda(\mathsf{q}.1^k))_{k\geq\mathsf{r.ind}(\mathcal{A})}. \quad (3.2)$$

    Then, we take

$$\pi(\mathcal{A}) = \left(\prod_{i=0}^{\mathsf{r.ind}(\mathcal{A})-1} \pi(\mathcal{A}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i)\lambda(\mathsf{q}.1^i)\right) \cdot \mathsf{id}(\mathcal{A}).$$

We also define the value of the irregular part of $\mathcal{A}$*:*

$$\pi_{\mathsf{irr}}(\mathcal{A}) = \prod_{i=0}^{\min\{\|\mathcal{A}\|,\mathsf{r.ind}(\mathcal{A})\}-1} \pi(\mathcal{A}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i)\lambda(\mathsf{q}.1^i).$$

If $\|\mathcal{A}\| < \infty$, *then we set* $\mathsf{id}(\mathcal{A}) = I$. *Using this notation, we have the equality*

$$\pi(\mathcal{A}) = \pi_{\mathsf{irr}}(\mathcal{A}) \cdot \mathsf{id}(\mathcal{A}). \quad (3.3)$$

The next result is a simple observation that we state for later reference.

Lemma 3.10. *Given a* DRH*-automaton* $\mathcal{A} = \langle V, \to, \mathsf{q}, F, \lambda_{\mathsf{H}}, \lambda\rangle$*, the following equalities hold:*

$$\begin{aligned}
\mathsf{lbf}_{i+1}(\pi(\mathcal{A})) &= \pi(\mathcal{A}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i)\lambda(\mathsf{q}.1^i),\ \textit{for}\ 0 \le i \le \|\mathcal{A}\| - 1;\\
\mathsf{lbf}_{i+1}(\pi(\mathcal{A})) &= I,\ \textit{for}\ i \ge \|\mathcal{A}\|;\\
\mathsf{irr}(\pi(\mathcal{A})) &= \pi_{\mathsf{irr}}(\mathcal{A});\\
\vec{c}(\mathcal{A}) &= \vec{c}(\pi(\mathcal{A})).
\end{aligned}$$

In particular, for a certain $u \in \overline{\Omega}_A\mathsf{DRH}$*, the elements* $\pi(\mathcal{A})$ *and* u *are* $\mathcal{R}$*-equivalent if and only if* $\pi_{\mathsf{irr}}(\mathcal{A}) = \mathsf{irr}(u)$ *and* $\mathsf{id}(\mathcal{A})\ \mathcal{R}\ \mathsf{reg}(u)$.

Proof. It follows from Properties (A.4), (A.5) and (A.6) that $\lambda(\mathsf{q}.1^i)$ does not belong to the set $c(\pi(\mathcal{A}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i))$, and $c(\pi(\mathcal{A}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i)\lambda(\mathsf{q}.1^i))$ contains $c(\pi(\mathcal{A}_{[i+1]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{i+1})\lambda(\mathsf{q}.1^{i+1}))$, for every i such that the expressions are defined. Therefore, the uniqueness of left basic factorizations in $\overline{\Omega}_A\mathsf{DRH}$ implies that

$$\begin{aligned}
\mathsf{lbf}_{i+1}(\pi(\mathcal{A})) &= \pi(\mathcal{A}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i)\lambda(\mathsf{q}.1^i),\ \text{for}\ 0 \le i \le \|\mathcal{A}\| - 1;\\
\mathsf{lbf}_{i+1}(\pi(\mathcal{A})) &= I,\ \text{for}\ i \ge \|\mathcal{A}\|.
\end{aligned}$$

In particular, we obtain that $\pi_{\mathsf{irr}}(\mathcal{A}) = \mathsf{irr}(\pi(\mathcal{A}))$. Furthermore, if $\|\mathcal{A}\| < \infty$, then both $\vec{c}(\pi(\mathcal{A}))$ and $\vec{c}(\mathcal{A})$ are the empty set, while if $\|\mathcal{A}\| = \infty$, then the following equalities hold

$$\begin{aligned}
\vec{c}(\pi(\mathcal{A})) &= c(\pi(\mathcal{A}_{[\mathsf{r.ind}(\mathcal{A})]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{\mathsf{r.ind}(\mathcal{A})})\lambda(\mathsf{q}.1^{\mathsf{r.ind}(\mathcal{A})}))\\
&= \lambda(\mathsf{q}.1^{\mathsf{r.ind}(\mathcal{A})}0\Sigma^*) \cup \{\lambda(\mathsf{q}.1^{\mathsf{r.ind}(\mathcal{A})})\}\\
&= \lambda(\mathsf{q}.1^{\mathsf{r.ind}(\mathcal{A})}\Sigma^*) = \vec{c}(\mathcal{A}).
\end{aligned}$$

Finally, for any $u \in \overline{\Omega}_A\mathsf{DRH}$, Lemma 2.27 yields that $\pi(\mathcal{A})$ and u are $\mathcal{R}$-equivalent if and only if $\mathsf{lbf}_\infty(u) = \mathsf{lbf}_\infty(\pi(\mathcal{A}))$, which in turn holds if and only if $\mathsf{irr}(u) = \mathsf{irr}(\pi(\mathcal{A}))$ and $\mathsf{reg}(u)\ \mathcal{R}\ \mathsf{reg}(\pi(\mathcal{A}))$. We already justified that $\mathsf{irr}(\pi(\mathcal{A})) = \pi_{\mathsf{irr}}(\mathcal{A})$ and it is clear that $\mathsf{reg}(\pi(\mathcal{A})) = \mathsf{id}(\mathcal{A})$. ☐

Example 3.11. Consider again the DRH-automaton sketched in Example 3.2. In order to compute its value $\pi(\mathcal{A})$, we start by computing the value of $\mathcal{A}_{\mathsf{q}.1}$. As we already observed, we have $\|\mathcal{A}_{\mathsf{q}.1}\| = \infty$. Hence, in order to calculate $\pi(\mathcal{A}_{\mathsf{q}.1})$, we first need to know the values of the subautomaton $(\mathcal{A}_{\mathsf{q}.1})_{[i]} = \mathcal{A}_{\mathsf{q}.1^{i+1}0}$, for each $i \ge 0$. For $i = 0$, we have that $\|\mathcal{A}\|_{\mathsf{q}.10} = 1$, and so, by definition,

$$\pi(\mathcal{A}_{\mathsf{q}.10}) = \pi(\mathcal{A}_{\mathsf{q}.100})\lambda_{\mathsf{H}}(\mathsf{q}.10)\lambda(\mathsf{q}.10) = a.$$

Analyzing Figure 3.1, we easily conclude that, for all $i \ge 1$, the value of $\mathcal{A}_{\mathsf{q}.1^{i+1}0}$ is always the same. Since $\left\|\mathcal{A}_{\mathsf{q}.1^{i+1}0}\right\| = \infty$, we need to compute all the elements $\pi(\mathcal{A}_{\mathsf{q}.1^{i+1}01^k0})$, for $k \ge 0$. Again, they are all equal, namely, I. Since the regularity index of $\mathcal{A}_{\mathsf{q}.1^{i+1}0}$ is 0, we have $\pi(\mathcal{A}_{\mathsf{q}.1^{i+1}0}) = \mathsf{id}(\mathcal{A}_{\mathsf{q}.1^{i+1}0})$, which in turn is the idempotent designated by the infinite product

$$(\pi(\mathcal{A}_{\mathsf{q}.1^{i+1}00})\lambda_{\mathsf{H}}(\mathsf{q}.1^{i+1}0)\lambda(\mathsf{q}.1^{i+1}0)\cdots\pi(\mathcal{A}_{1^{i+1}01^k0})\lambda_{\mathsf{H}}(\mathsf{q}.1^{i+1}01^k)\lambda(\mathsf{q}.1^{i+1}01^k))_{k\ge0} = (a^k)_{k\ge1}.$$

Clearly, the idempotent designated by this sequence is a^ω. Hence, the value of each $(\mathcal{A}_{\mathsf{q}.1})_{[i]}$ is a^ω. The same kind of reasoning allows us to conclude that the value of $\mathcal{A}_{\mathsf{q}.1}$ is $(ab(a^{\omega+1}b)^\omega)^\omega$. Similarly, we may compute $\pi(\mathcal{A}_{[0]}) = b^{\omega-1}a$, deriving that $\pi(\mathcal{A}) = b^{\omega-1}ac(ab(a^{\omega+1}b)^\omega)^\omega$.

Since the value of a DRH-automaton $\mathcal{A}$ depends only on the unique DRH-tree $\vec{\mathcal{A}}$ lying in the $\sim$-class of $\mathcal{A}$, there is a well defined map $\overline{\pi} : \mathbb{A}_A/{\sim} \to (\overline{\Omega}_A\mathsf{DRH})^I/\mathcal{R}$ which sends a class $\mathcal{A}/{\sim}$ to the $\mathcal{R}$-class of the value of $\vec{\mathcal{A}}$. This map is, in effect, a bijection.

Theorem 3.12. *The map*

$$\begin{aligned}\overline{\pi} : \mathbb{A}_A/{\sim} &\to (\overline{\Omega}_A\mathsf{DRH})^I/\mathcal{R}\\ \mathcal{A}/{\sim} &\mapsto [\pi(\vec{\mathcal{A}})]_{\mathcal{R}}\end{aligned}$$

is bijective.

Proof. To prove that $\overline{\pi}$ is injective, we consider two DRH-automata

$$\begin{aligned}\mathcal{A} &= \langle V, \to, \mathsf{q}, F, \lambda_{\mathsf{H}}, \lambda\rangle\\ \mathcal{A}' &= \langle V', \to', \mathsf{q}', F', \lambda'_{\mathsf{H}}, \lambda'\rangle,\end{aligned}$$

such that $\pi(\mathcal{A}) \mathrel{\mathcal{R}} \pi(\mathcal{A}')$ and we argue by induction on $|c(\pi(\mathcal{A}))| = |c(\pi(\mathcal{A}'))|$. If $|c(\pi(\mathcal{A}))| = 0$, then $\mathcal{A} = \mathbf{1} = \mathcal{A}'$ and there is nothing to prove. Suppose that $|c(\pi(\mathcal{A}))| > 0$. We claim that $\mathcal{A}_{[i]} = \mathcal{A}'_{[i]}$ for all $0 \le i \le \|\mathcal{A}\| - 1$. Indeed, by Lemma 2.27, the values $\pi(\mathcal{A})$ and $\pi(\mathcal{A}')$ lie in the same $\mathcal{R}$-class if and only if $\mathsf{lbf}_\infty(\pi(\mathcal{A})) = \mathsf{lbf}_\infty(\pi(\mathcal{A}'))$. Hence, by Lemma 3.10, we get the following equalities:

$$\begin{aligned}\|\mathcal{A}\| &= \|\mathcal{A}'\|,\\ \pi(\mathcal{A}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i) &= \pi(\mathcal{A}'_{[i]})\lambda'_{\mathsf{H}}(\mathsf{q}'.1^i),\ \text{for } 0 \le i \le \|\mathcal{A}\| - 1,\\ \lambda(\mathsf{q}.1^i) &= \lambda'(\mathsf{q}'.1^i),\ \text{for } 0 \le i \le \|\mathcal{A}\| - 1.\end{aligned} \tag{3.4}$$

Since, by (A.6), the inclusions $c(\lambda_{\mathsf{H}}(\mathsf{q}.1^i)) \subseteq \vec{c}(\pi(\mathcal{A}_{[i]}))$ and $c(\lambda'_{\mathsf{H}}(\mathsf{q}'.1^i)) \subseteq \vec{c}(\pi(\mathcal{A}'_{[i]}))$ hold, we also have $\pi(\mathcal{A}_{[i]}) \mathrel{\mathcal{R}} \pi(\mathcal{A}'_{[i]})$. By induction hypothesis, that implies $\mathcal{A}_{[i]} = \mathcal{A}'_{[i]}$ (recall that $\mathcal{A}_{[i]}$ and $\mathcal{A}'_{[i]}$ are both DRH-trees, and each equivalence class has a unique DRH-tree).

To conclude that $\overline{\pi}$ is injective, it remains to show that, for $0 \le i \le \|\mathcal{A}\| - 1$, the labels $\lambda_{\mathsf{H}}(\mathsf{q}.1^i)$ and $\lambda'_{\mathsf{H}}(\mathsf{q}'.1^i)$ coincide. If $\vec{c}(\mathcal{A}_{[i]}) = \emptyset = \vec{c}(\mathcal{A}'_{[i]})$, then Property (A.6) yields $\lambda_{\mathsf{H}}(\mathsf{q}.1^i) = I = \lambda'_{\mathsf{H}}(\mathsf{q}'.1^i)$. Otherwise, we have

$$\pi_{\mathsf{irr}}(\mathcal{A}_{[i]})\mathsf{id}(\mathcal{A}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i) = \pi(\mathcal{A}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i) \overset{(3.4)}{=} \pi(\mathcal{A}'_{[i]})\lambda'_{\mathsf{H}}(\mathsf{q}'.1^i) = \pi_{\mathsf{irr}}(\mathcal{A}'_{[i]})\mathsf{id}(\mathcal{A}'_{[i]})\lambda'_{\mathsf{H}}(\mathsf{q}'.1^i),$$

which in turn implies

$$\mathsf{id}(\mathcal{A}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i) = \mathsf{id}(\mathcal{A}'_{[i]})\lambda'_{\mathsf{H}}(\mathsf{q}'.1^i).$$

Since $\rho_{\mathsf{H}}(\mathsf{id}(\mathcal{A}_{[i]}))$ and $\rho_{\mathsf{H}}(\mathsf{id}(\mathcal{A}'_{[i]}))$ are both the identity of $\overline{\Omega}_A\mathsf{H}$, we obtain the equality

$$\lambda_{\mathsf{H}}(\mathsf{q}.1^i) = \lambda'_{\mathsf{H}}(\mathsf{q}'.1^i).$$

Let us prove that $\overline{\pi}$ is surjective. We proceed again by induction, this time on $|c(w)|$, for $w \in (\overline{\Omega}_A\mathsf{DRH})^I$. If $c(w)$ is the empty set, then we have $[w]_{\mathcal{R}} = \{I\} = \{\pi(\mathbf{1})\} = \overline{\pi}(\mathbf{1}/_{\sim})$. Otherwise, if $w \neq I$, then we let $w = w_0a_0 \cdots w_ka_kw'_k$ be the k-th iteration of the left basic factorization of w (whenever it is defined). For each $0 \leq i \leq \lceil w \rceil - 1$, we have $c(w_i) \subsetneq c(w)$ and so, by induction hypothesis, there exists a DRH-tree $\mathcal{A}_i = \langle V_i, \to_i, \mathsf{q}_i, F_i, \lambda_{i,\mathsf{H}}, \lambda_i \rangle$ such that $\pi(\mathcal{A}_i) \mathrel{\mathcal{R}} w_i$. In particular, the equality $\pi_{\mathsf{irr}}(\mathcal{A}_i) = \mathsf{irr}(w_i)$ holds and consequently, H satisfies

$$\pi(\mathcal{A}_i) \cdot \mathsf{reg}(w_i) = \pi_{\mathsf{irr}}(\mathcal{A}_i) \cdot \mathsf{id}(\mathcal{A}_i) \cdot \mathsf{reg}(w_i) = \mathsf{irr}(w_i) \cdot 1 \cdot \mathsf{reg}(w_i) = w_i. \tag{3.5}$$

On the other hand, since

$$c(\mathsf{reg}(w_i)) = c(\mathsf{id}(\mathcal{A}_i)) = \vec{c}(\mathsf{id}(\mathcal{A}_i)),$$

we deduce that $\mathsf{id}(\mathcal{A}_i) \cdot \mathsf{reg}(w_i)$ is $\mathcal{R}$-equivalent to $\mathsf{id}(\mathcal{A}_i)$. Therefore, w_i and $\pi(\mathcal{A}_i) \cdot \mathsf{reg}(w_i)$ are $\mathcal{R}$-equivalent as well. This relation together with (3.5) imply, by Lemma 2.32, that the equality $\pi(\mathcal{A}_i) \cdot \mathsf{reg}(w_i) = w_i$ holds.

Now, we construct a DRH-tree $\mathcal{A} = \langle V, \to, \mathsf{q}, F, \lambda_{\mathsf{H}}, \lambda \rangle$ as follows:

- $V = \begin{cases} \{\mathsf{v} \in V_i \colon i \geq 0\} \uplus \{\mathsf{v}_i \colon i \geq 0\}, & \text{if } \lceil w \rceil = \infty; \\ \{\mathsf{v} \in V_i \colon i = 0, \ldots, \lceil w \rceil - 1\} \uplus \{\mathsf{v}_i \colon i = 0, \ldots, \lceil w \rceil - 1\} \uplus \{\mathsf{v}_\varepsilon\}, & \text{if } \lceil w \rceil < \infty; \end{cases}$
- $\mathsf{q} = \mathsf{v}_0$;
- $F = \begin{cases} \{\mathsf{v} \in F_i \colon i \geq 0\}, & \text{if } \lceil w \rceil = \infty; \\ \{\mathsf{v} \in F_i \colon i = 0, \ldots, \lceil w \rceil - 1\} \uplus \{\mathsf{v}_\varepsilon\}, & \text{if } \lceil w \rceil < \infty; \end{cases}$
- $\lambda_{\mathsf{H}}(\mathsf{v}_i) = \rho_{\mathsf{H}}(\mathsf{reg}(w_i))$ and $\lambda(\mathsf{v}_i) = a_i$ for $i = 0, \ldots, \lceil w \rceil - 1$;
- $\lambda(\mathsf{v}_\varepsilon) = \varepsilon$, if $\lceil w \rceil$ is finite;
- $\mathsf{v}_i.0 = \mathsf{q}_i$ and $\mathsf{v}_i.1 = \begin{cases} \mathsf{v}_{i+1}, & \text{if } i < \lceil w \rceil - 1; \\ \mathsf{v}_\varepsilon, & \text{if } i = \lceil w \rceil - 1; \end{cases}$
- transitions and labelings on V_i are given by those of $\mathcal{A}_i$.

Then, it is easy to check that $\mathcal{A}$ is a DRH-tree and that, for all $0 \leq i < \lceil w \rceil$, the equality

$$\pi(\mathcal{A}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i)\lambda(\mathsf{q}.1^i) = w_ia_i$$

holds. Hence, the $\sim$-class of $\mathcal{A}$ is sent to the $\mathcal{R}$-class of w by $\overline{\pi}$. □

Given an element w of $(\overline{\Omega}_A\mathsf{S})^I$, the DRH-tree representing the $\sim$-class $\overline{\pi}^{-1}([\rho_{\mathsf{DRH}}(w)]_{\mathcal{R}})$ is denoted $\mathcal{T}(w)$. With a little abuse of notation, when $w \in (\overline{\Omega}_A\mathsf{DRH})^I$, we use $\mathcal{T}(w)$ to denote the unique DRH-tree in the $\sim$-class $\overline{\pi}^{-1}([w]_{\mathcal{R}})$. Later, we shall see that, for every κ-word w, there exists a finite DRH-automaton $\mathcal{A}$ in the $\sim$-class of $\mathcal{T}(w)$ (Corollary 3.24).

Suppose that we are given two DRH-automata $\mathcal{A}_i = \langle V_i, \to_i, \mathsf{q}_i, F_i, \lambda_{i,\mathsf{H}}, \lambda_i \rangle$, $i = 0, 1$, a letter $a \in A$ such that $\lambda_1(V_1) \subseteq \lambda_0(V_0) \uplus \{a\}$ and a pseudoword u such that $c(u) \subseteq \vec{c}(\mathcal{A}_0)$. Then, we denote by $(\mathcal{A}_0, u \mid a, \mathcal{A}_1)$ the DRH-automaton $\mathcal{A} = \langle V, \to, \mathsf{q}, F, \lambda_{\mathsf{H}}, \lambda \rangle$, where

- $V = V_0 \uplus V_1 \uplus \{\mathsf{q}\}$;
- $\mathsf{q}.0 = \mathsf{q}_0$ and $\mathsf{q}.1 = \mathsf{q}_1$;
- $F = F_0 \uplus F_1$;
- $\lambda_{\mathsf{H}}(\mathsf{q}) = \rho_{\mathsf{H}}(u)$ and $\lambda(\mathsf{q}) = a$;
- all the other transitions and labels are given by those of $\mathcal{A}_0$ and $\mathcal{A}_1$.

Lemma 3.13. *Let w be a pseudoword and write $\mathsf{lbf}(w) = (w_\ell, a, w_r)$. Then, we have the equality*

$$\mathcal{T}(w) = (\mathcal{T}(w_\ell), \mathsf{reg}(w_\ell) \mid a, \mathcal{T}(w_r)).$$

Proof. Write

$$\begin{aligned}
\mathcal{T} = (\mathcal{T}(w_\ell), \mathsf{reg}(w_\ell) \mid a, \mathcal{T}(w_r)) &= \langle V, \to, \mathsf{q}, F, \lambda_{\mathsf{H}}, \lambda \rangle;\\
\mathcal{T}(w_\ell) &= \langle V_0, \to_0, \mathsf{q}_0, F_0, \lambda_{0,\mathsf{H}}, \lambda_0 \rangle;\\
\mathcal{T}(w_r) &= \langle V_1, \to_1, \mathsf{q}_1, F_1, \lambda_{1,\mathsf{H}}, \lambda_1 \rangle.
\end{aligned}$$

The claim amounts to proving that $\pi(\mathcal{T}) \mathrel{\mathcal{R}} w$ modulo DRH. By definition of $\mathcal{T}$, we have $\|\mathcal{T}\| < \infty$ if and only if $\|\mathcal{T}(w_r)\| < \infty$. We start by proving that $\pi(\mathcal{T})$ and $\pi(\mathcal{T}(w_\ell))\rho_{\mathsf{H}}(\mathsf{reg}(w_\ell))a \cdot \pi(\mathcal{T}(w_r))$ belong to the same $\mathcal{R}$-class. It is worth noticing that, for every $1 \le i \le \|\mathcal{T}\|$, we have the following equality:

$$\mathcal{T}_{[i]} = \mathcal{T}_{\mathsf{q}.1^i0} = \mathcal{T}(w_r)_{\mathsf{q}_1.1^{i-1}0} = \mathcal{T}(w_r)_{[i-1]}. \tag{3.6}$$

First, assume that $\|\mathcal{T}\| < \infty$. Then, we have $\|\mathcal{T}\| = \|\mathcal{T}(w_r)\| + 1$. Following Definition 3.9 and the construction of $\mathcal{T}$, we may compute

$$\begin{aligned}
\pi(\mathcal{T}) &= \prod_{i=0}^{\|\mathcal{T}(w_r)\|} \pi(\mathcal{T}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i)\lambda(\mathsf{q}.1^i)\\
&= \pi(\mathcal{T}_{\mathsf{q}.0})\lambda_{\mathsf{H}}(\mathsf{q})\lambda(\mathsf{q}) \cdot \prod_{i=0}^{\|\mathcal{T}(w_r)\|-1} \pi(\mathcal{T}_{[i+1]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{i+1})\lambda(\mathsf{q}.1^{i+1})\\
&\overset{(3.6)}{=} \pi(\mathcal{T}(w_\ell))\rho_{\mathsf{H}}(\mathsf{reg}(w_\ell))a \cdot \pi(\mathcal{T}(w_r)).
\end{aligned} \tag{3.7}$$

Now, we suppose that $\|\mathcal{T}\| = \infty$. In that case, $\mathsf{r.ind}(\mathcal{T})$ is either $\mathsf{r.ind}(\mathcal{T}(\mathsf{w_r}))$ or $\mathsf{r.ind}(\mathcal{T}(\mathsf{w_r})) + 1$ according to whether $\rho_{\mathsf{DRH}}(w)$ is regular (in which case, it is 0) or not, respectively. Suppose that $\rho_{\mathsf{DRH}}(w)$ is not regular. We compute

$$\begin{aligned}
\pi(\mathcal{T}) &= \prod_{i=0}^{\mathsf{r.ind}(\mathcal{T}(\mathsf{w_r}))} \pi(\mathcal{T}_{[i]})\lambda_{\mathsf{H}}(\mathsf{q}.1^i)\lambda(\mathsf{q}.1^i) \cdot \mathsf{id}(\mathcal{T})\\
&= \pi(\mathcal{T}_{\mathsf{q}.0})\lambda_{\mathsf{H}}(\mathsf{q})\lambda(\mathsf{q}) \cdot \left(\prod_{i=0}^{\mathsf{r.ind}(\mathcal{T}(\mathsf{w_r}))-1} \pi(\mathcal{T}_{[i+1]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{i+1})\lambda(\mathsf{q}.1^{i+1}) \right) \cdot \mathsf{id}(\mathcal{T})\\
&\overset{(3.6)}{=} \pi(\mathcal{T}(w_\ell))\rho_{\mathsf{H}}(\mathsf{reg}(w_\ell))a \cdot \pi_{\mathsf{irr}}(\mathcal{T}(w_r)) \cdot \mathsf{id}(\mathcal{T}).
\end{aligned} \tag{3.8}$$

Now, $\mathsf{id}(\mathcal{T})$ is the idempotent designated by the infinite product

$$(\pi(\mathcal{T}_{[\mathsf{r.ind}(\mathcal{T})]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{\mathsf{r.ind}(\mathcal{T})})\lambda(\mathsf{q}.1^{\mathsf{r.ind}(\mathcal{T})})\cdots\pi(\mathcal{T}_{[k]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{k})\lambda(\mathsf{q}.1^{k}))_{k\geq\mathsf{r.ind}(\mathcal{T})},$$

which in turn, by (3.6), is the infinite product

$$(\pi(\mathcal{T}(w_r)_{[\mathsf{r.ind}(\mathcal{T}(\mathsf{w_r}))]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{\mathsf{r.ind}(\mathcal{T}(\mathsf{w_r}))})\lambda(\mathsf{q}.1^{\mathsf{r.ind}(\mathcal{T}(\mathsf{w_r}))})\cdots\pi(\mathcal{T}_{k})\lambda_{\mathsf{H}}(\mathsf{q}.1^{k})\lambda(\mathsf{q}.1^{k}))_{k\geq\mathsf{r.ind}(\mathcal{T}(\mathsf{w_r}))+1}.$$

Hence, we have $\mathsf{id}(\mathcal{T})=\mathsf{id}(\mathcal{T}(w_r))$, and so, the equality (3.8) yields

$$\pi(\mathcal{T})=\pi(\mathcal{T}(w_\ell))\rho_{\mathsf{H}}(\mathsf{reg}(w_\ell))a\cdot\pi(\mathcal{T}(w_r)). \tag{3.9}$$

If $\rho_{\mathsf{DRH}}(w)$ is regular, then $\pi(\mathcal{T})=\mathsf{id}(\mathcal{T})$. In this case, $\mathsf{id}(\mathcal{T})$ is the idempotent designated by the infinite product

$$(\pi(\mathcal{T}_{[0]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{0})\lambda(\mathsf{q}.1^{0})\cdot\pi(\mathcal{T}_{[1]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{1})\lambda(\mathsf{q}.1^{1})\cdots\pi(\mathcal{T}_{[k]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{k})\lambda(\mathsf{q}.1^{k}))_{k\geq 0}.$$

Again, (3.6) together with the construction of $\mathcal{T}$ give that this infinite product is precisely the product

$$(\pi(\mathcal{T}(w_\ell))\rho_{\mathsf{H}}(\mathsf{reg}(w_\ell))a\cdot\pi(\mathcal{T}(w_r)_{[0]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{0})\lambda(\mathsf{q}.1^{0})\cdots\pi(\mathcal{T}_{[k]})\lambda_{\mathsf{H}}(\mathsf{q}.1^{k})\lambda(\mathsf{q}.1^{k}))_{k\geq 0}.$$

Therefore, we may conclude that

$$\pi(\mathcal{T})=\mathsf{id}(\mathcal{T})\;\mathcal{R}\;\pi(\mathcal{T}(w_\ell))\rho_{\mathsf{H}}(\mathsf{reg}(w_\ell))a\cdot\mathsf{id}(\mathcal{T}(w_r))=\pi(\mathcal{T}(w_\ell))\rho_{\mathsf{H}}(\mathsf{reg}(w_\ell))a\cdot\pi(\mathcal{T}(w_r)). \tag{3.10}$$

Finally, we need to establish the equality $w_\ell=\pi(\mathcal{T}(w_\ell))\rho_{\mathsf{H}}(\mathsf{reg}(w_\ell))$. But, using Lemma 2.32, that is immediate, since $w_\ell\;\mathcal{R}\;\pi(\mathcal{T}(w_\ell))\;\mathcal{R}\;\pi(\mathcal{T}(w_\ell))\mathsf{reg}(w_\ell)$ modulo DRH and H satisfies

$$\begin{aligned}\pi(\mathcal{T}(w_\ell))\rho_{\mathsf{H}}(\mathsf{reg}(w_\ell))&=\pi_{\mathsf{irr}}(\mathcal{T}(w_\ell))\cdot\mathsf{id}(\mathcal{T}(w_\ell))\mathsf{reg}(w_\ell)\\&\overset{\text{Lemma 3.10}}{=}\mathsf{irr}(w_\ell)\cdot 1\cdot\mathsf{reg}(w_\ell)=w_\ell.\end{aligned}$$

Hence, it follows from (3.7), (3.9), and (3.10) that $w=w_\ell\cdot a\cdot w_r\;\mathcal{R}\;\pi(\mathcal{T})$, as intended. $\square$

The *value of a path* $\mathsf{q}_0\xrightarrow{\alpha_0}\mathsf{q}_1\xrightarrow{\alpha_1}\cdots\xrightarrow{\alpha_n}\mathsf{q}_{n+1}$ of a DRH-automaton $\mathcal{A}$ is given by

$$\prod_{i=0}^{n}(\alpha_i,\lambda_{\mathsf{H},\alpha_i}(\mathsf{q}_i),\lambda(\mathsf{q}_i))\in\left(\Sigma\times(\overline{\Omega}_A\mathsf{H})^I\times A\right)^+,\ \text{where }\lambda_{\mathsf{H},\alpha_i}(\mathsf{q}_i)=\begin{cases}\lambda_{\mathsf{H}}(\mathsf{q}_i), & \text{if }\alpha_i=0;\\ I, & \text{otherwise.}\end{cases}$$

Given a state v of $\mathcal{A}$, the *language associated to* v, $\mathcal{L}(\mathsf{v})\subseteq\left(\Sigma\times(\overline{\Omega}_A\mathsf{H})^I\times A\right)^+$, is the set of all values of successful paths of $\mathcal{A}_{\mathsf{v}}$. The *language* associated to $\mathcal{A}$, denoted $\mathcal{L}(\mathcal{A})$, is the language associated to its root. Finally, the *language associated to the pseudoword* w is $\mathcal{L}(w)=\mathcal{L}(\mathcal{T}(w))$.

The reader may wish to compare the next two results with [25, Lemma 3.23] and [25, Proposition 3.24], respectively.

Lemma 3.14. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be* DRH*-automata. Then, the languages $\mathcal{L}(\mathcal{A}_1)$ and $\mathcal{L}(\mathcal{A}_2)$ coincide if and only if the* DRH*-trees $\vec{\mathcal{A}}_1$ and $\vec{\mathcal{A}}_2$ are the same.*

Proof. Recall that, by Lemma 3.6, if $\vec{\mathcal{A}}_1 = \vec{\mathcal{A}}_2$, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent DRH-automata. Hence, Definition 3.4 makes clear the reverse implication. Conversely, consider two DRH-automata $\mathcal{A}_i = \langle V_i, \to_i, \mathsf{q}_{i,0}, F_i, \lambda_{i,\mathsf{H}}, \lambda_i\rangle$ ($i = 1,2$) such that $\mathcal{L}(\mathcal{A}_1) = \mathcal{L}(\mathcal{A}_2)$. We first observe that, for $i = 1,2$ and $\alpha \in \Sigma^*$, the state $\mathsf{q}_{i,0}.\alpha$ is defined if and only if there is an element in $\mathcal{L}(\mathcal{A}_i)$ of the form $(\alpha\beta, \_, \_)$, for a certain $\beta \in \Sigma^*$ (we are using the fact that DRH-automata are trim). Hence, the state $\mathsf{q}_{1,0}.\alpha$ is defined if and only if so is the state $\mathsf{q}_{2,0}.\alpha$. Choose an element $\alpha = \alpha_0\alpha_1\cdots\alpha_n \in \Sigma^*$, with each $\alpha_i \in \Sigma$, and such that $\mathsf{q}_{1,0}.\alpha$ is defined. If $\mathsf{q}_{1,0}.\alpha \in F_1$, then we have a successful path

$$\mathsf{q}_{1,0} \xrightarrow{\alpha_0} \mathsf{q}_{1,1} \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} \mathsf{q}_{1,n+1},$$

so that, the element $\prod_{i=0}^{n}(\alpha_i, \lambda_{1,\mathsf{H},\alpha_i}(\mathsf{q}_{1,i}), \lambda_1(\mathsf{q}_{1,i}))$ belongs to $\mathcal{L}(\mathcal{A}_1)$ and hence, to $\mathcal{L}(\mathcal{A}_2)$. But that implies that, in $\mathcal{A}_2$, there is a successful path

$$\mathsf{q}_{2,0} \xrightarrow{\alpha_0} \mathsf{q}_{2,1} \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} \mathsf{q}_{2,n+1},$$

which in turn yields that both $\mathsf{q}_{1,0}.\alpha$ and $\mathsf{q}_{2,0}.\alpha$ are terminal states. In particular the equalities in (3.1) hold. On the other hand, if $\mathsf{q}_{1,0}.\alpha$ is not a terminal state, then Property (A.3) implies that $\mathsf{q}_{1,0}.\alpha 0$ is defined. Let $\beta = \alpha_{n+2}\cdots\alpha_m \in \Sigma^*$ be such that

$$\mathsf{q}_{1,0} \xrightarrow{\alpha_0} \mathsf{q}_{1,1} \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} \mathsf{q}_{1,n+1} \xrightarrow{0} \mathsf{q}_{1,n+2} \xrightarrow{\alpha_{n+2}} \cdots \xrightarrow{\alpha_m} \mathsf{q}_{1,m+1} \tag{3.11}$$

is a successful path in $\mathcal{A}_1$. Again, since $\mathcal{L}(\mathcal{A}_1) = \mathcal{L}(\mathcal{A}_2)$, this determines a successful path in $\mathcal{A}_2$ given by

$$\mathsf{q}_{2,0} \xrightarrow{\alpha_0} \mathsf{q}_{2,1} \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} \mathsf{q}_{2,n+1} \xrightarrow{0} \mathsf{q}_{2,n+2} \xrightarrow{\alpha_{n+2}} \cdots \xrightarrow{\alpha_m} \mathsf{q}_{2,m+1},$$

with the same value as the path (3.11). In particular, the $(n+2)$-nd letter (which belongs to the alphabet $\Sigma \times (\overline{\Omega}_A\mathsf{H})^I \times A$) of that value is

$$(0, \lambda_{1,\mathsf{H},0}(\mathsf{q}_{1,n+1}), \lambda_1(\mathsf{q}_{1,n+1})) = (0, \lambda_{2,\mathsf{H},0}(\mathsf{q}_{2,n+1}), \lambda_2(\mathsf{q}_{2,n+1})).$$

But that means precisely that the desired equalities in (3.1) hold. Therefore, $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent and so, $\vec{\mathcal{A}}_1 = \vec{\mathcal{A}}_2$. □

Proposition 3.15. *Let $u, v \in \overline{\Omega}_A\mathsf{S}$. Then, the equality $\rho_{\mathsf{DRH}}(u) = \rho_{\mathsf{DRH}}(v)$ holds if and only if $\mathcal{L}(u) = \mathcal{L}(v)$ and* H *satisfies $u = v$.*

Proof. Let u and v be two equal pseudowords modulo DRH. In particular, the $\mathcal{R}$-classes $[\rho_{\mathsf{DRH}}(u)]_{\mathcal{R}}$ and $[\rho_{\mathsf{DRH}}(v)]_{\mathcal{R}}$ coincide and so, the DRH-trees $\mathcal{T}(u)$ and $\mathcal{T}(v)$ are the same, by Theorem 3.12. Therefore, we have

$$\mathcal{L}(u) = \mathcal{L}(\mathcal{T}(u)) = \mathcal{L}(\mathcal{T}(v)) = \mathcal{L}(v).$$

As H is a subpseudovariety of DRH, we also have $u =_{\mathsf{H}} v$. Conversely, suppose that $\mathcal{L}(u) = \mathcal{L}(v)$ and $u =_{\mathsf{H}} v$. By Lemma 3.14, it follows that $\mathcal{T}(u) = \mathcal{T}(v)$. Thus, by Theorem 3.12, the pseudovariety

DRH satisfies $u \mathcal{R} v$. As, in addition, the pseudowords u and v are equal modulo H, we conclude by Lemma 2.32 that DRH satisfies $u = v$. $\square$

3.2 A canonical form for κ-words over DRH

Throughout this section, we reserve the letter H to denote a pseudovariety of groups such that there exists a computable canonical form for the elements of $\Omega_A^\kappa \mathsf{H}$. We denote by $\mathsf{cf}_\mathsf{H}(w)$ the canonical form of $w \in \Omega_A^\kappa \mathsf{H}$ and set $\mathsf{cf}_\mathsf{H}(I) = I$. Our aim is to prove that this assumption on H is enough to define a canonical form for the elements of $\Omega_A^\kappa \mathsf{DRH}$.

Given a finite DRH-automaton $\mathcal{A} = \langle V, \to \mathsf{q}, F, \lambda_\mathsf{H}, \lambda \rangle$ such that $\lambda_\mathsf{H}(V) \subseteq (\Omega_A^\kappa \mathsf{H})^I$, let us define the expression $\pi_\mathsf{cf}(\mathcal{A})$ inductively on the number $|V|$ of states as follows.

- If $|V| = 1$, then $\mathcal{A} = \mathbf{1}$ and we take $\pi_\mathsf{cf}(\mathcal{A}) = I$.
- If $|V| > 1$ and $\|\mathcal{A}\| < \infty$, then we put

$$\pi_\mathsf{cf}(\mathcal{A}) = \prod_{i=0}^{\|\mathcal{A}\|-1} \pi_\mathsf{cf}(\mathcal{A}_{\mathsf{q}.1^i0})\mathsf{cf}_\mathsf{H}(\lambda_\mathsf{H}(\mathsf{q}.1^i))\lambda(\mathsf{q}.1^i).$$

- Finally, we suppose that $|V| > 1$ and $\|\mathcal{A}\| = \infty$. Since $\mathcal{A}$ is a finite automaton, we necessarily have a cycle of the form $\mathsf{q}.1^\ell \xrightarrow{1} \mathsf{q}.1^{\ell+1} \xrightarrow{1} \cdots \xrightarrow{1} \mathsf{q}.1^{\ell+n} \xrightarrow{1} \mathsf{q}.1^\ell$, where ℓ is a certain integer greater than or equal to $\mathsf{r.ind}(\mathcal{A})$. Choose ℓ to be the least possible. Then, we make $\pi_\mathsf{cf}(\mathcal{A})$ be given by

$$\prod_{i=0}^{\mathsf{r.ind}(\mathcal{A})-1} \pi_\mathsf{cf}(\mathcal{A}_{\mathsf{q}.1^i0})\mathsf{cf}_\mathsf{H}(\lambda_\mathsf{H}(\mathsf{q}.1^i))\lambda(\mathsf{q}.1^i)$$
$$\cdot \left(\prod_{i=\mathsf{r.ind}(\mathcal{A})}^{\ell-1} \pi_\mathsf{cf}(\mathcal{A}_{\mathsf{q}.1^i0})\mathsf{cf}_\mathsf{H}(\lambda_\mathsf{H}(\mathsf{q}.1^i))\lambda(\mathsf{q}.1^i) \left(\prod_{i=0}^{n} \pi_\mathsf{cf}(\mathcal{A}_{\mathsf{q}.1^{\ell+i}0})\mathsf{cf}_\mathsf{H}(\lambda_\mathsf{H}(\mathsf{q}.1^{\ell+i}))\lambda(\mathsf{q}.1^{\ell+i}) \right)^\omega \right)^\omega.$$

We point out that, by definition, the value of the κ-word over DRH naturally induced by $\pi_\mathsf{cf}(\mathcal{A})$ is precisely $\pi(\mathcal{A})$. On the other hand, it is easy to check that, for every $w \in \overline{\Omega}_A \mathsf{DRH}$, if $w \mathcal{R} \pi(\mathcal{A})$, then the identity $w = \pi(\mathcal{A})\mathsf{reg}(w)$ holds. Thus, in view of Theorem 3.12, we wish to standardize a choice of a finite DRH-automaton, say $\mathcal{A}(w)$, equivalent to $\mathcal{T}(w)$, for each $w \in \Omega_A^\kappa \mathsf{DRH}$. After that, we may let the canonical form of w be given by $\pi_\mathsf{cf}(\mathcal{A}(w))\mathsf{cf}_\mathsf{H}(\mathsf{reg}(w))$.

Example 3.16. Let $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}_2$ be the equivalent DRH-automata in Figures 3.1 and 3.2, respectively, and let q_i be the root of $\mathcal{A}_i$ ($i = 1,2$). Then, we may compute

$$\pi_\mathsf{cf}((\mathcal{A}_1)_{\mathsf{q}_1.1}) = (aba^\omega \mathsf{cf}_\mathsf{H}(a) b (a^\omega \mathsf{cf}_\mathsf{H}(a^{\omega+1}) b)^\omega)^\omega;$$
$$\pi_\mathsf{cf}((\mathcal{A}_2)_{\mathsf{q}_2.1}) = (ab((aa^\omega)^\omega \mathsf{cf}_\mathsf{H}(a) b)^\omega)^\omega.$$

Fix a DRH-automaton $\mathcal{A} = \langle V, \to, \mathsf{q}, F, \lambda_\mathsf{H}, \lambda \rangle$. We say that two states $\mathsf{v}_1, \mathsf{v}_2 \in V$ are *equivalent* if $\pi(\mathcal{A}_{\mathsf{v}_1})$ and $\pi(\mathcal{A}_{\mathsf{v}_2})$ lie in the same $\mathcal{R}$-class. Clearly, this defines an equivalence relation on V, say $\sim$ (it should be clear from the context when we are referring to this equivalence relation or to the equivalence relation on $\mathbb{A}_A$ introduced in Definition 3.4). We write $[\mathsf{v}]$ for the equivalence class of $\mathsf{v} \in V$.

Lemma 3.17. *Let $\mathcal{A} = \langle V, \to, \mathsf{q}, F, \lambda_{\mathsf{H}}, \lambda \rangle$ be a* DRH*-automaton and consider the equivalent class on V defined above. Then, for every $\mathsf{v}_1, \mathsf{v}_2 \in V \setminus F$, we have*

$$[\mathsf{v}_1] = [\mathsf{v}_2] \implies \begin{cases} [\mathsf{v}_1.0] = [\mathsf{v}_2.0] \text{ and } [\mathsf{v}_1.1] = [\mathsf{v}_2.1]; \\ \lambda_{\mathsf{H}}(\mathsf{v}_1) = \lambda_{\mathsf{H}}(\mathsf{v}_2) \text{ and } \lambda(\mathsf{v}_1) = \lambda(\mathsf{v}_2). \end{cases}$$

Proof. Let $\mathsf{v}_1, \mathsf{v}_2 \in V \setminus F$ be non-terminal states. By definition, the classes $[\mathsf{v}_1]$ and $[\mathsf{v}_2]$ coincide if and only if $\pi(\mathcal{A}_{\mathsf{v}_1}) \mathrel{\mathcal{R}} \pi(\mathcal{A}_{\mathsf{v}_2})$. Moreover, by Lemma 3.10, we have

$$\mathsf{lbf}(\pi(\mathcal{A}_{\mathsf{v}_1})) = (\pi(\mathcal{A}_{\mathsf{v}_1.0})\lambda_{\mathsf{H}}(\mathsf{v}_1), \lambda(\mathsf{v}_1), w_{1,r}),$$

where $w_{1,r}$ is $\mathcal{R}$-equivalent to $\pi(\mathcal{A}_{\mathsf{v}_1.1})$. Similarly, there exists $w_{2,r} \mathrel{\mathcal{R}} \pi(\mathcal{A}_{\mathsf{v}_2.1})$ such that

$$\mathsf{lbf}(\pi(\mathcal{A}_{\mathsf{v}_2})) = (\pi(\mathcal{A}_{\mathsf{v}_2.0})\lambda_{\mathsf{H}}(\mathsf{v}_2), \lambda(\mathsf{v}_2), w_{2,r}).$$

In particular, since we are assuming that $\pi(\mathcal{A}_{\mathsf{v}_1}) \mathrel{\mathcal{R}} \pi(\mathcal{A}_{\mathsf{v}_2})$, the relations $\pi(\mathcal{A}_{\mathsf{v}_1.0}) \mathrel{\mathcal{R}} \pi(\mathcal{A}_{\mathsf{v}_2.0})$ and $\pi(\mathcal{A}_{\mathsf{v}_1.1}) \mathrel{\mathcal{R}} \pi(\mathcal{A}_{\mathsf{v}_2.1})$ hold. But, that means that $[\mathsf{v}_1.0] = [\mathsf{v}_2.0]$ and $[\mathsf{v}_1.1] = [\mathsf{v}_2.1]$. Also, the mid components of $\mathsf{lbf}(\pi(\mathcal{A}_{\mathsf{v}_1}))$ and $\mathsf{lbf}(\pi(\mathcal{A}_{\mathsf{v}_2}))$ should coincide, that is, $\lambda(\mathsf{v}_1) = \lambda(\mathsf{v}_2)$. Finally, we may derive the equality $\lambda_{\mathsf{H}}(\mathsf{v}_1) = \lambda_{\mathsf{H}}(\mathsf{v}_2)$ as follows:

$$\begin{aligned}
& \pi(\mathcal{A}_{\mathsf{v}_1.0})\lambda_{\mathsf{H}}(\mathsf{v}_1) = \pi(\mathcal{A}_{\mathsf{v}_2.0})\lambda_{\mathsf{H}}(\mathsf{v}_2) \qquad \text{because } \pi(\mathcal{A}_{\mathsf{v}_1}) \mathrel{\mathcal{R}} \pi(\mathcal{A}_{\mathsf{v}_2}) \\
\iff & \pi_{\mathsf{irr}}(\mathcal{A}_{\mathsf{v}_1.0})\mathsf{id}(\mathcal{A}_{\mathsf{v}_1.0})\lambda_{\mathsf{H}}(\mathsf{v}_1) = \pi_{\mathsf{irr}}(\mathcal{A}_{\mathsf{v}_2.0})\mathsf{id}(\mathcal{A}_{\mathsf{v}_2.0})\lambda_{\mathsf{H}}(\mathsf{v}_2) \qquad \text{by (3.3)} \\
\implies & \mathsf{id}(\mathcal{A}_{\mathsf{v}_1.0})\lambda_{\mathsf{H}}(\mathsf{v}_1) = \mathsf{id}(\mathcal{A}_{\mathsf{v}_2.0})\lambda_{\mathsf{H}}(\mathsf{v}_2) \qquad \text{by Lemma 3.10 and Corollary 2.30} \\
\implies & \lambda_{\mathsf{H}}(\mathsf{v}_1) = \lambda_{\mathsf{H}}(\mathsf{v}_2) \qquad \text{because } \rho_{\mathsf{H}} \circ \mathsf{id}(\mathcal{A}) \text{ is always the identity of } \overline{\Omega}_A\mathsf{H}. \qquad \square
\end{aligned}$$

We define the *wrapping of a* DRH*-automaton* $\mathcal{A} = \langle V, \to, \mathsf{q}, F, \lambda_{\mathsf{H}}, \lambda \rangle$ to be the DRH-automaton $[\mathcal{A}] = \langle V/{\sim}, \to, [\mathsf{q}], F/{\sim}, \overline{\lambda}_{\mathsf{H}}, \overline{\lambda} \rangle$, where

- $[\mathsf{v}].0 = [\mathsf{v}.0]$ and $[\mathsf{v}].1 = [\mathsf{v}.1]$, for $\mathsf{v} \in V \setminus F$;
- $\overline{\lambda}_{\mathsf{H}}([\mathsf{v}]) = \lambda_{\mathsf{H}}(\mathsf{v})$ and $\overline{\lambda}([\mathsf{v}]) = \lambda(\mathsf{v})$, for $\mathsf{v} \in V$.

By Lemma 3.17, this automaton is well defined. Furthermore, its definition ensures that $\mathcal{A} \sim [\mathcal{A}]$. The *wrapped* DRH*-automaton of* $w \in \overline{\Omega}_A\mathsf{DRH}$ is $\mathcal{A}(w) = [\mathcal{T}(w)]$. Observe that, by Lemmas 2.26 and 3.13, the label λ_{H} of $\mathcal{T}(w)$ takes values in $\Omega_A^{\kappa}\mathsf{H}$ when w is a κ-word. Our next goal is to prove that $\mathcal{A}(w)$ is finite, provided w is a κ-word.

Example 3.18. Let $\mathcal{A}_2$ be the same as in Example 3.16. Then, we have the following identities:

$$\begin{aligned}
\pi((\mathcal{A}_2)_{\mathsf{q}_2.1}) &= (ab(a^{\omega+1}b)^{\omega})^{\omega}; \\
\pi((\mathcal{A}_2)_{\mathsf{q}_2.10}) &= a; \\
\pi((\mathcal{A}_2)_{\mathsf{q}_2.11}) &= (a^{\omega+1}b)^{\omega};
\end{aligned}$$

$$\pi((\mathcal{A}_2)_{\mathsf{q}_2.110}) = a^{\omega};$$
$$\pi((\mathcal{A}_2)_{\mathsf{q}_2.1101}) = a^{\omega}.$$

Hence, the wrapping of the DRH-automaton $(\mathcal{A}_2)_{\mathsf{q}_2.1}$ is obtained by merging the states $\mathsf{q}_2.1101$ and $\mathsf{q}_2.110$. The result is drawn in Figure 3.5.

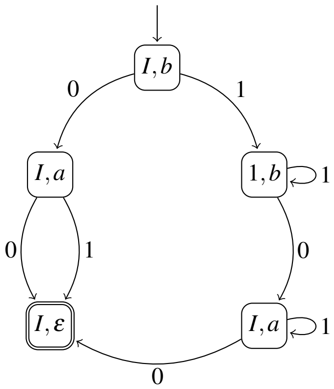

Fig. 3.5 The DRH-automaton $[(\mathcal{A}_2)_{\mathsf{q}_2.1}]$.

Let us associate to a pseudoword $w \in (\overline{\Omega}_A \mathsf{DRH})^I$ a certain set of its factors. For $\alpha \in \Sigma^*$, we define $f_\alpha(w)$ inductively on $|\alpha|$:

$$f_\varepsilon(w) = w;$$
$$(f_{\alpha 0}(w), a, f_{\alpha 1}(w)) = \mathsf{lbf}(f_\alpha(w)), \text{ for a certain } a \in A, \text{ whenever } f_\alpha(w) \neq I.$$

Then, the set of DRH-*factors* of w is given by

$$\mathcal{F}(w) = \{f_\alpha(w) \colon \alpha \in \Sigma^* \text{ and } f_\alpha(w) \text{ is defined}\}.$$

Example 3.19. Consider the κ-word $w = ab(a^{\omega+1}b)^{\omega}$. Then, the computation of $f_\alpha(w)$ is schematized in Figure 3.6. Thus, in this case, we have that the set of factors of w is given by

$$\mathcal{F}(w) = \{ab(a^{\omega+1}b)^{\omega}, a, I\} \cup \{(a^{\omega+1}b)^{\omega-k}, a^{\omega+1-k}\}_{k \geq 0}.$$

We further observe that there are only finitely many distinct $\mathcal{R}$-classes in $\mathcal{F}(w)$. As we shall prove, this happens in general, provided w is a κ-word (Proposition 3.23).

The relevance of the definition of the set $\mathcal{F}(w)$ is explained by the following result.

Lemma 3.20. *Let $w \in \overline{\Omega}_A \mathsf{DRH}$ and $\mathcal{T}(w) = \langle V, \to, \mathsf{q}, F, \lambda_{\mathsf{H}}, \lambda \rangle$. Then, for every $\alpha \in \Sigma^*$ such that $f_\alpha(w)$ is defined, the relation $f_\alpha(w) \mathrel{\mathcal{R}} \pi(\mathcal{T}(w)_{\mathsf{q}.\alpha})$ holds.*

Proof. We prove the statement by induction on $|\alpha|$. When $\alpha = \varepsilon$, the result follows from Theorem 3.12. Let $\alpha \in \Sigma^*$ and invoke the induction hypothesis to assume that $f_\alpha(w)$ and $\pi(\mathcal{T}(w)_{\mathsf{q}.\alpha})$ are $\mathcal{R}$-equivalent.

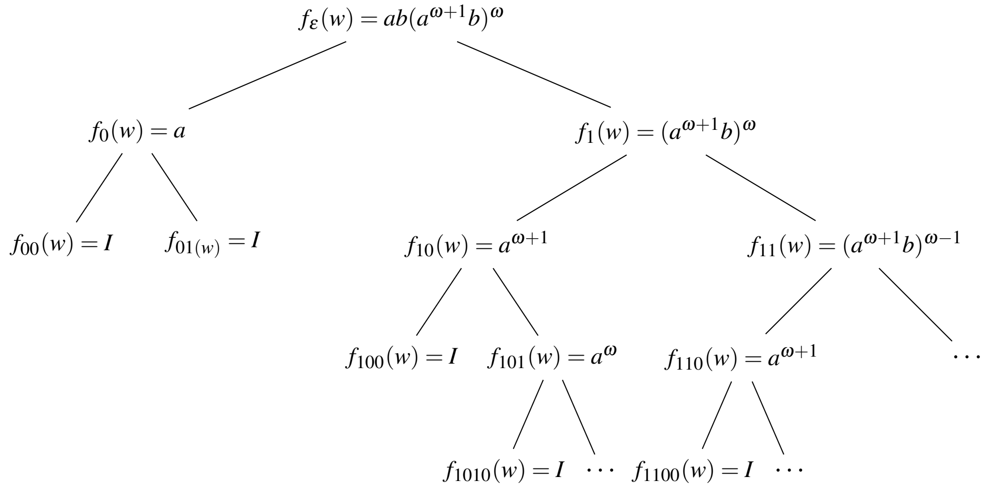

Fig. 3.6 Representation of $\mathcal{F}(ab(a^{\omega+1}b)^\omega)$.

Writing $\mathsf{lbf}(\pi(\mathcal{T}(w)_{\mathsf{q}.\alpha})) = (w_\ell, a, w_r)$, Lemma 3.10 yields the following relations:

$$w_\ell \;\mathcal{R}\; \pi(\mathcal{T}(w)_{\mathsf{q}.\alpha 0}),$$
$$w_r \;\mathcal{R}\; \pi(\mathcal{T}(w)_{\mathsf{q}.\alpha 1}).$$

On the other hand, since $\mathsf{lbf}(f_\alpha(w)) = (f_{\alpha 0}(w), b, f_{\alpha 1}(w))$, using Lemma 2.27 we deduce the equalities $f_{\alpha 0}(w) = w_\ell$ and $a = b$, and the relation $f_{\alpha 1}(w) \;\mathcal{R}\; w_r$, leading to the desired conclusion. □

Hence, in order to prove that $\mathcal{A}(w)$ is finite for every κ-word w, it suffices to prove that so is $\mathcal{F}(w)/\mathcal{R}$. The next two lemmas are useful to achieve that target.

Lemma 3.21. *Let w be a regular κ-word over* DRH. *Then, there exist κ-words x, y and z over* DRH *such that*

(a) $w = xy^{\omega-1}z$;

(b) $c(y) = c(w)$;

(c) $\vec{c}(x) \subsetneq c(w)$;

(d) *y is not regular.*

Proof. By definition of κ-word, we may write $w = w_1 \cdots w_n$, where each w_i is either a letter in A or an $(\omega-1)$-power of another κ-word. Since any letter of the cumulative content of w occurs in $\mathsf{lbf}_\infty(w)$ infinitely many times, there must be an $(\omega-1)$-power under which they all appear. Hence, since w is regular (and so, $c(w) = \vec{c}(w)$), there exists an index $i \in \{1, \ldots, n\}$ such that $w_i = v^{\omega-1}$ and $c(v) = c(w)$. Let j be the minimum such i. We have $w = u_0 v_0^{\omega-1} z_0$, where $u_0 = w_1 \cdots w_{j-1}$, $v_0^{\omega-1} = w_j$, and $z_0 = w_{j+1} \cdots w_n$. Also, minimality of j yields that $\vec{c}(u_0) \subsetneq \vec{c}(w) = c(w)$. So, if v_0 is not regular, then we just take $x = u_0$, $y = v_0$, and $z = z_0$. Suppose that v_0 is regular. Using the same reasoning, we may write $v_0 = u_1 v_1^{\omega-1} z_1$, with $\vec{c}(u_1) \subsetneq c(w)$ and $c(v_1) = c(v_0) = c(w)$. Again, if v_1 is not regular, then we may choose $x = u_0 u_1$, $y = v_1$ and $z = z_1 v_0^{\omega-2} z_0$. Otherwise, we repeat the process with v_1. Since w is a κ-word, there is only a finite number of occurrences of $(\omega-1)$-powers, so that, this iteration cannot run forever. Therefore, we eventually get κ-words x, y and z satisfying the desired properties (a)–(d). $\square$

Lemma 3.22. *Let $w \in \Omega_A^\kappa \mathsf{DRH}$ be regular. For each $m \geq 1$, let w'_m be the unique κ-word over* DRH *satisfying the equality $w = \mathsf{lbf}_1(w) \cdots \mathsf{lbf}_m(w) w'_m$. Then, both sets $\{\mathsf{lbf}_m(w) \colon m \geq 1\}$ and $\{[w'_m]_{\mathcal{R}} \colon m \geq 1\}$ are finite.*

Proof. Write $\mathsf{lbf}_m(w) = w_m a_m$, for every $m \geq 1$, and $w = xy^{\omega-1}z$, with x, y and z satisfying conditions (a)–(d) of Lemma 3.21. We define a sequence of pairs of possibly empty κ-words $\{(u_i, v_i)\}_{i\geq 0}$ and a strictly increasing sequence of non-negative integers $\{k_i\}_{i\geq 0}$ inductively as follows. We start with $(u_0, v_0) = (I, x)$ and we let k_0 be the maximum index such that $\mathsf{lbf}_1(w) \cdots \mathsf{lbf}_{k_0}(w)$ is a prefix of x. If x has no prefix of this form, then we set $k_0 = 0$. We also write $v_0 = v'_0 v''_0$, with $v'_0 = \mathsf{lbf}_1(w) \cdots \mathsf{lbf}_{k_0}(w)$ (by Proposition 2.13, given v'_0 there is only one possible value for v''_0). For each $i \geq 0$, we let u_{i+1} be such that $w_{k_i+1} = v''_i u_{i+1}$ and v_{i+1} is such that $y = u_{i+1} a_{k_i+1} v_{i+1}$. Observe that, by uniqueness of first-occurrences factorizations, there is only one pair (u_{i+1}, v_{i+1}) satisfying these conditions. The integer k_{i+1} is the maximum such that $\mathsf{lbf}_{k_i+2}(w) \cdots \mathsf{lbf}_{k_{i+1}}(w)$ is a prefix of v_{i+1} (or $k_{i+1} = k_i + 1$ if there is no such prefix) and we factorize $v_{i+1} = v'_{i+1} v''_{i+1}$, with $v'_{i+1} = \mathsf{lbf}_{k_i+2}(w) \cdots \mathsf{lbf}_{k_{i+1}}(w)$. By construction, for all $i \geq 0$, the pseudoidentity $w'_{k_i+1} = v_{i+1} y^{\omega-(i+2)} z$ holds. In particular, for every $m \geq 1$, there exist $i \geq 0$ and $\ell \in \{2, \ldots, k_{i+1} - k_i\}$ such that

$$w'_m = \mathsf{lbf}_{k_i+\ell}(w) \mathsf{lbf}_{k_i+\ell+1}(w) \cdots \mathsf{lbf}_{k_{i+1}}(w) v''_{i+1} y^{\omega-(i+2)} z. \tag{3.12}$$

On the other hand, for all $i \geq 0$, the factorization $y = u_{i+1} a_{k_i+1} v_{i+1}$ is such that $a_{k_i+1} \notin c(u_{i+1})$ (recall that $a_{k_i+1} \notin c(w_{k_i+1})$ and u_{i+1} is a factor of w_{k_i+1}). By uniqueness of first-occurrences factorization over DRH, it follows that the set $\{(u_i, v_i)\}_{i\geq 0}$ is finite. Consequently, the set

$$\{\mathsf{lbf}_{k_i+\ell}(w) \mathsf{lbf}_{k_i+\ell+1}(w) \cdots \mathsf{lbf}_{k_{i+1}}(w) v''_{i+1} \colon i \geq 0,\, \ell \in \{2, \ldots, k_{i+1} - k_i\}\}$$

is also finite. In particular, there is only a finite number of κ-words $\mathsf{lbf}_m(w)$. Finally, taking into account that $c(z) \subseteq c(y)$ and (3.12) we may conclude that there are only finitely many $\mathcal{R}$-classes of the form $[w'_m]_{\mathcal{R}}$ $(m \geq 1)$. $\square$

Now, we are able to prove that $\mathcal{F}(w)/\mathcal{R}$ is finite for every κ-word w over DRH.

Proposition 3.23. *Let w be a possibly empty κ-word over* DRH. *Then, the quotient $\mathcal{F}(w)/\mathcal{R}$ is finite.*

Proof. We prove the result by induction on $|c(w)|$. If $|c(w)| = 0$, then it is trivial. Suppose that $|c(w)| \geq 1$. We distinguish two possible scenarios.

Case 1. The κ-word w is not regular, that is, $\vec{c}(w) \subsetneqq c(w)$.

Then, there exists $k \geq 1$ such that $w = w_1 a_1 \cdots w_m a_m w'_m$, with $\mathsf{lbf}_k(w) = w_k a_k$, for $k = 1, \ldots, m$ and $c(w'_m) \subsetneqq c(w)$. By definition of $f_\alpha(w)$, we have the following identities:

$$\begin{aligned} f_{1^{k-1}0}(w) &= w_k, \text{ for } k = 1, \ldots, m; \\ f_{1^m}(w) &= w'_m. \end{aligned}$$

Hence, we may deduce that $\mathcal{F}(w)$ is the union of the sets $\mathcal{F}(w_k)$ (for $k = 1, \ldots, m$) together with $\mathcal{F}(w'_m)$. Using the induction hypothesis on each one of the intervening sets, we conclude that $\mathcal{F}(w)/\mathcal{R}$ is finite.

Case 2. The κ-word w is regular.

Again, write $\mathsf{lbf}_k(w) = w_k a_k$ and $w = \mathsf{lbf}_1(w) \cdots \mathsf{lbf}_k(w) w'_k$, for $k \geq 1$. Since $f_{1^{k-1}0}(w) = w_k$ and $f_{1^k}(w) = w'_k$, for every $k \geq 1$, by Lemma 3.22, we know that the sets $\{f_{1^{k-1}0}(w)\}_{k\geq 1}$ and $\{[f_{1^k}(w)]_{\mathcal{R}}\}_{k\geq 1}$ are both finite. Applying the induction hypothesis to each factor w_k, we derive that $\{[f_{1^{k-1}0\alpha}(w)]_{\mathcal{R}} \colon \alpha \in \Sigma^*\}_{k\geq 1}$ is also a finite set. Therefore, since any element of $\mathcal{F}(w)/\mathcal{R}$ is of one of the forms $[f_{1^{k-1}0\alpha}(w)]_{\mathcal{R}}$ and $[f_{1^k}(w)]_{\mathcal{R}}$, we conclude that $\mathcal{F}(w)/\mathcal{R}$ is finite as well. □

As an immediate consequence (recall Lemma 3.20), we obtain:

Corollary 3.24. *Let w be a possibly empty κ-word. Then, the wrapped* DRH*-automaton $\mathcal{A}(w)$ is finite.* □

Unlike the aperiodic case R, the converse of Corollary 3.24 does not hold in general. For instance, taking $\mathsf{H} = \mathsf{G}$, it is not hard to see that $\mathcal{A}(a^{p^\omega} b)$ (with p a prime number) is finite, although $a^{p^\omega} b$ is not a κ-word over DRG. A converse is achieved when we further require that the labels λ_{H} are valued by κ-words over H and that $\rho_{\mathsf{H}}(\mathsf{reg}(w))$ is itself a κ-word.

For a given $w \in (\Omega_A^\kappa \mathsf{DRH})^I$, we say that $\mathsf{cf}(w) = \pi_{\mathsf{cf}}(\mathcal{A}(w))\mathsf{cf}_{\mathsf{H}}(\rho_{\mathsf{H}}(\mathsf{reg}(w)))$ is the *canonical form of w*. We write $\mathsf{cf}(u) \equiv \mathsf{cf}(v)$ (with $u, v \in (\Omega_A^\kappa \mathsf{DRH})^I$) when both sides coincide. We have just proved that $\mathsf{cf}(\_)$ is well-defined for κ-words and thus, it determines a canonical form for the elements of $\Omega_A^\kappa \mathsf{DRH}$.

Theorem 3.25. *Let* H *be a pseudovariety of groups such that there exists a computable canonical form for the elements of* $\Omega_A^\kappa \mathsf{H}$*, say* $\mathsf{cf}_{\mathsf{H}}(\_)$*. Then, for all κ-words u and v over* DRH*, the equality $u = v$ holds if and only if* $\mathsf{cf}(u) \equiv \mathsf{cf}(v)$. □

3.3 $\langle\kappa\rangle$-terms seen as well-parenthesized words

In Section 3.1, we characterized $\mathcal{R}$-classes over DRH by means of certain equivalence classes of automata. In order to solve the κ-word problem over DRH, the next goal is to find an algorithm to construct such automata. This section serves the purpose of preparing that construction.

3.3.1 General definitions

Let B be a possibly infinite alphabet and consider the associated alphabet $B_{[]} = B \uplus \{[^q,]^q \colon q \in \mathbb{Z}\}$. We say that a word in $B_{[]}^*$ is *well-parenthesized over B* if it does not contain $[^q\,]^q$ as a factor and if it can be reduced to the empty word ε by applying the rewriting rules $[^q\,]^q \to \varepsilon$ and $a \to \varepsilon$, for $q \in \mathbb{Z}$ and $a \in B$. We denote the set of all well-parenthesized words over B by $\mathsf{Dyck}(B)$. The *content* of a well-parenthesized word x is the set of letters in B that occur in x and it is denoted $c(x)$.

To each $\langle\kappa\rangle$-term we may associate a well-parenthesized word over A inductively as follows:

$$\begin{aligned}\mathsf{word}(I) &= \varepsilon\\ \mathsf{word}(a) &= a, \quad \text{if } a \in A;\\ \mathsf{word}(u\cdot v) &= \mathsf{word}(u)\mathsf{word}(v), \quad \text{if } u \text{ and } v \text{ are } \langle\kappa\rangle\text{-terms;}\\ \mathsf{word}(u^{\omega+q}) &= [^q\mathsf{word}(u)]^q, \quad \text{if } u \text{ is a } \langle\kappa\rangle\text{-terms.}\end{aligned}$$

Conversely, we associate a κ-word to each well-parenthesized word over A as follows:

$$\begin{aligned}\mathsf{om}(\varepsilon) &= I\\ \mathsf{om}(a) &= a, \quad \text{if } a \in A;\\ \mathsf{om}(xy) &= \mathsf{om}(x)\cdot\mathsf{om}(y), \quad \text{if } x,y \in \mathsf{Dyck}(A);\\ \mathsf{om}([^qx]^q) &= \mathsf{om}(x)^{\omega+q}, \quad \text{if } x \in \mathsf{Dyck}(A).\end{aligned}$$

Note that, due to the associative property in both $\mathsf{Dyck}(A)$ and $\Omega_A^\kappa\mathsf{S}$, $\mathsf{om}(\_)$ is well-defined. With the aim of distinguishing the occurrences of each letter in A in a well-parenthesized word x over A, we assign to each $x \in \mathsf{Dyck}(A)$ a well-parenthesized word $x_{\mathbb{N}}$ over $A\times\mathbb{N}$ containing all the information about the position of the letters. With that in mind we define recursively the following family of functions $\{p_k : \mathsf{Dyck}(A) \to \mathsf{Dyck}(A\times\mathbb{N})\}_{k\geq 0}$:

$$\begin{aligned}p_k(a) &= (a,k+1), \quad \text{if } a \in A;\\ p_k([^q) &= [^q, \quad \text{if } q \in \mathbb{Z};\\ p_k(]^q) &=]^q, \quad \text{if } q \in \mathbb{Z};\\ p_k(ay) &= p_k(a)p_{k+1}(y), \quad \text{if } a \in A_{[]} \text{ and } y \in A_{[]}^*.\end{aligned}$$

We set $x_{\mathbb{N}} = p_0(x)$. For instance, if $x = a[^qb[^rca]^r]^qb$, then $x_{\mathbb{N}} = (a,1)[^q(b,2)[^r(c,3)(a,4)]^r]^q(b,5)$. It is often convenient to denote the pair (a,i) by a_i. Let $x \in \mathsf{Dyck}(A\times\mathbb{N})$. Then, we may associate to x two well-parenthesized words $\pi_A(x)$ and $\pi_{\mathbb{N}}(x)$ corresponding to the projection of x onto $A_{[]}^*$ and onto $\mathbb{N}_{[]}^*$, respectively. We denote $c_A(x) = c(\pi_A(x))$ and $c_{\mathbb{N}}(x) = c(\pi_{\mathbb{N}}(x))$. Given a $\langle\kappa\rangle$-term w, we denote by $\overline{w}$ the well-parenthesized word $0_0\mathsf{word}(w\#)_{\mathbb{N}}$ over the alphabet $(A\uplus\{0,\#\})\times\mathbb{N}$. The map $\eta : \mathsf{Dyck}(A\times\mathbb{N}) \to \Omega_A^\kappa\mathsf{S}$ assigns to each well-parenthesized word $x \in \mathsf{Dyck}(A\times\mathbb{N})$ the κ-word $\eta(x) = \mathsf{om}(\pi_A(x))$.

Example 3.26. Consider the $\langle\kappa\rangle$-term $w = (b^{\omega-1}\cdot(a\cdot c))\cdot((a\cdot b)\cdot(a^{\omega+1}\cdot b)^{\omega})^{\omega}$. Then, we have

$$\begin{aligned}\mathsf{word}(w) &= \mathsf{word}(b^{\omega-1}\cdot(a\cdot c))\mathsf{word}(((a\cdot b)\cdot(a^{\omega+1}\cdot b)^{\omega})^{\omega})\\ &= \mathsf{word}(b^{\omega-1})\mathsf{word}(a\cdot c)[^0\mathsf{word}((a\cdot b)\cdot(a^{\omega+1}\cdot b)^{\omega})]^0.\\ &= [^{-1}b]^{-1}ac[^0ab[^0[^1a]^1b]^0]^0.\end{aligned}$$

Conversely, if $x = a[^qb[^rca]^r]^qb$ as above, then we may compute

$$\mathsf{om}(x) = \mathsf{om}(a)\mathsf{om}([^qb[^rca]^r]^q)\mathsf{om}(b) = a(\mathsf{om}(b[^rca]^r))^{\omega+q}b$$

$$= a(\mathsf{om}(b)\mathsf{om}([^r ca]^r))^{\omega+q}b = a(b(ca)^{\omega+r})^{\omega+q}b.$$

Let x be a well-parenthesized word over $A\times\mathbb{N}$. We define its *tail* $\mathsf{t}_i(x)$ from position $i\in\mathbb{N}$ inductively as follows

$$\begin{aligned}
\mathsf{t}_i(\varepsilon) &= \varepsilon;\\
\mathsf{t}_i(yz) &= \mathsf{t}_i(z), \quad \text{if } y,z\in \mathsf{Dyck}(A\times\mathbb{N}) \text{ and } i\notin c_{\mathbb{N}}(y);\\
\mathsf{t}_i(a_i y) &= y, \quad \text{if } y\in \mathsf{Dyck}(A\times\mathbb{N});\\
\mathsf{t}_i([^q y]^q z) &= \mathsf{t}_i(y)[^{q-1} y]^{q-1} z, \quad \text{if } y,z\in \mathsf{Dyck}(A\times\mathbb{N}) \text{ and } i\in c_{\mathbb{N}}(y).
\end{aligned}$$

The *prefix* of $x\in\mathsf{Dyck}(A\times\mathbb{N})$ until $a\in A$ is defined by

$$\begin{aligned}
\mathsf{p}_a(\varepsilon) &= \varepsilon;\\
\mathsf{p}_a(yz) &= y\,\mathsf{p}_a(z), \quad \text{if } y,z\in \mathsf{Dyck}(A\times\mathbb{N}) \text{ and } a\notin c_A(y);\\
\mathsf{p}_a(a_i y) &= \varepsilon, \quad \text{if } y\in \mathsf{Dyck}(A\times\mathbb{N});\\
\mathsf{p}_a([^q y]^q z) &= \mathsf{p}_a(y), \quad \text{if } y,z\in \mathsf{Dyck}(A\times\mathbb{N}) \text{ and } a\in c_A(y).
\end{aligned}$$

The *factor* of a well-parenthesized word $x\in\mathsf{Dyck}(A\times\mathbb{N})$ *from* $i\in\mathbb{N}$ *until* $a\in A$ is given by

$$x(i,a) = \mathsf{p}_a(\mathsf{t}_i(x)).$$

If instead, we are given a $\langle\kappa\rangle$-term w, then we write $w(i,a)$ to mean the κ-word $\eta(\overline{w}(i,a))$. If a is a letter occurring in $\pi_A(x)$, for a well-parenthesized word x over $A\times\mathbb{N}$, then it is possible to write $x = ya_iz$ with y and z possibly empty not necessarily well-parenthesized words over $A\times\mathbb{N}$ such that $a\notin c_A(y)$. In this case we say that a_i is a *marker of* x. If a_i is the last first occurrence of a letter, that is, if the inclusion $c_A(z)\subseteq c_A(ya_i)$ holds, then we say that a_i is the *principal marker of* x.

Example 3.27. Set again $w = (((b^{\omega-1})\cdot(a\cdot c))\cdot(((a\cdot b)\cdot(((a^{\omega+1})\cdot b)^{\omega}))^{\omega}))$. From Example 3.26 we may easily conclude that

$$\overline{w} = 0_0[^{-1}b_1]^{-1}a_2c_3[^0a_4b_5[^0[^1a_6]^1b_7]^0]^0\#_8.$$

Then, 0_0, b_1, a_2, c_3 and $\#_8$ are the markers of $\overline{w}$, since they are the first occurrences of each letter in $\overline{w}$. The last first occurrence $\#_8$ is the principal marker of $\overline{w}$. Let us compute $\overline{w}(0,\#)$. Following the definitions, we have

$$\begin{aligned}
\overline{w}(0,\#) &= \mathsf{p}_\#(\mathsf{t}_0(0_0[^{-1}b_1]^{-1}a_2c_3[^0a_4b_5[^0[^1a_6]^1b_7]^0]^0\#_8))\\
&= \mathsf{p}_\#([^{-1}b_1]^{-1}a_2c_3[^0a_4b_5[^0[^1a_6]^1b_7]^0]^0\#_8)\\
&= [^{-1}b_1]^{-1}a_2c_3[^0a_4b_5[^0[^1a_6]^1b_7]^0]^0.
\end{aligned}$$

Hence, $w(0,\#)$ is the κ-word represented by w and the principal marker of $\overline{w}(0,\#)$ is c_3. For a less trivial example, we compute $\overline{w}(5,b)$.

$$\overline{w}(5,b) = \mathsf{p}_b(\mathsf{t}_5(0_0[^{-1}b_1]^{-1}a_2c_3[^0a_4b_5[^0[^1a_6]^1b_7]^0]^0\#_8))$$

$$
\begin{aligned}
&= \mathsf{p}_b(\,\mathsf{t}_5([^0 a_4 b_5 [^0 [^1 a_6]^1 b_7]^0]^0 \#_8)) \\
&= \mathsf{p}_b(\,\mathsf{t}_5(a_4 b_5 [^0 [^1 a_6]^1 b_7]^0)[^{-1} a_4 b_5 [^0 [^1 a_6]^1 b_7]^0]^{-1} \#_8) \\
&= \mathsf{p}_b([^0 [^1 a_6]^1 b_7]^0 [^{-1} a_4 b_5 [^0 [^1 a_6]^1 b_7]^0]^{-1} \#_8) \\
&= \mathsf{p}_b([^1 a_6]^1 b_7) \\
&= [^1 a_6]^1.
\end{aligned}
$$

Thus, we get $w(5,b) = a^{\omega+1}$.

3.3.2 Properties of tails and prefixes of well-parenthesized words

The next results state some properties concerning tails and prefixes of well-parenthesized words.

Lemma 3.28 (cf. [25, Lemma 5.3])**.** *Let $x \in \mathsf{Dyck}(A \times \mathbb{N})$ and let $a, b \in A$. Then*

$$b \in c_A(\,\mathsf{p}_a(x)) \implies \mathsf{p}_b(\,\mathsf{p}_a(x)) = \mathsf{p}_b(x).$$

Proof. We argue by induction on $|x|$. If $|x| = 0$, then the claim holds trivially. Let us suppose that $|x| \geq 1$ and $b \in c_A(\,\mathsf{p}_a(x))$. We consider the following different situations.

- If $x = yz$, with $y, z \in \mathsf{Dyck}(A \times \mathbb{N})$ and $a, b \notin c_A(y)$, then

  $$\mathsf{p}_b(\,\mathsf{p}_a(x)) = \mathsf{p}_b(y\,\mathsf{p}_a(z)) = y\,\mathsf{p}_b(\,\mathsf{p}_a(z)).$$

  Since $b \in c_A(\,\mathsf{p}_a(z))$ and z is a well-parenthesized word, it follows, by induction hypothesis, that

  $$\mathsf{p}_b(\,\mathsf{p}_a(x)) = y\,\mathsf{p}_b(\,\mathsf{p}_a(z)) = y\,\mathsf{p}_b(z) = \mathsf{p}_b(yz) = \mathsf{p}_b(x).$$

- Suppose that the first letter of $\pi_A(x)$ is either a or b. If it is a, then $x = a_i y$, which implies that $\mathsf{p}_a(x) = \varepsilon$ and so $b \notin c_A(\,\mathsf{p}_a(x))$. If it is $b \neq a$, then we may write $x = b_i y$. Thus, we have

  $$\mathsf{p}_b(\,\mathsf{p}_a(x)) = \mathsf{p}_b(b_i\,\mathsf{p}_a(y)) = \varepsilon = \mathsf{p}_b(x).$$

- Finally, suppose that $x = [^q y]^q z$, with $y, z \in \mathsf{Dyck}(A \times \mathbb{N})$ and $y \neq \varepsilon$. The situation that remains to be considered occurs when at least one of a and b belongs to $c_A(y)$. If $a \in c_A(y)$, then the equality $\mathsf{p}_a(x) = \mathsf{p}_a(y)$ holds and so, $b \in c_A(\,\mathsf{p}_a(y)) \subseteq c_A(y)$. Applying the induction hypothesis to y, we get

  $$\mathsf{p}_b(\,\mathsf{p}_a(x)) = \mathsf{p}_b(\,\mathsf{p}_a(y)) = \mathsf{p}_b(y) = \mathsf{p}_b(x).$$

  On the other hand, if $a \notin c_A(y)$ and $b \in c_A(y)$, then the equalities

  $$\mathsf{p}_b(\,\mathsf{p}_a(x)) = \mathsf{p}_b([^q y]^q\,\mathsf{p}_a(z)) = \mathsf{p}_b(y) = \mathsf{p}_b(x)$$

  hold. □

Lemma 3.29 (cf. [25, Lemma 5.4])**.** *Let $x \in \mathsf{Dyck}(A \times \mathbb{N})$ be such that $a \in c_A(x)$. If $k \in c_{\mathbb{N}}(\,\mathsf{p}_a(x))$, then $a \in c_A(\,\mathsf{t}_k(x))$.*

Proof. We argue by induction on $|x|$. If $|x| = 0$, then there is nothing to prove. Suppose that $|x| \geq 1$. We distinguish the three following cases.

- If $x = a_i y$, then $\mathsf{p}_a(x) = \varepsilon$ and the result is trivial again.
- If $x = yz$, with $a \notin c_A(y)$, $y, z \in \mathsf{Dyck}(A \times \mathbb{N})$ and $|y| \geq 1$, then $a \in c_A(z)$ and thus, we have $\mathsf{p}_a(x) = y\,\mathsf{p}_a(z)$. On the other hand, $\mathsf{t}_k(x)$ is either $\mathsf{t}_k(y)z$ or $\mathsf{t}_k(z)$ according to whether $k \in c_{\mathbb{N}}(y)$ or $k \notin c_{\mathbb{N}}(y)$, respectively. If the former situation happens, then the result is clear. Otherwise, we have $k \in c_{\mathbb{N}}(\mathsf{p}_a(z))$ and, since $|z| < |x|$, we may apply the induction hypothesis to z to obtain $a \in c_A(\mathsf{t}_k(z)) = c_A(\mathsf{t}_k(x))$.
- Finally, it remains to consider the case where $x = [^q y]^q$ with $a \in c_A(y)$. In this situation, we have $\mathsf{p}_a(x) = \mathsf{p}_a(y)$. It follows that $k \in c_{\mathbb{N}}(y)$ and therefore, $\mathsf{t}_k(x) = \mathsf{t}_k(y)[^{q-1}y]^{q-1}$. Thus, we get $a \in c_A(y) \subseteq c_A(\mathsf{t}_k(x))$. □

Lemma 3.30 (cf. [25, Lemma 5.5]). *Let $x \in \mathsf{Dyck}(A \times \mathbb{N})$, $a \in A$, and $k \in \mathbb{N}$. Then, we have*

$$k \in c_{\mathbb{N}}(\mathsf{p}_a(x)) \implies \mathsf{t}_k(\mathsf{p}_a(x)) = \mathsf{p}_a(\mathsf{t}_k(x)). \tag{3.13}$$

Proof. If $a \notin c_A(x)$, then the result holds, since $\mathsf{t}_k(\mathsf{p}_a(x)) = \mathsf{t}_k(x) = \mathsf{p}_a(\mathsf{t}_k(x))$. We suppose that $a \in c_A(x)$ and argue by induction on $|x|$. If $x = a_i$, then $\mathsf{p}_a(x) = \varepsilon$ and so, $k \notin c_{\mathbb{N}}(\mathsf{p}_a(x))$. Suppose that $|x| > 1$ and $k \in c_{\mathbb{N}}(\mathsf{p}_a(x))$. Let $x = yz$ be the product of two nonempty well-parenthesized words. If $a \notin c_A(y)$, then $\mathsf{p}_a(x) = y\,\mathsf{p}_a(z)$ and, consequently, k belongs to at least one of the sets $c_{\mathbb{N}}(y)$ and $c_{\mathbb{N}}(\mathsf{p}_a(z)) \subseteq c_{\mathbb{N}}(z)$. It follows that

$$\begin{aligned}
\mathsf{t}_k(\mathsf{p}_a(x)) = \mathsf{t}_k(y \cdot \mathsf{p}_a(z)) &= \begin{cases} \mathsf{t}_k(y) \cdot \mathsf{p}_a(z), & \text{if } k \in c_{\mathbb{N}}(y); \\ \mathsf{t}_k(\mathsf{p}_a(z)), & \text{otherwise;} \end{cases} \\
&= \begin{cases} \mathsf{p}_a(\mathsf{t}_k(y) \cdot z), & \text{if } k \in c_{\mathbb{N}}(y); \quad (\text{since } a \notin c_A(y) \supseteq c_A(\mathsf{t}_k(y))) \\ \mathsf{p}_a(\mathsf{t}_k(z)), & \text{otherwise;} \quad (\text{applying the induction hypothesis to } z) \end{cases} \\
&= \begin{cases} \mathsf{p}_a(\mathsf{t}_k(yz)), & \text{if } k \in c_{\mathbb{N}}(y); \\ \mathsf{p}_a(\mathsf{t}_k(yz)), & \text{otherwise;} \end{cases} \\
&= \mathsf{p}_a(\mathsf{t}_k(x)).
\end{aligned}$$

On the other hand, if $a \in c_A(y)$, then we have $\mathsf{p}_a(x) = \mathsf{p}_a(y)$ and so, $\mathsf{t}_k(\mathsf{p}_a(x)) = \mathsf{t}_k(\mathsf{p}_a(y))$. Since k belongs to the set $c_{\mathbb{N}}(\mathsf{p}_a(x)) = c_{\mathbb{N}}(\mathsf{p}_a(y))$, applying the induction hypothesis to y, we get

$$\mathsf{t}_k(\mathsf{p}_a(x)) = \mathsf{t}_k(\mathsf{p}_a(y)) = \mathsf{p}_a(\mathsf{t}_k(y)).$$

Moreover, as $k \in c_{\mathbb{N}}(\mathsf{p}_a(y))$, by Lemma 3.29, we have $a \in c_A(\mathsf{t}_k(y))$. Therefore, it follows that $\mathsf{p}_a(\mathsf{t}_k(y)) = \mathsf{p}_a(\mathsf{t}_k(y) \cdot z) = \mathsf{p}_a(\mathsf{t}_k(yz)) = \mathsf{p}_a(\mathsf{t}_k(x))$. It remains to consider the case where x is of the form $[^q y]^q$ for a well-parenthesized word y such that $a \in c_A(y)$. As $k \in c_{\mathbb{N}}(\mathsf{p}_a(x)) = c_{\mathbb{N}}(\mathsf{p}_a(y))$, we may apply the induction hypothesis to y to obtain that $\mathsf{t}_k(\mathsf{p}_a(x)) = \mathsf{t}_k(\mathsf{p}_a(y)) = \mathsf{p}_a(\mathsf{t}_k(y))$. Also, by

Lemma 3.29, we have $a \in c_A(\mathsf{t}_k(x))$. Hence, the equalities

$$\mathsf{p}_a(\mathsf{t}_k(x)) = \mathsf{p}_a(\mathsf{t}_k(y) \cdot [^{q-1}y]^{q-1}) = \mathsf{p}_a(\mathsf{t}_k(y))$$

are valid. So, we conclude the desired equality: $\mathsf{t}_k(\mathsf{p}_a(x)) = \mathsf{p}_a(\mathsf{t}_k(x))$. □

Lemma 3.31. *Let $\vec{x} = (x_j)_{j\geq 0}$ and $\vec{y} = (y_j)_{j\geq 0}$ be two sequences of possibly empty well-parenthesized words over $A \times \mathbb{N}$ such that $x_0y_0 \neq \varepsilon$, and for every $i, j \geq 0$, the index i occurs in $\pi_{\mathbb{N}}(x_0y_0x_1y_1\cdots x_jy_j)$ at most once. Let $\vec{q} = (q_j)_{j\geq 0}$ be a sequence of integers. For each $n \geq 0$, we define the well-parenthesized words $\mu_n(\vec{x},\vec{y},\vec{q})$ and $\xi_n(\vec{x},\vec{y},\vec{q})$ as follows:*

$$\begin{aligned} \mu_0(\vec{x},\vec{y},\vec{q}) &= x_0y_0 \\ \mu_{n+1}(\vec{x},\vec{y},\vec{q}) &= x_{n+1}[^{q_n}\mu_n(\vec{x},\vec{y},\vec{q})]^{q_n}y_{n+1}, \text{ if } n \geq 0 \\ \xi_n(\vec{x},\vec{y},\vec{q}) &= [^{q_n-1}\mu_n(\vec{x},\vec{y},\vec{q})]^{q_n-1}y_{n+1}, \text{ if } n \geq 0. \end{aligned}$$

Let i be a natural number and suppose that $i \in c_{\mathbb{N}}(x_\ell y_\ell)$ for a certain $\ell \geq 0$. Then, for every $n \geq \ell$, the following equality holds:

$$\mathsf{t}_i(\mu_n(\vec{x},\vec{y},\vec{q})) = \mathsf{t}_i(\mu_\ell(\vec{x},\vec{y},\vec{q})) \cdot \xi_\ell(\vec{x},\vec{y},\vec{q}) \cdot \xi_{\ell+1}(\vec{x},\vec{y},\vec{q}) \cdots \xi_{n-1}(\vec{x},\vec{y},\vec{q}). \tag{3.14}$$

Proof. We argue by induction on n. If $n = \ell$, then the result holds clearly, since the factor $\xi_\ell(\vec{x},\vec{y},\vec{q}) \cdot \xi_{\ell+1}(\vec{x},\vec{y},\vec{q}) \cdots \xi_{n-1}(\vec{x},\vec{y},\vec{q})$ vanishes in (3.14). Suppose that $n > \ell$ and that the result holds for any smaller n. We may compute

$$\begin{aligned} \mathsf{t}_i(\mu_n(\vec{x},\vec{y},\vec{q})) &= \mathsf{t}_i(x_n[^{q_{n-1}}\mu_{n-1}(\vec{x},\vec{y},\vec{q})]^{q_{n-1}}y_n) \\ &= \mathsf{t}_i(\mu_{n-1}(\vec{x},\vec{y},\vec{q})) \cdot [^{q_{n-1}-1}\mu_{n-1}(\vec{x},\vec{y},\vec{q})]^{q_{n-1}-1}y_n \quad \text{since } i \notin c_{\mathbb{N}}(x_n) \\ &\qquad \text{and } i \in c_{\mathbb{N}}(\mu_{n-1}(\vec{x},\vec{y},\vec{q})) \\ &= \mathsf{t}_i(\mu_{n-1}(\vec{x},\vec{y},\vec{q})) \cdot \xi_{n-1}(\vec{x},\vec{y},\vec{q}) \\ &= \mathsf{t}_i(\mu_\ell(\vec{x},\vec{y},\vec{q})) \cdot \xi_\ell(\vec{x},\vec{y},\vec{q}) \cdots \xi_{n-2}(\vec{x},\vec{y},\vec{q}) \cdot \xi_{n-1}(\vec{x},\vec{y},\vec{q}) \quad \text{by induction hypothesis} \end{aligned}$$

obtaining the desired equality (3.14). □

By successively applying Lemma 3.31, we obtain the next two results.

Corollary 3.32. *Using the same notation and assuming the same hypothesis as in the previous lemma, suppose that $k \in c_{\mathbb{N}}(y_0)$ and that $i \in c_{\mathbb{N}}(x_\ell)$ for a certain $\ell \geq 0$. Then, for every $n \geq \ell$ the following equality holds:*

$$\mathsf{t}_k(\mathsf{t}_i(\mu_n(\vec{x},\vec{y},\vec{q}))) = \mathsf{t}_k(y_0) \cdot \xi_0(\vec{x},\vec{y},\vec{q}) \cdot \xi_1(\vec{x},\vec{y},\vec{q}) \cdots \xi_{n-2}(\vec{x},\vec{y},\vec{q}) \cdot \xi_{n-1}(\vec{x},\vec{y},\vec{q}).$$

Proof. From Lemma 3.31 it follows that

$$\mathsf{t}_i(\mu_n(\vec{x},\vec{y},\vec{q})) = \mathsf{t}_i(\mu_\ell(\vec{x},\vec{y},\vec{q})) \cdot \xi_\ell(\vec{x},\vec{y},\vec{q}) \cdot \xi_{\ell+1}(\vec{x},\vec{y},\vec{q}) \cdots \xi_{n-1}(\vec{x},\vec{y},\vec{q}).$$

Since $i \in c_{\mathbb{N}}(x_\ell)$, we may compute $\mathtt{t}_i(\mu_\ell(\vec{x},\vec{y},\vec{q}))$ as follows

$$\mathtt{t}_i(\mu_\ell(\vec{x},\vec{y},\vec{q})) = \mathtt{t}_i(x_\ell[^{q_{\ell-1}}\mu_{\ell-1}(\vec{x},\vec{y},\vec{q})]^{q_{\ell-1}}y_\ell) = \mathtt{t}_i(x_\ell)[^{q_{\ell-1}}\mu_{\ell-1}(\vec{x},\vec{y},\vec{q})]^{q_{\ell-1}}y_\ell.$$

As $k \in c_{\mathbb{N}}(y_0)$, using again Lemma 3.31, we obtain

$$\begin{aligned}\mathtt{t}_k(\mathtt{t}_i(\mu_n(\vec{x},\vec{y},\vec{q}))) &= \mathtt{t}_k(\mathtt{t}_i(x_\ell)[^{q_{\ell-1}}\mu_{\ell-1}(\vec{x},\vec{y},\vec{q})]^{q_{\ell-1}}y_\ell\cdot\xi_\ell(\vec{x},\vec{y},\vec{q})\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q}))\\ &= \mathtt{t}_k(\mu_{\ell-1}(\vec{x},\vec{y},\vec{q}))\\ &\quad\cdot[^{q_{\ell-1}-1}\mu_{\ell-1}(\vec{x},\vec{y},\vec{q})]^{q_{\ell-1}-1}y_\ell\cdot\xi_\ell(\vec{x},\vec{y},\vec{q})\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})\\ &= \mathtt{t}_k(\mu_0(\vec{x},\vec{y},\vec{q}))\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{\ell-2}(\vec{x},\vec{y},\vec{z})\\ &\quad\cdot[^{q_{\ell-1}-1}\mu_{\ell-1}(\vec{x},\vec{y},\vec{q})]^{q_{\ell-1}-1}y_\ell\cdot\xi_\ell(\vec{x},\vec{y},\vec{q})\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})\\ &= \mathtt{t}_k(y_0)\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q}),\end{aligned}$$

as we required. □

Corollary 3.33. *Using again the notation and assuming the hypothesis of Lemma 3.31, suppose that $k \in c_{\mathbb{N}}(y_0)$ and that $i \in c_{\mathbb{N}}(y_\ell)$ for a certain $\ell \geq 0$. Then, for every $n \geq \ell$, if either $\ell = 0$ and $k \notin c_{\mathbb{N}}(\mathtt{t}_i(y_0))$ or $\ell \geq 1$, the following equality holds:*

$$\begin{aligned}\mathtt{t}_k(\mathtt{t}_i(\mu_n(\vec{x},\vec{y},\vec{q}))) &= \mathtt{t}_k(y_0)\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{\ell-1}(\vec{x},\vec{y},\vec{q})\\ &\quad\cdot[^{q_\ell-2}\mu_\ell(\vec{x},\vec{y},\vec{q})]^{q_\ell-2}y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q}).\end{aligned}$$

Proof. Lemma 3.31 yields that

$$\mathtt{t}_i(\mu_n(\vec{x},\vec{y},\vec{q})) = \mathtt{t}_i(\mu_\ell(\vec{x},\vec{y},\vec{q}))\cdot\xi_\ell(\vec{x},\vec{y},\vec{q})\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q}).$$

Computing $\mathtt{t}_i(\mu_\ell(\vec{x},\vec{y},\vec{q}))$, we obtain

$$\mathtt{t}_i(\mu_\ell(\vec{x},\vec{y},\vec{q})) = \mathtt{t}_i(x_\ell[^{q_{\ell-1}}\mu_{\ell-1}(\vec{x},\vec{y},\vec{q})]^{q_{\ell-1}}y_\ell) = \mathtt{t}_i(y_\ell).$$

Therefore, we have

$$\begin{aligned}\mathtt{t}_k(\mathtt{t}_i(\mu_n(\vec{x},\vec{y},\vec{q}))) &= \mathtt{t}_k(\mathtt{t}_i(y_\ell)\cdot\xi_\ell(\vec{x},\vec{y},\vec{q})\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q}))\\ &= \mathtt{t}_k([^{q_\ell-1}\mu_\ell(\vec{x},\vec{y},\vec{q})]^{q_\ell-1}y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q}))\\ &= \mathtt{t}_k(\mu_\ell(\vec{x},\vec{y},\vec{q}))\cdot[^{q_\ell-2}\mu_\ell(\vec{x},\vec{y},\vec{q})]^{q_\ell-2}y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q}).\end{aligned}\tag{3.15}$$

Finally, since $k \in c_{\mathbb{N}}(y_0)$, by Lemma 3.31 we have

$$\begin{aligned}\mathtt{t}_k(\mu_\ell(\vec{x},\vec{y},\vec{q})) &= \mathtt{t}_k(\mu_0(\vec{x},\vec{y},\vec{q}))\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{\ell-1}(\vec{x},\vec{y},\vec{q})\\ &= \mathtt{t}_k(y_0)\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{\ell-1}(\vec{x},\vec{y},\vec{q}),\end{aligned}$$

which in turn, substituting in (3.15), yields the desired equality. □

The reader may wish to compare the next result with [25, Lemma 5.8].

Lemma 3.34. *Let w be a $\langle\kappa\rangle$-term, $i \geq 0$, and $a \in c(w) \uplus \{\#\}$. Assume that b_k is the principal marker of $\overline{w}(i,a)$. Then, the following properties hold:*

(*a*) $\mathsf{p}_b(\overline{w}(i,a)) = \overline{w}(i,b)$;

(*b*) DRH *satisfies* $\eta(\mathsf{t}_k(\overline{w}(i,a)))\ \mathcal{R}\ w(k,a)$.

Moreover, if the projection of $w(i,a)$ onto $\overline{\Omega}_A$DRH is not regular, then the relation in (b) becomes an equality in $\overline{\Omega}_A$S.

Proof. By definition, we have $\overline{w}(i,a) = \mathsf{p}_a(\mathsf{t}_i(\overline{w}))$. Since $b \in c_A(\overline{w}(i,a))$, it follows from Lemma 3.28 that $\mathsf{p}_b(\overline{w}(i,a)) = \mathsf{p}_b(\mathsf{p}_a(\mathsf{t}_i(\overline{w}))) = \mathsf{p}_b(\mathsf{t}_i(\overline{w})) = \overline{w}(i,b)$.

Let us prove the second assertion. By definition of $\overline{w}$, we know that b_k appears exactly once in $\overline{w}$ and the same happens with the index i. Let $\overline{w} = x \cdot b_k \cdot y$. We distinguish the cases where x and y are both well-parenthesized words and where neither of x nor y is a well-parenthesized word. In the first case, since $b_k \in c(\overline{w}(i,a)) \subseteq c(\mathsf{t}_i(\overline{w}))$, the index i must belong to $c_{\mathbb{N}}(x)$. So, we get

$$\mathsf{t}_k(\overline{w}(i,a)) = \mathsf{t}_k(\mathsf{p}_a(\mathsf{t}_i(\overline{w}))) = \mathsf{t}_k(\mathsf{p}_a(\mathsf{t}_i(x)b_k y)).$$

Should a occur in $\mathsf{t}_i(x)b_k$, then b_k would not appear in $\overline{w}(i,a)$. So, it follows that

$$\mathsf{t}_k(\mathsf{p}_a(\mathsf{t}_i(x)b_k y)) = \mathsf{t}_k(\mathsf{t}_i(x)b_k\,\mathsf{p}_a(y)) = \mathsf{p}_a(y). \tag{3.16}$$

On the other hand, we have the equalities

$$\overline{w}(k,a) = \mathsf{p}_a(\mathsf{t}_k(\overline{w})) = \mathsf{p}_a(y) \overset{(3.16)}{=} \mathsf{t}_k(\overline{w}(i,a)),$$

and so the desired relation (b) follows.

Now, we suppose that $x = x_n[^{q_{n-1}}x_{n-1}\cdots[^{q_1}x_1[^{q_0}x_0$ and $b_k y = y_0]^{q_0}y_1]^{q_1}\cdots y_{n-1}]^{q_{n-1}}y_n$, where all the x_j's and y_j's are possibly empty well-parenthesized words, for $j = 0,\ldots,n$. We note that, since $k \in c_{\mathbb{N}}(\overline{w}(i,a)) = c_{\mathbb{N}}(\mathsf{p}_a(\mathsf{t}_i(\overline{w})))$, Lemma 3.30 yields the equalities

$$\mathsf{t}_k(\overline{w}(i,a)) = \mathsf{t}_k(\mathsf{p}_a(\mathsf{t}_i(\overline{w}))) = \mathsf{p}_a(\mathsf{t}_k(\mathsf{t}_i(\overline{w}))). \tag{3.17}$$

With that in mind, we start by computing the elements $\mathsf{t}_k(\overline{w})$ and $\mathsf{t}_k(\mathsf{t}_i(\overline{w}))$. Let

$$\begin{aligned}
\vec{x} &= (x_0, x_1, \ldots, x_n, \varepsilon, \varepsilon, \ldots);\\
\vec{y} &= (y_0, y_1, \ldots, y_n, \varepsilon, \varepsilon, \ldots);\\
\vec{q} &= (q_0, q_1, \ldots, q_{n-1}, 0, 0, \ldots)
\end{aligned}$$

and let $\ell \in \{0,1,\ldots,n\}$ be such that $i \in c_{\mathbb{N}}(x_\ell y_\ell)$. Noticing that $\overline{w} = \mu_n(\vec{x},\vec{y},\vec{q})$, k belongs to $c_{\mathbb{N}}(y_0)$, and using Lemma 3.31 we obtain

$$\begin{aligned}
\mathsf{t}_k(\overline{w}) &= \mathsf{t}_k(\mu_0(\vec{x},\vec{y},\vec{q}))\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})\\
&= \mathsf{t}_k(y_0)\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})
\end{aligned} \tag{3.18}$$

Now, we have two possible situations.

(i) $i \in c_{\mathbb{N}}(x_\ell)$, for a certain $\ell \in \{0,\dots,n\}$;

(ii) $i \in c_{\mathbb{N}}(y_\ell)$, for a certain $\ell \in \{n,\dots,0\}$.

If we are in Case (i), then we may use Corollary 3.32 and get

$$\mathsf{t}_k(\mathsf{t}_i(\overline{w})) = \mathsf{t}_k(y_0)\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-2}(\vec{x},\vec{y},\vec{q})\cdot\xi_{n-1}(\vec{x},\vec{y},\vec{q}).$$

Hence, we have an equality between $\mathsf{t}_k(\overline{w}(i,a)) = \mathsf{p}_a(\mathsf{t}_k(\mathsf{t}_i(\overline{w})))$ and $\overline{w}(k,a) = \mathsf{p}_a(\mathsf{t}_k(\overline{w}))$, thereby proving (b).

On the other hand, when the situation occurring is (ii), Corollary 3.33 yields

$$\begin{aligned}\mathsf{t}_k(\mathsf{t}_i(\overline{w})) = {} & \mathsf{t}_k(y_0)\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{\ell-1}(\vec{x},\vec{y},\vec{q})\\ & \cdot[^{q_\ell-2}\mu_\ell(\vec{x},\vec{y},\vec{q})]^{q_\ell-2}y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q}).\end{aligned}$$

If the first occurrence of a in $\mathsf{t}_k(\mathsf{t}_i(\overline{w}))$ is in $\mathsf{t}_k(y_0)\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{\ell-1}(\vec{x},\vec{y},\vec{q})$ or in $\mu_\ell(\vec{x},\vec{y},\vec{q})$, then the first occurrence of a in $\mathsf{t}_k(\overline{w})$ is also in one of these factors and we easily conclude that

$$\mathsf{p}_a(\mathsf{t}_k(\mathsf{t}_i(\overline{w}))) = \mathsf{p}_a(\mathsf{t}_k(y_0)\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{\ell-1}(\vec{x},\vec{y},\vec{q})\cdot\mu_\ell(\vec{x},\vec{y},\vec{q})) = \mathsf{p}_a(\mathsf{t}_k(\overline{w})),$$

thereby proving again an equality in (b).

Otherwise, the first occurrence of a in $\mathsf{t}_k(\mathsf{t}_i(\overline{w}))$ is in $y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})$. Analyzing the equality (3.18), we deduce that a occurs for the first time in $\mathsf{t}_k(\overline{w})$ also in the factor $y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})$. Then, we may compute

$$\begin{aligned}\mathsf{p}_a(\mathsf{t}_k(\mathsf{t}_i(\overline{w}))) = {} & \mathsf{t}_k(y_0)\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{\ell-1}(\vec{x},\vec{y},\vec{q})\cdot[^{q_\ell-2}\mu_\ell(\vec{x},\vec{y},\vec{q})]^{q_\ell-2}\\ & \cdot\mathsf{p}_a(y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})) \qquad (3.19)\\ \mathsf{p}_a(\mathsf{t}_k(\overline{w})) = {} & \mathsf{t}_k(y_0)\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{\ell-1}(\vec{x},\vec{y},\vec{q})\cdot[^{q_\ell-1}\mu_\ell(\vec{x},\vec{y},\vec{q})]^{q_\ell-1}\\ & \cdot\mathsf{p}_a(y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})). \qquad (3.20)\end{aligned}$$

Moreover, using again Lemma 3.31, we obtain

$$\begin{aligned}\overline{w}(i,a) &= \mathsf{p}_a(\mathsf{t}_i(\overline{w})) = \mathsf{p}_a(\mathsf{t}_i(\mu_n(\vec{x},\vec{y},\vec{q})))\\ &= \mathsf{p}_a(\mathsf{t}_i(\mu_\ell(\vec{x},\vec{y},\vec{q}))\cdot\xi_\ell(\vec{x},\vec{y},\vec{q})\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q}))\\ &= \mathsf{p}_a(\mathsf{t}_i(y_\ell)\cdot\xi_\ell(\vec{x},\vec{y},\vec{q})\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q}))\\ &= \mathsf{t}_i(y_\ell)[^{q_\ell-1}\mu_\ell(\vec{x},\vec{y},\vec{q})]^{q_\ell-1}\\ &\quad\cdot\mathsf{p}_a(y_{\ell+1}\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})) \quad \text{since } a\notin c_A(\mu_\ell(\vec{x},\vec{y},\vec{q}))\\ &= \mathsf{t}_i(y_\ell)[^{q_\ell-1}x_\ell[^{q_{\ell-1}}x_{\ell-1}[^{q_{\ell-2}}\cdots[^{q_0}x_0y_0]^{q_0}\cdots]^{q_{\ell-2}}y_{\ell-1}]^{q_{\ell-1}}y_\ell]^{q_\ell-1}\\ &\quad\cdot\mathsf{p}_a(y_{\ell+1}\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})) \qquad (3.21)\end{aligned}$$

Since b_k is the principal marker of $\overline{w}(i,a)$, we know that the following inclusion holds:

$$c_A(y_0y_1\cdots y_\ell\cdot \mathsf{p}_a(y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})))\subseteq c_A(\,\mathsf{t}_i(y_\ell)x_\ell\cdots x_0b_k).$$

Also, by definition of $\mu_\ell(\vec{x},\vec{y},\vec{q})$, we have an inclusion

$$c_A(\,\mathsf{t}_i(y_\ell)x_\ell\cdots x_0b_k)\subseteq c_A(\mu_\ell(\vec{x},\vec{y},\vec{q})).$$

Consequently, we obtain

$$c_A(\,\mathsf{p}_a(y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})))\subseteq c_A(\mu_\ell(\vec{x},\vec{y},\vec{q})).$$

Observing that

$$\begin{aligned}\vec{c}(\eta([^{q_\ell-2}\mu_\ell(\vec{x},\vec{y},\vec{q})]^{q_\ell-2}))=\vec{c}(\eta([^{q_\ell-1}\mu_\ell(\vec{x},\vec{y},\vec{q})]^{q_\ell-1}))&=c_A(\mu_\ell(\vec{x},\vec{y},\vec{q}))\\ &\supseteq c_A(\,\mathsf{p}_a(y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})))\\ &=c(\eta(\,\mathsf{p}_a(y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})))),\end{aligned}\tag{3.22}$$

we end up with the desired relations, which are valid in DRH:

$$\begin{aligned}\eta(\,\mathsf{t}_k(\overline{w}(i,a)))&\overset{(3.17),(3.19)}{=}\eta(\,\mathsf{t}_k(y_0)\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{\ell-1}(\vec{x},\vec{y},\vec{q}))\\ &\quad\cdot\eta([^{q_\ell-2}\mu_\ell(\vec{x},\vec{y},\vec{q})]^{q_\ell-2})\cdot\eta(\,\mathsf{p}_a(y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})))\\ &=\eta(\,\mathsf{t}_k(y_0)\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{\ell-1}(\vec{x},\vec{y},\vec{q}))\\ &\quad\cdot\eta(\mu_\ell(\vec{x},\vec{y},\vec{q}))^{\omega+q_\ell-2}\cdot\eta(\,\mathsf{p}_a(y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})))\\ &\overset{(3.22)}{\mathcal{R}}\ \eta(\,\mathsf{t}_k(y_0)\cdot\xi_0(\vec{x},\vec{y},\vec{q})\cdot\xi_1(\vec{x},\vec{y},\vec{q})\cdots\xi_{\ell-1}(\vec{x},\vec{y},\vec{q}))\\ &\quad\cdot\eta(\mu_\ell(\vec{x},\vec{y},\vec{q}))^{\omega+q_\ell-1}\cdot\eta(\,\mathsf{p}_a(y_{\ell+1}\cdot\xi_{\ell+1}(\vec{x},\vec{y},\vec{q})\cdots\xi_{n-1}(\vec{x},\vec{y},\vec{q})))\\ &\overset{(3.20)}{=}\eta(\overline{w}(k,a))=w(k,a).\end{aligned}$$

We finally observe that we actually proved an equality in $\overline{\Omega}_A\mathsf{S}$ rather than a relation modulo DRH, except in the last situation. But that scenario only occurs when $w(i,a)$ is regular modulo DRH. Indeed, since $b_k\in c(y_0)$ is the principal marker of $w(i,a)$, from the equality (3.21), we may deduce that

$$\vec{c}(w(i,a))=c(w(i,a)),$$

which by Proposition 2.15 implies that $\rho_{\mathsf{DRH}}(w(i,a))$ is regular. $\square$

For a well-parenthesized word x over $A\times\mathbb{N}$, we consider the following property:

$$\forall a,b\in A,\quad \forall i\in\mathbb{N},\quad a_i,b_i\in c(x)\implies a=b \tag{$H(x)$}$$

Lemma 3.35 (cf. [25, Lemma 5.9]). *Let $x \in \mathsf{Dyck}(A \times \mathbb{N}) \setminus \{\varepsilon\}$ satisfy $(H(x))$ and suppose that a_i is a marker of x. Then, the following equality holds:*

$$\eta(x) = \eta(\,\mathsf{p}_a(x) \cdot a_i \cdot \mathsf{t}_i(x)). \tag{3.23}$$

Proof. We proceed by induction on $|x|$. If $|x| = 1$, then we have $x = a_i$ and so, $\mathsf{p}_a(x) = \varepsilon = \mathsf{t}_i(x)$, yielding the result. If $|x| > 1$, then we may write $x = ya_iz$, with $a \notin c_A(y)$. Since Property $(H(x))$ holds, we also know that $i \notin c_{\mathbb{N}}(y)$. If y and z are both well-parenthesized words, then we have $\mathsf{p}_a(x) = y$ and $\mathsf{t}_i(x) = z$ and we trivially get (3.23). Otherwise, none of y and z is well-parenthesized and we may write $y = y_1\,[^q y_0$ and $z = z_0]^q z_1$, with y_1, z_1 possibly empty well-parenthesized words and $y_0a_iz_0$ a well-parenthesized word. Letting $w = y_0a_iz_0$, noticing that a_i being a marker of x implies that a_i is also a marker of w, and applying the induction hypothesis to w, we get

$$\eta(w) = \eta(\,\mathsf{p}_a(w) \cdot a_i \cdot \mathsf{t}_i(w)). \tag{3.24}$$

On the other hand, we also have

$$\mathsf{p}_a(x) = y_1 \cdot \mathsf{p}_a(w) \tag{3.25}$$

$$\mathsf{t}_i(x) = \mathsf{t}_i(w) \cdot [^{q-1}w]^{q-1} \cdot z_1. \tag{3.26}$$

Thus, we obtain

$$\begin{aligned}
\eta(x) = \eta(y_1[^q w]^q z_1) &= \eta(y_1) \cdot \eta(w)^{\omega+q} \cdot \eta(z_1) \\
&\overset{(3.24)}{=} \eta(y_1) \cdot \eta(\,\mathsf{p}_a(w) \cdot a_i \cdot \mathsf{t}_i(w)) \cdot \eta(w)^{\omega+q-1} \cdot \eta(z_1) \\
&= \eta(y_1 \cdot \mathsf{p}_a(w)) \cdot a \cdot \eta(\,\mathsf{t}_i(w)) \cdot \eta(w)^{\omega+q-1} \cdot \eta(z_1) \\
&\overset{(3.25)}{=} \eta(\,\mathsf{p}_a(x)) \cdot a \cdot \eta(\,\mathsf{t}_i(w) \cdot [^{q-1}w]^{q-1} z_1) \\
&\overset{(3.26)}{=} \eta(\,\mathsf{p}_a(x) \cdot a_i \cdot \mathsf{t}_i(x)),
\end{aligned}$$

as desired. □

Corollary 3.36 (cf. [25, Corollary 5.11]). *Let w be a $\langle\kappa\rangle$-term. Let $i \in \mathbb{N}$ and $a \in A \uplus \{\#\}$, and let b_k be the principal marker of $\overline{w}(i,a)$. Suppose that $\mathsf{lbf}(w(i,a)) = (w_\ell, m, w_r)$. Then, $m = b$ and DRH satisfies $w_\ell = w(i,b)$, and $w_r \mathrel{\mathcal{R}} w(k,a)$. Moreover, if $\rho_{\mathsf{DRH}}(w(i,a))$ is not regular, then we have an equality $\mathsf{lbf}(w(i,a)) = (w(i,b), b, w(k,a))$.*

Proof. As b_k is the principal marker of $\overline{w}(i,a)$, we can write $\overline{w}(i,a) = xb_ky$, where $c_A(y) \subseteq c_A(xb_k)$ and $b \notin c_A(x)$. Since $(H(\overline{w}(i,a)))$ holds, Lemma 3.35 yields

$$\eta(\overline{w}(i,a)) = \eta(\,\mathsf{p}_b(\overline{w}(i,a)) \cdot b_k \cdot \mathsf{t}_k(\overline{w}(i,a))) = \eta(\,\mathsf{p}_b(\overline{w}(i,a))) \cdot b \cdot \eta(\,\mathsf{t}_k(\overline{w}(i,a))).$$

Furthermore, since $b \notin c_A(x)$, we also have $c_A(\,\mathsf{p}_b(\overline{w}(i,a))) = c_A(x)$ and consequently, the left basic factorization of $w(i,a)$ is precisely $(\eta(\,\mathsf{p}_b(\overline{w}(i,a))), b, \eta(\,\mathsf{t}_k(\overline{w}(i,a))))$. In particular, we have $m = b$ and, by Lemma 3.34, the pseudovariety DRH satisfies $w_\ell = w(i,b)$ and $w_r \mathrel{\mathcal{R}} w(k,a)$, with an equality in S in the latter relation when $w(i,a)$ is not regular modulo DRH. □

Example 3.37. Consider $w = (b^{\omega-1} \cdot (a \cdot c)) \cdot ((a \cdot b) \cdot (a^{\omega+1} \cdot b)^{\omega})^{\omega}$. In Example 3.27, we observed that $\overline{w} = 0_0[^{-1}b_1]^{-1}a_2c_3[^0a_4b_5[^0[^1a_6]^1b_7]^0]^0\#_8$. We also calculated $\overline{w}(5,b) = [^1a_6]^1$, which has a_6 as a principal marker. Moreover, we have $\overline{w}(5,a) = \varepsilon$ and $\overline{w}(6,b) = [^0a_6]^0$. Although $w(5,b)$ is regular over DRH, we still have an equality $\mathsf{lbf}(w(5,b)) = (w(5,a),a,w(6,b))$. Let us now repeat this process. The principal marker of $\overline{w}(6,b)$ is again a_6. But now, $\mathsf{lbf}(w(6,b)) = (I,a,a^{\omega-1})$, while $(w(6,a),a,w(6,b)) = (I,a,a^{\omega})$.

3.3.3 Computing tails and prefixes of well-parenthesized words

In this subsection we show how one may effectively compute the elements $\overline{w}(i,a)$, that is, how to represent them by a well-parenthesized word.

Recall that, by definition, $\overline{w}(i,a) = \mathsf{p}_a(\mathsf{t}_i(\overline{w}))$. We exhibit two algorithms: given a nonempty well-parenthesized word x, Algorithm 3.1 computes the tail $\mathsf{t}_i(x)$ ($i \in c_{\mathbb{N}}(x)$), and Algorithm 3.2 computes the prefix $\mathsf{p}_a(x)$ ($a \in c_A(x)$). Combining both, we may then compute $\overline{w}(i,a)$.

Although the projections $\pi_{\mathbb{N}}(x)$ and $\pi_A(x)$ are only defined for $x \in \mathsf{Dyck}(A \times \mathbb{N})$, we agree that $\pi_{\mathbb{N}}([^q) = q = \pi_{\mathbb{N}}(]^q)$, $\pi_A([^q) = [$, and $\pi_A(]^q) =]$.

Lemma 3.38. *Algorithms 3.1 and 3.2 are correct and both run in $O(|x|)$-time. In particular, for every $x \in \mathsf{Dyck}(A \times \mathbb{N})$, $i \in c_{\mathbb{N}}(x)$, and $a \in c_A(x)$, the well-parenthesized word $x(i,a)$ may be calculated in linear time.*

Proof. Let $x = x_1 \cdots x_n$ be a well-parenthesized word over $A \times \mathbb{N}$, $i \in c_{\mathbb{N}}(x)$, and $a \in c_A(x)$.

We start by analyzing Algorithm 3.1. The first step is to scan the word until we find, for a certain $b \in A$, the character b_i (cycle **while** in lines 2–9), say that $x_k = b_i$. Meanwhile, we save in list L the positions of the parentheses that were open but not closed, in the order they appear. After that, we initialize the word y, that is intended to contain the final result $\mathsf{t}_i(x)$ (line 10). Then, we continue scanning the word, from the point where we found b_i. Whenever the current position x_j is not a parentheses matching one of the parentheses recorded in L, we add the character x_j to y (lines 14, 17, and 20). Simultaneously, we update the counter m in order to control the number of pending open brackets we read from x_k (lines 15 and 18). In this way, we know that a certain parenthesis $]^q$ closes a parenthesis referenced in L if and only if $m = 0$. If that is the case, say that the current position is $x_j =]^q$ and the last entry of L is ℓ (meaning that $x_\ell = [^q$ is the parenthesis matching x_j) then, according to the rules to compute $\mathsf{t}_i(x)$, we should add to y the word $[^{q-1}x_{\ell+1} \cdots x_{j-1}]^{q-1}$ (line 23).

To compute $\mathsf{p}_a(x)$, in Algorithm 3.2, we just need to copy the original word until we get to a letter a (line 3). The unique issue is that, we may end up with opened parenthesis that were not yet closed. For this reason, we keep a register of the parentheses we find in the way (lines 4–8). At the end, we erase the pending open brackets from our answer (line 11).

Finally, the linear time complexity comes from the fact that both algorithms only involve cycles **while** and **for**, whose number of iterations is bounded above by the size n of the input x. $\square$

Algorithm 3.1

Input: $x = x_1 \cdots x_n \in \mathsf{Dyck}(A \times \mathbb{N})$ and $i \in c_{\mathbb{N}}(x)$
Output: $\mathsf{t}_i(x)$

$k \leftarrow 1, L \leftarrow [\,]$
while $\pi_{\mathbb{N}}(x_k) \neq i$ or $\pi_A(x_m) \in \{[,]\}$ **do**
  if $\pi_A(x_k) = [$ **then**
    add k to the end of L
  else if $\pi_A(x_k) =]$ **then**
    delete the last entry of L
  end if
  $k \leftarrow k+1$
end while
$y \leftarrow \varepsilon$
$m \leftarrow 0$
for $j = k+1, \ldots, n$ **do**
  if $\pi_A(x_j) = [$ **then**
    $y \leftarrow yx_j$
    $m \leftarrow m+1$
  else if $\pi_A(x_j) =]$ and $m > 0$ **then**
    $y \leftarrow yx_j$
    $m \leftarrow m-1$
  else if $\pi_A(x_j) \neq]$ **then**
    $y \leftarrow yx_j$
  else
    $\ell \leftarrow$ last entry of L
    $y \leftarrow y[^{q-1} x_{\ell+1} \cdots x_{j-1}]^{q-1}$, where $q = \pi_{\mathbb{N}}(x_\ell)$
    delete the last entry of L
  end if
end for
return y

Algorithm 3.2

Input: $x = x_1 \cdots x_n \in \mathsf{Dyck}(A \times \mathbb{N})$ and $a \in c_A(x)$
Output: $\mathsf{p}_a(x)$

$k \leftarrow 1, L \leftarrow [\,], y \leftarrow \varepsilon$
while $\pi_A(x_k) \neq a$ **do**
  $y \leftarrow yx_k$
  if $\pi_A(x_k) = [$ **then**
    add k to the end of L
  else if $\pi_A(x_k) =]$ **then**
    remove the last entry of L
  end if
  $k \leftarrow k+1$
end while
$y \leftarrow$ word obtained when the m-letter is erased for each $m \in L$
return y

3.4 DRH-graphs and their computation

We begin this section with the definition of a DRH-graph. Through these structures, we are able to characterize when two pseudowords are $\mathcal{R}$-equivalent over DRH. If we further assume that the word problem is decidable in $\Omega_A^\kappa \mathsf{H}$, then the word problem is decidable in $\Omega_A^\kappa \mathsf{DRH}$ as well.

Definition 3.39. *Let w be a $\langle\kappa\rangle$-term. The* DRH-graph of w *is the finite* DRH*-automaton*

$$\mathcal{G}(w) = \langle V(w), \to, \mathsf{q}(0,\#), \{\varepsilon\}, \lambda_{\mathsf{H}}, \lambda\rangle,$$

defined as follows. The set of states is $V(w) = \{\mathsf{q}(i,a) \colon 0 \le i < |\overline{w}|, a \in c_A(\overline{w}) \text{ and } w(i,a) \neq I\} \uplus \{\varepsilon\}$. Given a state $\mathsf{q}(i,a) \in V(w) \setminus \{\varepsilon\}$, let b_k be the principal marker of $\overline{w}(i,a)$. The transitions of $\mathsf{q}(i,a)$ are $\mathsf{q}(i,a).0 = \mathsf{q}(i,b)$ and $\mathsf{q}(i,a).1 = \mathsf{q}(k,a)$. The labels are $\lambda_{\mathsf{H}}(\mathsf{q}(i,a)) = \rho_{\mathsf{H}}(\mathsf{reg}(w(i,b)))$ and $\lambda(\mathsf{q}(i,a)) = b$. If a state $\mathsf{q}(i,a)$ is not reached from the root $\mathsf{q}(0,\#)$, then we discard it from $V(w)$.

Remark 3.40. We point out that the DRH-graph $\mathcal{G}(w)$ rather than depending on the $\langle\kappa\rangle$-term w, it depends on the well-parenthesized word $\overline{w}$ that w defines.

Example 3.41. Recall that $\overline{w} = 0_0[^{-1}b_1]^{-1}a_2c_3[^0a_4b_5[^0[^1a_6]^1b_7]^0]^0\#_8$, for the same w of Example 3.26. The DRH-graph of w is drawn in Figure 3.7. The labels of the states are written in the second line

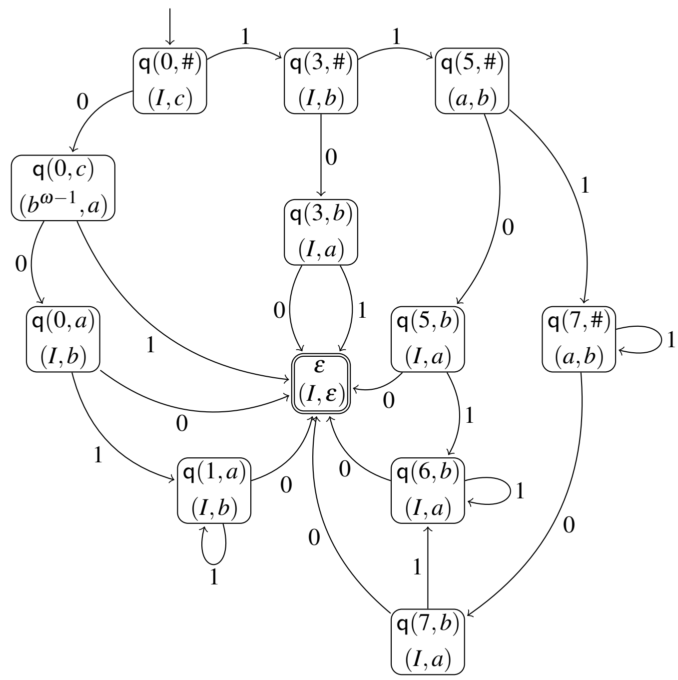

Fig. 3.7 DRH-graph of $w = (b^{\omega-1} \cdot (a \cdot c)) \cdot ((a \cdot b) \cdot (a^{\omega+1} \cdot b)^\omega)^\omega$.

of each state as a pair $(\lambda_{\mathsf{H}}(\_), \lambda(\_))$. The reader may wish to compare this DRH-graph with the equivalent DRH-automata in Figures 3.1 and 3.2.

The following result suggests that the construction of $\mathcal{G}(w)$ might be a starting point to solve the κ-word problem over DRH algorithmically.

Proposition 3.42. *For every $\langle\kappa\rangle$-term w, $\mathcal{G}(w)$ is a* DRH*-automaton equivalent to $\mathcal{T}(w(0,\#))$.*

Proof. Let $\mathcal{T}(w(0,\#)) = \langle V, \to_{\mathcal{T}}, \mathsf{q}, F, \lambda_{\mathcal{T},\mathsf{H}}, \lambda_{\mathcal{T}}\rangle$ and $\mathcal{G}(w) = \langle V(w), \to_{\mathcal{G}}, \mathsf{q}(0,\#), \{\varepsilon\}, \lambda_{\mathcal{G},\mathsf{H}}, \lambda_{\mathcal{G}}\rangle$. We first claim that, for every $\alpha \in \Sigma^*$, we have

$$\mathsf{q}(0,\#).\alpha = \mathsf{q}(i,a) \implies \mathcal{T}(w(0,\#))_{\mathsf{q}.\alpha} = \mathcal{T}(w(i,a)). \tag{3.27}$$

To prove this, we argue by induction on $|\alpha|$. If $|\alpha| = 0$, then the result holds trivially. Let $\alpha \in \Sigma^*$ be such that $|\alpha| \geq 1$ and suppose that the result holds for every other shorter word α. We can write $\alpha = \beta\gamma$, with $\gamma \in \{0,1\}$. Let $\mathsf{q}(0,\#).\beta = \mathsf{q}(i,a)$. By induction hypothesis, it follows the equality $\mathcal{T}(w(0,\#))_{\mathsf{q}.\beta} = \mathcal{T}(w(i,a))$. Let b_k be the principal marker of $\overline{w}(i,a)$. By definition of $\mathcal{G}(w)$, we have

$$\begin{aligned} \mathsf{q}(0,\#).\beta 0 &= \mathsf{q}(i,b) \\ \mathsf{q}(0,\#).\beta 1 &= \mathsf{q}(k,a). \end{aligned}$$

On the other hand, Lemma 3.13 gives that if $\mathsf{lbf}(w(i,a)) = (w_\ell, b, w_r)$, then

$$\mathcal{T}(w(i,a)) = (\mathcal{T}(w_\ell), \mathsf{reg}(w_\ell) \mid b, \mathcal{T}(w_r)),$$

which in turn, by Corollary 3.36, is equivalent to

$$\mathcal{T}(w(i,a)) = (\mathcal{T}(w(i,b)), \mathsf{reg}(w(i,b)) \mid b, \mathcal{T}(w(k,a))). \tag{3.28}$$

In particular, we conclude that

$$\mathcal{T}(w(0,\#))_{\mathsf{q}.\beta 0} = \mathcal{T}(w(i,b)) \text{ and } \mathcal{T}(w)_{\mathsf{q}.\beta 1} = \mathcal{T}(w(k,a)).$$

It is now enough to notice that, for each pair $(i,a) \in [0, |\overline{w}|[\times c_A(\overline{w})$, the labels of the node $\mathsf{q}(i,a)$ of $\mathcal{G}(w)$ and the labels of the root of $\mathcal{T}(w(i,a))$ coincide. In fact, if b_k is the principal marker of $\overline{w}(i,a)$, then the construction of $\mathcal{G}(w)$ yields the equalities

$$\begin{aligned} \lambda_{\mathcal{G}}(\mathsf{q}(i,a)) &= b \\ \lambda_{\mathcal{G},\mathsf{H}}(\mathsf{q}(i,a)) &= \rho_{\mathsf{H}}(\mathsf{reg}(w(i,b))) \end{aligned}$$

which, by (3.28), are precisely the labels of the root of $\mathcal{T}(w(i,a))$. $\square$

Imagine we are given a κ-word and let $w = a^{\omega+q}$ be one of its representations as a $\langle\kappa\rangle$-term, with q "very big". Then, we have $\overline{w} = 0_0[^q a_1]^q \#_2$ and so, $|\overline{w}| = 3$. Conceptually speaking, such a κ-word involves a "large" number of implicit operations of κ but the length of its representation $\overline{w}$ in $\mathsf{Dyck}(A \times \mathbb{N})$ is just 3. Therefore, allowing any representation of κ-words, we would not be able to get meaningful results for the efficiency of the forthcoming algorithms. Thus, it is reasonable to require that all κ-words are presented as κ-terms. We make that assumption from now on.

Consider a κ-term w. We may assume that w is given by a tree. For instance, if

$$w = ((b^{\omega-1} \cdot a) \cdot c) \cdot ((a \cdot b) \cdot a^{\omega-1})^{\omega-1},$$

then the tree representing w is depicted in Figure 3.8. Since from such a tree representation we

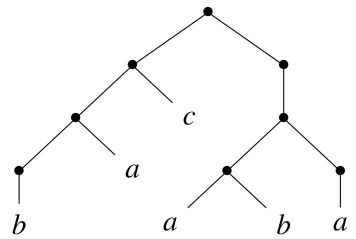

Fig. 3.8 The tree representing $w = ((b^{\omega-1} \cdot a) \cdot c) \cdot ((a \cdot b) \cdot a^{\omega-1})^{\omega-1}$.

may compute $\overline{w}$ in linear time, we assume that we are already given $\overline{w}$. If the tree representing w has n nodes then, following [25], we say that the *length* of w is $|w| = n+1$. It is clear that $O(|w|) = O(|\overline{w}|)$. To actually compute the DRH-graph $\mathcal{G}(w)$ we essentially need to compute the principal marker of the words $\overline{w}(i,a)$ as well as the regular parts of $w(i,a)$. Almeida and Zeitoun [25] exhibited an algorithm to compute the first occurrences of each letter of a well-parenthesized word x. Given a word x, $\mathsf{first}(x)$ consists of a list of the first occurrences of each letter in x. For instance, $\mathsf{first}([^{-1}b_1]^{-1}a_2c_3[^{-1}a_4b_5[^{-1}a_6]^{-1}]^{-1}) = [b_1, a_2, c_3]$. In particular, this computes the principal marker of x: it is the last entry of the outputted list. Moreover, if b_k is the principal marker of x, then the penultimate entry of the list is the principal marker of $\mathsf{p}_b(x)$, and so on. Hence, this is enough to almost compute $\mathcal{G}(w)$. More precisely, the knowledge of $\mathsf{first}(\overline{w}(i,a))$, for every pair (i,a), allows us to compute the reduct $\mathcal{G}_{\mathsf{R}}(w) = \langle V(w), \to, \mathsf{q}(0,\#), \{\varepsilon\}, \lambda \rangle$ in time $O(|w|\,|c(w)|)$.

Lemma 3.43 ([25, Lemma 5.15])**.** *Let w be a $\langle\kappa\rangle$-term. Then, one may compute in time $O(|w|\,|c(w)|)$ a table giving, for each i such there exists $a_i \in c(\overline{w}) \cap A \times \mathbb{N}$, the word* $\mathsf{first}(\overline{w}(i,\#))$.

It remains to find the labels of the states under λ_{H}. For that purpose, we observe that the regular part of a pseudoword u depends deeply on the content of the factors of the form $\mathsf{lbf}_k(u)$, which we may compute using Lemma 3.35; and of the cumulative content of u. Also, it follows from Lemma 3.10 and from Proposition 3.42 that the cumulative content of any pseudoword of the form $w(i,a)$ is completely determined by the reduct $\mathcal{G}_{\mathsf{R}}(w)$. Thus, we may start by computing the cumulative content of $w(i,a)$ and then compare it with the content of $\mathsf{lbf}_k(w(i,a))$, for increasing values of k. When we achieve an equality, we know what is the regular part of $w(i,a)$. Algorithm 3.3 does that job. We assume that we already have the table described in Lemma 3.43, so that, computing $c(w(i,a))$ and the principal marker of $\overline{w}(i,a)$ takes $O(1)$-time. Further, we may assume that we are given $\mathcal{G}_{\mathsf{R}}(w)$, since we already explained how to get it from the table of Lemma 3.43 in $O(|w|\,|c(w)|)$-time.

Lemma 3.44. *Algorithm 3.3 returns I if and only if $\vec{c}(w(i,a)) = \emptyset$. Otherwise, the value k outputted is such that* $\mathsf{reg}(w(i,a)) = w(k,a)$. *Moreover, the algorithm runs in linear time, provided we have the knowledge of* $\mathsf{first}(w(i,a))$.

Proof. By Property (A.3) of a DRH-automaton, and since there is only a finite number of possible states in $\mathcal{G}_{\mathsf{R}}(w)_{\mathsf{q}(i,a)}$, either there exists $k \geq 0$ such that $\mathsf{q}(i,a).1^k = \varepsilon$, or there exist $n > k \geq 0$ such that $\mathsf{q}(i,a).1^k = \mathsf{q}(i,a).1^n$. Therefore, the cycle **while** in line 2 does not run forever. If the occurring situation is the former, then $\vec{c}(\mathcal{G}(w)_{\mathsf{q}(i,a)}) = \emptyset$. On the other hand, by Proposition 3.42, we have $\mathcal{G}(w)_{\mathsf{q}(i,a)} \sim \mathcal{T}(w(i,a))$ which in turn, by Theorem 3.12, implies $\pi(\mathcal{G}(w)_{\mathsf{q}(i,a)}) \;\mathcal{R}\; w(i,a)$ modulo DRH.

Algorithm 3.3

Input: A $\langle\kappa\rangle$-term w and $(i,a) \in [0,|\overline{w}|[\, \times c_A(\overline{w})$ (with $\overline{w}(i,a) \neq \varepsilon$)
Output: $\mathsf{reg}(w(i,a)) = I$, if $\vec{c}(w(i,a)) = \emptyset$ or k such that $\mathsf{reg}(w(i,a)) = w(k,a)$, otherwise

1: $L \leftarrow \{\}$, $j \leftarrow i$
2: **while** $j \notin L$ and $\overline{w}(j,a) \neq \varepsilon$ **do**
3: $\quad j \leftarrow \pi_{\mathbb{N}}(\text{principal marker of } \overline{w}(j,a))$ $\quad\triangleright$ So that, if $\mathsf{q}(j,a).1 \neq \varepsilon$, then $\mathsf{q}(j,a) \leftarrow \mathsf{q}(j,a).1$
4: $\quad L \leftarrow L \cup \{j\}$
5: **end while**
6: **if** $\overline{w}(j,a) = \varepsilon$ **then**
7: $\quad$ **return** I
8: **else**
9: $\quad C \leftarrow c(w(j,a))$ $\quad\triangleright$ The set C is the cumulative content of $w(i,a)$
10: $\quad k \leftarrow i$
11: $\quad$ **while** $c_A(\overline{w}(k,a)) \neq C$ **do**
12: $\quad\quad k \leftarrow \pi_{\mathbb{N}}(\text{principal marker of } \overline{w}(k,a))$
13: $\quad$ **end while**
14: $\quad$ **return** k
15: **end if**

Also, Lemma 3.10 yields $\vec{c}(w(i,a)) = \vec{c}(\mathcal{G}(w)_{\mathsf{q}(i,a)}) = \emptyset$, and therefore, $\mathsf{reg}(w(i,a)) = I$. This is the case where the symbol I is returned in line 7.

Now, suppose that $n > k \geq 0$ are such that $\mathsf{q}(i,a).1^k = \mathsf{q}(i,a).1^n$. Then, the cycle **while** is exited because an index j is repeated. By Property (A.4), we have the following chain of inclusions:

$$\lambda(\mathcal{G}(w)_{\mathsf{q}(i,a).1^k}) \supseteq \lambda(\mathcal{G}(w)_{\mathsf{q}(i,a).1^{k+1}}) \supseteq \cdots \supseteq \lambda(\mathcal{G}(w)_{\mathsf{q}(i,a).1^n}).$$

As $\mathsf{q}(i,a).1^k = \mathsf{q}(i,a).1^n$, these inclusions are actually equalities, implying that k is greater than or equal to $\mathsf{r.ind}(\mathcal{G}(\mathsf{w})_{\mathsf{q(i,a)}})$. Combining again Proposition 3.42, Theorem 3.12 and Lemma 3.10, we may deduce that

$$\vec{c}(w(i,a)) = \vec{c}(\mathcal{G}(w)_{\mathsf{q}(i,a)}) = \lambda(\mathcal{G}(w)_{\mathsf{q}(i,a).1^k}),$$

where the last member is precisely $c(w(j,a))$ provided that $\mathsf{q}(i,a).1^k = \mathsf{q}(j,a)$. Therefore, in line 9 we assign to C the cumulative content of $w(i,a)$. Until now, since we are assuming that we are given all the information about $\mathcal{G}_{\mathsf{R}}(w)$, we only spend time $O(|w|)$, because that is the number of possible values of j that may appear in line 2.

Let us prove that, if we get to line 9, then the value k outputted in line 14 is such that

$$\mathsf{reg}(w(i,a)) = w(k,a).$$

We write

$$w(i,a) = \mathsf{lbf}_1(w(i,a)) \cdots \mathsf{lbf}_m(w(i,a)) w'_m,$$

for every $m \geq 1$ (notice that $\mathsf{lbf}_m(w(i,a))$ is defined for all $m \geq 1$ because we are assuming that $\vec{c}(w(i,a)) \neq \emptyset$). Then, the regular part of $w(i,a)$ is given by w'_ℓ, where

$$\ell = \min\{m \colon c(w'_m) = \vec{c}(w(i,a))\}.$$

In particular, the projection of w'_m onto $\overline{\Omega}_A\mathsf{DRH}$ is not regular, for every $m < \ell$. Set $(c_0,k_0) = (a,i)$ and, for $m \geq 0$, let (c_{m+1},k_{m+1}) be the principal marker of $\overline{w}(k_m,a)$. By Corollary 3.36, if $w(k_m,a)$ is not regular modulo DRH, then we have $\mathsf{lbf}(w(k_m,a)) = (w(k_m,c_{m+1}),c_{m+1},w(k_{m+1},a))$. Therefore, the equality $w'_m = w(k_m,a)$ holds, for every $m \leq \ell$. Thus, the value k returned in line 14 is precisely k_ℓ, implying that $\mathsf{reg}(w(i,a)) = w(k,a)$ as intended.

Since there are only $O(|\overline{w}|)$ possible values for k and we are assuming that we already know $\mathsf{first}(\mathsf{w}(\mathsf{i},\#))$ for all $i \in [0,|\overline{w}|[$, it follows that lines 8–15 run in time $O(|\overline{w}|)$.

Therefore, the overall time complexity of Algorithm 3.3 is $O(|w|)$. □

So far, we possess all the needed information for computing $\mathcal{G}(w)$. However, it is not reasonable to assume that, given a certain state $\mathsf{v} \in V(w)$, to know that $\lambda_\mathsf{H}(\mathsf{v}) = \rho_\mathsf{H}(w(i,a))$ is the same as actually knowing a representation of $\lambda_\mathsf{H}(\mathsf{v})$. A much more reasonable way of describing the label λ_H of $\mathcal{G}(w)$ is by means of well-parenthesized words. Algorithms 3.1 and 3.2 together do that job time linearly on $|w|$.

Theorem 3.45. *Given a κ-term w, it is possible to compute the* DRH*-graph of w in time $O(|w|^2\,|c(w)|)$.*

Proof. We already observed that $\mathcal{G}_\mathsf{R}(w)$ may be computed in $O(|w|\,|c(w)|)$-time. In fact, this is a consequence of [25, Theorem 5.16]. In order to completely determine $\mathcal{G}(w)$, it remains to compute the labels of the form $\lambda_\mathsf{H}(\_)$. Since each one of these labels is of the form $\rho_\mathsf{H}(\mathsf{reg}(w(i,a)))$, for a certain pair (i,a) computed in constant time, by Lemmas 3.38 and 3.44, that computation may be done taking $O(|w|)$ operations for each state. Therefore, the overall complexity is $O(|w|^2\,|c(w)|)$. □

The next question we should answer is how can we decide whether two DRH-graphs $\mathcal{G}(u)$ and $\mathcal{G}(v)$ represent the same $\mathcal{R}$-class of $\overline{\Omega}_A\mathsf{DRH}$, that is, whether $\mathcal{G}(u) \sim \mathcal{G}(v)$. A possible strategy consists in visiting states in both DRH-graphs, comparing their labels (in a certain order). When we find a pair of mismatching labels, we stop, concluding that $\mathcal{G}(u)$ and $\mathcal{G}(v)$ are not equivalent. Otherwise, we conclude that they are equivalent after visiting all the states. More precisely, starting in the roots of $\mathcal{G}(u)$ and $\mathcal{G}(v)$, we mark the current states, say $\mathsf{q}_u \in V(u)$ and $\mathsf{q}_v \in V(v)$, as visited, and then repeat the process relatively to the pairs of DRH-automata $(\mathcal{G}(u)_{\mathsf{q}_u.0},\mathcal{G}(v)_{\mathsf{q}_v.0})$ and $(\mathcal{G}(u)_{\mathsf{q}_u.1},\mathcal{G}(v)_{\mathsf{q}_v.1})$. For a better understanding of the procedure, we sketch it in Algorithm 3.4.

Lemma 3.46. *Algorithm 3.4 returns the logical value of "$\mathcal{G}_1 \sim \mathcal{G}_2$" for two input* DRH*-graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. Moreover, it runs in time $O(p\max\{|V_1|,|V_2|\})$, where p is such that the word problem modulo* H *for any pair of labels $\lambda_{1,\mathsf{H}}(\mathsf{v}_1)$ and $\lambda_{2,\mathsf{H}}(\mathsf{v}_2)$ (with $\mathsf{v}_1 \in V_1$ and $\mathsf{v}_2 \in V_2$) may be solved in time $O(p)$.*

Proof. The correctness follows straightforwardly from the definition of the relation $\sim$. On the other hand, it runs in time $O(p\max\{|V_1|,|V_2|\})$, since each call of the algorithm takes time $O(p)$ (line 5) and each pair of states of the form $(\mathsf{q}_1.\alpha,\ \mathsf{q}_2.\alpha)$ is visited exactly once. □

Given $\langle\kappa\rangle$-terms u and v, we use $p(u,v)$ to denote a function depending on some parameters associated with u and v (that may be, for instance, $|u|$, $|v|$ or $c(u)$, $c(v)$) and such that, the time for solving the word problem over H for any pair of factors of the form $u(i,a)$ and $v(j,b)$ is in $O(p(u,v))$. Observe that such a function is not unique, but the results we state are valid for any such function. Then, summing up the time complexities of all the intermediate steps considered above, we have just proved the following result.

Algorithm 3.4

Input: two DRH-graphs $\mathcal{G}_1 = \langle V_1, \to_1, \mathsf{q}_1, \lambda_{1,\mathsf{H}}, \lambda_1 \rangle$ and $\mathcal{G}_2 = \langle V_2, \to_2, \mathsf{q}_2, \lambda_{2,\mathsf{H}}, \lambda_2 \rangle$
Output: logical value of "$\mathcal{G}_1 \sim \mathcal{G}_2$"

```
if q1 = ε then
    return logical value of "q2 = ε"
else if q1 or q2 is unvisited then
    mark q1 and q2 as visited
    if λ1,H(q1) = λ2,H(q2) and λ1(q1) = λ2(q2) then
        return logical value of "(G1)q1.0 ∼ (G2)q2.0 and (G1)q1.1 ∼ (G2)q2.1"
    else
        return False
    end if
else
    return logical value of "(λ1,H(q1), λ1(q1)) = (λ2,H(q2), λ2(q2))"
end if
```

Theorem 3.47. *Let* H *be a* κ*-recursive pseudovariety of groups and let* u *and* v *be* κ*-terms. Then, the equality of the pseudowords represented by* u *and* v *over* DRH *can be tested in* $O((p(u,v)+m)m\,|A|)$*-time, where* $m = \max\{|u|, |v|\}$. $\square$

Observe that, in general, the complexity of an algorithm for solving the κ-word problem modulo H should depend on the length of the intervening $\langle\kappa\rangle$-terms. It is not hard to see that the length of the factors $w(i,a)$ grows quadratically on $|w|$ (we prove it below in Corollary 3.52). Hence, it is expected that, at least in most of the cases, m belongs to $O(p(u,v))$. Consequently, the overall time complexity stated in Theorem 3.47 becomes $O(p(u,v)m\,|A|)$. Since we are doing the same approach as in [25], this result is somehow the expected one. Roughly speaking, this may be interpreted as the time complexity of solving the word problem in R, together with a word problem in H for each state, that is, for each DRH-factor of the involved pseudowords (recall Lemmas 2.32 and 3.20).

Just as a complement, we mention that another possible approach would be to transform the DRH-graph $\mathcal{G}(w)$ in an automaton in the classical sense, say $\mathcal{G}'(w)$, recognizing the language $\mathcal{L}(w)$ (recall Proposition 3.15). That is easily done (time linear on the number of states), by moving the labels of a state to the arrows leaving it. More precisely, the automaton $\mathcal{G}'(w)$ shares the set of states with $\mathcal{G}(w)$ and each non terminal state $\mathsf{q}(i,a)$ has two transitions:

$$\begin{aligned} \mathsf{q}(i,a).(0, \lambda_{\mathsf{H}}(\mathsf{q}(i,a)), \lambda(\mathsf{q}(i,a))) &= \mathsf{q}(i,1).0, \\ \mathsf{q}(i,a).(1, I, \lambda(\mathsf{q}(i,a))) &= \mathsf{q}(i,a).1. \end{aligned}$$

Then, we could use the results in the literature in order to minimize the automaton, obtaining a unique automaton representing each $\mathcal{R}$-class of $(\overline{\Omega}_A\mathsf{DRH})^I$. The unique issue in that approach is that the algorithms are usually prepared to deal with alphabets whose members may be compared in constant time. Hence, we should previously prepare the input automaton by renaming the subset of the alphabet $\Sigma \times (\overline{\Omega}_A\mathsf{H})^I \times A$, in which the labels of transitions are being considered. Let $p(u,v)$ and m have the same meaning has in Theorem 3.47. Since, a priori, we do not possess any information about the possible values for λ_{H}, that would take $O(p(u,v)(m\,|A|)^2)$-time (each time we rename an element of

$(\overline{\Omega}_A \mathsf{H})^I$ we should first verify whether we already encountered another element with the same value over H). Thereafter, we could use the linear time algorithm presented in [26], which works for this kind of automaton.[1] Thus, a rough upper bound for the complexity of this method is $O(p(u,v)m^2\,|A|^2)$, which although a bit worse, is still polynomial.

In particular, Theorem 3.47 establishes that a pseudovariety of groups H being κ-recursive suffices for DRH to have the same property. Then, it is natural to ask whether the converse holds. It is not hard to see that if DRH is κ-recursive, then so is H. In fact, that is a result of any pair of κ-terms u and v being equal over H if and only if the κ-terms $(uv)^{\omega}u$ and $(uv)^{\omega}v$ coincide over DRH (recall Corollary 2.20).

Proposition 3.48. *Let* H *be a pseudovariety of groups. If the pseudovariety* DRH *is κ-recursive, then so is* H. □

The following result gives us a family of pseudovarieties of the form DRH that are κ-recursive.

Corollary 3.49. *Let p be a prime number. If* $\mathsf{H} \supseteq \mathsf{G}_p$ *is a pseudovariety of groups, then* DRH *is κ-recursive.*

Proof. Since the free group is residually in G_p [29], it follows that it is also residually in any pseudovariety of groups containing G_p. Consequently, $\mathrm{FG}_A = \Omega_A^{\kappa}\mathsf{H}$ and so, H is κ-recursive. The final conclusion is a consequence of Theorem 3.47. □

3.5 An application: solving the word problem over DRG

Let us illustrate the previous results by considering the particular case of the pseudovariety DRG. By Theorem 3.47, the time complexity of our procedure for testing identities of κ-words modulo DRG depends on a certain parameter $p(\_,\_)$. In order to discover that parameter, we should first analyze the (length of the) projection onto $\Omega_A^{\kappa}\mathsf{G} = \mathrm{FG}_A$ of the elements of the form $w(i,a)$, where w is an κ-term.

Consider the alphabets $B_1 = (A \times \mathbb{N}) \uplus \{[^{-1},]^{-1}\}$ and $B_2 = (A \times \mathbb{N}) \uplus \{[^{-1}, [^{-2},]^{-1},]^{-2}\}$. Let x be a well-parenthesized word over B_2. The *expansion* of x is the well-parenthesized word $\mathsf{exp}(x)$ obtained by successively applying the rewriting rule $[^{-2}y]^{-2} \to [^{-1}y]^{-1}[^{-1}y]^{-1}$, whenever y is a well-parenthesized word. It is clear that $\mathsf{om}(x)$ and $\mathsf{om}(\mathsf{exp}(x))$ represent the same κ-word and that x is a well-parenthesized word over B_1. Further, we have the following.

Lemma 3.50. *Let x be a nonempty well-parenthesized word over B_1 and $i \in c_{\mathbb{N}}(x)$. Then, $\mathsf{t}_i(x)$ is a well-parenthesized word over B_2 and $|\mathsf{exp}(\mathsf{t}_i(x))| \leq \frac{1}{2}(|x|^2 + 2\,|x| - 3)$. Moreover, this upper bound is tight for all odd values of $|x|$.*

Proof. The fact that $\mathsf{t}_i(x)$ is a well-parenthesized word over B_2 follows immediately from the definition of t_i. To prove the inequality, we proceed by induction on $|x|$. If $x = a_i$, then $\mathsf{t}_i(x)$ is the empty word and so, the result holds. Let x be a well-parenthesized word over B_1 such that $|x| > 1$. The inequality holds clearly, unless x is of the form $x = [^{-1}y]^{-1}z$, with y and z well-parenthesized words

[1] The referred algorithm applies to disjoint cycle automata. This is our case, since all transitions of a cycle in a DRH-automaton must be 1.

over B_1, y nonempty and $i \in c_{\mathbb{N}}(y)$. In that case, we have $\mathsf{t}_i(x) = \mathsf{t}_i(y)[^{-2}y]^{-2}z$. Therefore, we have the following (in)equalities:

$$\begin{aligned}
|\mathsf{exp}(\mathsf{t}_i(x))| &= \left|\mathsf{exp}(\mathsf{t}_i(y)[^{-2}y]^{-2}z)\right| \\
&= |\mathsf{exp}(\mathsf{t}_i(y))| + \left|[^{-1}y]^{-1}[^{-1}y]^{-1}z\right| \\
&= |\mathsf{exp}(\mathsf{t}_i(y))| + 2(|y|+2) + |z| \\
&\leq \frac{1}{2}(|y|^2 + 2|y| - 3) + 2(|y|+2) + |z| \quad \text{by induction hypothesis on } y \\
&= \frac{1}{2}(|y|^2 + 2|y| - 3) + |y| + 2 + |x| \quad \text{because } |x| = |y| + |z| + 2 \\
&= \frac{1}{2}(((|y|+2)^2 - 7) + 2 + |x| \\
&\leq \frac{1}{2}(((|x|-2+2)^2 - 7) + 2 + |x| \quad \text{because the quadratic function } (|y|+2)^2 \text{ is strictly} \\
&\qquad\qquad\qquad\qquad \text{increasing for } |y| \geq 1 \text{ and } |y| \text{ is, at most, } |x|-2 \\
&= \frac{1}{2}(|x|^2 + 2|x| - 3).
\end{aligned}$$

Finally, let $\vec{x} = (a_1, \varepsilon, \varepsilon, \ldots)$, $\vec{y} = (\varepsilon, \varepsilon, \ldots)$, $\vec{q} = (-1, -1, \ldots)$ and $u_{2n+1} = \mu_n(\vec{x}, \vec{y}, \vec{q})$ (recall the notation used in Lemma 3.31). Then, u_{2n+1} is a well-parenthesized word over B_1 of length $2n+1$. Moreover, using Lemma 3.31, we may compute

$$\begin{aligned}
|\mathsf{exp}(\mathsf{t}_1(u_{2n+1}))| &= |\mathsf{exp}(\mathsf{t}_1(\mu_0(\vec{x},\vec{y},\vec{q})) \cdot \xi_0(\vec{x},\vec{y},\vec{q}) \cdot \xi_1(\vec{x},\vec{y},\vec{q}) \cdots \xi_{n-1}(\vec{x},\vec{y},\vec{q}))| \\
&= \left|\mathsf{exp}\left([^{-2}\mu_0(\vec{x},\vec{y},\vec{q})]^{-2} \cdot [^{-2}\mu_1(\vec{x},\vec{y},\vec{q})]^{-2} \cdots [^{-2}\mu_{n-1}(\vec{x},\vec{y},\vec{q})]^{-2}\right)\right| \\
&= \sum_{k=0}^{n-1} 2(|\mu_k(\vec{x},\vec{y},\vec{q})| + 2) \\
&= 2n^2 + 4n \quad \text{because } |\mu_k(\vec{x},\vec{y},\vec{q})| = 2k+1 \\
&= \frac{1}{2}(|u_{2n+1}|^2 + 2|u_{2n+1}| - 3) \quad \text{substituting } n \text{ by } \frac{1}{2}(|u_{2n+1}| - 1)
\end{aligned}$$

and the result follows. □

Also, as a straightforward consequence of the definition of p_a, the following holds.

Lemma 3.51. *Let x be a nonempty well-parenthesized word over B_1 and $a \in A$. Then, $\mathsf{p}_a(x)$ is also a well-parenthesized word over B_1 and $|\mathsf{exp}(\mathsf{p}_a(x))| = |\mathsf{p}_a(x)| \leq |x|$.* □

Given a well-parenthesized word x over B_2, we define the *linearization over A* of x to be the word $\mathsf{lin}(x)$ over the alphabet $A \uplus A^{-1}$ obtained by applying the rewriting rules $[^{-1}a_i]^{-1} \to a^{-1}$, $[^{-1}yz]^{-1} \to [^{-1}z]^{-1}[^{-1}y]^{-1}$ and $[^{-2}y]^{-2} \to [^{-1}y]^{-1}[^{-1}y]^{-1}$ to x (with $a_i \in c(x)$ and y, z well-parenthesized words). It is easy to see that $\mathsf{lin}(x) = \mathsf{lin}(\mathsf{exp}(x))$ and that if x is a well-parenthesized word over B_1, then $O(|\mathsf{lin}(x)|) = O(|x|)$. Consequently, we have the next result.

Corollary 3.52. *Let w be an κ-term and $(i,a) \in [0, |\overline{w}|[\, \times c_A(\overline{w})$. Then, $|\mathsf{lin}(\overline{w}(i,a))|$ belongs to $O(|w|^2)$.*

Proof. First, we note that, since the representation of w only makes use of multiplications and $(\omega-1)$-powers, $\overline{w}$ is a well-parenthesized word over B_1. Using Lemmas 3.50 and 3.51, it follows that $|\mathsf{exp}(\overline{w}(i,a))| \in O(|w|^2)$. On the other hand, we already observed that $\mathsf{lin}(\overline{w}(i,a)) = \mathsf{lin}(\mathsf{exp}(\overline{w}(i,a)))$ and $O(|\mathsf{lin}(\mathsf{exp}(\overline{w}(i,a))|) = O(|\mathsf{exp}(\overline{w}(i,a)|)$, resulting that $|\mathsf{lin}(\overline{w}(i,a))| \in O(|w|^2)$ as desired. $\square$

Now, we wish to compute $\mathsf{lin}(x)$, for a given well-parenthesized word over B_2. Recall the tree representation of κ-terms exemplified in Figure 3.8. We may recover, in linear time, such a tree representation for $\mathsf{om}(x)$, for a well-parenthesized word x over B_1. Furthermore, if we are given a well-parenthesized word over B_2, we may compute, also in linear time, a tree representation for $\mathsf{om}(\mathsf{exp}(x))$. That amounts to, whenever we have a factor of the form $[^{-2}y]^{-2}$ in x, to include twice a subtree representing $[^{-1}y]^{-1}$.

Example 3.53. Let $z = ((b^{\omega-2}\cdot a)\cdot c)\cdot((a\cdot b)\cdot a^{\omega-1})^{\omega-2}$ be the $\langle\kappa\rangle$-term obtained by substituting in the κ-term w represented in Figure 3.8 some of the $(\omega-1)$-powers by an $(\omega-2)$-power. Then, the tree representation of z is drawn in Figure 3.9.

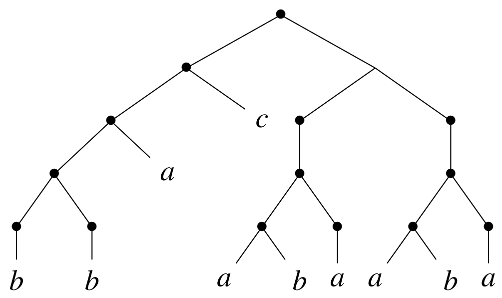

Fig. 3.9 Tree representation of $z = ((b^{\omega-2}\cdot a)\cdot c)\cdot((a\cdot b)\cdot a^{\omega-1})^{\omega-2}$.

On the other hand, since solving the word problem in FG_A (for words written over the alphabet $A\cup A^{-1}$) is a linear issue in the size of the input, by Corollary 3.52, we may take the parameter $p(u,v) = \max\{|u|^2, |v|^2\}$. Thus, we have proved the following.

Proposition 3.54. *The κ-word problem over* DRG *is decidable in $O(m^3\,|A|)$-time, where m is the maximum length of the inputs.* $\square$

Chapter 4

Reducibility of DRH with respect to certain classes of systems of equations

Since the pseudovariety DRH depends deeply on the pseudovariety of groups H, it is a natural question to ask how that dependency translates in terms of reducibility properties. More formally, let Δ and Ξ be two classes of finite systems of equations and σ an implicit signature. We would like to answer the following:

Question 1. *Does* H *being σ-reducible for Δ implies that* DRH *is σ-reducible for Ξ?*

Question 2. *Does* DRH *being σ-reducible for Δ implies that* H *is σ-reducible for Ξ?*

As some results are known about reducibility of pseudovarieties of groups, answers to Questions 1 and 2 allow us to deduce information about pseudovarieties DRH. Of course, ideally, the class Δ should not include "too many" systems while we aim the class Ξ to be as "big" as possible.

From now on, we fix a continuous homomorphism $\varphi : (\overline{\Omega}_A \mathsf{S})^I \to S^I$ into a finite semigroup S^I with a content function such that $\varphi^{-1}(I) = \{I\}$, a finite set of variables X and a map $\nu : X \to S$. We further fix an implicit signature σ.

4.1 Pointlike equations

Throughout this section, we shall assume that σ contains a non-explicit operation. In other words, that means that $\langle \sigma \rangle \neq \langle \{\_ \cdot \_\} \rangle$. Clearly, that is the case of the canonical implicit signature κ.

Propositions 2.4 and 2.9 motivate us to take for Ξ the class of all finite systems of pointlike equations. To guarantee that DRH is σ-reducible for Ξ, it suffices to suppose that H is σ-reducible for $\Delta = \Xi$ as well.

Theorem 4.1. *Let σ be an implicit signature containing a non-explicit operation, and assume that* H *is a pseudovariety of groups that is σ-reducible for finite systems of pointlike equations. Then, the pseudovariety* DRH *is also σ-reducible for finite systems of pointlike equations.*

Proof. Let $\mathcal{S} = \{x_{k,1} = \cdots = x_{k,n_k}\}_{k=1}^{N}$ be a finite system of pointlike equations in the set of variables X with constraints given by the pair (φ, ν). Without loss of generality, we may assume that, for all

$k,\ell \in \{1,\ldots,N\}$, with $k \neq \ell$, the subsets of variables $\{x_{k,1},\ldots,x_{k,n_k}\}$ and $\{x_{\ell,1},\ldots,x_{\ell,n_\ell}\}$ do not intersect. Further, with this assumption, we may also take $N = 1$. The general case is obtained by treating each system of equations $x_{k,1} = \cdots = x_{k,n_k}$ separately. Write $\mathcal{S} = \{x_1 = \cdots = x_n\}$ and suppose that the continuous homomorphism $\delta : \overline{\Omega}_X\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ is a solution modulo DRH of $\mathcal{S}$. To prove that $\mathcal{S}$ also has a solution in σ-words we argue by induction on $m = |c(\delta(x_1))|$.

If $m = 0$, then $\delta(x_i) = I$ for every $i = 1,\ldots,n$ and δ is already a solution in σ-words.

Suppose that $m > 0$ and that the statement holds for every system of pointlike equations with a smaller value of the parameter. Whenever the p-th iteration of the left basic factorization of $\delta(x_i)$ is nonempty, we write $\mathsf{lbf}_p(\delta(x_i)) = \delta(x_i)_p a_{i,p}$ and we let $\delta(x_i)'_p$ be such that

$$\delta(x_i) = \mathsf{lbf}_1(\delta(x_i))\cdots\mathsf{lbf}_p(\delta(x_i))\delta(x_i)'_p.$$

Notice that the uniqueness of left basic factorizations in $\overline{\Omega}_A\mathsf{DRH}$ entails the following properties

$$\begin{aligned} a_{1,p} &= \cdots = a_{n,p};\\ \delta(x_1)_p &=_{\mathsf{DRH}} \cdots =_{\mathsf{DRH}} \delta(x_n)_p;\\ \delta(x_1)'_p &=_{\mathsf{DRH}} \cdots =_{\mathsf{DRH}} \delta(x_n)'_p. \end{aligned} \tag{4.1}$$

If $\vec{c}(\delta(x_1)) \neq c(\delta(x_1))$, then we set $k = \ell = \min\{p \geq 1 \colon c(\delta(x_1)'_p) \subsetneqq c(\delta(x_1))\}$. Otherwise, since S is finite, there exist indices $k < \ell$ such that, for all $i = 1,\ldots,n$, we have

$$\varphi(\mathsf{lbf}_1(\delta(x_i))\cdots\mathsf{lbf}_k(\delta(x_i))) = \varphi(\mathsf{lbf}_1(\delta(x_i))\cdots\mathsf{lbf}_\ell(\delta(x_i))). \tag{4.2}$$

Let $\eta \in \langle\sigma\rangle$ be a non-explicit operation. Without loss of generality, we may assume that η is a unary operation. In particular, since S is finite, there is an integer M such that $\eta(s) = s^M$ for every $s \in S$. Then, equality (4.2) yields

$$\varphi(\delta(x_i)) = \varphi(\mathsf{lbf}_1(\delta(x_i))\cdots\mathsf{lbf}_k(\delta(x_i))\cdot\eta(\mathsf{lbf}_{k+1}(\delta(x_i))\cdots\mathsf{lbf}_\ell(\delta(x_i)))\delta(x_i)'_k). \tag{4.3}$$

Now, consider a new set of variables $X' = \{x_{i,p}, x'_i \colon i = 1,\ldots,n;\ p = 1,\ldots,\ell\}$ and a new system of pointlike equations

$$\mathcal{S}' = \begin{cases} \{x_{1,p} = \cdots = x_{n,p}\}_{p=1}^{\ell} \cup \{x'_1 = \cdots = x'_n\}, & \text{if } \vec{c}(\delta(x_1)) \neq c(\delta(x_1))\\ \{x_{1,p} = \cdots = x_{n,p}\}_{p=1}^{\ell}, & \text{if } \vec{c}(\delta(x_1)) = c(\delta(x_1)) \end{cases} \tag{4.4}$$

By (4.1), the continuous homomorphism $\delta' : \overline{\Omega}_{X'}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ assigning $\delta(x_i)_p$ to each variable $x_{i,p}$ and $\delta(x_i)'_\ell$ to each variable x'_i is a solution modulo DRH of $\mathcal{S}'$, with constraints given by (φ, ν'), where $\nu'(x_{i,p}) = \varphi(\delta(x_i)_p)$, and $\nu'(x'_i) = \varphi(\delta(x_i)'_k)$ ($i = 1,\ldots,n$ and $p = 1,\ldots,\ell$). Moreover, whatever is the system $\mathcal{S}'$ considered in (4.4), we decreased the induction parameter. By induction hypothesis, there exists a solution modulo DRH of $\mathcal{S}'$ in σ-words, say ε', keeping the values of the variables under φ. We distinguish between the case where $\vec{c}(\delta(x_1)) \neq c(\delta(x_1))$ and the case where $\vec{c}(\delta(x_1)) = c(\delta(x_1))$.

In the former, it is easy to check that the continuous homomorphism

$$\begin{aligned}\varepsilon : \overline{\Omega}_X \mathsf{S} &\to (\overline{\Omega}_A \mathsf{S})^I \\ x_i &\mapsto \varepsilon'(x_{i,1})a_{i,1}\cdots\varepsilon'(x_{i,\ell})a_{i,\ell}\varepsilon'(x_i')\end{aligned}$$

is a solution modulo DRH of $\mathcal{S}$. In the latter case, we consider the system of pointlike equations

$$\mathcal{S}_0 = \{x_1' = \cdots = x_n'\}.$$

From (4.1), it follows that δ' is a solution modulo H of $\mathcal{S}_0$. As we are taking for H a pseudovariety that is σ-reducible for systems of pointlike equations, there exists a solution modulo H of $\mathcal{S}_0$, say ε'', keeping the values of the variables under φ. Let $\varepsilon : \overline{\Omega}_X \mathsf{S} \to (\overline{\Omega}_A \mathsf{S})^I$ be given by

$$\varepsilon(x_i) = \varepsilon'(x_{i,1})a_{i,1}\cdots\varepsilon'(x_{i,k})a_{i,k}\cdot\eta(\varepsilon'(x_{i,k+1})a_{i,k+1}\cdots\varepsilon'(x_{i,\ell})a_{i,\ell})\varepsilon''(x_i').$$

Since ε' is a solution modulo DRH of $\mathcal{S}'$, η is non-explicit, and we are assuming that the semigroup S has a content function, it follows that, for all $i,j \in \{1,\ldots,n\}$, the pseudowords $\varepsilon(x_i)$ and $\varepsilon(x_j)$ are $\mathcal{R}$-equivalent modulo DRH. On the other hand, for all $i,j \in \{1,\ldots,n\}$, the following equalities are valid in H:

$$\begin{aligned}\varepsilon(x_i) &= \varepsilon'(x_{i,1})a_{i,1}\cdots\varepsilon'(x_{i,k})a_{i,k}\cdot\eta(\varepsilon'(x_{i,k+1})a_{i,k+1}\cdots\varepsilon'(x_{i,\ell})a_{i,\ell})\varepsilon''(x_i') \\ &= \varepsilon'(x_{j,1})a_{j,1}\cdots\varepsilon'(x_{j,k})a_{j,k}\cdot\eta(\varepsilon'(x_{j,k+1})a_{j,k+1}\cdots\varepsilon'(x_{j,\ell})a_{j,\ell})\varepsilon''(x_j') \\ &= \varepsilon(x_j).\end{aligned}$$

The second equality holds because ε' and ε'' are solutions modulo H of $\mathcal{S}'$ and $\mathcal{S}_0$, respectively. Therefore, Lemma 2.32 yields that DRH satisfies $\varepsilon(x_i) = \varepsilon(x_j)$. It remains to verify that the given constraints are still satisfied. But that is straightforwardly implied by (4.3). □

Remark 4.2. We observe that the construction performed in the proof of the previous theorem not only gives a solution modulo DRH in σ-terms of the original pointlike system of equations, but it also provides a solution keeping the cumulative content of each variable.

As a consequence of Proposition 2.9 and Theorem 4.1, we have the following.

Corollary 4.3. *If a pseudovariety of groups* H *is* σ*-reducible with respect to the equation* $x = y$*, then the pseudovariety* DRH *is* σ*-equational.* □

As far as we are aware, all known examples of pseudovarieties of groups that are σ-reducible with respect to systems of pointlike equations are also σ-reducible. For that reason, for now, we skip such examples, since they illustrate stronger results in the next section. We just point out the case of the pseudovariety Ab (recall Theorem 2.11). It is interesting to observe that, although $\overline{\mathsf{Ab}}$ is not κ-equational [20, Theorem 3.1], by Corollary 4.3 the pseudovariety $\mathsf{DRAb} = \mathsf{DRG} \cap \overline{\mathsf{Ab}}$ is.

On the other hand, taking into account the results of the previous chapter, we also have the following.

Corollary 4.4. *If* H *is a pseudovariety of groups that is* κ*-tame with respect to finite systems of pointlike equations, then so is* DRH.

Proof. Let H be a pseudovariety of groups κ-tame with respect to systems of pointlike equations. Then, Theorem 3.47 yields that DRH is κ-recursive and Theorem 4.1 yields that DRH is κ-reducible with respect to systems of pointlike equations. Hence, the pseudovariety DRH is κ-tame for systems of pointlike equations. □

Since, by Theorem 2.10, κ-tame pseudovarieties are hyperdecidable (with respect to a certain class $\mathcal{C}$), another application comes from Proposition 2.6 and Theorem 2.7.

Corollary 4.5. *Let* H *be a pseudovariety of groups that is* κ*-tame with respect to systems of pointlike equations. Then,*

- $\mathsf{DRH} * \mathsf{V}$ *is strongly decidable for every order-computable pseudovariety* V*;*
- $\mathsf{DRH} \vee \mathsf{V}$ *is strongly decidable for every order-computable pseudovariety* V. □

Still, we were not able to answer Question 2 for an arbitrary "nice" Δ. Nevertheless, we may prove that if we take for Δ the class of all finite systems of graph equations, then the answer becomes positive. We do not include that result here, since it appears as a particular case of a result in the next section, namely Proposition 4.15.

4.2 Graph equations

With the aim of proving tameness, we now let Ξ be the class of all systems of graph equations. Results on tameness of DRH also allow us to know more about pseudovarieties of the form $\mathsf{V} * \mathsf{DRH}$ and $\mathsf{DRH} \vee \mathsf{V}$ for certain pseudovarieties V (recall Theorems 2.5 and 2.7). We prove that, for an implicit signature σ containing a non-explicit operation, if H is a σ-reducible pseudovariety of groups, then so is DRH. To this end, we drew inspiration from [9]. Moreover, we assert the converse statement, which holds for every σ, thus answering Question 2.

Henceforth, we fix a finite graph $\Gamma = V \uplus E$ and a solution $\delta : \overline{\Omega}_\Gamma \mathsf{S} \to (\overline{\Omega}_A \mathsf{S})^I$ modulo DRH of $\mathcal{S}(\Gamma)$ such that, for every $x \in \Gamma$ the pseudoword $\delta(x)$ belongs to the clopen subset K_x of $(\overline{\Omega}_A \mathsf{S})^I$.

Let y be an edge of Γ, and let $x = \alpha(y)$ and $z = \omega(y)$. If $c(\delta(y)) \nsubseteq \vec{c}(\delta(x))$ then, by Corollary 2.14, we have unique factorizations $\delta(y) = u_y a v_y$ and $\delta(z) = u_z a v_z$ such that $c(u_y) \subseteq \vec{c}(\delta(x))$, $a \notin \vec{c}(\delta(x))$ and the pseudovariety DRH satisfies both $\delta(x) u_y = u_z$ and $v_y = v_z$. We refer to these factorizations as *direct* DRH*-splittings associated with the edge* y and we say that a is the corresponding *marker*. We call *direct* DRH*-splitting points* the triples (u_y, a, v_y) and (u_z, a, v_z).

The first remark spells out the relationship between the notion of a DRH-splitting factorization defined above and the notion of a splitting factorization in the context of [9] (in [9], a splitting factorization is defined as being an R-splitting factorization). It is a consequence of Corollary 2.14(b) applied to the pseudovariety DRH and to the pseudovariety R.

Remark 4.6. Let $y \in E$ be such that $c(\delta(y)) \nsubseteq \vec{c}(\delta(\alpha(y)))$. Consider factorizations $\delta(y) = u_y a v_y$ and $\delta(\omega(y)) = u_z a v_z$, with $c(u_y) \subseteq \vec{c}(\delta(\alpha(y)))$ and $a \notin \vec{c}(\delta(\alpha(y)))$, such that $\delta(\alpha(y)) u_y =_{\mathsf{DRH}} u_z$, as above. Then, these factorizations are direct R-splittings (note that δ is also a solution modulo

R of $\mathcal{S}(\Gamma)$ and so, it makes sense to refer to R-splitting factorizations) if and only if they are direct DRH-splittings.

We also define the *indirect* DRH-*splitting points* as follows. Let $t \in \Gamma$ and suppose that we have a factorization $\delta(t) = u_t a v_t$, with $a \notin \vec{c}(u_t)$. Then, one of the following three situations may occur.

- If there is an edge $y \in E$ such that $\alpha(y) = t$ and $\omega(y) = z$, then there is also a factorization $\delta(z) = u_z a v_z$ with DRH satisfying $u_t = u_z$ and $v_t\delta(y) = v_z$. In fact, this is a consequence of the pseudoidentity $\delta(t)\delta(y) = \delta(z)$ modulo DRH, which holds for every edge $t \xrightarrow{y} z$ in Γ.
- Similarly, if there is an edge $y \in E$ such that $\alpha(y) = x$ and $\omega(y) = t$ (and so, DRH satisfies $\delta(x)\delta(y) = \delta(t)$), then the factorization of $\delta(t)$ yields either a factorization $\delta(x) = u_x a v_x$ such that DRH satisfies $u_x = u_t$ and $v_x\delta(y) = v_t$, or a factorization $\delta(y) = u_y a v_y$ such that DRH satisfies $\delta(x)u_y = u_t$ and $v_y = v_t$.
- On the other hand, if t is itself an edge, say $\alpha(t) = x$ and $\omega(t) = z$, and if $\delta(x)u_t a$ is an end-marked pseudoword, then the factorization of $\delta(t)$ determines a factorization $\delta(z) = u_z a v_z$, such that DRH satisfies $\delta(x)u_t = u_z$ and $v_t = v_z$.

These considerations make clear the possible propagation of the DRH-direct splitting points. If the mentioned factorization of $\delta(t)$ comes from a DRH-(in)direct splitting factorization obtained through the successive factorization of the values of edges and vertices under δ in the way described above, then we say that each of the triples (u_x, a, x_x), (u_y, a, v_y) and (u_z, a, v_z) is an indirect DRH-splitting point *induced by the (in)direct* DRH-*splitting point* (u_t, a, v_t). In Figure 4.1 we schematize a propagation of splitting points arising from the direct DRH-splitting point associated with the edge y_1. We represent pseudowords by boxes, markers of splitting points by dashed lines and factors with the same value modulo DRH with the same filling pattern.

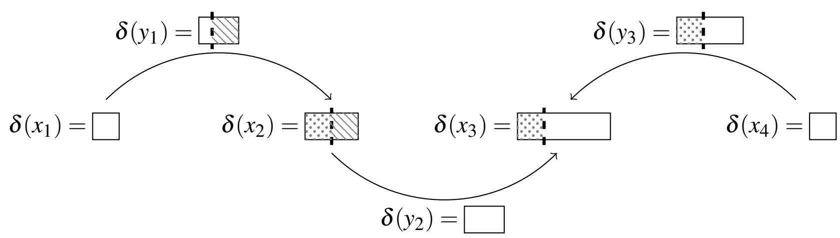

Fig. 4.1 Example of propagation of a direct splitting point.

Yet again, we obtain a nice relationship between the indirect DRH-splitting points just defined and the indirect splitting points introduced in [9] (which are the indirect R-splitting points). The reason is precisely the same as in Remark 4.6, together with the definition of indirect splitting points.

Remark 4.7. Let $t_0 \in \Gamma$ and $\delta(t_0) = u_0 a v_0$ be a direct R-splitting factorization and consider a subset $\{(u_i, a, v_i)\}_{i=1}^n \subseteq (\overline{\Omega}_A \mathsf{S})^I \times A \times (\overline{\Omega}_A \mathsf{S})^I$. Then, the following are equivalent:

(a) (u_i, a, v_i) is an indirect R-splitting point induced by (u_{i-1}, a, v_{i-1}), for every $i = 1, \ldots, n$;

(b) (u_i, a, v_i) is an indirect DRH-splitting point induced by (u_{i-1}, a, v_{i-1}), for every $i = 1, \ldots, n$.

The following lemma ensures that a direct R-splitting point does not propagate infinitely many times.

Lemma 4.8 ([9, Lemma 5.14])**.** *Given a solution δ over* R *of a system of graph equations, there is only a finite number of splitting points in the values of variables under δ.*

As an immediate consequence of Lemma 4.8 and of the relationship between (in)direct R-splitting points and (in)direct DRH-splitting points made explicit in Remarks 4.6 and 4.7 we have the following:

Corollary 4.9. *Given a solution δ over* DRH *of a system of graph equations, there is only a finite number of splitting points in the values of variables under δ.* $\square$

Taking into account Remarks 4.6 and 4.7, from now on we say (in)direct splitting point (respectively, factorization) instead of (in)direct DRH-splitting point (respectively, factorization).

Let Γ be a finite graph and consider the system of equations $\mathcal{S}(\Gamma)$. For each variable $x \in \Gamma$, let $\{(u_{x,i}, a_{x,i}, v_{x,i})\}_{i=1}^{m_x}$ be the (finite) set of splitting points of $\delta(x)$. By definition, each pseudoword $u_{x,i}a_{x,i}$ is an end-marked prefix of $\delta(x)$. By Proposition 2.21, we may assume, without loss of generality, the following relations:

$$u_{x,1} >_{\mathcal{R}} u_{x,2} >_{\mathcal{R}} \cdots >_{\mathcal{R}} u_{x,m_x}.$$

Hence, we have a reduced factorization (because the first letter of $\delta(x)[\alpha_{u_{x,k}}, \alpha_{u_{x,k+1}}[$ is $a_{x,k}$)

$$\delta(x) = \delta(x)[0, \alpha_{u_{x,1}}[\cdot \delta(x)[\alpha_{u_{x,1}}, \alpha_{u_{x,2}}[\cdots \delta(x)[\alpha_{u_{x,m_x-1}}, \alpha_{u_{x,m_x}}[\cdot \delta(x)[\alpha_{u_{x,m_x}}, \alpha_{\delta(x)}[\tag{4.5}$$

induced by the splitting points of $\delta(x)$. For each variable $x \in V$, we write the reduced factorization in (4.5) as $\delta(x) = w_{x,1} \cdot w_{x,2} \cdots w_{x,n_x}$ and, for each variable $y \in E$, we write that factorization as $\delta(y) = w_{y,0}w_{y,1} \cdots w_{y,n_y}$. Observe that, for $x \in V$, we have the equality $n_x = m_x + 1$, while for $y \in E$, we have $n_y = m_y$. Although this notation may not seem coherent, it is justified by property (c) of Lemma 4.10.

Lemma 4.10. *Let $xy = z$ be an equation of $\mathcal{S}(\Gamma)$. Using the above notation, the following holds:*

(a) $n_x + n_y = n_z$;

(b) DRH *satisfies* $\begin{cases} w_{x,k} = w_{z,k}, & \text{for } k = 1, \ldots, n_x - 1; \\ w_{x,n_x} w_{y,0} = w_{z,n_x}; \\ w_{y,k} = w_{z,n_x+k}, & \text{for } k = 1, \ldots, n_y; \end{cases}$

(c) $c(w_{y,0}) \subseteq \vec{c}(w_{x,n_x})$;

(d) *each of the following products is reduced:*

$$\begin{aligned} &w_{x,k} \cdot w_{x,k+1}\ (k = 1, \ldots, n_x - 1); \\ &(w_{x,n_x} w_{y,0}) \cdot w_{y,1}; \\ &w_{z,k} \cdot w_{z,k+1}\ (k = 1, \ldots, n_z - 1). \end{aligned}$$

Proof. As we already observed, the number of splitting points of $\delta(z)$ is $m_z = n_z - 1$. We distinguish between two situations.

- If $c(\delta(y)) \nsubseteq \vec{c}(\delta(x))$, then there are two direct splitting factorizations given by $\delta(y) = u_y a v_y$ and $\delta(z) = u_z a v_z$. So, by definition, the inclusion $c(u_y) \subseteq \vec{c}(\delta(x))$ holds. We notice that any other splitting point of $\delta(y)$, say (u'_y, b, v'_y), is necessarily induced by a splitting point of $\delta(z)$, say (u'_z, b, v'_z). Moreover, since the product $(\delta(x)u'_y) \cdot bv'_y$ is reduced (because so is $u'_z \cdot (bv'_z)$ and DRH satisfies $\delta(x)u'_y = u'_z$), the pseudoword u_y is a prefix of u'_y. On the other hand, the set of all splitting points of $\delta(z)$ induces a factorization of the pseudoword $\delta(x)\delta(y)$, namely,

$$\begin{aligned}\delta(x)\delta(y) &= (\delta(x)\delta(y))[0, \alpha_{w_{z,1}}[\cdot (\delta(x)\delta(y))[\alpha_{w_{z,1}}, \alpha_{w_{z,1}w_{z,2}}[\cdots \\ &\quad \cdot (\delta(x)\delta(y))[\alpha_{w_{z,1}\cdots w_{z,n_z-1}}, \alpha_{w_{z,1}\cdots w_{z,n_z}}[\\ &= w'_1 \cdot w'_2 \cdots w'_{n_z},\end{aligned} \tag{4.6}$$

  with DRH satisfying the pseudoidentity

$$w_{z,k} = (\delta(x)\delta(y))[\alpha_{w_{z,1}\cdots w_{z,k-1}}, \alpha_{w_{z,1}\cdots w_{z,k}}[= w'_k,$$

  for $k = 1, \ldots, n_z$. Of course, for each $k = 1, \ldots, n_x - 1$, the prefix $w'_1 \cdots w'_k$ of $\delta(x)\delta(y)$ corresponds to the first component of one of the splitting points of $\delta(x)$ (which is either induced by one of the splitting points of $\delta(z)$ or it induces a splitting point in $\delta(z)$). More specifically, the pseudoidentity $w_{z,k} = w'_k = w_{x,k}$ is valid in DRH. From the observation above, we also know that the first components of the indirect splitting points of $\delta(y)$ have u_y as a prefix. Therefore, we have $u_y = w_{y,0}$, the factor $w'_{n_x} = w_{z,n_x}$ coincides with $w_{x,n_x} w_{y,0}$ modulo DRH, and $c(w_{y,0}) = c(u_y) \subseteq \vec{c}(\delta(x)) = \vec{c}(w_{x,n_x})$. It also follows that $w'_{n_x+k} = w_{z,n_x+k} = w_{y,k}$ modulo DRH, for $k = 1, \ldots, n_y$. We just proved (b), (c) and (d). Finally, part (a) results from counting the involved factors in both sides of (4.6).

- If $c(\delta(y)) \subseteq \vec{c}(\delta(x))$, then $\delta(y)$ has no direct splitting points. As y is an edge, an indirect splitting point of $\delta(y)$ must be induced by some splitting point of $\delta(z)$. Suppose that (u_z, a, v_z) is a splitting point of $\delta(z)$ that induces a splitting point in $\delta(y)$, say (u_y, a, v_y). Then, we would have a reduced product $(\delta(x)u_y) \cdot (av_y)$, which contradicts the assumption $c(\delta(y)) \subseteq \vec{c}(\delta(x))$. Therefore, the pseudoword $\delta(y)$ has no splitting points at all. With the same kind of argument as the one above, we may derive the claims (a)–(d). □

Now, write $\mathcal{S}(\Gamma) = \{x_i y_i = z_i\}_{i=1}^N$. Note that $y_j \notin \{x_i, z_i\}$ for all i, j. We let $\mathcal{S}_1$ be the system of equations containing, for each $i = 1, \ldots, N$, the following set of equations:

$$\begin{aligned}(x_i)_k &= (z_i)_k, \text{ for } k = 1, \ldots, n_{x_i} - 1; \\ (x_i)_{n_{x_i}} y_{i,0} &= (z_i)_{n_{x_i}}; \\ y_{i,k} &= (z_i)_{n_{x_i}+k}, \text{ for } k = 1, \ldots, n_{y_i}.\end{aligned} \tag{4.7}$$

In the system $\mathcal{S}_1$, we are assuming that $(x_i)_k$ and $(x_j)_k$ (respectively, and $(z_j)_k$) represent the same variable whenever so do x_i and x_j (respectively, and z_j). By Lemma 4.10, it is clear that each solution modulo DRH of $\mathcal{S}_1$ yields a solution modulo DRH of $\mathcal{S}(\Gamma)$ and conversely. We next prove that, for a σ-reducible pseudovariety of groups H, if $\mathcal{S}_1$ has a solution modulo DRH, then it has a solution modulo DRH given by σ-words, thus concluding that the same happens with $\mathcal{S}(\Gamma)$. Before that, we establish the following.

Proposition 4.11. *Let σ be an implicit signature that contains a non-explicit operation. Let* H *be a σ-reducible pseudovariety of groups and let $\Gamma = V \uplus E$ be a finite graph. Suppose that there exists a solution $\delta : \overline{\Omega}_\Gamma \mathsf{S} \to (\overline{\Omega}_A \mathsf{S})^I$ modulo* DRH *of $\mathcal{S}(\Gamma)$ such that:*

(a) $\vec{c}(\delta(x)) \neq \emptyset$, for every vertex $x \in V$

(b) $c(\delta(y)) \subseteq \vec{c}(\delta(\alpha(y)))$, for every edge $y \in E$.

Then, $\mathcal{S}(\Gamma)$ has a solution modulo DRH *in σ-words, say ε, such that $\varphi(\varepsilon(x)) = \varphi(\delta(x))$, for all $x \in \Gamma$.*

Proof. Without loss of generality, we may assume that Γ has only one connected component (when disregarding the directions of the arrows). Otherwise, we may treat each component separately. Because of the hypothesis (b), the pseudowords $\delta(\alpha(y))$ and $\delta(\omega(y))$ are $\mathcal{R}$-equivalent modulo DRH for every edge $y \in E$. Since we are assuming that all vertices of Γ are in the same connected component, it follows that for all $x, z \in V$, the pseudowords $\delta(x)$ and $\delta(z)$ are $\mathcal{R}$-equivalent modulo DRH. Fix a variable $x_0 \in V$ and let u_0 be an accumulation point of $(\mathsf{lbf}_1(\delta(x_0)) \cdots \mathsf{lbf}_m(\delta(x_0)))_{m \geq 1}$ in $\overline{\Omega}_A \mathsf{S}$. Since, in DRH, the pseudowords u_0 and $\delta(x_0)$ are $\mathcal{R}$-equivalent, for each $x \in V$ there is a factorization $\delta(x) = u_x v_x$ (with v_x possibly empty) such that $c(v_x) \subseteq \vec{c}(u_x)$ and $u_x =_{\mathsf{DRH}} u_0$.

Consider the set $\widehat{V} = \{\widehat{x} \colon x \in V\}$ with $|V|$ distinct variables, disjoint from Γ, the system of equations $\mathcal{S}_0 = \{\widehat{x} = \widehat{z} \colon x, z \in V\}$ with variables in $\widehat{V}$, and let

$$\begin{aligned} \delta_0 : \overline{\Omega}_{\widehat{V} \uplus \Gamma} \mathsf{S} &\to (\overline{\Omega}_A \mathsf{S})^I \\ \widehat{x} &\mapsto u_x, \quad \text{if } \widehat{x} \in \widehat{V}; \\ x &\mapsto v_x, \quad \text{if } x \in V; \\ y &\mapsto \delta(y), \quad \text{otherwise.} \end{aligned}$$

By construction, the homomorphism δ_0 is a solution modulo DRH of $\mathcal{S}_0$ which is also a solution modulo H of $\mathcal{S}(\Gamma)$. Hence, on the one hand, Theorem 4.1 together with Remark 4.2 yield a solution $\varepsilon_0 : \overline{\Omega}_{\widehat{V}} \mathsf{S} \to \overline{\Omega}_A \mathsf{S}$ modulo DRH in σ-words of $\mathcal{S}_0$ such that

$$\begin{aligned} \varphi(\varepsilon_0(\widehat{x})) &= \varphi(\delta_0(\widehat{x})) = \varphi(u_x), \\ \vec{c}(\varepsilon_0(\widehat{x})) &= \vec{c}(\delta_0(\widehat{x})), \end{aligned}$$

for every $\widehat{x} \in \widehat{V}$. On the other hand, the fact that H is σ-reducible implies that there is a solution $\varepsilon' : \overline{\Omega}_\Gamma \mathsf{S} \to (\overline{\Omega}_A \mathsf{S})^I$ modulo H of $\mathcal{S}(\Gamma)$ given by σ-words satisfying

$$\varphi(\varepsilon'(x)) = \varphi(\delta_0(x)),$$

for every $x \in \Gamma$. Thus, we take $\varepsilon : \overline{\Omega}_\Gamma \mathsf{S} \to (\overline{\Omega}_A \mathsf{S})^I$ to be the continuous homomorphism defined by $\varepsilon(x) = \varepsilon_0(\widehat{x})\varepsilon'(x)$ if $x \in V$, and $\varepsilon(y) = \varepsilon'(y)$ otherwise. Taking into account that S has a content function, we may use Lemma 2.32 to deduce that ε is a solution modulo DRH of $\mathcal{S}(\Gamma)$ in σ-words. Additionally, the initial constraints for the variables of Γ are satisfied:

$$\begin{aligned}
\varphi(\varepsilon(x)) &= \begin{cases} \varphi(\varepsilon_0(\widehat{x})\varepsilon'(x)), & \text{if } x \in V \\ \varphi(\varepsilon'(x)), & \text{if } x \in E \end{cases} \\
&= \begin{cases} \varphi(\delta_0(\widehat{x}))\varphi(\delta_0(x)), & \text{if } x \in V \\ \varphi(\delta_0(x)), & \text{if } x \in E \end{cases} \\
&= \begin{cases} \varphi(u_x)\varphi(v_x), & \text{if } x \in V \\ \varphi(\delta(x)), & \text{if } x \in E \end{cases} \\
&= \varphi(\delta(x)).
\end{aligned}$$

This proves the result. □

Lemma 4.12. *Let $\mathcal{S}_1$ be the system of equations* (4.7) *with variables in X_1 and let $\delta_1 : \overline{\Omega}_{X_1}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ be its solution modulo* DRH. *Suppose that the implicit signature σ contains a non-explicit operation. If the pseudovariety* H *is σ-reducible, then $\mathcal{S}_1$ has a solution modulo* DRH *in σ-words.*

Proof. Analyzing the equations in (4.7), we easily conclude that there are no variables occurring simultaneously in two of the rows. Therefore, the system $\mathcal{S}_1$ can be thought as a system of pointlike equations $\mathcal{S}_2$ together with a system of graph equations $\mathcal{S}_3$ such that the conditions (a) and (b) of Proposition 4.11 hold and none of the variables occurring in $\mathcal{S}_2$ occurs in $\mathcal{S}_3$. Note that we are also including in $\mathcal{S}_2$ the equations in the second row of (4.7) such that the cumulative content of $\delta_1(x_i)_{n_{x_i}}$ is empty.

By Theorem 4.1 the system $\mathcal{S}_2$ has a solution modulo DRH in σ-words, while by Proposition 4.11 the system $\mathcal{S}_3$ has a solution modulo DRH in σ-words. Therefore, the intended solution for $\mathcal{S}_1$ also exists. □

We just proved the announced result.

Theorem 4.13. *When σ is an implicit signature containing a non-explicit operation, the pseudovariety* DRH *is σ-reducible if so is* H. □

We recall that, by Theorem 2.11, for every nontrivial extension closed pseudovariety of groups H, there is an implicit signature $\sigma(\mathsf{H}) \supseteq \kappa$ that turns H into a $\sigma(\mathsf{H})$-reducible pseudovariety. For instance, G_p and G_{sol} are both extension closed. Thus, DRG_p and DRG_{sol} are both σ-reducible for suitable signatures σ.

Yet again, using Theorem 4.13, some decidability properties may be deduced from the knowledge of κ-tameness of a pseudovariety of groups H.

Corollary 4.14. *Let* H *be a κ-tame pseudovariety of groups. Then,*

- DRH *is κ-tame;*
- $\mathsf{V} * \mathsf{DRH}$ *is decidable for every decidable pseudovariety* V *with finite rank, that contains the Brandt semigroup B_2;*

- $\mathsf{V} \vee \mathsf{DRH}$ *is hyperdecidable for every order-computable pseudovariety* V.

Proof. The first item follows immediately from Theorems 3.47 and 4.13. The last two assertions result from the first and from Theorem 2.10, together with Theorem 2.5 (for the second item) and with Theorem 2.7 (for the last). $\square$

We further prove that the converse of Theorem 4.13 also holds.

Proposition 4.15. *Let* H *be a pseudovariety of groups such that the pseudovariety* DRH *is* σ*-reducible. Then, the pseudovariety* H *is also* σ*-reducible.*

Proof. Let $\Gamma = V \uplus E$ be a graph such that $\mathcal{S}(\Gamma)$ admits $\delta : \overline{\Omega}_\Gamma\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ as a solution modulo H. We consider a new graph $\widehat{\Gamma} = \widehat{V} \uplus \widehat{E}$, where $\widehat{V} = \{\widehat{v} \colon v \in V\} \uplus \{v_0\}$ and $\widehat{E} = V \uplus E$. The functions $\widehat{\alpha}$ and $\widehat{\omega}$ of $\widehat{\Gamma}$ are given by $\widehat{\alpha}(v) = v_0$ and $\widehat{\omega}(v) = \widehat{v}$, for all $v \in V$ and by $\widehat{\alpha}(e) = \widehat{v}_1$ and $\widehat{\omega}(e) = \widehat{v}_2$ whenever $e \in E$ and $(\alpha(e), \omega(e)) = (v_1, v_2)$. The relationship between the graphs Γ and $\widehat{\Gamma}$ is depicted in Figure 4.2. Let $u \in \overline{\Omega}_A\mathsf{S}$ be a regular pseudoword modulo DRH such that $c(\delta(x)) \subseteq \vec{c}(u)$ for

$$\begin{tikzcd} v_1 \arrow[r, bend left, "e"] & v_2 \end{tikzcd} \qquad \begin{tikzcd} \widehat{v}_1 \arrow[rr, bend left, "e"] & & \widehat{v}_2 \\ & v_0 \arrow[ul, "v_1"] \arrow[ur, "v_2"'] & \end{tikzcd}$$

Fig. 4.2 On the left, an edge of Γ; on the right, the corresponding edges of $\widehat{\Gamma}$.

all $x \in \Gamma$. We take $\delta' : \overline{\Omega}_{\widehat{\Gamma}}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ to be the continuous homomorphism defined by $\delta'(e) = \delta(e)$, if $e \in E$; $\delta'(v) = \delta(v)$ and $\delta'(\widehat{v}) = u\delta(v)$, if $v \in V$; and $\delta'(v_0) = u$. Then, Lemma 2.32 combined with the fact that δ is a solution modulo H of $\mathcal{S}(\Gamma)$ imply that δ' is a solution modulo DRH of $\mathcal{S}(\widehat{\Gamma})$. Thus, since DRH is σ-reducible, there exists a solution in σ-words $\varepsilon : \overline{\Omega}_{\widehat{\Gamma}}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ modulo DRH of $\mathcal{S}(\widehat{\Gamma})$. In particular, for each edge $e \in E$ such that $\alpha(e) = v_1$ and $\omega(e) = v_2$, we have that $v_0v_1 = \widehat{v}_1$, $\widehat{v}_1e = \widehat{v}_2$, and $v_0v_2 = \widehat{v}_2$ are equations of $\mathcal{S}(\widehat{\Gamma})$. Therefore, the equalities

$$\varepsilon(v_0v_1e) = \varepsilon(\widehat{v}_1e) = \varepsilon(\widehat{v}_2) = \varepsilon(v_0v_2)$$

hold in DRH. Hence, H satisfies $\varepsilon(v_1e) = \varepsilon(v_2)$ and so, we may conclude that the restriction of ε to $\overline{\Omega}_\Gamma\mathsf{S}$ is a solution in σ-words modulo H of $\mathcal{S}(\Gamma)$. $\square$

Combined with Proposition 4.15, the results in the literature supply a family of pseudovarieties DRH that are not κ-reducible. Namely, DRG_p and DRH for every proper non locally finite subpseudovariety H of Ab (recall Theorem 2.11).

4.3 Idempotent pointlike equations

Theorem 2.8 provides a sufficient criterion for decidability of pseudovarieties of the form $\mathsf{V} \circledast \mathsf{DRH}$, whenever V is a decidable pseudovariety. With that fact in mind, we take for Ξ the class of all systems of idempotent pointlike equations. Although we have been unable to answer positively Question 1 for any $\Delta \subseteq \Xi$, we prove that such an answer is achieved by taking for Δ a still "satisfactory" class of systems, namely the class of all systems of graph equations. More precisely, we prove that, for an implicit signature σ satisfying certain conditions, if H is a σ-reducible pseudovariety of groups, then DRH is σ-reducible with respect to systems of idempotent pointlike equations.

In order to make the expression "reducible for systems of graph equations" more embracing, we first introduce a definition.

Definition 4.16. *Let V be a pseudovariety and $\mathcal{S}$ a finite system of equations in the set of variables X with certain constraints. We say that $\mathcal{S}$ is* V-equivalent to a system of graph equations *if there exists a graph Γ such that $X \subseteq \Gamma$ and such that every solution modulo V of $\mathcal{S}$ can be extended to a solution modulo V of $\mathcal{S}(\Gamma)$ (the constraints for the variables of $X \subseteq \Gamma$ are those given by the system $\mathcal{S}$). Moreover, whenever δ is a solution modulo V of $\mathcal{S}(\Gamma)$, the restriction $\delta|_{\overline{\Omega}_X\mathsf{S}}$ is a solution modulo V of $\mathcal{S}$. Each graph Γ with that property is said to be an* $\mathcal{S}$-graph *and we say that $\mathcal{S}$ is* V-equivalent to $\mathcal{S}(\Gamma)$ *for every $\mathcal{S}$-graph Γ.*

It is immediate from the definition that any σ-reducible pseudovariety V is σ-reducible for systems of equations that are V-equivalent to a system of graph equations. In the next few results we exhibit some systems of equations that are H-equivalent to a system of graph equations (for a pseudovariety of groups H).

Lemma 4.17. *Consider the system consisting of a single equation $\mathcal{S} = \{x_1 w_1 \cdots x_n w_n x_{n+1} = 1\}$, where x_i is a variable with $x_i \neq x_j$ whenever $i \neq j$, $\{w_i\}_{i=1}^n \subseteq A^*$, and the constraint of the variable x_i is given by the clopen subset $K_i \subseteq (\overline{\Omega}_A\mathsf{S})^I$. Then, for every pseudovariety of groups H, the system $\mathcal{S}$ is H-equivalent to a system of graph equations.*

Proof. We consider the finite graph $\Gamma = V \uplus E$, where the set of vertices and edges are, respectively, given by

$$\begin{aligned}
V &= \{y_i, z_i \colon i = 1, \dots, n+1\},\\
E &= \{x_0\} \uplus \{x_i \colon i = 1, \dots, n+1\} \uplus \{w_i \colon i = 1, \dots, n\}.
\end{aligned}$$

To define the mappings α and ω, we take

$$\begin{aligned}
(\alpha(x_0), \omega(x_0)) &= (y_1, z_{n+1});\\
(\alpha(x_i), \omega(x_i)) &= (y_i, z_i), \qquad \text{for } i = 1, \dots, n+1;\\
(\alpha(w_i), \omega(w_i)) &= (z_i, y_{i+1}), \qquad \text{for } i = 1, \dots, n\ ;
\end{aligned}$$

as shown in Figure 4.3.

$$\begin{tikzcd}
y_1 \arrow[r, bend left, "x_1"] \arrow[rrrrrrr, bend right, "x_0"'] & z_1 \arrow[r, bend left, "w_1"] & y_2 \arrow[r, bend left, "x_2"] & z_2 \arrow[r, bend left, "w_2"] & \cdots & z_n \arrow[r, bend left, "w_n"] & y_{n+1} \arrow[r, bend left, "x_{n+1}"] & z_{n+1}
\end{tikzcd}$$

Fig. 4.3 The graph Γ.

Let us now set the constraints, which are given by a clopen subset $K_x \subseteq (\overline{\Omega}_A\mathsf{S})^I$ for each $x \in \Gamma$. The constraint of each variable x_i should be the same as for the system $\mathcal{S}$, namely K_i. For each w_i, we set $K_{w_i} = \{w_i\}$, which is a clopen subset of $(\overline{\Omega}_A\mathsf{S})^I$ since w_i is a word. Finally, we take $K_{x_0} = \{I\}$, and $K_{y_i} = (\overline{\Omega}_A\mathsf{S})^I = K_{z_i}$ for $i = 1,\dots,n+1$.

If δ is a solution modulo H of $\mathcal{S}$, then we extend δ to $\overline{\Omega}_\Gamma\mathsf{S}$ by taking $\delta(x_0) = I$, $\delta(w_i) = w_i$, $\delta(y_1) = 1 = \delta(z_{n+1})$, $\delta(z_i) = \delta(y_i)\delta(x_i)$, and $\delta(y_{i+1}) = \delta(z_i)w_i$, for $i = 1,\dots,n$. The fact that the new homomorphism δ is a solution modulo H of $\mathcal{S}(\Gamma)$ follows immediately from construction. Conversely, suppose that δ' is a solution modulo H of $\mathcal{S}(\Gamma)$. Then, taking into account that $K_{w_i} = \{w_i\}$ implies $\delta'(w_i) = w_i$, we deduce that the following equalities are valid in H:

$$\begin{aligned}
\delta'(y_1x_1w_1x_2w_2\cdots x_nw_nx_{n+1}) &= \delta'(z_1w_1x_2w_2\cdots x_nw_nx_{n+1}) \quad \text{since } y_1x_1 = z_1 \in \mathcal{S}(\Gamma)\\
&= \delta'(y_2x_2w_2\cdots x_nw_nx_{n+1}) \quad \text{since } z_1w_1 = y_2 \in \mathcal{S}(\Gamma)\\
&= \delta'(z_2w_2\cdots x_nw_nx_{n+1}) = \cdots\\
&= \delta'(z_nw_nx_{n+1}) = \delta'(y_{n+1}x_{n+1}) = \delta'(z_{n+1}).
\end{aligned}$$

Besides, as $K_{x_0} = \{I\}$ and the equation $y_1x_0 = z_{n+1}$ belongs to $\mathcal{S}(\Gamma)$, the pseudovariety H also satisfies $\delta'(y_1) = \delta'(z_{n+1})$. Further, we are assuming that members of H are groups and so, it follows that H satisfies

$$\delta'(x_1w_1x_2w_2\cdots x_nw_nx_{n+1}) = 1.$$

Thus, the restriction of δ' to $\overline{\Omega}_X\mathsf{S}$ is a solution modulo H of $\mathcal{S}$. □

Lemma 4.18. *Let H be a pseudovariety of groups. If $\mathcal{S}$ is H-equivalent to a system of graph equations, x is a variable occurring in $\mathcal{S}$, and $\mathcal{S}_0 = \{x = x_1 = \cdots = x_n\}$, where $x_1,\dots,x_n$ are new variables, then $\mathcal{S}\cup\mathcal{S}_0$ is also H-equivalent to a system of graph equations.*

Proof. Let $\Gamma = V \uplus E$ be an $\mathcal{S}$-graph. Since x occurs in $\mathcal{S}$, either $x \in V$ or $x \in E$. If x is a vertex, then we define $\Gamma' = V' \uplus E'$, where $V' = V \uplus \{x_0\}$ and $E' = E \uplus \{x_i\}_{i=1}^n$. The maps α and ω for E' are kept unchanged in the subset $E \subseteq E'$, and are given by $(\alpha(x_i),\omega(x_i)) = (x_0,x)$, for $i = 1,\dots,n$. We also set $K_{x_0} = \{I\}$ (the constraints for the other variables are already determined by $\mathcal{S}$ and $\mathcal{S}_0$). Figure 4.4 illustrates Γ'. It is a routine matter to verify that the system $\mathcal{S}\cup\mathcal{S}_0$ is H-equivalent to $\mathcal{S}(\Gamma')$.

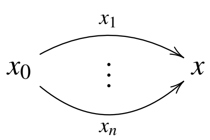

Fig. 4.4 The piece of the graph Γ' where it differs from Γ, when $x \in V$.

Otherwise, if x is an edge, say with $\alpha(x) = y$ and $\omega(x) = z$, then we define $\Gamma = V \uplus E'$, where E' is obtained from E by adding n edges $x_1,\dots,x_n$ and setting $(\alpha(x_i),\omega(x_i)) = (y,z)$, for $i = 1,\dots,n$ (see Figure 4.5). It is clear that any solution modulo H of $\mathcal{S}\cup\mathcal{S}_0$ may be extended to a solution modulo H of $\mathcal{S}(\Gamma')$. Conversely, let δ be a solution modulo H of $\mathcal{S}(\Gamma')$. Then, since $\Gamma \subseteq \Gamma'$ and Γ is an $\mathcal{S}$-graph, the restriction of δ to the variables occurring in $\mathcal{S}$ is a solution modulo H of $\mathcal{S}$. Moreover, the pseudovariety H satisfies $\delta(yx) = \delta(yx_1) = \cdots = \delta(yx_n) = \delta(z)$. The fact that $\overline{\Omega}_A\mathsf{H}$ is a group yields that H also satisfies $\delta(x) = \delta(x_1) = \cdots = \delta(x_n)$. Thus, the homomorphism δ induces a solution modulo H of $\mathcal{S}\cup\mathcal{S}_0$ by restriction. □

$$\begin{tikzcd} y \arrow[rr, bend left=60, "x"] \arrow[rr, "x_1"] \arrow[rr, phantom, shift right=2, "\vdots"] \arrow[rr, bend right=60, "x_n"'] & & z \end{tikzcd}$$

Fig. 4.5 The piece of the graph Γ' where it differs from Γ, when $x \in E$.

Lemma 4.19. *Let* H *be a pseudovariety of groups,* $\mathcal{S}$ *be a system of equations with variables in* X *that is* H*-equivalent to a system of graph equations, and* $\mathcal{S}_0 = \{x = x_1w_1 \cdots x_nw_nx_{n+1}\}$*, where* $x \in X$*,* $x_1, \ldots, x_n$ *are new variables,* x_{n+1} *is either the empty word or a new variable, and* $\{w_i\}_{i=1}^n \subseteq A^*$*. Then,* $\mathcal{S} \cup \mathcal{S}_0$ *is also* H*-equivalent to a system of graph equations.*

Proof. We start by observing that it really does not matter whether x_{n+1} is the empty word or a new variable. Indeed, if it is the empty word, then we just need to set a constraint $K_{x_{n+1}} = \{I\}$ for x_{n+1} and we may treat it as a variable.

Let $\Gamma = V \uplus E$ be an $\mathcal{S}$-graph. We construct a new graph $\Gamma' = V' \uplus E'$ depending on whether x is a vertex or an edge.

If x is a vertex, then we add to Γ a new path going from a new vertex y_1 to x, whose edges are labeled by $x_1, w_1, \ldots, x_n, w_n, x_{n+1}$ in this order, as depicted in Figure 4.6. Formally, this corresponds

$$y_1 \xrightarrow{x_1} z_1 \xrightarrow{w_1} y_2 \xrightarrow{x_2} z_2 \xrightarrow{w_2} \quad \cdots \quad z_n \xrightarrow{w_n} y_{n+1} \xrightarrow{x_{n+1}} x$$

Fig. 4.6 The new path in Γ if x is a vertex.

to setting

$$\begin{aligned}
V' &= V \uplus \{y_i\}_{i=1}^{n+1} \uplus \{z_i\}_{i=1}^{n};\\
E' &= E \uplus \{x_i\}_{i=1}^{n+1} \uplus \{w_i\}_{i=1}^{n};\\
(\alpha(x_i), \omega(x_i)) &= (y_i, z_i), \text{ for } i = 1, \ldots, n;\\
(\alpha(x_{n+1}), \omega(x_{n+1})) &= (y_{n+1}, x);\\
(\alpha(w_i), \omega(w_i)) &= (z_i, y_{i+1}), \text{ for } i = 1, \ldots, n.
\end{aligned}$$

We further take $K_{y_1} = \{I\}$, $K_{y_{i+1}} = (\overline{\Omega}_A \mathsf{S})^I = K_{z_i}$, and $K_{w_i} = \{w_i\}$ as the clopen sets defining the constraints for the new variables y_{i+1}, z_i, and w_i, respectively $(i = 1, \ldots, n)$. The constraints to be imposed to any other variable are those borrowed from the definition of $\mathcal{S}$ and $\mathcal{S}_0$. Let us check that Γ' is an $(\mathcal{S} \cup \mathcal{S}_0)$-graph. Given a solution δ modulo H of $\mathcal{S} \cup \mathcal{S}_0$, since Γ is an $\mathcal{S}$-graph, we may extend $\delta|_{\overline{\Omega}_X \mathsf{S}}$ to a solution δ_0 modulo H of $\mathcal{S}(\Gamma)$. Set

$$\begin{aligned}
\delta' : \overline{\Omega}_{\Gamma'} \mathsf{S} &\to (\overline{\Omega}_A \mathsf{S})^I\\
x &\mapsto \delta_0(x), \text{ if } x \in \Gamma;\\
x_i &\mapsto \delta(x_i), \text{ for } i = 1, \ldots, n+1;\\
w_i &\mapsto w_i, \text{ for } i = 1, \ldots, n;
\end{aligned}$$

$$\begin{aligned} y_1 &\mapsto I; \\ z_i, y_{i+1} &\mapsto \delta'(y_i)\delta(x_i), \delta'(z_i)w_i \text{ (respectively)}, \quad \text{for } i = 1, \ldots, n. \end{aligned}$$

Then, all the equations in $\mathcal{S}(\Gamma')$ but $y_{n+1}x_{n+1} = x$ are trivially satisfied by δ'. The satisfaction of the remaining equation may be deduced as follows:

$$\begin{aligned} \delta'(x) &= \delta(x) \quad \text{by definition of } \delta' \\ &=_{\mathsf{H}} \delta(x_1)w_1\delta(x_2)w_2\cdots\delta(x_n)w_n\delta(x_{n+1}) \quad \text{because } \delta \text{ is a solution modulo } \mathsf{H} \text{ of } \mathcal{S}_0 \\ &= \delta'(y_1)\delta(x_1)w_1\delta(x_2)w_2\cdots\delta(x_n)w_n\delta(x_{n+1}) \quad \text{because } \delta'(y_1) = I \\ &= \delta'(z_1)w_1\delta(x_2)w_2\cdots\delta(x_n)w_n\delta(x_{n+1}) \quad \text{by definition of } \delta' \\ &= \delta'(y_2)\delta(x_2)w_2\cdots\delta(x_n)w_n\delta(x_{n+1}) = \cdots \quad \text{by definition of } \delta' \\ &= \delta'(y_{n+1})\delta'(x_{n+1}). \end{aligned}$$

Conversely, take a solution δ modulo H of $\mathcal{S}(\Gamma')$. Since Γ is a subgraph of Γ' and an $\mathcal{S}$-graph, the restriction of δ to $\overline{\Omega}_X\mathsf{S}$ is a solution modulo H of $\mathcal{S}$. Furthermore, the path represented in Figure 4.6 translates in the equation $x_1w_1\cdots x_nw_nx_{n+1} = x$ (taking into account that $K_{y_1} = \{I\}$). Hence, the restriction of δ to $\overline{\Omega}_{\{x_1,\ldots,x_n\}}\mathsf{S}$ is also a solution modulo H of $\mathcal{S}_0$.

On the other hand, when x is an edge, say from v_1 to v_2, we simply obtain Γ' by adding a path in Γ from v_1 to v_2 with edges labeled by $x_1, w_1, \ldots, x_n, w_n, x_{n+1}$ (see Figure 4.7). A standard computation

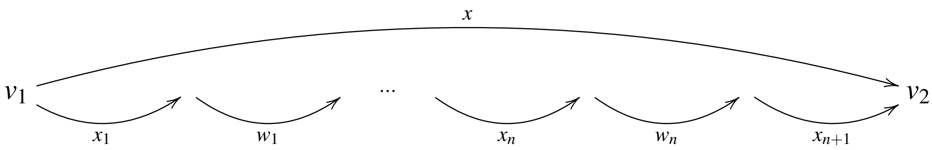

Fig. 4.7 The added path to Γ if x is an edge.

shows that any solution modulo H of $\mathcal{S} \cup \mathcal{S}_0$ yields a solution modulo H of $\mathcal{S}(\Gamma')$. Also, the fact that H is a pseudovariety of groups ensures that any solution modulo H of $\mathcal{S}(\Gamma')$ satisfies the equation $x = x_1w_1\cdots x_nw_nx_{n+1}$ modulo H. □

Corollary 4.20. *Let* H *be a pseudovariety of groups and let* $\mathcal{S}$ *be a system of equations* H*-equivalent to a system of graph equations and suppose that* $x_1, \ldots, x_N$ *are variables occurring in* $\mathcal{S}$*. Also, suppose that, for each* $i = 1, \ldots, N$*, the variables* $y_{i,1}, \ldots, y_{i,k_i}$ *and* $z_{i,1}, \ldots, z_{i,n_i}$ *are all distinct and do not occur in* $\mathcal{S}$*, and let* $\{w_{i,p} \colon i = 1, \ldots, N;\ p = 1, \ldots, k_i\} \subseteq A^*$*. We make each* t_i *be either the empty word or another different variable. Then, the system of equations*

$$\begin{aligned} \mathcal{S}' = \mathcal{S} &\cup \{x_i = y_{i,1}w_{i,1}\cdots y_{i,k_i}w_{i,k_i}t_i\}_{i=1}^N \\ &\cup \{t_i = z_{i,1} = \cdots = z_{i,n_i} \colon i = 1, \ldots, N \text{ and } t_i \text{ is not the empty word}\} \end{aligned}$$

is also H*-equivalent to a system of graph equations.*

Proof. The result follows immediately by successively applying Lemmas 4.18 and 4.19. □

The next statement consists of a general scenario that is instrumental for establishing the claimed answer to Question 1 mentioned in the beginning of this section.

Proposition 4.21. *Let* H *be a* σ*-reducible pseudovariety of groups, where* σ *is an implicit signature such that* $\langle\sigma\rangle$ *contains a non-explicit operation* η *such that* $\eta = 1$ *in* H. *Let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be finite systems of equations, such that* $\mathcal{S}_1$ *contains only pointlike equations, and both* $\mathcal{S}_1 \cup \mathcal{S}_2$ *and* $\mathcal{S}_2$ *are* H*-equivalent to systems of graph equations. Further assume that, if* X *is the set of variables occurring in* $\mathcal{S}_1 \cup \mathcal{S}_2$*, then the constraint on each variable* $x \in X$ *is given by a clopen subset* $K_x \subseteq (\overline{\Omega}_A\mathsf{S})^I$. *Then, the existence of a continuous homomorphism that is simultaneously a solution modulo* DRH *of* $\mathcal{S}_1$ *and a solution modulo* H *of* $\mathcal{S}_2$ *entails the existence of a continuous homomorphism in* σ*-words with the same property.*

Proof. Without loss of generality, we assume that η is a unary implicit operation. Let $\mathcal{S}_1 = \{x_{i,1} = \cdots = x_{i,n_i}\}_{i=1}^N$, with $x_{i,p} \neq x_{j,q}$, for all $i \neq j$. We consider a continuous homomorphism $\varphi : (\overline{\Omega}_A\mathsf{S})^I \to S^I$ such that each clopen set K_x is the preimage of a finite subset of S^I under φ (recall Remark 2.3). We proceed by induction on the parameter

$$M = \max\{|c(\delta(x_{i,p}))| : i = 1,\dots,N,\ p = 1,\dots,n_i\}.$$

If $M = 0$, then $\delta(x_{i,p}) = I$ for all $i = 1,\dots,N$ and $p = 1,\dots,n_i$ and therefore, every solution ε modulo H of $\mathcal{S}_2$ such that $\varepsilon(x_{i,p}) = I$ (for $i = 1,\dots,N$, and $p = 1,\dots,n_i$) is trivially a solution modulo DRH of $\mathcal{S}_1$. Since we are assuming that $\mathcal{S}_2$ is H-equivalent to a system of graph equations and we are taking for H a σ-reducible pseudovariety, there exists such an ε given by σ-words.

Suppose that $M > 0$ and that the result holds for any smaller parameter. If $\delta(x_{i,p})$ has empty cumulative content, then we let k_i be the maximum integer such that $\mathsf{lbf}_{k_i}(\delta(x_{i,p}))$ is nonempty and we write $\mathsf{lbf}_m(\delta(x_{i,p})) = \delta(x_{i,p})_m a_{i,m}$, for $m = 1,\dots,k_i$. Otherwise, for each $m \geq 1$, we consider the m-th iteration of the left basic factorization to the right of $\delta(x_{i,p})$, say

$$\delta(x_{i,p}) = \delta(x_{i,p})_1 a_{i,1} \cdots \delta(x_{i,p})_m a_{i,m} \delta(x_{i,p})'_m.$$

Since S and A are both finite, there are integers $1 \leq k < \ell$ such that, for all i, p satisfying $\vec{c}(\delta(x_{i,p})) \neq \emptyset$, we have

$$\vec{c}(\delta(x_{i,p})) = c(\delta(x_{i,p})_{k+1} a_{i,k+1});$$
$$\varphi(\delta(x_{i,p})_1 a_{i,1} \cdots \delta(x_{i,p})_k a_{i,k}) = \varphi(\delta(x_{i,p})_1 a_{i,1} \cdots \delta(x_{i,p})_\ell a_{i,\ell}),$$

which in turn implies

$$\varphi(\delta(x_{i,p})) = \varphi(\delta(x_{i,p})_1 a_{i,1} \cdots \delta(x_{i,p})_k a_{i,k} \cdot \eta(\delta(x_{i,p})_{k+1} a_{i,k+1} \cdots \delta(x_{i,p})_\ell a_{i,\ell}) \delta(x_{i,p})'_k).$$

Now, consider a new set of variables X' given by

$$\begin{aligned} X' = X &\uplus \{x_{i,p;m}, a_{i,m} \colon i = 1,\dots,N;\ p = 1,\dots,n_i;\ \vec{c}(\delta(x_{i,p})) = \emptyset;\ \text{and } m = 1,\dots,k_i\} \\ &\uplus \{x_{i,p;m}, a_{i,m}, x'_{i,p} \colon i = 1,\dots,N;\ p = 1,\dots,n_i;\ \vec{c}(\delta(x_{i,p})) \neq \emptyset;\ \text{and } m = 1,\dots,\ell\}, \end{aligned}$$

where all the introduced variables are distinct. In order to simplify the notation, we set $\ell_i = 0$ if $\vec{c}(\delta(x_{i,p})) = \emptyset$, and $k_i = k$ and $\ell_i = \ell$, otherwise. We further take the constraints on X' to be given by K_x if $x \in X$, and by the clopen sets $K_{x_{i,p;m}} = \varphi^{-1}(\varphi(\delta(x_{i,p})_m))$, $K_{a_{i,m}} = \{a_{i,m}\}$, and $K_{x'_{i,p}} = \varphi^{-1}(\varphi(\delta(x_{i,p})'_k))$ for the remaining cases.

Consider the system

$$\mathcal{S}'_1 = \{x_{i,1;m} = \cdots = x_{i,n_i;m} \colon i = 1,\ldots,N;\, m = 1,\ldots,\max\{k_i,\ell_i\}\}.$$

A new system $\mathcal{S}'_2$ is obtained from the system $\mathcal{S}_1 \cup \mathcal{S}_2$ (which is H-equivalent to a system of graph equations, by hypothesis) by adding two sets of equations:

- for each $i = 1,\ldots,N$, if there exists an index p such that $x_{i,p}$ is a variable occurring in $\mathcal{S}_2$, then we choose such an index, say p_i. Then, we add the equation

$$x_{i,p_i} = x_{i,p_i;1}a_{i,1}\cdots x_{i,p_i;k_i}a_{i,k_i}z_{i,p_i},$$

  where z_{i,p_i} stands for the empty word if $\ell_i = 0$, and for x'_{i,p_i} otherwise;

- and we add the set of equations

$$\{x'_{i,1} = \cdots = x'_{i,n_i} \colon i = 1,\ldots,N,\, \ell_i \neq 0\}.$$

By Corollary 4.20, the new system $\mathcal{S}'_2$ is still H-equivalent to a system of graph equations. Moreover, if we denote by X'_j the set of variables occurring in $\mathcal{S}'_j$ ($j = 1,2$), then the following equality holds:

$$X'_1 \cap X'_2 = \{x_{i,p_i;m} \colon i = 1,\ldots,N;\ p_i \text{ is defined; and } m = 1,\ldots,\max\{k_i,\ell_i\}\}.$$

Thus, Lemma 4.18 yields that the system $\mathcal{S}'_1 \cup \mathcal{S}'_2$ is H-equivalent to a system of graph equations as well. Let $\delta' : \overline{\Omega}_{X'}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ be the continuous homomorphism defined by

$$\begin{aligned}
\delta'(x_{i,p;m}) &= \delta(x_{i,p})_m, && \text{if } i = 1,\ldots,N;\ p = 1,\ldots,n_i; \text{ and } m = 1,\ldots,\max\{k_i,\ell_i\};\\
\delta'(x'_{i,p}) &= \delta(x_{i,p})'_k, && \text{if } i = 1,\ldots,N; \text{ and } p = 1,\ldots,n_i;\\
\delta'(x) &= \delta(x), && \text{otherwise.}
\end{aligned}$$

It follows from its definition that δ' is a solution modulo DRH of $\mathcal{S}'_1$ which is also a solution modulo H of $\mathcal{S}'_2$. Since the induction parameter corresponding to the triple $(\mathcal{S}'_1,\mathcal{S}'_2,\delta')$ is smaller than the one corresponding to the triple $(\mathcal{S}_1,\mathcal{S}_2,\delta)$, we may use the induction hypothesis to deduce the existence of a continuous homomorphism $\varepsilon' : \overline{\Omega}_{X'}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ in σ-words that is both a solution modulo DRH of $\mathcal{S}'_1$ and a solution modulo H of $\mathcal{S}'_2$.

Finally, we define ε as follows:

$$\begin{aligned}
\varepsilon : \overline{\Omega}_X\mathsf{S} &\to (\overline{\Omega}_A\mathsf{S})^I\\
x_{i,p} &\mapsto \varepsilon'(x_{i,p;1})a_{i,1}\cdots\varepsilon'(x_{i,p;k_i})a_{i,k_i}, \quad \text{if } \ell_i = 0;\\
x_{i,p} &\mapsto \varepsilon'(x_{i,p;1})a_{i,1}\cdots\varepsilon'(x_{i,p;k})a_{i,k}
\end{aligned}$$

$$\cdot\, \eta(\varepsilon'(x_{i,p;k+1})a_{i,k+1}\cdots\varepsilon'(x_{i,p;\ell})a_{i,\ell})\cdot\varepsilon'(x'_{i,p}), \quad \text{if } \ell_i \neq 0;$$
$$x \mapsto \varepsilon'(x), \quad \text{otherwise.}$$

Then, a straightforward computation shows that ε plays the desired role. □

We finally state and prove the result claimed at the beginning of the section.

Theorem 4.22. *Let σ be an implicit operation such that there exists $\eta \in \langle\sigma\rangle$ non-explicit, with $\eta = 1$ in H. If H is a σ-reducible pseudovariety of groups, then DRH is σ-reducible for idempotent pointlike systems of equations.*

Proof. Let $\mathcal{S} = \{x_1 = \cdots = x_n = x_n^2\}$ be an idempotent pointlike system of equations with constraints on the variables given by the pair (φ, ν), and let $\delta : \overline{\Omega}_{\{x_1,\ldots,x_n\}}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be a solution modulo DRH of $\mathcal{S}$. Suppose that $\delta(x_i) = u_i$. Then, by Proposition 2.15, DRH satisfies

$$u_1 = \cdots = u_n = u_n^2$$

if and only if the following conditions hold:

$$\begin{aligned} &c(u_n) = \vec{c}(u_n);\\ &u_1 =_{\mathsf{DRH}} \cdots =_{\mathsf{DRH}} u_n;\\ &u_n =_{\mathsf{H}} 1. \end{aligned} \tag{4.8}$$

For each $i \in \{1,\ldots,n\}$ and $m \geq 1$, let $u_i = \mathsf{lbf}_1(u_i)\cdots\mathsf{lbf}_m(u_i)u'_{i,m}$ and $\mathsf{lbf}_m(u_i) = u_{i,m}a_m$. Since S is finite, there are positive integers $k < \ell$ such that for all $i = 1,\ldots,n$ the equality

$$\varphi(\mathsf{lbf}_1(u_i)\cdots\mathsf{lbf}_k(u_i)) = \varphi(\mathsf{lbf}_1(u_i)\cdots\mathsf{lbf}_\ell(u_i))$$

holds. Take the set of variables

$$X = \{x_{i,p} \colon i = 1,\ldots,n;\ p = 1,\ldots,\ell\} \uplus \{x'_i \colon i = 1,\ldots,n\}$$

with constraints given by (φ, ν'), where $\nu'(x) = \varphi(u_{i,p})$ if $x = x_{i,p}$, and $\nu'(x) = \varphi(u'_{i,k})$ if $x = x'_i$. We consider the systems of equations

$$\begin{aligned} &\mathcal{S}_1 = \{x_{1,p} = \cdots = x_{n,p}\}_{p=1}^{\ell}\\ &\mathcal{S}_2 = \{x_{n,1}a_1\cdots x_{n,k}a_kx'_n = 1,\ x'_1 = \cdots = x'_n\}. \end{aligned}$$

Then, the homomorphism

$$\begin{aligned} \delta' : \overline{\Omega}_X\mathsf{S} &\to (\overline{\Omega}_A\mathsf{S})^I\\ x_{i,p} &\mapsto u_{i,p}, \quad \text{for } i = 1,\ldots,n;\ p = 1,\ldots,\ell;\\ x'_i &\mapsto u'_{i,k}, \quad \text{for } i = 1,\ldots,n; \end{aligned}$$

is a solution modulo DRH of $\mathcal{S}_1$ that is also a solution modulo H of $\mathcal{S}_2$. Besides that, since by Lemma 4.17 the system $\{x_{n,1}a_1\cdots x_{n,k}a_kx'_n = 1\}$ is H-equivalent to a system of graph equations, Lemma 4.18 yields that so is $\mathcal{S}_2$. In turn, again Lemma 4.18 implies that $\mathcal{S}_1 \cup \mathcal{S}_2$ is H-equivalent to a system of graph equations. Thus, we may invoke Proposition 4.21 to derive the existence of a continuous homomorphism $\varepsilon' : \overline{\Omega}_X\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ in σ-words that is a solution modulo DRH of $\mathcal{S}_1$, and a solution modulo H of $\mathcal{S}_2$.

Now, assuming that η is unary, we let $\varepsilon : \overline{\Omega}_{\{x_1,\dots,x_n\}}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be given by

$$\varepsilon(x_i) = \varepsilon'(x_{i,1})a_1\cdots\varepsilon'(x_{i,k})a_k \cdot \eta(\varepsilon'(x_{i,k+1})a_{k+1}\cdots\varepsilon'(x_{i,\ell})a_\ell)\varepsilon'(x'_i).$$

It is easily checked that

$$\begin{aligned}&\varepsilon(x_1) =_{\mathsf{DRH}} \cdots =_{\mathsf{DRH}} \varepsilon(x_n);\\ &\varepsilon(x_n) =_{\mathsf{H}} 1.\end{aligned}$$

Furthermore, by the choice of k and ℓ, we also know that $\varphi(\varepsilon(x_i)) = \varphi(\delta(x_i))$ and, as we are assuming that η is non-explicit and S has a content function, the equality $\vec{c}(\varepsilon(x_i)) = c(\varepsilon(x_i))$ holds. So, by (4.8), we may conclude that ε is a solution modulo DRH of $\mathcal{S}$ in σ-words that keeps the values under φ. $\square$

We observe that, whenever the ω-power belongs to $\langle\sigma\rangle$, the hypothesis of Theorem 4.22 concerning the implicit signature σ is satisfied. That is the case of the canonical implicit signature κ. Hence, we have the following.

Corollary 4.23. *Let* H *be a* κ*-tame pseudovariety of groups. Then,*

- DRH *is* κ*-tame with respect to finite systems of idempotent pointlike equations (similar to Corollary 4.4);*
- $\mathsf{V} \circledast \mathsf{DRH}$ *is decidable whenever* V *is a decidable pseudovariety (Theorems 2.8 and 2.10).* $\square$

In particular, the pseudovarieties DRG and DRAb are both κ-tame with respect to finite systems of idempotent pointlike equations and DRG_p and DRG_{sol} are σ-reducible with respect to the same class, for suitable implicit signatures σ (recall Theorem 2.11).

Unfortunately, Question 2 is again harder to answer. The best we are able to prove is that the result obtained by changing the places of the pseudovarieties H and DRH in Theorem 4.22 also holds. But that is just a consequence of Proposition 4.15, as systems of idempotent pointlike equations are H-equivalent to systems of graph equations. More precisely, the system $x_1 = \cdots = x_n = x_n^2$ is H-equivalent to the system of graph equations induced by the graph in Figure 4.8.

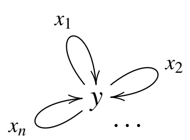

Fig. 4.8 A graph defining a system of equations H-equivalent to the system $x_1 = \cdots = x_n = x_n^2$.

Chapter 5

Complete κ-reducibility of DRH

The aim of this chapter is to identify conditions on a pseudovariety of groups H in order that the pseudovariety DRH be completely κ-reducible. However, we consider a scenario a little bit more general and we prove that, if H is a pseudovariety of groups that is σ-reducible for finite systems of κ-equations, then the pseudovariety DRH is σ-reducible for finite systems of κ-equations as well, whenever an implicit signature σ satisfies the condition (sig) and is such that $\kappa \subseteq \langle\sigma\rangle$.

Let us fix a finite alphabet A. Unless otherwise stated, we use H for an arbitrary pseudovariety of groups.

5.1 General simplifications

We carry on with some general simplifications on testing whether a pseudovariety is σ-reducible for finite systems of τ-equations, for some $\tau \subseteq \langle\sigma\rangle$. These first two results are merely a slight generalization of [10, Proposition 3.1] and [10, Proposition 3.2], respectively. Although their proofs are analogous, we include them for the sake of completeness.

Proposition 5.1 (cf. [10, Proposition 3.1])**.** *Let V be an arbitrary pseudovariety and σ, τ two implicit signatures such that $\tau \subseteq \langle\sigma\rangle$. If V is σ-reducible for finite systems of τ-equations without parameters, then V is σ-reducible for finite systems of τ-equations. In particular, by taking $\tau = \sigma$, we obtain that V is completely σ-reducible if and only if it is σ-reducible for finite systems of σ-equations without parameters.*

Proof. Let $\mathcal{S} = \{u_i = v_i\}_{i=1}^n$ be a finite system of τ-equations where $u_i, v_i \in \Omega^\tau_{X\cup P}\mathsf{S}$, P is the set of parameters evaluated by $\mathsf{ev} : P \to \Omega^\tau_A\mathsf{S}$, and X is the set of variables whose constraints are given by a clopen subset $K_x \subseteq \overline{\Omega}_A\mathsf{S}$, for each $x \in X$. Let $Y = X \uplus A$ be a new set of variables and let $\psi : \overline{\Omega}_{X\cup P} \to \overline{\Omega}_Y\mathsf{S}$ be the unique continuous homomorphism sending each variable $x \in X$ to itself and each parameter $p \in P$ to its evaluation $\mathsf{ev}(p)$. Finally, consider the system $\psi(\mathcal{S}) = \{\psi(u_i) = \psi(v_i)\}_{i=1}^n$ with constraints given by

$$K_y = \begin{cases} K_y, & \text{if } y \in X \\ \{y\}, & \text{if } y \in A \end{cases} \tag{5.1}$$

for each $y \in Y$. The set $\psi(\mathcal{S})$ is a finite system of τ-equations with empty set of parameters. Suppose that $\delta : \overline{\Omega}_{X\cup P}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ is a solution modulo V of $\mathcal{S}$. It is easy to check that the continuous homomorphism $\delta' : \overline{\Omega}_Y\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ defined by $\delta'(x) = \delta(x)$, for $x \in X$, and by $\delta'(a) = a$, for $a \in A$, is a solution modulo V of $\psi(\mathcal{S})$ satisfying the constraints given by (5.1). Moreover, by hypothesis, the pseudovariety V is σ-reducible for systems of τ-equations without parameters. Hence, there is a solution $\varepsilon' : \overline{\Omega}_Y\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ modulo V of $\psi(\mathcal{S})$ such that $\varepsilon'(Y) \subseteq \Omega_A^\sigma\mathsf{S}$. Now, let $\varepsilon : \overline{\Omega}_{X\cup P}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be the continuous homomorphism $\varepsilon' \circ \psi$. Clearly, we have $\varepsilon(X \cup P) \subseteq \Omega_A^\sigma\mathsf{S}$. Also, we may check that ε is a solution modulo V of the initial system $\mathcal{S}$. Indeed, for each $i = 1,\dots,n$, since ε' is a solution modulo V of $\psi(u_i) = \psi(v_i)$, the pseudovariety V satisfies $\varepsilon(u_i) = \varepsilon'(\psi(u_i)) = \varepsilon'(\psi(v_i)) = \varepsilon(v_i)$ and so (S.1) holds. Given a parameter $p \in P$, the condition (S.2) is satisfied because ψ sends a parameter p to its evaluation $\mathsf{ev}(p) \in \Omega_A^\tau\mathsf{S}$ and, on the other hand, the constraint $K_a = \{a\}$ for every variable $a \in A \subseteq Y$ guarantees that $\varepsilon'(a) = a$. Finally, the constraints for $x \in X$ are also satisfied, since $\varepsilon(x) = \varepsilon'(\psi(x)) = \varepsilon'(x) \in K_x$, getting (S.3). Thus, the pseudovariety V is σ-reducible for finite systems of τ-equations. □

We say that a pseudovariety V is *weakly cancellable* if whenever V satisfies the pseudoidentity $u_1au_2 = v_1av_2$ for some $u_1,u_2,v_1,v_2 \in \overline{\Omega}_{A\setminus\{a\}}\mathsf{S}$, it also satisfies the pseudoidentities $u_1 = v_1$ and $u_2 = v_2$. When V is a weakly cancellable pseudovariety, we may restrict our study to systems consisting of one single σ-equation without parameters.

Proposition 5.2 (cf. [10, Propoposition 3.2]). *Let V be a weakly cancellable pseudovariety and σ, τ implicit signatures such that $\tau \subseteq \langle\sigma\rangle$. If V is σ-reducible for systems consisting of a single τ-equation without parameters, then V is σ-reducible for finite systems of τ-equations. In particular, when $\tau = \sigma$, we get that V is completely σ-reducible if and only if it is σ-reducible for systems consisting of a single σ-equation.*

Proof. By Proposition 5.1 it is enough to prove that V is σ-reducible for finite systems of τ-equations without parameters. Let $\mathcal{S} = \{u_i = v_i\}_{i=1}^n$ be such a system, with variables in X, and constraints given by some clopen subsets $K_x \subseteq \overline{\Omega}_A\mathsf{S}$. Let $\delta : \overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be a solution modulo DRH of $\mathcal{S}$. Consider a new set of variables $Y = X \uplus \{\#_i\}_{i=1}^{n-1}$, and let $\{a_i\}_{i=1}^{n-1}$ be letters that do not belong to the alphabet A. Then, the continuous homomorphism $\delta' : \overline{\Omega}_Y\mathsf{S} \to \overline{\Omega}_{A\uplus\{a_i\}_{i=1}^{n-1}}\mathsf{S}$ whose restriction to $\overline{\Omega}_X\mathsf{S}$ is given by δ, and that maps each $\#_i$ to a_i is a solution modulo DRH of the τ-equation

$$u_1\#_1u_2\#_2\cdots\#_{n-1}u_n = v_1\#_1v_2\#_2\cdots\#_{n-1}v_n, \tag{5.2}$$

with the same constraints for the variables of X, and with constraint $K_{\#_i} = \{a_i\}$, for $i = 1,\dots,n-1$. Since we are assuming that V is σ-reducible for a single τ-equation without parameters, there exists a solution in σ-words $\varepsilon : \overline{\Omega}_Y\mathsf{S} \to \overline{\Omega}_{A\uplus\{a_i\}_{i=1}^{n-1}}\mathsf{S}$ modulo V of (5.2). Furthermore, since V is weakly σ-reducible, it follows that the restriction of ε to $\overline{\Omega}_X\mathsf{S}$ is a solution (in σ-words) modulo V of $\mathcal{S}$. Hence, V is σ-reducible for finite systems of τ-equations. □

Of course, the pseudovariety DRH is weakly cancellable. Indeed, weak cancellability is a particular instance of uniqueness of the first-occurrences factorization (recall Corollary 2.14(b)).

5.2 Simplifications for the pseudovariety DRH

In this section, we proceed with further simplifications for testing complete κ-reducibility of a pseudovariety of the form DRH. Similarly to the particular case of R (see [10, Lemmas 6.1 and 6.2]), in order to achieve complete κ-reducibility, it is enough to consider systems of word equations (without parameters).

Lemma 5.3. *Let $u,v \in \overline{\Omega}_A\mathsf{S}$. Then,* DRH *satisfies the pseudoidentity $u = v^{\omega-1}$ if and only if the equality $c(u) = c(v)$ holds, and the pseudoidentities $uvu = u$ and $uv = vu$ are valid in* DRH.

Proof. Suppose that DRH satisfies $u = v^{\omega-1}$. Since the semigroup $\overline{\Omega}_A\mathsf{DRH}$ has a content function, we have $c(u) = c(v^{\omega-1}) = c(v)$. In order to verify that the pseudoidentities $uvu = u$ and $uv = vu$ are valid in DRH, we may perform the following computations:

$$u =_{\mathsf{DRH}} v^{\omega-1} = v^{\omega-1}\,(vv^{\omega-1}) =_{\mathsf{DRH}} uvu,$$
$$uv =_{\mathsf{DRH}} v^{\omega-1}v = vv^{\omega-1} =_{\mathsf{DRH}} vu.$$

Conversely, suppose that DRH satisfies both $uvu = u$ and $uv = vu$, and the equality $c(u) = c(v)$ holds. Then, the following pseudoidentities are valid in DRH:

$$\begin{aligned} v^{\omega-1} &= v^{\omega-1}u^{\omega} && \text{by Corollary 2.20} \\ &= v^{\omega-1}u^{\omega-1}u = (uv)^{\omega-1}u && \text{because } uv =_{\mathsf{DRH}} vu \\ &= (uv)u && \text{because } uvu =_{\mathsf{DRH}} u \text{ implies } (uv)^{\omega-1} =_{\mathsf{DRH}} uv \\ &= u. \end{aligned}$$

This concludes the proof. □

Lemma 5.3 allows us to transform each κ-equation into a finite system of word equations.

Proposition 5.4 (cf. [10, Proposition 6.2])**.** *Let σ be an implicit signature such that $\kappa \subseteq \langle\sigma\rangle$. The pseudovariety* DRH *is σ-reducible for finite systems of κ-equations if and only if it is σ-reducible for a single word equation without parameters.*

Proof. Since DRH is a weakly cancellable pseudovariety, in view of Proposition 5.2 it is enough to prove that, given a κ-equation without parameters, it is possible to construct a finite system of word equations such that every solution modulo DRH of that system leads to a solution modulo DRH of the original equation and conversely. So, let $u = v$ be a κ-equation without parameters and $\delta : \overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ its solution modulo DRH. We set $\mathcal{S}_0 = \{u = v\}$ and we modify this system inductively. Let $i \geq 1$ and choose a subword of a member of an equation in $\mathcal{S}_{i-1}$ of the form $z_i^{\omega-1}$. We add to X a new variable x_i, with constraint given by the clopen subset $K_{x_i} \subseteq \overline{\Omega}_A\mathsf{S}$ that contains all the pseudowords whose content is $c(\delta(z_i))$ (note that, as $c = \rho_{\mathsf{Sl}}$ is a continuous homomorphism, K_{x_i} is indeed a clopen subset). We obtain $\mathcal{S}_i$ by adding to $\mathcal{S}_{i-1}$ the equations $x_iz_ix_i = x_i$ and $x_iz_i = z_ix_i$ and by substituting the subword $z_i^{\omega-1}$ by the variable x_i. Since the number of $(\omega-1)$-powers in the original equation $u = v$ is finite, this process eventually ends. Furthermore, if we extend δ by letting $\delta(x_i) = z_i^{\omega-1}$, then δ is a solution modulo DRH of the new system $\mathcal{S}_i$.

Conversely, if ε is a solution modulo DRH of $\mathcal{S}_i$ then, taking into account that ε satisfies the equality $c(\varepsilon(x_i)) = c(\delta(z_i))$, that $\{x_iz_ix_i = x_i,\, x_iz_i = z_ix_i\}$ is contained in $\mathcal{S}_i$, and applying Lemma 5.3, we conclude that ε is also a solution modulo DRH of $\mathcal{S}_{i-1}$. $\square$

Example 5.5. Consider the κ-equation $(yz)^{\omega-1}t = t(zt^{\omega-1})^{\omega-1}$. We start with

$$\mathcal{S}_0 = \{(yz)^{\omega-1}t = t(zt^{\omega-1})^{\omega-1}\}.$$

The subwords in $\mathcal{S}_0$ that are $(\omega-1)$-powers are $(yz)^{\omega-1}$, $(zt^{\omega-1})^{\omega-1}$, and $t^{\omega-1}$. We choose, for instance, the factor $(yz)^{\omega-1}$. Then, we introduce a new variable x_1 with constraint given by the clopen set $K_{x_1} = c^{-1}(c(\delta(yz)))$, and we take

$$\mathcal{S}_1 = \{x_1t = t(zt^{\omega-1})^{\omega-1}, x_1yzx_1 = x_1, x_1yz = yzx_1\}.$$

For the second step, we may choose the factor $(zt^{\omega-1})^{\omega-1}$ in $\mathcal{S}_1$. Then, we add a new variable x_2 with associated constraint $K_{x_2} = c^{-1}(c(\delta(zt^{\omega-1})))$ and $\mathcal{S}_2$ is given by

$$\mathcal{S}_2 = \{x_1t = tx_2, x_1yzx_1 = x_1, x_1yz = yzx_1, x_2zt^{\omega-1}x_2 = x_2, x_2zt^{\omega-1} = zt^{\omega-1}x_2\}.$$

Finally, it remains to take care of the subword $t^{\omega-1}$. We add a new variable x_3, set $K_{x_3} = c^{-1}(c(\delta(t)))$, and get

$$\mathcal{S}_3 = \{x_1t = tx_2, x_1yzx_1 = x_1, x_1yz = yzx_1, x_2zx_3x_2 = x_2, x_2zx_3 = zx_3x_2, x_3tx_3 = x_3, x_3t = tx_3\}.$$

Combining Propositions 5.2 and 5.4 and considering $\sigma = \kappa$, we obtain the following.

Corollary 5.6. *The pseudovariety* DRH *is completely κ-reducible if and only if it is κ-reducible for systems of one word equation without parameters.* $\square$

Suppose that there exists an implicit signature σ such that, for every n-ary implicit operation $\eta \in \sigma$, there exists a finite system $\mathcal{S}(\eta)$ of word equations in the set of variables $\{x_1, \ldots, x_n, x_{n+1}\}$ such that for all pseudowords $u_1, \ldots, u_n, u_{n+1}$ we have $\eta(u_1, \ldots, u_n) =_{\mathsf{DRH}} u_{n+1}$ if and only if the continuous homomorphism $\delta : \overline{\Omega}_{\{x_1,\ldots,x_n,x_{n+1}\}}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ sending each x_i to u_i is a solution modulo DRH of $\mathcal{S}(\eta)$. Then, the same kind of argument used in Proposition 5.4 allows us to prove a similar result, replacing κ by σ in the statement. The hope of finding such an implicit signature for which a certain pseudovariety of groups may be completely σ-reducible motivates the upcoming results for a generic implicit signature σ.

Let $u, v \in X^+$ and $\delta : \overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be a solution modulo DRH of $u = v$, subject to the constraints given by the pair $(\varphi : \overline{\Omega}_A\mathsf{S} \to S, \nu : X \to S)$. We say that δ is *reduced with respect to the equation* $u = v$ if whenever xy is a product of variables that is a factor of uv, the product $\delta(x)\cdot\delta(y)$ is reduced. The last simplification consists in transforming the word equation $u = v$ into a more convenient system of equations, namely, into a system that we denote by $\mathcal{S}_{u=v}$ and that is the union of systems $\{u' = v'\}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ with variables in X'. We construct $\mathcal{S}_{u=v}$ inductively as follows.

We use an auxiliary system $\mathcal{S}_0$ and start with $\mathcal{S}_0 = \mathcal{S}_1 = \mathcal{S}_2 = \emptyset$, $X' = X$, $u' = u\#$, and $v' = v\#$, where $\# \notin A$ is a parameter evaluated to itself. Since DRH is a weakly cancellable pseudovariety, the word equation $u = v$ is equivalent to the equation $u' = v'$.

If δ is not reduced with respect to $u' = v'$, then we pick a factor xy such that $\delta(x)\delta(y)$ is not a reduced product and we distinguish between two situations:

- If $c(\delta(y)) \subseteq \vec{c}(\delta(x))$, then we add a new variable z to X' and we put the equation $xy = z$ in $\mathcal{S}_1$. We also redefine u' and v' by substituting each occurrence of the product xy in the expression $u'v'$ by the variable z.

- If $c(\delta(y)) \nsubseteq \vec{c}(\delta(x))$, then we add three new variables y_1, y_2, and z to X' and we put the equations $y = y_1y_2$ and $z = xy_1$ in $\mathcal{S}_0$ and $\mathcal{S}_1$, respectively. We also redefine u' and v' by substituting the product xy in the expression $u'v'$ by the product of variables zy_2.

In both situations, we can factorize $\delta(y) = \delta(y)_1\delta(y)_2$, with $\delta(y)_2$ possibly an empty word, such that $c(\delta(y)_1) \subseteq \vec{c}(\delta(x))$ and the product $(\delta(x)\delta(y)_1) \cdot \delta(y)_2$ is reduced if $\delta(y)_2 \neq I$. We extend δ to $\overline{\Omega}_{X'}\mathsf{S}$ by letting $\delta(z) = \delta(x)\delta(y)_1$ and, whenever we are in the second situation, by letting $\delta(y_i) = \delta(y)_i$ $(i = 1,2)$. Of course, δ is a solution modulo DRH of the new system of equations $\{u' = v'\} \cup \mathcal{S}_0 \cup \mathcal{S}_1$.

We repeat the described process until the extended solution δ is reduced with respect to the equation $u' = v'$. Since u and v are both words, we have for granted that this iteration eventually ends. Yet, the extension of δ to $\overline{\Omega}_{X'}\mathsf{S}$ (which is a solution modulo DRH of $\{u' = v'\} \cup \mathcal{S}_0 \cup \mathcal{S}_1$) has the property of being reduced with respect to the equation $u' = v'$. We further observe that the resulting system $\mathcal{S}_1$ may be written as $\mathcal{S}_1 = \{x_{(i)}y_{(i)} = z_{(i)}\}_{i=1}^{N}$ and its extended solution δ satisfies the inclusion $c(\delta(y_{(i)})) \subseteq \vec{c}(\delta(x_{(i)}))$. For each variable $x \in X'$, we set $A_x = \vec{c}(\delta(x))$ and define $\mathcal{S}_2 = \{xa^{\omega} = x \colon a \in A_x\}_{x \in X'}$. The homomorphism δ is a solution modulo DRH of $\mathcal{S}_2$. Finally, since DRH is weakly cancellable and all the products $\delta(y_1) \cdot \delta(y_2)$ are reduced, we may assume that the satisfaction of the equations in $\mathcal{S}_0$ by δ is a consequence of the satisfaction of the equation $u' = v'$ by δ, without losing the reducibility of δ with respect to $u' = v'$. More specifically, if $y = y_1y_2$ is an equation of $\mathcal{S}_0$, then we take for u' the word $u'\#y$ and for v' the word $v'\#y_1y_2$, where $\#$ is a new symbol, working as a parameter evaluated to itself. In the same fashion, we may also assume that all the variables of X' occur in $u' = v'$. Although at the moment it may not be clear to the reader why we wish that all the variables in X' occur in the equation $u' = v'$, that becomes useful later, when dealing with certain systems of equations modulo H that intervene in the so-called "systems of boundary relations". The resulting system $\{u' = v'\} \cup \mathcal{S}_1 \cup \mathcal{S}_2$ is the one that we denote by $\mathcal{S}_{u=v}$ and it also has a solution modulo DRH. The constraints for the variables in X' are those defined by the described extension of δ to $\overline{\Omega}_{X'}\mathsf{S}$, namely, we put $\nu(x) = \varphi(\delta(x))$ for each $x \in X'$.

Conversely, suppose that $\mathcal{S}_{u=v}$ has a solution modulo DRH in σ-words, say ε. Then, it is easily checked that, by construction, the restriction of ε to $\overline{\Omega}_X\mathsf{S}$ is a solution modulo DRH of the original equation $u = v$. Moreover, by definition of $\mathcal{S}_2$, this solution is such that $\vec{c}(\varepsilon(x)) = \vec{c}(\delta(x))$, for all $x \in X'$. As, in addition, S has a content function, the satisfaction of the constraints yields that $c(\varepsilon(y_{(i)})) = c(\delta(y_{(i)}))$ and, in particular, the inclusion $c(\varepsilon(y_{(i)})) \subseteq \vec{c}(\varepsilon(x_{(i)}))$ holds for all the equations $x_{(i)}y_{(i)} = z_{(i)}$ in $\mathcal{S}_1$.

Taking into account Proposition 5.4, we have just proved the following result in which we use the above notation.

Proposition 5.7. *Let σ be an implicit signature such that $\kappa \subseteq \langle\sigma\rangle$ and suppose that the pseudovariety* DRH *is σ-reducible for systems of equations of the form*

$$\mathcal{S}_{u=v} = \{u' = v'\} \cup \mathcal{S}_1 \cup \mathcal{S}_2, \tag{5.3}$$

where $u' = v'$ is a word equation, $\mathcal{S}_1 = \{x_{(i)}y_{(i)} = z_{(i)}\}_{i=1}^{N}$, and $\mathcal{S}_2 = \{xa^{\omega} = x\colon a \in A_x\}_{x\in X}$, which have a solution δ modulo DRH *that is reduced with respect to the equation $u' = v'$ and satisfies $c(\delta(y_{(i)})) \subseteq \vec{c}(\delta(x_{(i)}))$, for $i = 1,\dots,N$. Then, the pseudovariety* DRH *is σ-reducible with respect to finite systems of κ-equations.* $\square$

Example 5.8. Consider, for instance, the word equation given by $xyzyz = w$ with solution modulo DRH given by

$$\begin{aligned} \delta : \overline{\Omega}_X\mathsf{S} &\to \overline{\Omega}_A\mathsf{S} \\ x &\mapsto (ab)^{\omega-1} \\ y &\mapsto a^{\omega-1} \\ z &\mapsto abc \\ w &\mapsto (ab)^{\omega-1}bca^{\omega}bc \end{aligned}$$

We set $\mathcal{S}_0 = \mathcal{S}_1 = \mathcal{S}_2 = \emptyset$. The products in the equation that are not reduced under δ are xy and yz. Depending on the product we choose first, the obtained result may be different. If we start by choosing the product $\delta(x)\delta(y)$ then, since $c(\delta(y)) \subseteq \vec{c}(\delta(x))$, we put in $\mathcal{S}_1$ the equation $t_{xy} = xy$, where t_{xy} is a new variable, and replace the original equation by the equation $t_{xy}zyz = w$. We extend the solution δ by letting $\delta(t_{xy}) = (ab)^{\omega-1}a^{\omega-1}$. Again, we have two factors to consider, namely, $t_{xy}z$ and yz. Since the product $\delta(t_{xy}z)\cdot\delta(yz)$ is reduced, it really does not matter which we consider first now, since they do not interact between them. Thus, we do the corresponding modifications of the system at the same time. This corresponds to factorizing $\delta(z) = ab\cdot c$, relatively to the factor $t_{xy}z$; and to factorizing $\delta(z) = a\cdot bc$, relatively to the factor yz. Then, we add to $\mathcal{S}_0$ the equations $z = z_1z_2$ and $z = z_1'z_2'$ and to $\mathcal{S}_1$ the equations $t_{t_{xy}z_1} = t_{xy}z_1$ and $t_{yz_1'} = yz_1'$, where $z_1, z_2, z_1', z_2', t_{t_{xy}z_1}, t_{yz_1'}$ are new variables. The equation $t_{xy}zyz = w$ is replaced by the equation $t_{t_{xy}z_1}z_2t_{yz_1'}z_2' = w$. The solution δ extends by letting $\delta(z_1) = ab$, $\delta(z_2) = c$, $\delta(z_1') = a$, $\delta(z_2') = bc$, $\delta(t_{t_{xy}z_1}) = (ab)^{\omega-1}a^{\omega-1}ab$, and $\delta(t_{yz_1'}) = a^{\omega-1}a$. At the end, we have $\mathcal{S}_0 = \{z = z_1z_2, z = z_1'z_2'\}$ and $\mathcal{S}_1 = \{t_{xy} = xy, t_{t_{xy}z_1} = t_{xy}z_1, t_{yz_1'} = yz_1'\}$ and the new equation is $t_{t_{xy}z_1}z_2t_{yz_1'}z_2' = w$. We also "include" $\mathcal{S}_0$ in the equation $t_{t_{xy}z_1}z_2t_{yz_1'}z_2' = w$, as well as all the variables that appear in $\mathcal{S}_1$, obtaining the equation

$$t_{t_{xy}z_1}z_2t_{yz_1'}z_2'\#_1z\#_2z\#_3x\#_4y\#_5t_{xy}\# = w\#_1z_1z_2\#_2z_1'z_2'\#_3x\#_4y\#_5t_{xy}\#,$$

where $\#_1$, $\#_2$, $\#_3$, $\#_4$ and $\#_5$ represent new parameters evaluated to themselves. Considering the cumulative content of the evaluation of each variable under δ, we conclude that the system $\mathcal{S}_2$ is given by

$$\mathcal{S}_2 = \{ta^{\omega} = t\colon t \in \{x, y, t_{xy}, t_{yz_1'}, t_{t_{xy}z_1}\}\} \cup \{tb^{\omega} = t\colon t \in \{x, t_{xy}, t_{t_{xy}z_1}\}\}.$$

On the other hand, if we choose first the factor yz, then the factorization of $\delta(z)$ we should consider is $\delta(z) = a \cdot bc$. Then, we put in $\mathcal{S}_0$ the equation $z = z_1 z_2$, in $\mathcal{S}_1$ the equation $t_{yz_1} = yz_1$, and we transform the initial equation in $xt_{yz_1} z_2 t_{yz_1} z_2 = w$, where z_1, z_2, t_{yz_1} are new variables. We set $\delta(z_1) = a$, $\delta(z_2) = bc$ and $\delta(t_{yz_1}) = a^{\omega-1}a$. It remains to consider one factor, namely, xt_{yz_1}. As $c(\delta(t_{yz_1})) \subseteq \vec{c}(\delta(x))$, we add to $\mathcal{S}_1$ the equation $t_{xt_{yz_1}} = xt_{yz_1}$, where $t_{xt_{yz_1}}$ is a new variable, and the initial equation turns into $t_{xt_{yz_1}} z_2 t_{yz_1} z_2 = w$. The extended solution δ sends $t_{xt_{yz_1}}$ to $(ab)^{\omega-1}a^{\omega-1}a$. By now, the systems $\mathcal{S}_0$ and $\mathcal{S}_1$ are given by $\mathcal{S}_0 = \{z = z_1 z_2\}$ and $\mathcal{S}_1 = \{t_{yz_1} = yz_1, t_{xt_{yz_1}} = xt_{yz_1}\}$. The system $\mathcal{S}_0$ and the variables appearing in the system $\mathcal{S}_1$ may be included in the equation $t_{xt_{yz_1}} z_2 t_{yz_1} z_2 = w$, by transforming it in the equation

$$t_{xt_{yz_1}} z_2 t_{yz_1} z_2 \#_1 z \#_2 x \#_3 y \# = w \#_1 z_1 z_2 \#_2 x \#_3 y \#,$$

where $\#_1$, $\#_2$ and $\#_3$ are new parameters evaluated to themselves. Finally, in this case, the system $\mathcal{S}_2$ is given by

$$\mathcal{S}_2 = \{ta^\omega = t \colon t \in \{x, y, t_{yz_1}, t_{xt_{yz_1}}\}\} \cup \{tb^\omega = t \colon t \in \{x, t_{xt_{yz_1}}\}\}.$$

We end this section with a result regarding reducibility of pseudovarieties of groups that is later used to derive reducibility properties of DRH.

Lemma 5.9. *Let σ be an implicit signature such that $\kappa \subseteq \langle \sigma \rangle$, H a pseudovariety of groups that is σ-reducible for finite systems of κ-equations, and $\mathcal{S}$ a finite system of κ-equations that admits the continuous homomorphism $\delta : \overline{\Omega}_X \mathsf{S} \to (\overline{\Omega}_A \mathsf{S})^I$ as a solution modulo H. Then, $\mathcal{S}$ has a solution modulo H in σ-words, say ε, such that $\vec{c}(\varepsilon(x)) = \vec{c}(\delta(x))$, for all $x \in X$.*

Proof. Let x be a variable of X. Given $i \leq \lceil \delta(x) \rceil$, we denote $\mathsf{lbf}_i(\delta(x))$ by $\delta(x)_i a_{x,i}$ and write $\delta(x) = \mathsf{lbf}_1(\delta(x)) \cdots \mathsf{lbf}_i(\delta(x)) \delta(x)'_i$. If $\vec{c}(\delta(x))$ is the empty set, then we have

$$\varphi(\delta(x)) = \varphi(\mathsf{lbf}_1(\delta(x)) \cdots \mathsf{lbf}_{\lceil \delta(x) \rceil}(\delta(x))). \tag{5.4}$$

For the remaining variables, since X, A, and S are finite, there are integers $1 < k < \ell$ such that

$$\begin{gathered} \vec{c}(\delta(x)) = c(\mathsf{lbf}_{k+1}(\delta(x))); \\ \varphi(\mathsf{lbf}_1(\delta(x)) \cdots \mathsf{lbf}_k(\delta(x))) = \varphi(\mathsf{lbf}_1(\delta(x)) \cdots \mathsf{lbf}_\ell(\delta(x))), \end{gathered}$$

for all $x \in X$ with $\vec{c}(\delta(x)) \neq \emptyset$. In particular, from the second equality we deduce

$$\varphi(\delta(x)) = \varphi(\mathsf{lbf}_1(\delta(x)) \cdots \mathsf{lbf}_k(\delta(x))) \varphi(\mathsf{lbf}_{k+1}(\delta(x)) \cdots \mathsf{lbf}_\ell(\delta(x)))^\omega \varphi(\delta(x)'_k). \tag{5.5}$$

We consider a new set of variables X' given by

$$\begin{aligned} X' = \{&y_{x,1}, b_{x,1}, \ldots, y_{x,\lceil \delta(x) \rceil}, b_{x,\lceil \delta(x) \rceil} \colon x \in X \text{ and } \vec{c}(\delta(x)) = \emptyset\} \\ &\uplus \{y_{x,1}, b_{x,1}, \ldots, y_{x,\ell}, b_{x,\ell}, y'_x \colon x \in X \text{ and } \vec{c}(\delta(x)) \neq \emptyset\} \end{aligned}$$

and a new system of equations $\mathcal{S}'$ with variables in X' obtained from $\mathcal{S}$ by substituting each variable x by the product

$$P_x = y_{x,1}b_{x,1}\cdots y_{x,\lceil\delta(x)\rceil}b_{x,\lceil\delta(x)\rceil}, \tag{5.6}$$

whenever $\vec{c}(\delta(x)) = \emptyset$, and by the product

$$P_x = y_{x,1}b_{x,1}\cdots y_{x,k}b_{x,k}(y_{x,k+1}b_{x,k+1}\cdots y_{x,\ell}b_{x,\ell})^{\omega}y'_x, \tag{5.7}$$

otherwise. The constraints for the variables in X' are given by the clopen sets $K_{y_{x,i}} = \varphi^{-1}(\varphi(\delta(x)_i))$, $K_{y'_x} = \varphi^{-1}(\varphi(\delta(x)'_k))$, and $K_{b_{x,i}} = \{a_{x,i}\}$. Finally, we let $\delta' : \overline{\Omega}_{X'}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be the homomorphism defined by $\delta'(y_{x,i}) = \delta(x)_i$, $\delta'(y'_x) = \delta(x)'_k$, and $\delta'(b_{x,i}) = a_{x,i}$. Since H satisfies $\delta'(P_x) = \delta(x)$, for every variable $x \in X$ (check (5.6) and (5.7)), the homomorphism δ' is a solution modulo H of $\mathcal{S}'$. Therefore, as we are assuming that the pseudovariety H is σ-reducible for finite systems of κ-equations, there is a solution $\varepsilon' : \overline{\Omega}_{X'}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ modulo H of $\mathcal{S}'$ such that $\varepsilon'(X') \subseteq \Omega_A^{\sigma}\mathsf{S}$. On the other hand, this homomorphism ε' defines a solution in σ-words modulo H of the original system $\mathcal{S}$, namely, by letting $\varepsilon(x) = \varepsilon'(P_x)$ for each $x \in X$. Moreover, since $K_{b_{x,i}} = \{a_{x,i}\}$ we necessarily have $\varepsilon'(b_{x,i}) = a_{x,i}$ and the fact that S has a content function entails that $c(\varepsilon'(y_{x,i})) = c(\delta'(y_{x,i})) = c(\delta(x)_i)$ and, similarly, that $c(\varepsilon'(y'_x)) = c(\delta'(y'_x)) = c(\delta(x)'_k)$. In particular, $a_{x,i}$ does not belong to $c(\delta(x)_i)$. So, the iteration of left factorization to the right of $\varepsilon(x)$ is the one induced by the product P_x, implying that $\vec{c}(\varepsilon(x)) = \vec{c}(\delta(x))$ as intended. Finally, we verify that the constraints on X are satisfied by ε:

$$\begin{aligned}
\varphi(\varepsilon(x)) = \varphi(\varepsilon'(P_x)) &= \begin{cases} \varphi(\varepsilon'(y_{x,1}b_{x,1}\cdots y_{x,\lceil\delta(x)\rceil}b_{x,\lceil\delta(x)\rceil})), & \text{if } \vec{c}(\delta(x)) = \emptyset \\ \varphi(\varepsilon'(y_{x,1}b_{x,1}\cdots y_{x,k}b_{x,k}(y_{x,k+1}b_{x,k+1}\cdots y_{x,\ell}b_{x,\ell})^{\omega}y'_x)), & \text{otherwise} \end{cases} \\
&= \begin{cases} \varphi(\delta(x)_1a_{x,1})\cdots\varphi(\delta(x)_{\lceil\delta(x)\rceil}a_{x,\lceil\delta(x)\rceil}), & \text{if } \vec{c}(\delta(x)) = \emptyset \\ \varphi(\delta(x)_1a_{x,1})\cdots\varphi(\delta(x)_k a_{x,k}) \\ \quad\cdot(\varphi(\delta(x)_{k+1}a_{x,k+1})\cdots\varphi(\delta(x)_\ell a_{x,\ell}))^{\omega}\varphi(\delta(x)'_k), & \text{otherwise} \end{cases} \\
&= \begin{cases} \varphi(\mathsf{lbf}_1(\delta(x))\cdots\mathsf{lbf}_{\lceil\delta(x)\rceil}(\delta(x))), & \text{if } \vec{c}(\delta(x)) = \emptyset \\ \varphi(\mathsf{lbf}_1(\delta(x))\cdots\mathsf{lbf}_k(\delta(x))) \\ \quad\cdot\varphi(\mathsf{lbf}_{k+1}(\delta(x))\cdots\mathsf{lbf}_\ell(\delta(x)))^{\omega}\varphi(\delta(x)'_k), & \text{otherwise} \end{cases} \\
&\overset{(5.4),(5.5)}{=} \varphi(\delta(x)).
\end{aligned}$$

Hence, the homomorphism ε plays the desired role. □

5.3 Periodicity modulo DRH

We proceed with the statement and proof of two results concerning a certain periodicity of members of $\overline{\Omega}_A\mathsf{DRH}$. We first need a few auxiliary lemmas.

Lemma 5.10 (cf. [10, Lemma 5.1]). *Let u, v be pseudowords such that $uv^{\omega} \mathcal{R} v^{\omega}$ modulo* DRH. *If the inclusion $c(u) \subsetneq c(v)$ holds, then the pseudovariety* DRH *satisfies $uv = v$.*

Proof. Let a be a letter in $c(v) \setminus c(u)$. We may factorize $v = v_1 a v_2$ with $a \notin c(v_1)$ (recall Corollary 2.14). Since DRH satisfies $uv^\omega = v^\omega$, it follows that it also satisfies $uv_1 a v_2 v^{\omega-1} = v_1 a v_2 v^{\omega-1}$. Since a does not belong to $c(uv_1)$, again Corollary 2.14 implies that DRH satisfies $uv_1 = v_1$, resulting in turn that it satisfies $uv = v$. □

Lemma 5.11 (cf. [10, Lemma 5.2]). *If $u, v \in \overline{\Omega}_A \mathsf{S}$ are such that* DRH *satisfies the pseudoidentity $uv^2 = v^2$, then it also satisfies $vu = u$.*

Proof. The fact that DRH satisfies $uv^2 = v^2$ implies that it also satisfies $uv^\omega = v^\omega$. Therefore, in the case where $c(u) \subsetneq c(v)$ we may use Lemma 5.10 to conclude that DRH satisfies $uv = v$. Let us suppose that $c(u) = c(v)$. Then, the pseudoidentity $uv^2 =_{\mathsf{DRH}} v^2$ yields $u^\omega v^2 = v^2$ modulo DRH. Since we are assuming the equality $c(u) = c(v)$, it follows that, $\vec{c}(v) = c(u) = c(v)$. Hence, the following holds modulo DRH:

$$uv \,\mathcal{R}\, uv^2 = v^2 \,\mathcal{R}\, v.$$

As, in addition H satisfies $uv = v$ (because $\mathsf{H} \supseteq \mathsf{DRH}$ and, consequently, $uv^2 =_{\mathsf{H}} v^2$), by Lemma 2.32, the pseudovariety DRH satisfies $uv = v$. □

Now, we are ready to prove the announced results on the periodicity in $\overline{\Omega}_A \mathsf{DRH}$.

Lemma 5.12 (cf. [10, Lemma 5.4]). *Let x and y be pseudowords such that $x^\omega = y^\omega$ modulo* DRH. *If the products $x \cdot x$ and $y \cdot y$ are reduced, then there are pseudowords $u \in \overline{\Omega}_A \mathsf{S}$ and $v, w \in (\overline{\Omega}_A \mathsf{S})^I$, and positive integers k, ℓ such that the following pseudoidentities hold in* DRH

$$\begin{aligned} x &= u^k v, \\ y &= u^\ell w, \\ u &= vu = wu, \end{aligned}$$

and all the products $u \cdot u$, $u \cdot v$, $u \cdot w$, $v \cdot u$, and $w \cdot u$ are reduced, whenever the second factor is nonempty.

Proof. We argue by transfinite induction on $\alpha = \max\{\alpha_x, \alpha_y\}$.

If $\alpha_x = \alpha_y$, since the products $x \cdot x$ and $y \cdot y$ are reduced, we then have $x = y$ in DRH, by Corollary 2.31. So, we may choose $u = x$, $v = w = I$, and $k = \ell = 1$.

From now on, we assume that the pseudovariety DRH does not satisfy $x = y$. Suppose, without loss of generality, that $\alpha_x < \alpha_y = \alpha$. Again, by Corollary 2.31, DRH satisfies

$$y = y^\omega[0, \alpha_y[= x^\omega[0, \alpha_y[= x^\omega[0, \alpha_x[x^\omega[\alpha_x, \alpha_y[= x x^\omega[\alpha_x, \alpha_y[$$

and so, x is a prefix of y modulo DRH. Thus, the set

$$P = \{m \geq 1 \colon \exists y_1, \ldots, y_m \in \overline{\Omega}_A \mathsf{S} \text{ such that } y \leq_{\mathcal{R}} y_1 \cdots y_m \text{ and } y_i =_{\mathsf{DRH}} x, \text{ for } i = 1, \cdots m\}$$

is nonempty. If it were unbounded then, since $x \cdot x$ is a reduced product and by definition of cumulative content, every letter of $c(x) = c(y_i)$ would be in the cumulative content of y, so that $\vec{c}(y) = c(x) = c(y)$, a contradiction with the hypothesis that $y \cdot y$ is a reduced product. We take $m = \max(P)$ and let

$y = y_1 \cdots y_m y'$, with $y_i =_{\mathsf{DRH}} x$, for $i = 1, \ldots, m$. Since $x^\omega =_{\mathsf{DRH}} y^\omega$, we deduce that DRH satisfies

$$x^\omega = y^\omega = y_1 \cdots y_m y' y^{\omega-1} = x^m y' y^{\omega-1}$$

which in turn, since the involved products are reduced, implies that DRH also satisfies

$$x^{\omega-m} = y' y^{\omega-1}.$$

In particular, as $y^\omega = x^\omega$ in DRH (and so, $c(x) = c(y)$), we may conclude that DRH satisfies

$$x^\omega = y' y^{\omega-1} x^m = y' x^\omega y^{\omega-1} x^m \mathrel{\mathcal{R}} y' x^\omega. \tag{5.8}$$

We now distinguish two cases.

- If $c(y') \subsetneqq c(x)$ then, by Lemma 5.10, the pseudovariety DRH satisfies $x = y'x$, so that we may choose $u = x$, $v = I$, $k = 1$, $w = y'$, and $\ell = m$.
- If $c(y') = c(x)$ then, successively multiplying by y' on the left the leftmost and rightmost sides of (5.8), we get that the relation $x^\omega \mathrel{\mathcal{R}} y'^\omega x^\omega = y'^\omega$ holds in DRH. As x^ω and y'^ω are both the identity in the same regular $\mathcal{R}$-class, hence in the same group, the mentioned relation is actually an equality: $x^\omega =_{\mathsf{DRH}} y'^\omega$. Furthermore, the product $y' \cdot y'$ is reduced because so is $y \cdot y$. Indeed, $\vec{c}(y') = \vec{c}(y)$, the first letters of y' and x coincide and, in turn, the first letter of x is the first letter of y. Consequently, y' and x verify the conditions of applicability of the lemma and have associated a smaller induction parameter. In fact, maximality of m guarantees that $\alpha_{y'} \leq \alpha_x < \alpha_y = \alpha$. By induction hypothesis, there exist $u \in \overline{\Omega}_A\mathsf{S}$, $v, w \in (\overline{\Omega}_A\mathsf{S})^I$, and $k, \ell > 0$ such that the identities
$$\begin{aligned} x &= u^k v, \\ y' &= u^\ell w, \\ u &= vu = wu \end{aligned} \tag{5.9}$$
are valid in DRH, and where all products, including $u \cdot u$ are reduced. The computation
$$y = x^m y' = (u^k v)^m u^\ell w = u^{km+\ell} w$$
modulo DRH justifies that, except for the value of ℓ, which now is $km + \ell$, the choice in (5.9) also fits the original pair x, y. □

Proposition 5.13 (cf. [10, Proposition 5.5]). *Let $x_0, x_1, \ldots, x_n \in \overline{\Omega}_A\mathsf{S}$ be such that $x_0^\omega = x_1^\omega = \cdots = x_n^\omega$ modulo* DRH *and suppose that, for $i = 0, 1, \ldots, n$, the product $x_i \cdot x_i$ is reduced. Then, there exist pseudowords $u \in \overline{\Omega}_A\mathsf{S}$, $v_0, v_1, \ldots, v_n \in (\overline{\Omega}_A\mathsf{S})^I$, and positive integers $p_0, p_1, \ldots, p_n$ such that the pseudovariety* DRH *satisfies*

$$\begin{aligned} x_i &= u^{p_i} v_i, \quad \textit{for } i = 0, 1, \ldots, n, \\ u &= v_i u, \quad \textit{for } i = 0, 1, \ldots, n, \end{aligned}$$

and all the products $u \cdot u$, $u \cdot v_i$, and $v_i \cdot u$ are reduced, whenever the second factor is nonempty.

Proof. We argue by induction on n. When $n = 1$, the claim amounts to the result of the preceding lemma. Suppose that $n > 1$. Applying Lemma 5.12 to each pair (x_0, x_i), for $i = 1, \ldots, n$, we obtain pseudowords $w_i \in \overline{\Omega}_A \mathsf{S}$, $r_i, s_i \in (\overline{\Omega}_A \mathsf{S})^I$, and positive integers k_i, ℓ_i such that DRH satisfies

$$x_0 = w_i^{k_i} r_i, \tag{5.10}$$

$$x_i = w_i^{\ell_i} s_i, \tag{5.11}$$

$$w_i = r_i w_i = s_i w_i, \tag{5.12}$$

and all the products of nonempty pseudowords are reduced. Hence, for $i = 1, \ldots, n$, $x_0^\omega = (w_i^{k_i} r_i)^\omega = w_i^\omega r_i$ modulo DRH. Also, as DRH $\supseteq$ H satisfies $w_i = r_i w_i$, the pseudovariety H satisfies $r_i = 1$. By Corollary 2.20, we deduce that $x_0^\omega = w_1^\omega = \cdots = w_n^\omega$ in DRH. Applying the induction hypothesis to $w_1, \ldots, w_n$ we get, in turn, the existence of pseudowords $y \in \overline{\Omega}_A \mathsf{S}$, $z_1, \ldots, z_n \in (\overline{\Omega}_A \mathsf{S})^I$, and of positive integers $m_1, \ldots, m_n$ such that

$$w_i = y^{m_i} z_i \tag{5.13}$$

$$y = z_i y \tag{5.14}$$

are equalities valid in DRH, and all the products of nonempty pseudowords are reduced. Now, we compute

$$\begin{aligned} x_0 &\overset{(5.10)}{=} w_i^{k_i} r_i \overset{(5.13)}{=} (y^{m_i} z_i)^{k_i} r_i \overset{(5.14)}{=} y^{m_i k_i} z_i r_i, \\ x_i &\overset{(5.11)}{=} w_i^{\ell_i} s_i \overset{(5.13)}{=} (y^{m_i} z_i)^{\ell_i} s_i \overset{(5.14)}{=} y^{m_i \ell_i} z_i s_i \end{aligned}$$

modulo DRH, for $i = 1, \ldots, n$. On the other hand, we also have the following modulo DRH:

$$\begin{aligned} y^{m_i} z_i \overset{(5.13)}{=} w_i \overset{(5.12)}{=} r_i w_i \overset{(5.13)}{=} r_i (y^{m_i} z_i) &\implies y^{m_i} z_i y^{2^{m_i} - m_i} = r_i y^{m_i} z_i y^{2^{m_i} - m_i} \\ &\overset{(5.14)}{\iff} y^{2^{m_i}} = r_i y^{2^{m_i}} \\ &\overset{\text{Lemma 5.11}}{\implies} y = r_i y \\ &\implies z_i y = z_i r_i y \\ &\overset{(5.14)}{\iff} y = z_i r_i y. \end{aligned}$$

Similarly, we may deduce that DRH satisfies $y = z_i s_i y$.

We just proved that it is enough to take

$$\begin{aligned} &u = y, \\ &v_0 = z_1 r_1, \text{ and } v_i = z_i s_i, \text{ for } i = 1, \ldots, n, \\ &p_0 = m_1 k_1, \text{ and } p_i = m_i \ell_i, \text{ for } i = 1, \ldots, n, \end{aligned}$$

in order to obtain the desired pseudoidentities. $\square$

5.4 Systems of boundary relations and their models

In this section, we define some tools that turn out to be useful when proving that DRH is completely κ-reducible. The original notion of a boundary equation was given by Makanin [50] and it was later adapted by Almeida, Costa and Zeitoun [10] to deal with the problem of complete κ-reducibility of the pseudovariety R. Here, we extend the definitions used in [10] to the context of the pseudovariety DRH, for any pseudovariety of groups H, and use them to prove that, under certain conditions, the pseudovariety DRH is σ-reducible for finite systems of κ-equations.

From hereon, we fix a word equation $u = v$ and a solution $\delta : \overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ modulo DRH of $\mathcal{S}_{u=v}$ (recall (5.3)) that satisfies the conditions stated in Proposition 5.7, and subject to the constraints given by the pair $(\varphi : \overline{\Omega}_A\mathsf{S} \to S, \nu : X \to S)$. We write $K_x = \varphi^{-1}(\nu(x))$. The implicit signature σ is assumed to satisfy the condition (sig) and be such that $\kappa \subseteq \langle\sigma\rangle$.

By a *system of boundary relations* we mean a tuple $\mathcal{S} = (\mathcal{X}, J, \zeta, M, \chi, \mathsf{right}, \mathcal{B}, \mathcal{B}_\mathsf{H})$ where

- $\mathcal{X}$ is a finite set equipped with an involution without fixed points $x \mapsto \bar{x}$, whose elements are called *variables*;
- J is a finite set equipped with a total order $\leq$, whose elements are called *indices*. If i and j are two consecutive indices, then we write $i \prec j$ and we denote i by j^-;
- $\zeta : \{(i,j) \in J \times J \colon i \prec j\} \to 2^{S \times S^I}$ is a function that is useful to deal with the constraints;
- $M : \{(i,j,\vec{s}) \in J \times J \times (S \times S^I) \colon i \prec j, \vec{s} \in \zeta(i,j)\} \to \omega \setminus \{0\}$ is a function that determines the number of different factorizations in $\overline{\Omega}_A\mathsf{S}$ modulo DRH that we assign to suitable segments of the solution;
- $\chi : \{(i,j) \in J \times J \colon i \prec j\} \to 2^A$ is a function whose aim is to control the cumulative content of suitable segments of the solution;
- $\mathsf{right} : \mathcal{X} \to J$ is a function that helps in defining the relations we need to attain our goal;
- $\mathcal{B}$ is a subset of $J \times \mathcal{X} \times J \times \mathcal{X}$, whose elements are of the form $(i,x,j,\bar{x})$. Moreover, if $(i,x,j,\bar{x})$ is an element of $\mathcal{B}$, then so is $(j,\bar{x},i,x)$. The elements of $\mathcal{B}$ are called *boundary relations* and the boundary relation $(j,\bar{x},i,x)$ is said to be the *dual boundary relation* of $(i,x,j,\bar{x})$. The pairs (i,x) and $(j,\bar{x})$ are *boxes of* $\mathcal{B}$. Together with the right function, the set $\mathcal{B}$ encodes the relations we want to be satisfied in DRH;
- finally, for each pair of indices i, j such that $i \prec j$, we consider a symbol $(i \mid j)$ and, for each pair $(\vec{s}, \mu) \in \zeta(i,j)$, we consider another symbol $\{i \mid j\}_{\vec{s},\mu}$. These symbols are understood as variables and we denote by $X_{(J,\zeta,M)}$ the set of those variables:

$$X_{(J,\zeta,M)} = \{(i \mid j) \colon i,j \in J, i \prec j\} \cup \{\{i \mid j\}_{\vec{s},\mu} \colon (i,j,\vec{s}) \in \mathsf{Dom}(M) \text{ and } \mu \in M(i,j,\vec{s})\} \tag{5.15}$$

Then, $\mathcal{B}_\mathsf{H}$ is a finite set of κ-equations with variables in $X_{(J,\zeta,M)}$ whose solutions are meant to be taken over H. If $i_0 \prec \cdots \prec i_n$ is a chain of indices in J, then we denote by $(i_0 \mid i_n)$ the product of variables $\prod_{k=1}^n (i_{k-1} \mid i_k)$.

Given a variable $x \in \mathcal{X}$, the *left of* x is $\mathsf{left}(x) = \min\{i \in I\colon$ there exists a box (i,x) in $\mathcal{B}\}$, when it is defined.

We let $\mathsf{prod} : \overline{\Omega}_A\mathsf{S} \times (\overline{\Omega}_A\mathsf{S})^I \to \overline{\Omega}_A\mathsf{S}$ be the function sending each pair of pseudowords (u,v) to its product uv.

A *model* of the system of boundary relations $\mathcal{S}$ is a triple $\mathcal{M} = (w, \iota, \Theta)$, where

- w is a possibly empty pseudoword;
- $\iota : J \to \alpha_w + 1$ is an injective function that preserves the order and such that, if J is not the empty set then ι sends $\min(J)$ to 0 and $\max(J)$ to α_w;
- for each triple $(i, j, \vec{s})$ in $\mathsf{Dom}(M)$ and each μ in $M(i, j, \vec{s})$, the element $\Theta(i, j, \vec{s}, \mu)$ is a pair $(\Phi(i, j, \vec{s}, \mu), \Psi(i, j, \vec{s}, \mu))$ of $\overline{\Omega}_A\mathsf{S} \times (\overline{\Omega}_A\mathsf{S})^I$ such that $c(\Psi(i, j, \vec{s}, \mu)) \subseteq \vec{c}(\Phi(i, j, \vec{s}, \mu))$.

Notation 5.14. *When there exists a map $\iota : J \to \alpha_w + 1$ as above, we may write $w(i, j)$ instead of $w[\iota(i), \iota(j)[$ (recall Notation 2.25).*

Moreover, the following properties are required for $\mathcal{M}$:

(M.1) if $(i, j, \vec{s}) \in \mathsf{Dom}(M)$ and $\mu \in M(i, j, \vec{s})$, then DRH satisfies $\mathsf{prod} \circ \Theta(i, j, \vec{s}, \mu) = w(i, j)$;

(M.2) if $(i, j, \vec{s}) \in \mathsf{Dom}(M)$, $\vec{s} = (s_1, s_2)$, and $\mu \in M(i, j, \vec{s})$, then

$$\varphi(\Phi(i, j, \vec{s}, \mu)) = s_1 \text{ and } \varphi(\Psi(i, j, \vec{s}, \mu)) = s_2;$$

(M.3) if $i \prec j$, then $\vec{c}(w(i, j)) = \chi(i, j)$;

(M.4) if $(i, x, j, \bar{x}) \in \mathcal{B}$, then DRH satisfies $w(i, \mathsf{right}(x)) \mathrel{\mathcal{R}} w(j, \mathsf{right}(\bar{x}))$;

(M.5) let $\mathcal{C} := (J, \iota, M, \Theta)$ and $\delta_{w,\mathcal{C}} : \overline{\Omega}_{X_{(J,\zeta,M)}}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be the unique continuous homomorphism defined by

$$\begin{aligned} \delta_{w,\mathcal{C}}(i \mid j) &= w(i, j); \\ \delta_{w,\mathcal{C}}(\{i \mid j\}_{\vec{s},\mu}) &= \Psi(i, j, \vec{s}, \mu). \end{aligned} \tag{5.16}$$

Then, $\delta_{w,\mathcal{C}}$ is a solution modulo H of $\mathcal{B}_{\mathsf{H}}$.

We say that $\mathcal{M}$ is a *model of* $\mathcal{S}$ *in* σ*-words* if $w \in (\Omega_A^\sigma\mathsf{S})^I$ and the coordinates of Θ are given by σ-words.

By Proposition 5.7, to prove that DRH is completely κ-reducible, it is enough to prove that DRH is κ-reducible for certain systems of equations of the form $\mathcal{S}_{u=v}$. With that in mind, we associate to such a system $\mathcal{S}_{u=v}$ a system of boundary relations, denoted $\overline{\mathcal{S}}_{u=v}$. Then, we construct a model of $\overline{\mathcal{S}}_{u=v}$ and prove that the existence of a model in κ-words entails the existence of a solution of the original system $\mathcal{S}_{u=v}$ also in κ-words (Proposition 5.16). Although we are mainly interested in the case where the implicit signature is κ, we formulate the results more generally for an arbitrary implicit signature σ, such that $\kappa \subseteq \langle\sigma\rangle$. Even though some of the results may still hold in general, since the condition (sig) is crucial for the validity of Theorem 5.26, we further require that σ satisfies it.

Let $\delta : \overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be a solution modulo DRH of $\mathcal{S}_{u=v} = \{u' = v'\} \cup \mathcal{S}_1 \cup \mathcal{S}_2$ such that δ is reduced with respect to $u' = v'$ and for every equation $xy = z$ of $\mathcal{S}_1$ we have $c(\delta(y)) \subseteq \vec{c}(\delta(x))$ (recall

Proposition 5.7). Suppose that $u' = x_1 \cdots x_r$ and $v' = x_{r+1} \cdots x_t$, and write $\mathcal{S}_1 = \{x_{(i)}y_{(i)} = z_{(i)}\}_{i=1}^N$ and $\mathcal{S}_2 = \{xa^\omega = x \colon a \in A_x\}_{x \in X}$. Let G be an undirected graph whose vertices are given by the set $\{1,\dots,t\}$ and that has an edge connecting the vertices p and q if and only if $p \neq q$ and either $x_p = x_q$ or $\{x_p, x_q\} = \{x_{(i)}, z_{(i)}\}$ for a certain i. Let $\widehat{G}$ be a spanning forest for G. We define

$$\overline{\mathcal{S}}_{u=v} = (\mathcal{X}, J, \zeta, M, \chi, \mathsf{right}, \mathcal{B}, \mathcal{B}_{\mathsf{H}}) \tag{5.17}$$

as follows:

- the set of variables is $\mathcal{X} = \{(p,q) \colon \text{ there is an edge in } \widehat{G} \text{ connecting } p \text{ and } q\} \uplus \{\mathsf{l}\} \uplus \{\mathsf{r}\}$. The involution in $\mathcal{X}$ is given by $\overline{(p,q)} = (q,p)$ and by $\bar{\mathsf{l}} = \mathsf{r}$;
- the set of indices is $J = \{i_0, \dots, i_t\}$ with $i_0 \prec \cdots \prec i_t$;
- the function ζ is defined by $\zeta(i_{p-1}, i_p) = \{(\nu(x_p), I)\}$ for every $p = 1, \dots, t$;
- we set $M(i_{p-1}, i_p, (\nu(x_p), I)) = 1$ for every $p = 1, \dots, t$;
- the function χ sends each pair (i_{p-1}, i_p) to the set A_{x_p};
- the right function is given by $\mathsf{right}(p,q) = i_p$, $\mathsf{right}(\mathsf{l}) = i_r$, and $\mathsf{right}(\mathsf{r}) = i_t$;
- the set of boundary relations $\mathcal{B}$ contains the boundary relations $(i_0, \mathsf{l}, i_r, \mathsf{r})$, and $(i_r, \mathsf{r}, i_0, \mathsf{l})$ plus all the boundary relations of the form $(i_{p-1}, (p,q), i_{q-1}, (q,p))$, where $(p,q) \in \mathcal{X}$;
- we put in $\mathcal{B}_{\mathsf{H}}$ the equation $(i_0 \mid i_r) = (i_r \mid i_t)$ and, for each $(p,q) \in \mathcal{X} \setminus \{\mathsf{l}, \mathsf{r}\}$, the equation $(i_{p-1} \mid i_p) = (i_{q-1} \mid i_q)$ if $x_p = x_q$, or the equation $(i_{p-1} \mid i_p)(i_{m-1} \mid i_m) = (i_{q-1} \mid i_q)$ if $x_p x_m = x_q$ belongs to $\mathcal{S}_1$.

Example 5.15. Let $X = \{x,y,z\}$, $u = xyx$, $v = x^2 z$, and let $\delta : \overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be defined by $\delta(x) = a$, $\delta(y) = (ab)^{p^\omega}$, and $\delta(z) = (ba)^{p^\omega}$. Clearly, the homomorphism δ is a solution modulo DRH of $u = v$ and the system $\mathcal{S}_{u=v} = \{u' = v'\} \cup \mathcal{S}_1 \cup \mathcal{S}_2$ is given by $u' = xt_{yx}\#_1 y\#$, $v' = x^2 z\#_1 y\#$, $\mathcal{S}_1 = \{t_{yx} = yx\}$, and $\mathcal{S}_2 = \{ya^\omega = y, yb^\omega = y, za^\omega = z, zb^\omega = z, t_{yx}a^\omega = t_{yx}, t_{yx}b^\omega = t_{yx}\}$. The extended solution δ is obtained by letting $\delta(t_{yx}) = (ab)^{p^\omega}a$. Then, the set of indices is $J = \{i_0, i_1, \dots, i_{11}\}$. In order to determine $\mathcal{B}$, we first construct the graph G, by identifying the variables that are either repeated in the equation $u' = v'$ or that are the first variable of both members of an equation in $\mathcal{S}_1$ (see Figure 5.1).

Considering $\widehat{G}$ as illustrated in Figure 5.1, we obtain that the set of variables in the system of boundary relations $\overline{\mathcal{S}}_{u=v}$ is the following

$$\{(1,6),(6,1),(6,7),(7,6),(2,4),(4,2),(4,10),(10,4),(3,9),(9,3),(5,11),(11,5),\mathsf{l},\mathsf{r}\}$$

and we schematize the set of boundary relations $\mathcal{B}$ in Figure 5.2.

Finally, the set $\mathcal{B}_{\mathsf{H}}$ contains the following equations:

- $(i_0 \mid i_1) = (i_5 \mid i_6) = (i_6 \mid i_7)$,
- $(i_3 \mid i_4) = (i_9 \mid i_{10})$,
- $(i_2 \mid i_3) = (i_8 \mid i_9)$,

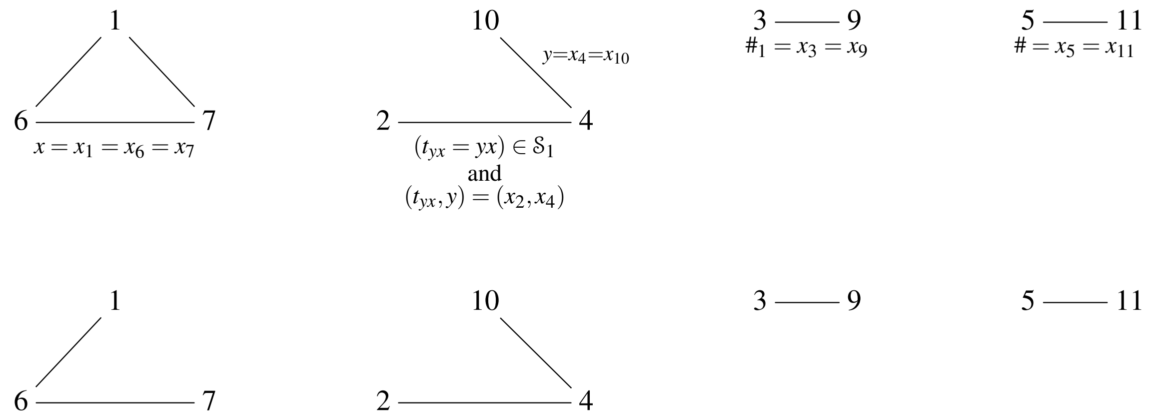

Fig. 5.1 The auxiliary graph G (above) and a possible choice for $\widehat{G}$ (below).

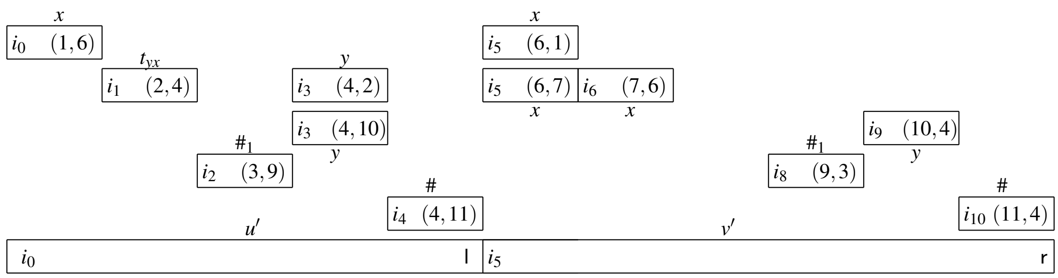

Fig. 5.2 The set of boundary relations $\mathcal{B}$.

- $(i_4 \mid i_5) = (i_{10} \mid i_{11})$,
- $(i_0 \mid i_5) = (i_5 \mid i_{11})$,
- $(i_1 \mid i_2) = (i_3 \mid i_4)(i_0 \mid i_1)$.

A candidate to be a model of $\overline{\mathcal{S}}_{u=v}$ is $\mathcal{M}_{u=v} = (w, \iota, \Theta)$, where $w = \delta(u'v')$, the mapping ι is given by $\iota(i_0) = 0$, and $\iota(i_p) = \alpha_{\delta(x_1 \cdots x_p)}$, for each $p = 1, \ldots, t$, and $\Theta(i_{p-1}, i_p, (\nu(x_p), I), 0) = (\delta(x_p), I)$, for $p = 1, \ldots, t$.

Proposition 5.16. *The tuple $\overline{\mathcal{S}}_{u=v}$ in (5.17) is a system of boundary relations which has $\mathcal{M}_{u=v}$ as a model. Moreover, if $\overline{\mathcal{S}}_{u=v}$ admits a model in σ-words, then the system of equations $\mathcal{S}_{u=v}$ has a solution modulo* DRH *in σ-words.*

Proof. For the first part, we notice that the Properties (M.1), (M.2) and (M.3) of the requirements for being a model are given for free from the construction. Let $(i, x, j, \overline{x})$ be a boundary relation. Since each equation $x_{(k)}y_{(k)} = z_{(k)}$ of $\mathcal{S}_1$ is such that the inclusion $c(\delta(y_{(k)})) \subseteq \vec{c}(\delta(x_{(k)}))$ holds, whenever an edge in the graph $\widehat{G}$ links two indices p and q, the elements $\delta(x_p) = \Phi(i_{p-1}, i_p, (\nu(x_p), I), 0)$ and $\delta(x_q) = \Phi(i_{q-1}, i_q, (\nu(x_q), I), 0)$ are $\mathcal{R}$-equivalent modulo DRH. Therefore, unless $(i, x, j, \overline{x})$ is one of the relations $(i_0, \mathsf{l}, i_r, \mathsf{r})$ or $(i_r, \mathsf{r}, i_0, \mathsf{l})$, the Property (M.4) is trivially satisfied. For those relations, we just need to observe that $w(i_0, \mathsf{right}(\mathsf{l})) = \delta(u')$ and $w(i_r, \mathsf{right}(\mathsf{r})) = \delta(v')$. The last Property (M.5) translates into the verification of pseudoidentities modulo H that are satisfied by the pseudovariety DRH by construction. This proves that $\mathcal{M}_{u=v}$ is a model of $\overline{\mathcal{S}}_{u=v}$.

For the second assertion, take $\mathcal{M}' = (w', \iota', \Theta')$ a model of $\overline{\mathcal{S}}_{u=v}$ in σ-words and let $\varepsilon : \overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be the continuous homomorphism that sends the variable x to $\mathsf{prod} \circ \Theta(i_{p-1}, i_p, (\nu(x_p), I), 0)$, where p is such that $x_p = x$. Such an x_p exists for every variable since we are assuming that all the variables occur in $u' = v'$. It is worth to mention that the value modulo DRH that we assign to $\varepsilon(x)$ when $x = x_p$ for some p does not depend on the chosen p. By Property (M.2), all the constraints imposed by $\mathcal{S}_{u=v}$ are satisfied by ε. The following computation shows that DRH satisfies $\varepsilon(u') = \varepsilon(v')$:

$$\begin{aligned}
\varepsilon(u') &= \varepsilon(x_1 \cdots x_r) = \varepsilon(x_1) \cdots \varepsilon(x_r)\\
&= \mathsf{prod} \circ \Theta'(i_0, i_1, (\nu(x_1), I), 0) \cdots \mathsf{prod} \circ \Theta'(i_{r-1}, i_r, (\nu(x_r), I), 0)\\
&\overset{\text{(M.1)}}{=} w'(i_0, i_1) \cdots w'(i_{r-1}, i_r) = w'(i_0, i_r)\\
&\overset{(*)}{=} w'(i_r, i_t) = w'(i_r, i_{r+1}) \cdots w'(i_{t-1}, i_t)\\
&\overset{\text{(M.1)}}{=} \mathsf{prod} \circ \Theta'(i_r, i_{r+1}, (\nu(x_{r+1}), I), 0) \cdots \mathsf{prod} \circ \Theta'(i_{t-1}, i_t, (\nu(x_t), I), 0)\\
&= \varepsilon(x_{r+1}) \cdots \varepsilon(x_t) = \varepsilon(x_{r+1} \cdots x_t) = \varepsilon(v').
\end{aligned}$$

The reason for $(*)$ is the fact that the relation $(i_0, \mathsf{l}, i_r, \mathsf{r})$ belongs to $\mathcal{B}$ and the equation $(i_0 \mid i_r) = (i_r \mid i_t)$ to $\mathcal{B}_{\mathsf{H}}$, together with Properties (M.4) and (M.5), and with Lemma 2.32. For the system $\mathcal{S}_2$, we point out that its only aim is to fix the cumulative content of the variables and Property (M.3) ensures that. Finally, let $x_p x_m = x_q$ be an equation of $\mathcal{S}_1$. Since for such an equation, we have a relation $(i_{p-1}, (p,q), i_{q-1}, (q,p))$ in $\mathcal{B}$ and an equation $(i_{p-1} \mid i_p)(i_{m-1} \mid i_m) = (i_{q-1} \mid i_q)$ in $\mathcal{B}_{\mathsf{H}}$, from (M.4) we deduce that $\varepsilon(x_p)$ and $\varepsilon(x_q)$ are $\mathcal{R}$-equivalent in DRH and from (M.5) that $\varepsilon(x_p)\varepsilon(x_m) = \varepsilon(x_q)$ is a valid pseudoidentity in H. In addition, the assumption that S has a content function together with Property (M.2) yield that $c(\delta(x)) = c(\varepsilon(x))$. In turn, we already observed that $\vec{c}(\delta(x)) = \vec{c}(\varepsilon(x))$. Therefore, as by construction of $\mathcal{S}_{u=v}$ we know that $c(\delta(x_m)) \subseteq \vec{c}(\delta(x_p))$, we have $\varepsilon(x_p)\varepsilon(x_m) \mathrel{\mathcal{R}} \varepsilon(x_q)$ modulo DRH, and from Lemma 2.32 we obtain that DRH satisfies $\varepsilon(x_p)\varepsilon(x_m) = \varepsilon(x_q)$. $\square$

The following criterion for having that a pseudovariety DRH is σ-reducible with respect to finite systems of κ-equations follows from Proposition 5.7 together with Proposition 5.16.

Corollary 5.17. *Let σ be an implicit signature such that $\kappa \subseteq \langle\sigma\rangle$. If every system of boundary relations which has a model also has a model in σ-words, then* DRH *is σ-reducible for finite systems of κ-equations.* $\square$

In particular, a pseudovariety DRH is completely κ-reducible provided every system of boundary relations which has a model also has a model in κ-words.

5.5 Factorization schemes

A *factorization scheme* for a pseudoword w is a tuple $\mathcal{C} = (J, \iota, M, \Theta)$, where:

- J is a totally ordered finite set;
- $\iota : J \to \alpha_w + 1$ is an injective function that preserves the order;
- $M : \{(i, j, \vec{s}) \in J \times J \times (S \times S^I) : i \prec j\} \to \omega \setminus \{0\}$ is a partial function;

- $\Theta: \{(i,j,\vec{s},\mu) \colon (i,j,\vec{s}) \in \mathsf{Dom}(M),\, \mu \in M(i,j,\vec{s})\} \to \overline{\Omega}_A \mathsf{S} \times (\overline{\Omega}_A \mathsf{S})^I$ is a function that sends the tuple $(i,j,\vec{s},\mu)$ to a pair $(\Phi(i,j,\vec{s},\mu), \Psi(i,j,\vec{s},\mu))$ satisfying $c(\Psi(i,j,\vec{s},\mu)) \subseteq \vec{c}(\Phi(i,j,\vec{s},\mu))$.

Moreover, if $(i,j,\vec{s}) \in \mathsf{Dom}(M)$ and $\mu \in M(i,j,\vec{s})$, then the following properties should be satisfied:

(FS.1) DRH satisfies $\mathsf{prod} \circ \Theta(i,j,\vec{s},\mu) = w[\iota(i), \iota(j)[$;

(FS.2) if $\vec{s} = (s_1, s_2)$, then $\varphi(\Phi(i,j,\vec{s},\mu)) = s_1$ and $\varphi(\Psi(i,j,\vec{s},\mu)) = s_2$.

We say that $\mathcal{C}$ is a *factorization scheme in σ-words* if the coordinates of Θ take σ-words as values. It is easy to check that, given a system of boundary relations $\mathcal{S}$ and a model $\mathcal{M}$ for $\mathcal{S}$, the pair $(\mathcal{S}, \mathcal{M})$ determines a factorization scheme for w, namely (J, ι, M, Θ), which we denote by $\mathcal{C}(\mathcal{S}, \mathcal{M})$. Furthermore, a factorization scheme $\mathcal{C}$ for a pseudoword w induces functions $\zeta_{w,\mathcal{C}}$ and $\chi_{w,\mathcal{C}}$ as follows

$$\begin{aligned} \zeta_{w,\mathcal{C}} : \{(i,j) \in J \times J \colon i \prec j\} &\to 2^{S \times S^I} \\ (i,j) &\mapsto \{\vec{s} \colon (i,j,\vec{s}) \in \mathsf{Dom}(M)\}, \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} \chi_{w,\mathcal{C}} : \{(i,j) \in J \times J \colon i \prec j\} &\to 2^{A} \\ (i,j) &\mapsto \vec{c}(w[\iota(i), \iota(j)[). \end{aligned} \tag{5.19}$$

The reason for using this notation becomes clear with the following lemma:

Lemma 5.18. *Let $\mathcal{S} = (\mathcal{X}, J, \zeta, M, \chi, \mathsf{right}, \mathcal{B}, \mathcal{B}_{\mathsf{H}})$ be a system of boundary relations, w a pseudoword, and $\mathcal{C} = (J, \iota, M, \Theta)$ a factorization scheme for w. We define $\mathcal{M} = (w, \iota, \Theta)$ as a candidate for a model of $\mathcal{S}$. If $\zeta = \zeta_{w,\mathcal{C}}$ and $\chi = \chi_{w,\mathcal{C}}$, then the Properties (M.1)–(M.3) are satisfied.*

Proof. We first notice that (5.18) guarantees that it is coherent to consider the same function M in both $\mathcal{S}$ and $\mathcal{C}$. Properties (M.1) and (M.2) are an immediate consequence of the Properties (FS.1) and (FS.2), respectively. Property (M.3) follows from the following computation:

$$\vec{c}(w(i,j)) \overset{\text{def.}}{=} \vec{c}(w[\iota(i), \iota(j)[) \overset{(5.19)}{=} \chi_{w,\mathcal{C}}(i,j). \quad \square$$

For $k = 1, 2$, let $\mathcal{C}_k = (J_k, \iota_k, M_k, \Theta_k)$ be a factorization scheme for w. We say that $\mathcal{C}_1$ is a *refinement* of $\mathcal{C}_2$ if the following properties are satisfied:

(R.1) $\mathsf{Im}(\iota_2) \subseteq \mathsf{Im}(\iota_1)$;

(R.2) there exists a function

$$\Lambda : \{(i,j,\vec{s},\mu) \colon (i,j,\vec{s}) \in \mathsf{Dom}(M_2),\, \mu \in M_2(i,j,\vec{s})\} \to \bigcup_{k \geq 1} (S \times S^I)^k \times \omega$$

such that, if $\Lambda(i,j,\vec{s},\mu) = ((\vec{t}_1, \ldots, \vec{t}_n), \mu')$, then the following holds:

(R.2.1) there are elements $i_0, \ldots, i_n$ in J_1 such that $i_0 \prec \cdots \prec i_n$, and the equalities $\iota_2(i) = \iota_1(i_0)$ and $\iota_2(j) = \iota_1(i_n)$ hold;

(R.2.2) $(i_{m-1}, i_m, \vec{t}_m) \in \mathsf{Dom}(M_1)$, for $m = 1, \ldots, n$;

(R.2.3) writing $\vec{s} = (s_1, s_2)$ and $\vec{t}_m = (t_{m,1}, t_{m,2})$ for $m = 1, \dots, n$, the following equalities hold:

$$s_1 = t_{1,1} t_{1,2} \cdots t_{n-1,1} t_{n-1,2} \cdot t_{n,1},$$
$$s_2 = t_{n,2};$$

(R.2.4) $\mu' \in M_1(i_{n-1}, i_n, \vec{t}_n)$ and $\Psi_2(i, j, \vec{s}, \mu) = \Psi_1(i_{n-1}, i_n, \vec{t}_n, \mu')$ modulo H.

We call the function Λ in (R.2) a *refining function from* $\mathcal{C}_2$ *to* $\mathcal{C}_1$.

Proposition 5.19 (cf. [10, Proposition 8.1]). *Let w be a pseudoword and let $\mathcal{C}_k = (J_k, \iota_k, M_k, \Theta_k)$ be a factorization scheme for w ($k = 1, 2$). Then, there is a factorization scheme $\mathcal{C}_3 = (J_3, \iota_3, M_3, \Theta_3)$ for w which is a common refinement of $\mathcal{C}_1$ and $\mathcal{C}_2$. Moreover, for every implicit signature σ satisfying the condition* (sig)*, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are both factorization schemes in σ-words, then we may choose $\mathcal{C}_3$ with the same property.*

Proof. Let $J_3 = \iota_1(J_1) \cup \iota_2(J_2)$ and $\iota_3 : J_3 \hookrightarrow \alpha_w + 1$ be the inclusion of ordinals. Starting with Θ_3 defined nowhere, we extend it inductively as follows. Fix $k, \ell \in \{1, 2\}$ with $k \neq \ell$, and let $i \prec j$ in J_k. Let $p_1, \dots, p_m \in J_\ell$ be the indices that are sent by ι_ℓ to an ordinal between $\iota_k(i)$ and $\iota_k(j)$ and suppose that $\{\beta_0, \beta_1, \dots, \beta_n\} = \iota_k(\{i, j\}) \cup \iota_\ell(\{p_1, \dots, p_m\})$ with $\beta_0 < \cdots < \beta_n$. Then, for $r = 1, \dots, n$, the

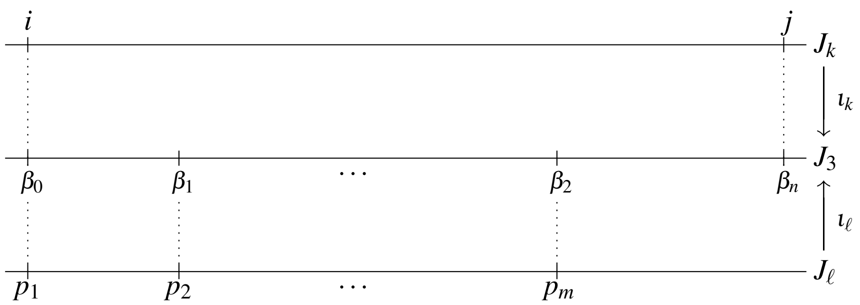

Fig. 5.3 The indices $i, j, p_1, \dots, p_m$ and the ordinals $\beta_0, \beta_1, \dots, \beta_n$.

relation $\beta_{r-1} \prec \beta_r$ holds in J_3. We fix $\vec{s} \in \zeta_{w, \mathcal{C}_k}(i, j)$, with $\vec{s} = (s_1, s_2)$. For each $r < n$, let

$$\vec{t}_r = (\varphi(\mathsf{prod} \circ \Theta_k(i, j, \vec{s}, 0)[\beta_{r-1}, \beta_r[), I),$$
$$\mu_r = \{\overline{\mu} \colon \Theta_3(\beta_{r-1}, \beta_r, \vec{t}_r, \overline{\mu}) \text{ is defined}\} + 1.$$

We set

$$\Theta_3(\beta_{r-1}, \beta_r, \vec{t}_r, \mu_r) = (\mathsf{prod} \circ \Theta_k(i, j, \vec{s}, 0)[\beta_{r-1}, \beta_r[, I).$$

For $r = n$, we take

$$\vec{t}_n = (\varphi(\Phi_k(i, j, \vec{s}, \mu))[\beta_{n-1}, \beta_n[, s_2).$$

Then, for each $\mu \in M_k(i,j,\vec{s})$, we set

$$\begin{aligned}\Theta_3(\beta_{n-1},\beta_n,\vec{t}_n,\mu') &= (\Phi_k(i,j,\vec{s},\mu)[\beta_{n-1},\beta_n[,\Psi_k(i,j,\vec{s},\mu)),\\ \Lambda_k(i,j,\vec{s},\mu) &= ((\vec{t}_1,\dots,\vec{t}_n),\mu'),\end{aligned}$$

where

$$\mu' = \{\overline{\mu}\colon \Theta_3(\beta_{n-1},\beta_n,\vec{t}_n,\overline{\mu}) \text{ is defined}\} + 1.$$

We repeat this process for all possible choices of k, ℓ, i, j, and $\vec{s}$. Finally, we set

$$M_3(\beta,\gamma,\vec{t}) = \{\mu\colon \Theta_3(\beta,\gamma,\vec{t},\mu) \text{ is defined}\}$$

whenever $\Theta_3(\beta,\gamma,\vec{t},0)$ is defined.

The way the construction was performed guarantees that $\mathcal{C}_3$ is actually a factorization scheme for w. Moreover, it follows from the fact that σ satisfies the condition (sig) that, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are both factorization schemes in σ-words, then so is $\mathcal{C}_3$.

Of course, Properties (R.1), (R.2.1) and (R.2.2) hold for each pair $(\mathcal{C}_k,\mathcal{C}_3)$. It remains to check Properties (R.2.3) and (R.2.4). For the former, given a pair $(\vec{s},\mu)$ of $\mathsf{Dom}(M_k)\times M(i,j,\vec{s})$ such that $\Lambda_k(i,j,\vec{s},\mu) = ((\vec{t}_1,\dots,\vec{t}_n),\mu')$ (with $\vec{t}_m = (t_{m,1},t_{m,2})$), and $\iota_k(i) = \beta_0 \prec \beta_1 \prec \cdots \prec \beta_n = \iota_k(j)$ in J_3, we may compute

$$\begin{aligned}\prod_{m=1}^{n-1} t_{m,1}t_{m,2}\cdot t_{n,1} &= \prod_{m=1}^{n-1}\varphi(\mathsf{prod}\circ\Theta_3(\beta_{m-1},\beta_m,\vec{t}_m,\mu_m))\cdot\varphi(\Phi_3(\beta_{n-1},\beta_n,\vec{t}_n,\mu'))\\ &= \prod_{m=1}^{n-1}\varphi(\Phi_k(i,j,\vec{s},0)[\beta_{m-1},\beta_m[)\cdot\varphi(\Phi_k(i,j,\vec{s},\mu)[\beta_{n-1},\beta_n[)\\ &= \varphi(\Phi_k(i,j,\vec{s},\mu))\\ &= s_1 \quad \text{by (FS.2) applied to } \mathcal{C}_k,\\ t_{n,2} &= s_2, \quad \text{by construction.}\end{aligned}$$

The latter property is assured by the construction, since we defined $\Psi_3(\beta_{n-1},\beta_n,\vec{t}_n,\mu') = \Psi_k(i,j,\vec{s},\mu)$. □

If $\mathcal{C}_1 = (J_1,\iota_1,M_1,\Theta_1)$ is a factorization scheme for w, then it induces a set of factorizations for w. However, it might be useful to consider the set of factorizations that we obtain by multiplying some of the adjacent factors. To this end, we define what is a *candidate for a refining function to* $\mathcal{C}_1$ *with respect to* J_2: given a totally ordered finite set J_2 and an order preserving injective function $\iota_2 : J_2 \to \alpha_w + 1$ such that $\mathsf{Im}(\iota_2) \subseteq \mathsf{Im}(\iota_1)$, it consists of a partial function

$$\Lambda : \{(i,j,\vec{s},\mu) \in J_2\times J_2\times(S\times S^I)\times\omega\colon i\prec j\} \to \bigcup_{k\geq 1}(S\times S^I)^k\times\omega$$

such that

(C.1) $\mathsf{Dom}(\Lambda)$ is finite;

(C.2) if $(i, j, \vec{s}, \mu) \in \mathsf{Dom}(\Lambda)$ and $\mu' \in \mu$, then $(i, j, \vec{s}, \mu') \in \mathsf{Dom}(\Lambda)$;

(C.3) If $(i, j, \vec{s}, \mu) \in \mathsf{Dom}(\Lambda)$ and $\Lambda(i, j, \vec{s}, \mu) = ((\vec{t}_1, \ldots, \vec{t}_n), \mu')$, then

(C.3.1) there exist $n+1$ elements in J_1, say $i_0, \ldots, i_n$, such that $i_0 \prec \cdots \prec i_n$, and the equalities $\iota_2(i) = \iota_1(i_0)$ and $\iota_2(j) = \iota_1(i_n)$ hold;

(C.3.2) writing $\vec{s} = (s_1, s_2)$ and $\vec{t}_m = (t_{m,1}, t_{m,2})$ for $m = 1, \ldots, n$, the following equalities hold:

$$
\begin{aligned}
s_1 &= t_{1,1}t_{1,2} \cdots t_{n-1,1}t_{n-1,2} \cdot t_{n,1}, \\
s_2 &= t_{n,2};
\end{aligned}
$$

(C.3.3) for $m = 1, \ldots, n$, $(i_{m-1}, i_m, \vec{t}_m) \in \mathsf{Dom}(M_1)$ and $\mu' \in M_1(i_{n-1}, i_n, \vec{t}_n)$.

Given a candidate Λ for a refining function to $\mathcal{C}_1$ with respect to J_2, we define a tuple $(J_2, \iota_2, M_2, \Theta_2)$ as follows:

- we let $\mathsf{Dom}(M_2) = \{(i, j, \vec{s}) \colon \exists \mu \in \omega \mid (i, j, \vec{s}, \mu) \in \mathsf{Dom}(\Lambda)\}$;
- if $(i, j, \vec{s}) \in \mathsf{Dom}(M_2)$, then we let $M_2(i, j, \vec{s}) = \{\mu \colon (i, j, \vec{s}, \mu) \in \mathsf{Dom}(\Lambda)\}$;
- let $(i, j, \vec{s}) \in \mathsf{Dom}(M_2)$ and $\mu \in M_2(i, j, \vec{s})$. If $\Lambda(i, j, \vec{s}, \mu) = ((\vec{t}_1, \ldots, \vec{t}_n), \mu')$ and $i_0 \prec \cdots \prec i_n$ in J_1 are such that $\iota_2(i) = \iota_1(i_0)$ and $\iota_2(j) = \iota_1(i_n)$, then we define

$$
\begin{aligned}
\Phi_2(i, j, \vec{s}, \mu) &= \left(\prod_{m=1}^{n-1} \mathsf{prod} \circ \Theta_1(i_{m-1}, i_m, \vec{t}_m, 0) \right) \cdot \Phi_1(i_{n-1}, i_n, \vec{t}_n, \mu'); \\
\Psi_2(i, j, \vec{s}, \mu) &= \Psi_1(i_{n-1}, i_n, \vec{t}_n, \mu').
\end{aligned}
$$

  We put $\Theta_2(i, j, \vec{s}, \mu) = (\Phi_2(i, j, \vec{s}, \mu), \Psi_2(i, j, \vec{s}, \mu))$.

We say that $\mathcal{C}_2 = (J_2, \iota_2, M_2, \Theta_2)$ is the *restriction of* $\mathcal{C}_1$ *to* J_2 *with respect to* Λ. The following result justifies this terminology.

Proposition 5.20. *Let* $\mathcal{C}_1$, $\mathcal{C}_2$ *and* Λ *be as above. Then,*

(a) $\mathcal{C}_2$ *is a factorization scheme for w;*

(b) $\mathcal{C}_1$ *is a refinement of* $\mathcal{C}_2$*;*

(c) Λ *is a refining function from* $\mathcal{C}_2$ *to* $\mathcal{C}_1$.

Moreover, if $\mathcal{C}_1$ *is a factorization scheme in* σ*-words, then so is* $\mathcal{C}_2$.

Proof. Once (a) is proved, the items (b) and (c) as well as the last assertion come all for free from the construction. Let $(i, j, \vec{s}) \in \mathsf{Dom}(M_2)$, $\mu \in M_2(i, j, \vec{s})$ and $\Lambda(i, j, \vec{s}, \mu) = ((\vec{t}_1, \ldots, \vec{t}_n), \mu')$. To prove (FS.1), we observe that the following pseudoidentities are valid in DRH:

$$\begin{aligned}
\mathsf{prod}\circ\Theta_2(i,j,\vec{s},\mu) &= \Phi_2(i,j,\vec{s},\mu)\Psi_2(i,j,\vec{s},\mu)\\
&= \left(\prod_{m=1}^{n-1}\mathsf{prod}\circ\Theta_1(i_{m-1},i_m,\vec{t}_m,0)\right)\cdot\Phi_1(i_{n-1},i_n,\vec{t}_n,\mu')\cdot\Psi_1(i_{n-1},i_n,\vec{t}_n,\mu')\\
&\overset{\text{(FS.1) for }\mathcal{C}_1}{=} \left(\prod_{m=1}^{n} w[\iota_1(i_{m-1}),\iota_1(i_m)[\right) = w[\iota(i),\iota(j)[.
\end{aligned}$$

For the Property (FS.2), we may compute:

$$\begin{aligned}
\varphi(\Phi_2(i,j,\vec{s},\mu)) &= \varphi\left(\left(\prod_{m=1}^{n-1}\mathsf{prod}\circ\Theta_1(i_{m-1},i_m,\vec{t}_m,0)\right)\cdot\Phi_1(i_{n-1},i_n,\vec{t}_n,\mu')\right)\\
&\overset{\text{(FS.2) for }\mathcal{C}_1}{=} t_{1,1}t_{1,2}\cdot t_{2,1}t_{2,2}\cdots t_{n-1,1}t_{n-1,2}\cdot t_{n,1}\\
&\overset{\text{(C.3.2)}}{=} s_1\\
\varphi(\Psi_2(i,j,\vec{s},\mu)) &= \varphi(\Psi_1(i_{n-1},i_n,\vec{t}_n,\mu')) \overset{\text{(FS.2) for }\mathcal{C}_1}{=} t_{n,2} \overset{\text{(C.3.2)}}{=} s_2.
\end{aligned}$$

This completes the proof. □

We proceed with a few notes describing general situations that appear repeatedly later.

Remark 5.21. Let w be a pseudoword and $\mathcal{C}=(J,\iota,M,\Theta)$ a factorization scheme for w. Suppose that $\mathcal{C}_1=(J_1,\iota_1,M_1,\Theta_1)$ is a refinement of the factorization scheme $\mathcal{C}$ and let Λ be a refining function from $\mathcal{C}$ to $\mathcal{C}_1$. Finally, suppose that $\mathcal{C}_1'=(J_1,\iota_1',M_1,\Theta_1')$ is a factorization scheme for another pseudoword w'. The function Λ is clearly a candidate for a refining function to $\mathcal{C}_1'$ with respect to J. Moreover, if $\mathcal{C}'=(J,\iota',M',\Theta')$ is the restriction of $\mathcal{C}_1'$ with respect to Λ, then $M'=M$.

Notation 5.22. *Suppose that $\mathcal{S}=(\mathcal{X},J,\zeta,M,\chi,\mathsf{right},\mathcal{B},\mathcal{B}_{\mathsf{H}})$ is a system of boundary relations that has $\mathcal{M}=(w,\iota,\Theta)$ as a model. Let $\mathcal{C}_1=(J_1,\iota_1,M_1,\Theta_1)$ be a refinement of $\mathcal{C}(\mathcal{S},\mathcal{M})$ and let Λ be a refining function from $\mathcal{C}(\mathcal{S},\mathcal{M})$ to $\mathcal{C}_1$. Define $\xi=\iota_1^{-1}\circ\iota$. We denote by $\xi_\Lambda(\mathcal{B}_{\mathsf{H}})$ the system of κ-equations with variables in $X_{(J_1,\zeta_{w,\mathcal{C}_1},M_1)}$ (recall* (5.15) *and* (5.18)*) obtained from $\mathcal{B}_{\mathsf{H}}$ by substituting each variable $(i\mid j)$ by $(\xi(i)\mid\xi(j))$ and each variable $\{i\mid j\}_{\vec{s},\mu}$ by $\{\xi(j)^-\mid\xi(j)\}_{\vec{t}_n,\mu'}$, where $\Lambda(i,j,\vec{s},\mu)=((\vec{t}_1,\ldots,\vec{t}_n),\mu')$.*

Remark 5.23. Using the notation above, the homomorphism $\delta_{w,\mathcal{C}_1}$ (recall (5.16)) is a solution modulo H of the system $\xi_\Lambda(\mathcal{B}_{\mathsf{H}})$.

Remark 5.24. Keeping again the notation, suppose that we are given a pseudoword w_1' and a factorization scheme $\mathcal{C}_1'=(J_1,\iota_1',M_1,\Theta_1')$ for w_1', such that $\delta_{w_1',\mathcal{C}_1'}$ is a solution modulo H of $\xi_\Lambda(\mathcal{B}_{\mathsf{H}})$. Further assume that there exists a factorization scheme of the form $\mathcal{C}'=(J,\iota',M,\Theta')$ for another pseudoword w' such that $\zeta_{w',\mathcal{C}'}=\zeta$ and the following pseudoidentities are valid in H, for every $(i\mid j),\{i\mid j\}_{\vec{s},\mu}\in X_{(J,\zeta,M)}$:

$$\begin{aligned}
w'(i,j) &= w_1'(\xi(i),\xi(j));\\
\Psi'(i,j,\vec{s},\mu) &= \Psi_1'(\xi(j)^-,\xi(j),\vec{t}_n,\mu').
\end{aligned}$$

Then, the homomorphism $\delta_{w',\mathcal{C}'}$ is a solution modulo H of $\mathcal{B}_{\mathsf{H}}$.

5.6 Proof of the main theorem

Suppose that DRH is a pseudovariety that is σ-reducible with respect to finite systems of κ-equations, and consider such a system $\mathcal{S} = \{u_i = v_i\}_{i=1}^n$ with variables in X and constraints given by the pair $(\varphi : \overline{\Omega}_A\mathsf{S} \to S, \nu : X \to S)$. Let $\delta : \overline{\Omega}_X\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be a solution modulo H of $\mathcal{S}$. For a new variable $x_0 \notin X$, we consider a new finite system of κ-equations given by $\mathcal{S}' = \{x_0u_i = x_0v_i\}_{i=1}^n$ and, writing $A = \{a_1,\dots,a_k\}$, we set the constraints on $X \cup \{x_0\}$ to be given by the pair (φ, ν'), where $\nu'|_X = \nu$ and $\nu'(x_0) = \varphi((a_1\cdots a_k)^\omega)$. By Corollary 2.20, the continuous homomorphism δ' defined by

$$\begin{aligned}\delta' : \overline{\Omega}_{X\uplus\{x_0\}}\mathsf{S} &\to \overline{\Omega}_A\mathsf{S}\\ x &\mapsto \delta(x), \text{ if } x \in X\\ x_0 &\mapsto (a_1\cdots a_k)^\omega\end{aligned}$$

is a solution modulo DRH of $\mathcal{S}'$. Since we are assuming that DRH is σ-reducible for systems of κ-equations, there exists a solution in σ-words modulo DRH of $\mathcal{S}'$. Of course, any solution modulo DRH of $\mathcal{S}'$ provides a solution modulo H of $\mathcal{S}$, by restriction to $\overline{\Omega}_X\mathsf{S}$. Hence, we proved the following.

Proposition 5.25. *If* DRH *is a pseudovariety σ-reducible for finite systems of κ-equations, then* H *is σ-reducible for finite systems of κ-equations as well.* □

Our next goal is to prove that H being σ-reducible for finite systems of κ-equations also suffices for so being DRH. With that in mind, throughout this section we fix a pseudovariety of groups H that is σ-reducible for finite systems of κ-equations. In view of Corollary 5.17, we should prove the following.

Theorem 5.26. *Suppose that σ is an implicit signature that satisfies the condition* (sig) *and such that $\kappa \subseteq \langle\sigma\rangle$. Let $\mathcal{S}$ be a system of boundary relations that has a model. Then, $\mathcal{S}$ has a model in σ-words.*

We fix the pair $(\mathcal{S}, \mathcal{M})$, where

$$\begin{aligned}&\mathcal{S} = (\mathcal{X}, J, \zeta, M, \chi, \mathsf{right}, \mathcal{B}, \mathcal{B}_{\mathsf{H}}) \text{ is a system of boundary relations,}\\ &\mathcal{M} = (w, \iota, \Theta) \text{ is a model of } \mathcal{S},\end{aligned} \tag{5.20}$$

and we define the parameter

$$[\mathcal{S}, \mathcal{M}] = (\alpha, n), \tag{5.21}$$

where α is the largest ordinal of the form $\iota(c)$ such that there exists a box (i,x) with $\mathsf{right}(x) = c$ if $\mathcal{B} \neq \emptyset$, and is 0 otherwise, and n is the number of boxes (i,x) such that $\iota(\mathsf{right}(x)) = \alpha$. We denote by r the index $\iota^{-1}(\alpha)$. In order to prove Theorem 5.26, we argue by transfinite induction on the parameter $[\mathcal{S}, \mathcal{M}]$, where the pairs (α, n) are ordered lexicographically. The induction step amounts to associating to each pair $(\mathcal{S}, \mathcal{M})$ a new pair $(\mathcal{S}_1, \mathcal{M}_1)$ such that the following properties are satisfied:

(P.1) $[\mathcal{S}_1, \mathcal{M}_1] < [\mathcal{S}, \mathcal{M}]$;

(P.2) if $\mathcal{S}_1$ has a model in σ-words, then $\mathcal{S}$ also has a model in σ-words.

Depending on the set of boundary relations $\mathcal{B}$, we consider the following cases:

Case 1. There is a box (i,x) in $\mathcal{B}$ such that $i = r = \mathsf{right}(x)$.

Case 2. There is a boundary relation $(i,x,i,\overline{x})$ such that $\mathsf{right}(x) = r = \mathsf{right}(\overline{x})$.

Case 3. There is a boundary relation $(i,x,j,\overline{x})$ such that $i < j$, $c(w(i,j)) \subsetneq c(w(i,\mathsf{right}(x)))$, and $\mathsf{right}(x) = r = \mathsf{right}(\overline{x})$.

Case 4. There is a boundary relation $(i,x,j,\overline{x})$ such that $\mathsf{right}(x) < \mathsf{right}(\overline{x}) = r$.

Case 5. There is a boundary relation $(i,x,j,\overline{x})$ such that $i < j$, $c(w(i,j)) = c(w(i,\mathsf{right}(x)))$, and $\mathsf{right}(x) = r = \mathsf{right}(\overline{x})$.

In each case, we assume that all the preceding cases do not apply. In [10, Section 9], where the analogous result for the pseudovariety R is proved, the cases that are considered are similar. However, the difference in definition of the induction parameter (5.21) justifies the fact of needing to deal with one less case in the present work.

5.6.1 Induction basis

If the induction parameter $[\mathcal{S},\mathcal{M}]$ is $(0,0)$, then $\mathcal{B} = \emptyset$ and so, Property (M.4) for a model of $\mathcal{S}$ becomes trivial. Hence, having a model in σ-words amounts to having, for each $(i,j,\vec{s}) \in \mathsf{Dom}(M)$ and each $\mu \in M(i,j,\vec{s})$, a pair of σ-words $(\Phi(i,j,\vec{s},\mu),\Psi(i,j,\vec{s},\mu))$ such that the Properties (M.1)– (M.3) and (M.5) are satisfied. Note that the Property (M.1) means that we should have

$$\Phi(i,j,\vec{s}_1,\mu_1)\Psi(i,j,\vec{s}_1,\mu_1) =_{\mathsf{DRH}} \Phi(i,j,\vec{s}_2,\mu_2)\Psi(i,j,\vec{s}_2,\mu_2)$$

for all $(i,j,\vec{s}_k) \in \mathsf{Dom}(M)$ and $\mu_k \in M(i,j,\vec{s}_k)$, $k = 1,2$. The following result entails the formalization of the induction basis step.

Proposition 5.27. *Let H be a pseudovariety that is σ-reducible for systems of κ-equations for a certain implicit signature σ satisfying $\kappa \subseteq \langle\sigma\rangle$. Let $\mathcal{S}_1 = \{x_{i,1}y_{i,1} = \cdots = x_{i,n_i}y_{i,n_i}\}_{i=1}^{N}$ and let $\mathcal{S}_2$ be a finite system of κ-equations (possibly with parameters in P). Let X be the set of variables occurring in $\mathcal{S}_1$ and $\mathcal{S}_2$ and suppose that the constraints for the variables are given by the pair (φ,ν). Let $\delta : \overline{\Omega}_{X\cup P}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ be a solution modulo DRH of $\mathcal{S}_1$ which is also a solution modulo H of $\mathcal{S}_2$ and such that, for $i = 1,\ldots,N$ and $p = 1,\ldots,n_i$, $c(\delta(y_{i,p})) \subseteq \vec{c}(\delta(x_{i,p}))$. Then, there exists a continuous homomorphism $\varepsilon : \overline{\Omega}_{X\cup P}\mathsf{S} \to (\overline{\Omega}_A\mathsf{S})^I$ such that*

(a) $\varepsilon(X) \subseteq (\Omega_A^{\sigma}\mathsf{S})^I$*;*

(b) *ε is a solution modulo DRH of $\mathcal{S}_1$;*

(c) *ε is a solution modulo H of $\mathcal{S}_2$;*

(d) $\vec{c}(\varepsilon(x)) = \vec{c}(\delta(x))$*, for all the variables $x \in X$.*

Proof. We argue by induction on $m = \max\{|c(\delta(x_{i,p}))| : i = 1,\ldots,N;\ p = 1,\ldots,n_i\}$. Note that, if $\delta(x_{i,1}) = I$, then we may discard the equations $x_{i,1}y_{i,1} = \cdots = x_{i,n_i}y_{i,n_i}$. Hence, when $m = 0$, the result amounts to proving the existence of ε satisfying (a), (c) and (d). But that comes for free from the fact that H is completely κ-reducible, together with Lemma 5.9.

Now, assume that $m \geq 1$ and suppose that $\delta(x_{i,p}) \neq I$, for all i,p. For each variable x and each $k \geq 1$ such that $\mathsf{lbf}_k(\delta(x))$ is nonempty we write

$$\begin{aligned}\mathsf{lbf}_k(\delta(x)) &= \delta(x)_k a_{x,k},\\ \delta(x) &= \mathsf{lbf}_1(\delta(x))\cdots\mathsf{lbf}_k(\delta(x))\delta(x)'_k.\end{aligned}$$

Since X, A and S are finite, there exist $1 \leq k < \ell$ such that, for all $x \in X$ with $\vec{c}(\delta(x)) \neq \emptyset$, the following equalities hold:

$$\begin{aligned}\vec{c}(\delta(x)) &= c(\mathsf{lbf}_{k+1}(\delta(x)));\\ \varphi(\mathsf{lbf}_1(\delta(x))\cdots\mathsf{lbf}_k(\delta(x))) &= \varphi(\mathsf{lbf}_1(\delta(x))\cdots\mathsf{lbf}_\ell(\delta(x))).\end{aligned}$$

In particular, the latter equality yields

$$\varphi(\delta(x)) = \varphi(\mathsf{lbf}_1(\delta(x))\cdots\mathsf{lbf}_k(\delta(x)))\varphi(\mathsf{lbf}_{k+1}(\delta(x))\cdots\mathsf{lbf}_\ell(\delta(x)))^{\omega}\varphi(\delta(x)'_k). \tag{5.22}$$

For $i = 1,\ldots,N$, set

$$\ell_i = \begin{cases}\ell, & \text{if } \vec{c}(\delta(x_{i,1})) \neq \emptyset,\\ \lceil \delta(x_{i,1}) \rceil, & \text{otherwise.}\end{cases}$$

We consider a new set of variables X' given by

$$\begin{aligned}X' = X &\uplus \{x_{i,p;j} \colon i = 1,\ldots,N;\ p = 1,\ldots,n_i;\ j = 1,\ldots,\ell_i\}\\ &\uplus \{x'_{i,p} \colon i = 1,\ldots,N;\ p = 1,\ldots,n_i;\ \vec{c}(\delta(x_{i,p})) \neq \emptyset\},\end{aligned}$$

where the variables $x_{i_1,p;j}$ and $x_{i_2,q;j}$, and the variables $x'_{i_1,p}$ and $x'_{i_2,p}$ (if defined) are the same, whenever the variables $x_{i_1,p}$ and $x_{i_2,q}$ are also the same. We also consider the following systems of equations with variables in X':

- $\mathcal{S}'_1 = \{x_{i,1;j} = \cdots = x_{i,n_i;j} \colon i = 1,\ldots,N;\ j = 1,\ldots,\ell_i\}$;

- $\mathcal{S}'_2$ is the system of equations obtained from $\mathcal{S}_2$ by substituting each one of the variables $x_{i,p}$ by the product $P_{i,p}$ given by

$$P_{i,p} = \begin{cases}x_{i,p;1}a_{x_{i,p},1}\cdots x_{i,p;k}a_{x_{i,p},k}x'_{i,p}, & \text{if } \vec{c}(\delta(x_{i,p})) \neq \emptyset,\\ x_{i,p;1}a_{x_{i,p},1}\cdots x_{i,p;\ell_i}a_{x_{i,p},\ell_i}, & \text{otherwise;}\end{cases}$$

- $\mathcal{S}_2'' = \{x'_{i,1} z_{i,1} = \cdots = x'_{i,n_i} z_{i,n_i} \colon i = 1,\dots,N;\ \vec{c}(\delta(x_{i,1})) \neq \emptyset\}$, where we take

$$z_{i,p} = \begin{cases} P_{j,q}, & \text{if } y_{i,p} = x_{j,q} \text{ for some } j = 1,\dots,N;\ q = 1,\dots,n_j, \\ y_{i,p}, & \text{otherwise.} \end{cases}$$

In the systems $\mathcal{S}_2'$ and $\mathcal{S}_2''$ the letters in A work as parameters evaluated to themselves, so that the system of equations $\mathcal{S}_2' \cup \mathcal{S}_2''$ has parameters in $P' = P \cup A$. We let the constrains for the variables be given by the pair (φ, ν'), where the map ν' is given by

$$\begin{aligned} \nu' : X' &\to S \\ x &\mapsto \nu(x), \quad \text{if } x \in X; \\ x_{i,p;j} &\mapsto \varphi(\delta(x_{i,p})_j), \quad \text{if } x_{i,p;j} \in X' \setminus X; \\ x'_{i,p} &\mapsto \varphi(\delta(x_{i,p})'_k), \quad \text{if } x'_{i,p} \in X' \setminus X; \end{aligned} \tag{5.23}$$

Let $\delta' : \overline{\Omega}_{X' \cup P'}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ be the continuous homomorphism defined by

$$\begin{aligned} \delta'(y) &= \delta(y), \quad \text{if } y \in X \cup P; \\ \delta'(x_{i,p;j}) &= \delta(x_{i,p})_j, \quad \text{if } i = 1,\dots,N;\ p = 1,\dots,n_i;\ j = 1,\dots,\ell_i; \\ \delta'(x'_{i,p}) &= \delta(x_{i,p})'_k, \quad \text{if } i = 1,\dots,N;\ p = 1,\dots,n_i;\ \vec{c}(\delta(x_{i,p})) \neq \emptyset\ ; \\ \delta'(a) &= a, \quad \text{if } a \in A. \end{aligned}$$

Then, δ' is a solution modulo DRH of $\mathcal{S}_1'$ which is also a solution modulo H of $\mathcal{S}_2' \cup \mathcal{S}_2''$. Since we decreased the induction parameter and the pair $(\mathcal{S}_1', \mathcal{S}_2' \cup \mathcal{S}_2'')$ satisfies the hypothesis of the proposition, we may invoke the induction hypothesis to derive the existence of a solution in σ-words modulo DRH of $\mathcal{S}_1'$, and modulo H of $\mathcal{S}_2' \cup \mathcal{S}_2''$, satisfying condition (d).

Now, we define the continuous homomorphism $\varepsilon : \overline{\Omega}_{X \cup P}\mathsf{S} \to \overline{\Omega}_A\mathsf{S}$ by:

$$\varepsilon(x_{i,p}) = \begin{cases} \varepsilon'(x_{i,p;1} a_{x_{i,p},1} \cdots x_{i,p;k} a_{x_{i,p},k}) \\ \quad \cdot \varepsilon'(x_{i,p;k+1} a_{x_{i,p},k+1} \cdots x_{i,p;\ell} a_{x_{i,p},\ell})^{\omega} \varepsilon'(x'_{i,p}), & \text{if } \vec{c}(\delta(x_{i,p})) \neq \emptyset; \\ \varepsilon'(P_{i,p}), \quad \text{if } \vec{c}(\delta(x_{i,p})) = \emptyset; \end{cases}$$

$$\varepsilon(x) = \varepsilon'(x), \quad \text{otherwise.}$$

Clearly, $\varepsilon(X) \subseteq \Omega_A^{\sigma}\mathsf{S}$. Moreover, since we are assuming that S has a content function, it follows from $\varphi \circ \varepsilon' = \varphi \circ \delta'$ that $\vec{c}(\varepsilon(x_{i,p})) = \vec{c}(\delta(x_{i,p}))$, for all i, p. For the other variables $x \in X$, the condition (d) for ε follows from the same condition for ε'.

Let us verify that ε is a solution modulo DRH of $\mathcal{S}_1$ and a solution modulo H of $\mathcal{S}_2$. Since ε' is a solution modulo DRH of $\mathcal{S}_1'$, for every pair of variables $x_{i,p}, x_{i,q}$, DRH satisfies $\varepsilon'(x_{i,p;j}) = \varepsilon'(x_{i,q;j})$, for $j = 1,\dots,\ell_i$. Further, since δ is a solution modulo DRH of $\mathcal{S}_1$ we also have $a_{x_{i,p;j}} = a_{x_{i,q;j}}$. Thus, we get

$$\varepsilon(x_{i,p}) = \begin{cases} \varepsilon'(x_{i,p;1}a_{x_{i,p},1}\cdots x_{i,p;k}a_{x_{i,p},k}) \\ \quad \cdot\varepsilon'(x_{i,p;k+1}a_{x_{i,p},k+1}\cdots x_{i,p;\ell}a_{x_{i,p},\ell})^{\omega}\varepsilon'(x'_{i,p}), & \text{if } \vec{c}(\delta(x_{i,p})) \neq \emptyset; \\ \varepsilon'(x_{i,p;1}a_{x_{i,p},1}\cdots x_{i,p;\ell_i}a_{x_{i,p},\ell_i}), & \text{if } \vec{c}(\delta(x_{i,p})) = \emptyset; \end{cases}$$

$$= \begin{cases} \varepsilon'(x_{i,q;1}a_{x_{i,q},1}\cdots x_{i,q;k}a_{x_{i,q},k}) \\ \quad \cdot\varepsilon'(x_{i,q;k+1}a_{x_{i,q},k+1}\cdots x_{i,q;\ell}a_{x_{i,q},\ell})^{\omega}\varepsilon'(x'_{i,p}), & \text{if } \vec{c}(\delta(x_{i,p})) \neq \emptyset; \\ \varepsilon'(x_{i,q;1}a_{x_{i,q},1}\cdots x_{i,q;\ell_i}a_{x_{i,q},\ell_i}), & \text{if } \vec{c}(\delta(x_{i,p})) = \emptyset. \end{cases}$$

In the second situation, when $\vec{c}(\delta(x_{i,p})) = \emptyset$, this means that DRH satisfies

$$\varepsilon(x_{i,p}y_{i,p}) = \varepsilon(x_{i,p}) = \varepsilon(x_{i,q}) = \varepsilon(x_{i,q}y_{i,q})$$

(notice that the former and latter equalities are justified by the assumption $c(\delta(y_{j,r})) \subseteq \vec{c}(\delta(x_{j,r}))$ for all $j = 1,\ldots,N$ and $r = 1,\ldots,n_j$). Otherwise, if $\vec{c}(\delta(x_{i,p})) \neq \emptyset$, the above equalities imply the relation $\varepsilon(x_{i,p}y_{i,p}) \mathrel{\mathcal{R}} \varepsilon(x_{i,q}y_{i,q})$ modulo DRH. Also, since ε' is a solution modulo H of $\mathcal{S}''_2$, we may use Lemma 2.32 to conclude that DRH satisfies $\varepsilon(x_{i,p}y_{i,p}) = \varepsilon(x_{i,q}y_{i,q})$. Thus, the homomorphism ε is a solution modulo DRH of $\mathcal{S}_1$. On the other hand, the pseudovariety H satisfies $\varepsilon(P_{i,p}) = \varepsilon(x_{i,p})$. By definition of $\mathcal{S}'_2$ it follows that ε is a solution modulo H of $\mathcal{S}_2$. Finally, it remains to verify that the constraints are satisfied. Since ε' satisfies the constraints, by (5.23), all the constraints but the ones for the variables of form $x_{i,p}$ with $\vec{c}(\delta(x_{i,p})) \neq \emptyset$ are trivially satisfied. For a variable $x_{i,p}$ with $\vec{c}(\delta(x_{i,p})) \neq \emptyset$, we may use (5.22) and (5.23) to obtain $\varphi(\varepsilon(x_{i,p})) = \varphi(\delta(x_{i,p}))$. Hence, ε is the required homomorphism. □

5.6.2 Factorization of a pair $(\mathcal{S},\mathcal{M})$

Instead of repeating the same argument several times, we use this subsection to describe a general construction that is performed later in some of the considered cases.

Let $\mathcal{E}$ be a subset of $\mathcal{B}$ such that, if $(i,x,j,\overline{x}) \in \mathcal{E}$, then $(j,\overline{x},i,x) \notin \mathcal{E}$. Suppose that we are given a set of pairs of ordinals $\Delta = \{(\beta_e,\gamma_e)\}_{e\in\mathcal{E}}$ such that, for each boundary relation $e = (i_e,x_e,j_e,\overline{x}_e) \in \mathcal{E}$, the following properties are satisfied:

(F.1) $\imath(i_e) < \beta_e < \imath(\mathsf{right}(x_e))$ and $\imath(j_e) < \gamma_e < \imath(\mathsf{right}(\overline{x}_e))$;

(F.2) DRH satisfies $w[\imath(i_e),\beta_e[= w[\imath(j_e),\gamma_e[$.

We say that the *factorization of* $(\mathcal{S},\mathcal{M})$ *with respect to* $(\mathcal{E},\Delta)$ is the pair $(\mathcal{S}_0,\mathcal{M}_0)$, where

$$\mathcal{S}_0 = (\mathcal{X}_0, J_0, \zeta_0, M_0, \chi_0, \mathsf{right}_0, \mathcal{B}_0, \mathcal{B}_{0,\mathsf{H}}) \text{ and } \mathcal{M}_0 = (w_0, \imath_0, \Theta_0),$$

are defined as follows:

- the set of variables $\mathcal{X}_0$ contains all the variables from $\mathcal{X}$ and a pair of new variables y_e, $\overline{y}_e$ for each relation $e \in \mathcal{E}$;

- we take $w_0 = w$;
- we let J_0, ι_0, M_0 and Θ_0 be determined by the factorization scheme $\mathcal{C}_0 = (J_0, \iota_0, M_0, \Theta_0)$, which is chosen to be a common refinement of the factorization schemes for w

$$\mathcal{C}(\mathcal{S},\mathcal{M}) \text{ and } (\{\beta_e,\gamma_e\}_{e\in\mathcal{E}}, \{\beta_e,\gamma_e\}_{e\in\mathcal{E}} \hookrightarrow \alpha_w + 1, \emptyset, \emptyset).$$

  We denote by ℓ_e and k_e the indices $\iota_0^{-1}(\beta_e)$ and $\iota_0^{-1}(\gamma_e)$ in J_0, respectively, by ξ the composite function $\iota_0^{-1} \circ \iota$, and we let

$$\Lambda : \{(i,j,\vec{s},\mu) \colon (i,j,\vec{s}) \in \mathsf{Dom}(M),\ \mu \in M(i,j,\vec{s})\} \to \bigcup_{k\geq 0} (S \times S^I)^k \times \omega \setminus \{0\}$$

  be a refining function from $\mathcal{C}(\mathcal{S},\mathcal{M})$ to $\mathcal{C}_0$;
- the maps ζ_0 and χ_0 are, respectively, $\zeta_{w_0,\mathcal{C}_0}$ and $\chi_{w_0,\mathcal{C}_0}$ (recall (5.18) and (5.19));
- the right_0 function assigns $\xi(\mathsf{right}(x))$ to each variable $x \in \mathcal{X}$ and, for each $e \in \mathcal{E}$, we let $\mathsf{right}_0(y_e) = \ell_e$ and $\mathsf{right}_0(\overline{y}_e) = k_e$;
- the set of boundary relations $\mathcal{B}_0$ is obtained by putting the boundary relation $(\xi(i), x, \xi(j), \overline{x})$ whenever $(i,x,j,\overline{x})$ neither belongs to $\mathcal{E}$ nor is the dual of a boundary relation of $\mathcal{E}$, and the boundary relations $(\xi(i_e), y_e, \xi(j_e), \overline{y}_e)$, $(\ell_e, x_e, k_e, \overline{x}_e)$ and their duals for each $e \in \mathcal{E}$;
- the set $\mathcal{B}_{0,\mathsf{H}}$ contains $\xi_\Lambda(\mathcal{B}_\mathsf{H})$ as well as the equation $(\xi(i_e) \mid \ell_e) = (\xi(j_e) \mid k_e)$, for each $e \in \mathcal{E}$.

The way we construct $\mathcal{B}_0$ is illustrated in Figure 5.4.

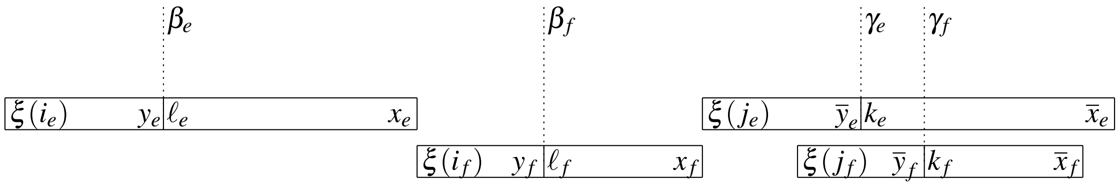

Fig. 5.4 Factorization of $(\mathcal{S},\mathcal{M})$, when $\mathcal{E} = \{e, f\}$.

Proposition 5.28. *The triple* $\mathcal{M}_0$ *is a model of* $\mathcal{S}_0$ *such that* $[\mathcal{S}_0,\mathcal{M}_0] = [\mathcal{S},\mathcal{M}]$ *and the Property (P.2) is satisfied.*

Proof. To check that $\mathcal{M}_0$ is a model of $\mathcal{S}_0$, it is enough to verify that the Properties (M.4) and (M.5) are satisfied, since the others are guaranteed by Lemma 5.18. All relations of $\mathcal{B}_0$ other than the ones of the form $(\xi(i_e), y_e, \xi(j_e), \overline{y}_e)$ or $(\ell_e, x_e, k_e, \overline{x}_e)$ and their duals, are those of $\mathcal{B}$ with renamed indices in the set J_0. Therefore, as $\mathcal{M}$ is a model of $\mathcal{S}$, Property (M.4) is satisfied by them. Let $e \in \mathcal{E}$. The relation $(\xi(i_e), y_e, \xi(j_e), \overline{y}_e)$ also satisfies (M.4) because it amounts to having the relation $w[\iota(i_e), \beta_e[\ \mathcal{R}\ w[\iota(j_e), \gamma_e[$ given by (F.2). In fact, this relation is actually an equality modulo DRH, which also proves (M.5) for the equation $(\xi(i_e) \mid \ell_e) = (\xi(j_e) \mid k_e)$. For the boundary relation

$(\ell_e, x_e, k_e, \overline{x}_e)$, taking into account that DRH satisfies $w(i_e, \mathsf{right}(x_e)) \mathrel{\mathcal{R}} w(j_e, \mathsf{right}(\overline{x}_e))$, we may use Corollary 2.28 to assert that $\beta_e - \iota(i_e) = \gamma_e - \iota(j_e)$ and then apply Corollary 2.31 to conclude that DRH satisfies $w_0(\ell_e, \mathsf{right}_0(x_e)) \mathrel{\mathcal{R}} w_0(k_e, \mathsf{right}_0(\overline{x}_e))$. At last, Remark 5.23 yields that $\delta_{w_0,\mathcal{C}_0}$ is a solution modulo H of the equations in $\xi_\Lambda(\mathcal{B}_{\mathsf{H}})$. This proves that $\mathcal{M}_0$ is a model of $\mathcal{S}_0$.

Relatively to the induction parameter it is clear that it keeps its value under the transformation $(\mathcal{S}, \mathcal{M}) \mapsto (\mathcal{S}_0, \mathcal{M}_0)$, since we do not change the number of boxes ending at r.

For Property (P.2), we suppose that $\mathcal{M}'_0 = (w'_0, \iota'_0, \Theta'_0)$ is a model of $\mathcal{S}_0$ in σ-words and we take $\mathcal{M}' = (w', \iota', \Theta')$, where $w' = w'_0$, $\iota' = \iota'_0 \circ \xi$, and Θ' is given by the factorization scheme $\mathcal{C}' = (J, \iota', M, \Theta')$ corresponding to the restriction of $\mathcal{C}(\mathcal{S}_0, \mathcal{M}_0)$ with respect to Λ (cf. Remark 5.21). We claim that $\mathcal{M}'$ is a model of $\mathcal{S}$ (in σ-words by Proposition 5.20). Properties (M.1) and (M.2) are a consequence of $\mathcal{C}'$ being a factorization scheme for w'. For Property (M.3), let $i \prec j$ in J. We compute

$$\vec{c}(w'(i,j)) = \vec{c}(w'_0[\iota'_0 \circ \xi(i), \iota'_0 \circ \xi(j)[) = \chi_0(\xi(i), \xi(j)) = \vec{c}(w[\iota(i), \iota(j)[) = \chi(i,j).$$

Property (M.4) is straightforward for all boundary relations except for the relations $(i_e, x_e, j_e, \overline{x}_e)$ and their duals. In this case, since $(\xi(i_e), y_e, \xi(j_e), \overline{y}_e)$ belongs to $\mathcal{B}_0$, $(\xi(i_e) \mid \ell_e) = (\xi(j_e) \mid k_e)$ belongs to $\mathcal{B}_{0,\mathsf{H}}$, and $\mathcal{M}'_0$ is a model of $\mathcal{S}_0$, we have

$$\begin{aligned} &w'_0(\xi(i_e), \ell_e) \mathrel{\mathcal{R}} w'_0(\xi(j_e), k_e) \text{ modulo DRH},\\ &w'_0(\xi(i_e), \ell_e) =_{\mathsf{H}} w'_0(\xi(j_e), k_e), \end{aligned}$$

and we invoke Lemma 2.32 to conclude that DRH satisfies $w'_0(\xi(i_e), \ell_e) = w'_e(\xi(j_e), k_e)$. On the other hand, the relation $(\ell_e, x_e, k_e, \overline{x}_e)$ also belongs to $\mathcal{B}_0$, so that $w'_0(\ell_e, \mathsf{right}_0(x_e)) \mathrel{\mathcal{R}} w'_0(k_e, \mathsf{right}_0(\overline{x}_e))$ modulo DRH. Thus, we obtain the following in DRH:

$$\begin{aligned} w'(i_e, \mathsf{right}(x_e)) &= w'_0(\xi(i_e), \ell_e) w'_0(\ell_e, \mathsf{right}_0(x_e))\\ &= w'_0(\xi(j_e), k_e) w'_0(\ell_e, \mathsf{right}_0(x_e))\\ &\mathrel{\mathcal{R}} w'_0(\xi(j_e), k_e) w'_0(k_e, \mathsf{right}_0(\overline{x}_e))\\ &= w'_0(\xi(j_e), \mathsf{right}_0(\overline{x}_e))\\ &= w'(j_e, \mathsf{right}(\overline{x}_e)). \end{aligned}$$

Finally, since $\xi_\Lambda(\mathcal{B}_{\mathsf{H}}) \subseteq \mathcal{B}_{0,\mathsf{H}}$, we may use Remark 5.24 to conclude that in order to prove Property (M.5) it is enough to show that the following identities hold in H:

$$\begin{aligned} w'(i,j) &= w'_0(\xi(i), \xi(j)), \quad \text{for all } i \prec j \text{ in } J;\\ \Psi'(i,j,\vec{s},\mu) &= \Psi'_0(\xi(j)^-, \xi(j), \vec{t}, \mu'), \quad \text{for all } (i,j,\vec{s},\mu) \in \mathsf{Dom}(M) \times M(i,j,\vec{s})\\ &\qquad \text{and } ((\ldots, \vec{t}), \mu') = \Lambda(i,j,\vec{s}). \end{aligned}$$

The first one follows from the definition of w'_0 and ι', while the second is implied by the fact that $\mathcal{C}'$ is the restriction of $\mathcal{C}'_0$ with respect to Λ. $\square$

5.6.3 Case 1

When we are in Case 1, we have at least one empty box (r,x). Since for every pseudoword w we have $w(r,\mathsf{right}(x)) = w(r,r) = I$, we may delete the boundary relations involving empty boxes. In this way we obtain a new system of boundary relations $\mathcal{S}_1$ which has exactly the same models as $\mathcal{S}$ and so, Property (P.2) is satisfied. Moreover, the parameter associated to $(\mathcal{S}_1,\mathcal{M})$ is smaller than the parameter associated to $(\mathcal{S},\mathcal{M})$ since we removed some boxes ending at r. Therefore, Property (P.1) also holds.

5.6.4 Case 2

In this case, there exists a boundary relation of the form $(i,x,i,\bar{x})$ with $\mathsf{right}(x) = r = \mathsf{right}(\bar{x})$. Since such a boundary relation yields a trivial relation in (M.4), we may argue as in the previous case and simply delete $(i,x,i,\bar{x})$ and its dual from $\mathcal{S}$ obtaining thus a new pair $(\mathcal{S}_1,\mathcal{M})$ satisfying (P.1) and (P.2).

5.6.5 Case 3

This is the case where we assume the existence of a boundary relation $(i_0,x_0,j_0,\bar{x}_0)$ such that $i_0 < j_0$, $\mathsf{right}(x_0) = r = \mathsf{right}(\bar{x}_0)$ and $c(w(i_0,j_0)) \subsetneqq c(w(i_0,\mathsf{right}(x_0)))$.

Let $a \in c(w(i_0,r)) \setminus c(w(i_0,j_0))$. Since $i_0 < j_0$, the letter a also belongs to $w(j_0,r)$. Therefore, by Corollary 2.14, there are unique factorizations $w(i_0,r) = u_i\, a\, v_i$ and $w(j_0,r) = u_j\, a\, v_j$ such that a belongs to neither of $c(u_i)$ nor $c(u_j)$, and DRH satisfies the pseudoidentity $u_i = u_j$ and the relation $v_i \mathrel{\mathcal{R}} v_j$. Thus, the decreasing of the induction parameter in this case is achieved by discarding the segment $[\iota(i_0)+\alpha_{u_i}, \iota(r)[$ in the boundary relation $(i_0,x_0,j_0,\bar{x}_0)$ as it is outlined in Figure 5.5 below.

Fig. 5.5 Discarding the segment $[\iota(i_0)+\alpha_{u_i}, \iota(r)[$ in the boundary relation $(i_0,x_0,j_0,\bar{x}_0)$.

Let $\mathcal{E} = \{(i_0,x_0,j_0,\bar{x}_0)\}$ and $\Delta = \{(\iota(i_0)+\alpha_{u_i}, \iota(i_0)+\alpha_{u_i})\}$. By the above, the pair $(\mathcal{E},\Delta)$ satisfies (F.1) and (F.2). Let $(\mathcal{S}_0,\mathcal{M}_0)$ be the factorization of $(\mathcal{S},\mathcal{M})$ with respect to $(\mathcal{E},\Delta)$. Then, the pair $(\mathcal{S}_0,\mathcal{M}_0)$ is covered by Case 2 and we may use it in order to decrease the induction parameter.

Before proceeding with Cases 4 and 5 we perform an auxiliary step that is useful in both of the remaining cases.

5.6.6 Auxiliary step

We are interested in modifying some of the boundary relations of the form $(i,x,j,\bar{x})$ such that $i < j$ and $\mathsf{right}(x) = r = \mathsf{right}(\bar{x})$, so we assume that there exists at least one. For each $i_0 \in \{i \in J \colon i < r\}$, let $\mathcal{E}(\mathcal{S},i_0) = \{(i,x,j,\bar{x}) \colon \mathsf{right}(x) = r = \mathsf{right}(\bar{x}),\ i < j,\ i \le i_0\}$. Our goal is to prove the existence of a new pair $(\mathcal{S}_1,\mathcal{M}_1)$ that keeps the induction parameter unchanged, satisfies Property (P.2), and such that $\mathcal{E}(\mathcal{S}_1, i_0) = \emptyset$. We first construct a pair $(\mathcal{S}_0, \mathcal{M}_0)$ satisfying the first two properties and such that $|\mathcal{E}(\mathcal{S}_0, i_0)| < |\mathcal{E}(\mathcal{S}, i_0)|$. Then we argue by induction to conclude the existence of such a pair $(\mathcal{S}_1, \mathcal{M}_1)$.

If $\mathcal{E}(\mathcal{S}, i_0) \neq \emptyset$, then we fix a boundary relation $(k_0, x_0, k_1, \bar{x}_0) \in \mathcal{E}(\mathcal{S}, i_0)$. Property (M.4) yields

$$w(k_0, k_1)w(k_1, r) = w(k_0, r) \mathrel{\mathcal{R}} w(k_1, r)$$

modulo DRH, which in turn implies that DRH satisfies

$$w(k_0, r) \mathrel{\mathcal{R}} w(k_0, k_1)^{\omega} w(k_1, r).$$

As we are assuming that the Case 3 does not hold, the contents of $w(k_0, k_1)$ and $w(k_1, r)$ are the same, and so, DRH satisfies

$$w(k_0, r) \mathrel{\mathcal{R}} w(k_0, k_1)^{\omega}. \tag{5.24}$$

Moreover, the fact that the relation $w(k_0, r) \mathrel{\mathcal{R}} w(k_1, r)$ holds in DRH implies that the first letter of $w(k_0, r)$ is the same as the first letter of $w(k_1, r)$. Since the product $w(k_0, k_1) \cdot w(k_1, r)$ is reduced by definition, the product $w(k_0, k_1) \cdot w(k_0, k_1)$ is also reduced. Consequently, we may use Corollary 2.28 and Theorem 2.24 to obtain

$$\alpha_{w(k_0,r)} = \alpha_{w(k_0,k_1)^{\omega}} = \alpha_{w(k_0,k_1)} \cdot \omega.$$

In particular, setting $\beta_p = \iota(k_0) + \alpha_{w(k_0,k_1)} \cdot p$ for every $p \geq 0$, the inequality $\beta_p < \alpha = \iota(r)$ holds. On the other hand, as $k_0 \leq i_0 < r$, we also have $\alpha_{w(k_0,i_0)} < \alpha_{w(k_0,r)} = \alpha_{w(k_0,k_1)} \cdot \omega$ and therefore there exists an integer $n \geq 1$ such that $\alpha_{w(k_0,i_0)} < \alpha_{w(k_0,k_1)} \cdot n$. We fix such an n and we take $\mathcal{E} = \{(k_0, x_0, k_1, \bar{x}_0)\}$ and $\Delta = \{(\beta_n, \beta_{n+1})\}$. Then, the pair $(\mathcal{E}, \Delta)$ not only satisfies (F.1) (we already observed that $\beta_p < \alpha$ for all $p \geq 0$), but it also satisfies (F.2). Indeed, we may compute

$$\begin{aligned}
\beta_n - \iota(k_0) &= (\iota(k_0) + \alpha_{w(k_0,k_1)} \cdot n) - \iota(k_0) = \alpha_{w(k_0,k_1)} \cdot n \\
&= (\iota(k_1) + \alpha_{w(k_0,k_1)} \cdot n) - \iota(k_1) \\
&= (\iota(k_0) + (\iota(k_1) - \iota(k_0)) + \alpha_{w(k_0,k_1)} \cdot n) - \iota(k_1) \\
&= (\iota(k_0) + \alpha_{w(k_0,k_1)} \cdot (n+1)) - \iota(k_1) \\
&= \beta_{n+1} - \iota(k_1)
\end{aligned}$$

and use Corollary 2.31 to conclude that $w[\iota(k_0), \beta_n[=_{\mathsf{DRH}} w[\iota(k_1), \beta_{n+1}[$. So, we let $(\mathcal{S}_0, \mathcal{M}_0)$ be the factorization of $(\mathcal{S}, \mathcal{M})$ with respect to $(\mathcal{E}, \Delta)$. Intuitively, the transformation performed in the step $(\mathcal{S}, \mathcal{M}) \mapsto (\mathcal{S}_0, \mathcal{M}_0)$ is represented in pictures 5.6 (before) and 5.7 (after).

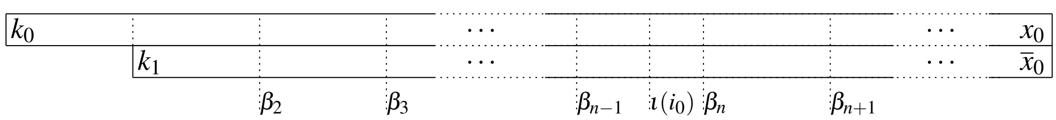

Fig. 5.6 Original relation $(k_0, x_0, k_1, \bar{x}_0)$ in the system of boundary relations $\mathcal{S}$.

We are now able to establish the desired result.

Lemma 5.29. *Let* $(\mathcal{S}_0, \mathcal{M}_0)$ *be the pair defined above. Then the following holds:*

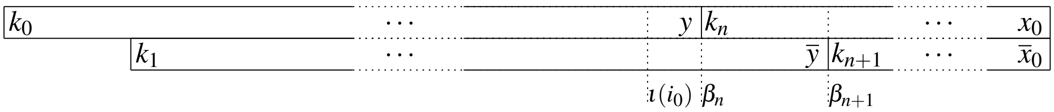

Fig. 5.7 Factorization of the relation $(k_0,x_0,k_1,\bar{x}_0)$ in the new system of boundary relations $\mathcal{S}_0$.

(a) *Cases 2 and 3 do not apply to the system of boundary relations* $\mathcal{S}_0$*;*

(b) *the inequality* $|\mathcal{E}(\mathcal{S}_0,i_0)| < |\mathcal{E}(\mathcal{S},i_0)|$ *holds.*

Proof. For the first part, we notice that (up to renaming indices) the boundary relations ending in r that belong to $\mathcal{B}_0$ and were not previously in $\mathcal{B}$ are $(k_n,x_0,k_{n+1},\bar{x}_0)$ and its dual (check construction in Subsection 5.6.2 and Figure 5.7). The non applicability of Case 2 is then immediate. Concerning Case 3, it follows from (5.24) that, modulo DRH, any finite power of $w(k_0,k_1)$ is a prefix of $w(k_0,r)$. In particular, we obtain the equalities $c(w(k_n,k_{n+1})) = c(w(k_0,k_1)) = c(w(k_1,r)) = c(w(k_{n+1},r))$.

Assertion (b) holds because the boundary relation $(k_0,x_0,k_1,\bar{x}_0)$ does not belong anymore to the set $\mathcal{E}(\mathcal{S}_0,i_0)$ and, on the other hand, we did not put any boundary relation in $\mathcal{E}(\mathcal{S}_0,i_0)$ that was not already in $\mathcal{E}(\mathcal{S},i_0)$. $\square$

Recall that, by Proposition 5.28, we also have $[\mathcal{S}_0,\mathcal{M}_0] = [\mathcal{S},\mathcal{M}]$ and Property (P.2) is satisfied by $(\mathcal{S}_0,\mathcal{M}_0)$. Thus, arguing by induction, we may assume, without loss of generality that, given a system $\mathcal{S}$ in Cases 4 or 5, we have $\mathcal{E}(\mathcal{S},i_0) = \emptyset$, for all $i_0 < r$ in J.

5.6.7 Case 4

In this case we suppose that the Cases 1, 2 and 3 do not hold and that there is a boundary relation $(i,x,j,\bar{x})$ such that $\mathsf{right}(\bar{x}) < \mathsf{right}(x) = r$. Consider the index

$$\ell = \min\{\mathsf{left}(x) \colon \mathsf{right}(\bar{x}) < \mathsf{right}(x) = r\}.$$

By the auxiliary step in Subsection 5.6.6, we may assume without loss of generality that all boundary relations $(i,x,j,\bar{x})$ satisfying $\mathsf{right}(x) = r = \mathsf{right}(\bar{x})$ are such that both i and j are greater than ℓ. Let $x_0 \in \mathcal{X}$ be such that $\mathsf{left}(x_0) = \ell$ and $\mathsf{right}(\bar{x}_0) < \mathsf{right}(x_0) = r$, and let $\ell^* \in J$ be such that $(\ell,x_0,\ell^*,\bar{x}_0) \in \mathcal{B}$. We set $r^* = \mathsf{right}(\bar{x}_0)$. Since Case 1 does not hold, we know that $\ell < r$. The intuitive idea consists in transferring all the information comprised in the factor $w(\ell,r)$ to the factor $w(\ell^*,r^*)$ in order to decrease the induction parameter by discarding the factors $w(r^-,r)$ and $w[\iota(\ell^*)+(\iota(r^-)-\iota(\ell)),\iota(r^*)[$ intervening in the boundary relation $(\ell,x_0,\ell^*,\bar{x}_0)$. See Figure 5.8.

More formally, we define the set of transport positions by

$$T = \{i \in J \colon \exists \text{ box } (i,x) \text{ such that } \mathsf{right}(x) = r\} \cup \{r^-,r\}.$$

Observe that $\min(T) = \ell$ and $\max(T) = r$. Hence, for $i \in T$ we may define the index

$$i^\circ = \iota(\ell^*) + (\iota(i) - \iota(\ell)).$$

Some useful properties of $\_^\circ$ are stated in the next lemma.

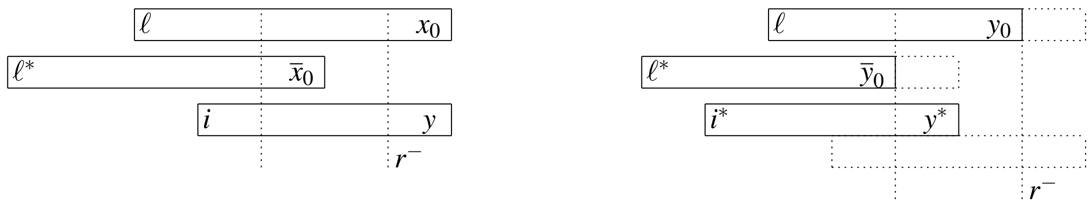

Fig. 5.8 Transferring the segment (ℓ, r) to the segment (ℓ^*, r^*) and discarding the final segments of the boxes (ℓ, x_0) and $(\ell^*, \bar{x}_0)$.

Lemma 5.30. *The function* $\_^\circ : T \to \alpha_w + 1$ *satisfies the following:*

(a) *it preserves the order and is injective;*

(b) *for every $i < j$ in T, the pseudovariety* DRH *satisfies the equality $w[i^\circ, j^\circ[= w(i, j)$ if $j < r$ and the relation $w[i^\circ, r^\circ[\mathrel{\mathcal{R}} w(i, r)$;*

(c) *for every $i \in T$, the inequality $i^\circ < \iota(i)$ holds.*

Proof. We omit the proofs of assertions (a) and (c) since they express properties of ordinal numbers and thus, are entirely analogous to the proofs of the corresponding properties in [10, Lemma 9.3].

Let us prove (b). Since $(\ell, x_0, \ell^*, \bar{x}_0)$ is a boundary relation in $\mathcal{B}$ and $\mathcal{M}$ is a model of $\mathcal{S}$, we have

$$w(\ell, r) = w(\ell, \mathsf{right}(x_0)) \mathrel{\mathcal{R}} w(\ell^*, \mathsf{right}(\bar{x}_0)) = w(\ell^*, r^*) \text{ modulo } \mathsf{DRH}.$$

Further, the equalities

$$\begin{aligned} \ell^\circ &= \iota(\ell^*) + (\iota(\ell) - \iota(\ell)) = \iota(\ell^*) \\ r^\circ &= \iota(\ell^*) + (\iota(r) - \iota(\ell)) = \iota(\ell^*) + \alpha_{w(\ell,r)} \overset{\text{Corollary 2.28}}{=} \iota(\ell^*) + \alpha_{w(\ell^*,r^*)} = \iota(r^*) \end{aligned}$$

imply that DRH satisfies $w(\ell, r) \mathrel{\mathcal{R}} w[\ell^\circ, r^\circ[$. On the other hand, since

$$j^\circ - i^\circ = (\iota(\ell^*) + (\iota(j) - \iota(\ell))) - (\iota(\ell^*) + (\iota(i) - \iota(\ell))) = \iota(j) - \iota(i),$$

we may use Corollary 2.31 twice to first conclude that, for $j < r$, DRH satisfies $w(\ell, j) = w[\ell^\circ, j^\circ[$ and then, that it satisfies the desired identity $w(i, j) = w[i^\circ, j^\circ[$. Similarly, when $j = r$, we get that DRH satisfies $w[i^\circ, r^\circ[\mathrel{\mathcal{R}} w(i, r)$. □

Before defining a new pair $(\mathcal{S}_1, \mathcal{M}_1)$, we still need to consider a factorization scheme for the pseudoword w, in order to memorize the information on constraints that we lose when transforming $\mathcal{S}$ according to Figure 5.8. We let $\mathcal{C}_0 = (J_0, \iota_0, M_0, \Theta_0)$ be defined as follows:

- $J_0 = \{i^\circ : i \in T\}$;
- $\iota_0 : J_0 \hookrightarrow \alpha_w + 1$ is the inclusion of ordinals;

- By Lemma 5.30(b) the pseudowords $w(r^-,r)$ and $w[(r^-)^\circ, r^\circ[$ are $\mathcal{R}$-equivalent modulo DRH. Therefore, since Property (M.1) holds for $(\mathcal{S},\mathcal{M})$, given $\vec{s} \in \zeta(r^-,r)$ and $\mu \in M(r^-,r,\vec{s})$ the pseudowords $\Phi(r^-,r,\vec{s},\mu)$ and $w[(r^-)^\circ, r^\circ[$ are $\mathcal{R}$-equivalent modulo DRH as well. For each such pair $(\vec{s},\mu)$, we fix a pseudoword $v_{\vec{s},\mu} \in (\overline{\Omega}_A\mathsf{S})^I$ such that

$$w[(r^-)^\circ, r^\circ[=_{\mathsf{DRH}} \Phi(r^-,r,\vec{s},\mu)\, v_{\vec{s},\mu}. \tag{5.25}$$

  In particular, it follows that $\Phi(r^-,r,\vec{s},\mu)\, v_{\vec{s},\mu}$ and $\Phi(r^-,r,\vec{s},\mu)$ are $\mathcal{R}$-equivalent modulo DRH. Combining Remark 2.16 with Lemma 2.27, we may deduce the inclusion

$$c(v_{\vec{s},\mu}) \subseteq \vec{c}(\Phi(r^-,r,\vec{s},\mu)) = \vec{c}(w[(r^-)^\circ, r^\circ[).$$

  Since $\zeta(r^-,r)$ is a finite set, we may write $\zeta(r^-,r) = \{\vec{s}_1,\dots,\vec{s}_m\}$. Let $\vec{s}_p = (s_{p,1}, s_{p,2})$ and denote by $\vec{t}_{p,\mu}$ the pair $(s_{p,1}, \varphi(v_{\vec{s}_p,\mu}))$ for each $\vec{s}_p \in \zeta(r^-,r)$ and $\mu \in M(r^-,r,\vec{s}_p)$. We define Θ_0 inductively as follows:

  - start with $\Theta_0 = \emptyset$;
  - for each $p \in \{1,\dots,m\}$ and $\mu \in M(r^-,r,\vec{s}_p)$, we set

$$\begin{aligned} \mu_{p,\mu} &= \{\overline{\mu}\colon \Theta_0((r^-)^\circ, r^\circ, \vec{t}_{p,\mu}, \overline{\mu}) \text{ is defined}\}; \\ \Theta_0((r^-)^\circ, r^\circ, \vec{t}_{p,\mu}, \mu_{p,\mu}) &= (\Phi(r^-,r,\vec{s}_p,\mu), v_{\vec{s}_p,\mu}). \end{aligned}$$

- the map M_0 is given by $M_0((r^-)^\circ, r^\circ, \vec{t}) = \{\mu'\colon \Theta_0(r^-,r,\vec{t},\mu') \text{ is defined}\}$, whenever $\vec{t} = \vec{t}_{p,\mu}$ for certain $p = 1,\dots,m$ and $\mu \in M(r^-,r,\vec{s}_p)$. Observe that we may have $\vec{t}_{p,\mu} = \vec{t}_{p',\mu'}$ with $(p,\mu) \neq (p',\mu')$.

Lemma 5.31. *The tuple $\mathcal{C}_0$ just constructed is a factorization scheme for w.*

Proof. Since $r^- \prec r$ in J, Lemma 5.30(a) yields $(r^-)^\circ \prec r^\circ$ in J_0. Therefore, the domain of Θ_0 is compatible with the definition of factorization scheme. Moreover, the definition of M_0 guarantees that the relationship between the domains of Θ_0 and of M_0 is the correct one. Let $\vec{s}_p \in \zeta(r^-,r)$ and $\mu \in M(r^-,r,\vec{s}_p)$. To prove (FS.1), we compute the following modulo DRH:

$$\begin{aligned} \mathsf{prod} \circ \Theta_0((r^-)^\circ, r^\circ, \vec{t}_{p,\mu}, \mu_{p,\mu}) &\overset{\text{def.}}{=} \Phi(r^-,r,\vec{s}_p,\mu)\, v_{\vec{s}_p,\mu} \\ &\overset{(5.25)}{=} w[(r^-)^\circ, r^\circ[. \end{aligned}$$

To prove (FS.2), we recall that $\vec{t}_{p,\mu} = (s_{p,1}, \varphi(v_{\vec{s}_p,\mu}))$. Since $\mathcal{M}$ is a model of $\mathcal{S}$, Property (M.2) yields

$$\varphi(\Phi_0((r^-)^\circ, r^\circ, \vec{t}_{p,\mu}, \mu_{p,\mu})) = \varphi(\Phi(r^-,r,\vec{s}_p,\mu)) = s_{p,1}$$

and by construction, we have

$$\varphi(\Psi_0((r^-)^\circ, r^\circ, \vec{t}_{p,\mu}, \mu_{p,\mu})) = \varphi(v_{\vec{s}_p,\mu}).$$

Thus, $\mathcal{C}_0$ is a factorization scheme for w. □

We are now ready to proceed with the construction of the new pair $(\mathcal{S}_1, \mathcal{M}_1)$, where

$$\mathcal{S}_1 = (\mathcal{X}_1, J_1, \zeta_1, M_1, \chi_1, \mathsf{right}_1, \mathcal{B}_1, \mathcal{B}_{1,\mathsf{H}}) \text{ and } \mathcal{M}_1 = (w_1, \iota_1, \Theta_1).$$

We take as set of variables $\mathcal{X}_1$ the old set $\mathcal{X}$ together with a pair of new variables y_i and $\overline{y}_i$, for each $i \in T \setminus \{r\}$. The pseudoword w_1 is w. Let $\mathcal{C}_1 = (J_1, \iota_1, M_1, \Theta_1)$ be a common refinement of $\mathcal{C}(\mathcal{S}, \mathcal{M})$ and $\mathcal{C}_0$. The elements J_1, M_1, ι_1 and Θ_1 are those given by $\mathcal{C}_1$. To simplify the notation, we set $\xi = \iota_1^{-1} \circ \iota$ and $i^\bullet = \iota_1^{-1}(i^\circ)$. The refining functions from $\mathcal{C}(\mathcal{S}, \mathcal{M})$ to $\mathcal{C}_1$ and from $\mathcal{C}_0$ to $\mathcal{C}_1$ are given, respectively, by

$$\Lambda : \{(i, j, \vec{s}, \mu) \colon (i, j, \vec{s}) \in \mathsf{Dom}(M), \mu \in M(i, j, \vec{s})\} \to \bigcup_{k \geq 0} (S \times S^I)^k \times \omega,$$

$$\Lambda_0 : \{((r^-)^\circ, r^\circ, \vec{t}_{p,\mu}, \mu_{p,\mu}) \colon (r^-, r, \vec{s}_p) \in \mathsf{Dom}(M), \mu \in M(r^-, r, \vec{s}_p)\} \to \bigcup_{k \geq 0} (S \times S^I)^k \times \omega.$$

The functions ζ_1 and χ_1 are the ones induced by $\mathcal{C}_1$, namely $\zeta_1 = \zeta_{w_1, \mathcal{C}_1}$ and $\chi_1 = \chi_{w_1, \mathcal{C}_1}$ (recall (5.18) and (5.19)). The right_1 function is given by

$$\begin{aligned} \mathsf{right}_1 : \mathcal{X}_1 &\to J_1 \\ x &\mapsto \xi(\mathsf{right}(x)), \quad \text{if } x \in \mathcal{X} \text{ and } \mathsf{right}(x) < r; \\ x &\mapsto r^\bullet, \quad \text{if } x \in \mathcal{X} \text{ and } \mathsf{right}(x) = r; \\ y_i &\mapsto \xi(i), \quad \text{if } i \in T \setminus \{r\}; \\ \overline{y}_i &\mapsto i^\bullet, \quad \text{if } i \in T \setminus \{r\}. \end{aligned}$$

We define $\mathcal{B}_1$ iteratively by:

(0) set $\mathcal{B}' = \mathcal{B} \setminus \{(\ell, x_0, \ell^*, \overline{x}_0), (\ell^*, \overline{x}_0, \ell, x_0)\}$;

(1) start with $\mathcal{B}_1 = \{(\xi(\ell), y_i, \ell^\bullet, \overline{y}_i), (\ell^\bullet, \overline{y}_i, \xi(\ell), y_i) \colon i \in T \setminus \{r\}\}$;

(2) for each variable $x \in \mathcal{X}$ such that $\mathsf{right}(x) = r$ and for each boundary relation $(i, x, j, \overline{x}) \in \mathcal{B}'$, we add to $\mathcal{B}_1$ two new boundary relations as follows:

  (a) if $\mathsf{right}(\overline{x}) < r$, then add the relations $(i^\bullet, x, \xi(j), \overline{x})$ and $(\xi(j), \overline{x}, i^\bullet, x)$;

  (b) if $\mathsf{right}(\overline{x}) = r$, then add the relations $(i^\bullet, x, j^\bullet, \overline{x})$ and $(j^\bullet, \overline{x}, i^\bullet, x)$;

(3) for each variable $x \in \mathcal{X}$ such that $\mathsf{right}(x) < r$ and $\mathsf{right}(\overline{x}) < r$ and for each boundary relation $(i, x, j, \overline{x}) \in \mathcal{B}'$, we add to $\mathcal{B}_1$ the boundary relations $(\xi(i), x, \xi(j), \overline{x})$ and $(\xi(j), \overline{x}, \xi(i), x)$.

Finally, in $\mathcal{B}_{1,\mathsf{H}}$ we include all the equations of the set $\xi_\Lambda(\mathcal{B}_\mathsf{H})$ as well as the following:

- $(\xi(\ell) \mid \xi(r^-)) = (\ell^\bullet \mid (r^-)^\bullet)$;

- $(\xi(r^-) \mid \xi(r)) = ((r^-)^\bullet \mid r^\bullet) \cdot \{(r^\bullet)^- \mid r^\bullet\}^{\omega-1}_{\vec{t}'_{p,\mu},\mu'_{p,\mu}} \cdot \{\xi(r)^- \mid \xi(r)\}_{\vec{s}'_p,\mu'}$, for each $\vec{s}_p \in \zeta(r^-,r)$ and $\mu \in M(r^-,r,\vec{s}_p)$. Here, we are writing

$$\Lambda(r^-,r,\vec{s}_p,\mu) = ((\ldots,\vec{s}'_p),\mu');$$
$$\Lambda_0((r^-)^\circ,r^\circ,\vec{t}_{p,\mu},\mu_{p,\mu}) = ((\ldots,\vec{t}'_{p,\mu}),\mu'_{p,\mu}).$$

Proposition 5.32. *The tuple $\mathcal{M}_1$ is a model of the system of boundary relations $\mathcal{S}_1$.*

Proof. Properties (M.1)–(M.3) are satisfied as a consequence of Lemma 5.18. For the remaining properties, we first observe that Lemma 5.30(b) implies that the pseudovariety DRH satisfies the pseudoidentity $w_1(\xi(i),\xi(j)) = w_1(i^\bullet,j^\bullet)$, if $i<j$ in $T \setminus \{r\}$. In particular, the boundary relations added to $\mathcal{B}_1$ in step (1) satisfy (M.4). For the rest of the boundary relations, we consider a box (i,x) in $\mathcal{B}'$ and we first compute $w_1(\xi(i),\mathsf{right}_1(x))$ or $w_1(i^\bullet,\mathsf{right}_1(x))$ according to whether $\mathsf{right}(x) \le r^-$ or $\mathsf{right}(x) = r$, respectively. If $\mathsf{right}(x) \le r^-$, then DRH satisfies

$$w_1(\xi(i),\mathsf{right}_1(x)) = w[\iota_1(\xi(i)),\iota_1(\xi(\mathsf{right}(x)))[= w[\iota(i),\iota(\mathsf{right}(x))[= w(i,\mathsf{right}(x)),$$

while, if $\mathsf{right}(x) = r$, then DRH satisfies

$$w_1(i^\bullet,\mathsf{right}_1(x)) = w[\iota_1(i^\bullet),\iota_1(r^\bullet)[= w[i^\circ,r^\circ[\overset{\text{Lemma 5.30}(b)}{\mathcal{R}} w(i,r) = w(i,\mathsf{right}(x)).$$

Since (M.4) holds for $(\mathcal{S},\mathcal{M})$, these relations and construction of $\mathcal{B}_1$ imply that (M.4) holds for $(\mathcal{S}_1,\mathcal{M}_1)$.

By Remark 5.23, the homomorphism $\delta_{w,\mathcal{C}_1} = \delta_{w_1,\mathcal{C}_1}$ is a solution modulo H of $\xi_\Lambda(\mathcal{B}_\mathsf{H})$. Also, the homomorphism $\delta_{w_1,\mathcal{C}_1}$ is a solution modulo H of the equation $(\xi(\ell) \mid \xi(r^-)) = (\ell^\bullet \mid (r^-)^\bullet)$ as a consequence of the fact that DRH satisfies $w_1(\xi(\ell),\xi(r^-)) = w_1(\ell^\bullet,(r^-)^\bullet)$, as already observed when proving (M.4). Finally, the equations of the form

$$(\xi(r^-) \mid \xi(r)) = ((r^-)^\bullet \mid r^\bullet) \cdot \{(r^\bullet)^- \mid r^\bullet\}^{\omega-1}_{\vec{t}'_{p,\mu},\mu'_{p,\mu}} \cdot \{\xi(r)^- \mid \xi(r)\}_{\vec{s}'_p,\mu'}$$

are satisfied by $\delta_{w_1,\mathcal{C}_1}$ modulo H since the following pseudoidentities are valid in H:

$$\begin{aligned}
\delta_{w_1,\mathcal{C}_1}(\xi(r^-) \mid \xi(r)) &= w_1(\xi(r^-),\xi(r)) = w(r^-,r) \\
&= \Phi(r^-,r,\vec{s}_p,\mu)\Psi(r^-,r,\vec{s}_p,\mu) \quad \text{by Property (M.1) for } (\mathcal{S},\mathcal{M}) \\
&= w[(r^-)^\circ,r^\circ[\, v^{\omega-1}_{\vec{s}_p,\mu}\, \Psi(r^-,r,\vec{s}_p,\mu) \quad \text{by (5.25)} \\
&= w_1((r^-)^\bullet,r^\bullet)\, \Psi_0((r^-)^\circ,r^\circ,\vec{t}_{p,\mu},\mu_{p,\mu})^{\omega-1}\, \Psi(r^-,r,\vec{s}_p,\mu) \\
&= w_1((r^-)^\bullet,r^\bullet) \\
&\quad \cdot \Psi_1((r^\bullet)^-,r^\bullet,\vec{t}'_{p,\mu},\mu'_{p,\mu})^{\omega-1}\, \Psi_1(\xi(r)^-,\xi(r),\vec{s}'_p,\mu') \quad \text{by (R.2.4) for } \Lambda \text{ and } \Lambda_0 \\
&= \delta_{w_1,\mathcal{C}_1}\Big(((r^-)^\bullet \mid r^\bullet) \cdot \{(r^\bullet)^- \mid r^\bullet\}^{\omega-1}_{\vec{t}'_{p,\mu},\mu'_{p,\mu}} \cdot \{\xi(r)^- \mid \xi(r)\}_{\vec{s}'_p,\mu'}\Big).
\end{aligned}$$

With this, we may conclude that $\mathcal{M}_1$ is a model of $\mathcal{S}_1$. $\square$

Proposition 5.33. *If σ satisfies the condition* (sig) *and $\kappa \subseteq \langle\sigma\rangle$, then Properties (P.1) and (P.2) are satisfied by the pairs* $(\mathcal{S},\mathcal{M})$ *and* $(\mathcal{S}_1,\mathcal{M}_1)$.

Proof. By construction, Property (P.1) holds because there are no boxes in $\mathcal{B}_1$ ending at $\iota(r)$, and so the first component of the induction parameter decreases.

For Property (P.2), we may let $\mathcal{M}_1' = (w_1', \iota_1', \Theta_1')$ be a model of $\mathcal{S}_1$ in σ-words and we construct a new triple $\mathcal{M}' = (w', \iota', \Theta')$ as follows. We fix a pair $(\vec{s}_q, \mu_0) \in \zeta(r^-, r) \times M(r^-, r, \vec{s}_q)$, for a certain $q \in \{1, \ldots, m\}$. We write

$$\Lambda(r^-, r, \vec{s}_q, \mu_0) = ((\ldots, \vec{s}\,'_q), \mu_0'),$$
$$\Lambda_0((r^-)^\circ, r^\circ, \vec{t}_{q,\mu_0}, \mu_{q,\mu_0}) = ((\ldots, \vec{t}\,'_{q,\mu_0}), \mu'_{q,\mu_0}).$$

The σ-word w' is given by

$$\begin{aligned} w' = {} & w_1'[0, \iota_1'(\xi(r^-))[\cdot w_1'((r^-)^\bullet, r^\bullet)\Psi_1'((r^\bullet)^-, r^\bullet, \vec{t}\,'_{q,\mu_0}, \mu'_{q,\mu_0})^{\omega-1}\Psi_1'(\xi(r)^-, \xi(r), \vec{s}\,'_q, \mu_0') \\ & \cdot w_1'[\iota_1'(\xi(r)), \alpha_{w_1'}[. \end{aligned}$$

Since we are assuming that σ satisfies the condition (sig) and $\kappa \subseteq \langle\sigma\rangle$, we have that w' is actually a σ-word. For $i \in J$, we let $\iota'(i)$ be given by

$$\iota'(i) = \begin{cases} \iota_1'(\xi(i)), & \text{if } i \le r^-; \\ \iota'(r^-) + (\iota_1'(r^\bullet) - \iota_1'((r^-)^\bullet)), & \text{if } i = r; \\ \iota'(r) + (\iota_1'(\xi(i)) - \iota_1'(\xi(r))), & \text{if } i > r. \end{cases}$$

Finally, we define Θ'. Given indices i, j such $i \prec j \le r^-$ or $r \le i \prec j$ in J, and a pair $(\vec{s}, \mu)$, where $\vec{s} \in \zeta(i,j)$ and $\mu \in M(i,j,\vec{s})$, let $\Lambda(i,j,\vec{s},\mu) = ((\vec{t}_1, \ldots, \vec{t}_n), \mu')$ and $\xi(i) = i_0 \prec i_1 \prec \cdots \prec i_n = \xi(j)$. Then, we take

$$\Theta'(i,j,\vec{s},\mu) = \left(\left(\prod_{k=1}^{n-1} \mathsf{prod} \circ \Theta_1'(i_{k-1}, i_k, \vec{t}_k, 0)\right)\Phi_1'(\xi(j)^-, \xi(j), \vec{t}_n, \mu'), \Psi_1'(\xi(j)^-, \xi(j), \vec{t}_n, \mu')\right).$$

On the other hand, when $(i,j) = (r^-, r)$, for each $\vec{s}_p \in \zeta(r^-, r)$ and $\mu \in M(r^-, r, \vec{s}_p)$, we write $\Lambda(r^-, r, \vec{s}_p, \mu) = ((\vec{t}_1, \ldots, \vec{t}_n), \mu')$, $\Lambda_0((r^-)^\circ, r^\circ, \vec{t}_{p,\mu}, \mu_{p,\mu}) = ((\vec{t}\,'_1, \ldots, \vec{t}\,'_n), \mu'_{p,\mu})$, and we let $i_0, \ldots, i_n$ be such that $(r^-)^\bullet = i_0 \prec i_1 \prec \cdots \prec i_n = r^\bullet$. We define

$$\Theta'(r^-, r, \vec{s}_p, \mu) = \left(\left(\prod_{k=1}^{n-1} \mathsf{prod} \circ \Theta_1'(i_{k-1}, i_k, \vec{t}\,'_k, 0)\right)\Phi_1'((r^\bullet)^-, r^\bullet, \vec{t}\,'_n, \mu'_{p,\mu}), \Psi_1'(\xi(r)^-, \xi(r), \vec{t}_n, \mu')\right).$$

It is worth observing that, since each component of Θ_1' is a σ-word, the components of Θ' are σ-words as well.

Let us verify that $\mathcal{M}'$ is a model of $\mathcal{S}$. For Properties (M.1) and (M.2), let $(i,j,\vec{s}) \in \mathsf{Dom}(M)$ and $\mu \in M(i,j,\vec{s})$. According to the construction of Θ', we distinguish whether $(i,j) \neq (r^-, r)$ or

$(i,j) = (r^-, r)$. In the former situation, DRH satisfies

$$\begin{aligned}
\mathsf{prod}\circ\Theta'(i,j,\vec{s},\mu) &= \left(\prod_{k=1}^{n-1}\mathsf{prod}\circ\Theta'_1(i_{k-1},i_k,\vec{t}_k,0)\right)\cdot\mathsf{prod}\circ\Theta'_1(\xi(j)^-,\xi(j),\vec{t}_n,\mu')\\
&= \left(\prod_{k=1}^{n-1}w'_1(i_{k-1},i_k)\right)\cdot w'_1(\xi(j)^-,\xi(j)) \quad \text{by Property (M.1) for } (\mathcal{S}_1,\mathcal{M}'_1)\\
&= w'_1(\xi(i),\xi(j)) = w'(i,j),
\end{aligned}$$

thereby obtaining that (M.1) is satisfied by $\mathcal{M}'$. For (M.2), we may compute

$$\begin{aligned}
\varphi(\Phi'(i,j,\vec{s},\mu)) &= \varphi\left(\left(\prod_{k=1}^{n-1}\mathsf{prod}\circ\Theta'_1(i_{k-1},i_k,\vec{t}_k,0)\right)\Phi'_1(\xi(j)^-,\xi(j),\vec{t}_n,\mu')\right)\\
&= \left(\prod_{k=1}^{n-1}t_{k,1}t_{k,2}\right)t_{n,1} \quad \text{by Property (M.2) for } (\mathcal{S}_1,\mathcal{M}'_1)\\
&= s_1 \quad \text{by Property (R.2.3) for } \Lambda;\\
\varphi(\Psi'(i,j,\vec{s},\mu)) &= \varphi(\Psi'_1(\xi(j)^-,\xi(j),\vec{t}_n,\mu'))\\
&= t_{n,2} \quad \text{by Property (M.2) for } (\mathcal{S}_1,\mathcal{M}'_1)\\
&= s_2 \quad \text{by Property (R.2.3) for } \Lambda.
\end{aligned}$$

When $(i,j) = (r^-, r)$, we suppose that $\vec{s} = \vec{s}_p$, for a certain p. Then, the following is valid in DRH:

$$\begin{aligned}
\mathsf{prod}\circ\Theta'(r^-,r,\vec{s}_p,\mu) &= \left(\prod_{k=1}^{n-1}\mathsf{prod}\circ\Theta'_1(i_{k-1},i_k,\vec{t}\,'_k,0)\right)\\
&\quad\cdot\Phi'_1((r^\bullet)^-,r^\bullet,\vec{t}\,'_n,\mu'_{p,\mu})\Psi'_1(\xi(r)^-,\xi(r),\vec{t}_n,\mu')\\
&= \prod_{k=1}^{n-1}w'_1(i_{k-1},i_k) \qquad\qquad \text{by (M.1) for } (\mathcal{S}_1,\mathcal{M}'_1)\\
&\quad\cdot w'_1((r^\bullet)^-,r^\bullet)\Psi'_1((r^\bullet)^-,r^\bullet,\vec{t}\,'_n,\mu'_{p,\mu})^{\omega-1}\Psi'_1(\xi(r)^-,\xi(r),\vec{t}_n,\mu')\\
&= w'_1((r^-)^\bullet,r^\bullet)\Psi'_1((r^\bullet)^-,r^\bullet,\vec{t}\,'_n,\mu'_{p,\mu})^{\omega-1}\Psi'_1(\xi(r)^-,\xi(r),\vec{t}_n,\mu').
\end{aligned} \tag{5.26}$$

In turn, since $c(\Psi'_1((r^\bullet)^-,r^\bullet,\vec{t}\,'_n,\mu'_{p,\mu})^{\omega-1}\Psi'_1(\xi(r)^-,\xi(r),\vec{t}_n,\mu')) \subseteq \vec{c}(w'_1((r^-)^\bullet,r^\bullet))$, DRH also satisfies

$$w'_1((r^-)^\bullet,r^\bullet)\Psi'_1((r^\bullet)^-,r^\bullet,\vec{t}\,'_n,\mu'_{p,\mu})^{\omega-1}\Psi'_1(\xi(r)^-,\xi(r),\vec{t}_n,\mu')\;\mathcal{R}\;w'_1((r^-)^\bullet,r^\bullet)\;\mathcal{R}\;w'(r^-,r). \tag{5.27}$$

On the other hand, since the equations

$$(\xi(r^-)\mid\xi(r)) = ((r^-)^\bullet\mid r^\bullet)\cdot\{(r^\bullet)^-\mid r^\bullet\}^{\omega-1}_{\vec{t}\,'_n,\mu'_{p,\mu}}\cdot\{\xi(r)^-\mid\xi(r)\}_{\vec{t}_n,\mu'}; \tag{5.28}$$

$$(\xi(r^-)\mid\xi(r)) = ((r^-)^\bullet\mid r^\bullet)\cdot\{(r^\bullet)^-\mid r^\bullet\}^{\omega-1}_{\vec{t}\,'_{q,\mu_0},\mu'_{q,\mu_0}}\cdot\{\xi(r)^-\mid\xi(r)\}_{\vec{s}\,'_q,\mu'_0} \tag{5.29}$$

belong to $\mathcal{B}_{1,\mathsf{H}}$, the pseudovariety H satisfies

$$\begin{aligned}
\mathsf{prod}\circ\Theta'(r^-,r,\vec{s}_p,\mu) &\overset{(5.26)}{=} w_1'((r^-)^\bullet,r^\bullet)\Psi_1'((r^\bullet)^-,r^\bullet,\vec{t}'_n,\mu'_{p,\mu})^{\omega-1}\Psi_1'(\xi(r)^-,\xi(r),\vec{t}_n,\mu')\\
&\overset{(5.28)}{=} w_1'(\xi(r^-),\xi(r))\\
&\overset{(5.29)}{=} w_1'((r^-)^\bullet,r^\bullet)\Psi_1'((r^\bullet)^-,r^\bullet,\vec{t}'_{q,\mu_0},\mu'_{q,\mu_0})^{\omega-1}\Psi_1'(\xi(r)^-,\xi(r),\vec{s}'_q,\mu'_0)\\
&\overset{\text{def.}}{=} w'(r^-,r).
\end{aligned}\tag{5.30}$$

Using (5.26), (5.27), (5.30) and Lemma 2.32, we finally get that DRH satisfies the pseudoidentity $\mathsf{prod}\circ\Theta'(r^-,r,\vec{s}_p,\mu)=w'(r^-,r)$, obtaining (M.1). Further, we may compute

$$\begin{aligned}
\varphi(\Phi'(r^-,r,\vec{s}_p,\mu)) &= \varphi\left(\left(\prod_{k=1}^{n-1}\mathsf{prod}\circ\Theta_1'(i_{k-1},i_k,\vec{t}'_k,0)\right)\Phi_1'((r^\bullet)^-,r^\bullet,\vec{t}'_n,\mu'_{p,\mu})\right)\\
&= \left(\prod_{k=1}^{n-1}t'_{k,1}t'_{k,2}\right)t'_{n,1} \quad \text{by (M.2) for } (\mathcal{S}_1,\mathcal{M}_1')\\
&= s_{p,1} \quad \text{by (R.2.3) for } \Lambda_0;\\
\varphi(\Psi'(r^-,r,\vec{s}_p,\mu)) &= \varphi(\Psi_1'(\xi(r)^-,\xi(r),\vec{t}_n,\mu')) = t_{n,2} \quad \text{by (M.2) for } (\mathcal{S}_1,\mathcal{M}_1')\\
&= s_{p,2} \quad \text{by (R.2.3) for } \Lambda.
\end{aligned}$$

This completes the proof of (M.1) and (M.2). For Property (M.3), let $i\prec j$ in J. Then, we have

$$\begin{aligned}
\vec{c}(w'(i,j)) &= \begin{cases}\vec{c}(w_1'(\xi(i),\xi(j))), & \text{if } (i,j)\neq(r^-,r);\\ \vec{c}(w_1'((r^-)^\bullet,r^\bullet)), & \text{if } (i,j)=(r^-,r);\end{cases}\\
&= \begin{cases}\chi_1(\xi(i),\xi(j)), & \text{if } (i,j)\neq(r^-,r);\\ \chi_1((r^-)^\bullet,r^\bullet), & \text{if } (i,j)=(r^-,r);\end{cases} \quad \text{by (M.3) for } (\mathcal{S}_1,\mathcal{M}_1')\\
&= \begin{cases}\vec{c}(w[\iota_1(\xi(i)),\iota_1(\xi(j))[), & \text{if } (i,j)\neq(r^-,r);\\ \vec{c}(w[\iota_1((r^-)^\bullet),\iota_1(r^\bullet)[), & \text{if } (i,j)=(r^-,r);\end{cases} \quad \text{by definition (5.19) of } \chi_1=\chi_{w,\mathcal{C}_1}\\
&= \begin{cases}\vec{c}(w(i,j)), & \text{if } (i,j)\neq(r^-,r);\\ \vec{c}(w[(r^-)^\circ,r^\circ[), & \text{if } (i,j)=(r^-,r);\end{cases} \quad \text{by definition of } _^\bullet,_^\circ \text{ and } \xi\\
&= \begin{cases}\vec{c}(w(i,j)), & \text{if } (i,j)\neq(r^-,r);\\ \vec{c}(w(r^-,r)), & \text{if } (i,j)=(r^-,r);\end{cases} \quad \text{by Lemma 5.30}(b)\\
&= \chi(i,j) \quad \text{by (M.3) for } (\mathcal{S},\mathcal{M}).
\end{aligned}$$

To prove that Property (M.4) holds, we first notice that, for every $i<j<r$ in T,

$$w_1'(\xi(i),\xi(j)) =_{\mathsf{DRH}} w_1'(i^\bullet,j^\bullet). \tag{5.31}$$

In fact, since $(\xi(\ell), y_{r^-}, \ell^\bullet, \bar{y}_{r^-})$ belongs to $\mathcal{B}_1$, $(\xi(\ell) \mid \xi(r^-)) = (\ell^\bullet \mid (r^-)^\bullet)$ is an equation of $\mathcal{B}_{1,\mathsf{H}}$ and $\mathcal{M}_1'$ is a model of $\mathcal{S}_1$, Lemma 2.32 implies

$$w_1'(\xi(\ell), \xi(r^-)) =_{\mathsf{DRH}} w_1'(\ell^\bullet, (r^-)^\bullet).$$

Since $(\xi(\ell), y_i, \ell^\bullet, \bar{y}_i)$ and $(\xi(\ell), y_j, \ell^\bullet, \bar{y}_j)$ are also relations of $\mathcal{B}_1$, by Corollary 2.28 we know that $\alpha_{w_1'(\xi(\ell),\xi(i))} = \alpha_{w_1'(\ell^\bullet,i^\bullet)}$ and $\alpha_{w_1'(\xi(\ell),\xi(j))} = \alpha_{w_1'(\ell^\bullet,j^\bullet)}$ and thus Corollary 2.31 yields (5.31). Now, let (i,x) be a box in $\mathcal{B}'$. Using the definitions of w' and of ι' we may compute

$$\begin{aligned}
w'(i, \mathsf{right}(x)) &= \begin{cases} w_1'(\xi(i), \xi(\mathsf{right}(x))), & \text{if } \mathsf{right}(x) \leq r^-; \\ w_1'(\xi(i), \xi(r^-))\, w_1'((r^-)^\bullet, r^\bullet) & \\ \quad \cdot \Psi_1'((r^\bullet)^-, r^\bullet, \vec{t}'_{q,\mu_0}, \mu'_{q,\mu_0})^{\omega-1} \Psi_1'(\xi(r)^-, \xi(r), \vec{s}'_q, \mu_0'), & \text{otherwise;} \end{cases} \\
&\overset{(5.31)}{=} \begin{cases} w_1'(\xi(i), \xi(\mathsf{right}(x))), & \text{if } \mathsf{right}(x) \leq r^-; \\ w_1'(i^\bullet, (r^-)^\bullet)\, w_1'((r^-)^\bullet, r^\bullet) & \\ \quad \cdot \Psi_1'((r^\bullet)^-, r^\bullet, \vec{t}'_{q,\mu_0}, \mu'_{q,\mu_0})^{\omega-1} \Psi_1'(\xi(r)^-, \xi(r), \vec{s}'_q, \mu_0'), & \text{otherwise;} \end{cases} \\
&\ \mathcal{R} \begin{cases} w_1'(\xi(i), \mathsf{right}_1(x)), & \text{if } \mathsf{right}(x) \leq r^-; \\ w_1'(i^\bullet, \mathsf{right}_1(x)), & \text{otherwise.} \end{cases}
\end{aligned}$$

Taking into account the steps (2) and (3) in the construction of $\mathcal{B}_1$, it is now easy to deduce that (M.4) holds for all the relations of $\mathcal{B}'$. It remains to verify that $w'(\ell, r)$ and $w'(\ell^*, r^*)$ are $\mathcal{R}$-equivalent modulo DRH. For that purpose, we show that the following relations hold in DRH:

$$\begin{aligned}
w'(\ell, r) &= w'(\ell, r^-) w'(r^-, r) \\
&\overset{\text{def.}}{=} w_1'(\xi(\ell), \xi(r^-))\, w_1'((r^-)^\bullet, r^\bullet) \Psi_1'((r^\bullet)^-, r^\bullet, \vec{t}'_{q,\mu_0}, \mu'_{q,\mu_0})^{\omega-1} \Psi_1'(\xi(r)^-, \xi(r), \vec{s}'_q, \mu_0') \\
&\overset{(5.31)}{=} w_1'(\ell^\bullet, (r^-)^\bullet)\, w_1'((r^-)^\bullet, r^\bullet) \Psi_1'((r^\bullet)^-, r^\bullet, \vec{t}'_{q,\mu_0}, \mu'_{q,\mu_0})^{\omega-1} \Psi_1'(\xi(r)^-, \xi(r), \vec{s}'_q, \mu_0') \\
&= w_1'(\ell^\bullet, r^\bullet) \Psi_1'((r^\bullet)^-, r^\bullet, \vec{t}'_{q,\mu_0}, \mu'_{q,\mu_0})^{\omega-1} \Psi_1'(\xi(r)^-, \xi(r), \vec{s}'_q, \mu_0') \\
&\ \mathcal{R}\ w_1'(\ell^\bullet, r^\bullet), \quad \text{because of the inclusion} \\
&\qquad\qquad c(\Psi_1'((r^\bullet)^-, r^\bullet, \vec{t}'_{q,\mu_0}, \mu'_{q,\mu_0})^{\omega-1} \Psi_1'(\xi(r)^-, \xi(r), \vec{s}'_q, \mu_0')) \subseteq \vec{c}(w_1'(\ell^\bullet, r^\bullet)) \\
&= w_1'(\iota_1^{-1}(\ell^\circ), \iota_1^{-1}(r^\circ)) = w_1'(\iota_1^{-1}(\iota(\ell^*)), \iota_1^{-1}(\iota(r^*))) \\
&= w_1'(\xi(\ell^*), \xi(r^*)) = w'(\ell^*, r^*).
\end{aligned}$$

Finally, since $\xi_\Lambda(\mathcal{B}_\mathsf{H}) \subseteq \mathcal{B}_{1,\mathsf{H}}$, in Remark 5.24 we observed that, in order to prove that Property (M.5) is satisfied, it is enough to prove that H satisfies

$$w'(i,j) = w_1'(\xi(i), \xi(j)) \tag{5.32}$$

$$\Psi'(i,j,\vec{s},\mu) = \Psi_1'(\xi(j)^-, \xi(j), \vec{s}', \mu'), \tag{5.33}$$

for every $(i,j,\vec{s},\mu) \in \mathsf{Dom}(M) \times M(i,j,\vec{s})$, where $\Lambda(i,j,\vec{s},\mu) = ((\ldots,\vec{s}'),\mu')$. The pseudoidentity (5.32) follows straightforwardly from the definition of w', except when $(i,j) = (r^-,r)$. In that case, by computing (5.32) modulo H, we get

$$\begin{aligned} w'(r^-,r) &= w'_1((r^-)^\bullet, r^\bullet)\Psi'_1((r^\bullet)^-, r^\bullet, \vec{t}'_{q,\mu_0}, \mu'_{q,\mu_0})^{\omega-1}\Psi'_1(\xi(r)^-, \xi(r), \vec{s}'_q, \mu'_0) \\ &= w'_1(\xi(r^-), \xi(r)), \end{aligned}$$

where the last equality holds because the equation

$$(\xi(r^-) \mid \xi(r)) = ((r^-)^\bullet \mid r^\bullet) \cdot \{(r^\bullet)^- \mid r^\bullet\}^{\omega-1}_{\vec{t}'_{q,\mu_0}, \mu'_{q,\mu_0}} \cdot \{\xi(r)^- \mid \xi(r)\}_{\vec{s}'_q, \mu'_0}$$

belongs to $\mathcal{B}_{1,\mathsf{H}}$ and $\mathcal{M}'_1$ is a model of $\mathcal{S}_1$. Lastly, the pseudoidentity (5.33) corresponds precisely to the definition of Θ'. Thus, $\mathcal{M}'$ is a model of $\mathcal{S}$ in σ-words and so, Property (P.2) holds for the pair $(\mathcal{S}_1, \mathcal{M}_1)$. $\square$

5.6.8 Case 5

Finally, it remains to consider the case where $\mathcal{B}$ has a boundary relation of the form $(i,x,j,\bar{x})$ with $\mathsf{right}(x) = r = \mathsf{right}(\bar{x})$ and none of the Cases 1–4 hold. In particular, the non occurrence of Cases 2, 3 and 4 implies that all the boundary relations $(i,x,j,\bar{x})$ verifying $i \le j$ and $\mathsf{right}(\bar{x}) = r$ are such that $i < j$, $\mathsf{right}(x) = r$ and the equality $c(w(i,j)) = c(w(i,r))$ holds.

We consider the index

$$c = \max\{\min(J), \max\{\mathsf{right}(x) \colon \mathsf{right}(x) < r\}, \max\{i \in J \colon i < r \text{ and there is no box } (i,x)\}\}$$

and we let $\mathcal{E} = \{(i,x,j,\bar{x}) \in \mathcal{B} \colon i < j;\ \mathsf{right}(x) = r = \mathsf{right}(\bar{x})\}$. By the auxiliary step, we may assume that all the boundary relations of $\mathcal{E}$ are such that $c < i, j < r$. Since the auxiliary step consists in successively factorizing a boundary relation from $\mathcal{E}$ with respect to a pair of ordinals both greater than $\iota(c)$ (recall Figure 5.7 and Lemma 5.29), it follows that for every index $c < i < r$ there exists a box (i,x) such that $\mathsf{right}(x) = r$. Observe that the choice of c guarantees that all the indices in the original set of boundary relations already satisfy this condition. Moreover, since $\mathcal{E}$ contains all the boxes ending in r, if (i,x) is a box such that $\mathsf{right}(x) = r$, then $c < i < r$.

Now, we let $\ell = \max\{i \in J \colon \text{ there exists } (i,x,j,\bar{x}) \in \mathcal{E}\}$. We use the construction presented in Subsection 5.6.2 to align the left of each variable intervening in $\mathcal{E}$ (as schematized in Figure 5.9). For

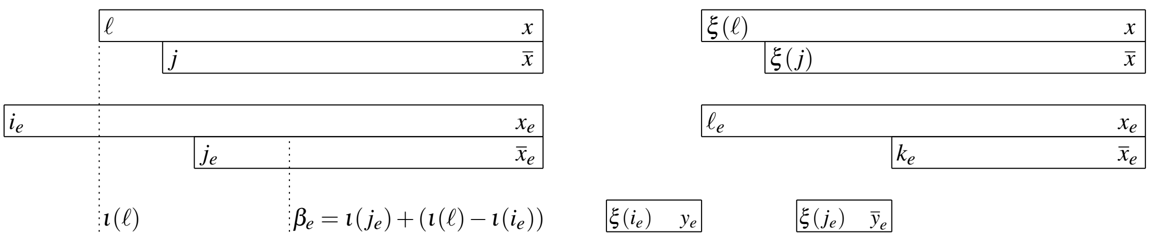

Fig. 5.9 Aligning a boundary relation on the left with ℓ.

each $e = (i_e, x_e, j_e, \bar{x}_e) \in \mathcal{E}$, let $\beta_e = \iota(j_e) + (\iota(\ell) - \iota(i_e))$. By Corollary 2.31, β_e is the unique ordinal such that the equality $w(i_e, \ell) = w[\iota(j_e), \beta_e[$ holds modulo DRH. Hence, if $\Delta = \{(\iota(\ell), \beta_e)\}_{e \in \mathcal{E}}$, then the pair $(\mathcal{E}, \Delta)$ satisfies (F.1) and (F.2). We let $(\mathcal{S}_0, \mathcal{M}_0)$ be the factorization of $(\mathcal{S}, \mathcal{M})$ with respect to $(\mathcal{E},\Delta)$, where $\mathcal{S}_0 = (\mathcal{X}_0, J_0, \zeta_0, M_0, \chi_0, \mathsf{right}_0, \mathcal{B}_0, \mathcal{B}_{0,\mathsf{H}})$ and $\mathcal{M}_0 = (w_0, \iota_0, \Theta_0)$. In the new set of boundary relations $\mathcal{B}_0$, any boundary relation such that one of its boxes ends at $\xi(r)$ is either of the form $(\xi(\ell), x_e, k_e, \bar{x}_e)$ or of the form $(k_e, \bar{x}_e, \xi(\ell), x_e)$, where $\xi = \iota_0^{-1} \circ \iota$, $k_e = \iota_0^{-1}(\beta_e)$ and $\mathsf{right}_0(x_e) = \xi(r) = \mathsf{right}_0(\bar{x}_e)$. In order to simplify the notation, we drop the index 0 in the pair $(\mathcal{S}_0, \mathcal{M}_0)$ and we simply assume that the given pair $(\mathcal{S}, \mathcal{M})$ is such that the set $\mathcal{E}$ defined above is given by

$$\mathcal{E} = \{(\ell, x_1, j_1, \bar{x}_1), \ldots, (\ell, x_n, j_n, \bar{x}_n)\}$$

with $j_1 \leq j_2 \leq \cdots \leq j_n$. We notice that, by definition of the index c, we have $j_n \prec r$ in J. Since $\mathcal{M}$ is a model of $\mathcal{S}$, the pseudovariety DRH satisfies

$$w(\ell, j_m) w(\ell, r) \mathrel{\mathcal{R}} w(\ell, j_m) w(j_m, r) = w(\ell, r),$$

for $m = 1, \ldots, n$. Multiplying successively by $w(\ell, j_m)$ on the left, we get

$$w(\ell, j_m)^{\omega} w(\ell, r) \mathrel{\mathcal{R}} w(\ell, r) \text{ modulo } \mathsf{DRH}.$$

Since $\vec{c}(w(\ell, j_m)^{\omega}) = c(w(\ell, j_m)) = c(w(\ell, r))$, it follows that

$$w(\ell, r) \mathrel{\mathcal{R}} w(\ell, j_1)^{\omega} \mathrel{\mathcal{R}} \cdots \mathrel{\mathcal{R}} w(\ell, j_n)^{\omega} \text{ modulo } \mathsf{DRH}. \tag{5.34}$$

But all the pseudowords $w(\ell, j_m)^{\omega}$ represent the identity in the same maximal subgroup of $\overline{\Omega}_A\mathsf{DRH}$ where they belong (recall Proposition 2.18). Therefore, all the elements $w(\ell, j_m)^{\omega}$ are the same over DRH. Then, Proposition 5.13 applied to the elements $w(\ell, j_1), \ldots, w(\ell, j_n)$ guarantees the existence of pseudowords $u \in \overline{\Omega}_A\mathsf{S}$, $v_1, \ldots, v_n \in (\overline{\Omega}_A\mathsf{S})^I$ and of positive integers $h_1, \ldots, h_n$ such that, for $m = 1, \ldots, n$ we have

$$\begin{aligned} w(\ell, j_m) &=_{\mathsf{DRH}} u^{h_m} v_m, \\ v_m u &=_{\mathsf{DRH}} u, \end{aligned} \tag{5.35}$$

where all the products $u \cdot u$, $u \cdot v_m$ and $v_m \cdot u$ are reduced, whenever the second factor is nonempty. Note that h_n is the maximum of $\{h_1, \ldots, h_n\}$. In fact, if we had, for a certain m, the inequalities $j_m < j_n$ and $h_m > h_n$ then, since DRH satisfies

$$w(\ell, j_m) = u^{h_m} v_m = u^{h_n} v_n u^{h_m - h_n} v_m = w(\ell, j_m) w(j_m, j_n) u^{h_m - h_n} v_m,$$

we would be able to use Corollary 2.28 and Theorem 2.24 to compute

$$\alpha_{w(\ell, j_m)} = \alpha_{w(\ell, j_m)} + \alpha_{w(j_m, j_n) u^{h_m - h_n} v_m} > \alpha_{w(\ell, j_m)},$$

which would yield a contradiction.

Now, we observe that the pseudoidentities in (5.35) imply that every finite power of u is a prefix of $w(\ell, j_m)^{\omega}$, which in turn, by (5.34), is $\mathcal{R}$-equivalent to $w(\ell, r)$ modulo DRH. Since the semigroup S where the constraints are defined is finite, this allows us to find some periodicity on them. With this in mind, to deal with the constraints, we consider a big enough direct power of the semigroup S, more specifically, the semigroup $T = S^K$, with $K = \sum_{\vec{s} \in \zeta(j_n, r)} M(j_n, r, \vec{s})$, and we take $N = |T| + 2$. Let us construct a new pair $(\mathcal{S}_1, \mathcal{M}_1)$ as follows:

$$\mathcal{S}_1 = (\mathcal{X}_1, J_1, \zeta_1, M_1, \chi_1, \mathsf{right}_1, \mathcal{B}_1, \mathcal{B}_{1,\mathsf{H}}) \text{ and } \mathcal{M}_1 = (w_1, \iota_1, \Theta_1),$$

where

- the set of variables is $\mathcal{X}_1 = \mathcal{X} \uplus \{y_q, \overline{y}_q\}_{q=1}^{h_n} \uplus \{z_m, \overline{z}_m\}_{m=1}^{n} \uplus \{f_i, \overline{f}_i\}_{i=1}^{N}$, where variables with different names are assumed to be distinct;
- the pseudoword in the model is $w_1 = w$;
- let O be the set containing the following ordinals:
  - $\beta_0 = \iota(\ell)$;
  - $\beta_q = \beta_0 + \alpha_u \cdot q$, for $q = 1, \dots, h_n + 1$;
  - $\gamma_m = \beta_0 + (\iota(j_m) - \beta_{h_m})$, for $m = 1, \dots, n$;
  - $\delta_p = \beta_0 + \alpha_u \cdot h_n p$, for $p = 0, \dots, N$.

The ordinals in O are depicted in Figure 5.10.

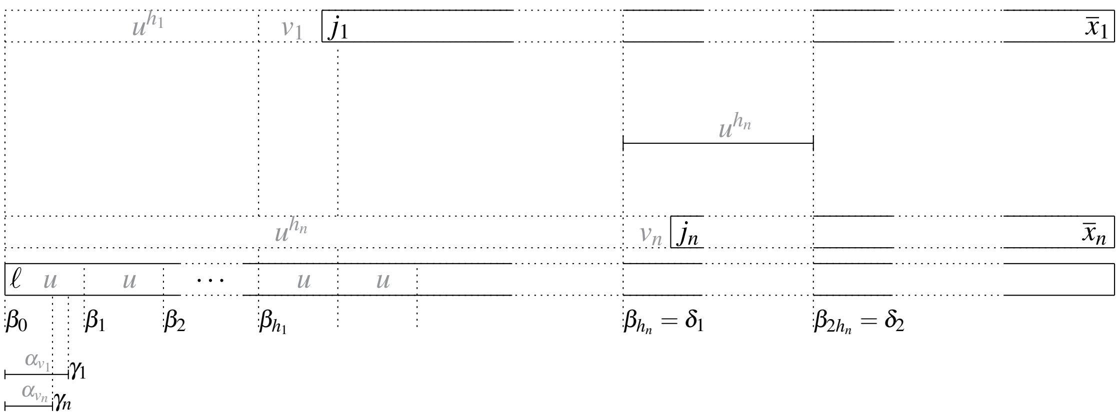

Fig. 5.10 The set of ordinals O.

We let $\mathcal{C}_1 = (J_1, \iota_1, M_1, \Theta_1)$ be a common refinement of the factorization schemes $\mathcal{C}(\mathcal{S}, \mathcal{M})$ and $(O, O \hookrightarrow \alpha_w + 1, \emptyset, \emptyset)$ for w and

$$\Lambda : \{(i, j, \vec{s}, \mu) : (i, j, \vec{s}) \in \mathsf{Dom}(M),\ \mu \in M(i, j, \vec{s})\} \to \bigcup_{k \in \mathbb{N}} (S \times S^I)^k \times \omega$$

be a refining function from $\mathcal{C}(\mathcal{S}, \mathcal{M})$ to $\mathcal{C}_1$. The factorization scheme $\mathcal{C}_1$ supplies the items J_1, ι_1, M_1 and Θ_1 and the items ζ_1 and χ_1 by taking $\zeta_1 = \zeta_{w_1, \mathcal{C}_1}$ and $\chi_1 = \chi_{w_1, \mathcal{C}_1}$ (recall (5.18) and (5.19)). We denote $b_q = \iota_1^{-1}(\beta_q)$, $c_m = \iota_1^{-1}(\gamma_m)$, $d_p = \iota_1^{-1}(\delta_p)$, and $\xi = \iota_1^{-1} \circ \iota$;

- the function right_1 is given by

$$\mathsf{right}_1(x) = \begin{cases} \xi(\mathsf{right}(x)), & \text{if } x \in \mathcal{X}; \\ b_q, & \text{if } x = y_q; \\ b_{q+1}, & \text{if } x = \overline{y}_q; \\ b_{h_m+1}, & \text{if } x \in \{z_m, \overline{z}_m\}; \\ d_p, & \text{if } x = f_p; \\ d_{p+1}, & \text{if } x = \overline{f}_p; \end{cases}$$

- in the set $\mathcal{B}_1$ we include the following boundary relations:
  - $(\xi(i), x, \xi(j), \overline{x})$, if $(i, x, j, \overline{x}) \in \mathcal{B} \setminus (\mathcal{E} \cup \{\text{dual of } e \colon e \in \mathcal{E}\})$;
  - $(b_{q-1}, y_q, b_q, \overline{y}_q)$ and $(b_q, \overline{y}_q, b_{q-1}, y_q)$, for $q = 1, \dots, h_n$;
  - $(b_{h_m}, z_m, \xi(j_m), \overline{z}_m)$ and $(\xi(j_m), \overline{z}_m, b_{h_m}, z_m)$, for $m = 1, \dots, n$;
  - $(d_{p-1}, f_p, d_p, \overline{f}_p)$ and $(d_p, \overline{f}_p, d_{p-1}, f_p)$, for $p = 1, \dots, N-1$;
- the set $\mathcal{B}_{1,\mathsf{H}}$ consists of the following equations:
  - all the equations of $\xi_\Lambda(\mathcal{B}_\mathsf{H})$;
  - $(b_0 \mid b_1) = (b_1 \mid b_2) = \cdots = (b_{h_n} \mid b_{h_n+1})$;
  - $(b_{h_m} \mid b_{h_m+1}) = (\xi(j_m) \mid b_{h_m+1})$, for $m = 1, \dots, n$;
  - $(d_0 \mid d_1) = (d_1 \mid d_2) = \cdots = (d_{N-1} \mid d_N)$.

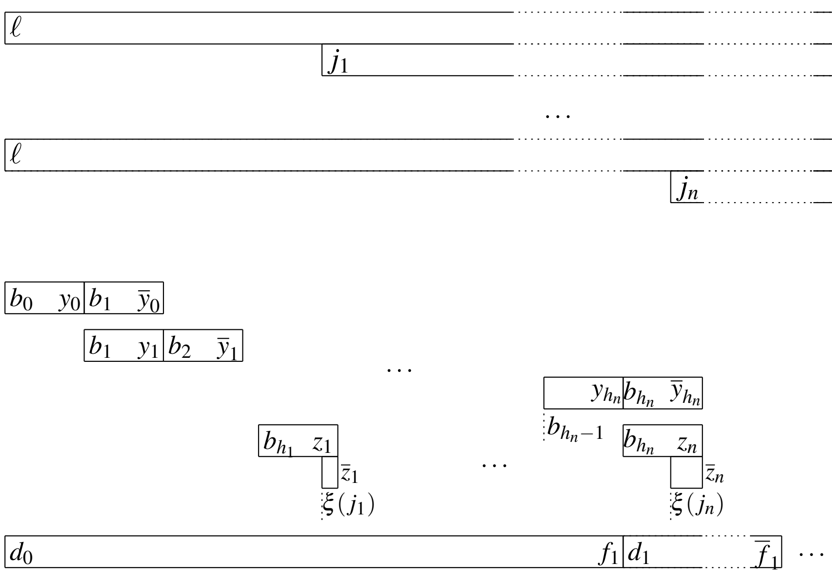

Fig. 5.11 Above, the old boundary relations; below, the new boundary relations.

Proposition 5.34. *Let* $(\mathcal{S}_1, \mathcal{M}_1)$ *be the pair defined above. Then,* $\mathcal{M}_1$ *is a model of* $\mathcal{S}_1$.

Proof. Properties (M.1), (M.2) and (M.3) follow from Lemma 5.18. Let us verify Properties (M.4) and (M.5). For the boundary relations of the form $(\xi(i),x,\xi(j),\bar{x})$, with $(i,x,j,\bar{x}) \in \mathcal{B}$, there is nothing to prove since, up to renaming of indices, they were already satisfied before changing the system. Because of (5.35), any power of u is a prefix of $w(\ell,r)$ modulo DRH. Therefore, all pseudowords $w(b_{q-1},b_q)$, for $q = 1,\dots,h_n+1$ represent the same element over DRH, namely u. This not only proves that Property (M.4) holds for all boundary relations of the form $(b_{q-1},y_q,b_q,\bar{y}_q)$, but also that (M.5) holds for the equations $(b_0 \mid b_1) = (b_1 \mid b_2) = \cdots = (b_{h_n} \mid b_{h_n+1})$. For the relations $(b_{h_m},z_m,\xi(j_m),\bar{z}_m)$, $m = 1,\dots,n$, we first observe that DRH satisfies

$$u = w[\beta_{h_m},\beta_{h_m+1}[\, = w[\beta_{h_m},\iota(j_m)[\,\cdot w[\iota(j_m),\beta_{h_m+1}[;$$
$$u = v_m \cdot u = w[\beta_{h_m},\iota(j_m)[\,\cdot w[\beta_{h_m},\beta_{h_m+1}[.$$

Thus, it follows from Corollary 2.31 that DRH satisfies $w[\beta_{h_m},\beta_{h_m+1}[\, = w[\iota(j_m),\beta_{h_m+1}[$ or, in other words, that DRH satisfies $w_1(b_{h_m},\mathsf{right}_1(z_m)) = w_1(\xi(j_m),\mathsf{right}_1(\bar{z}_m))$. Again, we also proved (M.5) for the equations $(b_{h_m} \mid b_{h_m+1}) = (\xi(j_m) \mid b_{h_m+1})$, $m = 1,\dots,n$. Finally, for the boundary relations $(d_{p-1},f_p,d_p,\overline{f}_p)$, with $p = 1,\dots,N-1$, taking into account the relationship between the ordinals β_q and δ_p, the pseudoidentity $w(d_{p-1},d_p) = u^{h_n}$ holds in $\overline{\Omega}_A\mathsf{DRH}$, for all $p = 1,\dots,N$. Therefore, all the elements $w_1(d_{p-1},\mathsf{right}_1(f_p))$ and $w_1(d_p,\mathsf{right}_1(\overline{f}_p))$ represent the same power of u modulo DRH. Yet again, Property (M.5) follows for the equations $(d_0 \mid d_1) = (d_1 \mid d_2) = \cdots = (d_{N-1} \mid d_N)$. Lastly, to complete the verification of Property (M.5), it remains to prove that $\delta_{w_1,\mathcal{C}_1}$ is a solution modulo H of $\xi_\Lambda(\mathcal{B}_\mathsf{H})$. But this is a consequence of Remark 5.23. □

Since in $\mathcal{B}_1$ there are no boxes ending at r, we decrease the first component of the induction parameter, and so, Property (P.1) holds. Before proving that Property (P.2) also holds, we define integers $1 \le H < L < N$ that later play an essential role.

Recall that, in J_1, we have $\xi(j_n) \prec b_{h_n+1} \preceq d_2 \prec d_3 \prec \cdots \prec d_N \prec \xi(r)$, where $b_{h_n+1} = d_2$ if and only if $h_n = 1$. Therefore, for each $(\vec{s},\mu) \in \zeta(j_n,r) \times M(j_n,r,\vec{s})$, the first component of $\Lambda(j_n,r,\vec{s})$ belongs to $(S \times S^I)^N$ if $h_n = 1$ or to $(S \times S^I)^{N+1}$, otherwise. We assume that $h_n > 1$. The same argument can be used when $h_n = 1$, simply by working with N instead of $N+1$. We may write

$$\Lambda(j_n,r,\vec{s},\mu) = \left(\left(\vec{t}_1^{\,(\vec{s},\mu)},\dots,\vec{t}_{N+1}^{\,(\vec{s},\mu)}\right),\mu_{\vec{s},\mu}\right), \tag{5.36}$$

with $\vec{t}_i^{\,(\vec{s},\mu)} = \left(t_{i,1}^{(\vec{s},\mu)},t_{i,2}^{(\vec{s},\mu)}\right)$. Let $\vec{t}_1,\dots,\vec{t}_N \in T$ satisfy the following properties:

- each element $\vec{t}_i$ is a tuple whose coordinates are of the form $t_{i,1}^{(\vec{s},\mu)}t_{i,2}^{(\vec{s},\mu)}$, for certain $\vec{s} \in \zeta(j_n,r)$ and $\mu \in M(j_n,r,\vec{s})$;
- for $i \in \{1,\dots,K\}$ and $k_1 \neq k_2$, if the k_1-th coordinate of $\vec{t}_i$ is $t_{i,1}^{(\vec{s}_1,\mu_1)}t_{i,2}^{(\vec{s}_1,\mu_1)}$ and the k_2-th coordinate of $\vec{t}_i$ is $t_{i,1}^{(\vec{s}_2,\mu_2)}t_{i,2}^{(\vec{s}_2,\mu_2)}$, then $(\vec{s}_1,\mu_1) \neq (\vec{s}_2,\mu_2)$;
- for $i \in \{2,\dots,K\}$, if the k-th coordinate of $\vec{t}_1$ is $t_{1,1}^{(\vec{s},\mu)}t_{1,2}^{(\vec{s},\mu)}$, then the k-th coordinate of $\vec{t}_i$ is $t_{i,1}^{(\vec{s},\mu)}t_{i,2}^{(\vec{s},\mu)}$.

Table 5.1 schematizes how the vectors $\vec{t}_i$ should be understood.

Pairs of the form $(\vec{s},\mu)$ in $\zeta(j_n,r)\times M(j_n,r,\vec{s})$:	$(\vec{s}_1,\mu_1)$	$(\vec{s}_2,\mu_2)$	$\cdots$	$(\vec{s}_K,\mu_K)$
$\vec{t}_1$:	$t_{1,1}^{(\vec{s}_1,\mu_1)}t_{1,2}^{(\vec{s}_1,\mu_1)}$	$t_{1,1}^{(\vec{s}_2,\mu_2)}t_{1,2}^{(\vec{s}_2,\mu_2)}$	$\cdots$	$t_{1,1}^{(\vec{s}_K,\mu_K)}t_{1,2}^{(\vec{s}_K,\mu_K)}$
$\vec{t}_2$:	$t_{2,1}^{(\vec{s}_1,\mu_1)}t_{2,2}^{(\vec{s}_1,\mu_1)}$	$t_{2,1}^{(\vec{s}_2,\mu_2)}t_{2,2}^{(\vec{s}_2,\mu_2)}$	$\cdots$	$t_{2,1}^{(\vec{s}_K,\mu_K)}t_{2,2}^{(\vec{s}_K,\mu_K)}$
		$\cdots$		
$\vec{t}_N$:	$t_{N,1}^{(\vec{s}_1,\mu_1)}t_{N,2}^{(\vec{s}_1,\mu_1)}$	$t_{N,1}^{(\vec{s}_2,\mu_2)}t_{N,2}^{(\vec{s}_2,\mu_2)}$	$\cdots$	$t_{N,1}^{(\vec{s}_K,\mu_K)}t_{N,2}^{(\vec{s}_K,\mu_K)}$

Table 5.1 The K coordinates of the vector $\vec{t}_i$ are represented in the i-th row.

Since $N-1>|T|$, there exist $1\leq H<L<N$ such that $\vec{t}_1\cdots\vec{t}_H=\vec{t}_1\cdots\vec{t}_L$, which implies that

$$\vec{t}_1\cdots\vec{t}_H\vec{t}_{H+1}\cdots\vec{t}_L=\vec{t}_1\cdots\vec{t}_H(\vec{t}_{H+1}\cdots\vec{t}_L)^{\omega}.$$

In particular, since Λ satisfies (R.2.3), given $\vec{s}=(s_1,s_2)\in\zeta(j_n,r)$ and $\mu\in M(j_n,r,\vec{s})$ the following equalities hold in the semigroup S:

$$\begin{aligned}
s_1 &= t_{1,1}^{(\vec{s},\mu)}t_{1,2}^{(\vec{s},\mu)}\cdot t_{2,1}^{(\vec{s},\mu)}t_{2,2}^{(\vec{s},\mu)}\cdots t_{H,1}^{(\vec{s},\mu)}t_{H,2}^{(\vec{s},\mu)}\cdot t_{H+1,1}^{(\vec{s},\mu)}t_{H+1,2}^{(\vec{s},\mu)}\cdots t_{L,1}^{(\vec{s},\mu)}t_{L,2}^{(\vec{s},\mu)}\cdot t_{L+1,1}^{(\vec{s},\mu)}t_{L+1,2}^{(\vec{s},\mu)}\cdots t_{N,1}^{(\vec{s},\mu)}t_{N,2}^{(\vec{s},\mu)}\cdot t_{N+1,1}^{(\vec{s},\mu)}\\
&= t_{1,1}^{(\vec{s},\mu)}t_{1,2}^{(\vec{s},\mu)}\cdot t_{2,1}^{(\vec{s},\mu)}t_{2,2}^{(\vec{s},\mu)}\cdots t_{H,1}^{(\vec{s},\mu)}t_{H,2}^{(\vec{s},\mu)}\cdot\left(t_{H+1,1}^{(\vec{s},\mu)}t_{H+1,2}^{(\vec{s},\mu)}\cdots t_{L,1}^{(\vec{s},\mu)}t_{L,2}^{(\vec{s},\mu)}\right)^{\omega+1}\\
&\quad\cdot t_{L+1,1}^{(\vec{s},\mu)}t_{L+1,2}^{(\vec{s},\mu)}\cdots t_{N,1}^{(\vec{s},\mu)}t_{N,2}^{(\vec{s},\mu)}\cdot t_{N+1,1}^{(\vec{s},\mu)},\\
s_2 &= t_{N+1,2}^{(\vec{s},\mu)}.
\end{aligned}$$

In order to ease the notation, we define

$$\begin{aligned}
s_1^{(\vec{s},\mu)} &= t_{1,1}^{(\vec{s},\mu)}t_{1,2}^{(\vec{s},\mu)}\cdot t_{2,1}^{(\vec{s},\mu)}t_{2,2}^{(\vec{s},\mu)}\cdots t_{H,1}^{(\vec{s},\mu)}t_{H,2}^{(\vec{s},\mu)},\\
s_2^{(\vec{s},\mu)} &= t_{H+1,1}^{(\vec{s},\mu)}t_{H+1,2}^{(\vec{s},\mu)}\cdots t_{L,1}^{(\vec{s},\mu)}t_{L,2}^{(\vec{s},\mu)},\\
s_{3,1}^{(\vec{s},\mu)} &= t_{L+1,1}^{(\vec{s},\mu)}t_{L+1,2}^{(\vec{s},\mu)}\cdots t_{N,1}^{(\vec{s},\mu)}t_{N,2}^{(\vec{s},\mu)}t_{N+1,1}^{(\vec{s},\mu)},\\
s_{3,2}^{(\vec{s},\mu)} &= t_{N+1,2}^{(\vec{s},\mu)}.
\end{aligned}\tag{5.37}$$

Hence, we have $s_1=s_1^{(\vec{s},\mu)}\cdot\left(s_2^{(\vec{s},\mu)}\right)^{\omega+1}\cdot s_{3,1}^{(\vec{s},\mu)}$ and $s_{3,2}^{(\vec{s},\mu)}=s_2$.

Next, we verify that Property (P.2) is satisfied, as claimed before.

Proposition 5.35. *Let σ be an implicit signature such that $\kappa\subseteq\langle\sigma\rangle$ satisfying* (sig)*, and suppose that there exists a model $\mathcal{M}_1'=(w_1',\iota_1',\Theta_1')$ of $\mathcal{S}_1$ in σ-words. Then, there is a model of $\mathcal{S}$ in σ-words as well.*

Proof. Let $\mathcal{M}'=(w',\iota',\Theta')$ be constructed as follows. The σ-word w' is set to be

$$w'=w_1'[0,\iota_1'(d_L)[\cdot(w_1'(d_H,d_L))^{\omega}w_1'(d_L,\xi(r))\cdot w_1'[\iota_1'(\xi(r)),\alpha_{w_1'}[.$$

The map ι' is defined by

$$\begin{aligned}
\iota' : J &\to \alpha_{w'} + 1\\
i &\mapsto \iota'_1 \circ \xi(i), \qquad \text{if } i < r,\\
r &\mapsto \alpha_{w'_1[0,\iota'_1(d_L)[\cdot(w'_1(d_H,d_L))^{\omega}}},\\
i &\mapsto \iota'(r) + (\iota'_1 \circ \xi(i) - \iota'_1 \circ \xi(r)), \qquad \text{if } i > r.
\end{aligned}$$

In order to define Θ', we first consider the following auxiliary pseudowords:

- for each $i \prec j \leq j_n$ and each $r \leq i \prec j$ in J, each $\vec{s} \in \zeta(i,j)$ and each $\mu \in M(i,j,\vec{s})$, if $\Lambda(i,j,\vec{s},\mu) = ((\vec{t}_1,\ldots,\vec{t}_k),\mu')$ and $\xi(i) = i_0 \prec i_1 \prec \cdots \prec i_k = \xi(j)$, then we take

$$\begin{aligned}
\Phi'_0(i,j,\vec{s},\mu) &= \left(\prod_{m=1}^{k-1} \mathsf{prod} \circ \Theta'_1(i_{m-1},i_m,\vec{t}_m,0)\right) \cdot \Phi'_1(\xi(j)^-,\xi(j),\vec{t}_k,\mu');\\
\Psi'_0(i,j,\vec{s},\mu) &= \Psi'_1(\xi(j)^-,\xi(j),\vec{t}_k,\mu');
\end{aligned}$$

- for each $\vec{s} \in \zeta(j_n,r)$ and $\mu \in M(j_n,r,\vec{s})$, we set (recall the notation in (5.36))

$$\begin{aligned}
\Phi'_0(j_n,H,\vec{s},\mu) &= \mathsf{prod} \circ \Theta'_1\left(\xi(j_n), b_{h_n+1}, \vec{t}_1^{(\vec{s},\mu)}, 0\right) \cdot \mathsf{prod} \circ \Theta'_1\left(b_{h_n+1}, d_2, \vec{t}_2^{(\vec{s},\mu)}, 0\right)\\
&\quad \cdot \prod_{m=3}^{H} \Theta'_1\left(d_{m-1}, d_m, \vec{t}_m^{(\vec{s},\mu)}, 0\right);\\
\Phi'_0(H,L,\vec{s},\mu) &= \prod_{m=H+1}^{L} \mathsf{prod} \circ \Theta'_1\left(d_{m-1}, d_m, \vec{t}_m^{(\vec{s},\mu)}, 0\right);\\
\Phi'_0(L,r,\vec{s},\mu) &= \left(\prod_{m=L+1}^{N} \mathsf{prod} \circ \Theta'_1\left(d_{m-1}, d_m, \vec{t}_m^{(\vec{s},\mu)}, 0\right)\right) \cdot \Phi'_1\left(d_N, \xi(r), \vec{t}_{N+1}^{(\vec{s},\mu)}, \mu_{\vec{s},\mu}\right);\\
\Psi'_0(L,r,\vec{s},\mu) &= \Psi'_1\left(d_N, \xi(r), \vec{t}_{N+1}^{(\vec{s},\mu)}, \mu_{\vec{s},\mu}\right).
\end{aligned}$$

Now, for $i \prec j$ in J, $(i,j,\vec{s}) \in \mathsf{Dom}(M)$ and $\mu \in M(i,j,\vec{s})$ we define

$$\begin{aligned}
\Theta'(i,j,\vec{s},\mu) &= (\Phi'_0(i,j,\vec{s},\mu), \Psi'_0(i,j,\vec{s},\mu)), \text{ whenever } j \neq r;\\
\Theta'(j_n,r,\vec{s},\mu) &= (\Phi'_0(j_n,H,\vec{s},\mu) \cdot \Phi'_0(H,L,\vec{s},\mu)^{\omega+1} \cdot \Phi'_0(L,r,\vec{s},\mu), \Psi'_0(L,r,\vec{s},\mu)).
\end{aligned}$$

Now, we verify that $\mathcal{M}'$ just defined is a model of $\mathcal{S}$. Let $(i,j,\vec{s}) \in \mathsf{Dom}(M)$ be such that $\vec{s} = (s_1,s_2)$, and $\mu \in M(i,j,\vec{s})$. We first suppose that $j \neq r$. Setting $\Lambda(i,j,\vec{s},\mu) = ((\vec{t}_1,\ldots,\vec{t}_k),\mu')$ and letting $\xi(i) = i_0 \prec i_1 \prec \cdots \prec i_k = \xi(j)$, the following holds modulo DRH

$$\begin{aligned}
\mathsf{prod} \circ \Theta'(i,j,\vec{s},\mu) &= \Phi'_0(i,j,\vec{s},\mu)\Psi'_0(i,j,\vec{s},\mu)\\
&= \left(\prod_{m=1}^{k-1} \mathsf{prod} \circ \Theta'_1(i_{m-1},i_m,\vec{t}_m,0)\right) \cdot \Phi'_1(\xi(j)^-,\xi(j),\vec{t}_k,\mu') \cdot \Psi'_1(\xi(j)^-,\xi(j),\vec{t}_k,\mu')\\
&= \left(\prod_{m=1}^{k-1} w'_1(i_{m-1},i_m)\right) \cdot w'_1(\xi(j)^-,\xi(j)) \quad \text{by Property (M.1) for } (\mathcal{S}_1,\mathcal{M}'_1)
\end{aligned}$$

$$
\begin{aligned}
&= w_1'(\xi(i), \xi(j)) \\
&= w'(i,j) \quad \text{by definition of } \iota',
\end{aligned}
$$

which proves (M.1). We deduce Property (M.2) from the same property for the pair $(\mathcal{S}_1, \mathcal{M}_1')$:

$$
\begin{aligned}
\varphi(\Phi'(i,j,\vec{s},\mu)) &= \varphi\left(\left(\prod_{m=1}^{k-1} \mathsf{prod} \circ \Theta_1'(i_{m-1}, i_m, \vec{t}_m, 0)\right) \cdot \Phi_1'(\xi(j)^-, \xi(j), \vec{t}_k, \mu')\right) \\
&= \left(\prod_{m=1}^{k-1} t_{m,1} t_{m,2}\right) \cdot t_{k,1} \quad \text{writing } \vec{t}_m = (t_{m,1}, t_{m,1}) \text{ and using (M.2) for } (\mathcal{S}_1, \mathcal{M}_1') \\
&= s_1 \quad \text{because } \Lambda \text{ satisfies (R.2.3)} \\
\varphi(\Psi'(i,j,\vec{s},\mu)) &= \varphi(\Psi_1'(\xi(j)^-, \xi(j), \vec{t}_k, \mu')) \overset{\text{(M.2)}}{=} t_{k,2} \\
&= s_2 \quad \text{because } \Lambda \text{ satisfies (R.2.3).}
\end{aligned}
$$

We justify (M.3) by observing that

$$
\begin{aligned}
\vec{c}(w'(i,j)) &= \vec{c}(w_1'(\xi(i), \xi(j))) \quad \text{by definition of } \iota' \text{ and } w' \\
&= \chi_1(\xi(i), \xi(j)) \quad \text{by (M.3) for } (\mathcal{S}_1, \mathcal{M}_1') \\
&= \vec{c}(w[\iota_1(\xi(i)), \iota_1(\xi(j))[) \quad \text{by definition (5.19) of } \chi_1 = \chi_{w_1, \mathcal{C}_1} \\
&= \vec{c}(w(i,j)) \\
&= \chi(i,j) \quad \text{by (M.3) for } (\mathcal{S}, \mathcal{M}).
\end{aligned}
$$

Now, consider the case where $i = j_n$ and $j = r$. The following pseudoidentities hold in DRH:

$$
\begin{aligned}
\mathsf{prod} \circ \Theta'(j_n, r, \vec{s}, \mu) &= \Phi_0'(j_n, H, \vec{s}, \mu) \cdot \Phi_0'(H, L, \vec{s}, \mu)^{\omega+1} \cdot \Phi_0'(L, r, \vec{s}, \mu) \cdot \Psi_0'(L, r, \vec{s}, \mu) \\
&= \mathsf{prod} \circ \Theta_1'\left(\xi(j_n), b_{h_n+1}, \vec{t}_1^{(\vec{s},\mu)}, 0\right) \cdot \mathsf{prod} \circ \Theta_1'\left(b_{h_n+1}, d_2, \vec{t}_2^{(\vec{s},\mu)}, 0\right) \\
&\quad \cdot \prod_{m=3}^{H} \Theta_1'\left(d_{m-1}, d_m, \vec{t}_m^{(\vec{s},\mu)}, 0\right) \cdot \left(\prod_{m=H+1}^{L} \mathsf{prod} \circ \Theta_1'\left(d_{m-1}, d_m, \vec{t}_m^{(\vec{s},\mu)}, 0\right)\right)^{\omega+1} \\
&\quad \cdot \prod_{m=L+1}^{N} \mathsf{prod} \circ \Theta_1'\left(d_{m-1}, d_m, \vec{t}_m^{(\vec{s},\mu)}, 0\right) \\
&\quad \cdot \Phi_1'\left(d_N, \xi(r), \vec{t}_{N+1}^{(\vec{s},\mu)}, \mu_{\vec{s},\mu}\right) \cdot \Psi_1'\left(d_N, \xi(r), \vec{t}_{N+1}^{(\vec{s},\mu)}, \mu_{\vec{s},\mu}\right) \\
&\overset{(*)}{=} w_1'\left(\xi(j_n), b_{h_n+1}\right) \cdot w_1'\left(b_{h_n+1}, d_2\right) \\
&\quad \cdot \prod_{m=3}^{H} w_1'\left(d_{m-1}, d_m\right) \cdot \left(\prod_{m=H+1}^{L} w_1'\left(d_{m-1}, d_m\right)\right)^{\omega+1} \\
&\quad \cdot \prod_{m=L+1}^{N} w_1'\left(d_{m-1}, d_m,\right) \cdot w_1'\left(d_N, \xi(r)\right) \\
&= w_1'(\xi(j_n), d_H) \cdot w_1'(d_H, d_L)^{\omega+1} \cdot w_1'(d_L, \xi(r)) \\
&= w'(j_n, r) \quad \text{by definition of } w',
\end{aligned}
$$

where step $(*)$ follows from $\mathcal{M}'_1$ being a model of $\mathcal{S}_1$, using Property (M.1). Moreover, since

$$\begin{aligned}
\varphi(\Phi'(j_n,r,\vec{s},\mu)) &= \varphi(\Phi'_0(j_n,H,\vec{s},\mu)\cdot\Phi'_0(H,L,\vec{s},\mu)^{\omega+1}\cdot\Phi'_0(L,r,\vec{s},\mu))\\
&= \varphi\left(\mathsf{prod}\circ\Theta'_1\left(\xi(j_n),b_{h_n+1},\vec{t}_1^{(\vec{s},\mu)},0\right)\cdot\mathsf{prod}\circ\Theta'_1\left(b_{h_n+1},d_2,\vec{t}_2^{(\vec{s},\mu)},0\right)\right)\\
&\quad\cdot\varphi\left(\prod_{m=3}^{H}\Theta'_1\left(d_{m-1},d_m,\vec{t}_m^{(\vec{s},\mu)},0\right)\cdot\left(\prod_{m=H+1}^{L}\mathsf{prod}\circ\Theta'_1\left(d_{m-1},d_m,\vec{t}_m^{(\vec{s},\mu)},0\right)\right)^{\omega+1}\right)\\
&\quad\cdot\varphi\left(\prod_{m=L+1}^{N}\mathsf{prod}\circ\Theta'_1\left(d_{m-1},d_m,\vec{t}_m^{(\vec{s},\mu)},0\right)\right)\cdot\varphi\left(\Phi'_1\left(d_N,\xi(r),\vec{t}_{N+1}^{(\vec{s},\mu)},\mu_{\vec{s},\mu}\right)\right)\\
&= t_{1,1}^{(\vec{s},\mu)}t_{1,2}^{(\vec{s},\mu)}\cdot t_{2,1}^{(\vec{s},\mu)}t_{2,2}^{(\vec{s},\mu)}\cdot\prod_{m=3}^{H}t_{m,1}^{(\vec{s},\mu)}t_{m,2}^{(\vec{s},\mu)}\cdot\left(\prod_{m=H+1}^{L}t_{m,1}^{(\vec{s},\mu)}t_{m,2}^{(\vec{s},\mu)}\right)^{\omega+1}\\
&\quad\cdot\prod_{m=L+1}^{N}t_{m,1}^{(\vec{s},\mu)}t_{m,2}^{(\vec{s},\mu)}\cdot t_{N+1,1}^{(\vec{s},\mu)}\quad\text{by (M.2) for }(\mathcal{S}_1,\mathcal{M}'_1)\\
&= s_1^{(\vec{s},\mu)}\cdot\left(s_2^{(\vec{s},\mu)}\right)^{\omega+1}\cdot s_{3,1}^{(\vec{s},\mu)} = s_1\quad\text{by (5.37)},\\
\varphi(\Psi'(j_n,r,\vec{s},\mu')) &= \varphi(\Psi'_0(L,r,\vec{s},\mu)) = \varphi\left(\Psi'_1\left(d_N,\xi(r),\vec{t}_{N+1}^{(\vec{s},\mu)},\mu_{\vec{s},\mu}\right)\right)\\
&= t_{N+1,2}^{(\vec{s},\mu)}\quad\text{by (M.2) for }(\mathcal{S}_1,\mathcal{M}'_1)\\
&= s_2\quad\text{by (5.37)},
\end{aligned}$$

we have Property (M.2). To establish (M.3), we observe that, since S has a content function and thanks to Property (M.2) for both pairs $(\mathcal{S}_1,\mathcal{M}_1)$ and $(\mathcal{S}_1,\mathcal{M}'_1)$, the content of the corresponding segments in $(w_1 = w)$ and in w'_1 does not change. Therefore, the equalities

$$\begin{aligned}
c(w'_1(\xi(j_n),d_L)) &= c(w[\iota(j_n),\delta_L[) = c(w[\beta_0,\beta_1[),\\
c(w'_1(d_H,d_L)) &= c(w[\delta_H,\delta_L[) = c(w[\beta_0,\beta_1[),\\
c(w'_1(d_L,\xi(r))) &= c(w[\delta_L,\iota(r)[) = c(w[\beta_0,\beta_1[)
\end{aligned}\tag{5.38}$$

hold. Thus, we also have

$$\begin{aligned}
\vec{c}(w'(j_n,r)) &= \vec{c}(w'_1(\xi(j_n),d_L)\cdot w'_1(d_H,d_L)^{\omega}\cdot w'_1(d_L,\xi(r))) = c(w[\beta_0,\beta_1[) = \vec{c}(w(j_n,r))\\
&= \chi(j_n,r).
\end{aligned}$$

It remains to verify that (M.4) and (M.5) are satisfied. For Property (M.4), all boundary relations but the ones of the form $(\ell,x_m,j_m,\bar{x}_m)$ are immediate. For those relations, we already observed in (5.38) that $c(w'_1(d_H,d_L)) = c(w'_1(d_L,\xi(r)))$, so that, $w'(j_m,r)$ and $w'_1(\xi(j_m),d_L)\cdot w'_1(d_H,d_L)^{\omega}$ lie in the same $\mathcal{R}$-class modulo DRH. Hence, the pseudovariety DRH satisfies

$$\begin{aligned}
w'(\ell,r) &= w'_1(d_0,d_L)\cdot w'_1(d_H,d_L)^{\omega}w'_1(d_L,\xi(r))\\
&\;\mathcal{R}\; w'_1(d_0,d_L)\cdot w'_1(d_H,d_L)^{\omega}\\
&= w'_1(b_0,b_1)^{h_nL}\cdot w'_1(d_H,d_L)^{\omega}
\end{aligned}$$

$$
\begin{aligned}
&= w'_1(b_{h_m}, b_{h_m+1}) w'_1(b_0, b_1)^{h_n L-1} \cdot w'_1(d_H, d_L)^{\omega} \\
&\overset{(*)}{=} w'_1(\xi(j_m), b_{h_m+1}) \cdot w'_1(b_0, b_1)^{h_n L-1} \cdot w'_1(d_H, d_L)^{\omega} \\
&\mathcal{R}\, w'_1(\xi(j_m), d_L) \cdot w'_1(d_H, d_L)^{\omega} \\
&\mathcal{R}\, w'(j_m, r).
\end{aligned}
$$

The validity of step $(*)$ is justified in view of $\mathcal{S}_1$ having $\mathcal{M}'_1$ as a model. More precisely, it follows from Property (M.4) for the relation $(b_{h_m}, z_m, \xi(j_m), \bar{z}_m)$ and from Property (M.5) for the equation $(b_{h_m} \mid b_{h_m+1}) = (\xi(j_m) \mid b_{h_m+1})$, together with Lemma 2.32. Finally, as the inclusion $\xi_\Lambda(\mathcal{B}_{\mathsf{H}}) \subseteq \mathcal{B}_{1,\mathsf{H}}$ holds, by Remark 5.23 it is enough to show that for all $(i, j, \vec{s}) \in \mathsf{Dom}(M)$ and $\mu \in M(i, j, \vec{s})$, if $\Lambda(i, j, \vec{s}, \mu) = ((\ldots, \vec{t}), \mu')$, then the pseudoidentities

$$
\begin{aligned}
w'(i, j) &= w'_1(\xi(i), \xi(j)); \\
\Psi'(i, j, \vec{s}, \mu) &= \Psi'_1(\xi(j)^-, \xi(j), \vec{t}, \mu')
\end{aligned}
$$

are valid in H. Analyzing the construction of Ψ', the second pseudoidentity becomes clear, since it is actually an equality of pseudowords. The first pseudoidentity $w'(i, j) = w'_1(\xi(i), \xi(j))$ is also immediate, whenever $j \neq r$, after noticing that $w'(i, j) = w'_1(\xi(i), \xi(j))$. It remains to prove that $w'(j_n, r) = w'_1(\xi(j_n), \xi(r))$ modulo H. That is made clear in the next computation modulo H:

$$
\begin{aligned}
w'(j_n, r) &= w'_1(\xi(j_n), d_L) \cdot w'_1(d_H, d_L)^{\omega} \cdot w'_1(d_L, \xi(r)) \\
&= w'_1(\xi(j_n), d_L) \cdot w'_1(d_L, \xi(r)) = w'_1(\xi(j_n), \xi(r)).
\end{aligned}
$$

This completes the proof. □

We have just completed the analysis of all the Cases 1–5. Thus, we proved Theorem 5.26. The announced result follows from Corollary 5.17.

Theorem 5.36. *Let σ be an implicit signature satisfying the condition* (sig) *and such that $\kappa \subseteq \langle \sigma \rangle$. Let* H *be a pseudovariety of groups. Then, the pseudovariety* DRH *is σ-reducible for finite systems of κ-equations if and only if the pseudovariety* H *is σ-reducible for finite systems of κ-equations.* □

Consequently, we have a characterization of the pseudovarieties of the form DRH that are completely κ-reducible in terms of reducibility properties for H.

Theorem 5.37. *The pseudovariety* DRH *is completely κ-reducible if and only if so is* H.

Illustrating the usefulness of these results, we may invoke the work of Almeida and Delgado [13, Theorem 6.2] to derive the complete κ-reducibility of the pseudovariety DRAb. Also, DRH is completely κ-reducible for every locally finite pseudovariety of groups H. On the other hand, by Theorems 5.37 and 2.11, the pseudovariety DRG is not completely κ-reducible.

Chapter 6

Further directions

The following is a summary of the main results of this thesis:

Chapter 3: • The existence of a canonical form for elements in $\Omega_A^\kappa \mathsf{H}$ yields the existence of a canonical form for elements in $\Omega_A^\kappa \mathsf{DRH}$ (Theorem 3.25).

• A pseudovariety of groups H has decidable κ-word problem if and only if the same happens for the pseudovariety DRH (Theorem 3.47 and Proposition 3.48).

Chapter 4: Let σ be an implicit signature such that $\langle\sigma\rangle \neq \langle\{\_\cdot\_\}\rangle$ and let H be a pseudovariety of groups. The following properties hold:

• if H is σ-reducible with respect to systems of pointlike equations, then so is DRH (Theorem 4.1);

• the pseudovariety H is σ-reducible if and only if so is DRH (Theorem 4.13 and Proposition 4.15);

• if $\langle\sigma\rangle$ contains a non-explicit operation η such that $\eta =_{\mathsf{H}} 1$ and H is σ-reducible, then DRH is σ-reducible with respect to systems of idempotent pointlike equations (Theorem 4.22).

Chapter 5: The pseudovariety DRH is completely κ-reducible if and only if the pseudovariety H enjoys the same property (Theorem 5.37).

Thereafter, several natural questions may arise. On the one hand, we may try to extend the results obtained and, on the other hand, we may try to apply different techniques for solving the same kind of problems. We proceed with the presentation of some of them.

6.1 Generalizing the results

1. We know that the pseudovariety G_p (with p prime) is not κ-reducible (Theorem 2.11). Hence, DRG_p is not κ-reducible. In [5], it was exhibited an implicit signature σ that makes G_p a σ-reducible pseudovariety and thus, also DRG_p is σ-reducible. Furthermore, the referred implicit signature σ is such that $\Omega_A^\sigma \mathsf{G}_p = \Omega_A^\kappa \mathsf{G}_p$, so that, G_p has decidable σ-word problem. It is then natural to ask whether DRG_p has decidable σ-word problem as well. A positive answer would imply σ-tameness of DRG_p. Also, one may try to determine whether the required conditions to apply the techniques used to deal with the complete κ-reducibility of DRH admit a generalization in order to deal with the complete σ-reducibility of DRH.

2. A natural generalization of a pseudovariety DRH is the pseudovariety $\mathsf{DO}\cap\overline{\mathsf{H}}$. In the same way that DRH may be considered a non aperiodic version of R, also $\mathsf{DO}\cap\overline{\mathsf{H}}$ may be seen as a non aperiodic version of DA. Since Moura [53] generalized the approach in [25] (the same in which our work was inspired) in order to solve the κ-word problem over DA, it should be possible to solve the κ-word problem over $\mathsf{DO}\cap\overline{\mathsf{H}}$ through a combination of her work with our own. Also, it is expected that a canonical form for the elements of $\Omega_A^\kappa\mathsf{DO}\cap\overline{\mathsf{H}}$ might be obtained from the knowledge of a canonical form for the elements of $\Omega_A^\kappa\mathsf{H}$.

3. Along the same lines of the preceding question, we may try to identify necessary and sufficient conditions on H in order to have the complete κ-reduciblity of $\mathsf{DO}\cap\overline{\mathsf{H}}$, through the generalization of the notions of "system of boundary relations" and respective "model". Also, the results of Section 4.2 are prone to be generalized with the same kind of techniques. Note that the problems corresponding to Sections 4.1 and 4.3 for the pseudovarieties $\mathsf{DO}\cap\overline{\mathsf{H}}$ were solved in [12].

6.2 The same problems, a different approach

4. Kufleitner and Wächter [49] proved that the κ-word problem is decidable over each pseudovariety in the *Trotter-Weil hierarchy*. In particular, that includes the decidability of the κ-word problem over R and over DA. Since their motivation arose from the *quantifier alternation hierarchy inside* FO^2 (two-variable first order logic), their approach was fairly combinatorial. It is then natural to ask whether such approach admits a generalization for proving decidability of the κ-word problem over the pseudovarieties DRH and $\mathsf{DO}\cap\overline{\mathsf{H}}$.

5. Makanin's algorithm [50] appeared as a means of establishing the decidability of the existential theory of equations over free semigroups. Roughly speaking, the idea of the algorithm is derived from the following property: "the existence of a solution of a given system of equations over the free semigroup entails the existence of a solution of that system of a special kind". That is precisely the idea behind the notion of reducibility. Concerning the pseudovarieties of the form DRH, there exist indeed some similarities between proving the existence of certain solutions of equations in the free semigroup and modulo DRH. That fact is witnessed by the possibility of adapting Makanin's algorithm in order to prove complete κ-reducibility of DRH under certain reasonable conditions on H (such adaptation is obtained by extending the constructions considered for proving complete κ-reducibility of R [10], which in turn were directly inspired by Makanin's algorithm). On the other hand, Plandowski and Rytter [56] came up with a different algorithm for deciding existence of solutions of equations over the free semigroup. Also, Jeż [44] proposed an alternative solution for the same problem. Both methods of Plandowski and Rytter, and Jeż can produce a representation of all solutions of a given word equation. Then, it may be interesting to understand how deep is the relationship between handling equations over the free semigroup and over DRH. In this case, that means to figure out whether the algorithms of Plandowski and

Rytter, and Jeż may be adapted in order to derive reducibility and/or hyperdecidability of DRH with respect to certain classes of systems.

Appendix A

Ordinal numbers

Here we recall some general facts about arithmetic of ordinal numbers.

Fact A.1. *Addition of ordinals is associative:*

$$(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma).$$

Fact A.2. *Addition of ordinals is strictly increasing on the right argument and increasing on the left:*

$$\alpha < \beta \implies \gamma+\alpha < \gamma+\beta,$$
$$\alpha < \beta \implies \alpha+\gamma \leq \beta+\gamma.$$

Fact A.3. *Addition of ordinals is left-cancellative:*

$$\alpha+\beta = \alpha+\gamma \implies \beta = \gamma.$$

Given two ordinal numbers α and β, if $\alpha \leq \beta$, then we denote by $\beta - \alpha$ the unique ordinal number γ such that $\alpha+\gamma = \beta$.

Fact A.4. *For ordinal numbers $\alpha \leq \beta$, the following equalities are a straightforward consequence of the definitions:*

$$(\alpha+\beta)-\alpha = \beta = \alpha+(\beta-\alpha).$$

Fact A.5. *Let β, γ and δ be ordinal numbers such that $\beta < \gamma < \delta$. Then, the following inequality holds:*

$$\gamma-\beta < \delta-\beta.$$

Fact A.6. *Let α, β, γ and δ be ordinals such that $\alpha < \beta < \gamma < \delta$. Then, the following equality holds:*

$$(\alpha+(\delta-\beta))-(\alpha+(\gamma-\beta)) = \delta-\gamma.$$

Proof. We first notice that Fact A.5 guarantees that all the involved subtractions are well defined. Then, saying that $(\alpha+(\delta-\beta))-(\alpha+(\gamma-\beta)) = \delta-\gamma$ is the same as saying that

$$(\alpha+(\gamma-\beta))+(\delta-\gamma) = (\alpha+(\delta-\beta)).$$

Associativity and left-cancellative properties of addition of ordinals together imply that this equality is equivalent to $(\gamma - \beta) + (\delta - \gamma) = (\delta - \beta)$. In turn, this means to have $\beta + (\gamma - \beta) + (\delta - \gamma) = \delta$. Now, using Fact A.4, we may compute $\beta + (\gamma - \beta) + (\delta - \gamma) = \gamma + (\delta - \gamma) = \delta$. $\square$

Appendix B

Implementation of the solution for the word problem in DRG

While in Chapter 3 we made a rather informal description of the steps to be taken in order to decide whether two given κ-words have the same value over DRG, in here we present the implementation in Python of the complete solution. The reader is referred to Chapter 3 for unexplained notation.

Let us describe the scenario considered in Sections B.1–B.3. We assume that the alphabet A is the set $\{1,2,\ldots,n\}$. A well-parenthesized word is represented by a list of pairs $[a,i]$, where a belongs to $\{0\}\cup A\cup\{n+1\}\cup\{[,]\}$ and i to $\mathbb{N}$. The pair $[a,i]$ represents a_i if $a\in A$, and it represents $[^i$ or $]^i$ if $a=[$ or $a=]$, respectively. Given a κ-term w, we use $n+1$ to denote the distinguished symbol # in the well-parenthesized word $\overline{w}$. We further assume that the κ-terms we receive as input are already given by a well-parenthesized word. In what follows, we use w to refer to such an arbitrary well-parenthesized word. Should no confusion arise from the context, we may abuse notation and still use w to represent one of the κ-terms that the well-parenthesized word w defines. For instance, we may write $\mathcal{G}(w)$ to mean the DRG-graph of such a κ-term. All our algorithms take the size n of the alphabet as input.

On the other hand, in Section B.4 we drop out all those technical assumptions and exhibit an algorithm taking strings in the input.

B.1 Preliminary computations

We start with Algorithm B.1 returning a table (that is, a list of lists) T of dimension $(|w|+1)\times(|A|+1)$, so that the entry $T[i][a]$ contains the word $\mathsf{first}(w(i,a))$ (also given by a list).

Once we possess all the information to construct $\mathcal{G}_{\mathsf{R}}(w)$, we should be able to determine the regular parts of each factor of the form $\mathsf{om}(w(i,a))$. Algorithm B.2 returns I if $\mathsf{reg}(\mathsf{om}(w(i,a)))=I$, and k if $\mathsf{reg}(\mathsf{om}(w(i,a)))=\mathsf{om}(w(k,a))$. It receives the result `FIRSTw = first(w, n)` of Algorithm B.1 as input. On the other hand, Algorithm B.3 computes the well-parenthesized word $w(i,a)$.

By now, we have all the theoretical data for computing $\mathcal{G}(w)$. Next step is to prepare the tools to solve the word problem over G when comparing two DRG-graphs. We do that in next section.

B.2 The word problem over G

We already know that all the κ-words we have to compare over G are of the form $\mathsf{om}(w(i,a))$. In turn, these factors are well-parenthesized words over B_2. We create a class `tree` of ternary trees, for representing such words and do a routine `wpwTOtree` to convert a well-parenthesized word w over B_2 into a tree (Algorithm B.4). Each attribute of a tree is given by a list $[a,s]$ (with $a \in A$ and $s \in \{-1,1\}$) that represents the letter a^s of $A \cup A^{-1}$. If we wish to recover a well-parenthesized word from a tree outputted by Algorithm B.4, then we shall get a well-parenthesized word over B_1. Thus, this step may be seen as the precomputation of the expansion of w.

A (binary) tree representing the linearization of a κ-term described by another tree is computed in Algorithm B.5. Finally, we may use Algorithm B.6 to obtain the word over $A \cup A^{-1}$ represented by a given binary tree and then, calculate its canonical form in the free group FG_A with Algorithm B.7.

B.3 Constructing DRG-graphs

Now, we are ready to construct DRG-graphs. We represent them by lists of `state`'s G. Each `state` has 5 attributes: two labels `l` and `lG` given by $a \in A$ and $k \in \mathbb{N} \cup \{I\}$, respectively, if $\lambda(\texttt{state}) = a$ and $\lambda_{\mathsf{G}}(\texttt{state}) = \rho_{\mathsf{G}}(w(k,a))$ (or $\lambda_{\mathsf{G}}(\texttt{state}) = I$ if $k = I$); two transitions `zero` and `one` representing transitions `state`.0 and `state`.1, respectively; and an optional `marker`. This is done in Algorithm B.8. The entry $G[i][a]$ encodes the state $\mathsf{q}(i,a)$ and each of the transitions `zero` and `one` is given by a list of the form $[j,b]$. We further leave the entry $G[i][a]$ empty if either the state is not reachable or is terminal. The routine `DRGgraph` owes its name to the fact that it constructs the DRG-graph of the κ-term corresponding to a given well-parenthesized word of the form $\overline{w}$. We point out that, although we are solving the word problem over DRG, since we are characterizing the nonempty labels $\lambda_{\mathsf{G}}(\mathsf{q})$ by an integer k such that $\mathsf{reg}(w(i,a)) = w(k,a)$ (for (i,a) correctly chosen depending on q), this routine actually returns the DRH-graph of w, independently of the pseudovariety H.

Lastly, we are able to decide whether two κ-words $\mathsf{om}(u)$ and $\mathsf{om}(v)$ are equal modulo DRG (for u and v well-parenthesized words over B_1). Algorithm B.9 arranges all the information and returns the logical value of $\mathsf{om}(u) =_{\mathsf{DGR}} \mathsf{om}(v)$, when called with the tuple

```
(DRHgraph(u,n,first(u)),[0,n+1],DRHgraph(v,n,first(v)),[0,n+1], n).
```

B.4 The solution

This last section is dedicated to an user friendly presentation of a routine to check whether two κ-words coincide over DRG. Algorithm B.11 is prepared to receive any pair of strings representing a κ-term, agreeing that any $(\omega-1)$-power is represented by `^w-1`. Unlike in the algorithms of previous sections, it is not necessary neither to assume that the alphabet is of the form $A = \{1, \cdots, n\}$, nor to input its size. The reason is that we use Algorithm B.10 to rename all the letters appearing in the given strings and to compute the corresponding well-parenthesized words. Thus, if we wish to decide, for instance, if $(ab^{\omega-1}a^{\omega-1})^{\omega-1}$ and $ab^{\omega-1}(aa^{\omega-1}b^{\omega-1})^{\omega-1}a$ are equal modulo DRG, then we just need to call

```
TESTmoduloDRG(‘‘(ab^w-1a^w-1)^w-1’’, ‘‘ab^w-1(aa^w-1b^w-1)^w-1a’’).
```

Algorithm B.1

```
def first(w,n):
    S = [];
    x = w[0:-1];
    wait = [[] for i in range (n+1)];
    res = [[] for i in range (len(x))];
    for i in range (len(x)):
        if x[i][0] == "[":
            S += [i];
            for a in range (n+1):
                wait[a] = [i] + wait[a];
        elif x[i][0] == "]":
            matchingOpen = S.pop();
            for a in range (n+1):
                if wait[a] != [] and wait[a][0] == matchingOpen:
                    wait[a] += [wait[a][0]];
                    del wait[a][0];
            line = res[matchingOpen];
            for k in range (len(line)):
                row = wait[line[k][0]];
                wait[line[k][0]] = [];
                for l in range (len(row)):
                    res[row[l]] += [line[k]];
            for a in range (n+1):
                wait[a] += [i];
        else:
            row = wait[x[i][0]];
            for j in range (len(row)):
                res[row[j]] += [x[i]];
            wait[x[i][0]] = [];
            for a in range (n+1):
                wait[a] += [i];
    for i in range (len(x)-1):
        if x[len(x) - i - 1][0] in ["[","]"]:
            del res[len(x) - i - 1]
    L = [[[] for i in range (n+2)] for j in range (len(res))];
    for i in range (len(L)):
        for j in range (len(res[i])):
            L[i][res[i][j][0]] = res[i][0:j]
        L[i][n+1] = res[i];
    return L
```

Algorithm B.2

```
def reg(w,i,a,n,FIRSTw):
    L = [];
    j = i;
    while j not in L and FIRSTw[j][a] != []:
        j = FIRSTw[j][a][-1][1];
        L += [j];
    if FIRSTw[j][a] == []:
        return "I"
    else:
        size_content = len(FIRSTw[j][a]);
        k = i;
        while len(FIRSTw[j][a]) != size_content:
            k = FIRSTw[k][a][-1][1];
        return k
```

Algorithm B.3

```
def factor(w,i,a):
    m = 0;
    L = [];
    while w[m][1] != i or w[m][0] in ["[","]"]:
        if w[m][0] == "[":
            L += [m];
        elif w[m][0] == "]":
            del L[-1];
        m += 1;
    x = [];
    n = 0;
    for j in range (m+1,len(w)):
        if w[j][0] == "[":
            x += [w[j]];
            n += 1;
        elif w[j][0] == "]" and n > 0:
            x += [w[j]];
            n -= 1;
        elif w[j][0] != "]":
            x += [w[j]];
        else:
            x += [["[",w[L[-1]][1]-1]] + w[L[-1]+1:j] +
                [["]",w[L[-1]][1]-1]];
            del L[-1];
    m = 0;
    L = [];
    y = [];
    while x[m][0] != a:
        y += [x[m]];
        if x[m][0] == "[":
            L += [m];
        elif x[m][0] == "]":
            del L[-1];
        m += 1;
    for i in range (len(L)):
        del y[L[i]-i];
    return y
```

Algorithm B.4

```
class tree:
    def __init__(self, left, mid, right):
        self.left = left
        self.mid = mid
        self.right = right
def wpwTOtree(w):
    L = [];
    T = 0;
    for i in range (len(w)):
        if w[i][0]== "[":
            L += [T];
            T = 0;
        elif w[i] == ["]",-1]:
            T = tree(L[-1],None,tree(None,T,None));
            del L[-1]
        elif w[i] == ["]",-2]:
            T0 = tree(None,T,None);
            T = tree(L[-1],None,tree(T0,None,T0));
            del L[-1]
        elif T != 0:
            T = tree(T,None,[w[i][0],1]);
        else:
            T = [w[i][0],1];
    return T
```

Algorithm B.5

```
def lin(T,sign):
    if not isinstance (T,tree):
        return [T[0],sign*T[1]]
    elif T.left == None:
        return lin(T.mid,-sign)
    elif sign == -1:
        return tree(lin(T.right ,sign),None,lin(T.left ,sign))
    else:
        return tree(lin(T.left ,sign),None,lin(T.right ,sign))
```

Algorithm B.6

```
def treeTOword(T):
    if not isinstance(T,tree):
        return [T]
    else:
        return treeTOword(T.left) + treeTOword(T.right)
```

Algorithm B.7

```
def cf(word):
    i = 0;
    while i < len(word)-1:
        if word[i] == [word[i+1][0],-word[i+1][1]]:
            word = word[0:i] + word[i+2:len(u)];
            i = max(0,i-1);
        else:
            i = i+1;
    return word[1:-1]
```

Algorithm B.8

```
class state:
    def __init__(self, l, lG, zero, one, marker = None):
        self.l = l
        self.lG = lG
        self.zero = zero
        self.one = one
        self.marker = marker
def areEqual(u,s1,v,s2):
    if s1 == []:
        return s2 == []
    elif s1.lG == "I":
        return s1.l == s2.l and s2.lG == "I"
    else:
        lGu = [[0,0]] + factor(u,s1.lG,s1.l);
        u0 = cf(treeTOword(lin(wpwTOtree(lGu),1)));
        lGv = [[0,0]] + factor(v,s2.lG,s2.l);
        v0 = cf(treeTOword(lin(wpwTOtree(lGv),1)));
        return s1.l == s2.l and u0 == v0
def DRGgraph(w,n,FIRSTw):
    G = [[[] for a in range (n+2)] for i in range (len(FIRSTw))];
    for i in range (len(FIRSTw)):
        for a in range (1,n+2):
            if FIRSTw[i][a] != []:
                PM = FIRSTw[i][a][-1];
                lG = reg(w,i,PM[0],n,FIRSTw)
                G[i][a]=state(PM[0],lG,[i,PM[0]],[PM[1],a])
    return G
```

Algorithm B.9

```
def compare(u,Gu,qu,v,Gv,qv,n):
    if Gu[qu[0]][qu[1]] == []:
        return Gv[qv[0]][qv[1]] == []
    elif Gu[qu[0]][qu[1]].marker == None or
       Gv[qv[0]][qv[1]].marker == None:
        Gu[qu[0]][qu[1]].marker = 1;
        Gv[qv[0]][qv[1]].marker = 1;
        if areEqual(u,Gu[qu[0]][qu[1]],v,Gv[qv[0]][qv[1]]):
            qu0 = Gu[qu[0]][qu[1]].zero;
            qu1 = Gu[qu[0]][qu[1]].one;
            qv0 = Gv[qv[0]][qv[1]].zero;
            qv1 = Gv[qv[0]][qv[1]].one;
            return compare(u,Gu,qu0,v,Gv,qv0,n) and
                compare(u,Gu,qu1,v,Gv,qv1,n)
        else:
            return False
    else:
        return areEqual(u,Gu[qu[0]][qu[1]],v,Gv[qv[0]][qv[1]])
```

Algorithm B.10

```
def rename(string, A):
    m = 0;
    L = [];
    i = 0;
    while i < len(string):
        if string[i] == "(":
            L.append(["[",-1]);
            i += 1;
        elif string[i] == ")":
            L.append(["]",-1]);
            i += 5;
        elif string[i] == "^":
            L.insert(-1,["[",-1]);
            L.append(["]",-1]);
            i += 4;
        else:
            m += 1;
            if string[i] in A:
                a = A.index(string[i]) + 1;
            else:
                A.append(string[i]);
                a = len(A);
            L.append([a,m]);
            i += 1;
    return [[[0,0]] + L + [[len(A)+1,m+1]], A]
```

Algorithm B.11

```
def TESTmoduloDRG(u,v):
    uu = rename(u,[]);
    vv = rename(v,uu[1]);
    n = len(vv[1]);
    WPWu = uu[0][:-1]+[[n+1,uu[0][-1][1]]];
    WPWv = vv[0];
    FIRSTu = first(WPWu,n);
    FIRSTv = first(WPWv,n);
    Gu = DRGgraph(WPWu,n,FIRSTu);
    Gv = DRGgraph(WPWv,n,FIRSTv);
    UmoduloG = cf(treeTOword(lin(wpwTOtree(WPWu),1)));
    VmoduloG = cf(treeTOword(lin(wpwTOtree(WPWv),1)));
    return compare(WPWu,Gu, [0,n+1],WPWv,Gv,[0,n+1], n) and
        UmoduloG == VmoduloG
```

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Notation