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MSc

Frankl Conjecture

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All modifications determined by the Jury, and only those, were made.

The President of the Jury,

Porto, ______/______/_________

MSc

In memory of my Grandad

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To my parents for everything.

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Abstract

This dissertation deals with a conjecture proposed in 1979 by Péter Frankl related to union-closed families of sets. It has been widely studied by several mathematicians from all around the world. There are many papers on the topic and several websites dedicated to its study. Despite the fact that the conjecture regards finite union-closed families of sets, which appear to be very simple objects, very little is known about them. We try to present in detail the main tools people have been using too approach the problem and unveil a little bit of the mystery behind these families.

Keywords: Union-closed families of sets, Frankl Conjecture, Union-closed sets conjecture, Up-compression, Lattice, Dual Families.

Resumo

Nesta dissertação é apresentada uma conjetura proposta em 1979 pelo matemático húngaro Péter Frankl relacionada com famílias de conjuntos fechadas para a reunião. Nos recentes anos, esta tem sido amplamente estudada por matemáticos de todo o mundo. Existem cerca de 50 artigos acerca da mesma e vários websites dedicados ao seu estudo. Apesar dos objetos do problema serem famílias finitas de conjuntos fechadas para a reunião, que são aparentemente simples, pouco se sabe ainda sobre os mesmos. Tentamos, neste trabalho, apresentar em detalhe as principais ferramentas usadas na abordagem do problema e desvendar um pouco o mistério escondido por detrás destas famílias.

Palavras-chave: Famílias de conjuntos fechadas para a reunião, Conjetura de Frankl, Conjetura dos conjuntos fechados para a reunião, Compressão, Reticulados, Famílias Duais.

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Introduction

In 1979, Peter Frankl formulated one of the most famous conjecture in extremal combinatorics, known as *Frankl Conjecture* or *Union-closed Sets Conjecture*. It has piqued the curiosity of some of the most influential combinatorialists and we hope that, after reading this dissertation, yours will be piqued as well! The main attraction of Frankl Conjecture is definitely the fact that it is very simple to state. It *only* claims that if we are given a finite family of sets $\mathcal{F} \subseteq \mathcal{P}(U)$, where U is a set, such that $S \in \mathcal{F} \wedge T \in \mathcal{F} \Rightarrow S \cup T \in \mathcal{F}$, then there is an element $x \in U$ such that x belongs to at least half the sets in $\mathcal{F}$.
In this dissertation, we present the main results and techniques used to study the conjecture. The goal is to familiarize the reader with the problem and show how the main partial results toward a proof of the conjecture are obtained. We also study in great detail the structure of some particular kinds of families.

In Chapter 1 we present the problem as well as some basic definitions about families of sets. We show how we can reduce the problem so that, instead of studying union-closed families in general, we only work with a special subclass of families, called *separating*. Some results about separating families are obtained. Also, a natural generalization of the problem is presented. We also show generalizations of the problem and some approaches that seem viable but are not.

In Chapter 2 it is shown how Frankl conjecture is equivalent to a problem in lattice theory, and we show the result for the class of lower semimodular lattices.

In Chapter 3 we study the properties that a hypothetical counterexample to the conjecture must have, specially the *smallest* counterexample. The results presented in this chapter are based essentially on the work of Giovanni Lo Faro, and allow us to have some bounds and some structural properties of an eventual counterexample.

In Chapter 4 we study in great detail a new formulation of the conjecture proposed by Salzborn, and mainly studied by Piotr Wójcik, and the different ways to construct dual families, which are the families dealt with in the new formulation. A proof of the equivalence between conjectures are given through the chapter, and some results about dual families are obtained using these techniques.

In Chapter 5 we show how compression techniques can be applied in the context of union-closed families of sets. The idea of these techniques is to transform the object of study, which is in our case union-closed families, into another one of a special type (in our case upward-closed families) and then study this new object to obtain results about the initial one. Also, some results about the upward-closed family that is originated by up-compressing the original family are shown. The concept of down-compression is also defined and shown to be equivalent to the concept of up-compression.

In Chapter 6 we present the concept of *Frankl Complete families* introduced by Sarvate and Renaud and later formalized by Poonen [18], and the characterization of such families. Some examples of those families are exhibited.

Notation

- $\mathcal{F}$:= family of sets
- $[n] = \{1, 2, \dots, n\}$
- $|S| := \#S$, where S is a set
- $\mathcal{P}(S)$ denotes the set of all subsets of S
- X^c = denotes the complement of X
- $\mathcal{F}^c = \{X^c \mid X \in \mathcal{F}\}$
- $U(\mathcal{F}) := \bigcup_{X \in \mathcal{F}} X$
- $\bigcap \mathcal{F} := \bigcap_{X \in \mathcal{F}} X$
- $\mathcal{F}_{\subseteq X} := \{A \in \mathcal{F} : A \subseteq X\}$
- $\mathcal{F}_{\nsubseteq X} := \{A \in \mathcal{F} : A \nsubseteq X\}$
- $\mathcal{F}_{\supseteq X} := \{A \in \mathcal{F} : A \supseteq X\}$
- $\mathcal{F}_{\nsupseteq X} := \{A \in \mathcal{F} : A \nsupseteq X\}$
- $\mathcal{F}_X := \{A \in \mathcal{F} : A \cap X \neq \varnothing\}$. In particular, we represent $\mathcal{F}_{\{a\}}$ by $\mathcal{F}_a$
- $\mathcal{F}_{\bar{a}} := \{X \in \mathcal{F} : a \notin X\}$
- $J(\mathcal{F})$ denotes the subfamily of $\cup$-irreducible sets in the family $\mathcal{F}$
- $\mathcal{F} \ominus S = \{X \backslash S : X \in \mathcal{F}\}$
- $\dot{\cup}$ denotes the disjoint union of sets
- When L is a lattice, L^* denotes the dual lattice of L

- $[a,b] := \{c \in P : a \leq c \leq b\}$, where P is a poset and $a, b \in P$
- $[a) := \{c \in P : a \leq c\}$
- $(a] := \{c \in P : c \leq a\}$
- $n_0 := |\mathcal{F}|$ where $\mathcal{F}$ is the minimal counter-example to the conjecture in terms of member-sets
- $q_0 := |U(\mathcal{F})|$ where $\mathcal{F}$ is the minimal counter-example to the conjecture with n_0 member-sets in terms of universe elements
- $\mathfrak{F}$ denotes the set of all counterexamples with n_0 sets and q_0 elements in its universe
- $m_{\mathcal{F}} := \min\{|\mathcal{F}_x| : x \in U(\mathcal{F})\}$
- $M := \max\{m_{\mathcal{F}} : \mathcal{F} \in \mathfrak{F}\}$
- $\mathfrak{F}^M := \{\mathcal{F} \in \mathfrak{F} : m_{\mathcal{F}} = M\}$
- $\mathfrak{F}^M_r := \{\mathcal{F} \in \mathfrak{F}^M \mid |\{x \in U(\mathcal{F}) : |\mathcal{F}_x| = M\}| = r\}$
- $r_0 := \min\{r \in \mathbb{N} : \mathfrak{F}^M_r \neq \varnothing\}$
- $\mathfrak{G} := \mathfrak{F}^M_{r_0}$
- $\partial_a \mathcal{F} := \{X \backslash \{a\} : X \in \mathcal{F}\}$
- $\mathcal{P}_a := \{X \in \mathcal{F}_{\bar{a}} : X \cup \{a\} \in \mathcal{F}\}$
- $\widehat{\mathcal{F}} := \{a \in U(\mathcal{F}) : |\mathcal{F}_a| \geq \frac{|\mathcal{F}|}{2}\}$
- If S and T are subsets of $[n]$, we put $[S,T] = \{X \subseteq [n] \mid S \subseteq X \subseteq T\}$ (which is empty when $S \nsubseteq T$)
- $\log$ will always denote $\log_2$
- $\mathfrak{S}_S$ is the group of permutations of elements in S
- We put $A \sqsubset B$ when there exists $i \in [n]$ such that $A \cup \{i\} = B$
- $E(\mathcal{F}) := \{(A,B) \in \mathcal{F}^2 : A \sqsubset B\}$
- $EB(\mathcal{F}) := \{(A,B) \in E(\mathcal{P}[n]) : A \notin \mathcal{F}, B \in \mathcal{F}\}$
- If $\mathcal{F}$ and $\mathcal{G}$ are two families of sets, we put $\mathcal{F} \uplus \mathcal{G} = \{S \cup T : S \in \mathcal{F}, T \in \mathcal{G}\}$

- $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^n$
- sgn is the sign function in $\mathbb{R}$ and is defined as usual: $sgn(x) = \frac{|x|}{x}$ if $x \neq 0$ and $sgn(0) = 0$
- $n_j^{\mathcal{F}}$ denotes the number of sets of cardinality j in $\mathcal{F}$

Chapter 1

The Conjecture and Basic Results

In this chapter, we introduce a popular conjecture in extremal combinatorics known as union-closed sets conjecture or Frankl conjecture. We will also present some basic definitions and results to help the reader to get familiar with the problem. In the last section, we present some statements that might look true, while being false and the reason why that is the case.

1.1 The Conjecture

The origin of the Frankl conjecture is somewhat mysterious. So much so, that in an article, Peter Wrinkler wrote that *"The 'union-closed sets conjecture' is well-known indeed, except from (1) its origin and (2) its answer!"*. Most authors attribute its formulation to Peter Frankl, while others refer to it as a "folklore conjecture" [12]. It is a fact that Frankl discovered the conjecture, and that popularized it, and so we will refer to it as Frankl conjecture, instead of the union-closed sets conjecture.

Although its origin is uncertain, its popularity is a certainty, which might come from the fact that the formulation of the conjecture is indeed very simple. So, in 1979 Frankl introduced the following problem:

Conjecture 1. *Let $\mathcal{F} \subseteq \mathcal{P}(U)$ be a finite family of sets, such that $S \in \mathcal{F} \wedge T \in \mathcal{F} \Rightarrow S \cup T \in \mathcal{F}$. Then, there is an element $x \in U$ such that x belongs to at least half the sets in $\mathcal{F}$.*

Despite having such a simple statement, and being studied by so many mathematicians throughout the past 35 years, very little is known about this problem. In the next section, we will present some basic definitions and results in order to study the conjecture in detail.

1.2 Basic Results

Let $\mathcal{F}$ be a finite family of sets. We say that $\mathcal{F}$ is *union-closed* if, for any $S, T \in \mathcal{F}$, we have $S \cup T \in \mathcal{F}$. We define the *universe* of a family as the union of all its member-sets and denote it by $U(\mathcal{F})$, so $U(\mathcal{F}) \in \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{P}(U(\mathcal{F}))$. Also, for $X \subseteq U(\mathcal{F})$ we define $\mathcal{F}_X = \{S \in \mathcal{F} : S \cap X \neq \varnothing\}$ and, for every $a \in U(\mathcal{F})$ denote $\mathcal{F}_{\{a\}}$ by $\mathcal{F}_a$. Similarly, we represent $\{S \in \mathcal{F} \mid a \notin \mathcal{F}\}$ by $\mathcal{F}_{\bar{a}}$.

We can now rewrite the problem using this terminology as follows.

Conjecture 1. *If a finite family of sets $\mathcal{F}$ is union-closed, then there is an element of its universe that belongs to at least half the sets of the family, i.e.,*

$$\exists a \in U(\mathcal{F}) : |\mathcal{F}_a| \geq \frac{|\mathcal{F}|}{2}.$$

Example 1. *Let $\mathcal{F} = \{\{1\}, \{2,3\}, \{1,2,4\}, \{1,2,3\}, \{1,2,3,4\}\}$. We have that $\mathcal{F}$ is union-closed, with $U(\mathcal{F}) = \{1,2,3,4\}$, and we have:*

- $\mathcal{F}_1 = \{\{1,2,4\}, \{1\}, \{1,2,3\}, \{1,2,3,4\}\}$;
- $\mathcal{F}_2 = \{\{1,2,4\}, \{2,3\}, \{1,2,3\}, \{1,2,3,4\}\}$;
- $\mathcal{F}_3 = \{\{2,3\}, \{1,2,3\}, \{1,2,3,4\}\}$;
- $\mathcal{F}_4 = \{\{1,2,4\}, \{1,2,3,4\}\}$

and so the conjecture is verified for this family, when we take $a = 1, 2$ or 3.

One might try to generalize the conjecture and allow the family to be infinite. However, in that case the conjecture is false (when our definition of half is reasonable) since we can take the family $\mathcal{F} = \{S_1, S_2, S_3 \ldots\}$, where $S_i = \{i, i+1, i+2, \ldots\}$ and all elements only belong to finitely many sets.

Despite being studied by many mathematicians in the past few years, there are not many strong results on this problem, and still very little is known about union-closed families. However, the conjecture is known to hold for some kinds of families such as:

- $\mathcal{F}$ such that $|U(\mathcal{F})| \leq 12$ (see [26]) or $|\mathcal{F}| \leq 50$;
- $\mathcal{F}$ such that $|\mathcal{F}|$ is *large* when compared to $|U(\mathcal{F})|$ in the sense that $|\mathcal{F}| \geq \frac{2}{3}2^{|U(\mathcal{F})|}$ (see [12]);
- $\mathcal{F}$ such that $|\mathcal{F}|$ is *small* when compared to $|U(\mathcal{F})|$ in the sense that $|\mathcal{F}| \leq 2|U(\mathcal{F})|$ (see [8]). For this result to hold, we also demand the families to be *separating*, which, as we will see towards this chapter, is not a very strong condition;
- $\mathcal{F}$ has a very particular structure, such as having a singleton or a set with only two elements.

We will now prove the last result.

Lemma 1.2.1. *If, in $\mathcal{F}$, there is a set with only one element, then that element belongs to at least half of the sets in $\mathcal{F}$.*

Proof. Suppose that $\{a\} \in \mathcal{F}$. Clearly, $\mathcal{F} = \mathcal{F}_a \dot{\cup} \mathcal{F}_{\bar{a}}$, and the map $\mathcal{F}_{\bar{a}} \to \mathcal{F}_a$ given by $B \mapsto B \cup \{a\}$ is injective. It follows that $|\mathcal{F}_a| \geq |\mathcal{F}_{\bar{a}}|$, and thus $|\mathcal{F}_a| \geq \frac{|\mathcal{F}|}{2}$. ∎

Let W be the set of words w in the alphabet $\Sigma = U(\mathcal{F}) \dot{\cup} \{\bar{a} : a \in U(\mathcal{F})\}$ such that, for each $a \in U(\mathcal{F})$, at most one of the symbols $a, \bar{a}$ appears in w. We will abuse notation by indistinctly dealing with an element of W as a word and as a subset of Σ. For every $w \in W$, set

$$\mathcal{F}_w = \bigcap_{a \in w} \mathcal{F}_a,$$

and

$$f_w = |\mathcal{F}_w|.$$

Also, for $a \in U(\mathcal{F})$, set $|\bar{a}| = |a| = a$, $\bar{\bar{a}} = a$, and, for $w = \sigma_1\sigma_2\sigma_3\cdots\sigma_k \in W$, put $\bar{w} = \bar{\sigma}_1\bar{\sigma}_2\bar{\sigma}_3\cdots\bar{\sigma}_k$ and $|w| = |\sigma_1||\sigma_2||\sigma_3|\cdots|\sigma_k|$.

Example 2. *If $w = a\bar{b}c$, then $\mathcal{F}_w = \{S \in \mathcal{F} : a, c \in S \wedge b \notin S\}$, $|w| = abc$, and $\bar{w} = \bar{a}b\bar{c}$.*

Remark 1. *For all $a \in U(\mathcal{F}), w \in W$ such that neither a nor $\bar{a}$ appear in w, one has*

- $\mathcal{F}_{aw} \dot{\cup} \mathcal{F}_{\bar{a}w} = \mathcal{F}_w$.

Proposition 1.2.2. *For each $w \in W$ such that $|w| \in \mathcal{F}$, one has $f_w \le f_{|w|}$.*

Proof. The map

$$\begin{array}{rcl} \mathcal{F}_w & \to & \mathcal{F}_{|w|} \\ S & \mapsto & S \cup \{a \in U(\mathcal{F}) : \bar{a} \in w\} \end{array}$$

is injective, since, for $S, T \in \mathcal{F}_w$, one has that $\bar{a} \in w \Rightarrow a \notin S \cup T$, and therefore,

$$S \cup \{a \in U(\mathcal{F}) : \bar{a} \in w\} = T \cup \{a \in U(\mathcal{F}) : \bar{a} \in w\} \Leftrightarrow S = T.$$

■

Corollary 1.2.3. *Let $\mathcal{F}$ be an union-closed family of sets. If $\{a, b\} \in \mathcal{F}$, then $f_a \ge \frac{|\mathcal{F}|}{2} \vee f_b \ge \frac{|\mathcal{F}|}{2}$.*

Proof. It follows from the previous result that $f_{\bar{a}\bar{b}} \le f_{ab}$. But then $f_{\bar{a}} = f_{\bar{a}\bar{b}} + f_{\bar{a}b} \le f_{ab} + f_{\bar{a}b} = f_b$, i.e. $f_{\bar{a}} \le f_b$.

■

We will now present some results that allow us to restrict the problem to smaller classes of families.

Proposition 1.2.4. *It suffices to prove the conjecture for union-closed families $\mathcal{F}$ such that $\varnothing \in \mathcal{F}$.*

Proof. Consider an union-closed family $\mathcal{F}$ such that $\varnothing \notin \mathcal{F}$. Now, take the family $\mathcal{F}' = \mathcal{F} \cup \{\varnothing\}$. If the conjecture holds for $\mathcal{F}'$, there exists $a \in U(\mathcal{F}') = U(\mathcal{F})$ such that $|\mathcal{F}'_a| \ge \frac{|\mathcal{F}'|}{2}$. In that case, $|\mathcal{F}_a| = |\mathcal{F}'_a| \ge \frac{|\mathcal{F}'|}{2} = \frac{|\mathcal{F}|+1}{2} \ge \frac{|\mathcal{F}|}{2}$ and so the conjecture also holds for $\mathcal{F}$. ■

Definitions 1.2.5. *A family $\mathcal{F}$ is said to be* separating *if for every two elements of its universe a and b, there is a set $X \in \mathcal{F}$ such that $|(X \cap \{a, b\})| = 1$, i.e., $\mathcal{F}_a \ne \mathcal{F}_b$. Similarly, we call a family* totally separating *if for every pair of elements of its universe a and b, there are sets X and Y in $\mathcal{F}$ such that $a \in X \backslash Y$, and $b \in Y \backslash X$, i.e., $\mathcal{F}_a$ and $\mathcal{F}_b$ are not $\subseteq$-comparable.*

Example 3. *Let $\mathcal{F} = \{\{1\}, \{2, 3\}, \{1, 2, 4\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$. As we checked in Example 1, for all $a, b \in U(\mathcal{F})$, one has $a \ne b \Rightarrow \mathcal{F}_a \ne \mathcal{F}_b$. Thus $\mathcal{F}$ is separating.*
Now let $\mathcal{G} = \{\{3\}, \{1, 2, 4\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$.Then $\mathcal{G}$ is not separating because $\mathcal{G}_1 = \mathcal{G}_2 = \{\{1, 2, 4\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$.

Proposition 1.2.6. *It suffices to prove the conjecture for separating families.*

Proof. Let $\mathcal{F}$ be a non separating union-closed family of sets and define an equivalence relation $\sim$ in $U(\mathcal{F})$ by $a \sim b$ iff $\mathcal{F}_a = \mathcal{F}_b$. Then, we replace the occurrence of a class in each set by a chosen representative element. The new family $\mathcal{F}'$ is obviously still union-closed, and $|\mathcal{F}| = |\mathcal{F}'|$. Now, if the conjecture holds for $\mathcal{F}'$, it also holds for $\mathcal{F}$ (for the same element). Obviously with this process the family $\mathcal{F}'$ we obtain is separating, and so the result follows. ∎

We will now present some results that help us understand a little bit more about the structure of separating families. When there is no doubt about which family we are referring to, we sometimes denote $U(\mathcal{F}_{\bar{a}})$ by U_a.

Proposition 1.2.7. *A family $\mathcal{F}$ is separating if and only if $U_a \neq U_b,\ \forall a, b \in U(\mathcal{F}),$ with $a \neq b.$*

Proof. Suppose $\mathcal{F}$ is separating and let $a, b \in U(\mathcal{F}),$ with $a \neq b$. Then, assume w.l.o.g. that we have a set $S \in \mathcal{F}$ such that $a \in S$ and $b \notin S$. Then, $a \in S \subseteq U_b$ and $a \notin U_a$. Hence, $U_a \neq U_b$.

Conversely, if $\mathcal{F}$ is not separating, there are $a, b \in U(\mathcal{F}),\ a \neq b$, such that $\mathcal{F}_a = \mathcal{F}_b$ and so $\mathcal{F}_{\bar{a}} = \mathcal{F}_{\bar{b}}$ and therefore $U_a = U_b$. ∎

Corollary 1.2.8. *A separating union-closed family of sets with universe $[n]$ has at least n sets.*

When we stated the conjecture we said nothing about the finiteness of the sets in the family. It is an easy corollary of the Proposition 1.2.6 that we can assume all sets to be finite as well, and that is what we will do from now on. We will now show that we could also assume otherwise, that all sets are infinite, but we don't seem to get any advantage by doing that.

Let $\mathcal{F}$ be an union-closed family of finite sets such that $U(\mathcal{F}) = [n]$. Let p_a denote the a-th prime number greater than n. Consider the family

$$\mathcal{F}' = \{X \cup \{p_a^k \mid k \geq 1, a \in X\} \mid X \in \mathcal{F}\}$$

The new family $\mathcal{F}'$ is union-closed since if we have $S, T \in \mathcal{F}'$ then, $S = S' \cup S''$ and $T = T' \cup T''$, where $S', T' \in \mathcal{F}$ and S'', T'' are sets of prime powers. We have

$S' \cup T' \in \mathcal{F}$ and $\{p_a^k \mid k \geq 1, a \in S' \cup T'\} = \{p_a^k \mid k \geq 1, a \in S'\} \cup \{p_a^k \mid k \geq 1, a \in T'\} = S'' \cup T''$. Hence, we have $S \cup T = (S' \cup T') \cup (S'' \cup T'') \in \mathcal{F}'$. If $\mathcal{F}'$ satisfies the union-closed sets conjecture, then either an element of $[n]$ or some p_a^k, for some $a \in [n], k \in \mathbb{N}$ is in at least half the sets of $\mathcal{F}'$. Since $\mathcal{F}'_{p_a^k} = \mathcal{F}'_a$ we have that there is an element of $[n]$ in at least half the sets of $\mathcal{F}'$ and so, in at least half the sets of $\mathcal{F}$, since $|\mathcal{F}| = |\mathcal{F}'|$.

The next result shows us another class of families the conjecture is known to hold for. We say that a family is *intersection-closed* if the intersection of two of its sets is also a set in the family.

Proposition 1.2.9. *Let $\mathcal{F}$ be an union-closed and intersection-closed family of sets. Then $\mathcal{F}$ satisfies the conjecture.*

Proof. Arguing as above, we can assume w.l.o.g. that $\mathcal{F}$ is separating. Let S be a minimal nonempty set in $\mathcal{F}$. For every $T \in \mathcal{F}, S \neq T$ we have $S \cap T \in \mathcal{F}$ and by minimality of S, we have $S \cap T = S$ or $S \cap T = \varnothing$. Now, suppose $|S| \geq 2$ and let $a, b \in S,\ a \neq b$. Now, for every $X \in \mathcal{F}_a$, we have $X \cap S = S$ and so $b \in X$, i.e., $X \in \mathcal{F}_b$. Hence, $\mathcal{F}_a \subseteq \mathcal{F}_b$. Similarly, we can see that $\mathcal{F}_b \subseteq \mathcal{F}_a$ and so $\mathcal{F}_a = \mathcal{F}_b$, which is absurd since $\mathcal{F}$ is separating. From Lemma 1.2.1 the result follows. ■

Definitions 1.2.10. *Given a finite family $\mathcal{G}$ of sets we consider the union-closed family* ∪-generated *by $\mathcal{G}$ defined by $\mathcal{F} = \{\bigcup_{X \in G'} X \mid G' \subseteq \mathcal{G}\}$ and denote it by $\mathcal{F} = \langle \mathcal{G} \rangle$. Given an union-closed family $\mathcal{F}$, we call a set $X \in \mathcal{F}$* ∪-irreducible *if having $X = S \cup T$ for some $S, T \in \mathcal{F}$ implies $S = X$ or $T = X$.*

It is easy to see that $J(\mathcal{F}) = \{X \in \mathcal{F} \mid X \text{is } \cup\text{-irreducible}\}$ is the (unique) minimal generating set of the family $\mathcal{F}$.

Example 4. *Let $\mathcal{F} = \langle \{\{4\}, \{1,2\}, \{3,5\}, \{1,2,4\}, \{3,4,5\}\} \rangle$.*
Then $\mathcal{F} = \{\{4\}, \{1,2\}, \{3,5\}, \{1,2,4\}, \{3,4,5\}, \{1,2,3,5\}, \{1,2,3,4,5\}\}$. This set of generators is not minimal since $\{3,4,5\} = \{3,5\} \cup \{4\}$. It is easy to see that, in this case, $J(\mathcal{F}) = \{\{4\}, \{1,2\}, \{3,5\}, \{1,2,4\}\}$.

In [13], the author proposes the following generalization to the conjecture:

Conjecture 2. *Let $\mathcal{F}$ be an union-closed family of sets with universe $\mathcal{U}$ and let $n = |\mathcal{U}|$. For any $k \leq n$ positive integer, there exists at least one set $S \subset \mathcal{U}$ of size k such that it is contained in at least $2^{-k}|\mathcal{F}|$ of the sets in $\mathcal{F}$.*

It is obvious that, if this conjecture is true, then so is the union-closed sets conjecture. The next theorem shows that Conjecture 2 is in fact equivalent to the union-closed sets conjecture.

Theorem 1.2.11. *Let $\mathcal{F}$ be an union-closed family of sets with universe $\mathcal{U}$ and let $n = |\mathcal{U}|$. If the union-closed sets conjecture holds then for any $k \leq n$ positive integer there are sets $S_k \subset \mathcal{U}$ such that $|S_k| = k$, $S_k \subset S_{k+1}$ and such that S_k is contained in at least $2^{-k}|\mathcal{F}|$ of the sets in $\mathcal{F}$. Hence, in that case, Conjecture 2 also holds.*

Proof. We prove the theorem by induction on k. The case $k = 1$ is trivial since in that case Conjecture 2 states the same as the union-closed sets conjecture. Now, assume that we have some set $S_k \subset \mathcal{U}$ such that S_k is contained in at least $2^{-k}|\mathcal{F}|$ of the sets in $\mathcal{F}$ and consider a new family $\mathcal{G} = \{X \in \mathcal{F} : S_k \subseteq X\}$. We know that $|\mathcal{G}| \geq 2^{-k}|\mathcal{F}|$ and also that $\mathcal{G}$ is an union-closed set since if $S, T \in \mathcal{G} \subseteq \mathcal{F}$ then $S_k \subset S \cup T \in \mathcal{F}$. Now take the family $\mathcal{G} \ominus S_k = \{X \backslash S_k : X \in \mathcal{G}\}$. This new family is still union-closed since if we let $A, B \in \mathcal{G} \ominus S_k$ we have $(A \cup S_k) \cup (B \cup S_k) = (A \cup B) \cup S_k \in \mathcal{G}$, and, $A \cup B \in \mathcal{G} \ominus S_k$. Also, $|\mathcal{G} \ominus S_k| = |\mathcal{G}|$. Let $\mathcal{V}$ be the universe of $\mathcal{G} \ominus S_k$. Since we assume the union-closed sets conjecture is valid, we know there exists an element $x \in \mathcal{V} \subset \mathcal{U} - S_k$ such that x is in at least $\frac{|\mathcal{G} \ominus S_k|}{2} = \frac{|\mathcal{G}|}{2} \geq 2^{-(k+1)}|\mathcal{F}|$ sets. Now just take the set $S_{k+1} = S_k \cup \{x\}$. We have $|S_{k+1}| = |S_k| + 1 = k + 1$, and S_{k+1} is contained in at least $2^{-(k+1)}|\mathcal{F}|$ of the sets in $\mathcal{F}$. This proves the theorem. ■

1.3 Appealing Assumptions and an Infinity of Examples

In this section, we present several natural conjectures that one may be tempted to do and the reason why they are false. In some cases, we will find convenient to represent a family of sets $\mathcal{F}$ as a matrix of 0's and 1's. If $\mathcal{F}$ is a family of n sets, $X_1, \ldots, X_n$ in $[m]$, we represent $\mathcal{F}$ by an $n \times m$ matrix, so that the entry (i, j) is 1 if $j \in X_i$ and 0 otherwise. Define an operation $\star$ in $\mathbb{Z}_2^n$ as $x \star y = x + y - xy$, where xy is the product component-wise. It is easy to see that an union-closed family is represented by a matrix such that the set of its lines is $\star$-closed, because given a row i, then the entry (i, j) is $\mathbb{1}_{X_i}(j)$ and we know that $\mathbb{1}_{A \cup B}(j) = \mathbb{1}_A(j) + \mathbb{1}_B(j) - \mathbb{1}_{A \cap B}(j)$, where $\mathbb{1}_S$ is the indicator function of the set S. Given a set X of lines of a matrix, the matrix generated by them is the matrix such that its lines are the elements of the smallest $\star$-closed set containing X.

Proposition 1.3.1. *There is an union-closed family of sets $\mathcal{F}$, such that the average frequency of its elements is less than $\frac{|\mathcal{F}|}{2}$.*

In fact, we present a more general proposition.

Proposition 1.3.2. *There exists a separating union-closed family of sets, such that the average frequency of its elements is as low as we want.*

Proof. For a fixed n consider the families $\mathcal{G}_k := \langle\{[1],[2],\cdots,[n],\{2\},\{3\},\cdots,\{k+1\}\}\rangle$, for some $k < n$. When we see $\mathcal{G}_k$ as a matrix and study the average frequency, what we want to determine is the ratio of 1's in the matrix. The matrix G_k is the one generated by the following $(n+k)\times n$ matrix

$$A = \left[\begin{array}{ccccccccc}
1&0&0&0&0&\cdots&0&0&0\\
1&1&0&0&0&\cdots&0&0&0\\
1&1&1&0&0&\cdots&0&0&0\\
1&1&1&1&0&\cdots&0&0&0\\
\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\
\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\
1&1&1&1&1&\cdots&1&1&0\\
1&1&1&1&1&\cdots&1&1&1\\
\hline
0&1&0&0&\cdots&\multicolumn{1}{c|}{0}&&&\\
0&0&1&0&\cdots&\multicolumn{1}{c|}{0}&&&\\
0&0&0&1&\cdots&\multicolumn{1}{c|}{0}&&\Large 0&\\
\vdots&\vdots&\ddots&\vdots&\ddots&\multicolumn{1}{c|}{\vdots}&&&\\
0&0&0&\cdots&0&\multicolumn{1}{c|}{1}&&&
\end{array}\right]$$

We will call the first n rows of A the *first part* of the matrix and we will refer to the other rows as the *second part* of the matrix. We claim that when we $\star$-generate the matrix above we get a $(n+2^{k+1}-k-2)\times n$ matrix. We have n rows in the first part of the matrix, which is $\star$-closed and when we $\star$-generate the second part, we obtain 2^k-1 different sets (we exclude the empty set). Now, we have to add the sets that result from considering rows from different parts. For this, it is enough to consider the first row of the second part, because, letting l_i be the i–th row of the first part, and letting x be a row of the second part, after generating, if $i > k$, then $l_i \star x = l_i$;

if not, then $l_i \star x = l_1 \star y$, for some y in the first part, since we have every possible row that only have 1's in entries of index $m \leq k+1$. So, we have $2^k - 1$ sets when we consider rows from different parts, but we have some sets that were already counted, namely the sets in the first part, that will be generated again which are the ones with more than one and less than $k+2$ ones (exactly k sets). So, we have $n + 2^{k+1} - k - 2$ sets.

Now, we count the number of $1's$. Coming from the first part we have $\frac{n(n+1)}{2}$. In the second part we have $k2^{k-1}$, and coming from union of sets of both parts we have $k2^{k-1} + 2^k - 1$ which are the sets that are the union of the first row of the first part with sets from the second one, but again we are counting twice the sets of the first part that have 0 in all entries of index $m > k+1$, except the row $[1\,0\,0\cdots0]$ because we do not consider the empty set as one of the second part. Therefore, the number of ones counted repeatedly is $\frac{(k+1)(k+2)-2}{2}$.

If we calculate the ratio of ones we get

$$\frac{n^2 + n + (k+1)2^k - (k+1)(k+2)}{2n^2 + 2n(2^{k+1} - (k+2))} = \frac{1 + \frac{1}{n} + \frac{1}{n^2}((k+1)(2^{k+1} - k - 2))}{2 + \frac{2}{n}(2^{k+1} - k - 2)}.$$

We can now build a family such the average frequency is smaller than $\frac{1}{t}$ for any $t \in \mathbb{N}$.

By forcing

$$\begin{aligned} 1 + \frac{1}{n} + \frac{1}{n^2}(k+1)\left(2^{k+1} - k - 2\right) < \frac{1}{t}\left(2 + \frac{2}{n}\left(2^{k+1} - k - 2\right)\right) \Leftrightarrow \\ \Leftrightarrow t - 2 + \frac{t}{n} + \frac{t}{n^2}(k+1)\left(2^{k+1} - k - 2\right) < \frac{2}{n}\left(2^{k+1} - k - 2\right) \Leftrightarrow \\ \Leftrightarrow (t-2)n + t + \frac{t}{n}(k+1)\left(2^{k+1} - k - 2\right) < 2\left(2^{k+1} - k - 2\right) \end{aligned}$$

we get the result. This can be achieved by choosing k such that $2^{k+1} - k - 2 > (t-2)t(k+1) + t$ and $n = t(k+1)$ since in that case

$$\begin{aligned} (t-2)n + t + \frac{t}{n}(k+1)(2^{k+1} - k - 2) < 2(2^{k+1} - k - 2) \Leftrightarrow \\ \Leftrightarrow (t-2)t(k+1) + t + 2^{k+1} - k - 2 < 2(2^{k+1} - k - 2) \Leftrightarrow \\ \Leftrightarrow (t-2)t(k+1) + t < 2^{k+1} - k - 2. \end{aligned}$$

■

Corollary 1.3.3. *There is a separating union-closed family of sets $\mathcal{F}$, such that the average frequency of its elements is less than $\frac{|\mathcal{F}|}{2}$.*

In the first version of the preprint [3] the author claims that one can suppose that every element of $U(\mathcal{F})$ has the same frequency when studying the conjecture. We cannot prove that this can never be done. What we show is why the proof in the preprint was incorrect, together with some examples.

The author of [3] argued that if we are given a family $\mathcal{F}$ such that the element with biggest frequency appears in j sets, then for every other element in the universe i we could, starting by the maximal sets X in $\mathcal{F}$ such that $i \notin X$ and $X \cup \{i\} \notin \mathcal{F}$, add the element i to the sets until the element i has frequency j. If this could be done keeping the union-closed property, we would obtain a new union-closed family of sets such that every element has frequency j, and if for this new family the union-closed sets conjecture statement holds, then it also holds for the first one, because that means $j \geq \frac{\mathcal{F}}{2}$. First of all, this process can not be done in every family for every element of its universe, since we might not have enough sets to add the element to. A simple example is the family $\mathcal{F} = \{\{1,2\},\{2,3\},\{1,2,3\}\}$. Here, we can not add the element 1 to any set and so its frequency can not raise. The same happens with the element 3.
The examples given above do not apply if we consider $\mathcal{F}$ to be a counterexample to the conjecture. In this case the statement would be: "If there is a counterexample to the conjecture, then there is also one counterexample in which all elements of the universe have equal frequency." The problem above is no longer a concern since if $\mathcal{F}$ is a counterexample and a is an element of its universe with frequency i, then we have $|\mathcal{F}| - i$ sets that do not contain $\{a\}$. From these, at most i are sets $S \in \mathcal{F}_{\bar{a}}$ such that $S \cup \{a\} \in \mathcal{F}$ and so, we can add a to at least $|\mathcal{F}| - 2i$ sets. Now, let j be the maximum frequency out of all elements in the universe. We have $i \leq j < \frac{|\mathcal{F}|}{2}$ and so $i + j < |\mathcal{F}|$. Hence, $|\mathcal{F}| - 2i = (|\mathcal{F}| - i) - i > j - i$. However, it is not clear that we end up with an union-closed family. Obviously, there isn't any known counterexample to the conjecture in which it fails that we know of, but it $seems$ to fail in general. When we add an element to a set X it might happen that $X = S \cup T$, for some $S, T \in \mathcal{F}$ and that neither S nor T are inflated, and so the family will have the original sets S and T but not $S \cup T$. An example of this is given by the family

$$\{\{3\},\{4\},\{5\},\{1,2\},\{3,4\},\{3,5\},\{4,5\},\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,5\},\{3,4,5\},\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{1,2,3,4,5\}\}.$$

The element with highest frequency belongs to 10 sets. When we try to apply the process above in order to add the element 2 until its frequency is 10, we start by adding it to the set $\{3,4,5\}$ (notice we can't add to the sets $\{1,3,5\}$ and $\{1,3,4,5\}$

since these sets are 2–problematic). We must add it to one more set. It can be the set $\{3,4\},\{3,5\}$, or the set $\{4,5\}$. In every case, the resulting family will not be union-closed, since the sets $\{3\},\{4\}$ and $\{5\}$ belong to the family.

Proposition 1.3.4. *There exists an union-closed family $\mathcal{F}$ such that its smallest set does not contain an element in at least half the sets.*

Proof. In [6], the author presents the following example of a family with 28 sets and universe $[9]$ such that the smallest set of $\mathcal{F}$ is the set $\{1,2,3\}$ and each of its elements is present in only 13 sets. We will denote by B_S the set $[9]\backslash S$. Let

$$\mathcal{F} = \{\varnothing,\{1,2,3\},\{4,5,6,7,8\},\{4,5,6,7,9\},\{4,5,6,8,9\},\{4,5,7,8,9\},\\ \{4,6,7,8,9\},\{5,6,7,8,9\},B_{\{1,2,3\}},B_{\{1,2,8\}},B_{\{1,2,9\}},B_{\{1,3,6\}},\\ B_{\{1,3,7\}},B_{\{2,3,4\}},B_{\{2,3,5\}},B_{\{1,2\}},B_{\{1,3\}},B_{\{2,3\}},B_{\{1\}},\\ B_{\{2\}},B_{\{3\}},B_{\{4\}},B_{\{5\}},B_{\{6\}},B_{\{7\}},B_{\{8\}},B_{\{9\}},[9]\}.$$

Then one has that

$$\mathcal{F}_1 = \{\{1,2,3\},B_{\{2,3,4\}},B_{\{2,3,5\}},B_{\{2,3\}},B_{\{2\}},B_{\{3\}},B_{\{4\}},B_{\{5\}},B_{\{6\}},B_{\{7\}},B_{\{8\}},B_{\{9\}},[9]\},$$

$$\mathcal{F}_2 = \{\{1,2,3\},B_{\{1,3,6\}},B_{\{1,3,7\}},B_{\{1,3\}},B_{\{1\}},B_{\{3\}},B_{\{4\}},B_{\{5\}},B_{\{6\}},B_{\{7\}},B_{\{8\}},B_{\{9\}},[9]]\}.$$

and

$$\mathcal{F}_3 = \{\{1,2,3\},B_{\{1,2,8\}},B_{\{1,2,9\}},B_{\{1,2\}},B_{\{1\}},B_{\{2\}},B_{\{4\}},B_{\{5\}},B_{\{6\}},B_{\{7\}},B_{\{8\}},B_{\{9\}},[9]\}.$$

∎

In [11], Gil Kalai conjectured that for every union-closed family $\mathcal{F}$, there is always an injective map from $\mathcal{F}_{\bar{x}}$ to $\mathcal{F}_x$ such that each set in $\mathcal{F}_{\bar{x}}$ is a subset of its image, for some $x \in U(\mathcal{F})$. That would be a generalization of the original conjecture but was proved false by Alec Edgington, with the following example.

Example 5. *Consider the family $\mathcal{F}$ with universe $\mathcal{U} = \{0,1,2,3,4\}$ consisting of the empty set and all the sets of the form $\{x,\cdots,x+a\}$, $a = 1,2,3,4$, $x \in \mathcal{U}$, with $+$ meaning the addition in $\mathbb{Z}_5$. This family has 17 sets, since the sets $\{x,\cdots,x+a\}$, $a = 1,2,3$, which are all different, for $x \in \mathbb{Z}_5$, and we have the empty set and the universe. Also, this family is union-closed: we can see that the union of a 4-set with another set* *is either itself or the universe, thus an element of the family; The union of a* $3-$ *set with another set is either itself, the universe or a* $4-$*set and all* $4-$*sets belong to the family; finally, the union of a* $2-$*set, say* $\{x, x+1\}$*, with another set we never get* $\{x, x+1, x+3\}$*, which is the only set having* x *and* $x+1$ *that is not in the family. But, it does not exist an injective map from* $\mathcal{F}_{\bar{x}}$ *to* $\mathcal{F}_x$ *such that each set in* $\mathcal{F}_{\bar{x}}$ *is a subset of its image, because if there was one then, there would be one for* $x = 0$*, by symmetry. In that case, the set* $\{1, 2, 3, 4\}$ *would have to be mapped to the set* $\{0, 1, 2, 3, 4\}$*; the set* $\{2, 3, 4\}$ *would have to be mapped to the set* $\{0, 2, 3, 4\}$*, because the map is injective and the target set is* $\mathcal{F}_x$*, and the set* $\{1, 2, 3\}$ *would have to be mapped to* $\{0, 1, 2, 3\}$*. But then, the set* $\{2, 3\}$ *would have to be mapped to one set already used, since* $\{0, 2, 3\} \notin \mathcal{F}$*.*

Chapter 2

Lattice formulation

In this chapter we present two equivalent formulations of the conjecture and study with more emphasis on one of them, that concerns lattice theory. Some definitions about lattices are exposed and the conjecture is proved for the largest class of lattices the conjecture is known to hold.

2.1 Preliminaries

Union-closed sets conjecture has also a dual formulation that is sometimes useful to consider defined in intersection-closed families, where a family is said to be intersection-closed if the intersection of any two member sets is still a member set of the family.

Conjecture 3. *Let $\mathcal{F}$ be a finite family of finite sets closed under intersection. Then*

$$\exists a \in U(\mathcal{F}) : |\mathcal{F}_a| \leq \frac{|\mathcal{F}|}{2}.$$

Proposition 2.1.1. *Conjectures 3 and 1 are equivalent.*

Proof. Suppose Conjecture 1 holds and let $\mathcal{F}$ be an intersection-closed family. Take the family $\mathcal{F}^c = \{X^c : X \in \mathcal{F}\}$. Since $\mathcal{F}$ is intersection-closed we have that $\forall X^c, Y^c \in \mathcal{F}^c$, $X^c \cup Y^c = (X \cap Y)^c \in \mathcal{F}^c$ and so $\mathcal{F}^c$ is union-closed and so it verifies union-closed sets conjecture. By definition of $\mathcal{F}^c$, the element which belongs to at least half the sets of $\mathcal{F}^c$ is in at most half the sets of $\mathcal{F}$ and so, Conjecture 3 holds.

The converse is analogous. ∎

Proposition 2.1.2. *It suffices to prove Conjecture 3 for intersection-closed families $\mathcal{F}$ that have $U(\mathcal{F})$ and $\varnothing$ as elements, where $\varnothing$ is the intersection of all member sets.*

Proof. Consider an intersection-closed family $\mathcal{F}$ such that the intersection of all its sets, $\bigcap\mathcal{F}$, is nonempty and take the family $\mathcal{F}' = \{X \backslash \bigcap\mathcal{F} : X \in \mathcal{F}\}$. It is obvious that $|\mathcal{F}'| = |\mathcal{F}|$ and so $\mathcal{F}'$ verifies the conjecture if and only if $\mathcal{F}$ does.

The second part is analogous to the proof of Proposition 1.2.4. ∎

This problem has already been approached in several different ways having a graph theoretical based formulation, and also one based on lattice theory. The existence of all these formulations is very important since they provide new tools to attack the problem and strengthen the conviction that the conjecture holds. In this chapter we will emphasize the lattice theoretical formulation of the union-closed sets conjecture (actually we will present an equivalent formulation to the intersection-closed sets conjecture, which is equivalent to the union-closed sets conjecture) and we will present a proof that lower semimodular lattices verify the conjecture, originally presented by Reinhold [20].

2.2 Lattices and Frankl Conjecture

A lattice is a poset $(P, \leq)$ such that each pair of elements s and t have a meet and a join, denoted by $s \wedge t$ and $s \vee t$, respectively.

A finite intersection-closed family $\mathcal{F}$ that has the empty set and the universe as elements defines a finite lattice L with partial order given by $\subseteq$ since given s and t we have $s \wedge t = s \cap t \in L$ and the fact that $U(\mathcal{F}) \in \mathcal{F}$ guarantees the existence of a join because we can take the intersection of all sets greater or equal to s and t which is a nonempty element of the lattice and is still greater or equal to both elements. Notice that the join of two elements might not be their union. The definitions about lattices will be based on the present in [24].

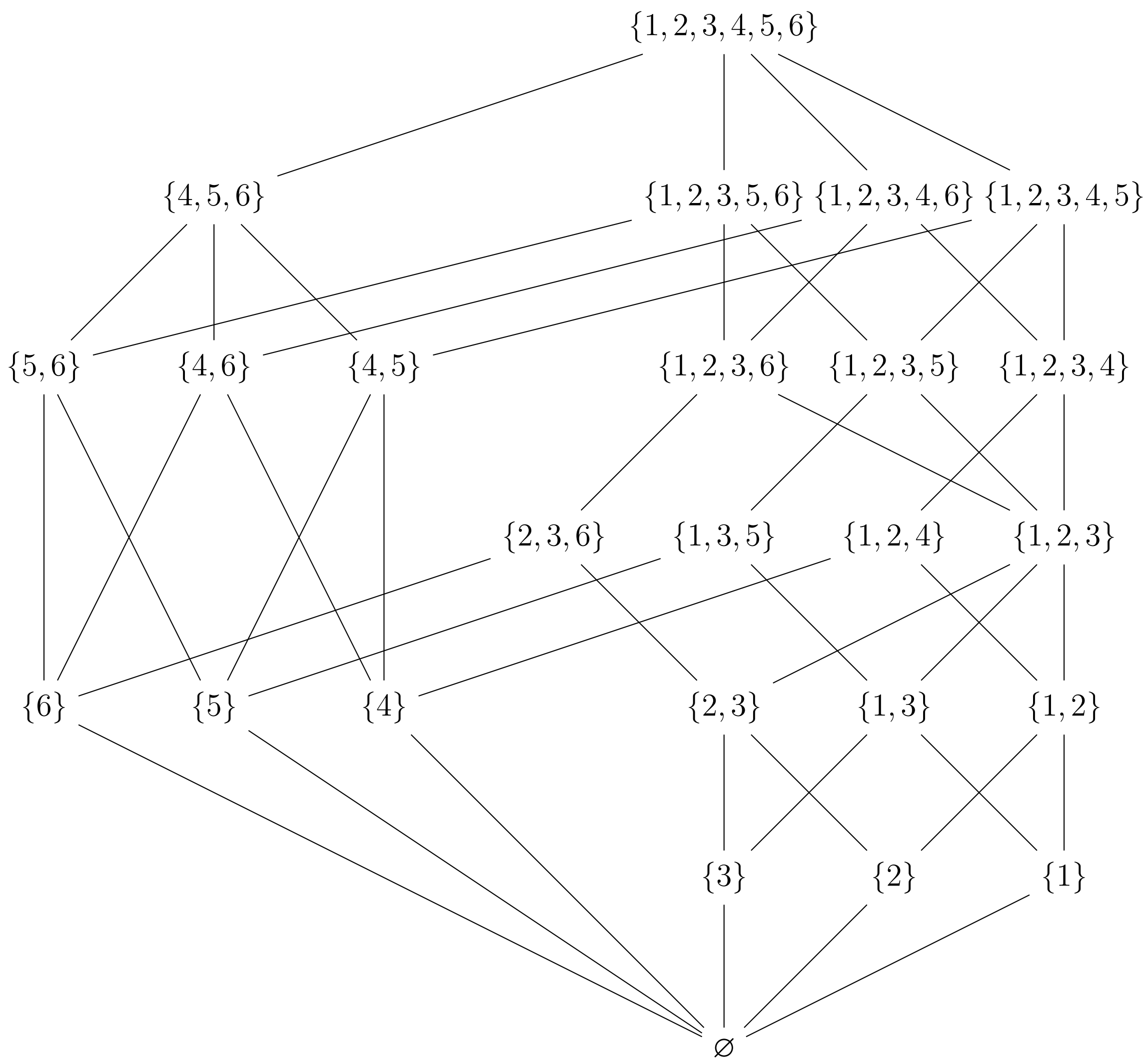

Figure 2.1: Example of an intersection-closed family seen as a lattice

Definitions 2.2.1. *We call a poset P graded of rank n if every maximal chain in P has size n. In that case, there exists a unique* rank function $\rho : P \Rightarrow \{0, 1, \ldots, n\}$ *such that $\rho(s) = 0$ if s is a minimal element of P and $\rho(t) = \rho(s) + 1$ if t covers s.*

Notation 1. *Let P be a poset and $a, b \in P$. We define*

- $[a, b] \coloneqq \{c \in P : a \leq c \leq b\}$.
- $[a) \coloneqq \{c \in P : a \leq c\}$.
- $(a] \coloneqq \{c \in P : c \leq a\}$.

Definition 2.2.2. *The length of a finite poset P is $l(P) \coloneqq max\{l(C) : C$ is a chain of $P\}$, where $l(C) = |C| - 1$ if C is a chain of P. We will denote the length of an interval $[s, t]$ by $l(s, t)$.*

Next we will see an important result that characterizes the class of lower semimodular lattices, for which we intend to prove the conjecture.

Theorem 2.2.3. *Let L be a finite lattice. The following conditions are equivalent:*

i. L is graded and the rank function ρ satisfies:

$$\rho(s) + \rho(t) \leq \rho(s \wedge t) + \rho(s \vee t), \forall s, t \in L.$$

ii. If $s \vee t$ covers s then t covers $s \wedge t$.

iii. If $s \vee t$ covers s and t then both s and t cover $s \wedge t$.

Proof. $(i) \Rightarrow (ii)$ Suppose that $s \vee t$ covers s. Then $\rho(s) = \rho(s \vee t) - 1$. Using $i.$ we have that $\rho(t) - 1 \leq \rho(s \wedge t)$. Since $\rho(s \wedge t) < \rho(t)$ we have that $\rho(s \wedge t) = \rho(t) - 1$ and so t covers $s \wedge t$.
$(ii) \Rightarrow (iii)$ Suppose that $s \vee t$ covers s and t. Then by $ii.$, we have the intended.
$(iii) \Rightarrow (i)$ Suppose L is not graded and let $[u, v]$ be a non graded interval with minimal size. There are s_1 and s_2 in $[u, v]$ covered by v such that all maximal chains of $[u, s_i], i = 1, 2$ have size l_i and $l_1 \neq l_2$. By $iii.$ there are saturated chains in $[u, s_i]$ in this form $u = t_1 < t_2 < \cdots < t_k < s_1 \wedge s_2 < s_i$, contradicting $l_1 \neq l_2$.. Hence, L is graded.

Suppose there are s and t in L such that:

$$\rho(s) + \rho(t) > \rho(s \wedge t) + \rho(s \vee t) \tag{2.1}$$

and we choose s and t with minimal $l(s \wedge t, s \vee t)$ and then, among those ones with maximal $\rho(s)+\rho(t)$. By $iii.$ we can not have s and t (both) covered by $s \vee t$. Therefore, we will assume that $s < s' < s \vee t$. By the minimality of $l(s \wedge t, s \vee t)$ and maximality of $\rho(s) + \rho(t)$ we have that:

$$\rho(s') + \rho(t) \leq \rho(s' \wedge t) + \rho(s' \vee t). \tag{2.2}$$

We then have $s' \vee t = s \vee t$ and so 2.1 and 2.2 imply $\rho(s) + \rho(s' \wedge t) > \rho(s') + \rho(s \wedge t)$.

It is obvious that $s \wedge (s' \wedge t) = s \wedge t$ and $s \vee (s' \wedge t) \leq s'$. So, defining $S = s$ and $T = s' \wedge t$, we find S and T in L satisfying $\rho(S) + \rho(T) > \rho(S \wedge T) + \rho(S \vee T)$ and $l(S \wedge T, S \vee T) < l(s \wedge t, s \vee t)$, which is absurd. ∎

Definition 2.2.4. *A lattice L is said to be lower semimodular if one of the (equivalent) conditions of the previous theorem holds for L.*

Definition 2.2.5. *A lattice L is said to be upper semimodular if its dual lattice L^* is lower semimodular. Moreover, a lattice L is said to be modular if it is lower and upper semimodular.*

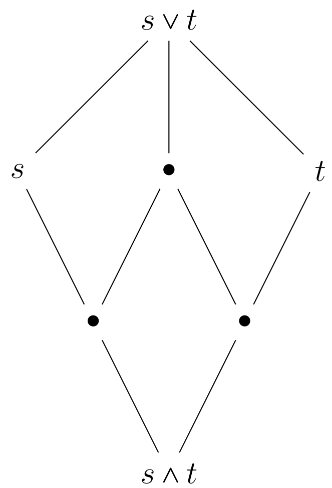

Figure 2.2: Example of an upper semimodular but not modular lattice

Let's denote the greatest element of a lattice L by $\top$ and the lowest by $\bot$. An element $a \in L$ is said to be an *atom* if a covers $\bot$ and a *coatom* if $\top$ covers a.

Definition 2.2.6. *We call an element $a \in L$ $\vee-$irreducible if having $a = b \vee c$ implies that $a = b$ or $a = c$. Analogously, we call $a \in L$ $\wedge-$irreducible if having $a = b \wedge c$ implies that $a = b$ or $a = c$.*

The next lemma gives us a characterization of the elements of the lattice based on the $\vee-$irreducible elements lower or equal than them, essential to establish the correspondence between the union-closed sets conjecture and lattice theory.

Lemma 2.2.7. *In a finite lattice, each element is the join of the $\vee-$irreducible elements lower or equal to it.*

Proof. Let L be a finite lattice and $P \in L$. We will prove the lemma by induction on $\#(P]$. If $P =\bot$ or P is $\vee-$irreducible the result holds. Suppose the result holds for all elements x such that $|(x]| < |(P]|$. If P is not $\vee-$irreducible then it is the join of some Q and R, with $Q, R < P$ and so $|(Q]|, |(R]| < |(P]|$. Let A, B and C be the set of the $\vee-$irreducible elements lower than P, Q and R, respectively. By induction hypothesis we have that Q and R are the join of B and C, respectively. Therefore, we have that P is the join of $B \cup C$. But P is an upper bound of A and $B \cup C \subseteq A$, since $Q, R < P$. Hence, P is the join of A. ∎

Conjecture 4. *In every finite lattice L such that $|L| \geq 2$ there exists a $\vee$−irreducible element a such that $|[a)| \leq \frac{|L|}{2}$.*

Theorem 2.2.8. *Conjectures 3 and 4 are equivalent. Consequently, Conjectures 1 and 4 are equivalent.*

Proof. Let's suppose Conjecture 4 holds. Let $\mathcal{F}$ be an intersection-closed family having $\varnothing$ and $U(\mathcal{F})$ as member sets and consider the lattice associated to $\mathcal{F}$ as seen before. There exists a $\vee$−irreducible element $J \in \mathcal{F}$ such that $|[J)| \leq \frac{|\mathcal{F}|}{2}$.

Suppose every element in J is an element of some proper subset of J that belongs to $\mathcal{F}$. Then we have:

$$\bigcup_{\substack{A \subset J \\ A \in \mathcal{F}}} A = J \quad \text{and} \quad \bigcup_{\substack{A \subset J \\ A \in \mathcal{F}}} A \subseteq \bigvee_{\substack{A \subset J \\ A \in \mathcal{F}}} A.$$

It follows that

$$\bigvee_{\substack{A \subset J \\ A \in \mathcal{F}}} A = J.$$

But J is $\vee$−irreducible, so there exists some $x \in J$ that is not an element of any proper subset of J in $\mathcal{F}$. Let $A \in \mathcal{F}$ be a set of which x is an element. We have that $J \cap A = J$ since $J \cap A$ is a subset of J that has x as an element. Therefore, $J \subseteq A$, i.e., $A \in [J)$. Since $|[J)| \leq \frac{|\mathcal{F}|}{2}$, then x is an element of at most half the sets of the family $\mathcal{F}$ and so, Conjecture 3 holds.

To prove the converse we suppose Conjecture 3 holds. Let L be a finite lattice and we associate at each $x \in L$ the set $S(x)$ of the $\vee$−irreducible elements z such that $z \leq x$. We have that, for $x, y \in L$, $S(x \wedge y) = S(x) \cap S(y)$ and $\mathcal{F} = \{S(x) : x \in L\}$ is an intersection-closed family of sets. By the previous lemma, every element $P \in L$ is the join of the $\vee$−irreducible elements that belong to $S(P)$, so $S(P) = S(Q)$ implies that $P = Q$ and $|L| = |\mathcal{F}|$. By hypothesis, there is a $\vee$−irreducible element x in at most half the sets of $\mathcal{F}$. Therefore, for every $y \geq x$ we have $x \in S(y)$ and so $|[x)|$ is bounded by the number of sets in $\mathcal{F}$ that have x,i.e.,$|[x)| \leq \frac{|L|}{2}$. Hence, Conjecture 4 holds. ∎

It is worth pointing out that if we have a family and consider the lattice associated to it, then if the lattice verifies Conjecture 4, the family verifies 3 but it is not clear from the proof that if the family verifies 3 then the lattice verifies 4 because the proof is not direct in that sense. However the proof shows that if one conjecture holds for *all* families then the other holds for *all* lattices.

Abe and Nakano [2] proved that Conjecture 4 holds for modular lattices. This result was generalized by Reinhold [20] that proved the conjecture for lower semimodular lattices. This is the strongest known result about this conjecture.

Let $(P, \leq)$ and $(P', \leq')$ be posets. We say that a map $\varphi : P \to P'$ is *order-preserving* if the implication $a \leq b \Rightarrow \varphi(a) \leq' \varphi(b)$ is verified and that is *order-embedding* if $a \leq b \Leftrightarrow \varphi(a) \leq' \varphi(b)$.

Remark 2. *If $\varphi : P \to P'$ is order-embedding then it is one-to-one since $\varphi(a) = \varphi(b) \Leftrightarrow \varphi(a) \leq' \varphi(b)$ and $\varphi(b) \leq' \varphi(a) \Leftrightarrow a \leq b$ and $b \leq a \Leftrightarrow a = b$.*

Now we will see a new characterization of semimodular lattices that will be particularly useful in the proof of the main result.

Lemma 2.2.9. *A lattice L is lower semimodular if and only if $\forall a, b \in L$ such that $b \vee a$ covers b, the map*

$$\begin{aligned} \varphi : [a, b \vee a] &\to [b \wedge a, b] \\ x &\mapsto b \wedge x \end{aligned}$$

is order-embedding.

Proof. Let L be a lower semimodular lattice. Consider $a, b \in L$ such that $b \vee a$ covers b and the map $\varphi : [a, b \vee a] \to [b \wedge a, b]$, $x \mapsto b \wedge x$. Let $x, y \in [a, b \vee a]$ such that $x \leq y$. In that case, $b \wedge x \leq x \leq y$ and $b \wedge x \leq b$ and so $b \wedge x \leq b \wedge y$. We then have φ is order preserving.

Now, let $x, y \in [a, b \vee a]$ be such that $x \nleq y$ and suppose $\varphi(x) \leq \varphi(y)$. Then:

$$b \wedge x = \varphi(x) = \varphi(x) \wedge \varphi(y) = b \wedge x \wedge y \leq x \wedge y < x. \tag{2.3}$$

Since $x \in [a, b \vee a]$ we have that $b \vee a = b \vee x$ because $b \vee a \geq x$ and $b \vee a \geq b$, so $b \vee a \geq b \vee x$ and also $b \vee x \geq x \geq a$ and $b \vee a \geq b$. Therefore, $b \vee x \geq b \vee a$. We then have that $b \vee x$ covers b and since L is lower semimodular we have that x covers $b \wedge x$. This way, in 2.3 we have equality between $b \wedge x$ and $x \wedge y$, because $b \wedge x < x \wedge y < x$ is impossible.

Hence, $a \leq x \wedge y = b \wedge x \leq b$ which contradicts the fact that $b \vee a$ covers b.

Now, let's suppose that L is not a lower semimodular lattice. Then, we have $a, b, c \in L$ such that $b \vee c$ covers b and $b \wedge c < a < c$. Then $b \vee a = b \vee c$ and so $b \vee a$ covers b. But $a \leq c \leq b \vee a$ and $\varphi(c) = b \wedge c < a$. Hence, $b \wedge c \leq b \wedge a \Rightarrow b \wedge c = b \wedge a,$ because $a \leq c$. Then $\varphi(a) = \varphi(c)$, and so, $\varphi : [a, b \vee a] \to [b \wedge a, b], x \mapsto b \wedge x$ is not order-embedding. ∎

Theorem 2.2.10. *Every lower semimodular lattice L with $|L| \geq 2$ satisfies Conjecture 4.*

Proof. Let L a lower semimodular lattice with $|L| \geq 2$. We will prove that there exists a $\vee-$irreducible element a such that $|[a)| \leq \frac{|L|}{2}$. If $\top$ is $\vee-$irreducible then L satisfies the conjecture. If not, there exists a coatom b and a $\vee-$irreducible element a with $a \nleq b$. Then $\top = b \vee a$ covers b. By the previous lemma, $\varphi : [a) \to [b \wedge a, b],\ x \mapsto b \wedge x$ is order-embedding, hence injective, and so $\#[b \wedge a, b] \geq \#[a)$.

Since $[b \wedge a, b] \subseteq L \backslash [a)$, it follows that $|[a)| \leq \frac{|L|}{2}$.

∎

The case of the upper semimodular lattices seems much more difficult. Poonen [18] proved that geometric lattices verified the conjecture, with these being a particular case of the upper semimodular lattices in which every element is the join of a set of atoms. Abe [1] considers a different subclass of the upper semimodular lattices, the one of the strong semimodular lattices and proves the conjecture in that case. There are still results from Czédli and Schmidt [5] that prove the result for *large* upper semimodular lattices in the sense that $|L| > \frac{5}{8}2^m$, where m represents the number of $\vee-$irreducible elements in L. However, a proof of the general case remains unknown.

Chapter 3

Properties of a possible counterexample

In this chapter, we study some properties that an eventual counterexample to the conjecture must have, studying in greater detail the case when the counterexample is minimal in some sense that will later be made explicit. This approach provided some bounds on the largest family size and on the largest universe size for which the conjecture is known to hold, that were later surpassed by the ones obtained using other techniques. Still, this approach might be improved and provide better bounds in the future. We follow mostly the approach of Giovanni Lo Faro ([9],[10]).

Throughout this chapter, $\mathcal{F}$ will represent, unless said otherwise, a minimal counter-example to the conjecture, first in terms of $|\mathcal{F}|$ and then in terms of $|U(\mathcal{F})|$. We define $n_0 := |\mathcal{F}|$, where $\mathcal{F}$ is a minimal counter-example to the conjecture in terms of number of member-sets, and $q_0 := |U(\mathcal{F})|$ for $\mathcal{F}$ a minimal counter-example to the conjecture with n_0 member-sets in terms of number of universe elements.

3.1 Basic properties

Theorem 3.1.1. *n_0 is odd.*

Proof. Suppose n_0 is even, let $\mathcal{F}$ be a minimal counterexample, and M a $\cup-$irreducible element in $\mathcal{F}$. Then $\mathcal{F}' = \mathcal{F}\backslash\{M\}$ is also union-closed. We have that $\forall x \in U(\mathcal{F}), |\mathcal{F}_x| \leq \frac{n_0-2}{2}$. Since $|\mathcal{F}'_x| \leq |\mathcal{F}_x| \leq \frac{n_0-2}{2} < \frac{|\mathcal{F}'|}{2}$, we have that $\mathcal{F}'$ is also a counterexample to the conjecture, which is absurd by the minimality of $\mathcal{F}$. ∎

We introduce the concept of derivative of a family with respect to an element of its universe. It is a way to reduce a family into a new family with smaller universe and potentially less sets, and it allows us to obtain some structural results on the original family.

Definition 3.1.2. *Let $\mathcal{F}$ be a family of sets and $a \in U(\mathcal{F})$. We define the* derivative of $\mathcal{F}$ with respect to a *as the family* $\partial_a \mathcal{F} := \{S \backslash \{a\} : S \in \mathcal{F}\}$.

Remark 3. *The name* derivative *seemed adequate due to the following properties:*

- $\partial_a(\mathcal{F} \cup \mathcal{G}) = \partial_a \mathcal{F} \cup \partial_a \mathcal{G}$
- $\partial_{(a,b)}(\mathcal{F} \times \mathcal{G}) = \partial_a \mathcal{F} \times \mathcal{G} \cup \mathcal{F} \times \partial_b \mathcal{G}$, *where* $\mathcal{F} \times \mathcal{G} = \{S \times T \mid S \in \mathcal{F}, T \in \mathcal{G}\}$

The next proposition shows two basic properties of the derivative of an union-closed family.

Proposition 3.1.3. *Let $\mathcal{F}$ be an union-closed family and $a \in U(\mathcal{F})$.*

i. $\partial_a \mathcal{F}$ *is union-closed.*

ii. $J(\partial_a \mathcal{F}) \subseteq \partial_a(J(\mathcal{F}))$.

Proof.

i. Let $S, T \in \partial_a \mathcal{F}$. Obviously $a \notin S \cup T$. Now, either $S \cup T \in \mathcal{F}$ or $S \cup T \cup \{a\} \in \mathcal{F}$. In both cases $S \cup T \in \partial_a \mathcal{F}$.

ii. Let $S \in J(\partial_a \mathcal{F})$. If $S \in \mathcal{F}$, then obviously $S \in J(\mathcal{F})$ and $S \backslash \{a\} = S \in \partial_a(J(\mathcal{F}))$. If $S \notin \mathcal{F}$, then $S = T \backslash \{a\}$, for some $T \in \mathcal{F}_a$. If $T = K \cup L$, for some $K, L \in \mathcal{F} \backslash \{T\}$, then $S = (K \backslash \{a\}) \cup (L \backslash \{a\})$ and $K \backslash \{a\}, L \backslash \{a\} \in \partial_a \mathcal{F} \backslash \{S\}$, which is absurd since S is $\cup$–irreducible. ∎

Remark 4. *The reciprocal inclusion in $ii.$ does not hold. Take, for example, the family $\mathcal{F} = \{\{2\}, \{1,2\}, \{1,3\}, \{1,2,3\}\{1,2,3,4\}\}$. In this case, $J(\mathcal{F}) = \mathcal{F} \backslash \{\{1,2,3\}\}$ and so, $\partial_3(J(\mathcal{F})) = \{\{2\}, \{1,2\}, \{1\}, \{1,2,3\}\{1,2,4\}\}$, which is different from $J(\partial_3 \mathcal{F})$, since it is not a family of $\cup$-irreducible elements, because $\{1,2\} = \{1\} \cup \{2\}$.*

Example 6. *Let $\mathcal{F} = \{\{1,2,4\},\{3,5\},\{1,2\},\{4\},\{3,4,5\},\{1,2,3,5\},\{1,2,3,4,5\}\}$ be the union-closed family in Example 4. We will see that in this case, the reciprocal inclusion of the one in $ii.$ of the previous proposition holds. Recall that we have $J(\mathcal{F}) = \{\{1,2,4\},\{3,5\},\{1,2\},\{4\}\}$. Then $\partial_4\mathcal{F} = \{\{3,5\},\{1,2\},\varnothing,\{1,2,3,5\}\}$ is union-closed and $J(\partial_4\mathcal{F}) = \{\{3,5\},\{1,2\},\varnothing\}$.*

Definition 3.1.4. *Given a family of sets $\mathcal{F}$ and an element $a \in U(\mathcal{F})$, we call a set $S \in \mathcal{F}$ $a-$*problematic *if $a \notin S$ and $S \cup \{a\} \in \mathcal{F}$.*
Also, we define $\mathcal{P}_a^{\mathcal{F}} := \{S \in \mathcal{F} : S$ is $a-$problematic$\}$ (usually there is no doubt about which family we are referring to, so we just denote it by $\mathcal{P}_a$).

The following theorem gives a property of a minimal counterexample and of its derivatives

Theorem 3.1.5. *If $\mathcal{F}$ is a minimal counterexample to the conjecture, then:*

i. $|\partial_a\mathcal{F}| < |\mathcal{F}|, \forall a \in U(\mathcal{F})$;

ii. $\mathcal{F}_a \neq \mathcal{F}_b, \forall a,b \in U(\mathcal{F})$, *i.e., $\mathcal{F}$ is separating.*

Proof.

i. Clearly $|\partial_a\mathcal{F}| \leq |\mathcal{F}|,\ \forall a \in U(\mathcal{F})$. Suppose there is an element $b \in U(\mathcal{F})$ such that $|\partial_b\mathcal{F}| = |\mathcal{F}|$. Since $\partial_b\mathcal{F}$ is also union-closed and its universe has cardinality $q_0 - 1$ there would exist another element $z \in U(\partial_b\mathcal{F})$ such that $|(\partial_b\mathcal{F})_z| > \frac{n_0}{2}$. But then we would have $|\mathcal{F}_z| > \frac{n_0}{2}$ which is absurd.

ii. Note that *i.* is equivalent to say that there exists, for any a, $X \in \mathcal{F}_{\bar{a}} : X \cup \{a\} \in \mathcal{F}, i.e., \mathcal{P}_a \neq \varnothing$. Suppose there are elements $a,b \in U(\mathcal{F}), a \neq b$ such that $\mathcal{F}_a = \mathcal{F}_b$ and let $P \in \mathcal{P}_a$. We have $b \in P \cup \{a\}$, which implies that $b \in P$, and so, $a \in P$ which is absurd. ∎

We will now see that the minimal counterexample must *almost* satisfy the conjecture for at least three elements, which was proved by Norton and Sarvate in [21].

Theorem 3.1.6. *There are at least three distinct elements $x_1, x_2, x_3 \in U(\mathcal{F})$ such that $|\mathcal{F}_{x_i}| = \frac{n_0-1}{2}$, $i = 1,2,3$.*

Proof. Let $a \in U(\mathcal{F})$ and choose a $\cup$–irreducible set M_a such that $a \in M_a$. We know that $\mathcal{F}^a = \mathcal{F}\backslash\{M_a\}$ is also union-closed and so it satisfies the conjecture. Let x_1 be such that $|\mathcal{F}^a_{x_1}| \geq \frac{n_0-1}{2}$. Since $|\mathcal{F}_{x_1}| \leq \frac{n_0-1}{2}$, it follows that $|\mathcal{F}_{x_1}| = \frac{n_0-1}{2}$ and $x_1 \notin M_a$. Consider now the family $\mathcal{F}^{x_1} = \mathcal{F}\backslash\{M_1\}$, for some $\cup$–irreducible set M_1 such that $x_1 \in M_1$. Then, we have x_2 such that $x_2 \neq x_1$ and $|\mathcal{F}_{x_2}| = \frac{n_0-1}{2}$. Then, taking the family $\mathcal{F}^{x_1,x_2} = \mathcal{F}\backslash\{M_1, M_2\}$, for some $\cup$–irreducible set M_2 such that $x_2 \in M_2$, we get $x_3 \neq x_1, x_2$ and $|\mathcal{F}_{x_3}| = \frac{n_0-1}{2}$, since n_0 is odd. ∎

Proposition 3.1.7. *Consider the set* $H = \{x \in U(\mathcal{F}) \mid |\mathcal{F}_x| = \frac{n_0-1}{2}\}$. *One has*

i. $H \subseteq U(\mathcal{F}_{\bar{x}}),\ \forall x \in U(\mathcal{F})\backslash H$

ii. $H\backslash\{x\} \subseteq U(\mathcal{F}_{\bar{x}}),\ \forall x \in H.$

Proof. If $a \in H$ and $x \notin H$, then we have that x belongs to less sets than a. Hence there is a set S having a and not having x and so $a \in S \subseteq U(\mathcal{F}_{\bar{x}})$. Similarly, since $\mathcal{F}$ is separating, $H\backslash\{x\} \subseteq U(\mathcal{F}_{\bar{x}}),\ \forall x \in H.$ ∎

Proposition 3.1.8. *Let* $\mathcal{C} = \{U(\mathcal{F}_{\bar{x}}) \mid x \in U(\mathcal{F})\}$. *Then* $\mathcal{C} \cap J(\mathcal{F}) = \varnothing$.

Proof. Consider the set H defined in Proposition 3.1.7. Suppose we have $a \in H$ such that $U(\mathcal{F}_{\bar{a}}) \in J(\mathcal{F})$. In that case, let $J \in J(\mathcal{F})$ be such that $a \in J$ and consider the family $\mathcal{G} = \mathcal{F}\backslash\{J, U(\mathcal{F}_{\bar{a}})\}$. Then $\mathcal{G}$ is an union-closed family of $n_0 - 2$ sets and no element is in more that $\frac{n_0-3}{2}$ sets, since $H\backslash\{a\} \subseteq U(\mathcal{F}_{\bar{a}})$ and $a \in J$. That contradicts the minimality of $\mathcal{F}$.
Now, suppose we have $a \in U(\mathcal{F})\backslash H$ such that $U(\mathcal{F}_{\bar{a}}) \in J(\mathcal{F})$. In that case, $\mathcal{G} = \mathcal{F}\backslash\{U(\mathcal{F}_{\bar{a}})\}$ is an union-closed family of $n_0 - 1$ sets and no element is in more that $\frac{n_0-3}{2}$ sets, since $H \subseteq U(\mathcal{F}_{\bar{a}})$. Again, this contradicts the minimality of $\mathcal{F}$. ∎

Given a family of sets $\mathcal{F}$, we define $\widehat{\mathcal{F}} := \left\{x \in U(\mathcal{F}) : |\mathcal{F}_x| \geq \frac{|\mathcal{F}|}{2}\right\}$.

Remark 5. *It is obvious that both* $\widehat{\mathcal{P}_a}$ *and* $\widehat{\partial_a\mathcal{F}}$ *are non-empty sets since both families are smaller union-closed families than* $\mathcal{F}$.

Theorem 3.1.9. *We have that* $\widehat{\mathcal{P}_a} \cap \widehat{\partial_a\mathcal{F}} = \varnothing,\ \forall a \in U(\mathcal{F})$.

Proof. Let $k = |\mathcal{P}_a|$ and suppose there exists $b \in \widehat{\mathcal{P}_a} \cap \widehat{\partial_a\mathcal{F}}$. We have $|(\partial_a\mathcal{F})_b| \geq \frac{|\partial_a\mathcal{F}|}{2} = \frac{|\mathcal{F}|-k}{2}$, since there are k a-problematic sets and $\{a\} \notin \mathcal{F}$. Also, $|(\mathcal{P}_a)_b| \geq \frac{|\mathcal{P}_a|}{2} = \frac{k}{2}$.

But we know that if $b \in X$, for some $X \in \mathcal{P}_a$, then $b \in X \cup \{a\} \in \mathcal{F} \backslash \partial_a \mathcal{F}$. Thus, $|(\mathcal{P}_a)_b| \leq |(\mathcal{F} \backslash \partial_a \mathcal{F})_b|$.
It is now clear that $b \in \widehat{\mathcal{P}_a} \cap \widehat{\partial_a \mathcal{F}} \Rightarrow |\mathcal{F}_b| = |(\partial_a \mathcal{F})_b| + |\mathcal{F} \backslash (\partial_a \mathcal{F})_b| \geq |(\partial_a \mathcal{F})_b| + |(\mathcal{P}_a)_b| \geq \frac{|\mathcal{F}|}{2}$, which is absurd. ∎

We will now present several results about the structure of a counterexample that will later be used to obtain some bounds on its size and on the frequency of its elements.

Theorem 3.1.10. $U(\mathcal{F}_{\bar{a}}) \in \mathcal{P}_a,\ \forall a \in U(\mathcal{F})$.

Proof. Let $P \in \mathcal{P}_a$. Since $a \notin P$ we have $P \subseteq U(\mathcal{F}_{\bar{a}})$. We also know that $P \cup \{a\} \in \mathcal{F}$ and so $U(\mathcal{F}_{\bar{a}}) \cup P \cup \{a\} = U(\mathcal{F}_{\bar{a}}) \cup \{a\} \in \mathcal{F}$. Hence, $U(\mathcal{F}_{\bar{a}}) \in \mathcal{P}_a$. ∎

The following corollary will help us simplify the proof of Theorem 3.1.16 below, which is Theorem 9 in [9].

Corollary 3.1.11. *For each $a \in U(\mathcal{F})$ there are at least three a-problematic sets.*

Proof. Consider $a \in U(\mathcal{F})$ and take $b \in \widehat{\partial_a \mathcal{F}}$. Since $\mathcal{F}_{\bar{a}} \subseteq \partial_a \mathcal{F}$, $|\mathcal{F}_{\bar{a}}| > \frac{|\partial_a \mathcal{F}|}{2}$, and $b \in \widehat{\partial_a \mathcal{F}}$, it follows that b must belong to at least one set of $\mathcal{F}_{\bar{a}}$, and so $b \in U(\mathcal{F}_{\bar{a}})$. Then, from Theorem 3.1.10 we have that b is an element of at least one a-problematic set. On the other hand, Theorem 3.1.9, we must have at least two a–problematic sets that do not contain $\{b\}$. The result follows. ∎

Remark 6. *Note that we can consider $U(\mathcal{F}) = [n]$ with $|\mathcal{F}_1| \leq |\mathcal{F}_2| \leq \cdots \leq |\mathcal{F}_n|$, which we assume from now on, up to the end of this chapter. Notice that the previous corollary implies $|\mathcal{F}_1| \geq 3$.*

Definition 3.1.12. *Let $a, b \in U(\mathcal{F})$. We say b is dominated by a if $\mathcal{F}_b \subsetneq \mathcal{F}_a$. Notice that this is equivalent to $b \notin U(\mathcal{F}_{\bar{a}})$.*

Theorem 3.1.13. *There are at least three distinct elements $x_1, x_2, x_3 \in U(\mathcal{F})$ such that $U(\mathcal{F}_{\bar{x}_i}) = U(\mathcal{F}) \backslash \{x_i\}$, $i = 1, 2, 3$, i.e., there are three elements that don't dominate any other element.*

Proof. We can take $x_1 = 1$, since, if 1 dominated some other element a, then by the previous remark we would have $\mathcal{F}_1 = \mathcal{F}_a$, and that is absurd because $\mathcal{F}$ is

separating. Now, take $X \in \mathcal{F}_{x_1}$ such that $X \neq U(\mathcal{F})$ (its existence is granted by the previous remark). Take $y \in U(\mathcal{F})\backslash X$. If $U(\mathcal{F}_{\bar{y}}) = U(\mathcal{F})\backslash\{y\}$ then take $x_2 = y$. Else, take $z \in U(\mathcal{F})\backslash U(\mathcal{F}_{\bar{y}})$, with $z \neq y$. If $U(\mathcal{F}_{\bar{z}}) = U(\mathcal{F})\backslash\{z\}$ take $x_2 = z$. Else, notice that, since $U(\mathcal{F}_{\bar{y}}) \cup \{y\} \in \mathcal{F}$ (by Theorem 3.1.10) and $z \notin U(\mathcal{F}_{\bar{y}})$, we have $x_1 \in X \subsetneq U(\mathcal{F}_{\bar{y}}) \cup \{y\} \subseteq U(\mathcal{F}_{\bar{z}})$. Obviously, continuing this process we find $x_2 \in U(\mathcal{F}), x_2 \neq x_1$ such that $U(\mathcal{F}_{\bar{x}_2}) = U(\mathcal{F})\backslash\{x_2\}$.
Let M_{x_1} and M_{x_2} be $\cup$–irreducible sets such that $x_1 \in M_{x_1}$ and $x_2 \in M_{x_2}$. We can prove that $M_{x_1} \cup M_{x_2} \neq U(\mathcal{F})$ since otherwise we would consider the union closed family $\mathcal{F}' = \mathcal{F} - \{M_{x_1}, M_{x_2}\}$ which satisfies the conjecture and so there exists an element $z \in U(\mathcal{F}')$ such that z belongs to at least $\frac{n_0-1}{2}$ sets and to at least one of M_{x_1}, M_{x_2} and so, to at least $\frac{n_0+1}{2}$ sets of $\mathcal{F}$ and that is absurd. If we argue as above considering the set $M_{x_1} \cup M_{x_2}$ we can complete the proof. ∎

Theorem 3.1.14. $|\mathcal{F}_1| \geq 5$.

Proof. $|\mathcal{F}_1| \geq 3$ is obvious from Corollary 3.1.11. Suppose $|\mathcal{F}_1| = 4$. By the previous theorem we have $\mathcal{F}_1 = \{U(\mathcal{F}), U(\mathcal{F})\backslash\{x_2\}, U(\mathcal{F})\backslash\{x_3\}, B\}$, for some $B \subseteq U(\mathcal{F})$. Set $z \in \widehat{\mathcal{F}_{\bar{1}}}$. Since $|(\mathcal{F}_1)_z| \geq 2$ we have $z \in \widehat{\mathcal{F}}$ which is a contradiction. ∎

Theorem 3.1.15. *Let $x_1 = 1, x_2, x_3$ as in Theorem 3.1.13. There is an element $x_4 \in U(\mathcal{F})\backslash\{x_1, x_2, x_3\}$ such that $U(\mathcal{F})\backslash\{x_3, x_4\} \subseteq U(\mathcal{F}_{\bar{x}_4})$.*

Proof. We start by proving that there are sets $X_1 \in \mathcal{F}_{x_1}$ and $X_2 \in \mathcal{F}_{x_2}$ such that $U(\mathcal{F})\backslash\{x_3\} \nsubseteq X_1 \cup X_2$. Suppose otherwise, i.e., suppose that $U(\mathcal{F})\backslash\{x_3\} \subseteq X_1 \cup X_2$ for all $X_i \in \mathcal{F}_{x_i}, i = 1, 2$. Take M_1, M_2 be $\cup$–irreducible elements such that $x_i \in M_i, i = 1, 2$ and consider the family $\mathcal{F}^{x_1,x_2} = \mathcal{F}\backslash\{M_1, M_2\}$. Let $z \in \widehat{\mathcal{F}^{x_1,x_2}}$. Since $U(\mathcal{F})\backslash\{x_3\} \subseteq X_1 \cup X_2$, one has that $\widehat{\mathcal{F}^{x_1,x_2}} = \{x_3\}$, because if there exists $y \in \widehat{\mathcal{F}^{x_1,x_2}}\backslash\{x_3\}$ then $y \in M_1 \cup M_2$ and so $y \in \widehat{\mathcal{F}}$, which cannot happen. Hence, we have $|\mathcal{F}_{x_3}| = \frac{n_0-1}{2}$. Now, we claim that there exists $w \notin \{x_1, x_2, x_3\}$ such that $w \notin M_1$, because if that was not the case, we would have that $U(\mathcal{F})\backslash\{x_2, x_3\} \subseteq Y$, for all $Y \in \mathcal{F}_{x_1}$, which is absurd since there are only 4 supersets of $U(\mathcal{F})\backslash\{x_2, x_3\}$ and by the previous theorem, one has that $|\mathcal{F}_{x_1}| \geq 5$. Now, we know that $U(\mathcal{F})\backslash\{x_3\} \subseteq M_1 \cup X_2$, $\forall X_2 \in \mathcal{F}_{x_2}$ which implies that $x_4 \in X_2$, $\forall X_2 \in \mathcal{F}_{x_2}$, i.e., that $\mathcal{F}_{x_2} \subseteq \mathcal{F}_{x_4}$. Since $|\mathcal{F}_{x_2}| = \frac{n_0-1}{2}$, we have that $\mathcal{F}_{x_2} = \mathcal{F}_{x_4}$, which is impossible.

So, take $X_1 \in \mathcal{F}_{x_1}$ and $X_2 \in \mathcal{F}_{x_2}$ such that $U(\mathcal{F})\backslash\{x_3\} \nsubseteq X_1 \cup X_2$. Then there is $y_1 \neq x_3$ such that $y_1 \notin X_1 \cup X_2$ and we proceed as in Theorem 3.1.13 in order to get x_4 such that $X_1 \cup X_2 \subseteq U(\mathcal{F}_{\bar{x}_4})$ and $U(\mathcal{F})\backslash\{x_3, x_4\} \subseteq U(\mathcal{F}_{\bar{x}_4})$. The idea is that if $U(\mathcal{F})\backslash\{y_1, x_3\} \subseteq U(\mathcal{F}_{\bar{y}_1})$ then we take $x_4 = y_1$ and we are done. Otherwise, there is $z_1 \notin \{x_3, y_1\}$ such that $U(\mathcal{F}_{\bar{y}_1}) = U(\mathcal{F})\backslash\{y_1, z_1, \cdots, z_s\}$ and we have $X_1 \cup X_2 \subsetneq U(\mathcal{F}_{\bar{y}_1}) \cup \{y_1\} \subseteq U(\mathcal{F}_{\bar{z}_1})$. ■

Theorem 3.1.16. $|\mathcal{F}_1| \geq 9$.

Proof. Suppose $|\mathcal{F}_1| = 5$. We have $\mathcal{F}_1 = \{U(\mathcal{F}), U(\mathcal{F})\backslash\{x_2\}, U(\mathcal{F})\backslash\{x_3\}, B_1, B_2\}$, for some B_1, $B_2 \subseteq U(\mathcal{F})$. From the previous theorem we can take $x_4 \notin \{1, x_2, x_3\}$ such that $U(\mathcal{F})\backslash\{x_3, x_4\} \subseteq U(\mathcal{F}_{\bar{x}_4}) \subseteq U(\mathcal{F})\backslash\{x_4\}$ and $B_1 = U(\mathcal{F}_{\bar{x}_4})$.

Let $z \neq 1$ such that $|(\mathcal{F}_{\bar{1}})_z| \geq \frac{n-5}{2}$. Since $|\mathcal{F}_z| \leq \frac{n-1}{2}$ we have $|(\mathcal{F}_1)_z| \leq 2$. It is obvious that $z = x_2 \vee z = x_3$ since any other element belongs to $U(\mathcal{F}), U(\mathcal{F})\backslash\{x_2\}$ and $U(\mathcal{F})\backslash\{x_3\}$. In the case $z = x_2$ we have $z \in U(\mathcal{F}), U(\mathcal{F})\backslash\{x_3\}, B_1$ which is absurd. Then one has that $z = x_3 \notin B_1, B_2$; $B_1 = U(\mathcal{F})\backslash\{x_3, x_4\}$; and $|\mathcal{F}_{x_3}| = \frac{n-1}{2}$. Since $x_3 \notin B_1 = U(\mathcal{F}_{\bar{x}_4})$ then $\mathcal{F}_{x_3} \subseteq \mathcal{F}_{x_4}$, thus $\mathcal{F}_{x_3} = \mathcal{F}_{x_4}$ because $|\mathcal{F}_{x_3}| = \frac{n-1}{2}$. That is absurd from Theorem 3.1.5.

The case $|\mathcal{F}_1| = 6$ is similar.

Suppose $|\mathcal{F}_1| = 7$. We have $\mathcal{F}_1 = \{U(\mathcal{F}), U(\mathcal{F})\backslash\{x_2\}, U(\mathcal{F})\backslash\{x_3\}, B_1, B_2, B_3, B_4\}$, for some B_1, B_2, B_3, $B_4 \subseteq U(\mathcal{F})$. From the previous theorem we can take $x_4 \notin \{1, x_2, x_3\}$ such that $U(\mathcal{F})\backslash\{x_3, x_4\} \subseteq U(\mathcal{F}_{\bar{x}_4}) \subseteq U(\mathcal{F})\backslash\{x_4\}$ and $B_1 = U(\mathcal{F}_{\bar{x}_4})$.

Let $z \neq 1$ such that $|(\mathcal{F}_{\bar{1}})_z| \geq \frac{n_0-7}{2}$. Since $|\mathcal{F}_z| \leq \frac{n-1}{2}$ we have $|(\mathcal{F}_1)_z| \leq 3$. It can be easily seen that $z \in \{x_2, x_3, x_4\}$ as above.

Suppose now $B_1 = U(\mathcal{F})\backslash\{x_4\}$, then $z \notin B_2 \cup B_3 \cup B_4$ because z belongs to $U(\mathcal{F})$ and to at least two of $U(\mathcal{F})\backslash\{x_2\}, U(\mathcal{F})\backslash\{x_3\}, B_1$. Without loss of generality assume $z = x_2$. If there exists $B_i (i = 2, 3, 4)$ such that $U(\mathcal{F})\backslash\{x_2, x_3, x_4\} \nsubseteq B_i$ then there exists $y \notin B_i \cup \{x_2, x_3, x_4\}$. Then $U(\mathcal{F})\backslash\{x_3\} \subseteq B_i \cup X_2$ or $U(\mathcal{F})\backslash\{x_4\} \subseteq B_i \cup X_2$, $\forall X_2 \in \mathcal{F}_{x_2}$, since $B_i \cup X_2 \in \mathcal{F}_1$ and $x_2 \in B_i \cup X_2$, thus $B_i \cup X_2 \neq B_j$, $\forall j \in \{2, 3, 4\}$. So $y \in X_2$, $\forall X_2 \in \mathcal{F}_{x_2}$, hence $\mathcal{F}_y = \mathcal{F}_{x_2}$, a contradiction. If $U(\mathcal{F})\backslash\{x_2, x_3, x_4\} \subseteq B_i$, for each $i = 2, 3, 4$, then

$$\begin{aligned}\mathcal{F}_1 = \{&U(\mathcal{F}), U(\mathcal{F})\backslash\{x_2\}, U(\mathcal{F})\backslash\{x_3\}, U(\mathcal{F})\backslash\{x_4\},\\ &U(\mathcal{F})\backslash\{x_2, x_3\}, U(\mathcal{F})\backslash\{x_2, x_4\}, U(\mathcal{F})\backslash\{x_2, x_3, x_4\}\}.\end{aligned}$$

Let M_{x_2} be a $\cup$-irreducible set such that $x_2 \in M_{x_2}$. If we consider the union-closed family $\mathcal{F}' = \mathcal{F}\backslash(\{M_{x_2}\} \cup \mathcal{F}_1)$ we have an element in at least $\frac{n_0-7}{2}$ and that contradicts the fact that $\mathcal{F}$ is a counter-example.

Suppose now $B_1 = U(\mathcal{F})\backslash\{x_3, x_4\}$ and obviously $U(\mathcal{F})\backslash\{x_4\} \notin \mathcal{F}_1$. We consider several cases.

Suppose $z = x_2$. Again, as above $x_2 \notin B_2 \cup B_3 \cup B_4$, $|\mathcal{F}_{x_2}| = \frac{n-1}{2}$ and there exists $y \notin B_i \cup \{x_2, x_3, x_4\}$ since otherwise we would have

$$\mathcal{F}_1 = \{U(\mathcal{F}), U(\mathcal{F})\backslash\{x_2\}, U(\mathcal{F})\backslash\{x_3\}, U(\mathcal{F})\backslash\{x_3, x_4\},\\ U(\mathcal{F})\backslash\{x_2, x_3\}, U(\mathcal{F})\backslash\{x_2, x_4\}, U(\mathcal{F})\backslash\{x_2, x_3, x_4\}\}$$

and that is absurd since, $(U(\mathcal{F})\backslash\{x_2, x_4\}) \cup (U(\mathcal{F})\backslash\{x_3, x_4\}) = U(\mathcal{F})\backslash\{x_4\} \notin \mathcal{F}_1$.

We then obtain $\mathcal{F}_y = \mathcal{F}_{x_2}$, which is absurd.

Suppose $z = x_4$. This case is similar to the previous.

Suppose $z = x_3$. Since $x_3 \in U(\mathcal{F})$ and $x_3 \in U(\mathcal{F})\backslash\{x_2\}$ we have $|\mathcal{F}_{x_3}| \geq \frac{n_0-3}{2}$. Since $U(\mathcal{F})\backslash\{x_4\} \notin \mathcal{F}_1$, then $U(\mathcal{F})\backslash\{x_3, x_4\} \cup X_3 = U(\mathcal{F}),\ \forall X_3 \in \mathcal{F}_3$, which means $\mathcal{F}_{x_3} \subsetneq \mathcal{F}_{x_4}$. Then, we have $|\mathcal{F}_{x_3}| = \frac{n_0-3}{2}$ and $|\mathcal{F}_{x_4}| = \frac{n_0-1}{2}$. It follows that $U(\mathcal{F})\backslash\{x_3\}$ is the only set having x_4 and not having x_3. But, from Corollary 3.1.11, one has that there are at least two x_3-problematic sets different from $U(\mathcal{F})\backslash\{x_3\}$. Let P be one x_3-problematic set different from $U(\mathcal{F})\backslash\{x_3\}$. Then $P \cup \{x_3\} \in \mathcal{F}$ and since $\mathcal{F}_{x_3} \subset \mathcal{F}_{x_4}$, we have $x_4 \in P$ and x_3 not in P, which is absurd.

The case $|\mathcal{F}_1| = 8$ is similar. ∎

3.2 A relation between n_0 and q_0

Now we present one of the most important results about Frankl conjecture that is obtained by studying the properties of an eventual counterexample and that has not been improved since 1994. Lo Faro proved it in [10] and it was later rediscovered by Roberts and Simpson in [22]. However, we will show that the proof in [22] is flawed.

Theorem 3.2.1. *i. If $\forall x \in U(\mathcal{F})$, we have $U(\mathcal{F}_{\bar{x}}) = U(\mathcal{F})\backslash\{x\}$, then $n_0 \geq 4q_0 - 1$.*

ii. If there exists $x \in U(\mathcal{F})$ such that $U(\mathcal{F}_{\bar{x}}) \neq U(\mathcal{F})\backslash\{x\}$, then $n_0 \geq 4q_0 + 1$.

To prove this, one needs to understand how dominance between elements can be established. We define some concepts and prove some results to help us

understand it.
We are considering minimal counterexamples to the conjecture but there might be several counterexamples with n_0 sets and with q_0 elements in its universe. Now, we will try to *organize* them and select a more specific counterexample among those. Let $\mathfrak{F}$ be the set of all counterexamples in these conditions.
The first result will be stated without proof.

Theorem 3.2.2. *Let $x, y \in [q_0]$, with $x \neq y$. If $|\mathcal{F}_y| + 4 \geq |\mathcal{F}_x|$, then $y \in U(\mathcal{F}_{\bar{x}})$.*

Given $\mathcal{F} \in \mathfrak{F}$, we set:

$$\begin{aligned} m_{\mathcal{F}} &:= \min\{|\mathcal{F}_x| : x \in U(\mathcal{F})\} \\ M &:= \max\{m_{\mathcal{F}} : \mathcal{F} \in \mathfrak{F}\} \\ \mathfrak{F}^M &:= \{\mathcal{F} \in \mathfrak{F} : m_{\mathcal{F}} = M\} \\ \mathfrak{F}^M_r &:= \{\mathcal{F} \in \mathfrak{F}^M \mid |\{x \in U(\mathcal{F}) : |\mathcal{F}_x| = M\}| = r\} \\ r_0 &:= \min\{r \in \mathbb{N} : \mathfrak{F}^M_r \neq \varnothing\} \\ \mathfrak{G} &:= \mathfrak{F}^M_{r_0} \end{aligned}$$

It is clear that there exists a counterexample to the conjecture if and only if $\mathfrak{G} \neq \varnothing$. Now, we define an equivalence relation $\approx$ on $\mathfrak{G}$ by $\mathcal{F} \approx \mathcal{G} \Leftrightarrow |\mathcal{F}_x| = |\mathcal{G}_x|, \forall x \in [q_0]$ and denote the class of $\mathcal{F}$ by $[\mathcal{F}]$ and a total ordering $<$ on $\mathfrak{G}/\approx$ by $[\mathcal{F}] < [\mathcal{G}] \Leftrightarrow |\mathcal{F}_k| < |\mathcal{G}_k|$, where $k = \min\{i \in [q_0] : |\mathcal{F}_i| \neq |\mathcal{G}_i|\}$. Note that $<$ is well defined since k does not depend on the representative chosen. We now consider $\mathcal{F} \in [\mathcal{F}]$, where $\mathcal{F}$ is the maximum in $(\mathfrak{G}/\approx, \leq)$.

Theorem 3.2.3. *Let $x, y \in U(\mathcal{F})$, with $x \neq y$. If y dominates x, i.e., if $x \notin U(\mathcal{F}_{\bar{y}})$, then $S \cup \{x\} \in \mathcal{F}$, for all $S \in \mathcal{F}_y$.*

Proof. Let $\mathcal{A} = \{S \in \mathcal{F}_y : S \cup \{x\} \notin \mathcal{F}\}$. We want to prove that $\mathcal{A} = \varnothing$. To do so, define $\mathcal{F}' = (\mathcal{F} \backslash \mathcal{A}) \cup (\mathcal{A} \uplus \{\{x\}\})$. It is easy to see that $\mathcal{F}'$ is union-closed and $\mathcal{F}' \in \mathfrak{F}$. One has $|\mathcal{F}_z| = |\mathcal{F}'_z|, \forall z \in [q_0] \backslash \{x\}$ and also $|\mathcal{F}_x| \leq |\mathcal{F}'_x|$ and $|\mathcal{F}_x| = |\mathcal{F}'_x| \Leftrightarrow \mathcal{A} = \varnothing$. Assume for a contradiction that $|\mathcal{F}_x| < |\mathcal{F}'_x|$. We consider two cases:
$\textbf{Case 1}$: Suppose $|\mathcal{F}_x| = |\mathcal{F}_1| = M$.
If $r_0 = 1$, then $x = 1$ and $m_{\mathcal{F}'} > m_{\mathcal{F}} = M$ since $m_{\mathcal{F}'} = |\mathcal{F}'_x| > |\mathcal{F}_x|$ or $\exists z \in [q_0] \backslash \{x\} : m_{\mathcal{F}'} = |\mathcal{F}'_z| = |\mathcal{F}_z| > |\mathcal{F}_x|$, because $r_0 = 1$. That contradicts the maximality of M.
If, on the other hand, $r_0 > 1$, then there is an element $z \in [q_0] \backslash \{x\}$ such that $|\mathcal{F}'_z| = |\mathcal{F}_z| = M$ and so $\mathcal{F}' \in \mathfrak{F}^M$. Also, $\{x \in [q_0] : |\mathcal{F}'_x| = M\} = \{x \in [q_0] : |\mathcal{F}_x| = M\} \backslash \{x\}$ and so $\mathcal{F}' \in \mathfrak{F}^M_{r_0 - 1}$, which contradicts the minimality of r_0.

$\mathbf{Case\ 2}:$ Suppose $|\mathcal{F}_x| > M$.
In this case $\mathcal{F}' \in \mathfrak{G}$. If necessary, reorder the elements in $\mathcal{F}'$ to have $|\mathcal{F}'_1| \leq |\mathcal{F}'_2| \cdots \leq |\mathcal{F}'_{q_0}|$. We have that $|\mathcal{F}'_z| = |\mathcal{F}_z|$, $\forall z \in [q_0]\backslash\{x\}$, but $|\mathcal{F}'_x| < |\mathcal{F}_x|$ and so $[\mathcal{F}] < [\mathcal{F}']$, which is absurd since $[\mathcal{F}]$ is the maximum. Hence, we have $\mathcal{A} = \varnothing$, as desired. ∎

Corollary 3.2.4. *Let $x, y \in U(\mathcal{F})$, with $x \neq y$. If y dominates x, then $|\mathcal{F}_y| \leq 2|\mathcal{F}_x|$.*

Proof. By the previous theorem, we have an injection $\varphi : \mathcal{F}_y\backslash\mathcal{F}_x \to \mathcal{F}_x$ given by $S \mapsto S \cup \{x\}$. Hence, $|\mathcal{F}_y| = |\mathcal{F}_x| + |\mathcal{F}_y\backslash\mathcal{F}_x| \leq 2|\mathcal{F}_x|$. ∎

Corollary 3.2.5. *For each $x \in [q_0]$, x can only be dominated by at most one element.*

Proof. Suppose there are two different elements $y, z \in [q_0]\backslash\{x\}$ such that $x \notin U(\mathcal{F}_{\bar{y}})$ and $x \notin U(\mathcal{F}_{\bar{z}})$. By the Theorem 3.2.3, we have that $S \cup \{x\} \in \mathcal{F}_x \subseteq \mathcal{F}_z, \forall S \in \mathcal{F}_y$, which implies $S \in \mathcal{F}_z$ and so $\mathcal{F}_y \subseteq \mathcal{F}_z$. Analogously, we get $\mathcal{F}_z \subseteq \mathcal{F}_y$ and so $\mathcal{F}_y = \mathcal{F}_z$, which is absurd from Theorem 3.1.5. ∎

Remark 7. *One can write the previous corollary as follows: for each $x \in [q_0]$, we have $1 \leq |\{i \in [q_0] : x \notin U(\mathcal{F}_{\bar{i}})\}| \leq 2.$*

Corollary 3.2.6. $M = |\mathcal{F}_1| \geq 2q_0 - 5.$

Proof. Consider the union-closed family $\mathcal{F}_{\bar{1}}$. Since $\mathcal{F}$ is a minimal counterexample to the conjecture, there exists $z \in [q_0]\backslash\{1\}$ such that $|(\mathcal{F}_{\bar{1}})_z| \geq \frac{n_0 - M}{2}$. By the previous corollary, we have that $|\{i \in [q_0]\backslash\{1,z\} : \{1,z\} \subseteq U(\mathcal{F}_{\bar{i}})\}| \geq q_0 - 4.$
So z belongs to at least $q_0 - 3$ sets in $\mathcal{F}_1$, because $\{1, z\}$ is also contained in $U(\mathcal{F}) = [q_0]$, and to at least $\frac{n_0 - M}{2}$ sets in $\mathcal{F}_{\bar{1}}$. Since $\mathcal{F}$ is a counterexample to the conjecture, one has $|\mathcal{F}_z| \leq \frac{n_0 - 1}{2}$. It follows that $\frac{n_0 - M}{2} + q_0 - 3 \leq \frac{n_0 - 1}{2}$. Since

$$\begin{aligned}
&\frac{n_0 - M}{2} + q_0 - 3 \leq \frac{n_0 - 1}{2} \Leftrightarrow \\
\Leftrightarrow\ &\frac{n_0 - M}{2} \leq \frac{n_0 + 5 - 2q_0}{2} \Leftrightarrow \\
\Leftrightarrow\ &n_0 - M \leq n_0 + 5 - 2q_0 \Leftrightarrow \\
\Leftrightarrow\ &M \geq 2q_0 - 5,
\end{aligned}$$

we have the desired result. ∎

Now, we are ready to prove Theorem 3.2.1, which we recall.

Theorem 3.2.1. *i. If $\forall x \in U(\mathcal{F})$, we have $U(\mathcal{F}_{\bar{x}}) = U(\mathcal{F})\backslash\{x\}$, then $n_0 \geq 4q_0 - 1$.*

ii. If there exists $x \in U(\mathcal{F})$ such that $U(\mathcal{F}_{\bar{x}}) \neq U(\mathcal{F})\backslash\{x\}$, then $n_0 \geq 4q_0 + 1$.

Proof.

i. We repeat the argument used to prove the last corollary, only here we have $|\{i \in [q_0]\backslash\{1,z\} : \{1,z\} \subseteq U(\mathcal{F}_{\bar{i}})\}| = q_0 - 2$ and so we can prove that $M \geq 2q_0 - 1$. Hence, $\frac{n_0-1}{2} \geq M \geq 2q_0 - 1$ and so $n_0 \geq 4q_0 - 1$.

ii. Let x be such that $U(\mathcal{F}_{\bar{x}}) \neq U(\mathcal{F})\backslash\{x\}$ and let $y \neq x$ such that $y \notin U(\mathcal{F}_{\bar{x}})$. By Theorem 3.2.2 we have that $|\mathcal{F}_y| + 5 \leq |\mathcal{F}_x|$. Thus, $\frac{n_0-1}{2} \geq |\mathcal{F}_x| \geq |\mathcal{F}_y| + 5 \geq |\mathcal{F}_1| + 5 \geq 2q_0$, by the previous corollary and the result follows. ∎

As pointed out earlier, this result was rediscovered in a slightly different form by Roberts and Simpson [22]. In the paper they prove that if q is the minimum cardinality of $U(\mathcal{F})$ taken over all counterexamples (assuming the conjecture fails), then any other counterexample must have at least $4q - 1$ sets. However, to prove this, the authors prove some auxiliary results and in the proof of Theorem 3 there is a flaw. The authors define the subfamily $\mathcal{D} := \mathcal{F}\backslash\{U(\mathcal{F})\}\backslash\mathcal{C}$ and claim that, if $x \in H$ and $a \notin H$, then $|\mathcal{D}_x| = n - q$ and $|\mathcal{D}_a| < n - q$, where $n = \frac{|\mathcal{F}|-1}{2}$ and $q = |U(\mathcal{F})|$. To prove so, they assume w.l.o.g. that $|\mathcal{D}_a|$ is maximal for $a \in S\backslash H$. Then they consider two different cases, but there is one case missing. First, he obtains a contradiction if there exists $b \neq a$ such that $a \notin U(\mathcal{F}_{\bar{b}})$. Since the contradiction is obtaining using the maximality of a, then it means that the $b \in S\backslash H$. But in the second case, the authors consider is when $a \in U(\mathcal{F}_{\bar{b}})$, for all $b \neq a$, with $b \in S$. Nothing is said about the case when there is $b \in H$ such that $a \notin U(\mathcal{F}_{\bar{b}})$.
Still, this result is strong because improving a lower bound on the universe of the family for which the conjecture is known to hold also improves the lower bound on member-sets for which the conjecture is known to hold.

Chapter 4

The Salzborn Formulation

In this chapter, we present a very surprising equivalent formulation of the union-closed sets conjecture attributed to Salzborn by Piotr Wójcik [27]. The main advantage that this formulation seems to have over the usual one is that it only concerns a subclass of union-closed families, which we call *normalized*. In Section 4.2 we study the structure of normalized families and then we present the main result, which is the new formulation of the conjecture. The ideas for the latter part of the section are based on the ones in Piotr Wójcik's paper [27]. In the last section, we present different ways of formalizing the ideas of the previous section based on the ideas from Vaughan and Johnson [14] and some other results concerning normalized families and the conjecture are obtained.

4.1 Notation

Throughout this chapter, we use the following notation: Given a family of sets $\mathcal{F}$, and $X \subseteq U(\mathcal{F})$ we set

- $\mathcal{F}_{\subseteq X} := \{A \in \mathcal{F} : A \subseteq X\}$
- $\mathcal{F}_{\nsubseteq X} := \{A \in \mathcal{F} : A \nsubseteq X\}$
- $\mathcal{F}_{\supseteq X} := \{A \in \mathcal{F} : A \supseteq X\}$
- $\mathcal{F}_{\nsupseteq X} := \{A \in \mathcal{F} : A \nsupseteq X\}$
- $\mathcal{F}_{X} := \{A \in \mathcal{F} : A \cap X \neq \varnothing\}$

- The family of all $\cup$–irreducible sets of a family $\mathcal{F}$ is denoted by $J(\mathcal{F})$.
- As done before, given a set $S \subseteq U(\mathcal{F})$, $\mathcal{F} \ominus S$ will denote the family $\{X \backslash S : X \in \mathcal{F}\}$. Also, $\mathcal{F}^c = \{X^c \mid X \in \mathcal{F}\}$ and we will call U_i to $U(\mathcal{F}_{\bar{i}})$, as done in Chapter 1.

4.2 The Salzborn formulation

We now define a subclass of families that will play a central role in this chapter.

Definition 4.2.1. *A family of sets $\mathcal{F}$ is said to be* normalized *if it's a separating union-closed family such that $\emptyset \in \mathcal{F}$ and $|U(\mathcal{F})| = |\mathcal{F}| - 1$.*

We will also call normalized to families which are separating, without the empty set and such that $|U(\mathcal{F})| = |\mathcal{F}|$.

Corollary 1.2.8 reinforces the idea that the condition of being normalized is a strong one, in the sense that a separating family must have at least n sets and normalized families have exactly $n + 1$ sets. In fact, every set of a normalized family $\mathcal{F}$ is either the universe of the family or a set of the form U_i, for some $i \in U(\mathcal{F})$. Since none of the sets U_i is equal to $U(\mathcal{F})$ because every element of the universe must belong to some set in the family, we have that there exists $x \in U(\mathcal{F})$ such that $U_x = \emptyset$, which means that x belongs to every set in the family.

Proposition 4.2.2. *Let $\mathcal{F}$ be a separating union-closed family of sets. There is an element $x \in U(\mathcal{F})$ such that $U(\mathcal{F}) \backslash \{x\} \in \mathcal{F}$.*

Proof. We will deal with the family $\mathcal{F}^c$ and prove it has a singleton. We start by observing that $\mathcal{F}^c$ is also separating since if there is a set in $\mathcal{F}$ such that $x_1 \in S$ and $x_2 \notin S$, for some $x_1, x_2 \in U(\mathcal{F})$, the $S^c \in \mathcal{F}^c$ and $x_2 \in S^c$, $x_1 \notin S^c$. Obviously, we can assume that there is a set $S \in \mathcal{F}$ such that $|S| > 1$. Let $S = \{x_1, \dots, x_k\}$ be an arbitrary set in $\mathcal{F}^c$. Let $T \in \mathcal{F}^c$ such that $x_1 \in T$ and $x_2 \notin T$ or vice-versa. Then $S \cap T$ is a set that belongs to the family (since $\mathcal{F}^c$ is intersection-closed) whose cardinality is smaller than the cardinality of S, but still greater or equal to one, since x_1 or x_2 belong to $S \cap T$, depending on which of the elements belongs to T. Iterating this process, it is obvious that there must be a singleton in $\mathcal{F}^c$. ■

Similarly, we see that in a totally separating union-closed family, all sets of the form $U(\mathcal{F})\backslash\{x\}$, $x \in U(\mathcal{F})$ belong to the family, which in particular implies that there are no normalized totally separating union-closed families of sets.

Proposition 4.2.3. *Let $\mathcal{F}$ be an union-closed family and $i, j, k \in U(\mathcal{F})$. If $U_i = U_j \cup U_k$, then $\mathcal{F}_i \subseteq \mathcal{F}_j$ and $\mathcal{F}_i \subseteq \mathcal{F}_k$. In particular, $|\mathcal{F}_i| \leq |\mathcal{F}_j|$ and $|\mathcal{F}_i| \leq |\mathcal{F}_k|$.*

Proof. Suppose we have $U_i = U_j \cup U_k$. Let $S \in \mathcal{F}_{\bar{j}}$ and suppose $i \in S$. Then, $i \in S \subseteq U_j \subseteq U_i$, which is absurd since $i \notin U_i$. Hence, $S \in \mathcal{F}_{\bar{j}} \Rightarrow i \notin S$. We can proceed similarly for k. Then we have $i \in S \Rightarrow \{j, k\} \subseteq S$ and the result follows. ∎

Now, we present a concept strictly related with the class of normalized families.

Definition 4.2.4. *Given a family of sets $\mathcal{F}$, the family $\mathcal{F}^* = \{\mathcal{F}_X : X \subseteq U(\mathcal{F})\}$ is called the dual family of $\mathcal{F}$.*

Now, we present some lemmas that give us structural properties of the dual family of an union-closed family $\mathcal{F}$.

Lemma 4.2.5. *Let $\mathcal{F}$ be any family of sets. Then:*

i. *$\mathcal{F}^*$ is union-closed ;*

ii. *$\varnothing \in \mathcal{F}^*$;*

iii. *$U(\mathcal{F}^*) = \mathcal{F}\backslash\{\varnothing\}$.*

Proof. Let $S, T \in \mathcal{F}^*$. By definition, $S = \mathcal{F}_X$ and $T = \mathcal{F}_Y$, for some $X, Y \subseteq U(\mathcal{F})$. Obviously, $X \cup Y \subseteq U(\mathcal{F})$ and we have $\mathcal{F}_{X \cup Y} = \mathcal{F}_X \cup \mathcal{F}_Y$ since the intersection of a set with $X \cup Y$ is nonempty if and only if its intersection with X or Y is nonempty. Hence, $i.$ holds. Also, $\mathcal{F}_\varnothing = \varnothing$ and so $ii.$ holds. It is clear that $U(\mathcal{F}^*) \subseteq \mathcal{F}\backslash\{\varnothing\}$ and since $\varnothing \neq X \in \mathcal{F}$, implies $X \in \mathcal{F}_X \subseteq U(\mathcal{F}^*)$, one sees that $\mathcal{F}\backslash\{\varnothing\} \subseteq U(\mathcal{F}^*)$, and so $iii.$ holds. ∎

Lemma 4.2.6. *Let $\mathcal{F}$ be an union-closed family with $\varnothing \in \mathcal{F}$ and let $\mathcal{G}$ be a generating subfamily of $\mathcal{F}$, i.e. every set in $\mathcal{F}$ is the union of sets from $\mathcal{G}$. Then:*

i. *$\mathcal{G}^* = \{\mathcal{G}_{\notin X} \mid X \in \mathcal{F}\}$,* *ii.* $|\mathcal{G}^*| = |\mathcal{F}|$,

iii. $\mathcal{G}^*$ *is separating, i.e., for every* $X, Y \in U(\mathcal{G}^*), X \neq Y,$ *there exists a set in* $\mathcal{G}^*$ *which separates* X *and* Y.

iv. $J(\mathcal{G}^*) \subseteq \{\mathcal{G}_x | x \in U(\mathcal{G})\} \cup \{\varnothing\}$.

Proof.

i. We have $U(\mathcal{G}) = U(\mathcal{F})$. Also, notice that $\mathcal{G}_{\nsubseteq C} = \mathcal{G}_{C^c}$. By definition, $\mathcal{G}^* = \{\mathcal{G}_X : X \subseteq U(\mathcal{G})\} = \{\mathcal{G}_X : X \subseteq U(\mathcal{F})\} = \{\mathcal{G}_{X^c} : X \subseteq U(\mathcal{F})\} = \{\mathcal{G}_{\nsubseteq X} : X \subseteq U(\mathcal{F})\}$. We will now see that for each $X \subseteq U(\mathcal{F})$ and $G \in \mathcal{G}$, one has $G \subseteq X$ iff $G \subseteq U(\mathcal{G}_{\subseteq X})$. Let X and G in the conditions defined. It is obvious that if $G \subseteq X$, then $G \subseteq U(\mathcal{G}_{\subseteq X})$ since $G \in \mathcal{G}_{\subseteq X}$. Conversely, we have $G \subseteq U(\mathcal{G}_{\subseteq X}) \subseteq X$. Hence, $\mathcal{G}^* = \{\mathcal{G}_{\nsubseteq X} \mid X \subseteq U(\mathcal{F})\} = \{\mathcal{G}_{\nsubseteq U(\mathcal{G}_{\subseteq X})} \mid X \subseteq U(\mathcal{F})\}$.

   We now claim that $\{U(\mathcal{G}_{\subseteq X}) \mid X \subseteq U(\mathcal{F})\} = \mathcal{F}$, and $i.$ follows. Since $\mathcal{F}$ is union-closed, it is clear that $\{U(\mathcal{G}_{\subseteq X}) \mid X \subseteq U(\mathcal{F})\} \subseteq \mathcal{F}$. Conversely, let $S \in \mathcal{F}$. Then $S = U(\mathcal{G}_{\subseteq S})$, because $\mathcal{G}$ is a subfamily of generators.

ii. It is obvious from $i.$ that $|\mathcal{G}^*| \leq |\mathcal{F}|$. But it is true that for every $X, Y \in \mathcal{F}$, $X \neq Y \Rightarrow \mathcal{G}_{\nsubseteq X} \neq \mathcal{G}_{\nsubseteq Y}$, since if $\mathcal{G}_{\nsubseteq X} = \mathcal{G}_{\nsubseteq Y}$ then $X = U(\mathcal{G}_{\subseteq X}) = U(\mathcal{G}_{\subseteq Y}) = Y$, and so, $ii.$ holds.

iii. Let $X, Y \in U(\mathcal{G}^*) = \mathcal{G} \backslash \varnothing$, $X \neq Y$. Assume that $X \nsubseteq Y$ (the reciprocal case is analogous). Then $X \in \mathcal{G}_{\nsubseteq Y}$ and $Y \notin \mathcal{G}_{\nsubseteq Y}$. So, $\mathcal{G}_{\nsubseteq Y}$ separates X and Y. By $i.$ we have $\mathcal{G}_{\nsubseteq Y} \in \mathcal{G}^*$.

iv. Note that $\mathcal{G}_{X \cup Y} = \mathcal{G}_X \cup \mathcal{G}_Y$. Let $M \in J(\mathcal{G}^*)$. Then, by definition of $\mathcal{G}^*$, $M = \mathcal{G}_X$, for some $X \subseteq U(\mathcal{G})$. If $|X| = 1$, then we are done. Else, $|X| = s \geq 2$, say $X = \{x_1, x_2, \cdots, x_s\}$, then $M = \bigcup_{i=1}^{s} \mathcal{G}_{x_i}$. If there exists x_i such that $M = \mathcal{G}_{x_i}$ we are done. If not, we have that M is not an element in $J(\mathcal{G}^*)$ because $\mathcal{G}_{x_i} \in \mathcal{G}^*$, $\forall i \in [s]$. ∎

Remark 8. *Let* $X, Y \in \mathcal{F}$. *If* $X \subseteq Y$, *then for every set* Z *such that* $Z \nsubseteq Y$, *one has* $Z \nsubseteq X$, *which means that in this case,* $\mathcal{G}_{\nsubseteq Y} \subseteq \mathcal{G}_{\nsubseteq X}$. *From* $i.$ *of the previous lemma, it follows that the lattice of* $\mathcal{G}^*$ *is the dual lattice of* $\mathcal{F}$. *Also, in* $ii.$, *we show that the map from* $\mathcal{F}$ *to* $\mathcal{G}^*$ *that maps* X *to* $\mathcal{G}_{\nsubseteq X}$ *is a bijection.*

Lemma 4.2.7. *If* $\mathcal{F}$ *is a union-closed family of sets with* $\varnothing \in \mathcal{F}$, *then* $\mathcal{F}^*$ *is a normalized family and* $|\mathcal{F}^*| = |\mathcal{F}|$.

Proof. Follows from previous lemmas with $\mathcal{G} = \mathcal{F}$ in the previous lemma. ∎

The next lemma exhibits a property that suggests that normalized families have a particularly regular structure, in some sense.

Lemma 4.2.8. *If $\mathcal{F}$ is a normalized family of sets, then for every $X \in \mathcal{F}, |\mathcal{F}_{\nsupseteq X}| = |X|$.*

Proof. For each $x \in U(\mathcal{F})$, let $T_{\mathcal{F}}(x) = U(\mathcal{F}_{\bar{x}})$. We will show that for each $X \in \mathcal{F}$ and $x \in U(\mathcal{F})$ we have:

i. $x \in X$ iff $X \nsubseteq T_{\mathcal{F}}(x)$,

ii. $T_{\mathcal{F}}$ is a bijection from $U(\mathcal{F})$ to $\mathcal{F} \backslash U(\mathcal{F})$ and

iii. $T_{\mathcal{F}}(X) = \mathcal{F}_{\nsupseteq X}$.

Clearly, Lemma 4.2.8 follows from properties $ii.$ and $iii.$ since from $ii.$ one has $|T_{\mathcal{F}}(X)| = |X|$ and from $iii.$ it follows that $|T_{\mathcal{F}}(X)| = |\mathcal{F}_{\nsupseteq X}|$.

Suppose $x \in X$. Since $x \notin T_{\mathcal{F}}(x)$, then $X \nsubseteq T_{\mathcal{F}}(x)$. Now, suppose $X \nsubseteq T_{\mathcal{F}}(x)$. Then $X \notin \mathcal{F}_{\bar{x}}$ and so $x \in X$. So $i.$ holds.

To check ii., let $y, z \in U(\mathcal{F}), y \neq z$. Since $\mathcal{F}$ is separating we can consider $A \in \mathcal{F}$ that separates y and z. We can assume without loss of generality $y \in A$ (and so, $z \notin A$). Then, by i. we have $A \nsubseteq T_{\mathcal{F}}(y)$ and $A \subseteq T_{\mathcal{F}}(z)$, so $T_{\mathcal{F}}(z) \neq T_{\mathcal{F}}(y)$. Thus $T_{\mathcal{F}}$ is an injection.
Given that $|U(\mathcal{F})| = |(\mathcal{F} \backslash U(\mathcal{F}))|$, since $\mathcal{F}$ is normalized and that $T_{\mathcal{F}}(U(\mathcal{F})) \subseteq \mathcal{F} \backslash U(\mathcal{F})$, because for all $x \in U(\mathcal{F})$, $T_{\mathcal{F}}(x) = U(\mathcal{F}_{\bar{x}}) \in \mathcal{F}$ and $x \notin T_{\mathcal{F}}(x)$, one has ii..

By i., $T_{\mathcal{F}}(X) \subseteq \mathcal{F}_{\nsupseteq X}$. Now if $x \in T_{\mathcal{F}}^{-1}(\mathcal{F}_{\nsupseteq X})$, then $X \nsubseteq T_{\mathcal{F}}(x)$ and so, by i. we have $x \in X$. Hence, $T_{\mathcal{F}}^{-1}(\mathcal{F}_{\nsupseteq X}) \subseteq X$. The last inclusion yields $\mathcal{F}_{\nsupseteq X} \subseteq T_{\mathcal{F}}(X)$. Therefore, iii. holds. ∎

The next theorem provides an equivalent formulation of the union-closed sets conjecture that concerns solely normalized families, which sounds very surprising because normalization is a very strong condition to impose on the families to study. However, this formulation hasn't yet produced any strong new results about union-closed families.

Theorem 4.2.9. *Let $n \geq 2$ and $k \geq 0$. The following assertions are equivalent:*

i. For every union-closed family $\mathcal{F}$ with $|\mathcal{F}| = n$ there exists an element $x \in U(\mathcal{F})$ such that $|\mathcal{F}_x| \geq k$.

ii. For every union-closed family $\mathcal{F}$ with $|\mathcal{F}| = n$ and $\varnothing \in \mathcal{F}$ there exists a $\cup$−irreducible set M such that $|\mathcal{F}_{\not\supseteq M}| \geq k$.

iii. For every normalized family $\mathcal{F}$ with $|\mathcal{F}| = n$ there exists a $\cup$−irreducible set M of $\mathcal{F}$ such that $|\mathcal{F}_{\not\supseteq M}| \geq k$.

iv. For every normalized family $\mathcal{F}$ with $|\mathcal{F}| = n$ there exists a $\cup$−irreducible set M of $\mathcal{F}$ such that $|M| \geq k$.

Proof. Let us fix n and k. Clearly $ii.$ implies $iii.$. The equivalence of $iii.$ and $iv.$ follows from the previous lemma. So we will only prove $i. \Rightarrow ii.$ and $iv. \Rightarrow i.$.

$i. \Rightarrow ii.$. Suppose $i.$ holds. Let $\mathcal{F}$ be an union-closed family of sets such that $\varnothing \in \mathcal{F}$ and let $n = |\mathcal{F}|$ and $\mathcal{G} = J(\mathcal{F})$. It is obvious that $\mathcal{G}$ satisfies the conditions of Lemma 4.2.6. Applying Lemmas 4.2.5 and 4.2.6 ($i.$ of the former $ii.$ of the latter, in particular) we get that $\mathcal{G}^*$ satisfies assumptions of $i.$. Therefore, there exists an element $G \in U(\mathcal{G}^*)$ such that $|\mathcal{G}^*|_{\ni G} \geq k$. So $G \in \mathcal{G}$ by Lemma 4.2.5 ($iii.$). Hence, Lemma 4.2.6 ($i.$) gives $\mathcal{G}^*_{\ni G} = \{\mathcal{G}_{\not\supseteq X} | X \in \mathcal{F}$ and $G \in \mathcal{G}_{\not\supseteq X}\} = \{\mathcal{G}_{\not\supseteq X} | X \in \mathcal{F}_{\not\supseteq G}\}$.

Thus, $|\mathcal{F}_{\not\supseteq G}| = |\mathcal{G}^*_{\ni G}| \geq k$ and $ii.$ follows.

$iv. \Rightarrow i.$. Suppose now that $iv.$ holds. Let $\mathcal{F}$ be an union-closed family of sets such that $\varnothing \in \mathcal{F}$ and let $|\mathcal{F}| = n$. By Lemma 4.2.7, $\mathcal{F}^*$ satisfies the conditions of $iv.$. From $iv.$ and Lemma 4.2.6 ($iv.$) it follows that there exists $x \in U(\mathcal{F})$ such that $\mathcal{F}_x \in J(\mathcal{F}^*)$ and $|\mathcal{F}_x| \geq k$. So $i.$ holds. ∎

Note that in case $k = \frac{n}{2}$ we get different formulations of the union-closed sets conjecture. The one in $iv.$ is known as the Salzborn formulation of Frankl conjecture.

4.3 Other constructions

In the previous section, we present a way to build a dual family of $\mathcal{F}$, $\mathcal{F}^*$. Here we present an alternative way to build similar dual families as the one above, and some ideas based on the ones proposed by Vaughan and Johnson in [14]. Consider an union-closed family $\mathcal{F} = \{X_1, \cdots, X_n\}$ with universe $[m]$ and a subfamily $\mathcal{H} =$

$\{H_1, \cdots, H_s\} \subseteq \mathcal{F}$. For each $x \in [m]$, set $H(x) = \{i \in [s] : x \in H_i\}$, $\bar{H}(x) = \{i \in [s] : x \notin H_i\}$ and $H'(\mathcal{F}) = \{H(x) : x \in U(\mathcal{H})\}$. Then we can define $\mathcal{H}^+$ as the family generated by $H'(\mathcal{F})$. If we recall $iv.$ of Lemma 4.2.6, it becomes clear that when $\mathcal{H}$ is a subfamily of generators, $\mathcal{H}^+$ and $\mathcal{H}^*$ are basically the *same* except that we replace each set by its index. It is very easy to see that the proof of the equivalence between the Salzborn formulation and the usual one could be adapted using this new way of constructing the sets. This is specially useful because it allows us to easily create normalized families with any size we want by constructing $\mathcal{F}^+$, which is always normalized, like $\mathcal{F}^*$.

Example 7. *Suppose we want to construct a normalized family of sets with size* 6*. To do so, we need an union-closed family with* 6 *sets, for example,* $\mathcal{F} = \mathcal{P}([3])\backslash\{\varnothing, \{1\}\} = \{\{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$*. One has* $F(1) = \{3,4,6\}$*,* $F(2) = \{1,3,5,6\}$ *and* $F(3) = \{2,4,5,6\}$*. Now, we simply build*

$$\begin{aligned}\mathcal{F}^+ &= \langle\{\{3,4,6\}, \{1,3,5,6\}, \{2,4,5,6\}\}\rangle \\ &= \{\{3,4,6\}, \{1,3,5,6\}, \{2,4,5,6\}, \{1,3,4,5,6\}, \{2,3,4,5,6\}, \{1,2,3,4,5,6\}\},\end{aligned}$$

which is normalized with size 6*.*

Let $\mathcal{F}$ be an union-closed family of sets and $\mathcal{G} = \{G_1, \cdots, G_t\}$ a subfamily of generators. The following results appear in [14] in the particular case where the family of generators is $J(\mathcal{F})$. Since these results hold in general we present the general case.

Lemma 4.3.1. *Let* $X \in \mathcal{F}$ *and suppose* $X \neq U(\mathcal{F})$*, say* $X = \mathcal{U}\backslash\{y_1, \cdots y_k\}$*. Put* $I = \bigcap_{i=1}^{k} \bar{G}(y_i)$*. Then* $X = \bigcup_{j \in I} G_j$*.*

Proof. Let J be the set of all indices i such that $G_i \subseteq X$. Then, since $X \in \mathcal{F}$, $X = \bigcup_{j \in J} G_j$. Set $I = \bigcap_{i=1}^{k} \bar{G}(y_i)$. If $j \in J$ then $G_j \subseteq X$, which means $j \in \bar{G}(y_i), \forall i \in [k]$, i.e., $j \in I$. So, $J \subseteq I$. Conversely, $j \in I \Rightarrow j \in \bar{G}(y_i), \forall i \in [k] \Rightarrow \forall y_i,\, y_i \notin G_j \Rightarrow G_j \subseteq X \Rightarrow j \in J$. Hence, we have $I = J$ and the lemma follows. ∎

Next, we define the correspondence β from $\mathcal{F}\backslash\{U(\mathcal{F})\}$ to $(\mathcal{G}^+)^c\backslash\{\varnothing\}$ that maps $X = U(\mathcal{F})\backslash\{y_1, \cdots, y_k\}$ to $\bigcap_{i=1}^{k} \bar{G}(y_i)$.

Lemma 4.3.2. *Let* $X, Y \in \mathcal{F}$*.*

i. $X = \bigcup_{i \in \beta(X)} G_i$ *;*

ii. $\beta(X) = \beta(Y)$ *if and only if* $X = Y$ *;*

iii. If $X \subseteq Y$*, then* $\beta(X) \subseteq \beta(Y)$ *;*

iv. $\beta(X) \cup \beta(Y) \subseteq \beta(X \cup Y)$*.*

Proof.

i. This follows from the previous lemma.

ii. This follows from $i.$.

iii. Let $X = U(\mathcal{F})\backslash\{y_1, \ldots, y_k\}$ for some k. and suppose $X \subseteq Y$. Then, one has that $Y = U(\mathcal{F})\backslash\{y_1, \ldots, y_r\}$ for some $r \leq k$. Then $\beta(X) = \bigcap_{i=1}^{k} \bar{G}(y_i) \subseteq \bigcap_{i=1}^{r} \bar{G}(y_i) = \beta(Y)$.

iv. From $iii.$ we know that $\beta(X) \subseteq \beta(X \cup Y)$ and $\beta(Y) \subseteq \beta(X \cup Y)$. Therefore, we have $\beta(X) \cup \beta(Y) \subseteq \beta(X \cup Y)$. ∎

So, we have that this correspondence is order-preserving (from $iii.$) and one-to-one (from $ii.$). Now, for every nonempty set $S \in (\mathcal{G}^+)^c$, we have $S = (\bigcup_{x \in X} G(x))^c = \bigcap_{x \in X} \bar{G}(x)$ for some $X \subseteq U(\mathcal{F})$. We can also see that $S = \beta(\bigcup_{j \in S} G_j)$ and so the correspondence is onto. Let us call $T = \bigcup_{j \in S} G_j$. We have $\beta(\bigcup_{j \in S} G_j) = \bigcap_{t \in T^c} \bar{G}(t)$. Now, if $t \in T^c$, then $S \subseteq \bar{G}(t)$ because t is not element of any G_i, $i \in S$ and so $S \subseteq \beta(\bigcup_{j \in S} G_j)$. Conversely, we have $x \in X$, and so $x \in T^c$, since an element in X does not belong to any G_i, $i \in S$. Therefore, $X \subseteq T^c \Rightarrow \beta(\bigcup_{j \in S} G_j) = \bigcap_{t \in T^c} \bar{G}(t) \subseteq \bigcap_{t \in X} \bar{G}(t) = S$. The following theorem follows trivially:

Theorem 4.3.3. *Let* $\mathcal{F}$ *be an union-closed family of sets and* $\mathcal{G}$ *a subfamily of generators. Then,* $|\mathcal{F}| = |\mathcal{G}^+|$*.*

Proof. This follows immediately form the previous lemma and the observation above. ∎

This is not surprising because we knew from the previous section that this was the case (Lemma 4.2.6, $ii.$). Also, the fact that the correspondence between $\mathcal{F}$ and $(\mathcal{G}^+)^c$ is order preserving implies that both of these families have the same lattice structure, which we already knew from the previous section (the lattice correspondent to $\mathcal{F}^+$ is the dual lattice of the one corresponding to $\mathcal{F}$).

It is now very easy to show that the usual formulation of Frankl conjecture implies the Salzborn formulation using the particular case where $\mathcal{G} = J(\mathcal{F})$.

Theorem 4.3.4. *If $\mathcal{F}$ is a normalized family of n sets and $\mathcal{F}^+$ satisfies the union-closed sets conjecture, then $\mathcal{F}$ satisfies the Salzborn formulation of the conjecture.*

Proof. Let $\mathcal{F}$ be a normalized family of n sets and let $J(\mathcal{F}) = \{M_i : i \in [t]\}$, where $t = |J(\mathcal{F})|$. We have that $J'(\mathcal{F})$ is a family of n sets in $[t]$ and since $|\mathcal{F}^+| = n$ we know that $\mathcal{F}^+ = J'(\mathcal{F})$, and so we have that $J'(\mathcal{F})$ is union-closed. If $\mathcal{F}^+ = J'(\mathcal{F}) = \{X_1, \cdots X_n\}$ satisfies the union-closed sets conjecture, then there is an element $a \in [t]$ in at least half the sets of $\mathcal{F}^+$. Now by definition of $J'(\mathcal{F})$ $a \in X_i \Leftrightarrow i \in M_a$. Therefore, if we have a in at least $\frac{n}{2}$ sets of $\mathcal{F}^+$, we have $|M_a| \geq \frac{n}{2}$ and so $\mathcal{F}$ satisfies the Salzborn formulation of the union-closed sets conjecture. ∎

Corollary 4.3.5. *If $\mathcal{F}$ is an union-closed normalized family with universe $\mathcal{U}$ such that $\varnothing \notin \mathcal{F}$, then there is $x \in \mathcal{U}$ such that x is an element of every set of the family.*

Proof. Since $\mathcal{F}$ is normalized, we have that $J'(\mathcal{F})$ is union-closed. That means that $[t] = U(J'(\mathcal{F})) \in J'(\mathcal{F})$, where $t = |J(\mathcal{F})|$ and from the definition of $J'(\mathcal{F})$ used as above follows that there is an element a in every $\cup$–irreducible set, and that implies that a is in every set. ∎

We now present a generalization of the concept of separation in union-closed families, which we call independence. Independent families will have a relevant role in the construction of dual families.

Definition 4.3.6. *A family $\mathcal{F}$ of sets on $[n]$ is called* independent *if, $\forall i \in [n], \forall S \subseteq [n] - \{i\}$, it satisfies one of the following conditions :*

- *There is a set T in $\mathcal{F}$ such that $S \cap T \neq \varnothing \wedge i \notin T$;*
- *There is a set T in $\mathcal{F}$ such that $S \cap T = \varnothing \wedge i \in T$.*

We say that a family is *dependent* if it is not independent.

It is easy to see by the definition of independent families that they are in particular separating, when we have $|S| = 1$, the definition coincides with the definition of separating family. The following proposition will help understanding the relevance of this concept when studying dual families.

Proposition 4.3.7. *Let $\mathcal{I}$ be a family of sets. Then $\mathcal{I}$ is dependent if and only if there exists $i \in U(\mathcal{I})$ and $S \subseteq U(\mathcal{I}) - \{i\}$ such that $\mathcal{I}_i = \bigcup_{j \in S} \mathcal{I}_j$.*

Proof. Let $\mathcal{I}$ be a dependent family of sets. Then $\exists i \in U(\mathcal{I}), S \subseteq U(\mathcal{I}) \backslash \{i\}$ such that given a set $T \in \mathcal{I}$, one has that $S \cap T = \varnothing \vee i \in T$ and $S \cap T \neq \varnothing \vee i \notin T$. Hence, if $i \in T$, then $S \cap T \neq \varnothing$. If not, $S \cap T = \varnothing$. So, we have that if $T \in \mathcal{I}_i$, then there is $j \in S \cap T$, which implies that $T \in \mathcal{I}_j$ and so $T \in \bigcup_{k \in S} \mathcal{I}_k$. Thus $\mathcal{I}_i \subseteq \bigcup_{j \in S} \mathcal{I}_j$. Analogously, one has $\bigcup_{j \in S} \mathcal{I}_j \subseteq \mathcal{I}_i$, and so $\mathcal{I}_i = \bigcup_{j \in S} \mathcal{I}_j$.
Conversely, suppose there exists $i \in U(\mathcal{I})$ and $S \subseteq U(\mathcal{I}) \backslash \{i\}$ such that $\mathcal{I}_i = \bigcup_{j \in S} \mathcal{I}_j$. From $\mathcal{I}_i \subseteq \bigcup_{j \in S} \mathcal{I}_j$ it follows that $i \in T \Rightarrow \exists k \in S : k \in T$. From $\bigcup_{j \in S} \mathcal{I}_j \subseteq \mathcal{I}_i$, it follows that $\exists k \in S : k \in T \Rightarrow i \in T$. Hence, $\mathcal{I}$ is dependent. ∎

It is now easy to see the relation between independent families and dual families. If we have an independent family $\mathcal{F}$ and we construct $\mathcal{F}^+$ (or $\mathcal{G}^+$ for a $\cup$-generating subfamily $\mathcal{G}$), what we have before generating is a family of $\cup$-irreducible sets.

Given a totally separating union-closed family $\mathcal{F}$ we can easily see that $(J(\mathcal{F}))^+$ is generated by an antichain $\mathcal{A}$ in $[t]$ where t is the number of $\cup$-irreducible sets of $\mathcal{F}$ and $|\mathcal{A}| = |U(\mathcal{F})|$.

Remark 9. *If the family $\mathcal{F} = \{X_1, \cdots, X_n\}$ is totally separating, then $J(\mathcal{F}^+)$ is an antichain since $J(\mathcal{F}^+) \subseteq \{F(x) : x \in U(\mathcal{F})\}$, where $F(x) = \{t : x \in X_t\}$ and $\{F(x) : x \in U(\mathcal{F})\}$ is an antichain because $\mathcal{F}$ is totally separating. In this case, we can see that the join-irreducible sets of the normalized family $\mathcal{F}^+$ are exactly the sets of the form U_k, with $|\mathcal{F}_{\bar{k}}| = 1$. It is clear that, if $|\mathcal{F}_{\bar{k}}| = 1$, then U_k is join-irreducible. If we have a join-irreducible set J, then it does not have any proper subset in the family, because if it did, then it had a proper subset which is a join-irreducible set and that is not possible, since $J(\mathcal{F}^+)$ is an antichain. So, if some U_k is a join-irreducible set and has no subsets, then it is the only set that does not have k as its element.*

It follows the following theorem.

Theorem 4.3.8. *If $\mathcal{F}$ is a totally separating union-closed family with $|U(\mathcal{F})| = m$ and t $\cup$-irreducible sets, then we have*

$$m \leq \binom{t}{\lfloor \frac{t}{2} \rfloor}.$$

Also, this condition is sharp.

Proof. The result follows from the observations above, together with Sperner's theorem that states that the largest antichain in $[t]$ has $\binom{t}{\lfloor \frac{t}{2} \rfloor}$ sets.
To see that the bound is sharp, given $t \in \mathbb{N}$, take $\mathcal{A}$ as the set of subsets of $[t]$ with size $\lfloor \frac{t}{2} \rfloor$. Consider $\mathcal{A}^+$. It is not hard to see that $\mathcal{A}$ is independent and so $\mathcal{A}^+$ has t $\cup$-irreducible sets. ∎

Theorem 4.3.9. *[Kleitman, [7]] If $\mathcal{F}$ is a separating union-closed family with $|U(\mathcal{F})| = m$ and t $\cup$-irreducible sets, then we have*

$$t \leq \binom{m}{\lfloor \frac{m}{2} \rfloor} + \frac{2^m}{m}.$$

Corollary 4.3.10. *If $\mathcal{I}$ is a family such that $|\mathcal{I}| = m$ and $|U(\mathcal{I})| = t \geq \binom{m}{\lfloor \frac{m}{2} \rfloor} + \frac{2^m}{m}$, then $\mathcal{I}$ is not independent.*

Proof. Notice that if $\mathcal{I}$ was independent, then $\mathcal{I}^+$ would contradict Theorem 4.3.9. ∎

Note that any improvement on this bound will result on an improvement of the previous one.

It is very easy to give examples of a normalized family of any size, if we consider the *staircase family* $\{[1], \cdots, [n]\}$. But the example above shows us a process that can be used to build every normalized family with size n by choosing different original union-closed families as the next lemma shows.

Lemma 4.3.11. *Let $\mathcal{G}$ be a normalized family of sets. Then $\mathcal{G} = \mathcal{F}^+$ for some $\mathcal{F}$ independent union-closed family of sets.*

Proof. Let $\mathcal{G} = \{G_1, \cdots, G_n\}$ be a normalized family of n sets and $J(\mathcal{G}) = \{M_1, \cdots, M_t\}$. We will see that $\mathcal{G} = (J(\mathcal{G})^+)^+$. The family $(J(\mathcal{G})^+)^+$ is union-closed. Therefore, to show that $\mathcal{G} \subseteq (J(\mathcal{G})^+)^+$ it is enough to show $J(\mathcal{G}) \subseteq (J(\mathcal{G})^+)^+$. Fixed $i \in [t]$, let $\{x_1, \cdots, x_k\} = M_i \in J(\mathcal{G})$ and consider the family $J(\mathcal{G})^+ = \langle \{Y_1, \cdots, Y_n\} \rangle$, where $Y_i = \{r \in [t] \mid i \in M_r\}$. Since $|J(\mathcal{G})^+| = |\mathcal{G}| = n$, one has that $J(\mathcal{G})^+ = \{Y_1, \cdots, Y_n\}$. By construction, $J(\mathcal{G})^+_i = \{Y_{x_1}, \cdots Y_{x_k}\}$ and analogously one gets that $M_i = \{x_1, \cdots, x_k\} \in (J(\mathcal{G})^+)^+$. We have proved that $\mathcal{G} \subseteq (J(\mathcal{G})^+)^+$.

Conversely, we have to show that $J((J(\mathcal{G})^+)^+) \subseteq \mathcal{G}$. Since $(J(\mathcal{G})^+)_i = \{Y_{x_1}, \cdots Y_{x_k}\}$, we have that $(J(\mathcal{G})^+)^+ = \langle \{M_1, \cdots, M_t\} \rangle$ and thus $J((J(\mathcal{G})^+)^+) = \{M_1, \cdots, M_t\} \subseteq \mathcal{G}$. Also, since $\{M_1, \cdots, M_t\}$ is a family of $\cup$-irreducible sets, we have that $J(\mathcal{G})^+$ is independent and the result follows. ∎

Example 8. *Let $\mathcal{F}$ be the following normalized family of size 7:*

$$\mathcal{F} = \{\{1,4,6,7\},\{2,5,6,7\},\{3,4,5,6\},[7]-\{3\},[7]-\{2\},[7]-\{1\},[7]\}.$$

We have $J(\mathcal{F}) = \{\{1,4,6,7\},\{3,4,5,6\},\{2,5,6,7\}\}$. We will build $(J(\mathcal{F})^+)^+$ and see that it coincides with $\mathcal{F}$ as the previous lemma guarantees.

$$\begin{aligned} J(\mathcal{F})^+ &= \langle\{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}\rangle \\ &= \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}. \end{aligned}$$

Now, we see that $(J(\mathcal{F})^+)^+ = \langle\{\{1,4,6,7\},\{2,5,6,7\},\{3,4,5,6\}\}\rangle = \mathcal{F}$, as expected.

To emphasize how the concept of independence is stronger than the concept of separation we will show that there is only one independent normalized family with n sets, while by definition all normalized families are separating.

Proposition 4.3.12. *The only independent normalized family with n sets is the* staircase *family $\mathcal{F} = \{[1],\cdots,[n]\}$.*

Proof. It is easy to see that the staircase family is independent for every $n \in \mathbb{N}$. Now let $\mathcal{F}$ be an independent normalized family of sets. Then, $J'(\mathcal{F})$ is union-closed and, by independence, all its sets are $\cup$–irreducible. Let $X, Y \in J'(\mathcal{F})$. Then, $X \cup Y = X$ and so $Y \subseteq X$ or $X \cup Y = Y$ and so $X \subseteq Y$, by the irreducibility of sets in $J'(\mathcal{F})$. Then $J'(\mathcal{F})$ is a chain. Since that is the case, it cannot have two elements of the same rank in the lattice of $\mathcal{P}([n])$. Since it has n sets in $[t]$, where $t = |J(\mathcal{F})| \leq n$, then $t = n$ and it has exactly one element of each rank and it is a chain. Then it must be the staircase family. ∎

We already know that it suffices to prove Frankl Conjecture for separating union-closed families. We know prove a stronger result, that states that it is enough to prove the conjecture for independent families.

Theorem 4.3.13. *It is enough to prove the union-closed sets conjecture for independent families.*

Proof. Let $\mathcal{F}$ be an union-closed family of sets with universe $\mathcal{U}$. If $\mathcal{F}$ is not independent, then, by Proposition 4.3.7, there is some $X \subseteq U(\mathcal{F})$ and some $y \notin X$ such that $J(y) = \bigcup_{x \in X} J(x)$. Now, consider the family $\mathcal{F} \ominus \{y\}$. It is obvious the family is still union-closed and since we have for every $S \in \mathcal{F}$, $y \in S \Leftrightarrow S \cap X \neq \emptyset$ we know that the correspondence $S \mapsto S \backslash \{y\}$ is one-to-one and so $|\mathcal{F} \ominus \{y\}| = |\mathcal{F}|$. So, if the new family satisfies the conjecture, so does $\mathcal{F}$. If we do this for every element of the universe under the conditions of the theorem we get an independent family and the result follows. ∎

In Theorem 4.3.4, we proved that the usual formulation of Frankl conjecture implies the Salzborn formulation. Now it is easier to see a link between both formulations. It is clear that the size of the sets of the family we will generate corresponds to the frequency of elements in the original family. Although it is not true that the sets we get to generate are always $\cup-$ irreducible sets, that is the case if the original family is independent. So, if we have an independent union-closed family $\mathcal{G}$ and build $\mathcal{G}^{+}$, the sets we have before generating, $G(x)$ for $x \in U(G)$, are exactly $J(\mathcal{G}^{+})$. It now becomes *obvious* that the Salzborn formulation implies the usual one.

Chapter 5

Up-compression

In this chapter, we see how compression techniques can be used to study union-closed families. The idea of these techniques is to transform the object of study, which is in our case union-closed families, into another one of a special type (in our case upward-closed families) and then study this new object to obtain results about the initial one. Compression may also be referred to as shifting. This approach is typical in extremal combinatorics but regarding the study of Frankl Conjecture it was introduced by Reimer in [19], whose results we present in the first part of the chapter, and then followed by Rodaro in [23] and Balla, Bollobás and Eccles in [12]. In order to get an idea of how powerful the concept of shifting is, we recommend reading Gil Kalai's post in [15].

5.1 Notation

Throughout this chapter we use the following notation:

- If S and T are subsets of $[n]$ we denote by $[S,T]$ the set $\{X \subseteq [n] \mid S \subseteq X \subseteq T\}$ (which is empty when $S \nsubseteq T$);
- We denote $\log_2$ by $\log$;
- We denote the group of permutations of elements in S by $\mathfrak{S}_S$.

5.2 Upward-Closed Families

In this section, we consider a new class of families, which we call *upward-closed* families and present some of its properties

Definition 5.2.1. *A family of sets $\mathcal{F}$ is said to be* upward-closed *if, given a set $S \in \mathcal{F}$, we have that $[S, U(\mathcal{F})]$, i.e., every superset of a set of the family is itself a set of the family, as long as it is a subset of the universe. These families have also been called* up-closed *or* up-sets*. Analogously, a family $\mathcal{F}$ is called* downward-closed *if, given a set $S \in \mathcal{F}$, we have that $T \subseteq S \Rightarrow T \in \mathcal{F}$, i.e., every subset of a set of the family is itself a set of the family.*

It is obvious that an upward-closed family $\mathcal{F}$ is itself an union-closed family, since given two sets $A, B \in \mathcal{F}$, we have $A \cup B$ is a superset of A (and B) contained in $U(\mathcal{F})$ and so $A \cup B \in \mathcal{F}$. Also, it is easy to see that Conjecture 1 holds for every element in the universe of this families because the map $\varphi_a : \mathcal{F}_{\bar{a}} \to \mathcal{F}_a$ defined by $S \mapsto S \cup \{a\}$ is well defined and injective for all $a \in U(\mathcal{F})$.

Definition 5.2.2. *Given a family $\mathcal{G}$ of sets, the union-closed family* ↑-generated *by $\mathcal{G}$ is defined by $\mathcal{F} = \{S \subseteq U(\mathcal{G}) \mid \exists G \in \mathcal{G}$ such that $G \subseteq S\}$ and denoted by $\mathcal{F} = \langle \mathcal{G} \rangle_\uparrow$. It is easy to see that the set of minimal sets in $\mathcal{F}$ is the (unique) minimal ↑-generating set of the family $\mathcal{F}$.*

A family of sets (not necessarily union-closed) $\mathcal{F}$ such that $U(\mathcal{F}) = [n]$ can be seen as a directed graph whose vertices are the sets, and whose edges are pairs of sets that differ by exactly one element, directed from the smaller set to the larger set.

Notation 2. *Put $A \sqsubset B$ when there exists $i \in [n]$ such that $A \cup \{i\} = B$.*

Definition 5.2.3. *Let $A, B \in \mathcal{F}$. The* graph associated to *$\mathcal{F}$ has set of vertices $V(\mathcal{F})$ and set of edges $E(\mathcal{F})$, where*

$$V(\mathcal{F}) := \mathcal{F},$$
$$E(\mathcal{F}) := \{(A, B) \in \mathcal{F}^2 : A \sqsubset B\}.$$

We also define the edge boundary *of an upward closed family $\mathcal{F}$ as*

$$EB(\mathcal{F}) := \{(A, B) \in E(\mathcal{P}[n]) : A \notin \mathcal{F}, B \in \mathcal{F}\}.$$

Example 9. *Let* $\mathcal{F} = \{\{2\},\{1,3\},\{2,4\},\{1,2,4\}\}$*.One has*

$$E(\mathcal{F}) = \{(\{2\},\{2,4\}),(\{2,4\},\{1,2,4\})\},$$

and

$$EB(\mathcal{F}) = \{(\varnothing,\{2\}),(\{1\},\{1,3\}),(\{3\},\{1,3\}),\\ (\{4\},\{2,4\}),(\{1,4\},\{1,2,4\}),(\{1,2\},\{1,2,4\})\}.$$

Lemma 5.2.4. *For every upward closed family of sets* $\mathcal{F}$*, one has*

$$2\sum_{F\in\mathcal{F}}|F| = n|\mathcal{F}| + |EB(\mathcal{F})|$$

Proof. Start by noticing that for every $F\in\mathcal{F}$, $|F| = \sum_{G\sqsubset F}1$. Similarly, we have for every $F\in\mathcal{F}$, $n-|F| = \sum_{F\sqsubset G}1$. Now we have,

$$2\sum_{F\in\mathcal{F}}|F| = \sum_{F\in\mathcal{F}}|F| + \sum_{F\in\mathcal{F}}\sum_{G\in\mathcal{F}:G\sqsubset F}1 + \sum_{F\in\mathcal{F}}\sum_{G\notin\mathcal{F}:G\sqsubset F}1$$

Now, $\sum_{F\in\mathcal{F}}\sum_{G\in\mathcal{F}:G\sqsubset F}1 = \sum_{G\in\mathcal{F}}\sum_{G\sqsubset F}1$ since $\mathcal{F}$ is upward-closed. Hence,

$$\begin{aligned}2\sum_{F\in\mathcal{F}}|F| &= \sum_{F\in\mathcal{F}}|F| + \sum_{G\in\mathcal{F}}\sum_{G\sqsubset F}1 + |EB(\mathcal{F})|\\ &= \sum_{G\in\mathcal{F}}|G| + \sum_{G\in\mathcal{F}}(n-|G|) + |EB(\mathcal{F})|\\ &= \sum_{G\in\mathcal{F}}n + |EB(\mathcal{F})|\\ &= n|\mathcal{F}| + |EB(\mathcal{F})|.\end{aligned}$$

∎

Remark 10. *The number of ways we have to choose a set of (minimal) generators of an upward closed family with universe* $[m]$ *is the number of ways we have to choose a totally separating family with* m *sets, since those are exactly the antichains (see Remark 9).*

5.3 Compression

We start by defining a *rising function* that subjects a given family to small changes until we obtain an upward-closed family.

Definition 5.3.1. *Given a family of sets $\mathcal{F}$ with universe $[n]$, for $A \in \mathcal{F}$ and $i \in [n]$ define*

$$R_{\mathcal{F},i}(A) := \begin{cases} A, & \text{if } A \cup \{i\} \in \mathcal{F}, \\ A \cup \{i\}, & \text{if } A \cup \{i\} \notin \mathcal{F}. \end{cases}$$

Then define the i-rising of $\mathcal{F}$ by

$$R_i(\mathcal{F}) := \{R_i(A) : A \in \mathcal{F}\}.$$

When there is no doubt about which family we are referring to, we will simply write $R_i(A)$ instead of $R_{\mathcal{F},i}(A)$. It is easy to see that given a family $\mathcal{F}$, the map $A \mapsto R_i(A)$ is one-to-one. Now, given a family of sets $\mathcal{F}$ and a set $A \in \mathcal{F}$, define

$$\mathcal{F}_i := \begin{cases} \mathcal{F}, & \text{if } i = 0, \\ R_i(\mathcal{F}_{i-1}), & \text{if } i > 0. \end{cases}$$

$$A_i := (R_{\mathcal{F}_{i-1},i} \circ R_{\mathcal{F}_{i-2},i-1} \circ \cdots \circ R_{\mathcal{F}_0,1})(A).$$

If $A_i \neq A_{i-1}$ we say A_{i-1} rises.

Example 10. *When we apply up-compression to the union closed family*

$$\mathcal{F} = \{\{4\},\{1,2\},\{3,5\},\{1,2,4\},\{3,4,5\},\{1,2,3,5\},\{1,2,3,4,5\}\},$$

we obtain

- $\mathcal{F}_1 = \{\{1,4\},\{1,2\},\{1,3,5\},\{1,2,4\},\{1,3,4,5\},\{1,2,3,5\},\{1,2,3,4,5\}\}$
- $\mathcal{F}_2 = \{\{1,4\},\{1,2\},\{1,3,5\},\{1,2,4\},\{1,3,4,5\},\{1,2,3,5\},\{1,2,3,4,5\}\}$
- $\mathcal{F}_3 = \{\{1,3,4\},\{1,2,3\},\{1,3,5\},\{1,2,3,4\},\{1,3,4,5\},\{1,2,3,5\},\{1,2,3,4,5\}\}$
- $\mathcal{F}_4 = \{\{1,3,4\},\{1,2,3\},\{1,3,5\},\{1,2,3,4\},\{1,3,4,5\},\{1,2,3,5\},\{1,2,3,4,5\}\}$
- $\mathcal{F}_5 = \{\{1,3,4\},\{1,2,3\},\{1,3,5\},\{1,2,3,4\},\{1,3,4,5\},\{1,2,3,5\},\{1,2,3,4,5\}\}$

Lemma 5.3.2. *Given $A_{i-1} \in \mathcal{F}_{i-1}$, one has $A_{i-1} \cup \{i\} \in \mathcal{F}_i$ and $A_i \cup \{i\} \in \mathcal{F}_i$.*

Proof. If A_{i-1} rises, then $A_i = A_{i-1} \cup \{i\} \in \mathcal{F}_i$ and also $i \in A_i$, thus $A_i \cup \{i\} = A_i \in \mathcal{F}_i$. If, on the other hand, $A_i = A_{i-1}$, i.e., $R_{\mathcal{F}_{i-1},i}(A_{i-1}) = A_{i-1}$ one has $A_{i-1} \cup \{i\} \in \mathcal{F}_{i-1}$, by definition of the rising function. Also, $A_i \cup \{i\} = A_{i-1} \cup \{i\} = R_{\mathcal{F}_{i-1},i}(A_{i-1} \cup \{i\}) \in \mathcal{F}_i$. ∎

Lemma 5.3.3. *If $\mathcal{F}_{i-1}$ is union-closed and $C_{i-1}, D_{i-1} \in \mathcal{F}_{i-1}$ are such that $D_{i-1} = C_{i-1} \cup X$ for some $X \subseteq [n]$, then $C_i \cup X \in \mathcal{F}_i$. Furthermore, $C_{i-1} \neq C_{i-1} \cup X$ implies $C_i \neq C_i \cup X$.*

Proof. Suppose we have $C_i \cup X \notin \mathcal{F}_i$. We know that $C_{i-1} \cup X \cup \{i\} \in \mathcal{F}_i$ and so $C_{i-1} \cup X \cup \{i\} \neq C_i \cup X$. Therefore, C_{i-1} does not rise, i.e., $C_{i-1} = C_i$, and one has that C_i, X and D_{i-1} do not contain i. We know, from the previous lemma, that $C_{i-1} \cup \{i\} \in \mathcal{F}_{i-1}$, and by closure $(C_{i-1} \cup \{i\}) \cup D_{i-1} = D_{i-1} \cup \{i\} \in \mathcal{F}_{i-1}$. Hence, D_{i-1} does not rise and $C_i \cup X = C_{i-1} \cup X = D_{i-1} = D_i \in \mathcal{F}_i$, a contradiction.

Now, if $C_{i-1} \neq C_{i-1} \cup X$ and $C_i = C_i \cup X$ then we obviously have $C_i \neq C_{i-1}$ and C_{i-1} rises. But $X \subseteq C_{i-1} \cup \{i\}$ and $X \nsubseteq C_{i-1}$ implies $i \in X$ and so $C_i = (C_{i-1} \cup \{i\}) \cup X = C_{i-1} \cup X = D_{i-1} \in \mathcal{F}_{i-1}$, and one concludes that C_{i-1} does not rise, which is absurd. ∎

The following lemma shows us that applying the rising operator to a given union-closed family keeps the structure of the family in the sense that it preserves the union-closure property.

Lemma 5.3.4. *If $\mathcal{F}$ is union-closed, then $\mathcal{F}_i$ is union-closed for all $i \in \{0, 1, \cdots, n\}$.*

Proof. We will prove this by induction on i. In the case $i = 0$ there is nothing to prove. Now assume $\mathcal{F}_j$ is union-closed for all $j \in \{0, 1, \cdots, i-1\}$ and let A_i, $B_i \in \mathcal{F}_i$. Set $C_{i-1} = A_{i-1}$ and $D_{i-1} = A_{i-1} \cup B_{i-1} \in \mathcal{F}_{i-1}$ in the previous lemma. Then $A_i \cup D_{i-1} = A_i \cup A_{i-1} \cup B_{i-1} = A_i \cup B_{i-1} \in \mathcal{F}_i$. Now, $A_i \cup B_i$ is either $A_i \cup D_{i-1}$ or $A_i \cup D_{i-1} \cup \{i\}$. But, by Lemma 5.3.2, $A_i \cup D_{i-1} \in \mathcal{F}_i$ implies $A_i \cup D_{i-1} \cup \{i\} \in \mathcal{F}_i$ and we are done. ∎

The goal of the compression technique is to obtain an upward-closed family after applying the rising operator to an union-closed family $\mathcal{F}$. We will now see that this is the case, since $\mathcal{F}_n$ is always upward-closed. But first, we present an auxiliary Lemma, which is a consequence of Lemma 5.3.3.

Lemma 5.3.5. *Let $i, j \in [n]$, where $i < j$. Let C_i, $D_i \in \mathcal{F}_i$ such that $D_i = C_i \cup X$ for some $X \subseteq [n]$. Then $C_j \cup X \in \mathcal{F}_j$. Furthermore, $C_i \neq C_i \cup X$ implies $C_j \neq C_j \cup X$.*

Proof. This result is obtained by repeated application of Lemma 5.3.3. ∎

Proposition 5.3.6. *$\mathcal{F}_n$ is an upward closed family of sets.*

Proof. It is enough to show that $A_n \cup \{i\} \in \mathcal{F}_n$, for all $A \in \mathcal{F}$, $i \in [n]$. Let $i \in [n]$. We know, from Lemma 5.3.2, that $A_i \in \mathcal{F}_i$ implies $A_i \cup \{i\} \in \mathcal{F}_i$. Letting $C_i = A_i$, $X = \{i\}$ and $j = n$ in Lemma 5.3.5, one concludes that $A_n \cup \{i\} \in \mathcal{F}_n$. ∎

The next lemma, together with Lemma 5.2.4, will play a crucial role in order to obtain a lower bound for the average set size of a given union-closed family. To prove the next lemma we will use Jensen's inequality, which we recall.

Remark 11. *Jensen's inequality states that for a real convex function φ, numbers $x_1, x_2, \ldots, x_n$ in its domain, and positive weights a_i, one has*

$$\varphi\left(\frac{\sum a_i x_i}{\sum a_i}\right) \leq \frac{\sum a_i \varphi(x_i)}{\sum a_i}.$$

Lemma 5.3.7. *Let $\mathcal{F}$ be an union-closed family with $U(\mathcal{F}) = [n]$. Then, the map from $\mathcal{F}$ to $\mathcal{F}_n$, defined by $A \mapsto A_n$ is a bijection. Also, we have:*

1. $A \subseteq A_n$;
2. $A, B \in \mathcal{F}$, $A \neq B \Rightarrow [A, A_n] \cap [B, B_n] = \varnothing$;
3. $\sum_{A \in \mathcal{F}} |A_n - A| \leq |\mathcal{F}| \, (n - \log(|\mathcal{F}|))$.

Proof. It is obvious that $A \mapsto A_n$ is a bijection from $\mathcal{F}$ to the upward closed family $\mathcal{F}_n$ since it is the composition of bijections $R_{\mathcal{F}_{n-1},n} \circ R_{\mathcal{F}_{n-2},n-1} \circ \cdots \circ R_{\mathcal{F}_0,1}$. We now need to prove parts 1–3 of the lemma.

1. For every $i \in [n]$, $A_{i-1} \subseteq A_i$ by definition, so $A = A_0 \subseteq A_n$.
2. Let $A, B \in \mathcal{F}$ and suppose that $[A, A_n] \cap [B, B_n] \neq \varnothing$, i.e., there exists $Y \in [A, A_n] \cap [B, B_n]$. Since $A, B \subseteq Y$ we have $A \cup B \subseteq Y \subseteq A_n$. Hence, $A \cup B \in [A, A_n]$. By closure, $A \cup B \in \mathcal{F}$ and $[A, A_n] \cap [A \cup B, (A \cup B)_n] \neq \varnothing$. If $B \nsubseteq A$, then $A \neq A \cup B$. So, if we replace B with $A \cup B$ we have $A \subset B$. Therefore, if 2. does not hold, then it does not hold for some $A, B \in \mathcal{F}$ such that $A \subset B$, thus we may assume that.

   Let $C_i = A$, $X = B$, $i = 0$ and $j = n$ in Lemma 5.3.5. Since $A \neq A \cup B$, we get $A_n \neq A_n \cup B$ which implies $B \nsubseteq A_n$ and $B \notin [A, A_n]$, which is a contradiction since $B \subseteq Y \subseteq A_n$.

3. Since $A \subseteq A_n$, we know $|[A, A_n]| = 2^{|A_n \setminus A|}$. We have $\dot{\bigcup}_{A \in \mathcal{F}}[A, A_n] \subseteq \mathcal{P}(n)$, with the intervals being disjoint by 2., so

$$\sum_{A \in \mathcal{F}} |[A, A_n]| \leq 2^n$$

and obviously $\sum_{A \in \mathcal{F}} 2^{|A_n \setminus A|} = \sum_{A \in \mathcal{F}} |[A, A_n]| \leq 2^n$. Applying Jensen's inequality to the function $\varphi(x) = 2^x$, which is convex, with the weights being $a_i = 1, \forall i \in [|\mathcal{F}|]$, we get

$$2^{(|\mathcal{F}|^{-1} \sum_{A \in \mathcal{F}} |A_n \setminus A|)} \leq |\mathcal{F}|^{-1} \sum_{A \in \mathcal{F}} 2^{|A_n \setminus A|} \leq \frac{2^n}{|\mathcal{F}|}.$$

Then we have

$$|\mathcal{F}|^{-1} \sum_{A \in \mathcal{F}} |A_n - A| \leq n - \log(|\mathcal{F}|),$$

and so

$$\sum_{A \in \mathcal{F}} |A_n - A| \leq |\mathcal{F}|(n - \log(|\mathcal{F}|)).$$

■

Remark 12. *It is clear that, given an union-closed family $\mathcal{F}$ such that the average frequency of the elements is greater than half the size of the family, then there is at least one element in half the sets, and so Frankl conjecture holds for such $\mathcal{F}$, i.e., if $\frac{1}{|U(\mathcal{F})|} \sum_{x \in U(\mathcal{F})} |\mathcal{F}_x| \geq \frac{|\mathcal{F}|}{2}$, then $\mathcal{F}$ satisfies Frankl conjecture. In Section 1.3, we showed an example of an union-closed family $\mathcal{F}$ such that the average frequency is lower than $\frac{|\mathcal{F}|}{2}$. Since $\sum_{S \in \mathcal{F}} |S| = \sum_{x \in U(\mathcal{F})} |\mathcal{F}_x|$, then if the average set size is at least half the size of the universe, i.e., $\frac{1}{|\mathcal{F}|} \sum_{S \in \mathcal{F}} |S| \geq \frac{1}{2}|U(\mathcal{F})|$, then $\mathcal{F}$ satisfies Frankl conjecture.*

The theorem that follows shows how we can use up-compression in order to obtain a bound on the average set size of an union-closed family.

Theorem 5.3.8. *If $\mathcal{F}$ is an union-closed family with universe $U(\mathcal{F}) = [n]$, then*

$$\frac{\sum_{A \in \mathcal{F}} |A|}{|\mathcal{F}|} \geq \frac{1}{2} \log(|\mathcal{F}|).$$

When $\mathcal{F} = \mathcal{P}([n])$, both sides of the inequality become $\frac{n}{2}$ and hence the result is sharp.

Proof. We start by considering the set $E(\mathcal{P}([n]))\}$ defined in 5.2.3. For each $A \in \mathcal{F}$, define the edge set $E_A := \{(B, A_n) \in E(\mathcal{P}([n])) : B = A_n - \{i\},\ i \in A_n - A\}$. Obviously, $|E_A| = |A_n - A|$. Note that the sets E_A are disjoint.
Now, we claim that $E_A \subseteq E([A, A_n]) \cap EB(\mathcal{F}_n)$. We have $A \subseteq A_n$ and so, $A \subseteq A_n - \{i\}, \forall i \in A_n - A$, thus $E_A \subseteq E([A, A_n])$. On the other hand, $A_n - \{i\} \notin \mathcal{F}_n$, for $i \in A_n - A$, because if that was not the case, we would have $B \in \mathcal{F}$ such that $B_n = A_n - \{i\}$ and $A, B \in \mathcal{F}$, $A \neq B$ such that $A_n - \{i\} \in [A, A_n] \cap [B, B_n]$ which contradicts the second condition of Lemma 5.3.7. Hence, $E_A \subseteq EB(\mathcal{F}_n)$. It follows that $\sum_{A\in\mathcal{F}} |E_A| \leq |EB(\mathcal{F}_n)|$, or equivalently $\sum_{A\in\mathcal{F}} |A_n - A| \leq |EB(\mathcal{F}_n)|$. Apply Lemma 5.2.4 to $\mathcal{F}_n$ noticing that $\sum_{A\in\mathcal{F}} |A_n| = \sum_{A\in\mathcal{F}} |A| + \sum_{A\in\mathcal{F}} |A_n - A|$. We have

$$2\sum_{A\in\mathcal{F}} |A| + 2\sum_{A\in\mathcal{F}} |A_n - A| = n\,|\mathcal{F}| + |EB(\mathcal{F}_n)|,$$
$$2\sum_{A\in\mathcal{F}} |A| + \sum_{A\in\mathcal{F}} |A_n - A| \geq n\,|\mathcal{F}|$$

By part 3. of Lemma 5.3.7, we get

$$2\sum_{A\in\mathcal{F}} |A| + |\mathcal{F}|n - |\mathcal{F}|\log(|\mathcal{F}|) \geq |\mathcal{F}|n,$$
$$\frac{\sum_{A\in\mathcal{F}} |A|}{|\mathcal{F}|} \geq \frac{1}{2}\log(|\mathcal{F}|).$$

■

In [23], Rodaro slightly changes the way we consider the rising function by allowing the family to be risen with respect to the elements of the universe but taken in different orders. So he defines $\mathcal{F}_j = \varphi_j(\mathcal{F}_{j-1})$, where $\varphi_j = R_{\mathcal{F}_{j-1}, a_j} \circ \varphi_{j-1}$ is called the *rising function with respect to the word* $w = a_1 a_2 \ldots a_n$ of the family $\mathcal{F}$, and we denote it by φ_w. He then uses the following result about the upward-closed family $F = \varphi_w(\mathcal{F})$ to generalize the second statement of 5.3.7, by showing that it holds no matter the order of the elements by which we raise the family.

To do so, we define the notion of *principal ideal of a set* and present an auxiliary result that appears in [23] needed to prove the proposition.

Definition 5.3.9. *Let $\mathcal{F}$ be an union-closed family of sets with universe $\mathcal{U}$ and S a subset of $\mathcal{U}$. We call the* principal ideal of S *to the set of all the elements of $\mathcal{F}$ containing S and denote it by $\mathcal{F}[S]$. This can be extended to any subfamily $\mathcal{G} \subset \mathcal{F}$ and so we define the* principal ideal generated by $\mathcal{G}$ *as the set* $\mathcal{F}[\mathcal{G}] = \bigcup_{S\in\mathcal{G}} \mathcal{F}[S]$.

Remark 13. *Clearly, if $S \in \mathcal{F}$, then $\mathcal{F}[S] = \{S \cup T,\ T \in \mathcal{F}\}$.*

The author also proves that for each $\mathcal{G} \subseteq \mathcal{F}$, one has that $\varphi_w : \mathcal{F}[\mathcal{G}] \to F[\mathcal{G}]$ is a bijection, and that the inverse of $\varphi_w : \mathcal{F} \to F$ is given by

$$\varphi_w^{-1}(\eta) = \bigcup_{\{f \in \mathcal{F} : f \subseteq \eta\}} f.$$

Proposition 5.3.10. *Let $f, g \in \mathcal{F}$ and $\sigma, \theta \in \mathfrak{S}_{\mathcal{U}}$. Then $f \neq g$ if and only if $[f, \varphi_{w\theta}(f)] \cap [g, \varphi_{w\sigma}(g)] = \varnothing$.*

Proof. Suppose that $S \in [f, \varphi_{w\theta}(f)] \cap [g, \varphi_{w\sigma}(g)] \neq \varnothing$. In this case we obviously have $f \subseteq \varphi_{w\sigma}(g)$ because $f \subseteq S \subseteq \varphi_{w\sigma}(g)$. By the previous theorem, we know $g = \bigcup_{\{h \in \mathcal{F} : h \subseteq \varphi_{w\sigma}(g)\}} h$ and so, f is necessarily a subset of g since g is the union of f with another set. Changing g with f we get the other inclusion $g \subseteq f$. Hence $f = g$. Clearly the other side of the implication holds since $f = g \Rightarrow [f, \varphi_{w\theta}(f)] \cap [g, \varphi_{w\sigma}(g)] \neq \varnothing$ since f is its element. ■

Example 11. *Consider the union-closed family*

$$\mathcal{F} = \{\{3,4\},\{2,3,5\},\{1,2,3,4\},\{1,3,5,6\},\{2,3,4,5\},\\ \{1,2,3,4,5\},\{1,2,3,5,6\},\{1,3,4,5,6\},\{1,2,3,4,5,6\}\}.$$

When we apply up-compression to this family defined in the usual way, i.e., with respect to the word 123456 we obtain the upward-closed family

$$\mathcal{G} = \{\{1,3,5,6\},\{1,2,3,5\},\{1,3,4,5\},\{1,2,3,4,6\},\{1,2,3,5,6\},\\ \{1,3,4,5,6\},\{2,3,4,5,6\},\{1,2,3,4,5\},\{1,2,3,4,5,6\}\}.$$

When, for example we apply compression with respect to the word 213456, we get the upward-closed family

$$\mathcal{H} = \{\{1,3,5,6\},\{1,2,3,5\},\{2,3,4,6\},\{1,2,3,4,6\},\{1,2,3,5,6\},\\ \{1,3,4,5,6\},\{2,3,4,5,6\},\{1,2,3,4,5\},\{1,2,3,4,5,6\}\}.$$

Notice that these families are different in the sense that there is no way to rename the elements of $\mathcal{G}$ that makes it equal to $\mathcal{H}$, since 1 has frequency 7 in $\mathcal{H}$ and there is no element in $\mathcal{G}$ with frequency 7.

We know that for each word w, the family $\varphi_w(\mathcal{F})$ is an upward-closed family of sets. However this families are not necessarily equal for different words w as the example above shows. Since the union of upward-closed families with the same universe is still an upward-closed family of sets, we define for each union-closed family $\mathcal{F}$ with universe $\mathcal{U}$ the upward-closed family

$$\mathbf{U}(\mathcal{F}) = \bigcup_{\vartheta \in \mathfrak{S}_{\mathcal{U}}} \varphi_{w\vartheta}(\mathcal{F})$$

where $w = a_1 \dots a_n$ which we call the *invariant upward-closed family* associated to $\mathcal{F}$.

Using these techniques, in [23], the author obtained an upper bound on the number of $\cup-$irreducible sets of an union-closed family, proving that for every union-closed family $\mathcal{F}$, with $|U(\mathcal{F})| = n$, one has that $|J(\mathcal{F})| \le 2\binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$. Despite being a worse bound than Kleitman's (see Theorem 4.3.9) it is possible that some better bounds can be obtained using up-compression.
Also, in [12] the authors prove, using averaging and up-compression techniques, that Frankl conjecture holds for union-closed families with at least $\frac{2}{3}2^{|U(\mathcal{F})|}$ sets.

Analogously we can define the $i-$lowering of $\mathcal{F}$

$$L_i(\mathcal{F}) := \{L_i(S) : S \in \mathcal{F}\},$$

where

$$L_i(S) := \begin{cases} S, & \text{if } S \backslash \{i\} \in \mathcal{F}, \\ S \backslash \{i\}, & \text{if } S \backslash \{i\} \notin \mathcal{F} \end{cases},$$

and

$$\mathcal{F}^i := \begin{cases} \mathcal{F}, & \text{if } i = 0, \\ L_i(\mathcal{F}^{i-1}), & \text{if } i > 0. \end{cases}$$

The next theorem gives us a nice relation between up and down-compression and shows that it both techniques are equivalent in some sense.

Theorem 5.3.11. *The following diagram*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\ \cdot^c\ } & \mathcal{F}^c \\ {\scriptstyle R_i}\big\downarrow & & \big\downarrow{\scriptstyle L_i} \\ \mathcal{F}_i & \xrightarrow{\ \cdot^c\ } & \mathcal{F}^i \end{array}$$

commutes, where $\cdot^c$ *is the map* $\mathcal{F} \mapsto \mathcal{F}^c$.

Proof. Let $A \in \mathcal{F}$ and so $A^c \in \mathcal{F}^c$. It is enough to see $(R_i(A))^c = L_i(A^c)$. Suppose now $A \cup \{i\} \in \mathcal{F}$. In that case $(R_i(A))^c = A^c$. Also, notice that $A^c - \{i\} = A^c \cap \{i\}^c = (A \cup \{i\})^c \in \mathcal{F}^c$ since $A \cup \{i\} \in \mathcal{F}$ and so $L_i(A^c) = A^c = (R_i(A))^c$.

Now, if $A \cup \{i\} \notin \mathcal{F}$, we have $(R_i(A))^c = (A \cup \{i\})^c = A^c - \{i\}$. Since $A \cup \{i\} \notin \mathcal{F}$, we know $A^c - \{i\} = (A \cup \{i\})^c \notin \mathcal{F}^c$ and so $L_i(A^c) = A^c - \{i\} = (R_i(A))^c$.

■

Corollary 5.3.12. *Let $\mathcal{F}$ be an intersection-closed family. Then $\mathcal{F}^n$ is a downward-closed family of sets.*

Chapter 6

FC-families

In Chapter 1, we proved that a family containing a singleton or a set with two elements satisfies the conjecture, and that in the first case the element in the singleton was the one present in at least half the sets in the family, while in the second case one of the two elements was present in at least half the sets of the family.

In this section, we present a formalization of that concept and a characterization of those families due to Poonen [18]. The proof is surprising because it uses discrete geometry tools to attack this problem, which was something that had not been done before. We give a couple examples and prove that the result announced for families with a singleton or with a 2-set cannot be generalized to families having a 3-set. After Poonen, Vaughan [25] and Morris [17] also explored this concept and the latter presents a full list of all FC-families with universe having at most 5 elements and proves the conjecture for families with an universe having at most 9 elements.

6.1 Notation

Throughout this chapter, we will use the following notation:

- If $\mathcal{F}$ and $\mathcal{G}$ are two families of sets, we put $\mathcal{F} \uplus \mathcal{G} = \{S \cup T : S \in \mathcal{F}, T \in \mathcal{G}\}$;
- $\langle \cdot, \cdot \rangle$ will denote the usual inner product in $\mathbb{R}^n$;
- sgn is the sign function in $\mathbb{R}$ and is defined as usual: $sgn(x) = \frac{|x|}{x}$ if $x \neq 0$ and $sgn(0) = 0$;

- $n_j^{\mathcal{F}}$ denotes the number of sets of cardinality j in $\mathcal{F}$.

6.2 FC-Families

Definition 6.2.1. *A family of sets $\mathcal{F}'$ with universe $U(\mathcal{F}') = [k]$ is called* FC (*or* Frankl Complete) *if for every union closed family $\mathcal{F}$ containing $\mathcal{F}'$, there exists $i \in [k]$ such that $|\mathcal{F}_i| \geq \frac{|\mathcal{F}|}{2}$.*

Theorem 6.2.2. *(Bjorn Poonen) The following are equivalent, for a family $\mathcal{F}'$ with $U(\mathcal{F}') = [k]$:*

1. *$\mathcal{F}'$ is FC.*
2. *There exist nonnegative real numbers $c_1, \cdots, c_k$ with sum 1 such that, for every union-closed family $\mathcal{G} \subseteq \mathcal{P}([k])$ with $\mathcal{F}' \uplus \mathcal{G} = \mathcal{G}$, one has $\sum_{i=1}^k c_i |\mathcal{G}_i| \geq \frac{|\mathcal{G}|}{2}$.*

Proof. Throughout this proof, we will call $\mathcal{F}'$-closed to an union-closed family $\mathcal{G}$ allowed in (2), i.e., if $\mathcal{G} \subseteq \mathcal{P}([k])$ and $\mathcal{F}' \uplus \mathcal{G} = \mathcal{G}$.
$(1) \Rightarrow (2)$ For each $\mathcal{F}'$-closed family $\mathcal{G}$, we set

$$X(\mathcal{G}) = \left(|\mathcal{G}_1| - \frac{|\mathcal{G}|}{2}, \cdots, |\mathcal{G}_k| - \frac{|\mathcal{G}|}{2} \right) \in \mathbb{R}^k.$$

Let C be the convex hull of these points, and let

$$\mathcal{N} = \{(x_1, \cdots, x_k) \in \mathbb{R}^k \mid x_i < 0, \text{ for all } i = 1, \ldots, k\}.$$

We will prove that $C \cap \mathcal{N} = \varnothing$. Suppose the opposite. Then, one has

$$\sum_{j=1}^{r} w_j X(\mathcal{G}^j) \in \mathcal{N}$$

for some $\mathcal{F}'$-closed families $\mathcal{G}^1$, $\mathcal{G}^2 \cdots$, $\mathcal{G}^r$, and for some nonnegative real numbers $w_1, w_2, \cdots, w_r$ with sum 1. Since $\mathcal{N}$ is open, we may assume the w_j are rational and so, we can suppose that the w_j are nonnegative integers with nonzero sum w (no longer 1), by multiplying by a common denominator. So we have $\mathcal{F}'$-closed families $\mathcal{H}^1, \cdots, \mathcal{H}^w$ such that

$$\sum_{j=1}^{w} X(\mathcal{H}^j) \in \mathcal{N},$$

where $\mathcal{H}^1 = \cdots = \mathcal{H}^{w_1} = \mathcal{G}^1, \mathcal{H}^{w_1+1} = \cdots = \mathcal{H}^{w_1+w_2} = \mathcal{G}^2$, etc.

For a positive integer d, let $B = \{\alpha_1, \alpha_2, \cdots, \alpha_{wd}\}$ be a set of wd elements disjoint from $[k]$, and for $1 \leq s \leq wd$, let $B_s = B \backslash \{\alpha_s\}$. Note that $B_s \cup B_t = B,\ \forall s \neq t$. Consider the following subset of $\mathcal{P}([k] \cup B)$:

$$\begin{aligned}\mathcal{F} = \mathcal{F}' &\cup (\{B_1, B_2, \cdots, B_d\} \uplus \mathcal{H}^1)\\ &\cup (\{B_{d+1}, B_{d+2}, \cdots, B_{2d}\} \uplus \mathcal{H}^2)\\ &\qquad\vdots\\ &\cup (\{B_{(w-1)d+1}, B_{(w-1)d+2}, \cdots, B_{wd}\} \uplus \mathcal{H}^w)\\ &\cup (\{B\} \uplus \mathcal{P}([k])).\end{aligned}$$

We will check that $\mathcal{F}$ is an union-closed family of sets. Let $X, Y \in \mathcal{F}$. It is obvious that each of the families $\mathcal{F}'$, and $\{B\} \uplus \mathcal{P}([k])$, is union-closed, and so if X and Y are both in one of these families, then $X \cup Y \in \mathcal{F}$. We now consider all the other cases:

- If $X, Y \in \{B_{id+1}, B_{id+2}, \cdots, B_{(i+1)d}\} \uplus \mathcal{H}^{i+1}, i \in \{0, 1, \cdots, w-1\}$, for some $i \in \{0, \ldots, w-1\}$, then $X \cup Y \in \{B\} \uplus \mathcal{P}([k]) \subseteq \mathcal{F}$, since $B_s \cup B_t = B,\ \forall s \neq t$;

- If $X \in \mathcal{F}'$ and $Y \in \{B_{id+1}, B_{id+2}, \cdots, B_{(i+1)d}\} \uplus \mathcal{H}^{i+1}$, for some $i \in \{0, 1, \cdots, w-1\}$ then let $Y = B_j \cup Z$, for some $j \in \{id+1, id+2, \cdots, (i+1)d\}$ and $Z \in \mathcal{H}^{i+1}$. We have $X \cup Z \in \mathcal{F}' \uplus \mathcal{H}^{i+1} = \mathcal{H}^{i+1}$ (by hypothesis), and so $X \cup Y = X \cup Z \cup B_j \in \mathcal{H}^{i+1} \uplus \{B_{id+1}, B_{id+2}, \cdots, B_{(i+1)d}\} \subseteq \mathcal{F}$.

- If $X \in \mathcal{F}'$ and $Y \in \{B\} \uplus \mathcal{P}([k])$, then let $Y = B \cup Z$, for some $Z \subseteq [k]$. We have that $X \cup Z \subseteq [k]$ and so $X \cup Y = B \cup X \cup Z \in \{B\} \uplus \mathcal{P}([k]) \subseteq \mathcal{F}$.

- If $X \in \{B_{id+1}, B_{id+2}, \cdots, B_{(i+1)d}\} \uplus \mathcal{H}^{i+1}$ and $Y \in \{B_{jd+1}, B_{jd+2}, \cdots, B_{(j+1)d}\} \uplus \mathcal{H}^{j+1}$, for some $i, j \in \{0, 1, \cdots, w-1\}, i \neq j$, then let $X = B_m \cup Z$, for some $m \in \{id+1, id+2, \cdots, (i+1)d\}$ and $Z \in \mathcal{H}^{i+1}$ and let $Y = B_n \cup W$, for some $n \in \{jd+1, jd+2, \cdots, (j+1)d\}$ and $W \in \mathcal{H}^{j+1}$. We have $X \cup Y = B_n \cup B_m \cup Z \cup W = B \cup Z \cup W \in \{B\} \uplus \mathcal{P}([k]) \subseteq \mathcal{F}$.

- If $X \in \{B_{id+1}, B_{id+2}, \cdots, B_{(i+1)d}\} \uplus \mathcal{H}^{i+1}$ and $Y \in \{B\} \uplus \mathcal{P}([k])$, let $X = B_j \cup Z$, for some $j \in \{id+1, id+2, \cdots, (i+1)d\}$ and $Z \in \mathcal{H}^{i+1}$ and $Y = B \cup W$, for some $W \subseteq [k]$. Then, $X \cup Y = B \cup Z \cup W \in \{B\} \uplus \mathcal{P}([k]) \subseteq \mathcal{F}$.

If $i \in [k]$, then

$$|\mathcal{F}_i| - \frac{|\mathcal{F}|}{2} = \left(|\mathcal{F}'_i| - \frac{|\mathcal{F}'|}{2}\right) + d\left[\left(|\mathcal{H}^1_i| - \frac{|\mathcal{H}^1|}{2}\right) + \cdots + \left(|\mathcal{H}^w_i| - \frac{|\mathcal{H}^w|}{2}\right)\right].$$

One has that $\left[\left(|\mathcal{H}^1_i| - \frac{|\mathcal{H}^1|}{2}\right) + \cdots + \left(|\mathcal{H}^w_i| - \frac{|\mathcal{H}^w|}{2}\right)\right]$ is negative, since it is the ith coordinate of the point $\sum_{j=1}^{w} X(\mathcal{H}^j)$, which is in $\mathcal{N}$. So for sufficiently large d, we obtain for all $i \in [k]$, $|\mathcal{F}_i| - \frac{|\mathcal{F}|}{2} < 0$, which is absurd since $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{F}'$ is FC by hypothesis.

So, one has that $C \cap \mathcal{N} = \varnothing$. But C and $\mathcal{N}$ are convex, with $\mathcal{N}$ open, so by the separating hyperplane theorem, that can be found in [16] (p.6), there exists a nonzero vector $v = (c_1, \cdots, c_k)$ and a real number c such that

$$\langle x, v\rangle < c \text{ and } \langle y, v\rangle \geq c, \forall x \in \mathcal{N}, y \in C.$$

We will now prove that we then must also have such a vector that satisfies the inequalities for $c = 0$. We start by seeing that $c \geq 0$. Assume otherwise, and take $x = \left(\frac{c}{2\ell c'_1}, \cdots, \frac{c}{2\ell c'_k}\right)$, where ℓ is the number of nonzero entries in v and $c'_i = |c_i|$ if $c_i \neq 0$, and 1 otherwise. It is clear that $x \in \mathcal{N}$ and that if we consider $c_{i_1}, \cdots, c_{i_l}$ to be the nonzero coordinates of v we have

$$\langle x, v\rangle = sgn(c_{i_1})\frac{c}{2\ell} + \cdots + sgn(c_{i_\ell})\frac{c}{2\ell} \geq \sum_{i=1}^{\ell} \frac{c}{2\ell} = \frac{c}{2} > c.$$

Therefore, c must be nonnegative. We will see that if we have a vector v satisfying both inequalities for a positive real number c then that same vector satisfies the inequalities for $c = 0$. Obviously if we have $c > 0$ and $\langle y, v\rangle \geq c, \forall y \in C$, then $\langle y, v\rangle \geq 0, \forall y \in C$. If $c_i \geq 0, \forall i \in [k]$, then $\langle x, v\rangle < 0, \forall x \in \mathcal{N}$. If there exists $i \in [k]$ such that $c_i < 0$, taking $x = \left(\frac{(sgn(c_1)-1)\ell c - c}{\ell c'_1}, \cdots, \frac{(sgn(c_k)-1)\ell c - c}{\ell c'_k}\right) \in \mathcal{N}$, we have

$$\langle x, v\rangle = sgn(c_{i_1})\frac{(sgn(c_{i_1}) - 1)\ell c - c}{\ell} + \cdots + sgn(c_{i_\ell})\frac{(sgn(c_{i_\ell}) - 1)\ell c - c}{\ell}$$

Note that each summand is either $\frac{-c}{l}$ or $\frac{2\ell+1}{\ell}$, and the smallest value this sum can take is $\frac{(2\ell+1)c}{\ell} + \sum_{i=1}^{\ell-1} \frac{-c}{\ell}$, when only one summand is $\frac{2\ell+1}{\ell}$, since we are assuming that there exists $i \in [k]$ such that $c_i < 0$. Hence,

$$\langle x, v\rangle \geq \frac{(2\ell+1)c}{\ell} + \sum_{i=1}^{\ell-1} \frac{-c}{\ell} = \frac{(1-\ell)c}{\ell} + \frac{(2\ell+1)c}{\ell} = \frac{(\ell+2)c}{\ell} > c.$$

So, we have that there exists a vector $v = (c_1, \cdots, c_k)$, with $c_i \geq 0, \forall i \in [k]$ and such that $c_1x_1 + \cdots + c_kx_k$ is nonnegative for $(x_1, \ldots, x_k) \in C$. By scaling, assume $c_1 + \cdots + c_k = 1$. For each $\mathcal{F}'$-closed family $\mathcal{G}, X(\mathcal{G}) \in C$, one then has

$$c_1\left(|\mathcal{G}_1| - \frac{|\mathcal{G}|}{2}\right) + \cdots + c_k\left(|\mathcal{G}_k| - \frac{|\mathcal{G}|}{2}\right) \geq 0,$$

which implies

$$c_1|\mathcal{G}_1| + \cdots + c_k|\mathcal{G}_k| \geq (c_1 + \cdots + c_k)\frac{|\mathcal{G}|}{2} = \frac{|\mathcal{G}|}{2}.$$

So (2) holds.

$(2) \Rightarrow (1)$ Let $\mathcal{F}$ be an union-closed family of sets such that $\mathcal{F}' \subseteq \mathcal{F}$ and let $B = U(\mathcal{F}) - [k]$. We have that each set in $\mathcal{F}$ can be written as $S \cup T$, with $S \subseteq B$ and $T \subseteq [k]$. Write

$$\mathcal{F} = \bigcup_{S \subseteq B} \{S\} \uplus \mathcal{G}^S,$$

where $\mathcal{G}^S \subseteq \mathcal{P}([k])$ for each S and consists on the family of sets $X \subset [k]$ such that $X \cup S \in \mathcal{F}$. Since $\mathcal{F}$ is union-closed and $\mathcal{F}' \subseteq \mathcal{F}$, we have that if $X \in \mathcal{G}^S$ and $Y \in \mathcal{F}'$, then $X \cup Y \in \mathcal{G}^S$ because $X \cup S \in \mathcal{F}$ and $Y \in \mathcal{F}$, so $(X \cup Y) \cup S \in \mathcal{F}$. Therefore $X \cup Y \in \mathcal{G}^S$ and we have $\mathcal{G}^S \uplus \mathcal{F}' = \mathcal{G}^S$ for each $S \subseteq B$. Also, $\mathcal{G}^S \uplus \mathcal{G}^S = \mathcal{G}^S$ and so, $\mathcal{G}^S$ is either union-closed or empty (note that we cannot have $\mathcal{G}^S = \{\varnothing\}$, since in that case we would have $\mathcal{G}^S \uplus \mathcal{F}' = \mathcal{F}' \neq \{\varnothing\}$ and by definition every FC-family has a nonempty universe). By (2), we have nonnegative real numbers $c_1, \cdots, c_k$ such that

$$\sum_{i=1}^{k} c_i |\mathcal{G}_i^S| \geq \frac{|\mathcal{G}^S|}{2}$$

for each S (this also holds when $\mathcal{G}^S = \varnothing$). Then

$$\begin{aligned}
\sum_{i=1}^{k} c_i |\mathcal{F}_i| &= \sum_{i=1}^{k} c_i \sum_{S \subseteq B} |\mathcal{G}_i^S| \\
&= \sum_{S \subseteq B} \sum_{i=1}^{k} c_i |\mathcal{G}_i^S| \\
&\geq \sum_{S \subseteq B} \frac{|\mathcal{G}^S|}{2} \\
&= \frac{|\mathcal{F}|}{2}.
\end{aligned}$$

But if a weighted average of the $|\mathcal{F}_i|$ is at least $\frac{|\mathcal{F}|}{2}$, then for some i, $|\mathcal{F}_i| \geq \frac{|\mathcal{F}|}{2}$ and so (1) holds. ■

Note that, for fixed $\mathcal{F}'$, there are only finitely many $\mathcal{F}'$-closed families, which leads to a finite system of linear inequalities in $c_1, \cdots, c_k$. A terminating algorithm can thus determine whether this system has a solution, and therefore this theorem gives a method for determining whether a subfamily $\mathcal{F}'$ is enough to guarantee an element in half of the sets. Also, if a family $\mathcal{G}$ is $\mathcal{F}'$-closed, we have $U(\mathcal{G}) = U(\mathcal{F}')$, since $\mathcal{G} \subseteq \mathcal{P}([k])$ and so $U(\mathcal{G}) \subseteq U(\mathcal{F}')$ and $\mathcal{F}' \uplus \mathcal{G} = \mathcal{G}$ implies that we have, in particular, $[k] \cup X \in \mathcal{G}$ for every $X \in \mathcal{G}$, and so $U(\mathcal{G}) \supseteq U(\mathcal{F}')$. The lemma that follows will be very useful in applying the previous theorem.

Lemma 6.2.3. *Suppose* $U(\mathcal{G}) = [m]$*, for some* $m < 1$*. If* $\sum_{j=1}^{m-1}(j - \frac{m}{2})n_j^{\mathcal{G}} \geq 0$*, then* $\frac{1}{m}\sum_{i=1}^{m}|\mathcal{G}_i| \geq \frac{|\mathcal{G}|}{2}$.

Proof. Writing n_j for $n_j^{\mathcal{G}}$, one has that

$$\sum_{j=0}^{m}\left(j - \frac{m}{2}\right)n_j = \frac{m}{2}(n_m - n_0) + \sum_{j=1}^{m-1}\left(j - \frac{m}{2}\right)n_j \geq 0,$$

since $n_m = 1$ and n_0 is either 0 or 1 depending on the presence or absence of $\varnothing$ on $\mathcal{G}$. Hence,

$$\sum_{i=1}^{m}|\mathcal{G}_i| = \sum_{S\in\mathcal{G}}|S| = \sum_{j=0}^{m}n_j j = \sum_{j=0}^{m}\left(j - \frac{m}{2}\right)n_j + \frac{m}{2}\sum_{j=0}^{m}n_j \geq 0 + \frac{m}{2}|\mathcal{G}|,$$

and dividing by m gives the desired result. ■

We now present an alternative proof of Lemma 1.2.1 and Corollary 1.2.3 using the previous theorem.

Corollary 6.2.4. *If an union-closed family* $\mathcal{F}$ *has a set* S *with only one or two elements, some element of* S *is in at least half the elements of* $\mathcal{F}$*, i.e.,* $\{S\}$ *is FC.*

Proof. Let $S = [m]$, where $m = 1$ or $m = 2$, and $\mathcal{F}' = \{S\}$ in Theorem 6.2.2. If $\mathcal{G}$ is $\mathcal{F}'$-closed, and $\mathcal{G} \neq \varnothing$, then $S = U(\mathcal{F}') = U(\mathcal{G}) \in \mathcal{G} \subset \mathcal{P}(S)$ and we can apply the previous lemma. The hypothesis there is automatically satisfied when $m = 1$ or $m = 2$. So in (2) of the Theorem 6.2.2, we can take $c_1 = \cdots = c_m = \frac{1}{m}$. ■

Corollary 6.2.5. *There exists an union-closed family* $\mathcal{F}$ *having a set* S *with three elements, such that no element of* S *is in at least half the sets of* $\mathcal{F}$.

Proof. Let $S = \{1, 2, 3\}$ and let $\mathcal{F}' = \{\varnothing, S\}$. We will prove that $\mathcal{F}'$ is not an FC-family and so the result will follow. Suppose $\mathcal{F}'$ is an FC-family. From Theorem 6.2.2 we have real numbers c_1, c_2 and c_3 satisfying

$$c_1 + c_2 + c_3 = 1$$
$$2c_1 + c_2 + c_3 \geq \frac{3}{2} \quad \text{(from } \mathcal{G} = \{\varnothing, \{1\}, \{1,2,3\}\}\text{)}$$
$$c_1 + 2c_2 + c_3 \geq \frac{3}{2} \quad \text{(from } \mathcal{G} = \{\varnothing, \{2\}, \{1,2,3\}\}\text{)}$$
$$c_1 + c_2 + 2c_3 \geq \frac{3}{2} \quad \text{(from } \mathcal{G} = \{\varnothing, \{3\}, \{1,2,3\}\}\text{)}$$

Adding the last three inequalities, and dividing by four we get $c_1 + c_2 + c_3 \geq \frac{9}{8}$, which contradicts the first equation. So by Theorem 6.2.2, we get that $\mathcal{F}'$ is not FC, thus there is an union-closed family having S as a member and such that no element of S is in at least half the sets of $\mathcal{F}$. ∎

A concrete example of such a family was given in Section 1.3.

FC-families are called *proper* if it contains no strictly smaller FC-family. From the results above, it is clear that there are no proper FC-families whose universe has only 3 elements. In [17], FC-families are defined with the additional condition that they must be union-closed. This is not a restriction since any union-closed family containing an FC-family must also contain its $\cup$-closure.

In [25] a list of FC-families is given, and in [17] the author provides a characterization of all proper (union-closed) FC-families whose universe has at most 5 elements which we present next (without proof). He is also able to prove that the conjecture holds for families whose universe has 9 elements.

Theorem 6.2.6. *A family $\mathcal{F}$ is a proper union-closed FC-family if and only if it is generated by one of the following set systems (under some permutation of $\{1,2,3,4,5\}$).*

- *Any three of the subsets of $\{1,2,3,4,5\}$ with 3 elements,*
- $\{1,2,3\}$, $\{1,2,4\}$ *and* $\{1,3,4,5\}$*,*
- $\{1,2,3\}$, $\{1,4,5\}$ *and* $\{2,3,4,5\}$*,*
- $\{1,2,3\}$, $\{1,4,5\}$, $\{1,2,3,4\}$, $\{1,2,3,5\}$, $\{1,2,4,5\}$ *and* $\{1,3,4,5\}$*,*
- $\{1,2,3\}$, $\{1,2,4,5\}$, $\{1,3,4,5\}$ *and* $\{2,3,4,5\}$*,*
- *All five subsets of $\{1,2,3,4,5\}$ with 4 elements.*

Conclusion

In this dissertation, we covered some of the main techniques used to approach Frankl conjecture, illustrating them with several examples and developed some of these techniques a bit further in order to clarify some results and also to obtain some other ones.

In Chapter 1 we presented with great detail the concept of separation in union-closed families, and obtained some results about separating union-closed families. Finally, in Section 1.3, we provided some counterexamples to some tempting generalizations of the problem and one of a separating family such that the average frequency of its elements is as low as we want, which we weren't able to find in the literature.
In Chapter 2, we presented a conjecture related with lattice theory that is equivalent to Frankl conjecture and show it for lower semimodular lattices.
We were able to prove in Chapter 3 that there are at least 3 a-problematic sets for every $a \in U(\mathcal{F})$, where $\mathcal{F}$ represents a minimal hypothetical counterexample to the conjecture, as a corollary of two results, Theorems 3.1.9 and 3.1.10, that help us understand the family of a-problematic sets, $\mathcal{P}_a$, which are results that we weren't able to find in the literature.
We believe that in Chapter 4 we made clear the relation between the dual families defined by Johnson and Vaughan [14] and the ones defined by Piotr Wójcik in [27]. Also, the concept of independence was studied and formalized in detail and proved that there is only one independent normalized family. Structural properties of normalized families were also obtained, and we proved that every normalized family can be seen as the dual of a regular union-closed family.
In Chapter 5, we studied upward-closed families and provided a series of examples of the up-compression technique. Also, we proved that up-compression and down-compression are symmetrical processes, in the sense of Theorem 5.3.11.
In Chapter 6 we show a full proof of Poonen's Theorem which characterizes FC-

families as well as some examples.

Bibliography

[1] Tetsuya Abe. Strong semimodular lattices and Frankl's conjecture. *Algebra univers.*, 44:379–382, 2000.

[2] Tetsuya Abe and Bumpei Nakano. Frankl's conjecture is true for modular lattices. *Graphs Comb. 14*, pages 305–311, 1998.

[3] Vladimir Blinovsky. Proof of Union-Closed Sets Conjecture. *http://arxiv.org/abs/1507.01270v3*, 2015.

[4] Henning Bruhn and Oliver Schaudt. The journey of the union- closed sets conjecture. *arXiv:1309.3297v2*, 30 October 2013.

[5] Gábor Czédli and E. Tamás Schmidt. Frankl's conjecture for large semimodular and planar semimodular lattices. *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica*, 47:47–53, 2008.

[6] D.G.Sarvate and J-C. Renaud. Improved bouds for the union-closed sets conjecture. *Ars Combin.*, pages 149–154, 1989.

[7] D.J.Kleitman. Extremal properties of collections of subsets containing no two sets and their union. *J. Combin. Theory (Series A)*, 20:390–392, 1976.

[8] V. Falgas-Ravry. Minimal weight in union-closed families. *Electron. J. Combin.*, 18, 2011.

[9] Giovanni Lo Faro. A note on the union-closed sets conjecture. *J. Austral. Math. Soc. (Series A)*, 57:230–236, 1994.

[10] Giovanni Lo Faro. Union-closed sets conjecture: Improved bounds. *Journal of Combinatorial Mathematics and Combinatorial Computing*, pages 97–102, 1994.

[11] Timothy Gowers. Gowers's Weblog. *https://gowers.wordpress.com*.

[12] Béla Bollobás Igor Balla and Tom Eccles. Union-closed families of sets. *Journal of Combinatorial Theory, Series A*, 120:531–544, 2013.

[13] Yunjiang Jiang. A generalization of the union-closed set conjecture. *http://math.stanford.edu/~jyj/Union_closed_set_conj.pdf*, 2009.

[14] Robert T. Johnson and Therese P. Vaughan. On union-closed families, I. *Journal of Combinatorial Theory (Series A)*, 84(242-249), 1998.

[15] Gil Kalai. Extremal Combinatorics IV:Shifting. *https://gilkalai.wordpress.com/2008/10/06/extremal-combinatorics-iv-shifting/*.

[16] Jiří Matoušek. *Lectures on Discrete Geometry*, volume 212. Springer, 2002.

[17] Robert Morris. FC-families, and improved bounds for Frankl's conjecture. *Europ. J. Combin.*, 27:269–282, 2006.

[18] Bjorn Poonen. Union-closed families. *Journal of Combinatorial Theory (Series A)*, 59:253–269, 1992.

[19] David Reimer. An average set size theorem. *Combinatorics, Probability and Computing.*, 12:89–93, 2003.

[20] Jurgen Reinhold. Frankl's conjecture is true for lower semimodular lattices. *Graphs and Combinatorics*, 16:115–116, 2000.

[21] R.M.Norton and D.G.Sarvate. A note on the union-closed sets conjecture. *J. Austral. Math. Soc. (Series A)*, 55:411–413, 1993.

[22] Ian Roberts and Jamie Simpson. A note on the union-closed sets conjecture. *Australas. J. Combin.*, 47:265–267, 2010.

[23] Emanuele Rodaro. Union-closed vs upward-closed families of finite sets. *arXiv:math/1208.5371v2*, 2012.

[24] Richard P. Stanley. *Enumerative Combinatorics*, volume I. Wadsworth & Brooks/Cole, 1986.

[25] Therese P. Vaughan. Families implying the Frankl conjecture. *Europ. J. Combin.*, 23:851–860, 2002.

[26] B. Vučković and M. Zivković. The 12 element case of frankl's conjecture. *preprint*, 2012.

[27] Piotr Wójcik. Union-closed families of sets. *Discrete Mathematics*, 199:173–182, 1999.