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Bifurcations in the Footloose Entrepreneur Model

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Master degree dissertation in Economics

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Biography

Ana Margarida Jesus Ferreira, Portuguese, was born on 10 March of 1990.

She lived in Arouca until 2008, after which she moved to OPorto to get her Bachelor degree in Economics at the School of Economics of the University of Porto (FEP). She graduated in 2011 with a final average of fourteen (14) points out of twenty (20). Immediately after the Bachelor degree and with a willingness to learn and improve her skills she joined the Master of Science in Economics, specialization of Economic Analysis, at the same institution.

During this time, she has complemented her higher education with other activities no less important to their personal development. She has knowledge of English and she accepted the challenge to learn German not for professional reasons but for the sheer challenge. Actually, she is doing a part-time job as a cashier in a food retail market.

Acknoledgements

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Abstract

We study the drastic changes in agglomeration patterns in a solvable core-periphery model with 2 asymmetric regions. For this purpose, we extend the Footloose Entrepreneur model developed by Forslid and Ottaviano (2003) by considering a possibly asymmetric distribution of unskilled workers (which are immobile) and a possible preference by skilled workers (which are mobile) for living in a particular region - which may be due to its scenic beauty, climate or cultural factors. The modified model reproduces all the features of the original model, including analytical solvability. Our main aim is to study how these asymmetries interfere with the agglomeration forces to generate agglomeration patterns. In particular, we provide evidence which suggests that our extension of the model of Forslid and Ottaviano (2003) constitutes a universal unfolding. This means that the two perturbations (of the distribution of unskilled workers and of the preferences of skilled workers) generate all the possible qualitative behaviour of the model. Other perturbations of the model would not lead to new qualitative behaviour. We also show that the tomahawk bifurcation is not robust to generate a Core-Periphery pattern - disappearing with the introduction of asymmetries between regions. Many examples of bifurcation diagrams are used to illustrate the behaviour of our model.

Keywords: New Econmic Geography, Core-Periphery, Footloose Entrepreneur, Bifurcation; migration dynamics; genericity analysis
JEL Classification Numbers: R12, R13, C62, F12

Resumo

Estudamos as mudanças nos padrões de aglomeração numa versão resolúvel do modelo Core-Periphery de Krugman com duas regiões assimétricas. Para isso, ampliamos o modelo "Footloose entrepreneur" desenvolvido por Forslid e Ottaviano (2003) considerando uma possível distribuição assimétrica de trabalhadores não qualificados (que são imóveis) e uma possível preferência por parte trabalhadores qualificados (que são móveis) para viver numa determinada região - seja devido à sua paisagem, clima ou fatores culturais. O modelo modificado reproduz todas as características do modelo original, incluindo a sua solvabilidade analítica. O nosso principal objetivo é estudar como essas assimetrias interferem com as forças de aglomeração para gerar padrões de aglomeração. Em particular, fornecemos evidências que sugerem que a nossa extensão do modelo de Forslid e Ottaviano (2003) constitui um "desdobramento universal". Isto significa que as duas perturbações (da distribuição dos trabalhadores não qualificados e das preferências dos trabalhadores qualificados) geram todos os comportamentos qualitativos possíveis. Outras perturbações do modelo não acrescentarão qualquer comportamento qualitativo novo. Demonstramos também que o "tomahawk" não é robusto para gerar um padrão de centro-periferia - desaparecendo com a introdução das assimetrias entre as regiões. Usamos alguns exemplos de diagramas de bifurcação para ilustrar o comportamento do nosso modelo.

Palavras-chave: Nova Economia Geográfica; Centro-Periferia; "footloose entrepreneur"; Bifurcações; Dinâmica de Migração; Análise de Generacidade
Código JEL: R12, R13, C62, F12

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1 Introduction

Economic Geography is the field that studies the distribution of economic activities and the reasons that underlie this distribution. There are many equilibrium models which consider space and geography, like those that descend from the Von Thünen model of land use. However, in the past two decades, following the work of Krugman (1991), Economic Geography has gained more relevance as several New Economic Geography (NEG) models have been developed. The model of Krugman (1991) is considered the benchmark of NEG models, being based on the notion of monopolistic competition à la Dixit and Stiglitz (1977). The outcome is the well-known Core-Periphery (CP) model, where trade costs are crucial to explain the geographic distribution of economic activity between two regions. When trade costs are high, there is geographic dispersion, but when trade costs are low there is geographic concentration. In this original NEG model, there are two absolutely identical regions, and two sectors (the manufacturing sector and the agricultural sector). Manufacturing firms only employ the labour of industrial workers who are mobile between regions, while the agricultural sector only uses the labour of agricultural workers who are immobile.

The mechanics of the CP model is very simple. Spatial concentration may itself create an environment that encourages spatial concentration, being driven by three effects: "market size effect", "cost of living effect" and "market-crowding effect". The first effect describes the tendency of firms to locate their production in the big market and export to small markets due to the positive impact on demand per firm as long as the income generated by a new entrant is mostly spent locally, thus increasing local expenditures. The second effect reflects the impact of a decrease in price indices motivated by the entrance of a new firm, which results in a positive impact on consumer surplus. Indeed, consumers will import a narrower range of products in order to avoid trade costs. Finally, the third effect concerns the negative impact on demand per firms due to the decrease in price indices caused by the entrance of a new firm. The first two effects encourage spatial concentration, while the third discourages it. Therefore, trade costs are crucial because they determine the relative strength of that agglomeration and dispersion effects.

The Krugman's CP model captures important aspects for the explanation of spatial distribution evolution patterns relying on a number of assumptions, which, if relaxed, may lead to additional interesting results. Other studies have been focusing on the possibility of obtaining more results from existing NEG models or on the possibility to relate them with other subjects, extending their basic framework. For instance, Baldwin and Forslid (1999) present a modified CP model that shows how long-run growth can be a powerful centripetal force.

There are many works based on dropping the assumption of non-existence of direct negative externalities between firms (for example due to pollution or congestion). An example is the work of Brakman *et al.* (1994) who introduced negative technological externalities using also a multi-region version of CP model. Congestion effects are usually introduced by making the fixed and/or the marginal costs depend positively on the number of firms in a region. For this reason, agglomeration may not be complete, as firms will realize that it is more profitable to move to the less congested periphery. This explains why a complete concentration of manufacturing in a region rarely occurs. This issue is also well discussed in the work of Lanaspa and Sanz (2001) where there are non-constant transport costs. These transport costs increase with the size of regions (congestion concept). On the other hand, the authors also assume the existence of infrastructures which require a certain dense population in the region in question to be executed.[1] This causes the agricultural population to divide unevenly between the two regions and the appearance of asymmetric stable equilibria.

Adding more regions to the model, leads to more complex agglomeration patterns, and it also makes the framework applicable to new questions. Several authors have been contributing to this investigation branch. Castro *et al.* (2012) developed a core-periphery model with three equidistant regions. They have concluded that 3-region model favours the agglomeration of economic activity while partial agglomeration equilibrium is excluded as a stable outcome. That is, when there are three regions available, a distribution of industrial activity in which one of the regions is empty and the other two share the workers equally is always unstable. They have also obtained some results regarding the n-region model. Gaspar *et al.* (2013) also provided a 3-region model using as a base model the Footloose Entrepreneur model of Forslid and Ottaviano (2003).[2]

Another important offshoot of this research theme, is to consider that the dynamics of migration choice can be derived explicitly from forward-looking optimizing behavior because the results of the CP model can be strongly dependent on the adjustment process considered. An example of a work that dropped this assumption is by Oyama (2006) who considers a migration dynamic with forward-looking agents.

[1] See also Martin and Rogers (1995) for an extended overview of industrial location and public infrastructure.

[2] We will next describe the Footloose Entrepreneur model of Forslid and Ottaviano (2003) in more detail.

Using a solvable variant of the CP model with multi-regions that are asymmetric with respect to their import barriers and market sizes, the author considers that migration should be seen as an investment decision because individuals are naturally interested in current utility but also with expected future utility, which depends on future location patterns. Baldwin (1999) also presented a model where agglomeration arises from endogenous capital with forward looking agents.

Research had also raised in what concerns the question on if a region has a superior resource base or technology. This way, some interesting efficiency questions emerge, which are absent in the basic model. Efficiency questions are discussed in Lanaspa and Sanz (1999) where regions have different natural resources and land quality, which causes the agricultural population to divide itself unequally between the two existing regions. The introduction of intermediate goods also leads to the emergence of new backward and forward linkages. Venables (1995) modified Krugman's model considering firms in an upstream and downstream industries and, where there is no labour mobility. Nevertheless, the concentration of the manufacturing industry sales in one region may result in benefits for both. Upstream firms benefit from being in a region with many downstream firms, because this way they can serve customers more cheaply. Downstream firms benefit from being in a region with many upstream firms because this decreases input costs. Gaigné and Thisse (2013) explain the differences in the economic performance of regions by the behaviours of households and firms that are located within them. Unlike the standard core-periphery model where regions are like spaceless places, Gaigné and Thisse (2013) take into account a land market because they recognize that when a considerable number of workers settles in a region, it takes the form of a city where economic agents compete for land. One of their main conclusions is that the relative position of a city within the whole network of interactions matters because households and firms relocate between and within cities in response to major changes in economic environment.

Last, but not least, we can still relax the assumption of the CP model relative to preferences for living in the two (or more) regions. If households are not indifferent between regions offering identical wages, because they take into account issues such as pollution or the value of amenities (landscape, climate, etc), new centrifugal forces can appear. Tabuchi and Thisse (2002) argue that although the standard assumption of identical regions is convenient to isolate the effects generated by the interplay between agglomeration and dispersion forces, it does not permit to study the impact of differential amenities. In their model, they show how preference heterogeneity is a strong dispersion force and that dispersed equilibrium is generally asymmetric with the region that has a higher level of amenities being larger than the other one. Mori and Turrini (2005) add skill heterogeneity on mobile labour induced by pecuniary externalities and show that the sustainability of a symmetric location pattern breaks. Thus, they prove that regional inequality is inevitable since more highly skilled workers are attracted by wealthier locations.

Everyone knows that Krugman was, and has been, actually, a very diligent author in this field of Economic Geography[3] but, unfortunately, the CP model he developed is difficult to manipulate analytically and most results in the literature are actually obtained through numerical simulations. In the past few decades, there have been several theoretical advances in this field, thanks not only to new techniques developed in economic modeling and numerical methods, but also in other fields such as industrial economics. Forslid and Ottaviano (2003) modified the CP model in order to render it analytically solvable. The modification consists in assuming that the variable input in manufacturing is unskilled (geographically immobile) labour. As a result, the only role of skilled (geographically mobile) labour is as the fixed input in manufacturing. Their model, called the Footloose Entrepreneur model, reproduces all the features of Krugman's CP model, but allows them to obtain additional analytical results. Furthermore, they investigate the implications of asymmetries between regions by introducing different regional endowments of unskilled workers.

Most of the existing studies were carried out for symmetric CP models, that is, considering identical regions. An exception is the work of Sidorov (2011) who, considering that «*Asymmetric cases remain weakly studied*», studied the CP model of Krugman (1991) with asymmetric regions. Assuming that the share of agricultural population in a region may be greater than in the other region, he showed how the bifurcation diagrams change from the usual tomahawk to other configurations (see also Baldwin *et al.* (2003) and Forslid and Ottaviano (2003)). Berliant and Kung (2009) also studied an asymmetric Core-Periphery model extended with three exogenous parameters (two that parametrize regional fixed inputs and another that parametrizes regional amenity), and argue that the choice of parameters affects the equilibrium diagrams. They provide analytic proofs that the use of pitchfork or tomahawk bifurcations (or other that have crossing in the equilibrium loci) are not robust to generate a core-periphery pattern. As Pflüger and Südekum (2008) highlight *putting too much emphasis on the particular implications of the tomahawk bifurcation type [...] is, therefore, unwarranted because this bifurcation type will only result under certain conditions of the underlying individual preferences.* These

[3] See, for instance, the survey of Schmutzler (1999) that well summarizes and discusses the research already done in this area of NEG models.

authors have discussed how the results of three different Footloose-Entrepreneur new economic geography models with regard to the shape of the location pattern can be reconciled suggesting that certain insights are robust but others are not.

Our aim is, first of all, to study drastic changes in agglomeration patterns in a solvable core-periphery model with two asymmetric regions, namely their predictions concerning the shape of the location pattern. For this purpose, we extend the Footloose Entrepreneur model of Forslid and Ottaviano (2003) by assuming that the distribution of unskilled workers between regions may be non-symmetric and that workers can prefer to live in one of the regions relatively to the other. This preference for one of the regions can reflect regional amenities like scenic beauty, climate or cultural factors. As Baldwin *et al.* (2003) argue in their work, if an exogenous asymmetry in the regional endowments of immobile unskilled workers is considered, catastrophic agglomeration still occurs, but the tomahawk bifurcation disappears. So there is much evidence in the literature that this type of bifurcation is not robust to perturbations that break the exogenous symmetry between regions. Note also that, as Tabuchi and Thisse (2002) argue, regions are not necessarily similar because there are different natural and cultural features and people tend to value differently these local amenities. Empirical evidence shows that natural amenities, such as a coastal location, beautiful scenic and good climate, explain the spatial distribution of industrial activities (Perloff *et al.* (1960); Mills (1972); Black and Henderson (1999); Gallup *et al.* (1999)).[4]

Our model can be seen as a perturbation of the Footloose Entrepreneur model of Forslid and Ottaviano (2003). Therefore, the question of whether this perturbation is generic or not emerges. We provide evidence that it is in fact a universal unfolding, which means that any smooth perturbation that might be added in the model would not lead to new qualitative behaviour.

This dissertation is structured as follows. We begin in section 2 by extending the Footloose Entrepreneur model of Forslid and Ottaviano (2003) and deriving the nominal wages as a new explicit function of the spatial distribution of skilled workers. In section 3, we analyse the conditions for long-run equilibria and their stability concerning the dynamical system, setting the differences with the original model. Section 4 provides a way to recognize our extended model as generic perturbation. We prove, using mathematical properties and some results from bifurcation theory, that our generic perturbation is a universal unfolding.[5] In section 5 a detailed

[4]References cited in Tabuchi and Thisse (2002).

[5]See Golubitsky and Schaeffer (1985) for an extensive overview about characterization and recognition of Universal Unfoldings.

analysis of persistent bifurcation diagrams concerning the dynamics of our extended model is made. We make some concluding remarks in Section 6.

2 The Model

This section describes the Footloose Entrepreneur model developed by Forslid and Ottaviano (2003), with some remarks about the assumption of the symmetry of that model. Thus all comments made here about this model are based on Forslid and Ottaviano (2003) since a good deal of our model is a replication of the construction made by them with the certainty that the modified model reproduces all the features of the original one. To this purpose, we have changed the assumption of symmetry in the distribution of unskilled workers between regions and we have introduced a new variable to measure the attractiveness of a region as Berliant and Kung (2009) have already suggested.[6] This changes allow us to study the changes in agglomeration patterns and they are explicitly and clearly detailed in the considerations that follow.

2.1 General assumptions

In this model, the economy comprises two regions (1 and 2) and two sectors (called *agriculture* and *industry*) that we describe next. In the industrial sector there is monopolistic competition, while in the agricultural sector there is perfect competition. There are two types of labour, skilled labour and unskilled labour, with the assumption that each worker provides one unit of his type of labour inelastically. Introducing some notation, we define $H = H_1 + H_2$ as the allocation of skilled labour and $L = L_1 + L_2$ the allocation of unskilled labour, with L_i and H_i the total allocation of both factors in region $i, i = 1, 2$.

Unskilled workers, unlike what happens in Forslid and Ottaviano (2003) where they are evenly spread across regions, are distributed by the two regions, in the following way: $L_1 = kL$ and $L_2 = (1 - k)L$, with $k \in (0, 1)$ the percentage of unskilled workers in region 1. If $k > \frac{1}{2}$, the market provided by unskilled labour in region 1 is bigger than region 2. If $k < \frac{1}{2}$, region 1 is smaller than region 2. Skilled workers are like individual entrepreneurs who move freely between the two regions. Unskilled workers do not move between the regions. For $k = \frac{1}{2}$ we recover the model of Forslid and Ottaviano (2003).

In the agricultural sector a homogeneous good (A) is produced under constant returns to scale using as production factor only the immobile workers. The industrial sector produces a horizontally differentiated product (X) using skilled labour (fixed cost) and unskilled labour (variable cost) under increasing returns to scale.

[6]Berliant and Kung (2009) suggest the introduction of a parameter γ as a measure of workers' preferences for a particular region due to its climate and landscape.

The transportation of the agricultural good between the two regions is costless, whereas for industrial products there is a cost associated with their transportation, called *iceberg costs*: for one unit of the differentiated good to reach the destination region, $\tau \in (1, +\infty)$ units must be shipped from the region where it is produced.

2.2 Demand

All workers have the same preferences regarding the consumption of industrial goods and agricultural goods. The utility function of a representative consumer in each region is given by:

$$U_1 = (1+\gamma)X_1^{\mu} A_1^{1-\mu}, \tag{2.1}$$

$$U_2 = X_2^{\mu} A_2^{1-\mu}, \tag{2.2}$$

where $\mu \in (0,1)$ is the fraction of expenditure on industrial goods; A_i is the consumption of agricultural goods; and X_i represents a composite index of the consumption of industrial goods defined as a function of constant elasticity of substitution (CES):

$$X_i = \left[\int_{s \in N} d_i(s)^{\frac{\sigma}{\sigma-1}} ds \right]^{\frac{\sigma}{\sigma-1}}, \tag{2.3}$$

where $d_i(s)$ is the consumption of variety s of good X; N is the mass of varieties produced; and, finally, $\sigma > 1$ represents, not only the demand elasticity of any variety, but also the elasticity of substitution between any two varieties.

The consumer's problem involves the maximization of (2.1) and (2.2) subject to the following budget restriction:

$$P_i X_i + p_i^A A_i = Y_i, \tag{2.4}$$

where p_i^A is the price of the agricultural good; P_i is the price index of the industrial goods; Y_i is the local income, comprising wages of skilled (w_i) and unskilled (w_i^L) workers, given by:

$$Y_i = w_i H_i + w_i^L L_i. \tag{2.5}$$

The resolution of (2.4) demonstrates that (Fujita *et al.*, 2001, pp. 46-47), for a given value of the index of industrial goods consumption, X_i, the consumption of each variety, $d(s)$, is chosen to minimize the cost of obtaining X_i, whereby:

$$d_{ji}(s) = \frac{p_{ji}(s)^{-\sigma}}{P_i^{1-\sigma}} \mu Y_i, \ \ i,j = 1,2, \tag{2.6}$$

where $d_{ji}(s)$ is the consumption by residents in region i of a variety produced in region j; p_{ji} is the consumer price of a variety produced in region j and sold in region i; and P_i is the local price index linked to (2.3) defined by:

$$P_i = \left[\int_{n_i} p_{ii}(s)^{1-\sigma} ds + \int_{n_j} p_{ji}(s)^{1-\sigma} ds \right]^{\frac{1}{1-\sigma}}, \tag{2.7}$$

where n_i are the varieties produced in region $i, i = 1, 2$, such that $n_1 + n_2 = N$. In addition, from utility maximization of (2.4) we obtain the consumption of agricultural goods in region i. The fraction of consumers' income spent on consumption of agricultural goods is $(1 - \sigma)$, and therefore:

$$A_i = (1 - \mu) \frac{Y_i}{P_i^A}. \tag{2.8}$$

2.3 Supply

Firms in the agricultural sector produce a homogeneous good under constant returns to scale using only unskilled workers. For the production of each unit of the good, one unit of labour is required. Thus, the unit cost of production and the marginal cost of a firm in the agricultural sector equal the wage of unskilled workers. As already mentioned, firms in this sector operate on the rules of perfect competition which justifies that the price equals marginal cost. Moreover, once the agricultural good has transportation costs associated, its price will be the same in both regions. These considerations justify the wages equality in both regions which, without loss of generality, is assumed to take the unit value: $p_i^A = w_i^L = 1$. Therefore, considering the parameter k, equation (2.5) can be rewritten:

$$\begin{cases} Y_1 = w_1 H_1 + kL \\ Y_2 = w_2 H_2 + (1 - k)L. \end{cases} \tag{2.9}$$

At this point, we distinguish what relates to the model by Forslid and Ottaviano (2003) where $k = 0.5$ and $\gamma = 0$ from the current new model, dependent on the values of k and γ. We use the superscript 0 to indicate quantities in Forslid and Ottaviano (2003) and no superscript for quantities in the current model. Thus, $Y_i^0 = w_i^0 H_i + \frac{L}{2}$ is the local income in region $i, (i = 1, 2)$, in Forslid and Ottaviano

(2003).[7]

In the industrial sector, horizontally differentiated goods are produced ensuring a one-to-one relationship between firms and varieties, i.e, each firm produces only one variety of good X. Thus, each firm produces $x(s)$ units of variety s of the industrial good. Regarding production costs, each firm supports a fixed cost corresponding to α units of skilled labour and a marginal cost corresponding to β units of unskilled labour *per* unit produced:

$$TC_i(s) = w_i\alpha + \beta x_i(s). \tag{2.10}$$

In this point, the model of Forslid and Ottaviano (2003) differs from the model of Krugman (1991). Krugman (1991) assumes that skilled labour is not only an integral part of the fixed cost but also of the variable cost of each firm. According to the costs in (2.10), the profit function of each firm in the industrial sector located in region i is:

$$\Pi(s) = p_{ii}(s)d_{ii}(s) + p_{ij}(s)d_{ij}(s) - \beta[d_{ii}(s) + \tau d_{ii}(s)] - \alpha w_i, \tag{2.11}$$

where $\tau d_{ij}(s)$ is the total supply of the variety s to the region j including the fraction of products that are lost in transport due to the iceberg costs. So, firms choose quantities that maximize their profits (2.11). To solve this maximization problem, the first order conditions are:

$$p_{ii}(s) = \frac{\beta\sigma}{\sigma - 1} \text{ and } p_{ij}(s) = \frac{\tau\beta\sigma}{\sigma - 1}. \tag{2.12}$$

Given the fixed input α required in production, the equilibrium in the skilled labour market implies that, in equilibrium, the number of firms is such that the number of active firms in region i is proportional to the number of skilled workers that live there:

$$n_i = \frac{H_i}{\alpha}. \tag{2.13}$$

2.4 Short-run equilibrium

In short-run equilibrium, given the distribution of industrial workers, demand equals supply in all markets, labour and goods. The industry price index in equation

[7] Note that $w_i \equiv w_i(h, \phi, k)$, whereas $w_i^0 \equiv w_i^0(h, \phi)$.

(2.7), using conditions in (2.12), becomes:

$$P_i = \frac{\beta\sigma}{\sigma - 1} \left[n_i + \phi\ n_j\right]^{\frac{1}{1-\sigma}}, \tag{2.14}$$

where $\phi = \tau^{1-\sigma}$ is a measure of economic integration ("freeness of trade") in the industrial sector. If $\phi = 1$, there are no transportation costs ($\tau = 1$), so there is full economic integration. Conversely, if $\phi = 0$, transportation costs are infinite, so we can conclude that there is no economic integration.

Using (2.13), equation (2.14) can be rewritten and we obtain the two price indices P_1 and P_2:

$$P_1 = \frac{\beta\sigma}{\sigma - 1} \left(\frac{H}{\alpha}\right)^{1/(1-\sigma)} \left[h + \phi(1-h)\right]^{1/(1-\sigma)}, \tag{2.15}$$

$$P_2 = \frac{\beta\sigma}{\sigma - 1} \left(\frac{H}{\alpha}\right)^{1/(1-\sigma)} \left[1 - h + \phi h\right]^{1/(1-\sigma)}, \tag{2.16}$$

In the absence of entry barriers in the manufacturing industry, in equilibrium, profits are zero. Therefore, the wage of skilled workers is given by:

$$w_i = \frac{\mu}{\sigma} \left[\frac{Y_i}{H_i + \phi H_j} + \frac{\phi Y_j}{\phi H_i + H_j}\right]. \tag{2.17}$$

The short-run equilibrium is determined by solving the system, for $i = 1, 2$, composed by equations (2.9), (2.12), (2.13) and (2.17). The system determines, for a given distribution of skilled workers, the variables n_i, p_i, w_i and Y_i. Substituting (2.9) in (2.17), we obtain the equilibrium nominal wage of skilled workers due to the spatial distribution of them:

Proposition 2.1. *The nominal skilled wages are given by:*

$$\begin{cases} w_1 = \mu\ L \dfrac{\sigma\phi H_1 + \left[(\sigma-\mu)k - ((\sigma-\mu)k - \sigma)\phi^2\right] H_2}{\sigma\phi(\sigma-\mu)(H_1^2 + H_2^2) + \left[(\sigma-\mu)^2 + (\sigma^2 - \mu^2)\phi^2\right] H_1 H_2} \\ \\ w_2 = \mu\ L \dfrac{\sigma\phi H_2 + \left[(\sigma-\mu)(1-k) + ((\sigma-\mu)k + \mu)\phi^2\right] H_1}{\sigma\phi(\sigma-\mu)(H_1^2 + H_1^2) + \left[(\sigma-\mu)^2 + (\sigma^2 - \mu^2)\phi^2\right] H_1 H_2}. \end{cases} \tag{2.18}$$

Proof. See Appendix A. □

In this case, we can achieve the wage equation obtained by Forslid and Ottaviano (2003) if we consider $k = 0.5$. Moreover, setting $h = H_1/H$, as the percentage of skilled workers in region 1, and, implicitly, $(1-h)$ as percentage of workers in region 2, it follows that:

$$\frac{w_1}{w_2} = \frac{\sigma\phi h + \left[(\sigma-\mu)k - ((\sigma-\mu)k-\sigma)\phi^2\right](1-h)}{\sigma\phi(1-h) + \left[(\sigma-\mu)(1-k) + ((\sigma-\mu)k+\mu)\phi^2\right]h}. \tag{2.19}$$

In Figure 1 we can observe how the wages ratio reacts to different values of k. For instance, if region 1 is bigger than region 2 ($k > \frac{1}{2}$), the red line tells us that the real wage in region 1 increases relative to that in region 2. In this case, in equilibrium, there would be a greater fraction of skilled workers preferring region 1.

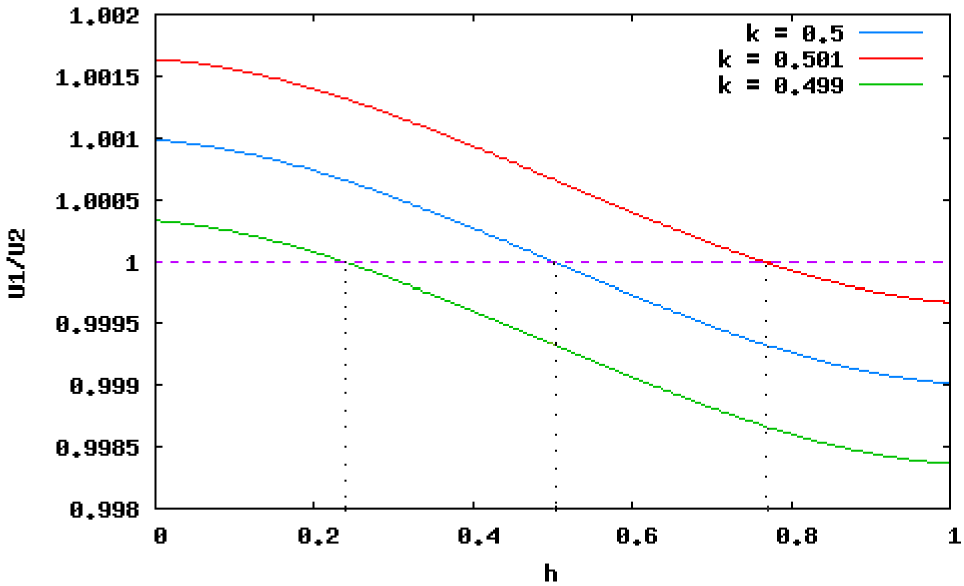


Figure 1: Indirect utility for different values of k and $\gamma = 0$.

3 Long-run equilibrium and stability

A long-run equilibrium is a spatial distribution of skilled workers that remains unchanged over time, i.e., such that there are no incentives for skilled workers to migrate across regions. A long-run equilibrium is stable if, after a small deviation from the equilibrium distribution, the spatial distribution of skilled workers returns to the initial one.

In the long-run, skilled workers choose their location in order to maximize their indirect utility. They take into account not only the real wage that they earn, but also the regional amenities, such as scenic beauty, climate or cultural factors. These amenities are captured by the parameter γ. As we have already said the two regions are not, necessarily, equally attractive. A positive value of γ means that region 1 is more attractive than region 2. When $\gamma = 0$, the regions are symmetric in terms of amenities. So, choosing $\gamma = 0$ is a necessary and sufficient condition for both regions to be identical, as in Forslid and Ottaviano (2003).

The direction of skilled worker migration depends on the sign of the indirect utility differential between the two regions. The speed of migration is, for our purposes, irrelevant. As Forslid and Ottaviano (2003), we assume that it is equal to the indirect utility differential:

$$\dot{h} = \begin{cases} W(h,\phi), & \text{if } 0 < h < 1 \\ \min\{0, W(h,\phi)\}, & \text{if } h = 1 \\ \max\{0, W(h,\phi)\}, & \text{if } h = 0 \end{cases}, \tag{3.1}$$

where,

$$W(h,\phi) = \eta \left[(1+\gamma)\frac{w_1}{P_1^u} - \frac{w_2}{P_2^u} \right], \tag{3.2}$$

$\eta = \mu^\mu (1-\mu)^{(1-\mu)}$ and γ is the parameter that measures the relative attractiveness of region 1.

The dynamics of migration in (3.1) capture interior and boundary dynamics. The spatial equilibrium implies that $\dot{h} = 0$. If $W(h,\phi)$ is positive, then if there are skilled workers in region 2, they will have greater utility in region 1, and thus some workers will move from region 2 to region 1 until real wages in the two regions are balanced and the equilibrium is achieved again. If $W(h,\phi)$ is negative, the opposite will happen.

Note that it is possible to have $h = 0$ while $\dot{h} \neq 0$, unlike the dynamics referred to in Berliant and Kung (2009) and Castro *et al.* (2013).

As we can see $W(h,\phi)$ is the focal point of migration dynamics. So, substituting (2.15) and (2.16) in (3.2) we obtain:

$$W(h,\phi) = \Omega(h,\phi)V(h,\phi), \tag{3.3}$$

where,

$$\Omega(h,\phi) = \eta\frac{\mu L}{H}\frac{1}{\sigma\phi(\sigma-\mu)[h^2+(1-h)^2]+\left[(\sigma-\mu)^2+\phi^2(\sigma^2-\mu^2)\right]h(1-h)}\left(\frac{\beta\sigma}{\sigma-1}\right)^{-\mu}\left(\frac{H}{\alpha}\right)^{\frac{-\mu}{1-\sigma}},$$

and,

$$V(h,\phi) = (1+\gamma)\frac{\sigma\phi h+\left[(\sigma-\mu)k-\phi^2((\sigma-\mu)k-\sigma)\right](1-h)}{[h+\phi(1-h)]^{\frac{\mu}{1-\sigma}}} - \tag{3.4}$$
$$\frac{\sigma\phi(1-h)+\left[(\sigma-\mu)(1-k)-\phi^2((\sigma-\mu)k+\mu)\right]h}{[1-h+\phi h]^{\frac{\mu}{1-\sigma}}}.$$

Similarly to what happens in Forslid and Ottaviano (2003), equation (3.3) shows that all that matters to the determination of equilibria, that is, to $W(h,\phi)=0$, is $V(h,\phi)$ because $\Omega(h,\phi)$ is a positive bundling for all values of parameters and variables.

We are next interested in studying the dispersion and the agglomeration configurations. Dispersion occurs when none of the regions is empty. This includes the extreme situation where we have one of the regions with only one skilled worker and all other in another region. On the other hand, agglomeration occurs when all skilled workers are concentrated in one region in such a way that the other region has no skilled worker.[8]

3.1 Stable Dispersion

One of the two existing equilibria is that for which all skilled workers are dispersed across two regions in such a way that no region is without skilled workers. This type of equilibria, also called interior equilibria, are solutions to $V(h,\phi)=0$. This condition ensures that each skilled worker has no incentive to migrate to the other region because they look to the other and note that both regions provide them with the same indirect utility.

[8]We stress that we use the terms agglomeration and concentration as equivalent.

Interior equilibria ($0 < h < 1$) are stable if, and only if, the slope of $W(h, \phi)$ is non-positive in the neighbourhood of equilibrium.

Definition 3.1. *A dispersion equilibrium distribution is an* $h^* \in (0, 1)$ *such that* $V(h^*, \phi) = 0$. *It is stable if* $\frac{\partial W(h,\phi)}{\partial h} < 0$.

Considering a dispersion equilibrium with $0 < h < 1$ if the differential real wages $W(h, \phi)$ are decreasing in h, then continuity of real wages in the share of skilled workers ensures that if there was a marginal and exogenous migration of skilled workers to the other region all the skilled workers return to their region of origin.

Proposition 3.2. *A sufficient condition for the stability of a dispersion equilibrium is:*

$$\frac{\partial V(h, \phi)}{\partial h}(h^*, \phi) < 0. \tag{3.5}$$

Proof. By definition 3.1, the stability of a dispersion equilibrium is given by:

$$\frac{\partial W(h^*, \phi)}{\partial h} = \frac{\partial \Omega}{\partial h} V(h^*, \phi) + \Omega \frac{\partial V(h^*, \phi)}{\partial h}.$$

An equilibrium implies that $V(h^*, \phi) = 0$. Then, since $\Omega > 0$, we have that $\frac{\partial V(h^*,\phi)}{\partial h} < 0 \Rightarrow \frac{\partial W(h^*,\phi)}{\partial h} < 0$. □

From the moment that we introduced asymmetries it makes no sense to use the concept of break point of Fujita *et al.* (1999). This is because the asymmetries introduce a discontinuity in the equilibrium branches. Then, there are situations when for particular values of ϕ there are no interior equilibria.[9] But, solving $V(h, \phi) = 0$ and defining h as a function of ϕ, if there exists a ϕ_f such that $\frac{\partial h}{\partial \phi}(\phi_{\phi_f}) = 0$ we have a point where the interior equilibrium looses stability as we increase continuously the value of ϕ. We call this point a *Fold Point.*[10] Because of the existence of boundaries this point mostly does not appear in our bifurcation diagrams. But we know it exists as we can verify in section 5.

[9]See for instance the bifurcation diagram in Figure 4(b) where there is no point where interior equilibria switches from stable to unstable.

[10]Under some genericity conditions, namely $\frac{\partial^2 h}{\partial \phi^2} \neq 0$, it is related to a fold bifurcation.

3.2 Stable Agglomeration

Another existing equilibrium configuration is that for which all skilled workers are concentrated in one of the two regions and the other region is without skilled workers. Given the dynamical system referred to in (3.1), these configurations occur when $h = 0$ and $h = 1$ and are equilibria if and only if $V(0, \phi) < 0$ and $V(1, \phi) > 0$, respectively. If $V(0, \phi) > 0$ and $V(1, \phi) < 0$, those total agglomeration configurations are not equilibria. It is because workers only have incentive to remain in a region if the indirect utility they obtain in that region is higher than in the other. Otherwise they want to migrate to the region that offers them higher indirect utility.

Definition 3.3. *Agglomeration in region 1 (region 2) is an equilibrium distribution if and only if $V(1, \phi) \geq 0$ ($V(0, \phi) \leq 0$). It is stable if the inequality is strict.*

Unlike what happens in Forslid and Ottaviano (2003), here $V(0, \phi) \neq -V(1, \phi)$ if $k \neq 0.5$ or $\gamma \neq 0$. This shows how the new assumptions introduced in the model can perturb not only the interior equilibria but also the boundary equilibria, if they exist. We have that:

$$V(0, \phi) = (1+\gamma)\frac{(\sigma-\mu)k - \phi^2((\sigma-\mu)k - \sigma)}{\phi^{\frac{\mu}{1-\sigma}}} - \sigma\phi, \tag{3.6}$$

and

$$V(1, \phi) = (1+\gamma)\sigma\phi - \frac{(\sigma-\mu)(1-k) + \phi^2((\sigma-\mu)k + \mu)}{\phi^{\frac{\mu}{1-\sigma}}}. \tag{3.7}$$

If we set $k = 0.5$ and $\gamma = 0$, we certainly obtain that:

$$V(0, \phi) = -V(1, \phi) = \frac{0.5(\sigma-\mu) + (0.5\sigma + 0.5\mu)\phi^2}{\phi^{\frac{\mu}{1-\sigma}}} - \sigma\phi. \tag{3.8}$$

As mentioned previously, an equilibrium is stable if, for any small deviation from equilibrium, the spatial distribution of workers is pulled back to the initial one. According to the dynamical system in (3.1) the total agglomeration solutions ($h = 0$ and $h = 1$) are always stable as long as they are equilibria.

Fujita *et al.* (1999) named *sustain point* the threshold value of ϕ above which concentration is a stable equilibrium. Because in (3.1) stability and existence of a concentration equilibrium coincide, we refer to a sustain point as a value of ϕ such that concentration is an equilibrium on only one side of ϕ. In our modified model, due to the presence of the parameters γ and k, which introduce some asymmetries between both regions, we will have two *sustain points*, one for agglomeration in region 1 ($h = 1$) and another for the agglomeration in region 2 ($h = 0$) which are, respectively and implicitly defined by ϕ_{s_1} and ϕ_{s_2}:

$$(1+\gamma)\sigma\phi_{s_1}^{\frac{\mu}{1-\sigma}+1} - (\sigma-\mu)(1-k) - \phi_{s_1}^2((\sigma-\mu)k+\mu) = 0, \tag{3.9}$$

$$(1+\gamma)\left[(\sigma-\mu)k - \phi_{s_2}^2((\sigma-\mu)k-\sigma)\right] - \sigma\phi_{s_2}^{\frac{\mu}{1-\sigma}+1} = 0. \tag{3.10}$$

In the event of the existence of two solutions in one of these equations, then as we shall see in section 5, Figures 3.1(b), 6 and 8(a), it can occur that agglomeration is stable for $\phi \in (\phi_{s_1}, \phi_{s_2})$, where ϕ_{s_1} and ϕ_{s_2} are those two solutions.

4 Genericity

When perturbing an existing model, there is always the question of determining how generic the perturbation is or should be. In this section, we use results from bifurcation theory to show that our perturbation of Forslid and Ottaviano (2003) describes qualitatively all possible perturbations of their model with a minimum number of parameters. Therefore, the bifurcation diagrams we exhibit in section 5 are generically the only ones appearing in any perturbation of Forslid and Ottaviano (2003).

Let $g(x, \lambda)$ be a smooth map for which a bifurcation occurs at (x_0, λ_0) and let $G(x, \lambda, \alpha, \beta)$ be a two-parameter family of perturbations of g. This means $G(x, \lambda, 0, 0) = g(x, \lambda)$. We say (see Golubitsky and Schaeffer (1985), Definition 1.3) that $G(x, \lambda, \alpha, \beta)$ is a *versal unfolding* of g if any perturbation of g can be transformed, through a convenient smooth change of coordinates, into $G(x, \lambda, \alpha, \beta)$. If two is the minimum number of unfolding parameters in a versal unfolding then G is said to be *universal*.

A universal unfolding is a parametrized family including all the possible perturbations with a minimum number of parameters of a given problem. This means that any smooth perturbation that might be added in the model would not lead to new qualitative behaviour, that is, to behaviour not already present for the proposed two-parameter family of perturbations (in our case, γ and k).

According to Golubitsky and Schaeffer (1985), any bifurcation problem $g(x, \lambda)$ which at a specific point satisfies $g_x = g_{xx} = g_\lambda = 0$, $g_{xxx} \neq 0$ and $g_{\lambda x} \neq 0$ is equivalent to the normal form for the pitchfork bifurcation which is given by $f(x, \lambda) = \pm x^3 \pm \lambda x$. Moreover, according to Proposition 4.4 of Golubitsky and Schaeffer (1985), if $G(x, \lambda, \alpha, \beta)$ is a two-parameter unfolding of a map g equivalent to $f(x, \lambda) = \pm x^3 \pm \lambda x$, then G is a universal unfolding of g if and only if:

$$\det \begin{pmatrix} 0 & 0 & g_{x\lambda} & g_{xxx} \\ 0 & g_{\lambda x} & g_{\lambda\lambda} & g_{\lambda xx} \\ G_\alpha & G_{\alpha x} & G_{\alpha\lambda} & G_{\alpha xx} \\ G_\beta & G_{\beta x} & G_{\beta\lambda} & G_{\beta xx} \end{pmatrix} \neq 0,$$

at the bifurcation point of the unperturbed problem.[11]

[11] In this section the subscript denotes the partial differentiation with respect to the corresponding argument.

Let us say that G is, in our context, the dynamics function $W(h,\phi)$ in our extended model [referred to in section 2, equation (3.2)] and g is the corresponding function in Forslid in Ottaviano (2003) in such a way that $W(h,\phi,\frac{1}{2},0) = W^0(h,\phi)$. [12] So, k and γ are, in this context, the two unfolding parameters. In this way, we have that:

$$G(h,\phi,k,\gamma) = W(h,\phi) = \eta\left((1+\gamma)\frac{w_1}{P_1^u} - \frac{w_2}{P_2^u}\right), \tag{4.1}$$

and

$$g(h,\phi) = W^0(h,\phi) = \eta\left(\frac{w_1^0}{P_1^\mu} - \frac{w_2^0}{P_2^\mu}\right). \tag{4.2}$$

Furthermore, we can say that g in Forslid and Ottaviano (2003) is equivalent to $f(h,\phi) = h^3 + \phi h$ because g satisfies the conditions $g_h = g_{hh} = g_\phi = 0$ and $g_{hhh} > 0$ and $g_{\phi h} > 0$ at the specific bifurcation point $(1/2, \phi_b^0)$ on Forslid and Ottaviano (2003).[13]

Thus, to prove the existence of a universal unfolding we need to calculate the following determinant and verify that it is non-zero:

$$D = \det\begin{pmatrix} 0 & 0 & g_{h\phi} & g_{hhh} \\ 0 & g_{\phi h} & g_{\phi\phi} & g_{\phi hh} \\ G_\gamma & G_{\gamma h} & G_{\gamma\phi} & G_{\gamma hh} \\ G_k & G_{kh} & G_{k\phi} & G_{khh} \end{pmatrix}, \tag{4.3}$$

at the bifurcation point of the unperturbed problem: $(h,\ \phi,\ k,\ \gamma)$=$(\frac{1}{2},\ \phi_b^0,\ \frac{1}{2},\ 0)$, where $k = \frac{1}{2}$ and $\gamma = 0$ correspond to the absence of perturbation.[14]

Here, it should be noted that G_k and G_γ are the partial derivatives of $G(h,\phi,k,\gamma)$ with respect to unfolding parameters k and γ, respectively. The signs of these derivatives are useful in the calculation of the sign of D and also provide information on how the current indirect utility differential reacts to a variation in each parameter.

Lemma 4.1. *An increase in agricultural population in region 1 relatively to region 2 causes an increase in real wages of industrial workers in region 1.*

[12]For more details about the function g see Forslid and Ottaviano (2003), p. 235.

[13]In Forslid and Ottaviano (2003), p. 236, these conditions are confirmed and the expression of ϕ_b^0 is also specified.

[14]See the expressions for the derivatives in Appendix B.

Proof. It is sufficient to show that $G_k > 0$ for any values of the parameters:

$$G_k(h,\phi,k,\gamma) = \frac{\eta L\mu \left(\frac{(\gamma+1)(1-h)(\sigma-\mu)(1-\phi^2)}{(h+\phi(1-h))^{\frac{\mu}{1-\sigma}}} + \frac{(h(\sigma-\mu)(1-\phi^2))}{(\phi h+1-h)^{\frac{\mu}{1-\sigma}}} \right)}{\left(\frac{\beta\sigma}{\sigma-1}\right)^{\mu}\left(\frac{H}{\alpha}\right)^{\frac{\mu}{1-\sigma}}\left[(1-h)h(\phi^2(\sigma^2-\mu^2)+(\sigma-\mu)^2)+\phi(h^2+(1-h)^2)\sigma(\sigma-\mu)\right]},$$

$$sign(G_k(h,\phi,k,\gamma)) = sign\left(\frac{(\gamma+1)(1-h)(\sigma-\mu)(1-\phi^2)}{(h+\phi(1-h))^{\frac{\mu}{1-\sigma}}} + \frac{(h(\sigma-\mu)(1-\phi^2))}{(\phi h+1-h)^{\frac{\mu}{1-\sigma}}} \right), \tag{4.4}$$

since $\sigma > \mu$, $h \in [0,1]$ and $\phi \in (0,1)$, we have $G_k(h,\phi,k,\gamma) > 0$. □

Lemma 4.2. *An increase in workers' preferences for region 1 relatively to region 2 causes an increase in real wages of industrial workers in region 1.*

Proof. It is sufficient to show that $G_\gamma > 0$ for any values of the parameters:

$$G_\gamma = \frac{\left(\frac{\beta\sigma}{\sigma-1}\right)^{\mu}\left(\frac{H}{\alpha}\right)^{\frac{\mu}{1-\sigma}} L\eta\mu\left((1-h)\left[k(\sigma-\mu)(1-\phi^2)+\sigma\phi^2\right]+\phi h\sigma\right)}{\left[h+\phi(1-h)\right]^{\frac{\mu}{1-\sigma}}\left[(1-h)h(\phi^2(\sigma^2-\mu^2)+(\sigma-\mu)^2)+\phi\sigma(h^2+(1-h)^2)(\sigma-\mu)\right]},$$

$$sign(G_\gamma(h,\phi,k,\gamma)) = sign\left((1-h)\left[k(\sigma-\mu)(1-\phi^2)+\sigma\phi^2\right]+\phi\sigma h\right), \tag{4.5}$$

since $\sigma > \mu$, $h \in [0,1]$ and $\phi \in (0,1)$, we have $G_\gamma(h,\phi,k,\gamma) > 0$. □

We can also prove that $g_{\phi\phi} = 0$ and $g_{\phi hh} = 0$ at the bifurcation point $(h, \phi, k, \gamma)=(\frac{1}{2}, \phi_b^0, \frac{1}{2}, 0)$, which simplifies the calculation of the determinant D in (4.3).

Lemma 4.3. *The derivatives $g_{\phi\phi} = 0$ and $g_{\phi hh} = 0$ at $(h, \phi, k, \gamma)=(\frac{1}{2}, \phi_b^0, \frac{1}{2}, 0)$.*

Proof. See Appendix C. □

In this way, the determinant we have to calculate simplifies to:

$$D = \det\begin{pmatrix} 0 & 0 & g_{h\phi} & g_{hhh} \\ 0 & g_{\phi h} & 0 & 0 \\ G_\gamma & G_{\gamma h} & G_{\gamma\phi} & G_{\gamma hh} \\ G_k & G_{kh} & G_{k\phi} & G_{khh} \end{pmatrix}.$$

By inspection we observe that all entries in D have the common factor:

$$\frac{L\eta\mu}{\left(\frac{\beta\sigma}{\sigma-1}\right)^{\frac{\mu}{1-\sigma}}\left(\frac{H}{\alpha}\right)^{\frac{\mu}{1-\sigma}}}.$$

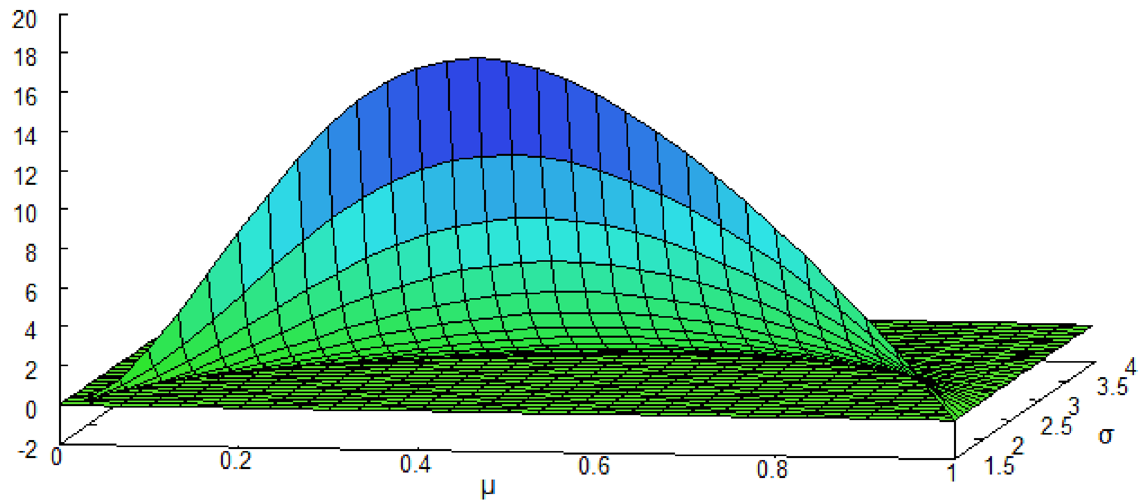
Then the determinant comes:

$$D=\left(\frac{L\eta\mu}{\left(\frac{\beta\sigma}{\sigma-1}\right)^{\frac{\mu}{1-\sigma}}\left(\frac{H}{\alpha}\right)^{\frac{\mu}{1-\sigma}}}\right)^{4}\times\overline{D},$$

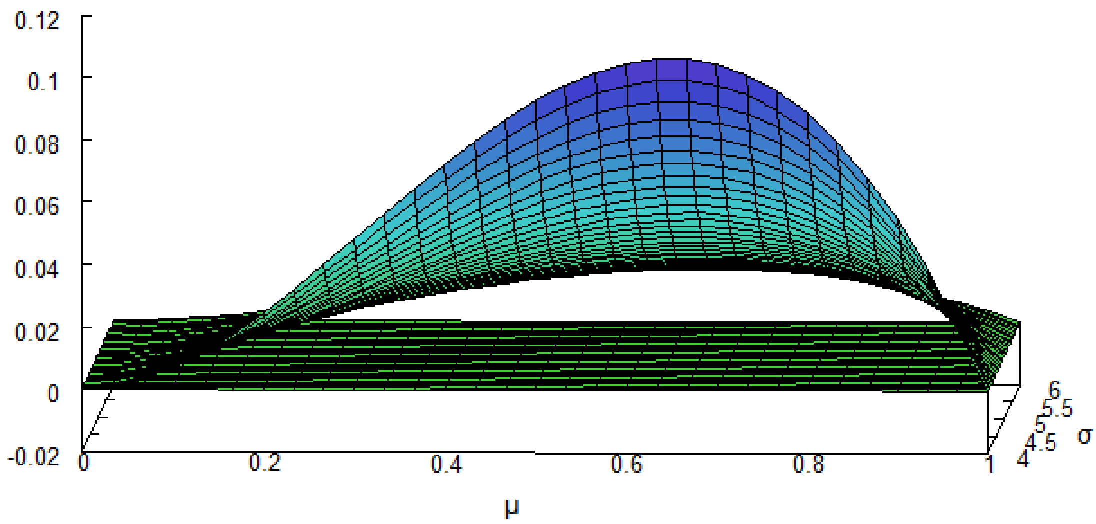
and $D\neq 0 \Leftrightarrow \overline{D}\neq 0$.

In Figures 2(a) and 2(b) we plot $\overline{D}$ to see that it is positive for all values of $\mu\in(0,1)$ and $\sigma\in(1,+\infty)$.

We succeeded in proving that our bifurcation problem is a universal unfolding of that in Forslid and Ottaviano (2003). So this proves that the tomahawk bifurcation and others that have crossing equilibrium loci are not robust to generate a core-periphery pattern under general perturbations because of what we know about the bifurcation diagrams of the unfolded problem. Berliant and Kung (2009) also argue this fact. They extended the model with three exogenous parameters (one that parametrizes the workers' preferences over regions and two other that parametrize "regional fixed inputs") and proved that there is a path set of parameters where those classes of bifurcations do not appear. But their model is a particular case, in terms of qualitative behaviour, of our current model. The reason is that our modified model is a universal unfolding. This means that from our extended model we can obtain the same qualitative behaviour in bifurcation diagrams that they may eventually achieve, but the opposite may not be possible. That is, our universal unfolding allow us to find particular agglomeration patterns they may not be able to achieve perturbing the parameters of their model. Furthermore, the way they achieve that conclusion is redundant because there is no need to introduce three parameters in the model. As a convenience matter on choosing parameters we get that conclusion using only two parameters (the minimum number of unfolding parameters in our versal unfolding) and therefore the term *universal*. In this way, we shall see in the next section that any smooth perturbation in the unfolding parameters generates different agglomeration patterns as the symmetry between regions and distribution of agricultural population cease to exist.



(a) determinant for $\sigma \in (1, 4)$.



(b) determinant for $\sigma \in (4, 6)$.

Figure 2: Determinant $\overline{D}$.

5 Robustness and bifurcation diagrams

In this section we will address the issue of bifurcation, that is, we will study the changes that occur in terms of equilibria and their stability depending on the parameters. Obviously, we are particularly interested in economic implications in terms of the population migration dynamics. In particular, we study the change in agglomeration patterns as ϕ varies for several combinations of the symmetry-breaking unfolding parameters, k and γ. In this way, we will describe the configurations of long-term equilibria and the respective diagrams that are obtained by giving different combinations of values to parameters. As we will see, we have several examples of bifurcations that do not have crossing of the equilibrium loci, which reinforces the conclusion we drew in the previous section.

We recall that characterizing the behaviour of our model with such asymmetry requires numerical simulation. So, all bifurcation diagrams we plot were obtained by numerical computation. Furthermore, hereinafter the stable long-run equilibria are illustrated with green lines, while the unstable long-run equilibria are marked by red lines.

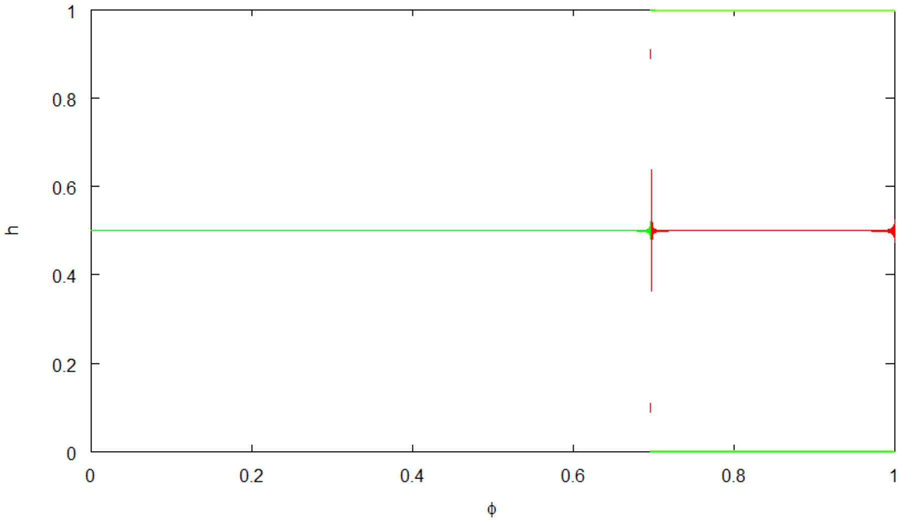
All of the bifurcation diagrams we plot here were obtained by setting $\sigma = 4$ and $\mu = 0.3$, the values that Krugman (1991) uses in his benchmark model of *New Economic Geography*.

First of all it is important to show that if we consider the values that turn us to the Forslid and Ottaviano (2003) model ($k = 0.5$ and $\gamma = 0$) we obtain the expected bifurcation diagram in Figure 3(a).

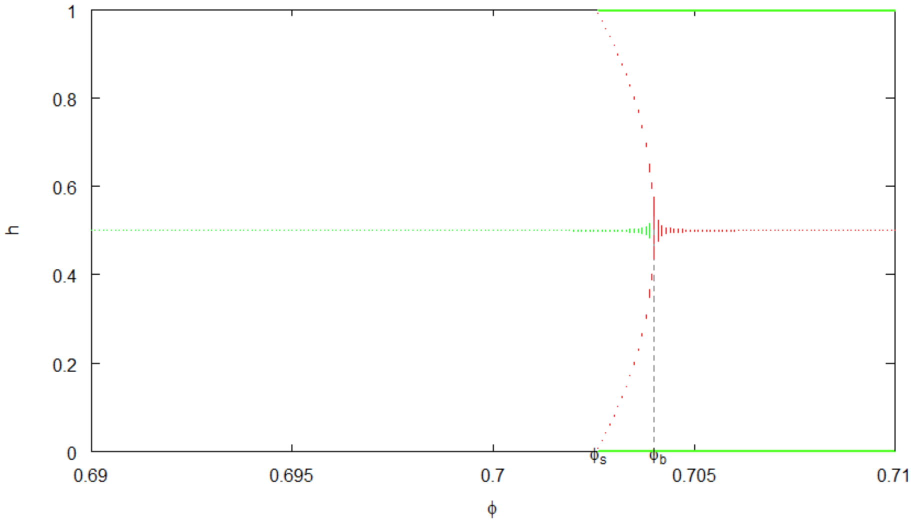
If we set a particular zoom for the convenient values of ϕ we can better observe in Figure 3(b) the tomahawk that we know always happens in the symmetric Footloose Entrepreneur model.

Let ϕ_s and ϕ_b be the values of ϕ corresponding, respectively, to the break-point and sustain-point in Forslid and Ottaviano (2003). As expected, in the symmetric case, for $\phi < \phi_s$, $h = \frac{1}{2}$ is the only stable equilibrium. For $\phi > \phi_b$ there are three equilibria, full agglomeration configurations $h = 0$ and $h = 1$ and symmetric outcome $h = \frac{1}{2}$, but only the agglomeration ones are stable. Finally, for $\phi_s < \phi < \phi_b$ there are five equilibria. Two are the agglomeration configurations which are stable, two other are interior asymmetric equilibria which are unstable and the last one is the symmetric outcome and it is also stable.

Including boundaries in h on the tomahawk diagram the hysteresis effect emerges. Hysteresis means that the equilibrium is sensitive, not only to the change in value of the parameter ϕ, but also to the way in which this change occurs, that is, increasing



(a)



(b)

Figure 3: Bifurcation diagram for the unperturbed model ($k = \frac{1}{2}$ and $\gamma = 0$).

and decreasing the parameter does not lead to the same result. Whenever $\phi > \phi_s$, the bifurcation diagram features multiple stable long-run equilibria. This means that temporary shocks, for example, temporary policy changes, may have hysteresis effects on the location of the industrial workers. For instance, let us consider a temporary policy change that takes ϕ to a value between the sustain point and the break point. If we consider a larger past value of ϕ, this decrease on ϕ leads to agglomeration equilibria. But if we consider a smaller past value, then the increase on ϕ leads to a dispersion equilibrium (see Figure 3(b)).

We also plot the bifurcation diagrams with exogenous asymmetries between regions. Like Forslid and Ottaviano (2003) do, we study the case where there are only differences in the regional endowment of immobile unskilled workers, that is $k \neq \frac{1}{2}$. Recalling, if $k > (<)\frac{1}{2}$ region 1 is bigger (smaller) than region 2. If we gradually decrease the value of k, region 2 will become increasingly bigger than region 1 and we obtain the bifurcation diagrams like these in Figure 4.

As we mentioned in section 2 there will be two sustain points (ϕ_{s_1} for agglomeration in region 1 and ϕ_{s_2} for the agglomeration in region 2). A key contrast between Figure 4 and the respective diagram in Forslid and Ottaviano (2003)[15] is the existence of a unique unstable asymmetric equilibrium. Introducing boundaries on h the asymmetric equilibria in first branch on their diagram disappear. Consequently the *break point* also ceases to exist. According to Figure 4(a), where $k = 0.499$, for $\phi < \phi_{s_2}$ there is only one equilibrium which is stable and for which there are more firms in the biggest region. For $\phi_{s_2} < \phi < \phi_{s_1}$ the agglomeration in region 2, where there are more unskilled workers, is the only equilibrium. Only when $\phi > \phi_{s_1}$ there are also an agglomeration equilibrium in region 1 and an unstable asymmetric equilibrium with partial agglomeration in region 1.

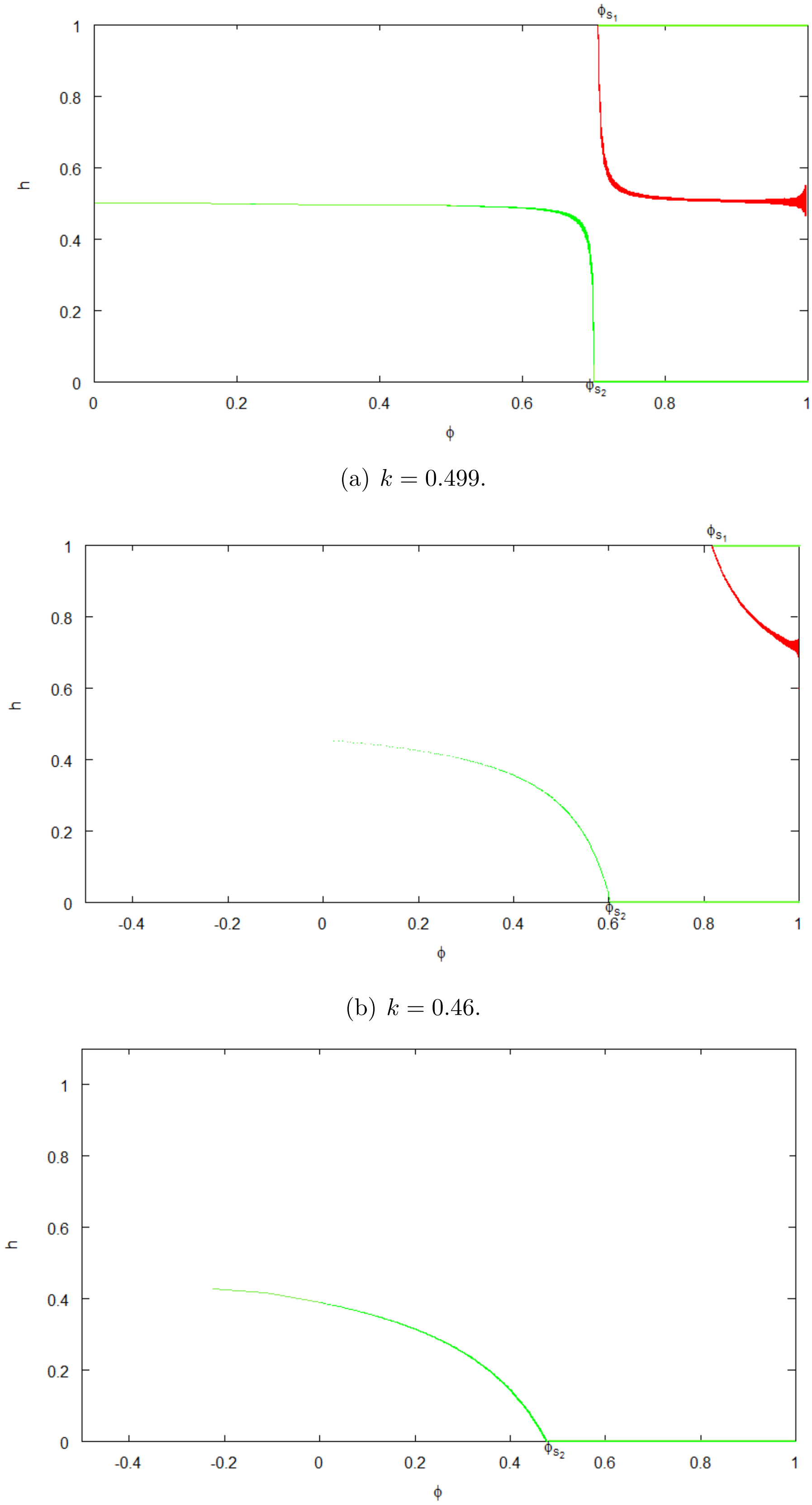
If we decrease the value of k to $k = 0.46$ and then to $k = 0.4$ we obtain the bifurcation diagrams in Figures 4(b) and 4(c), respectively. These show that it is necessary to have more of economic integration for the agglomeration equilibrium in region 1 to be stable.[16] Furthermore, as economic integration proceeds (that is, as ϕ increases), the smaller the region 1 is, the more quickly the agglomeration equilibrium in region 2 is reached. There is a value of k for which region 1 becomes so small that there is no longer agglomeration equilibrium (stable or unstable) in that region and the greater fraction of population chooses to be in the biggest region. For high transportation costs, that is, when there is a small level of economic integration, the only equilibrium is the asymmetric stable equilibrium where most of population concentrates in the region 2. As the transportation costs decrease towards more economic integration population tends to concentrate increasingly in the region 2. For $\phi > \phi_{s_2}$ the only equilibrium is the stable agglomeration in the region 2.

We can conclude that the distribution of unskilled workers substantially disturbs the equilibria of the model and perturbing only the parameter k there is a value of k beyond which the region with more unskilled workers is the optimal choice to all workers.

If we focus on the perturbations only in the parameter γ we observe that the bifurcation diagrams also change significantly. That is perturbing singly each of our parameters produces different effects. In Figure 5 we can observe the effects of an

[15] See Figure 2 on Forslid and Ottaviano (2003) p. 239.

[16] See how ϕ_{s_1} increases as region 2 increases its agricultural population relative to region 1.



(a) $k = 0.499$.

(b) $k = 0.46$.

(c) $k = 0.4$.

Figure 4: Bifurcation diagram perturbing only the parameter k.

increase in γ where region 1 becomes more attractive than region 2.

A smooth increase in the preferences of workers for a particular region, in the case of the Figure 5(a), where $\gamma = 0.001$, for region 1, exhibits for high transportation costs an asymmetric stable equilibrium with a greater fraction of firms operating in the region 1. Furthermore, that increase in the preferences for lower transportation costs also leads to the emergence of a stable full agglomeration equilibrium in region 2 at the same time as an unstable dispersion equilibrium with more firms operating in region 2. When there is a sufficiently low level of trade costs, the workers prefer the less attractive region. That is, if the attractiveness level of region 1 is not sufficiently high then when the transportation costs are low there can also exist an agglomeration equilibrium in the other region. This shows that as the economic integration proceeds, different levels of preferences for a region lead to different impacts on diagrams.

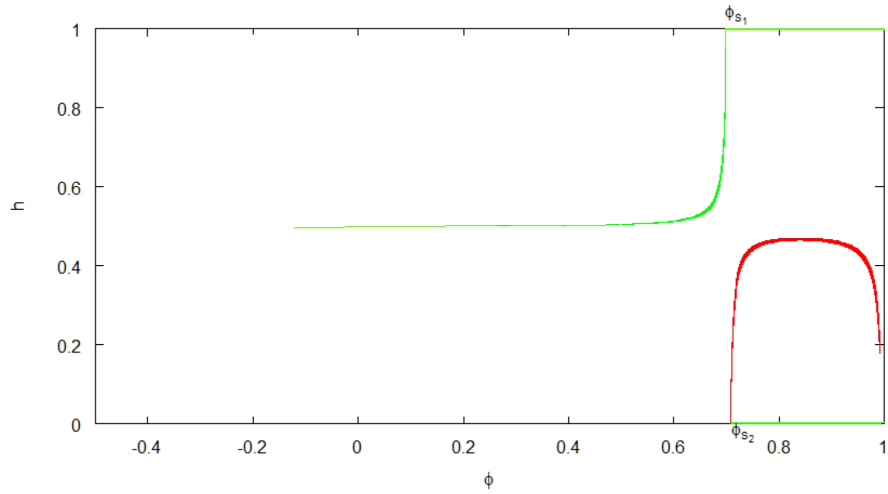
If we gradually increase the value of γ, region 1 will become increasingly more attractive than region 2 and we obtain the bifurcation diagrams like those in Figures 5(b) and 5(c) where we assume $\gamma = 0.01$ and $\gamma = 0.02$, respectively. Looking at these bifurcation diagrams we can note that, as economic integration proceeds, the more attractive region 1 is, the more quickly the agglomeration stable equilibrium in that region is reached.

A new feature appears as illustrated in Figure 5(b): as expected concentration becomes an equilibrium for high ϕ but, in this case, concentration in region 2 (the least preferred) is no longer an equilibrium when economic integration is close to perfect. In this case, workers move to the most preferred region. This phenomenon occurs also for situations depicted in other figures.

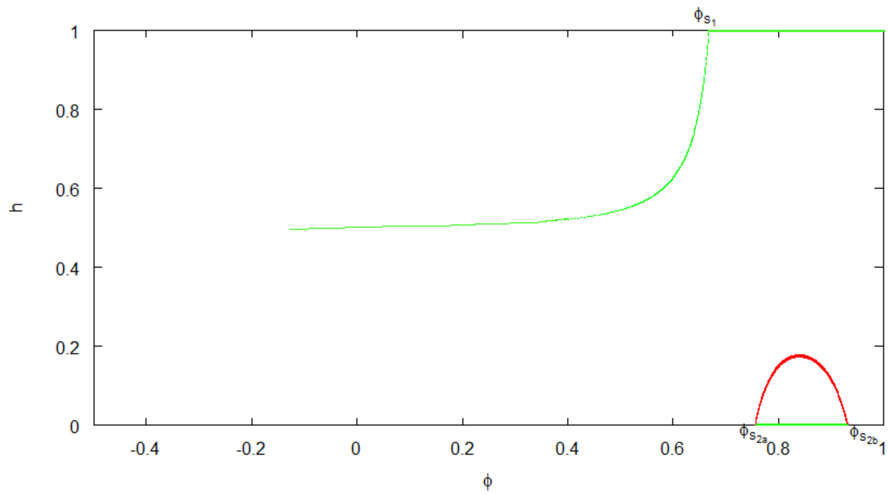
According to Figure 5(c) where the attractiveness measure is sufficiently high we can see that the second branch of equilibria disappears (the asymmetric unstable equilibrium and the agglomeration in the region 2). It is important to note that the parameter γ is more sensitive than k. In Figures 4 and 5 we can verify that the same smooth perturbation in γ leads to stronger changes in equilibria configurations than to the same smooth perturbation in k.

However, we are also interested in configurations arising from perturbations of both parameters simultaneously. Here we have to distinguish two situations. One when we perturb both parameters in such a way that one of the regions is favoured (for example when we set $k > \frac{1}{2}$ and $\gamma > 0$ the region 1 is bigger and more attractive). On the other hand we also show a situation when there is compensation between parameters (for example if we set region 1 bigger but region 2 more attractive).

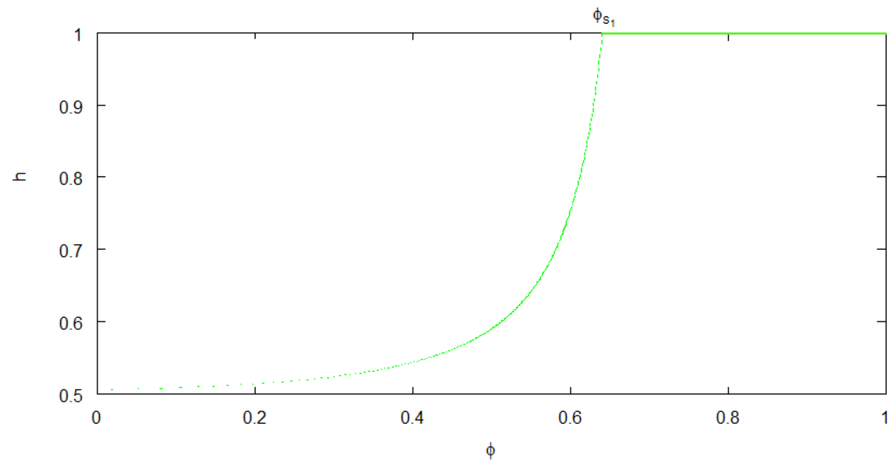
The first situation is shown in the bifurcation diagram in Figure 6. This Figure



(a) $\gamma = 0.001$.



(b) $\gamma = 0.01$.



(c) $k = 0.02$.

Figure 5: Bifurcation diagram perturbing only the parameter γ.

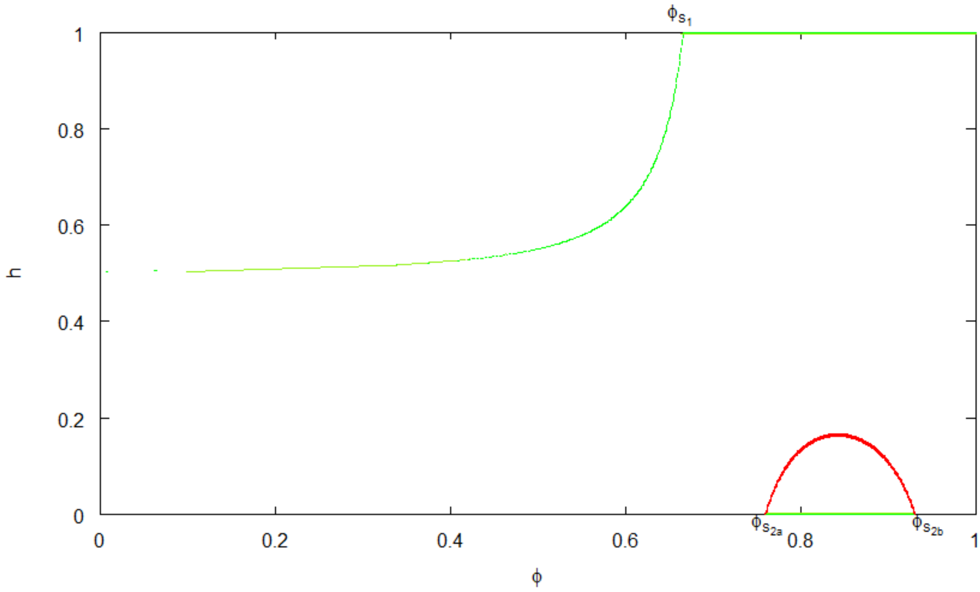


Figure 6: Bifurcation diagram for the perturbed model ($k = 0.501$ and $\gamma = 0.01$).

is similar to Figure 5(b) where the asymmetry is caused by γ only. This does not mean that the parameter k does not add new qualitative behavior. In fact, Figure 4 shows that perturbing only the parameter k the agglomeration patterns change significantly.

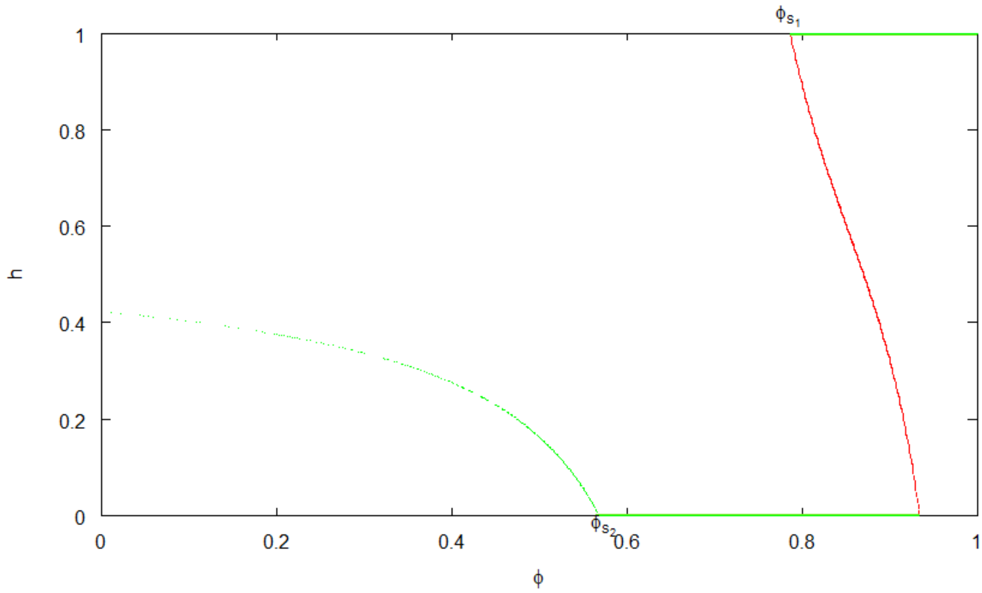


Figure 7: Bifurcation diagram for the perturbed model ($k = 0.426$ and $\gamma = 0.0194$).

In Figure 7 we show the second situation. Although region 1 is smaller we compensate by making that region more attractive. It is interesting to observe that despite the slight increase in preferences for region 1, the smaller fraction of unskilled labour in region 1 encourages more firms to settle in region 2. This allows us to conclude about the existence of a strong market access advantage of the larger region, that is, larger immobile demand in one region attracts more mobile demand as Forslid and Ottaviano (2003) also argue. Thence, only for a particular set of ϕ when transportation costs are sufficiently low, an agglomeration stable equilibrium in region 1 emerges.

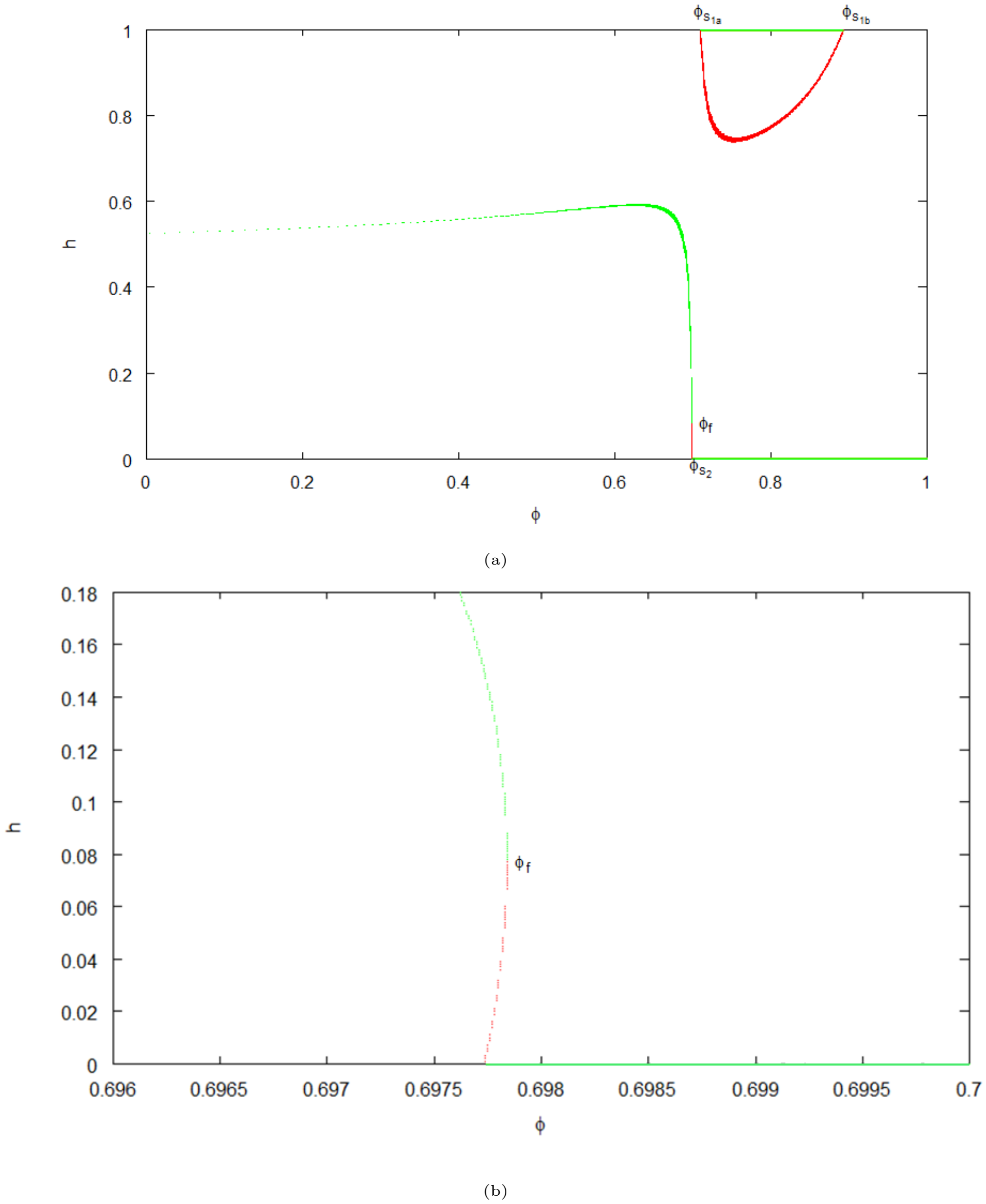


Figure 8: Bifurcation diagram for the perturbed model ($k = 0.528$ and $\gamma = -0.0192$).

We also plot two situations where the emergence of the *fold point* mentioned in section 3 occurs. This is the case of Figure 8. In this Figure, $k = 0.528$ and $\gamma = -0.0192$, thus region 1 is bigger but, at the same time, is less attractive than region 2. In this case, we have a stable dispersion equilibrium for $\phi < \phi_f$. As transportation costs decrease the skilled workers, who initially were mostly in region 1, have incentive to migrate to region 2. Then, for $\phi > \phi_{s_2}$ there is a stable agglomeration equilibrium in region 2. But, there is also another stable agglomeration equilibrium. When $\phi_{s_1a} < \phi < \phi_{s_1b}$ agglomeration in region 1 is also stable. For this range of transport costs we can say that maybe workers do not value significantly the relative unattractiveness of the region 1.

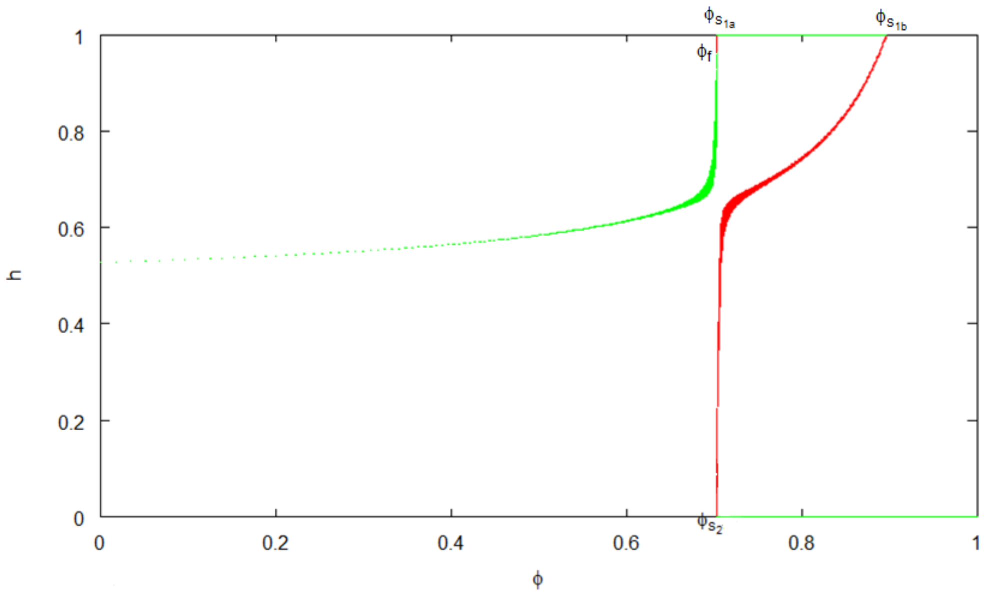


Figure 9: Bifurcation diagram for the perturbed model ($k = 0.53$ and $\gamma = -0.0192$).

The fold point we mentioned above is better observed if we set a particular zoom for the convenient values of ϕ. See this case in Figure 8(b).

But making a smooth increase in k, from $k = 0.528$ to $k = 0.53$, see Figure 9, there is also a *fold point*, but the bifurcation diagram changes significantly. That is, with a small increase of unskilled workers in region 1, for $\phi < \phi_f$, as transportation costs decrease, there are more and more skilled workers moving to region 1 (the big one). We can say that in this case, for high transportation costs, the workers tend to value region 1 (the larger region) more than region 2 (the smaller region). But if economic integration be sufficiently installed in the economy, workers want to migrate to the most beautiful region. This is because people can live in the region that provides them more glee and travel to the other region to buy goods without incurring in significant transport costs.

6 Conclusion

We have extended the 2-region Footloose Entrepreneur model by Forslid and Ottaviano (2003) by introducing two additional parameters: one that describes the possibly asymmetric distribution of unskilled workers between regions; and another that describes a possible asymmetry in the level of regional amenities, which may reflect the preferences of workers for scenic, climate or cultural factors. These amenities exert influence over the location decisions of workers, whose relevance and intensity are not neutral to the size of the regions.

We prove that our extended model is a generic perturbation in the sense that it describes all possible perturbations, in terms of qualitative behaviour, of the model of Forslid and Ottaviano (2003). This means that any smooth perturbation that might be added in the model would not lead to new qualitative behaviour (that is, to behaviour not already present for the proposed two-parameter family of perturbations). For this reason, and because we attain this with a minimum number of parameters, our extended model is a universal unfolding of the FE model of Forslid and Ottaviano (2003). That is because our modified model is a perturbation with the fewest parameters that describes all possible qualitative perturbations of the original Footloose Entrepreneur model of Forslid and Ottaviano (2003). Therefore, the bifurcation diagrams we have exhibited illustrate, in a generic way, the admissible perturbations that may appear in a perturbation of the FE model.

Another important result of our approach is that slight changes of the possible values of the two unfolding parameters give rise to considerable changes regarding the resulting agglomeration patterns. That is, with certain specific values of the unfolding parameters, we obtain new patterns of skilled workers distribution. Perturbing singly each of the parameters produces agglomeration patterns that are already present in the existing literature. But perturbing both parameters together gives rise to appreciable changes in agglomeration patterns, which, to the best of our knowledge, have not been properly discussed in the literature.

Appendix A

In order to study the dynamics of the modified model it is necessary to find the new expression for nominal wages of skilled workers w_i. That is because skilled workers, comparing real wages in the two regions, make their migration decisions. So, the region where the real wage is higher earns them a higher indirect utility and, naturally, they decide to move there. Substituting (2.9) in (2.17), we obtain the equilibrium nominal wage of skilled workers given their spatial distribution:

$$
\begin{aligned}
&\begin{cases} w_1 = \frac{\mu}{\sigma}\left[\frac{Y_1}{H_1+\phi H_2} + \frac{\phi Y_2}{\phi H_1+H_2}\right] \\ \\ w_2 = \frac{\mu}{\sigma}\left[\frac{Y_2}{H_2+\phi H_1} + \frac{\phi Y_1}{\phi H_2+H_1}\right] \end{cases} \\
\Leftrightarrow &\begin{cases} w_1 = \frac{\mu}{\sigma}\left[\frac{w_1 H_1+kL}{H_1+\phi H_2} + \frac{\phi w_2 H_2+\phi(1-k)L}{\phi H_1+H_2}\right] \\ \\ w_2 = \frac{\mu}{\sigma}\left[\frac{w_2 H_2+(1-k)L}{H_2+\phi H_1} + \frac{\phi w_1 H_1+\phi kL}{\phi H_2+H_1}\right] \end{cases} \\
\Leftrightarrow &\begin{cases} w_1 = \frac{1}{\sigma(H_1+\phi H_2-\mu H_1)}\left\{\frac{\mu kL(\phi H_1+H_2)+\mu[\phi w_2 H_2+\phi(1-k)L][H_1+\phi H_2]}{\phi H_1+H_2}\right\} \\ \\ w_2 = \frac{1}{\sigma(H_2+\phi H_1-\mu H_2)}\left\{\frac{\mu(1-k)L(\phi H_2+H_1)+\mu[\phi w_1 H_1+\phi kL][H_2+\phi H_1]}{\phi H_2+H_2}\right\} \end{cases},
\end{aligned}
$$

which becomes after solving the system:

$$
\begin{cases}
w_1 = \frac{\mu}{[\sigma(H_1+\phi H_2)-\mu H_1][\sigma(H_2+\phi H_1)-\mu H_2]-\mu^2\phi^2 H_1 H_2} * \\
\left[\frac{kL\left[\mu H_2\phi^2-\mu H_2+\sigma(H_2+\phi H_1)\right][\phi H_1+H_2]+[\phi(1-k)L(\sigma(H_2+\phi H_1)-\mu H_2)+\phi H_2\mu L(1-k)][\phi H_2+H_1]}{\phi H_1+H_2}\right] \\
\\
w_2 = \frac{\mu}{[\sigma(H_2+\phi H_1)-\mu H_2][\sigma(H_1+\phi H_2)-\mu H_1]-\mu^2\phi^2 H_1 H_2} * \\
\left[\frac{(1-k)L\left[\mu H_1\phi^2-\mu H_1+\sigma(H_1+\phi H_2)\right][\phi H_2+H_1]+[\phi kL(\sigma(H_1+\phi H_2)-\mu H_1)+\phi H_1\mu kL][\phi H_1+H_2]}{\phi H_2+H_1}\right]
\end{cases}, \tag{6.1}
$$

After some manipulation, the system simplifies to:

$$\begin{cases} w_1 = \mu L \dfrac{\sigma\phi H_1 + \left[(\sigma-\mu)k - ((\sigma-\mu)k-\sigma)\phi^2\right] H_2}{\sigma\phi(\sigma-\mu)(H_1^2+H_2^2) + \left[(\sigma-\mu)^2 + (\sigma^2-\mu^2)\phi^2\right] H_1 H_2} \\ \\ w_2 = \mu L \dfrac{\sigma\phi H_2 + \left[(\sigma-\mu)(1-k) + ((\sigma-\mu)k+\mu)\phi^2\right] H_1}{\sigma\phi(\sigma-\mu)(H_1^2+H_2^2) + \left[(\sigma-\mu)^2 + (\sigma^2-\mu^2)\phi^2\right] H_1 H_2} \end{cases} \tag{6.2}$$

Appendix B

This Appendix includes the expressions of partial derivatives needed to prove the existence of a universal unfolding.

The partial derivatives are calculated at the bifurcation point of the unperturbed problem: $(h, \phi, k, \gamma) = (\frac{1}{2}, \phi_b^0, \frac{1}{2}, 0)$

To simplify writing, we call *num* and *den* to the following expressions, respectively:

$$num = \frac{\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)} + \frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2} - 0.5(\sigma-\mu).$$

$$den = \frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)^2}{(\mu+\sigma-1)(\mu+\sigma)} + 0.25\left(\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\sigma^2-\mu^2)}{(\mu+\sigma-1)^2(\mu+\sigma)^2} + (\sigma-\mu)^2\right).$$

$$g_{h\phi}\Big|_{(\frac{1}{2},\phi_b^0)} = \frac{\frac{2(\sigma-\mu-1)(\sigma-\mu)}{(\mu+\sigma-1)} - 2\sigma}{den\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}} -$$

$$\frac{\mu\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\left[0.5\sigma-\frac{(0.5(\sigma-\mu)-\sigma)(\sigma-\mu-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]}{(1-\sigma)den} -$$

$$\frac{0.5\mu\left[\frac{0.5(\sigma-\mu-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\left[\frac{\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}-0.5(\sigma-\mu)\right]}{(1-\sigma)den} -$$

$$\frac{\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)(\sigma^2-\mu^2)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\sigma(\sigma-\mu)\right]\left[\frac{\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}-0.5(\sigma-\mu)\right]}{[den]^2\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}} +$$

$$\frac{\mu\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(0.5(\sigma-\mu)-\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right]}{(1-\sigma)den} +$$

$$\frac{\mu}{(1-\sigma)den}\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)(\sigma^2-\mu^2)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\sigma(\sigma-\mu)\right]\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\times$$
$$\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(0.5(\sigma-\mu)-\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right] -$$

$$\frac{0.5\mu}{(1-\sigma)den}\left(\frac{-\mu}{1-\sigma}-1\right)\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-2}\times$$
$$\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(0.5(\sigma-\mu)-\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right] +$$

$$\frac{\mu\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right)\right]}{(1-\sigma)den} -$$

$$\frac{\mu}{(1-\sigma)den}\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)(\sigma^2-\mu^2)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\sigma(\sigma-\mu)\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\left[\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}-1\right]\times$$
$$\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right)\right] +$$

$$\frac{0.5\mu}{(1-\sigma)den}\left(\frac{-\mu}{1-\sigma}-1\right)\left(\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right)^{\frac{-\mu}{1-\sigma}-2}\left(\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}-1\right)\times$$
$$\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2[\mu+0.5(\sigma-\mu)]}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right)\right] +$$

$$\frac{0.5\mu\left(\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right)^{\frac{-\mu}{1-\sigma}-1}\left(\frac{-\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2[\mu+0.5(\sigma-\mu)]}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right)}{(1-\sigma)den} +$$

$$\frac{\left(\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)(\sigma^2-\mu^2)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\sigma(\sigma-\mu)\right)\left(\frac{-\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2[\mu+0.5(\sigma-\mu)]}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right)}{[den]^2\left(\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right)^{\frac{\mu}{1-\sigma}}} +$$

$$\frac{\mu\left(\frac{(-\mu+\sigma-1)(\sigma-\mu)[\mu+0.5(\sigma-\mu)]}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\sigma\right)\left(\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right)^{\frac{-\mu}{1-\sigma}-1}\left(\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}-1\right)}{(1-\sigma)den}.$$

$$
g_{hhh}\Big|_{(\frac{1}{2},\phi_b^0)} = \frac{-3\mu\left(\frac{-\mu}{1-\sigma}-1\right)\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]^2\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-2}[num]}{(1-\sigma)\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)^2}{(\mu+\sigma-1)(\mu+\sigma)}+0.25\left(\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\sigma^2-\mu^2)}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+(\sigma-\mu)^2\right)\right]} -
$$

$$
\frac{3\left[\frac{4\sigma(-\mu+\sigma-1)(\sigma-\mu)^2}{(\mu+\sigma-1)(\mu+\sigma)}-\frac{2(-\mu+\sigma-1)^2(\sigma-\mu)^2(\sigma^2-\mu^2)}{(\mu+\sigma-1)^2(\mu+\sigma)^2}-2(\sigma-\mu)^2\right][num]}{\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)^2}{(\mu+\sigma-1)(\mu+\sigma)}+0.25\left(\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\sigma^2-\mu^2)}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+(\sigma-\mu)^2\right)\right]^2\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\mu/(1-\sigma)}} +
$$

$$
\frac{3\mu}{(1-\sigma)\,[den]^2}\left[\frac{4\sigma(-\mu+\sigma-1)(\sigma-\mu)^2}{(\mu+\sigma-1)(\mu+\sigma)}-\frac{2(-\mu+\sigma-1)^2(\sigma-\mu)^2(\sigma^2-\mu^2)}{(\mu+\sigma-1)^2(\mu+\sigma)^2}-2(\sigma-\mu)^2\right]\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]\times
$$

$$
\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(0.5(\sigma-\mu)-\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right]+
$$

$$
\frac{-\mu\left(\frac{-\mu}{1-\sigma}-2\right)\left(\frac{-\mu}{1-\sigma}-1\right)}{(1-\sigma)\,[den]}\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]^3\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-3}\times
$$

$$
\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(0.5(\sigma-\mu)-\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right]+
$$

$$
\frac{\mu\left(\frac{-\mu}{1-\sigma}-2\right)\left(\frac{-\mu}{1-\sigma}-1\right)}{(1-\sigma)\,[den]}\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-3}\times
$$

$$
\left[\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}-1\right]^3\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right)\right]+
$$

$$
\frac{-3}{(1-\sigma)\,[den]}\left[\frac{4\sigma(-\mu+\sigma-1)(\sigma-\mu)^2}{(\mu+\sigma-1)(\mu+\sigma)}-\frac{2(-\mu+\sigma-1)^2(\sigma-\mu)^2(\sigma^2-\mu^2)}{(\mu+\sigma-1)^2(\mu+\sigma)^2}-2(\sigma-\mu)^2\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\times
$$

$$
\left[\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}-1\right]\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right)\right]+
$$

$$
\frac{3}{[den]^2\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}}\left[\frac{4\sigma(-\mu+\sigma-1)(\sigma-\mu)^2}{(\mu+\sigma-1)(\mu+\sigma)}-\frac{2(-\mu+\sigma-1)^2(\sigma-\mu)^2(\sigma^2-\mu^2)}{(\mu+\sigma-1)^2(\mu+\sigma)^2}-2(\sigma-\mu)^2\right]\times
$$

$$
\left[\frac{-\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right]+
$$

$$
\frac{3\mu\left(\frac{-\mu}{1-\sigma}-1\right)}{(1-\sigma)\,[den]}\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-2}\left[\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}-1\right]^2\times
$$

$$
\left[\frac{-\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right].
$$

$$g_{\phi h}\Big|_{(\frac{1}{2},\phi_b^0)} = \frac{-\left[\frac{2(-\mu+\sigma-1)(\sigma-\mu)(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)(\mu+\sigma)}-\sigma\right]}{[den]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}} + \frac{\frac{2(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+\sigma}{[den]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}} +$$

$$\frac{-\mu\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\left[0.5\sigma-\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]}{(1-\sigma)\,[den]} +$$

$$\frac{-0.5\mu\,[num]}{(1-\sigma)\,[den]}\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1} + \frac{-\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)(\sigma^2-\mu^2)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\sigma(\sigma-\mu)\right][num]}{[den]^2\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}} +$$

$$\frac{\mu}{(1-\sigma)\,[den]}\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1} \times$$
$$\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(0.5(\sigma-\mu)-\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right] +$$

$$\frac{\mu}{(1-\sigma)\,[den]^2}\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)(\sigma^2-\mu^2)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\sigma(\sigma-\mu)\right]\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right] \times$$
$$\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(0.5(\sigma-\mu)-\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right] +$$

$$\frac{-0.5\mu\left(\frac{-\mu}{1-\sigma}-1\right)}{(1-\sigma)\,[den]}\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-2} \times$$
$$\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(0.5(\sigma-\mu)-\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right] +$$

$$\frac{\mu}{(1-\sigma)\,[den]}\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1} \times$$
$$\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right)\right] +$$

$$\frac{-\mu}{(1-\sigma)\,[den]^2}\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)(\sigma^2-\mu^2)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\sigma(\sigma-\mu)\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1} \times$$
$$\left[\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}-1\right]\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right)\right] +$$

$$\frac{0.5\mu\left(\frac{-\mu}{1-\sigma}-1\right)}{(1-\sigma)\,[den]}\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-2}\left[\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}-1\right] \times$$
$$\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right)\right] +$$

$$\frac{0.5\mu\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\left[\frac{-\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right]}{(1-\sigma)\,[den]} +$$

$$\frac{\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)(\sigma^2-\mu^2)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\sigma(\sigma-\mu)\right]}{[den]^2\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}} \times$$
$$\left[\frac{-\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^2(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+0.5(\sigma-\mu)\right] +$$
$$\frac{\mu\left[\frac{(-\mu+\sigma-1)(\sigma-\mu)(\mu+0.5(\sigma-\mu))}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\sigma\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\left[\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}-1\right]}{(1-\sigma)\,[den]}.$$

$$G_{\gamma}\Big|_{(\frac{1}{2},\phi_b^0,\frac{1}{2},0)} = \frac{\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)} + 0.5\left[0.5(\sigma-\mu) - \frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right]}{[den]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}}.$$

$$G_{\gamma h}\Big|_{(\frac{1}{2},\phi_b^0,\frac{1}{2},0)} = \frac{\frac{\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)} + \frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2} - 0.5(\sigma-\mu)}{[den]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}} -$$

$$\frac{\mu}{(1-\sigma)\,[den]}\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\times$$
$$\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)} + 0.5\left(0.5(\sigma-\mu) - \frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right].$$

$$G_{\gamma\phi}\Big|_{(\frac{1}{2},\phi_b^0,\frac{1}{2},0)} = \frac{0.5\sigma - \frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}}{[den]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}} -$$

$$\frac{0.5\mu}{(1-\sigma)\,[den]}\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\times$$
$$\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)} + 0.5\left(0.5(\sigma-\mu) - \frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right]+$$

$$\frac{-\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)(\sigma^2-\mu^2)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\sigma(\sigma-\mu)\right]}{[den]^2\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}}\times$$
$$\left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)} + 0.5\left(0.5(\sigma-\mu) - \frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right].$$

$$
\begin{aligned}
G_{\gamma hh}\Big|_{(\frac{1}{2},\phi_b^0,\frac{1}{2},0)} &= \frac{-2\mu\,[num]\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}}{(1-\sigma)\,[den]}- \\
&\quad \frac{\mu\left(\frac{-\mu}{1-\sigma}-1\right)}{(1-\sigma)\,[den]}\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]^2\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-2}\times \\
&\quad \left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(0.5(\sigma-\mu)-\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right]+ \\
&\quad \frac{-\left[\frac{4\sigma(-\mu+\sigma-1)(\sigma-\mu)^2}{(\mu+\sigma-1)(\mu+\sigma)}-\frac{2(-\mu+\sigma-1)^2(\sigma-\mu)^2(\sigma^2-\mu^2)}{(\mu+\sigma-1)^2(\mu+\sigma)^2}-2(\sigma-\mu)^2\right]}{[den]^2\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}}\times \\
&\quad \left[\frac{0.5\sigma(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\left(0.5(\sigma-\mu)-\frac{(0.5(\sigma-\mu)-\sigma)(-\mu+\sigma-1)^2(\sigma-\mu)^2}{(\mu+\sigma-1)^2(\mu+\sigma)^2}\right)\right].
\end{aligned}
$$

$$
G_k\Big|_{(\frac{1}{2},\phi_b^0,\frac{1}{2},0)} = \frac{-\left[\frac{(-u+s-1)^2(s-u)^3}{(u+s-1)^2(u+s)^2}+u-s\right]}{[den]\left[\frac{0.5(-u+s-1)(s-u)}{(u+s-1)(u+s)}+0.5\right]^{\frac{u}{1-s}}}.
$$

$$
\begin{aligned}
G_{kh}\Big|_{(\frac{1}{2},\phi_b^0,\frac{1}{2},0)} &= \frac{-\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^3}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+\mu-\sigma}{[den]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}}+\frac{-\left[\frac{-(-\mu+\sigma-1)^2(\sigma-\mu)^3}{(\mu+\sigma-1)^2(\mu+\sigma)^2}-\mu+\sigma\right]}{[den]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}}+ \\
&\quad \frac{-0.5\mu}{(1-\sigma)\,[den]}\left[\frac{-(-\mu+\sigma-1)^2(\sigma-\mu)^3}{(\mu+\sigma-1)^2(\mu+\sigma)^2}-\mu+\sigma\right]\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}+ \\
&\quad \frac{0.5\mu}{(1-\sigma)\,[den]}\left[\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^3}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+\mu-\sigma\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\left[\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}-1\right] \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
G_{k\phi}\Big|_{(\frac{1}{2},\phi_b^0,\frac{1}{2},0)} &= \frac{-2(-\mu+\sigma-1)(\sigma-\mu)^2}{(\mu+\sigma-1)(\mu+\sigma)\,[den]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}}+ \\
&\quad \frac{\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)(\sigma^2-\mu^2)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\sigma(\sigma-\mu)\right]\left[\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^3}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+\mu-\sigma\right]}{[den]^2\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}}.
\end{aligned}
$$

$$G_{khh}\Big|_{(\frac{1}{2},\phi_b^0,\frac{1}{2},0)} = \frac{2\mu\left[\frac{-(-\mu+\sigma-1)^2(\sigma-\mu)^3}{(\mu+\sigma-1)^2(\mu+\sigma)^2}-\mu+\sigma\right]\left[1-\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}}{(1-\sigma)\,[den]}+$$

$$\frac{\left[\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^3}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+\mu-\sigma\right]\left[\frac{4\sigma(-\mu+\sigma-1)(\sigma-\mu)^2}{(\mu+\sigma-1)(\mu+\sigma)}-\frac{2(-\mu+\sigma-1)^2(\sigma-\mu)^2(\sigma^2-\mu^2)}{(\mu+\sigma-1)^2(\mu+\sigma)^2}-2(\sigma-\mu)^2\right]}{[den]^2\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{\mu}{1-\sigma}}}+$$

$$\frac{2\mu\left[\frac{(-\mu+\sigma-1)^2(\sigma-\mu)^3}{(\mu+\sigma-1)^2(\mu+\sigma)^2}+\mu-\sigma\right]\left[\frac{0.5(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}+0.5\right]^{\frac{-\mu}{1-\sigma}-1}\left[\frac{(-\mu+\sigma-1)(\sigma-\mu)}{(\mu+\sigma-1)(\mu+\sigma)}-1\right]}{(1-\sigma)\,[den]}.$$

Appendix C

We prove Lemma 4.3 which we restate here for completeness:

The derivatives $g_{\phi\phi} = 0$ *and* $g_{\phi hh} = 0$ *at* $(h,\, \phi,\, k,\, \gamma)$=$(\frac{1}{2},\, \phi_b^0,\, \frac{1}{2},\, 0)$.

Proof of Lemma 4.3:

$$
\begin{aligned}
g_{\phi\phi}\Big|_{(\frac{1}{2},\phi_b^0)} &= -\frac{0.5(0.5\phi+0.5)^{-\mu/(1-\sigma)-1}[0.5(0.5(\sigma-\mu)-\phi^2(0.5(\sigma-\mu)-\sigma))+0.5\phi\sigma]\mu^2}{(1-\sigma)[0.25(\phi^2(\sigma^2-\mu^2)+(\sigma-\mu)^2)+0.5\phi\sigma(\sigma-\mu)]}+ \\
&\frac{0.5(0.5\phi+0.5)^{-\mu/(1-\sigma)-1}[0.5(\phi^2(\mu+0.5(\sigma-\mu))+0.5(\sigma-\mu))+0.5\phi\sigma]\mu^2}{(1-\sigma)[0.25(\phi^2(\sigma^2-\mu^2)+(\sigma-\mu)^2)+0.5\phi\sigma(\sigma-\mu)]}- \\
&\frac{\mu[\phi(\mu+0.5(\sigma-\mu))+0.5\sigma]}{(0.5\phi+0.5)^{\mu/(1-\sigma)}[0.25(\phi^2(\sigma^2-\mu^2)+(\sigma-\mu)^2)+0.5\phi\sigma(\sigma-\mu)]}+ \\
&\frac{\mu[0.5\sigma-\phi(0.5(\sigma-\mu)-\sigma)]}{(0.5\phi+0.5)^{\mu/(1-\sigma)}[0.25(\phi^2(\sigma^2-\mu^2)+(\sigma-\mu)^2)+0.5\phi\sigma(\sigma-\mu)]}+ \\
&\frac{\mu[0.5(\phi^2(\mu+0.5(\sigma-\mu))+0.5(\sigma-\mu))+0.5\phi\sigma][0.5\phi(\sigma^2-\mu^2)+0.5\sigma(\sigma-\mu)]}{(0.5\phi+0.5)^{\mu/(1-\sigma)}[0.25(\phi^2(\sigma^2-\mu^2)+(\sigma-\mu)^2)+0.5\phi\sigma(\sigma-\mu)]^2}- \\
&\frac{\mu[0.5(0.5(\sigma-\mu)-\phi^2(0.5(\sigma-\mu)-\sigma))+0.5\phi\sigma][0.5\phi(\sigma^2-\mu^2)+0.5\sigma(\sigma-\mu)]}{(0.5\phi+0.5)^{\mu/(1-\sigma)}[0.25(\phi^2(\sigma^2-\mu^2)+(\sigma-\mu)^2)+0.5\phi\sigma(\sigma-\mu)]^2} \\
&= 0.5(0.5\phi+0.5)^{\frac{-\mu}{1-\sigma}-1}\mu^2[0.5(\phi^2(\mu+0.5(\sigma-\mu))+0.5(\sigma-\mu))+0.5\sigma\phi- \\
&0.5(0.5(\sigma-\mu)-\phi^2(0.5(\sigma-\mu)-\sigma))-0.5\phi\sigma]- \\
&\mu(\phi(\mu+0.5(\sigma-\mu))+0.5\sigma)+(0.5\sigma-\phi(0.5(\sigma-\mu)-\sigma))\mu- \\
&\mu(0.5\phi(\sigma^2-\mu^2)+0.5\sigma(\sigma-\mu))[0.5(\phi^2(\mu+0.5(\sigma-\mu))+0.5(\sigma-\mu))+0.5\sigma\phi- \\
&0.5(0.5(\sigma-\mu)-\phi^2(0.5(\sigma-\mu)-\sigma))+0.5\sigma\phi] \\
&= 0
\end{aligned}
$$

Proving that $g_{\phi hh} = 0$, we have that:

$$
g_\phi(\frac{1}{2}+x) = -g_\phi(\frac{1}{2}-x), \tag{6.3}
$$

where,

$$
\begin{aligned}
g_\phi(\frac{1}{2}+x) =& \frac{\partial}{\partial \phi}\left[\frac{w_1^0}{P_1}\left(\frac{1}{2}+x,\frac{1}{2}-x\right)\right] - \frac{\partial}{\partial \phi}\left[\frac{w_2^0}{P_2}\left(\frac{1}{2}+x,\frac{1}{2}-x\right)\right] \\
=& \frac{\partial}{\partial \phi}\left[\frac{w_2^0}{P_2}\left(\frac{1}{2}-x,\frac{1}{2}+x\right)\right] - \frac{\partial}{\partial \phi}\left[\frac{w_1^0}{P_1}\left(\frac{1}{2}-x,\frac{1}{2}+x\right)\right] \\
=& -\left\{\frac{\partial}{\partial \phi}\left[\frac{w_1^0}{P_1}\left(\frac{1}{2}-x,\frac{1}{2}+x\right)\right] - \frac{\partial}{\partial \phi}\left[\frac{w_2^0}{P_2}\left(\frac{1}{2}-x,\frac{1}{2}+x\right)\right]\right\} \\
=& -g_\phi(\frac{1}{2}-x), by\ \ (6.3).
\end{aligned}
$$

If $\frac{\partial}{\partial x} g_\phi(\frac{1}{2}+x) = \frac{\partial}{\partial u} g_\phi(u)$,

and $\frac{\partial}{\partial x} g_\phi(\frac{1}{2}-x) = \frac{\partial}{\partial u} g_\phi(u)(-1)$,

then $\frac{\partial}{\partial x} g_\phi(\frac{1}{2}+x) = -\frac{\partial}{\partial x} g_\phi(\frac{1}{2}-x) \Leftrightarrow \frac{\partial}{\partial u} g_\phi(u) = -\left[-\frac{\partial}{\partial u} g_\phi(u)\right]$.

Differentiating twice, we have that:

if $\frac{\partial^2}{\partial x^2} g_\phi(\frac{1}{2}+x) = \frac{\partial^2}{\partial u^2} g_\phi(u)$,

and $\frac{\partial^2}{\partial x^2} g_\phi(\frac{1}{2}-x) = \frac{\partial^2}{\partial u^2} g_\phi(u)(-1)(-1) = \frac{\partial^2}{\partial u^2} g_\phi(u)$,

then by (6.3): $\frac{\partial^2}{\partial x^2} g_\phi(\frac{1}{2}+x) = -\frac{\partial^2}{\partial x^2} g_\phi(\frac{1}{2}-x) \Leftrightarrow \frac{\partial^2}{\partial u^2} g_\phi(u) = 0$.

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