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Density of First Poincaré Returns and Periodic Orbits

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Densidade dos Primeiros Retornos de Poincaré e Órbitas Periódicas

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I race to win. If I am on the bike or in a car it will always be the same.

Valentino Rossi

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I dedicate this work to my mother, my sister and my godfather.

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Abstract

It is known that unstable periodic orbits of a given map give all information about the natural measure of a chaotic attractor. There are conditions that need to be satisfied for the existence of such measure but if it exists then the natural measure of some subset S is, by definition, the fraction of iterates of the orbit $\{f^i(x_0)\}$, for almost every point x_0 (Lebesgue), lying in S. So, we can look to this measure as a density of periodic returns.

This work tries to characterize the density function of the first Poincaré returns in terms of unstable periodic orbits. We present a conjecture on how periodic orbits may be used to compute the density of the first Poincaré returns and we present numerical results that support the conjecture for some well known dynamical systems. We prove, in the case of Markov transformation under some conditions, that the density function of the first Poincaré returns is completely determined by the unstable periodic points for an element or for a perfect union of elements of the Markov partition of the map. We also discuss the extension to a more general subset S of the phase space. Unlike the work of Grebogi, Ott and York to calculate the natural measure, to characterize the density function of the first Poincaré returns we need not all but just some special unstable periodic orbits.

The close relation between periodic orbits and the Poincaré returns allows for estimates of relevant quantities in dynamical systems, as the Kolmogorov-Sinai entropy. Since return times can be trivially observed and measured, this work has also application to the treatment of experimental systems.

Resumo

É sabido que é possível calcular a medida natural usando as órbitas periódicas instáveis de uma dada aplicação que exiba um atractor chaótico. Existem condições a serem satisfeitas para que exista tal medida mas no caso de ela existir então ela é, por definição, a fracção de vezes que a órbita de quase todo o ponto (Lebesgue) visita o conjunto que queremos medir. Assim sendo, esta medida pode ser vista como uma densidade de retornos periódicos.

Neste trabalho tenta-se caracterizar a densidade dos primeiros retornos de Poincaré usando as órbitas periódicas instáveis do sistema. É conjecturada uma forma desta densidade ser calculada usando as órbitas periódicas bem como apresentadas simulações numéricas, usando sistemas dinâmicos clássicos, que sustentam a conjectura. É provada a conjectura para a classe das transformações de Markov sob certas condições e são também discutidas as possíveis extensões do resultado. Ao contrário do que apresenta o trabalho do Grebogi, Ott e Yorke (onde caracterízam a medida natural usando todas as órbitas periódicas instáveis de um certo período), para caracterizar a densidade dos primeiros retornos de Poincaré basta considerar algumas e não todas as órbitas periódicas instáveis.

Esta relação próxima entre as órbitas periódicas e a densidade dos primeiros retornos permite estimativas de quantidades relevantes em sistemas dinâmicos como por exemplo a entropia de Kolmogorov-Sinai. Como os tempos de retorno podem ser trivialmente observados e medidos então este trabalho tem também uma forte aplicação ao tratamento de dados experimentais.

Résumé

On connaît déjà que les órbites périodiques d'une systm̀e nous donnent des informations à propos de la mesure naturelle d'un attractor chaotique. Il y a des conditions qui assurent l'existence de cette mesure. Cependant si la mesure existe, elle correspond à la mesure naturelle pour un sous-ensemble S. Cette mesure est, par définition, la fraction des itérés que l'orbite $\{f^i(x_0)\}_{i\in N}$ visite S, pour presque tous les points x_0. Par consequence, il est possible de considerer cette mesure comme une mesure des densités des retours périodiques.

Dans cette thèse, nous ensayerons d'étudier la fonction densité des primiers retours de Poincaré en usant les orbites périodiques instables. Nous suggerons un conjecture pour calculer la fonction densité des premiers retours de Poincaré et nous présenterons des simulations associés à des systemes dynamiques classiques, supportant la conjecture. Nous démontrerons que, sous certaines conditions, dans le cas des transformations de Markov, la fonction de premier retour est complement déterminé par les points périodiques instables associés a un element ou par l'union parfaite des élèments de la partition de Markov.

Nous discuterons l'extension de ce résultat pour un sous-ensemble de l'espace de phase. Malgré le fact que Grebogi, Ott et York caracterisent la mesure naturelle en usant toutes les orbites périodiques instables, pour caractérizer la fonction densité des primiers retourns de Poincaré il faut seulement certaines órbites périodiques instables spéciales.

Cette rélation proche entre les órbites périodiques et les retours de Poincaré est valide pour les estimatives de quantités centrales aux systémes dinamiques, comme l'entropie de Kolmogorov-Sinai. Étant donné que les temps de retours sont computables et mensurables, cette thèse a une application aux traitement des systèmes expérimentaux.

Contents

Introduction

State of Art

In 60s, Lorenz presented to the world a very simple dynamical system with sensitivity to initial conditions[12]. Then, there were many attempts to explain Lorenz's observation. Among these, Li and Yorke[11] proved a theorem on sensitivity to initial conditions for 1-dimensional maps, the well known result "period three implies chaos". Here the definition of chaos is not only about sensitivity to initial conditions. More precisely, $f : V \to V$ is chaotic on V if

1. f has sensitive dependence on initial conditions,
2. f is topologically transitive,
3. periodic points are dense in V.

On the literature there are some different ways to define "chaotic attractors". Some of them are not equivalent but in general, a chaotic attractor is a chaotic set (a forward limit set of a chaotic orbit) that is an attractor (a forward limit set which attracts a set of initial values that has nonzero Lebesgue measure). By chaotic orbit of some point x we mean an orbit that is bounded, is not asymptotically periodic and the orbit has a positive Lyapunov exponent. In this thesis we adopt the above definitions of "chaotic map", "chaotic attractor" and "chaotic orbit".

In 70s and 80s, considerable attention was given to "chaotic attractors" in dynamical systems. Some important results were proved for instance the Lasota and Yorke theorem[10] about the existence of invariant measures in such attractors. In the end of the 80's, Grebogi, Ott and Yorke[7] published a paper about unstable periodic orbits and the dimensions of multifractal chaotic attractors. Among other results proved in this article, there is one particular result about natural measure and periodic points. Essentially, they proved that the natural measure of a chaotic attractor, over some conditions on the dynamical system, is totally described by the unstable periodic orbits. More precisely:

Consider a d-dimensional C^2 map of the form $x_{n+1} = F(x_n)$, where $x \in R^d = \Omega$ and Ω represents the phase space of the system. Consider $A \subset \Omega$ to represent an attractor.

For a subset S of the phase space and an initial condition x_0 in the basin of attraction of A, we define $\mu(x_0, S)$ as the fraction of time the trajectory originating at x_0 spends in S in the limit that the length of the trajectory goes to infinity. So,

$$\mu(x_0, S) = \lim_{n\to\infty} \frac{\sharp\{F^i(x_0) \in S,\ 0 \le i \le n\}}{n}. \tag{1}$$

Definition 1 *If $\mu(x_0, S)$ has the same value for almost every x_0 in the basin of attraction of A, then we call the value $\mu(S)$ the **natural measure of** S.*

By almost every x_0 we mean with respect to the Lebesgue measure.

For now we assume that our attractor A has always a natural measure associated to it and is mixing: given two subsets, B_1 and B_2, in A, we have:

$$\lim_{n\to\infty} \mu(B_1 \cap F^{-n}(B_2)) = \mu(B_1)\mu(B_2).$$

In addition, we consider A to be a hyperbolic set: Let $F : R^n \to R^n$ be a diffeomorphism, and let $M \subset R^n$ be a compact invariant set for f. We denote by TM the restriction of the tangent bundle TR^n to M.

Definition 2 *F is uniformly hyperbolic on M if for every $x \in M$ the tangent space T_xM splits into a direct sum $T_xM = E^u_x \oplus E^s_x$ such that*

$$DF(x)E^u_x = E^u_{F(x)},$$
$$DF(x)E^s_x = E^s_{F(x)},$$

and there are constants $c > 0$ and $0 < \theta < 1$ independent of x such that

$$||DF^n(x)v|| < c\theta^n||v|| \quad for\ v \in E^s_x,$$
$$||DF^{-n}(x)v|| < c\theta^n||v|| \quad for\ v \in E^u_x,$$

holds for $n \ge 0$.

The eigenvalues of the Jacobian matrix of the n-th iterate, F^n, at the jth fixed point x^j of F^n are denoted by $\lambda_{1j}, \lambda_{2j}, ..., \lambda_{uj}, \lambda_{(u+1)j}, ..., \lambda_{dj}$, where we order the eigenvalues from the biggest, in magnitude, to the lowest and the number of the unstable eigenvalues is u. Let $L_j(n)$ be the product of absolute values of the unstable eigenvalues at x^j. Then it was proved by Bowen in 1972[4] and also by Grebogi, Ott and Yorke in 1988[7] the following:

Theorem 3 *For mixing hyperbolic attractors, the natural probability measure of the attractor contained in some closed subset S of the d-dimensional phase space is*

$$\mu(S) = \lim_{n\to\infty} \sum_j L_j^{-1}(n), \tag{2}$$

where the summation is taken over all the fixed points $x^j \in S$ of F^n.

So, this formula is essentially the representation of the natural measure in terms of the periodic orbits embedded in the attractor.

For nonhyperbolic systems there is no such result. However, on the paper [9], they test the goodness of such a periodic-orbit characterization of the natural measure for nonhyperbolic systems from unstable periodic orbits. They suggest that the previous result is typically valid for nonhyperbolic systems.

Contribution of the present thesis

This work consists essentially on proving the existence of a strong relation between the periodic orbits of a given chaotic map and the density of the first returns. As it was referred before, the natural measure is characterized, under some assumptions on the map, by the unstable periodic orbits. The goal here is similar but for the density function of the first returns. The first challenge was to identify which unstable periodic orbits will characterize such density since we already knew that, for a sufficiently large period, all orbits characterize the natural measure of a given subset of the phase space.

Chapter 2 consists of two articles ([3],[14]) with numerical simulations. These articles suggest the type of unstable periodic orbits, defined in section 1.1, to estimate the density of the first returns. The conjecture, which is presented in section 1.2 with more detail, is tested in some well known chaotic dynamical systems as the logistic map, the Henon map and the Chua's circuit and the results suggest that the conjecture is plausible. Also, in [14], is presented an application of the conjecture where we use it to calculate an approximate value for the Kolmogorov-Sinai entropy of the logistic family and, again, the numerical simulations suggest that the conjecture is plausible. These articles are presented as a motivation for the results in Chapter 3, that are the main results in the thesis.

Having tested the conjecture numerically we proceeded to its analytical treatment. Chapter 3 is dedicated to the proof of this fact for some class of dynamical systems and for some special subsets of the phase space. In section 3.2 is presented the definition of Markov transformation and it is proved the conjecture on the elements of the Markov partition for the linear case. We start, in section 3.3, with the description of the density function of the first return on the space of sequences and with properties of the space of sequences that will be used to prove the result. Markov transformations allows us to construct a topological conjugacy between the original map and the shift acting on the space of the sequences. Under some conditions on this class of dynamical systems, we prove, by theorem 23, the conjecture for elements of the Markov partition. Also we prove (theorem 31 and 32) the conjecture for subsets that are represented by a union of elements of the Markov partition. Finally we discuss the extension of this result to more general subsets of phase space and we present some numerical simulation to estimate the error, since theorem 23 and theorem 31 are not true anymore, only in an approximate sense.

Chapter 4 is dedicated to the main conclusions of this work and to future work.

Chapter 1

Preliminaries

1.1 Definitions

Consider some discrete dynamical system generated by the map $F : I \to I$, where I is a compact metric space. We assume that we have always a chaotic attractor, A, that is dense in I.

Definition 4 *A* ***non-recurrent*** *periodic point of F with period $p > 1$, with respect to a set $S \subset I$, is a periodic point of period p inside S that only returns to S after p iterations. When this fails because $F^j(x) \in S$ for some $1 < j < p$ with $x \in S$ and $F^p(x) = x$, the periodic point is called* ***recurrent****.*

For an initial condition x_0 in the basin of attraction of A, we define $\mu(x_0, S)$ as

$$\mu(x_0, S) = \lim_{n\to\infty} \frac{\#\{F^i(x_0) \in S,\ 0 \le i \le n\}}{n}. \tag{1.1}$$

Definition 5 *If $\mu(x_0, S)$ has the same value for Lebesgue almost every x_0 in the basin of attraction of A, then we denote this value by $\mu(S)$ and say that $\mu(S)$ is the* ***natural measure*** *of the attractor inside S.*

Observe that μ, defined as before, is always F-invariant: for almost all $x_0 \in I$

$$\mu(F^{-1}(S)) = \lim_{n\to\infty} \frac{\#\{F^i(x_0) \in F^{-1}(S),\ 0 \le i \le n\}}{n} =$$

$$= \lim_{n\to\infty} \frac{\#\{F^{i+1}(x_0) \in S,\ 0 \le i \le n\}}{n} =$$

$$= \lim_{n\to\infty} \frac{\#\{F^i(x_0) \in S,\ 1 \le i \le n+1\}}{n} =$$

$$= \lim_{n\to\infty} \frac{\#\{F^i(x_0) \in S,\ 0 \le i \le n\} - r_1 + r_2}{n} =$$

$$= \mu(S) + \lim_{n\to\infty} \frac{r_2 - r_1}{n} = \mu(S),$$

where $r_j \in \{0,1\}\ \forall j \in \{1,2\}$.

Definition 6 *A natural number τ, $\tau > 0$, is the* ***first Poincaré return*** *to S of a point $x_0 \in S$ if $F^\tau(x_0) \in S$ and there is no other $\tau^* < \tau$ such that $F^{\tau^*}(x_0) \in S$.*

Definition 7 *The* ***density function of the first return*** *of length p for some subset S, denoted by $\rho(p,S)$, is defined as the fraction, in measure, of points inside S that have first Poincaré returns of length p. Equivalently,*

$$\rho(p,S) = \frac{\mu(S')}{\mu(S)},$$

where $S' = F^{-p}(S) \cap S - (\cup_{i=1}^{p-1} F^{-i}(S) \cap S)$ and μ is the natural measure.

Definition 8 *For a given natural number p we define*

$$\mu_{NR}(p,S) = \sum_j L_{uj}^{-1},$$

where it is considered in the summation all non-recurrent periodic points with period p, with respect to S, and L_{uj} is the product of the absolute values of unstable eigenvalues of $DF^p(x_j)$ for the jth non-recurrent periodic point, x_j, inside S.

In particular, for a 1-dimensional expanding map, for some subset S of the phase space and for a periodic point $x_j \in S$ of period p, we have $L_{uj}(x_j) = |(F^p)'(x_j)|$.

The measure of recurrent points, $\mu_R(p,S)$, may be defined in a similar way using the recurrent unstable periodic points.

1.2 Conjecture

For a chaotic attractor A generated by a mixing uniformly hyperbolic map F, for an open ball S on the basin of attraction of A we have that

$$\rho(\tau,S) = \mu_{NR}(\tau,S). \tag{1.2}$$

Essentially we conjecture that between all orbits, that the Grebogi, Ott and Yorke formula uses to calculate the natural measure, only the non-recurrent ones will give us the information about the frequency of the first Poincaré return on a subset S of the basin of attraction of the chaotic attractor A.

Obviously the conjecture is false for a general subset S with at least one non-recurrent periodic point of period τ inside it since we can simply consider $S' = S - \{periodic\ points\ of\ period\ \tau\}$ and the result is false for S' and τ. However we would like to know how common are the sets for which it holds.

Chapter 2

Numerical Evidences

2.1 Article 1 - Kolmogorov-Sinai entropy from recurrence times

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KOLMOGOROV-SINAI ENTROPY FROM RECURRENCE TIMES

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Abstract. Observing how long a dynamical system takes to return to some state is one of the most simple ways to model and quantify its dynamics from data series. This work proposes two formulas to estimate the KS entropy and a lower bound of it, a sort of Shannon's entropy per unit of time, from the recurrence times of chaotic systems. One formula provides the KS entropy and is more theoretically oriented since one has to measure also the low probable very long returns. The other provides a lower bound for the KS entropy and is more experimentally oriented since one has to measure only the high probable short returns. These formulas are a consequence of the fact that the series of returns do contain the same information of the trajectory that generated it. That suggests that recurrence times might be valuable when making models of complex systems.

1. Introduction

Recurrence times measure the time interval a system takes to return to a neighborhood of some state, being that it was previously in some other state. Among the many ways time recurrences can be defined, two approaches that have recently attracted much attention are the first Poincaré recurrence times (FPRs) [1] and the recurrence plots (RPs) [2].

While Poincaré recurrences refer to the sequence of time intervals between two successive visits of a trajectory (or a signal) to one particular interval (or a volume if the trajectory is high dimensional), a recurrence plot refers to a visualization of the values of a square array which indicates how much time it takes for two points in a trajectory with M

points to become neighbors again. Both techniques provide similar results but are more appropriately applicable in different contexts. While the FPRs are more appropriated to obtain exact dynamical quantities (Lyapunov exponents, dimensions, and the correlation function) of dynamical systems [3], the RPs are more oriented to estimate relevant quantities and statistical characteristics of data coming from complex systems [4].

The main argument in order to use recurrence times to model complex systems [5] is that one can easily have experimental access to them. In order to know if a model can be constructed from the recurrence times, it is essential that at least the series of return times contains the same amount of information generated by the complex system, information being quantified by the entropy.

Entropy is an old thermodynamic concept and refers to the disorganized energy that cannot be converted into work. It was first mathematically quantified by Boltzmann in 1877 as the logarithm of the number of microstates that a gas occupies. More recently, Shannon [6] proposed a more general way to measure entropy H_S in terms of the probabilities ρ_i of all possible i states of a system:

$$H_S = -\sum_i \rho_i \log(\rho_i). \tag{1}$$

Applied to non-periodic continuous trajectories, e.g. chaotic trajectories, H_S is an infinite quantity due to the infinitely many states obtained by partitioning the phase space in arbitrarily small sites. Therefore, for such cases it is only meaningful to measure entropy relative to another trajectory. In addition, once a dynamical system evolves with time, it is always useful for comparison reasons to measure its entropy production per unit of time.

Such an ideal entropy definition for a dynamical system was introduced by Kolmogorov in 1958 [7] and reformulated by Sinai in 1959. It is known as the Kolmogorov-Sinai (KS) entropy, denoted by H_{KS}, basically the Shannon's entropy of the set per unit of time [8], and it is the most successful invariant quantity that characterize a dynamical system [9]. However, the calculation of the KS entropy to systems that might possess an infinite number of states is a difficult task, if not impossible. For a smooth chaotic system [10] (typically happens for dissipative systems that present an attractor), Pesin [11] proved an equality between H_{KS} and the sum of all the positive Lyapunov exponents. However, Lyapunov exponents are difficult or even impossible to be calculated in systems whose equations of motion are unknown. Therefore, when treating data coming from complex systems, one should use alternative ways to calculate the KS entropy, instead of applying Pesin's equality.

Methods to estimate the correlation entropy, K_2, a lower bound of H_{KS}, and to calculate H_{KS} from time series were proposed in Refs. [12, 13]. In Ref. [12] K_2 is estimated from the correlation decay and in

Ref. [13] by the determination of a generating partition of phase space that preserves the value of the entropy. But while the method in Ref. [12] unavoidably suffers from the same difficulties found in the proper calculation of the fractal dimensions from data sets, the method in Ref. [13] requires the knowledge of the generating partitions, information that is not trivial to be extracted from complex data [14]. In addition, these two methods and similar others as the one in Ref. [15] require the knowledge of a trajectory. Our work is devoted to systems whose trajectory cannot be measured.

A convenient way of determining all the relevant states of a system and their probabilities (independently whether such a system is chaotic) is provided by the FPRs and the RPs. In particular to the Shannon's entropy, in Refs. [16, 17, 18, 4] ways were suggested to estimate it from the RPs. In Refs. [16, 17, 4] a subset of all the possible probabilities of states, the probabilities related to the level of coherence/correlation of the system, were considered in Eq. (1). Therefore, as pointed out in Ref. [18], the obtained entropic quantity does not quantify the level of disorganization of the system. Remind that unavoidably Shannon's entropy calculated from RPs or FPRs depends on the resolution with which the returns are measured.

The main result of this contribution is to show how to easily estimate the KS-entropy from return times, without the knowledge of a trajectory. We depart from similar ideas as in Refs. [16, 17, 18, 4] and show that the KS entropy is the Shannon entropy [in Eq. (1)] calculated considering the probabilities of all the return times observed divided by the length of the shortest return measured. This result is corroborated with simulations on the logistic map, the Hénon map, and coupled maps. We also show how to estimate a lower bound for the KS entropy using for that the returns with the shortest lengths (the most probable returns), an approach oriented to the use of our ideas in experimental data. Finally, we discuss in more details the intuitive idea of Lettelier [18] to calculate the Shannon's entropy from a RP and show the relation between Letellier's result and the KS entropy.

2. Estimating the KS entropy from time returns

Let us start with some definitions. By measuring two subsequent returns to a region, one obtains a series of time intervals (FPRs) denoted by τ_i (with $i = 1, \dots, N$). The characterization of the FPRs is done by the probability distribution $\rho(\tau, \mathcal{B})$ of τ_i, where $\mathcal{B}$ represents the volume within which the FPRs are observed. In this work, $\mathcal{B}$ is a D-dimensional box, with sides ϵ_1, and D is the phase space dimension of the system being considered. We denote the shortest return to the region $\mathcal{B}$ as $\tau_{min}(\mathcal{B})$.

Given a trajectory $\{\mathbf{x_i}\}_{\mathbf{i=1}}^{\mathbf{M}}$, the recurrence plot is a two-dimensional graph that helps the visualization of a square array R_{ij}:

$$R_{ij} = \theta(\epsilon_2 - \|\mathbf{x_i} - \mathbf{x_j}\|) \tag{2}$$

where ϵ_2 is a predefined threshold and θ is the Heaviside function [2]. In the coordinate (i, j) of the RP one plots a black point if $R_{ij} = 1$, and a white point otherwise.

There are many interesting ways to characterize a RP, all of them related to the lengths (and their probabilities of occurrence) of the diagonal, horizontal, and vertical segments of *recurrent* points (black points) and of *nonrecurrent* points (white points). Differently from Ref. [18] where it was used the nonrecurrent diagonal segments, we consider here the vertical nonrecurrent and recurrent segments because they provide a direct link to the FPRs [19].

Given a column i, a vertical segment of Q white points starting at $j = p$ and ending at $j = p + Q - 1$, indicates that a trajectory previously in the neighborhood of the point $\mathbf{x}_i$ returns to it firstly after $Q + 1$ iterations in the neighborhood of the point $\mathbf{x}_i$, basically the same definition as the FPR to a volume centered at $\mathbf{x}_i$. However, the white points represent returns to the neighborhood of $\mathbf{x}_i$ which are larger than 1. In order to obtain the returns of length 1, one needs to use the recurrent segments, the segments formed by black points. A recurrent vertical segment at the column i, starting at $j = p$ and ending at $j = p + Q$, means that it occurred Q first returns of length 1 to the neighborhood of the point $\mathbf{x}_i$. The probability density of the return times observed in the RP is represented also by $\rho(\tau, \mathcal{B})$. It is constructed considering the first returns observed in all columns of the RP and it satisfies $\int \rho(\tau, \mathcal{B}) d\tau = 1$.

Notice that the Shannon's entropy of first returns of non-periodic continuous systems becomes infinite [20] as the size ϵ of the volume $\mathcal{B}$ approaches zero. For chaotic systems (as well as for stochastic systems) the reason lies on the fact that the probability density $\rho(\tau, \mathcal{B})$ approaches the exponential form $\mu e^{-\mu\tau}$ [21], where μ is the probability of finding the trajectory within the volume $\mathcal{B}$.

Placing in Eq. (1) the probabilities of returns $\rho(\tau, \mathcal{B})$, we can write that $H_{KS} = H_S/T$, where T is some characteristic time of the returns [8] that depends on how the returns are measured. For the FPRs there exists three characteristic times: the shortest, the longest and the average return. The quantity T cannot be the longest return since it is infinite. It cannot be the average return, since one would arrive to $H_{KS} \cong \mu \log(\mu)$ which equals zero as $\epsilon \to 0$. Therefore, $T = \tau_{min}$ is the only remaining reasonable characteristic time to be used which lead us to

$$H_{KS}(\mathcal{B}[\epsilon]) = \frac{1}{\tau_{min}(\mathcal{B}[\epsilon])} \sum_{\tau} \rho(\tau, \mathcal{B}[\epsilon]) \log\left(\frac{1}{\rho(\tau, \mathcal{B}[\epsilon])}\right). \tag{3}$$

For uniformly hyperbolic chaotic systems (tent map, for example), we can prove the validity of Eq. (3). From Ref. [26] we have that

$$H_{KS} = -\lim_{\epsilon\to 0}\frac{1}{\tau_{min}}\log(\rho(\tau_{min},\mathcal{B}[\epsilon])) \tag{4}$$

a result derived from the fact that the KS entropy exponentially increases with the number of unstable periodic orbits embedded in the chaotic attractor. Since $\rho(\tau,\epsilon)\to\mu e^{-\mu\tau}$ as $\epsilon\to 0$, assuming τ_{min} to be very large, and noticing that $\int -\mu e^{-\mu\tau} log[\mu e^{-\mu\tau}]d\tau = -log[\mu]+1$, assuming that $\tau_{min}\to\infty$ and noticing that for such systems $\mu[\mathcal{B}] = \rho(\tau_{min},\epsilon)$, we finally arrive that

$$-\frac{1}{\tau_{min}}\log\left[\rho(\tau_{min})\right] = -\frac{1}{\tau_{min}}\sum_{\tau}\rho(\tau)\log\left[\rho(\tau)\right] \tag{5}$$

and therefore, the right-hand side of Eq. (3) indeed reflects the KS entropy. But notice that Eq. (3) is being applied not only to non-uniformly hyperbolic systems (Logistic and Hénon maps) but also to higher dimensional systems (two coupled maps).

This result can also be derived from Ref. [27] where it was shown that the positive Lyapunov exponent λ in hyperbolic 1D maps is

$$\lambda = \lim_{\epsilon\to 0}\frac{-log[\mu(\epsilon)]}{\tau_{min}(\mathcal{B}[\epsilon])}. \tag{6}$$

Since $\rho(\tau,\epsilon)\to\mu e^{-\mu\tau}$ as $\epsilon\to 0$, and using that $\lambda = H_{KS}$ (Pesin's equality), and finally noticing that $\int -\mu e^{-\mu\tau} log[\mu e^{-\mu\tau}]d\tau = -log[\mu]+1$, one can arrive to the conclusion that $T=\tau_{min}$ in Eq. (3).

The quantity in Eq. (3) is a local estimation of the KS entropy. To make a global estimation we can define the average

$$\langle H_{KS}\rangle = \frac{1}{L}\sum_{\mathcal{B}(\epsilon)} H_{KS}[\mathcal{B}(\epsilon)] \tag{7}$$

representing an average of $H_{KS}[\mathcal{B}(\epsilon)]$ calculated considering L different regions in phase space.

In order to estimate the KS entropy in terms of the probabilities obtained from the RPs, one should use $T=\langle\tau_{min}\rangle$, i.e., replace τ_{min} in Eq. (3) by $\langle\tau_{min}\rangle$, where $\langle\tau_{min}\rangle = \frac{1}{M}\sum_i \tau_{min}(i)$, the average value of the shortest return observed in every column of the RP. The reason to work with an average value instead of using the shortest return considering all columns of the RP is that every vertical column in the RP defines a shortest return $\tau_{min}(i)$ $(i=1,\ldots,M)$, and it is to expect that there is a nontypical point i for which $\tau_{min}(i)=1$.

Imagining that the RP is constructed considering arbitrarily small regions ($\epsilon_2\to 0$) and that we could treat an arbitrarily long data set, the column of the RP which would produce $\tau_{min}=1$ would be just one out of infinite others which produce $\tau_{min}>>1$. There would be also a finite number of columns which would produce τ_{min} of the order of one (but larger than one), but also those could be neglect when estimating the KS-entropy from the RPs. The point we want to make in here is that the possible existence of many columns for which one has $\tau_{min} = 1$ are a consequence of the finite resolution with which one constructs a RP. In order to minimize such effect in our calculation we just ignore the fact that we have indeed found in the RP $\tau_{min} = 1$, and we consider as τ_{min} any return time longer than 1 as the minimal return time. In fact, neglecting the existence of returns of length one is a major point in the work of Ref. [18], since there only the nonrecurrent diagonal segments are considered [19], and thus, the probability of having a point returning to its neighborhood after one iteration is zero.

From the conditional probabilities of returns, a lower bound for the KS entropy can be estimated in terms of the FPRs and RPs by

$$H_{KS}(\mathcal{B}[\epsilon]) \geq -\frac{1}{n}\sum_{i=1}^{n}\frac{1}{P_i}\frac{\rho(\tau_i + P_i)}{\rho(\tau_i)}\log\left[\frac{\rho(\tau_i + P_i)}{\rho(\tau_i)}\right] \tag{8}$$

where we consider only the returns τ_i for which $\rho(\tau_i + P_i)/\rho(\tau_i) > 0$ and $\tau_i + P_i < 2\tau_{min}$, with $P_i \in \mathcal{N}$.

The derivation of Eq. (8) is not trivial because it requires the use of a series of concepts and quantities from the Ergodic Theory. In the following, we describe the main steps to arrive at this inequality.

First we need to understand the way the KS-entropy is calculated via a spatial integration. In short, the KS-entropy is calculated using the Shannon's entropy of the conditional probabilities of trajectories within the partitions of the phase space as one iterates the chaotic system backward [2]. More rigorously, denote a phase space partition δ_N. By a partition we refer to a space volume but that is defined in terms of Markov partitions. Denote S as $S = S_0 \cap S_1 \cap S_{k-1}$ where $S_j \in F^{-j}\delta_N$ $(j = 0, \ldots, k-1)$, where F is a chaotic transformation. Define $h_N(k) = \frac{\mu(S\cap S_k)}{\mu(S)} log \frac{\mu(S\cap S_k)}{\mu(S)}$ and $\mu(S)$ represents the probability measure of the set S. The KS-entropy is defined as $H_{KS} = \lim_{l\to\infty}\frac{1}{l}\sum_{k=0}^{l-1}\int\rho(dx)h_N(k)$, where the summation is taken over l iterations.

Assume now that the region $\mathcal{B}$ represents the good partition δ_N. The region S_j is the result of $F^{-j}\delta_N$, i.e., a j-th backward iteration of $\mathcal{B}$. So, clearly, if one applies j forward iterations to S_j, then $F^jS_j \to \mathcal{B}$. The quantities $\mu(S \cap S_k)$ and $\mu(S)$ refer to the measure of the chaotic attractor inside $S \cap S_k$ and S, respectivelly. By measure we mean the natural measure, i.e. the frequency with which a typical trajectory visits a region. $\mu(S \cap S_k)$ refers to the measure that remained in $\mathcal{B}$ after k iterations and $\mu(S)$ the measure that remained in $\mathcal{B}$ after $k-1$ iterations.

For $k \to \infty$, we have that $\frac{\mu(S\cap S_k)}{\mu(\mathcal{B})} \to \mu(\mathcal{B})$. Also for finite values of k, one has that $\frac{\mu(S\cap S_k)}{\mu(\mathcal{B})} \approx \mu(\mathcal{B})$. For any finite k, we can split this fraction into two components: $\frac{\mu(S\cap S_k)}{\mu(\mathcal{B})} = \mu_{REC}(k,\mathcal{B}) + \mu_{NR}(k,\mathcal{B})$. μ_{REC} refers

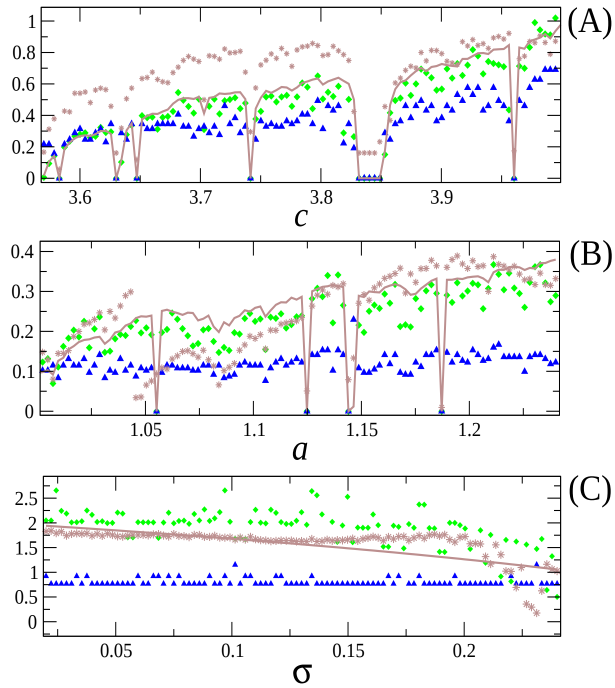

FIGURE 1. [color online] Results from Eq. (3) and (6). The probability function $\rho(\tau, \mathcal{B})$ of the FPRs (RPs) were obtained from a series of 500.000 FPRs (from a trajectory of length 15.000 points). The brown line represents the values of the positive Lyapunov exponent. In (A) we show results for the Logistic map as we vary the parameter c, $\epsilon_2 = 0.002$ for the brown stars and ϵ_1=0.001 for the green diamonds. In (B) we show results for the Hénon map as we vary the parameter a for b=0.3, $\epsilon_2 = [0.002 - 0.03]$ for the brown stars and ϵ_1=0.002 for all the other results, and in (C) results for the coupled maps as we vary the coupling strength σ, ϵ_2=0.05 for the brown stars and ϵ_1=0.02 for green diamonds.

to the measure in $\mathcal{B}$ associated with unstable periodic orbits (UPOs) that return to $\mathcal{B}$, after k iteration of F, at least twice or more times. μ_{NR} refers to the measure in $\mathcal{B}$ associated with UPOs that return to $\mathcal{B}$ only once.

As it is shown in Ref. [26], $\rho(\tau, \mathcal{B}) = \mu_{NR}(\tau, \mathcal{B})$, which in other words means that the probability density of the FPRs in $\mathcal{B}$ is given by $\mu_{NR}(k, \mathcal{B})$. But, notice that for $\tau < 2\tau_{min}$, $\mu_{REC}(k, \mathcal{B}) = 0$ since only returns associated with UPOs that return once can be observed inside $\mathcal{B}$, and therefore $\rho(\tau, \mathcal{B}) = \frac{\mu(S \cap S_\tau)}{\mu(\mathcal{B})}$, if $\tau < 2\tau_{min}$. Consequently, we have that $\frac{\mu(S \cap S_\tau)}{\mu(S)} = \frac{\rho(\tau, \mathcal{B})}{\rho(\tau-1, \mathcal{B})}$, since $\frac{\mu(S \cap S_\tau)}{\mu(\mathcal{B})} = \rho(\tau, \mathcal{B})$ and $\frac{\mu(S)}{\mu(\mathcal{B})} = \rho(\tau - 1, \mathcal{B})$.

The remaining calculations to arrive in Eq. (8) consider the measure of the region $S_\tau \cap S_{\tau+P}$ (instead of $S \cap S_\tau$) in order to have a positive condition probability, i.e. $\frac{\mu(S_\tau \cap S_{\tau+P})}{\mu(S_\tau)} > 0$, with $\mu(S_\tau)$ representing the measure of the trajectories that return to $\mathcal{B}$ after τ iterations and $\mu(S_\tau \cap S_{\tau+P})$ the measure of the trajectories that return to $\mathcal{B}$ after $\tau + P$ iterations. The inequality in Eq. (8) comes from the fact that one neglects the infinitely many terms coming from the measure $\mu_{REC}(\tau, \mathcal{B})$ that would contribute positively to this summation.

3. Estimation of errors in H_{KS} and $\langle H_{KS} \rangle$

In order to derive Eq. (5), we have assumed that $\int -\mu e^{-\mu\tau} log[\mu e^{-\mu\tau}] d\tau = -\log[\mu] + 1$, which is only true when τ_{min}=0. In reality, for $\tau_{min} > 0$, we have $\int_{\tau_{min}}^{\infty} -\mu e^{-\mu\tau} \log[\mu e^{-\mu\tau}] d\tau = e^{-\mu\tau_{min}}[\mu\tau_{min} - \log\mu] + 1$, but as ϵ tends to zero $\mu\tau_{min} \rightarrow 0$ and therefore, as assumed $\int -\mu e^{-\mu\tau} \log[\mu e^{-\mu\tau}] d\tau \approx -\log[\mu] + 1$.

Making the same assumptions as before that $\rho(\tau, \epsilon) \rightarrow \mu e^{-\mu\tau}$ as $\epsilon \rightarrow 0$, and using Eq. (6), then Eq. (3) can be written as

$$H_{KS}(\mathcal{B}[\epsilon]) \approx \lambda + \frac{1}{\tau_{min}(\mathcal{B}[\epsilon])}. \tag{9}$$

Theoretically, one can always imagine a region ϵ with an arbitrarily small size, which would then make the term $\frac{1}{\tau_{min}}$ to approach zero. But, in practice, for the considered values of ϵ, we might have (for atypical intervals) shortest returns as low as $\tau_{min} = 4$. As a result, we expect that numerical calculations of the quantity in Eq. (3) would lead us to a value larger than the positive Lyapunov exponent, as estimated from the returns of the trajectory to a particular region.

Naturally, $\frac{1}{\tau_{min}}$ would provide a local deviation of the quantity in Eq. (3) with respect to the KS entropy. To have a global estimation of the error we are making by estimating the KS entropy, we should consider the error in the average quantity $\langle H_{KS} \rangle$ which is given by

$$E = \sum_{\mathcal{B}(\epsilon)} \frac{1}{\tau_{min}(\mathcal{B}[\epsilon])} \tag{10}$$

where the average is taken over L different regions in phase space, and thus for chaotic systems with no more than one positive Lyapunov exponent

$$\langle H_{KS} \rangle \approx \lambda + E \tag{11}$$

To generalize this result to higher dimensional systems, we make the same assumptions as the ones to arrive to Eq. (9), but now we use Eq. (5). We arrive that

$$\langle H_{KS}(\mathcal{B}[\epsilon]) \rangle \approx H + E, \tag{12}$$

where H denotes the exact value of the KS entropy.

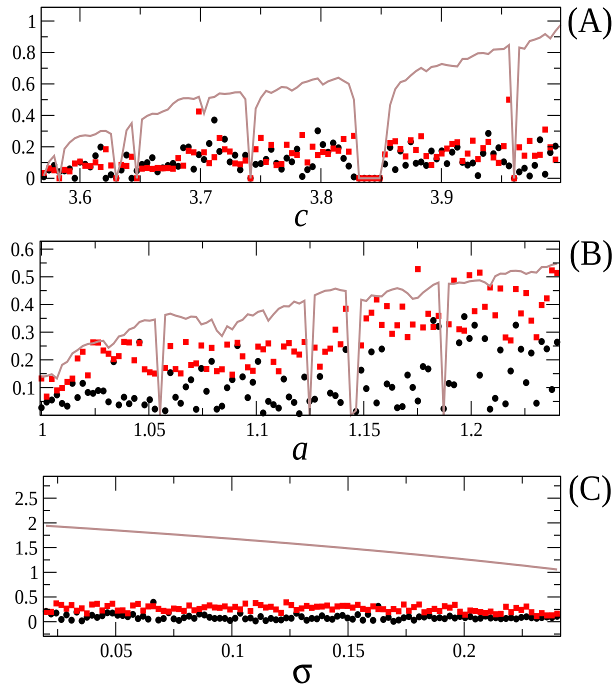

FIGURE 2. [color online] Results from Eq. (8). The probability function $\rho(\tau, \mathcal{B})$ of the FPRs (RPs) were obtained from a series of 500.000 FPRs (from a trajectory of length 15.000 points). The brown line represents the values of the positive Lyapunov exponent. In (A) we show results for the Logistic map as we vary the parameter c, $\epsilon_2 = 0.002$ for the black circles and ϵ_1=0.001 for the red squares. In (B) we show results for the Hénon map as we vary the parameter a for b=0.3, $\epsilon_2 = [0.002 - 0.03]$ for the black circles and ϵ_1=0.002 for the red squares, and in (C) results for the coupled maps as we vary the coupling strength σ, ϵ_2=0.05 for the black circles and ϵ_1=0.02 the red squares.

Finally, it is clear from Eq. (12) that $\langle H_{KS}(\mathcal{B}[\epsilon])\rangle$ is an upper bound for the KS entropy. Thus,

$$H \leq \langle H_{KS}(\mathcal{B}[\epsilon])\rangle. \tag{13}$$

4. Estimating the KS entropy and a lower bound of it in maps

In order to illustrate the performance of our formulas we use the Logistic map $[x_{n+1} = cx_n(1 - x_n)]$, the Hénon map $[x_{n+1} = a - x_n^2 + by_n$, and $y_{n+1} = x_n]$, and a system of two mutually coupled linear maps $[x_{n+1} = 2x_n - 2\sigma(y_n - x_n)$ and $y_{n+1} = 2y_n - 2\sigma(x_n - y_n)$, $mod(1)]$,

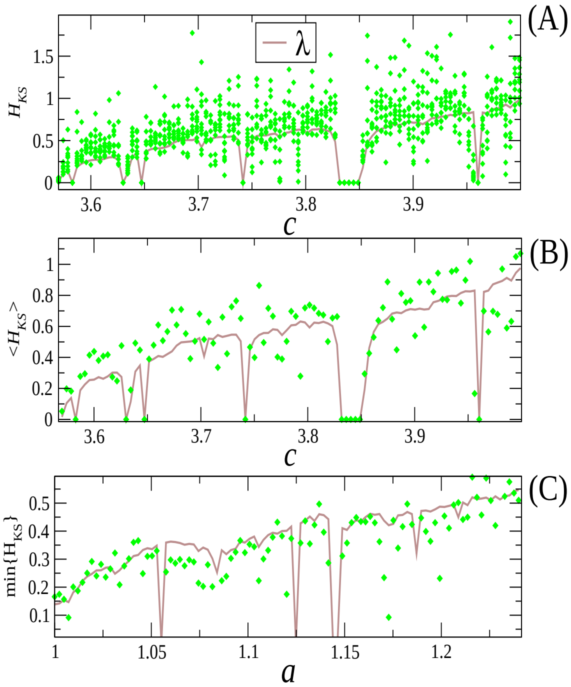

FIGURE 3. [color online] Results from Eq. (3) applied to the FPRs coming from the Logistic map (A-B), as we vary the parameter c and ϵ_1=0.00005, and from the Hénon map (C), as we vary the parameter a and $\epsilon_1 = 0.001$. These quantities were estimated considering 10 randonmly selected regions. The brown line represents the values of the positive Lyapunov exponent. The probability density function $\rho(\tau, \mathcal{B})$ was obtained from a series of 500.000 FPRs. Green diamonds represent in (A) the values of H_{KS} calculated for each one of the 10 randonmly selected regions, in (B) the average value $\langle H_{KS} \rangle$ and in (C) the minimal value of H_{KS}.

systems for which Pesin's equality holds. The parameter σ in the coupled maps represents the coupling strength between them, chosen to produce a trajectory with two positive Lyapunov exponents.

Using Eqs. (3) and (6) to estimate H_{KS} and λ furnishes good values if the region $\mathcal{B}$ where the returns are being measured is not only sufficiently small but also well located such that τ_{min} is sufficiently large. In such a case the trajectories that produce such a short return visit the whole chaotic set [28]. For that reason we measure the FPRs for 50 different regions with a sufficiently small volume dimension, denoted by ϵ_1, and use the FPRs that produce the largest τ_{min}, minimizing H_{KS}. Since the lower bound of H_{KS} in Eq. (8) is a minimal bound for the KS entropy, the region chosen to calculate it is the one for which the lower bound is maximal. This procedure makes H_{KS} and its lower bound (calculated using the FPRs) not to depend on $\mathcal{B}$.

As pointed out in Ref. [18], one should consider volume dimensions (also known as thresholds) which depend linearly on the size of the attractor [28], in order to calculate the Shannon's entropy. In this work, except for the Hénon map, we could calculate well H_{KS}, λ and a lower bound for H_{KS} from the FPRs and RPs, considering for every system fixed values ϵ_1 and ϵ_2. For the Hénon map, as we increase the parameter b producing more chaotic attractors, we increase linearly the size of the volume dimension ϵ_2 within the interval $[0.002 - 0.03]$.

We first compare H_{KS} (see Fig. 1), calculated from Eq. (3) in terms of the probabilities coming from the FPRs and RPs, in green diamonds and brown stars, respectively, with the value of the KS entropy calculated from the sum of the positive Lyapunov exponents, represented by the brown straight line. As expected H_{KS} is close to the sum of all the positive Lyapunov exponents. When the attractor is a stable periodic orbit we obtain that H_{KS} is small if calculated from the RPs. In such a case, we assume that $H_{KS} = 0$ if calculated from the FPRs. This assumption has theoretical grounds, since if the region is centered in a stable periodic attractor and $\epsilon_1 \rightarrow 0$ (what can be conceptually make), one will clearly obtain that the attractor is periodic.

The value of the Lyapunov exponent calculated from the formula (6) is represented in Fig. 1 by the blue up triangles. As it can be checked in this figure, Eq. (6) holds only for 1D hyperbolic maps. So, it works quite well for the logistic map (a 1D "almost" uniformly hyperbolic map) and somehow good for the Hénon map. However, it is not appropriate to estimate the sum of the positive Lyapunov exponents coming from 2D coupled systems. This formula assumes sufficient hyperbolicity and one-dimensionality such that $e^{\tau_{min}\lambda} = 1/\epsilon$.

To compare our approach with the method in Ref. [12], we consider the Hénon map with a=1.4 and $b = 0.3$ for which the positive Lyapunov exponent equals 0.420. Therefore, by using Ruelle equality, $H_{KS} = 0.420$. In Ref. [12] it is obtained that the correlation entropy K_2 equals 0.325, with $H_{KS} \geq K_2$ and in Ref. [13] $H_{KS} = 0.423$. From Eq. (3), we obtain $H_{KS} = 0.402$ and from Eq. (8), we obtain $H_{KS} \geq 0.342$, for ϵ_1=0.01.

In Fig. 2(A-C), we show the lower bound estimation of H_{KS} [in Eq. (8)] in terms of the RPs (black circles) and in terms of FPRs (red squares). As expected, both estimations follow the tendency of H_{KS} as we increase a.

Another possible way Eq. (3) can be used to estimate the value of the KS-entropy is by averaging all the values obtained for different intervals, the quantity $\langle H_{KS} \rangle$ in Eq. (7). In Fig. 3(A), we show the values of H_{KS} as calculated from Eq. (3) considering a series of FPRs with 500.000 returns of trajectories from the Logistic map. For

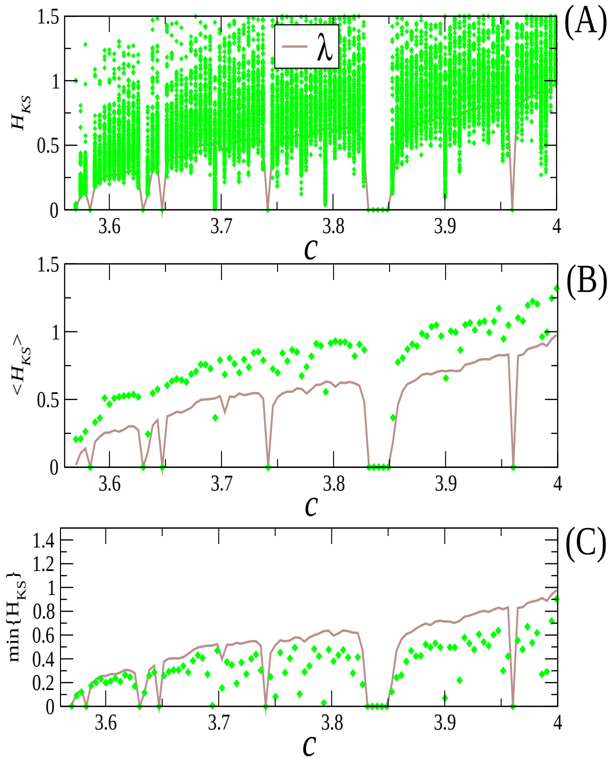

FIGURE 4. [color online] The same quantities shown in Fig. 3, but now considering only the Logistc map, with ϵ_1=0.0002 and 500 randonmly selected regions.

each value of the control parameter c, we randomnly pick 10 different intervals with ϵ_1=0.00005. The average $\langle H_{KS} \rangle$ is shown in Fig. 3(B). As one can see, $\langle H_{KS} \rangle$ is close to the Lyapunov exponent λ. Notice that from Fig. 3(A) one can see that the minimal value of H_{KS} (obtained for the largest τ_{min}) approaches well the value of λ.

In order to have a more accurate estimation of the KS-entropy for the Hénon map, we have used in Figs. 1(B) and 2(B) a varying ϵ_2 depending on the value of the parameter a, exactly as suggested in [18], but similar results would be obtained considering a constant value. As an example, in Fig. 3(C) we show the minimal value of H_{KS} considering regions with $\epsilon_1 = 0.001$, for a large range of the control parameter a.

In order to illustrate how the number of regions as well as the size of the regions alter the estimation of the KS-entropy, we show, in Fig. 4(A-C), the same quantities shown in Fig. 3(A-B), but now from FPRs exclusively coming from the Logistic map, considering 500 randonmly selected regions all having sizes ϵ_1=0.0002. Recall that in Figs. 1 and 3, the minimal value of H_{KS} was chosen out of no more than 50 randonmly selected regions. Comparing Figs. 3(B) and 4(B) one notices that an increase in the number of selected regions is responsible to smooth the curve of $\langle H_{KS} \rangle$ with respect to c. Concerning the minimal value of H_{KS}, the use of intervals with size $\epsilon_1 = 0.0002$ provides values close to the Lyapunov exponent if this exponent is sufficiently low (what happens for $b < 3.7$). Otherwise, these values deviate when this exponent is larger (what happens for $b > 3.7$). This deviation happens because for these chaotic attractors the size of the chosen interval was not sufficiently small [28].

Notice that the estimated KS entropy deviates from λ. See, for example, Figs. 3(B) and 4(B). One sees two main features in these figures. The first is that for most of the simulations, $\langle H_{KS} \rangle > \lambda$. The second is that the larger λ is, the larger the deviation is. The reason for the first feature can be explained by Eqs. (11) and (13). The reason for the second is a consequence of the fact that the larger the Lyapunov exponent is, the smaller τ_{min} is, and therefore the larger the error in the estimation of the KS entropy.

To see that our error estimate provides reasonable results, we calculate the quantities $\langle H_{KS} \rangle$ (green diamonds in Fig. 5), for the Logistic map considering a series of 250.000 FPRs to L=100 randomly selected regions of size $\epsilon_1 = 0.0002$, and the average error E, in Eq. (11) [shown in Fig. 5 by the error bars]. The value of the positive Lyapunov exponent is shown in the full brown line.

The error in our estimation is inversely proportional to the shortest return. Had we considered smaller ϵ regions, τ_{min} would be typically larger and as a consequence we would obtain a smaller error E in our estimation for the KS entropy. Had we consider a larger number of FPRs, the numerically obtained value of τ_{min} would be typically slightly smaller, making the error E to become slightly larger. So, the reason of why the positive Lyapunov exponent in Fig. 5 is located bellow the error bars for the quantity $\langle H_{KS} \rangle$ is a consequence of the fact that we have only observed 250.000 returns, producing an overestimation for the value of τ_{min}. Had we considered a larger number of FPRs would make the error E to become slightly larger.

The considered maps are Ergodic. And therefore, the more (less) intervals used, the shorter (the longer) the time series needed in order to calculate the averages from the FPR as well as from the RP, as the average $\langle H_{KS} \rangle$.

5. Conclusions

Concluding, we have shown how to estimate the Kolmogorov-Sinai entropy and a lower bound of it using the Poincaré First Return Times (FPRs) and the Recurrence Plots. This work considers return times in discrete systems. The extension of our ideas to systems with a continuous description can be straightforwardly made using the ideas in Ref. [29].

We have calculated the expected error in our estimation for the KS entropy and shown that this error appears due to the fact that FPRs

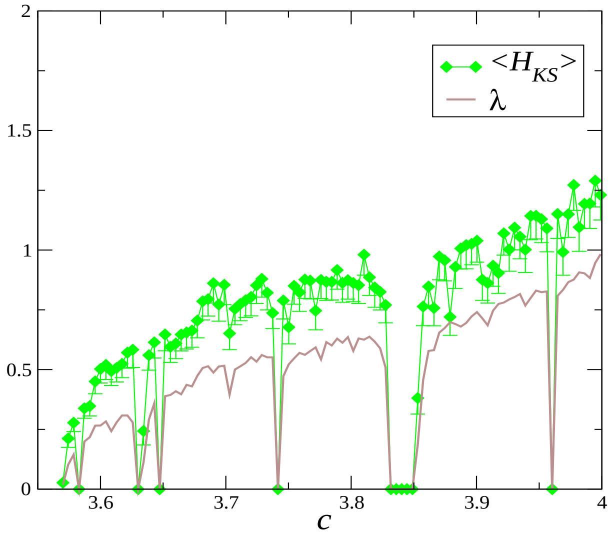

FIGURE 5. [color online] Results obtained considering FPRs coming from the Logistic map, as we vary the parameter c and ϵ_1=0.0002. The probability density function $\rho(\tau, \mathcal{B})$ was obtained from a series of 250.000 FPRs. Green diamonds represent the values of $\langle H_{KS} \rangle$ calculated for each one of the 100 randomly selected regions. The error bar indicates the value of the average error E in Eq. (11). These quantities were estimated considering 100 randomly selected regions. The brown line represents the values of the positive Lyapunov exponent.

can only be physically measured considering finite sized regions and only a finite number of FPRs can be measured. This error is not caused by any fundamental problems in the proposed Eq. (3). Nevertheless, even for when such physical limitations are present, the global estimator of the KS entropy [Eq. (7)] can be considered as an upper bound for the KS entropy [see Eq. (13)].

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[8] The time version of the KS entropy is calculated from the average value of the difference between the Shannon entropies (per unit of time) of a trajectory with a length $n\delta t$ and of a trajectory with a length $(n+1)\delta t$, for all possible values of n. More rigorously, $\lim_{n\to\infty} \lim_{(\epsilon,\delta t)\to 0} \frac{1}{n\delta\tau} \sum_{i=0}^{n-1} [K_{i+1} - K_i]$, where K_i is the shannon's entropy of a trajectory that visits $(n+1)$ volumes of sides ϵ in phase space. In each volume the trajectory remains during a time interval δt. As $i \to \infty$, the terms K_{i+1} are infinities, but not the difference $[K_{i+1} - K_i]$. Since $\sum_{i=0}^{n-1} [K_{i+1} - K_i] = K_n - K_0$ and that $K_0/n\delta\tau \to 0$, then $H_{KS} = \lim_{n\to\infty} \lim_{(\epsilon,\delta t)\to 0} \frac{1}{n\delta\tau} K_n$. So, basically the time version of the KS-entropy can be thought as the Shannon's entropy divided by a characteristic time, T, yet to be determined.

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[19] While the white vertical segments correspond to the first Poincaré returns to an interval centered at a point, the white diagonal segments provide the n-th Poincaré returns. As an example, imagine that there are two black points placed at the coordinates (10,20) and (20,30) in the RP. In the white diagonal segment connecting these two black points there are no black points, which means that we have a nonrecurrent diagonal segment of length 10. That can only be possible if two first Poincaré returns of length 10 happened, or if one second Poincaré return of length 10 happened.

[20] Assuming that $\rho(\tau, \mathcal{B}) = \mu e^{-\mu\tau}$, this leads $H_S = -\log(\mu) + 1$. Since $\mu \sim \epsilon$, if $\epsilon \to 0$, then $H_S \to \infty$.

[21] The exponential form of ρ relies on the fact that the first returns can be imagined to be uncorrelated random variables due to the fast decay of correlation that chaotic systems have. The exponential form of ρ to arbitrarely small volumes is proved for a large class of uniformly hyperbolic maps (see Ref. [22]), for one-dimensional non-uniform hyperbolic maps (unimodal maps, see Ref. [23] and multimodal maps, see [24]). For finite sized volumes ρ still preserves the exponential form [25].

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[28] Usually, the larger an attractor, the larger the Lyapunov exponent. As discussed in [23], $1 = \epsilon e^{\tau_{min}\lambda}$. Assuming that a sufficiently small threshold ϵ provides a sufficiently large τ_{min}, this leads to $\epsilon = e^{-\tau_{min}\lambda}$ and therefore the more chaotic an attractor, the larger ϵ must be in order to have a sufficiently large τ_{min}. Since the series of FPRs can be (in principle theoretically) calculated from arbitrarily large trajectories, we can consider (from a theoretical perspective) regions sufficiently small so that one can obtain FPRs with sufficiently large τ_{min}, even for different values of the control parameters. On the other hand, RPs are constructed with trajectories (time series) shorter than the ones used for the FPRs. In order to have a RP for which $\langle \tau_{min} \rangle$ is sufficiently large and at the same time producing a reasonable continuous distribution $\rho(\tau)$, the volume dimensions ϵ_2 considered to construct the RPs should be reasonably larger than ϵ_1. In addition, as one changes the control parameters producing more complex chaotic attractors, we might have to increase the value of ϵ_2.

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2.2 Article 2 - Density of first Poincaré returns, periodic orbits and Kolmogorov-Sinai entropy

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DENSITY OF FIRST POINCARÉ RETURNS, PERIODIC ORBITS, AND KOLMOGOROV-SINAI ENTROPY

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Abstract. It is known that unstable periodic orbits of a given map give information about the natural measure of a chaotic attractor. In this work we show how these orbits can be used to calculate the density function of the first Poincaré returns. The close relation between periodic orbits and the Poincaré returns allows for estimates of relevant quantities in dynamical systems, as the Kolmogorov-Sinai entropy, in terms of this density function. Since return times can be trivially observed and measured, our approach to calculate this entropy is highly oriented to the treatment of experimental systems. We also develop a method for the numerical computation of unstable periodic orbits.

1. Introduction

Knowing how often a dynamical system returns to some place in phase space is fundamental to understand dynamics. There is a well established way to quantify that: the **first Poincaré return (FPR)**, which measures how much time a trajectory of a dynamical system takes to make two consecutive returns to a given region. Due to their stochastic behaviour, given a return time it is not feasible to predict the future return times and for that reason one is usually interested in calculating the frequency with which the Poincaré returns happen, the **density of the first Poncaré returns** (DFP).

This work explains the existence of a strong relationship between unstable periodic orbits (UPOs) and the first Poincaré returns in chaotic attractors. Unstable orbits and first Poincaré returns have been usually employed as a tool to analyse and characterise dynamical systems. With our novel approach we can calculate how frequently returns happen by knowing only a few unstable periodic orbits. Additionally, such relation allows us to easily estimate other fundamental quantities of dynamical systems such as the Kolmogorov-Sinai entropy.

Our motivation to search for a theoretical and simple way of calculating the distribution of Poincaré return times comes from the fact that they can be simply and quickly accessible in experiments and also due to the wide range of complex systems that can be characterized by such a distribution. Among many examples, in Ref. [1] the return times were used to characterize a experimental chaotic laser, in Refs. [2, 3] they were used to characterize extreme events, in Refs. [4, 5] they were used to characterize fluctuations in fusion plasmas, and in Ref. [6] a series of application to complex data analysis were described.

In addition, relevant quantifiers of low-dimensional chaotic systems may be obtained by the statistical properties of the FPR such as the dimensions and Lyapunov exponents [7, 8] and the extreme value laws [9]. For most of the rigorous results concerning the FPR, in particular the form of the DFP [10], one needs to consider very long returns to arbitrarily small regions in phase space, a condition that imposes limitations into the real application to data sets.

We first show how the DFP can be calculated from only a few UPOs inside a finite region. Then, we explain how the DFP can be used to calculate quantities as the Kolmogorov-Sinai entropy, even when only short return times are measured in finite regions of the phase space.

Our work is organized as follows. We first introduce the work of Ref. [11], which relates the natural measure of a chaotic attractor to the UPOs embedded in a chaotic attractor. The measure of a chaotic attractor refers to the frequency of visits that a trajectory makes to a portion of the phase space. This measure is called natural when it is invariant for typical initial conditions. This appears in Sec. 2, along with the relevant definitions. In Sec. 3 we define $\rho(\tau, S)$ the density of first Poincaré returns for a time τ to a subset S of phase space and we study the relation between the UPOs and this function. This can be better understood if we classify the UPOs inside S as recurrent and non-recurrent. Recurrent are those UPOs that return more than once to the subset S before completing its cycle. Non-recurrent are UPOs that visits the subset S only once in a period. While in the calculation of the natural measure of S one should consider the two types of UPOs with a given large period inside it, for the calculation of the DFP for a time τ one should consider only non-recurrent UPOs with a period τ. Sec. 4 is mostly dedicated to show how to calculate $\rho(\tau, S)$ even when not all non-recurrent UPOs of a large period are known. Such a situation typically arises when the time τ is large. We have numerically shown that the error of our estimation becomes smaller, the longer the period of the UPOs and the larger the number of UPOs considered.

Throughout the paper we illustrate results by presenting the calculations for the *tent map*. Finally, in Sec. 6 we show numerical results on the *logistic map* that support our approach. In particular, we obtain numerical estimates of the Kolmogorov-Sinai entropy, the most successful invariant in dynamics, so far. The estimates are obtained considering the density of only short first return times, as discussed in Sec. 5. The UPOs of period p are computed numerically as stable periodic orbits of a system of p coupled cells, a method described in 6.5.

2. Definitions and results

Consider a d-dimensional C^2 map of the form $x_{n+1} = F(x_n)$, where $x \in \Omega \subset R^n$ and Ω represents the phase space of the system. Consider $A \subset \Omega$ to represent a chaotic attractor. By chaotic attractor we mean an attractor that has at least one positive Lyapunov exponent.

For a subset S of the phase space and an initial condition x_0 in the basin of attraction of A, we define $\mu(x_0, S)$ as the fraction of time the trajectory originating at x_0 spends in S in the limit that the length of the trajectory goes to infinity. So,

$$\mu(x_0, S) = \lim_{n\to\infty} \frac{\sharp\{F^i(x_0) \in S,\ 0 \le i \le n\}}{n}. \tag{1}$$

Definition 2.1. *If $\mu(x_0, S)$ has the same value for almost every x_0 (with respect to the Lebesgue measure) in the basin of attraction of A, then we call the value $\mu(S)$ the **natural measure of** S.*

For now we assume that our chaotic attractor A has always a natural measure associated to it, normalized to have $\mu(A) = 1$. In particular this means that the attractor is ergodic[11].

We also assume that the chaotic attractor A is mixing: given two subsets, B_1 and B_2, in A, we have:

$$\lim_{n\to\infty} \mu(B_1 \cap F^{-n}(B_2)) = \mu(B_1)\mu(B_2).$$

In addition, we consider A to be a hyperbolic set.

The eigenvalues of the Jacobian matrix of the n-th iterate, F^n, at the jth fixed point x_j of F^n are denoted by $\lambda_{1j}, \lambda_{2j}, ..., \lambda_{uj}, \lambda_{(u+1)j}, ..., \lambda_{dj}$, where we order the eigenvalues from the biggest, in magnitude, to the lowest and the number of the unstable eigenvalues is u. Let $L_j(n)$ be the product of absolute values of the unstable eigenvalues at x_j.

Then it was proved by Bowen in 1972 [12] and also by Grebogi, Ott and Yorke in 1988 [11] the following:

Theorem 2.1. *For mixing hyperbolic chaotic attractors, the natural probability measure of some closed subset S of the d-dimensional phase space is*

$$\mu(S) = \lim_{n\to\infty} \sum_{x_j} L_j^{-1}(n), \tag{2}$$

where the summation is taken over all the fixed points $x_j \in S$ *of* F^n*.*

This formula is the representation of the natural measure in terms of the periodic orbits embedded in the chaotic attractor. To illustrate how it works let us take a simple example like the tent map:

Example 2.1. *Let us consider* $F : [0,1] \to [0,1]$ *such that*

$$F(x) = \begin{cases} 2x, & if\ x \in [0,1/2] \\ 2-2x, & if\ x \in]1/2,1] \end{cases}$$

For this map there is only one unstable direction. Since the absolute value of the derivative is constant in $[0,1]$ *we have* $L_j(\tau) = L(\tau) = 2^\tau$*.*

For the tent map, periodic points are uniformly distributed in $[0,1]$*. Using this fact together with some of the ideas of G.H. Gunaratne and I. Procaccia* [13]*, it is reasonable to write the natural measure of a subset* S *of* $[0,1]$ *as:*

$$\mu(S) = \lim_{\tau\to\infty} \frac{N(\tau,S)}{N(\tau)}, \tag{3}$$

where $N(\tau,S)$ *is the number of fixed points of* F^τ *in* S *and* $N(\tau)$ *is the number of fixed points of* F^τ *in all space* $[0,1]$*. For this particular case we have* $N(\tau) = L(\tau) = L_j(\tau)$ *and so*

$$\mu(S) = \lim_{\tau\to\infty} \frac{N(\tau,S)}{N(\tau)} = \lim_{\tau\to\infty} \frac{N(\tau,S)}{L(\tau)} = \lim_{\tau\to\infty} \sum_{j=1}^{N(\tau,S)} \frac{1}{L_j(\tau)}$$

and we obtain the Grebogi, Ott and Yorke formula.

3. Density of first returns and UPOs

In this section we relate the DFP, $\rho(\tau,S)$, and the UPOs of a chaotic attractor. We show in Eq. (10) that $\rho(\tau,S)$ can also be calculated in terms of the UPOs but one should consider in Eq. (2) only the non-recurrent ones.

3.1. **First Poincaré returns.** Consider a map F that generates a chaotic attractor $A \subset \Omega$, where Ω is the phase space. The *first Poincaré return* for a given subset $S \subset \Omega$ such that $S \cap A \neq \emptyset$ is defined as follows.

Definition 3.1. *A natural number* τ*,* $\tau > 0$*, is the* ***first Poincaré return*** *to* S *of a point* $x_0 \in S$ *if* $F^\tau(x_0) \in S$ *and there is no other* $\tau^* < \tau$ *such that* $F^{\tau^*}(x_0) \in S$*.*

A trajectory generates an infinite sequence, $\tau_1, \tau_2, ..., \tau_i$, of first returns where $\tau_1 = \tau$ and τ_i is the first Poincaré return of $F^{n_i}(x_0)$ with $n_i = \sum_{n=1}^{i-1} \tau_n$.

The subset S' of points in $S \subset \Omega$ that produce FPRs of length τ to S is given by

$$S' = S'(\tau, S) = \left(F^{-\tau}(S) \cap S\right) - \bigcup_{0<j<\tau} \left(F^{-j}(S) \cap S\right). \tag{4}$$

3.2. **Density function.** In this work, we are concerned with systems for which the DFP decreases exponentially as the length of the return time goes to infinity. Such systems have mixing properties and as a consequence we expect to find $\rho(\tau, S) \approx \mu(S)(1 - \mu(S))^{\tau-1}$, where $(1-\mu(S))^{\tau-1}$ represents the probability of a trajectory remaining $\tau-1$ iterations out of the subset S. We are interested in systems for which the decay of $\rho(\tau)$ is exponential, i.e., $\rho(\tau) \propto e^{-\alpha\tau}$.

The usual way of defining $\rho(\tau, S)$, for a given subset $S \subset \Omega$, is by measuring the fraction of returns to S that happen with a given length τ with respect to all other possible first returns [see Eq. (27)]. It is usually required for a density that

$$\int \rho(\tau, S) d\tau = 1.$$

In this work, we also adopt a more appropriate definition for $\rho(\tau, S)$ in terms of the natural measure. We define the function $\rho(\tau, S)$ as the natural measure of the set of orbits that makes a first return τ to S divided by the natural measure in S. More rigorously

Definition 3.2. *The* ***density function*** *of the first Poincaré return τ for a particular subset $S \subset \Omega$ such that $\mu(S) \neq 0$ is defined as*

$$\rho(\tau, S) = \frac{\mu(S')}{\mu(S)}, \tag{5}$$

where $S' = S'(\tau, S) \subset S$ is the subset of points that produce FPRs of length τ defined in Eq. (4).

Even for a simple dynamical system as the tent map, the analytical calculation of $\rho(\tau, S)$ is not trivial. However, an upper bound for this function can be easily derived as in the following example:

Example 3.1. *Consider the tent map defined in example 2.1, for which the natural measure coincides with the Lebesgue measure λ, and let $S \subset [0, 1]$ be a non-trivial closed interval.*

To have a return to S we only need to know the natural number n^* such that $F^{n^*}(S) = [0, 1]$. Since F is an expansion, this natural number always exists. To find it when $\lambda(S) = \epsilon > 0$, we first solve the equation $2^{x^*} = 1/\epsilon$ and get $x^* = -\log(\epsilon)/\log(2)$, so we take $n^* = [-\log(\epsilon)/\log(2)] + 1$, where $[x]$ represents the integer part of x. Then n^* is an upper bound for τ_{min}, the shortest first return to S.

Most intervals S of small measure have large values of τ_{min} and $\tau_{min} \approx n^*$ is a good approximation. A sharper upper bound for τ_{min} to S is the lowest period of an UPO that visits it.

The set $D = F^{-n^*}(S) \cap S \neq \emptyset$ *represents the fraction of points in* S *that return to* S *(not necessarily first return) after* n^* *iterations. Using Eq. (5) and since* $S' \subset D$ *we have*

$$\rho(n^*, S) \leq \frac{\lambda(D)}{\lambda(S)} \leq \frac{\epsilon \frac{1}{2^{n^*}}}{\epsilon} = 2^{-n^*}.$$

It is natural to expect that for τ *of the order of* n^* *and close to* τ_{min} *we have* $\rho(\tau, S) \leq 2^{-\tau}$.

We can write this equation as $\rho(\tau, S) \leq e^{(-\tau \log(2))} = e^{(-\tau\lambda_1)}$, *where* $\lambda_1 = \log(2)$ *is the Lyapunov exponent for the tent map. In fact, in 1991, G. M. Zaslavsky and M. K. Tippett*

[14][15] *presented one formula for the exact value of*

$\rho(\tau, S)$. *That result can only be valid under the same conditions that we have used previously, i.e.* $\tau \approx \tau_{min}$ *and for most sets of sufficiently small measure* ϵ, *so that* $\tau_{min} \approx n^*$.

3.3. **Density function in terms of recurrent and non-recurrent UPOs.** Since our chaotic attractor A is mixing, the natural measure associated with A satisfies, for any subset S of nonzero measure:

$$\mu(S) = \lim_{\tau \to \infty} \frac{\mu(S \cap F^{-\tau}(S))}{\mu(S)}.$$

We can write the right hand side of the last equation, for any positive τ, in two terms:

$$\frac{\mu(S \cap F^{-\tau}(S))}{\mu(S)} = \frac{\mu(S')}{\mu(S)} + \frac{\mu(S^*)}{\mu(S)} \tag{6}$$

with S' as defined in Eq. (4) and where $S^* = S^*(S, \tau)$ is the set of points in S that are mapped to S after τ iterations but for which τ is not the FPR to S, so $S' \cup S^* = (S \cap F^{-\tau}(S))$ and $S' \cap S^* = \emptyset$.

An UPO of period τ is *recurrent* with respect to a set $S \subset \Omega$ if there is a point $x_0 \in S$ in the UPO with $F^n(x_0) \in S$ for $0 < n < \tau$. In other words, its FPR is less than its period. Thus, the UPOs in the set S^* are all recurrent. We refer to them as the recurrent UPOs *inside* S.

Associated with the recurrent UPOs in S we define

$$\mu_R(\tau, S) = \sum_j \frac{1}{L_j^R(\tau)} \tag{7}$$

and associated with the non-recurrent UPOs in S we define

$$\mu_{NR}(\tau, S) = \sum_j \frac{1}{L_j^{NR}(\tau)} \tag{8}$$

where $L_j^R(\tau)$ and $L_j^{NR}(\tau)$ refer, respectively, to the product of the absolute values of the unstable eigenvalues of recurrent and non-recurrent UPOs of period τ that visit S.

Notice that , if $\mu(S) \neq 0$,

$$\lim_{\tau \to \infty} \frac{\mu(S^*)}{\mu(S)} = \lim_{\tau \to \infty} \mu_R(\tau, S)$$

and

$$\lim_{\tau \to \infty} \frac{\mu(S')}{\mu(S)} = \lim_{\tau \to \infty} \mu_{NR}(\tau, S) \tag{9}$$

since $\mu(S^*)/\mu(S)$ measures the frequency with which chaotic trajectories that are associated with the recurrent UPOs visit S and $\mu(S')/\mu(S)$ measures the frequency with which chaotic trajectories that are associated with the non-recurrent UPOs visit S.

Comparing Eqs. (5), (6) and (9) we obtain the following:

Main Idea: *For a chaotic attractor A generated by a mixing uniformly hyperbolic map F, for a small subset $S \subset A$, generated by a Markov partition and such that the measure in S is provided by the UPOs inside it, we have that*

$$\rho(\tau, S) \approx \mu_{NR}(\tau, S), \tag{10}$$

for a sufficiently large τ. Moreover,

$$\mu(S) = \lim_{\tau \to \infty} [\rho(\tau, S) + \mu_R(\tau, S)].$$

A Markov partition is a very special splitting of the phase space. For the purpose of better justifying Eq. (10), if a region $C(\tau)$ belongs to a Markov partition of order τ, then there is a sub-interval $\tilde{C}(\tau)$ of $C(\tau)$ that after τ iterations is mapped exactly over $C(\tau)$. Moreover, points inside $\tilde{C}(\tau)$ make first returns to $C(\tau)$ after τ iterations. Then, $\mu_R(\tau, C(\tau))$=0. As a consequence, for sufficiently large τ we can write that $\mu[C(\tau)] \to \rho[\tau, C(\tau)]$.

But approximation (10) remains valid for a small nonzero τ. The reason for that is the following: Notice that from the way Kac's lemma is derived (see Sec. 8.1), Eq. (2) can be written as

$$\mu(S) = \frac{\int_{\tau_{min}}^{\infty} \rho(\tau, S) d\tau}{< \tau >},$$

where $< \tau >$ represents the average of the FPRs inside S, since $\int_{\tau_{min}}^{\infty} \rho(\tau, S) d\tau = 1$. This equation illustrates that any possible existing error in the calculation of $\mu(S)$ by Eq. (2) is a summation over all errors coming from $\rho(\tau, S)$ for all values of τ that we are considering. As shown in Ref. [11], $\mu(S)$ can be calculated by Eq. (2) using UPOs with a small and finite period p. This period is of the order of the time that the Perron-Frobenius operator converges and thus linearization around UPOs can be used to calculate the measure associated with them. As a consequence, if $\mu(S)$ can be well estimated for $p \approx 30$ then $\rho(\tau, S)$

can be well estimated for $\tau << p$. As we will observe, considering τ small, of the order of 5, we get a very good estimation for $\rho(\tau, S)$.

In addition, we observe in our numerical simulation that S does not need to be a cell in a Markov partition but just a small region located in an arbitrary location in Ω.

We say that an UPO has FPRs associated with it if the UPO is non-recurrent. See that for every UPO there is a neighborhood containing no other UPO with the same period. If the UPO is non-recurrent then all points inside a smaller neighborhood will produce FPRs associated with this UPO in the sense that their FPR coincides with the UPO's. Consider τ_{min} as the shortest first return in S.

Case $\tau < 2\tau_{min}$

UPOs of period τ are non-recurrent. This is illustrated in Fig. 1 (A), where $\tau_{min} = 7$, for the logistic map ($c = 4$). In that picture we observe that for $\tau \leq 14$ all FPRs are associated with UPOs. Because of this fact $\mu(S^*) = 0$ and then all the chaotic trajectories that return to S are associated with non-recurrent UPOs. So, $\rho(\tau, S) \approx \mu(S)$ and thus, $\rho(\tau, S) \approx \mu_{NR}(\tau, S)$.

Case $\tau \geq 2\tau_{min}$

We can have recurrent UPOs of period τ, that do not have first returns associated with them. As a consequence $\mu(S^*) > 0$ and recurrent UPOs contribute to the measure of S. This is illustrated in Fig. 1 (B), when $\tau = 16$.

4. How to calculate the density of first Poincaré returns

A practical issue is how to calculate $\mu_{NR}(\tau, S)$. There are two relevant cases: All UPOs can be calculated; only a few can be calculated.

Assuming τ to be sufficiently small such that all UPOs of period τ can be calculated and sufficiently large so that Eq. (10) is reasonably valid, $\mu_{NR}(\tau, S)$ can be exactly calculated and we can easily estimate $\rho(\tau, S)$ from Eq.(10), using $\rho(\tau, S) \approx \mu_{NR}(\tau, S)$.

When τ is large then, typically, only a few UPOs can be calculated. For this case, it is difficult to use Eq. (10) to estimate $\rho(\tau, S)$ since there will be too many UPOs. In order to calculate $\rho(\tau, S)$ using $\mu_{NR}(\tau, S)$ we do the following. First notice that

$$\mu(S) = \lim_{\tau \to \infty} (\mu_{NR}(\tau, S) + \mu_R(\tau, S)). \tag{11}$$

Considering then τ sufficiently large we have that

$$\mu(S) \approx \mu_{NR}(\tau, S) + \mu_R(\tau, S)$$

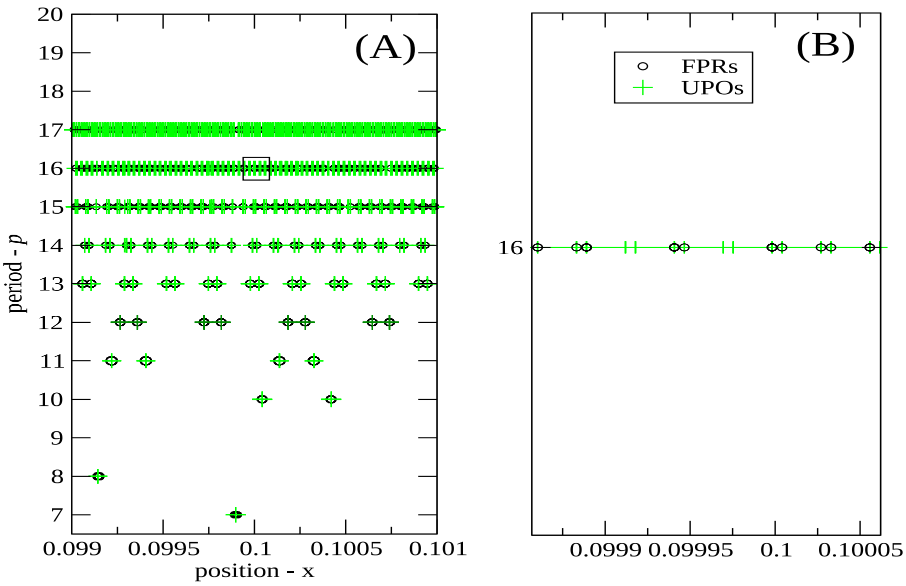


FIGURE 1. This picture shows some UPOs inside $S \subset [0,1]$ and first Poincaré returns for the logistic map, $[x_{n+1} = 4x_n(1-x_n)]$. In this example $\tau_{min} = 7$. For $\tau < 14$ all UPOs have FPRs associated with them. For $\tau \geq 14$ (as in (B) for $\tau = 16$) some UPOs are recurrent. Picture (B) is a zoom of picture (A).

which can be rewritten [using Eq. (10) which says that $\rho(\tau, S) \approx \mu_{NR}(\tau, S)$, for finite τ] as

$$\rho(\tau, S) \approx \mu(S) - \mu_R(\tau, S) = \mu(S)\left(1 - \frac{\mu_R(\tau, S)}{\mu(S)}\right). \tag{12}$$

This equation allows us to reproduce, approximately, the function $\rho(\tau, S)$, for any sufficiently large τ, only using the estimated value of the quotient

$$\frac{\mu_R(\tau, S)}{\mu(S)}$$

that is easy to obtain numerically, since not all UPOs should be calculated but just a few ones with period τ. We discuss this in 4.1 below.

4.1. **How can we estimate $\mu_R(\tau, S)/\mu(S)$?** Considering a subset S and fixing τ, we calculate a number t of different UPOs with period τ (say, $t = 50$) inside S (It is explained in Sec. 6.5 how to calculate numerically UPOs with any period of a given map). These UPOs are calculated from randomly selected symbolic sequences for which the generated UPOs visit S. See that, for example, in the tent map, for

$\tau = 10$ and $S = [0, \frac{1}{8}]$, we may have $2^{10}/8$ UPOs inside S and so, here 50 UPOs inside S is, in fact, a very small number of UPOs.

Now, we separate all the t UPOs that visit S into recurrent and non-recurrent ones and suppose that we have r recurrent and nr non-recurrent such that $r + nr = t$. So, r and nr depend on t and S. With these particular $r(t, S)$ recurrent UPOs we use Eq. (7) and we obtain

$$\tilde{\mu}_R[\tau, S, r(t, S)] = \sum_{j=1}^{r(t,S)} \frac{1}{L_j^R(\tau)}$$

where $L_j^R(\tau)$ represents the product of the absolute values of the unstable eigenvalues of the j-th recurrent UPO within the set of $r(t, S)$ recurrent UPOs. See that this quantity is not equal to $\mu_R(\tau, S)$ since we are not considering all recurrent UPOs inside S but just a small number $r(t, S)$ of them. We do the same thing with the $nr(t, S)$ non-recurrent UPOs and obtain the quantity $\tilde{\mu}_{NR}[\tau, S, nr(t, S)]$.

Finally, we observe that, for a sufficiently large t, we have

$$\frac{\tilde{\mu}_R[\tau, S, r(t, S)]}{\tilde{\mu}(\tau, S, t)} \approx \frac{\mu_R(\tau, S)}{\mu(S)},$$

where $\tilde{\mu}(\tau, S, t) = \tilde{\mu}_R[\tau, S, r(t, S)] + \tilde{\mu}_{NR}[\tau, S, nr(t, S)]$. Therefore, with only a few UPOs inside S we calculate an estimated value for $\rho(\tau, S)$. This estimation is represented by ρ_M and is given by

$$\rho_M[\tau, S, r(t, S)] = \mu(S) \left(1 - \frac{\tilde{\mu}_R[\tau, S, r(t, S)]}{\tilde{\mu}(\tau, S, t)}\right) \tag{13}$$

Notice that, for a large τ we will have more recurrent UPOs than non-recurrent ones and therefore the larger τ is, the larger is the contribution of the recurrent UPOs to the measure inside S.

4.2. **Error in the estimation.** To study how much our estimation in Eq. (13) depends on the number t of UPOs, we first assume that if all UPOs are known, the calculated distribution in Eq. (10) is "exact", or in other words it has a neglectable error as when compared to the real distribution provided by Eq. (5).

Then, the error in Eq. (13) will depend on the deviation of the quotient

$$q_1 = \frac{\tilde{\mu}_R[\tau, S, r(t, S)]}{\tilde{\mu}(\tau, S, t)}, \tag{14}$$

calculated when only t UPOs are known, to the quotient

$$q_2 = \frac{\tilde{\mu}_R[\tau, S, r(t = N(\tau, S), S)]}{\tilde{\mu}(\tau, S, t = N(\tau, S))}, \tag{15}$$

calculated when all the $N(\tau, S)$ UPOs are known.

Thus, the amount of error that our estimate [Eq. (13)] has as when compared to the "exact" value of ρ (when all the UPOs are known) can be calculated by

$$E[\tau, S, t] = \frac{|q_1 - q_2|}{q_2} \tag{16}$$

which means that the quantity E gives the amount of deviation, in a scale from 0 to 1, of ρ_M [Eq. (13)] as when compared to the "exact" value of ρ [Eq. (10)]. Notice that in Eq. (16), the quantity $100E$ corresponds to the percentage of error that our estimation has.

4.3. **Uniformly distributed UPOs.** There is another way to estimate the value of $\rho(\tau, S)$ in terms of the number of UPOs in a subset S of a chaotic attractor A. We define $N(\tau)$ as the number of fixed points of F^τ in A, $N(\tau, S)$ as the number of fixed points of F^τ in S, $N_R(\tau, S)$ as the number of fixed points of F^τ in S whose orbit under F is recurrent and $N_{NR}(\tau, S)$ as the number of fixed points of F^τ in S whose orbit under F is non-recurrent. Then, for a sufficiently large τ and for a uniformly hyperbolic dynamical system for which periodic points are uniformly distributed in A, we have

$$\mu_R(\tau, S) \approx \frac{N_R(\tau, S)}{N(\tau)}, \quad \mu_{NR}(\tau, S) \approx \frac{N_{NR}(\tau, S)}{N(\tau)}.$$

Using the previous approximations we can write

$$\mu(S) \approx \frac{N_R(\tau, S)}{N(\tau)} + \frac{N_{NR}(\tau, S)}{N(\tau)} = \frac{N(\tau, S)}{N(\tau)}.$$

By Eq. (10) we may write $\rho(\tau, S) \approx \mu_{NR}(\tau, S)$ and we have that

$$\rho(\tau, S) \approx \mu(S) - \frac{N_R(\tau, S)}{N(\tau)}. \tag{17}$$

which can be written as

$$\rho(\tau, S) \approx \mu(S) \left(1 - \frac{N_R(\tau, S)}{N(\tau, S)}\right). \tag{18}$$

Again, we have an expression with a quotient

$$\frac{N_R(\tau, S)}{N(\tau, S)}$$

that is, again, easy to obtain numerically by the same technique from which μ_R/μ can be estimated and therefore we can obtain an estimation for $\rho(\tau, S)$, represented by ρ_N, by

$$\rho_N[\tau, S, r(t, S)] = \mu(S) \left(1 - \frac{r(t, S)}{t}\right) \tag{19}$$

where $r(t, S)$ represents the number of recorrent UPOs out of a total of t UPOs, exactly as previously defined.

5. Kolmogorov-Sinai entropy

In 1958 Kolmogorov introduced the concept of entropy into ergodic theory and this has been the most successful invariant so far[16]. In this section we explain how to use the density of first Poincaré returns to estimate the Kolmogorov-Sinai entropy H_{KS}.

The exposition here does not aim to be rigorous, only to explain how we have arrived at the numerical estimates for the logistic map of Sec. 6. which is a non uniformly hyperbolic map.

It is known that[17]

$$N(\tau) \propto \exp(\tau H_{KS}). \tag{20}$$

Consider F as a dynamical system that has the following property:

$$\frac{N_{NR}(\tau, S)}{N(\tau)} \approx \mu_{NR}(\tau, S) \approx \rho(\tau, S),$$

for a sufficiently large τ. For example, dynamical systems for which periodic points are uniformly distributed on the chaotic attractor A have this property.

Considering the tent map and $S \subset [0,1]$ such that $N_{NR}(\tau, S) = 1$ (if there is more that one non-recurrent UPO of period τ inside S we shrink S to have only one), we have $\rho(\tau, S) \approx \frac{1}{2^\tau}$ that agrees with example 3.1, for τ close to τ_{min} and for most intervals S. For other non-uniformly hyperbolic systems as the logistic the Hénon maps, this property holds in an approximate sense and this approximation is better the larger τ is and the closer the interval S is to a Markov partition.

Using the last approximation together with Eq. (20) we may write

$$\frac{N_{NR}(\tau, S)}{\rho(\tau, S)} \approx b \exp(\tau H_{KS}),$$

for some positive constant $b \in R$. So, we have that

$$H_{KS} \approx \frac{1}{\tau} \log\left(\frac{N_{NR}(\tau, S)}{b\rho(\tau, S)}\right) = \frac{1}{\tau} \log\left(\frac{N_{NR}(\tau, S)}{\rho(\tau, S)}\right) - \frac{\log(b)}{\tau}. \tag{21}$$

We define the quantity $H(\tau, S)$ as

$$H(\tau, S) = \frac{1}{\tau} \log\left(\frac{N_{NR}(\tau, S)}{\rho(\tau, S)}\right) \tag{22}$$

and then, for $b \geq 1$, it is clear that

$$H_{KS} \approx \frac{1}{\tau} \log\left(\frac{N_{NR}(\tau, S)}{b\rho(\tau, S)}\right) \leq H(\tau, S),$$

so $H(\tau, S)$ is a local upper bound for the approximation of H_{KS}, considering a sufficiently large τ.

Supposing that there is at least one non-recurrent UPO inside S, then for large τ we have $\frac{N_{NR}(\tau,S)}{\rho(\tau,S)} \gg b$, as b is constant. Thus, the term

$$\frac{1}{\tau}\log\left(\frac{N_{NR}(\tau,S)}{\rho(\tau,S)}\right)$$

dominates the expression (21), for longer times.

This equation allows us to obtain an upper bound for $\rho(\tau,S)$. See that $\rho(\tau,S) \leq N_{NR}(\tau,S)\exp(-\tau H_{KS})$ and if $\tau \approx \tau_{min}$ then $N_{NR}(\tau,S) \approx 1$ and we obtain $\rho(\tau,S) \leq \exp(-\tau H_{KS})$ as in example 3.1.

Equation (22) depends on the choice of the subset S and is then a local estimation for H_{KS}. To have a global estimate we take a finite number, n, of subsets S_i in the chaotic attractor and make a space average as

$$\frac{1}{\tau n}\sum_{i=1}^{n}\log\left(\frac{N_{NR}(\tau,S_i)}{\rho(\tau,S_i)}\right). \tag{23}$$

Better results are obtained taking the average over pairwise disjoint subsets S_i that are well distributed over A.

When we consider $N_{NR}(\tau,S) = 1$ this means that we have only one non-recurrent UPO, with period τ, inside S. In general, for sufficiently small subsets, S_i, we may have $N_{NR}(\tau,S_i) = 1\ \forall i$ and we obtain an approximation that only depends on the density function of the first Poincaré returns

$$H_{KS} \approx \frac{1}{\tau n}\sum_{i}\log\left(\frac{1}{\rho(\tau,S_i)}\right). \tag{24}$$

An equation which can be trivially used from the experimental point of view since we just need to estimate $\rho(\tau,S_i)$ and we do not need to know the UPOs. For practical purposes, we consider in Eqs. (22), (23) and (24) that $\tau = \tau_{min}$.

6. Numerical results

6.1. **Calculating ρ when all UPOs are known.** The logistic family $F : [0,1] \to [0,1]$ is

$$F(x) = cx(1-x), \tag{25}$$

were $c \in R$. There are many biological motivations to study this family of maps[18]. The maps that we obtain when the parameter c is varied have interesting mathematical properties. It is therefore of relevant use for mathematical and biological study.

For most numerical simulations in this section we take $c = 4$ in Eq. (25), for which the map is chaotic and the chaotic attractor is compact.

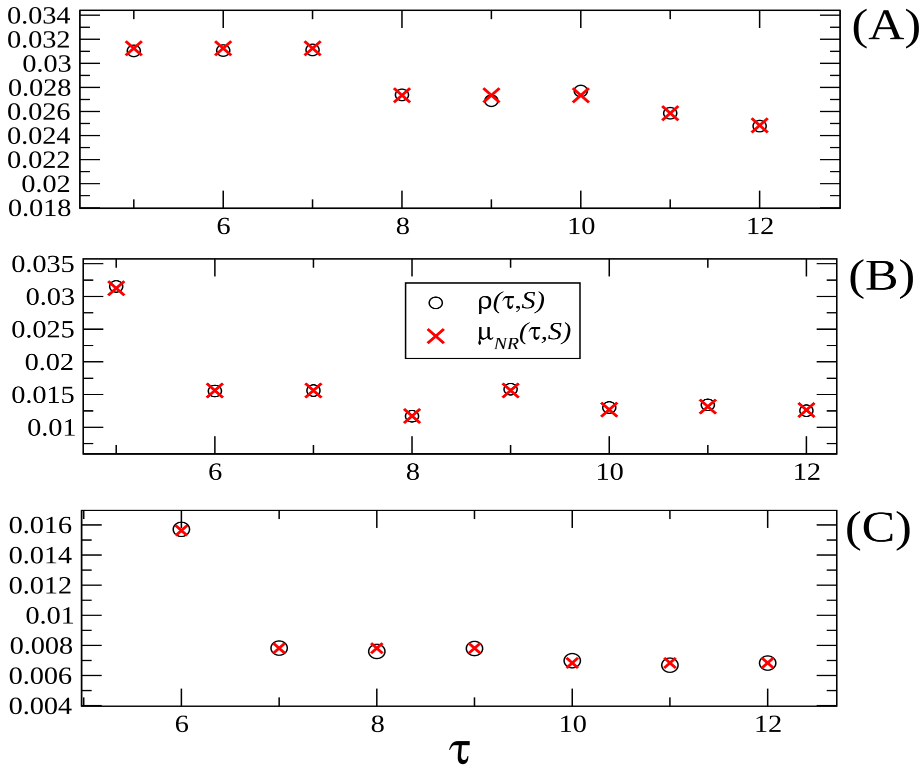


FIGURE 2. Density function of the FPRs, $\rho(\tau, S)$, as empty circles and the measure of the non-recurrent periodic orbits, $\mu_{NR}(\tau, S)$, as crosses, considering the following intervals: (A), $S = [0.3 - 0.05, 0.3 + 0.05]$; (B), $S = [0.3 - 0.01, 0.3 + 0.01]$; (C), $S = [0.3 - 0.005, 0.3 + 0.005]$.

6.2. **Calculating ρ when not all UPOs are known.** Figure 2 shows the function $\rho(\tau, S)$ calculated by Eq. (27) and the values of $\mu_{NR}(\tau, S)$ calculated by Eq. (8), for some subsets S. See that the DFP can be almost exactly obtained if all the non-recurrent UPOs inside S with period τ can be calculated: In Sec. 3 we concluded that $\rho(\tau, S) \approx \mu_{NR}(\tau, S)$.

Figure 3 shows the approximations for $\rho(\tau, S)$ using Eqs. (13) and (19). In (B), comparing with (A), we consider longer first return times. We only use Eqs. (13) and (19) for $\tau > 2\tau_{min}$.

6.3. **Error of our estimation when not UPOs are known.** To numerically calculate the error [Eq. (16)] of our estimation in Eq. (13), we only consider UPOs with a period smaller than 20. The reason is because in order to calculate the quotient q_2 in Eq. (15), all the UPOs

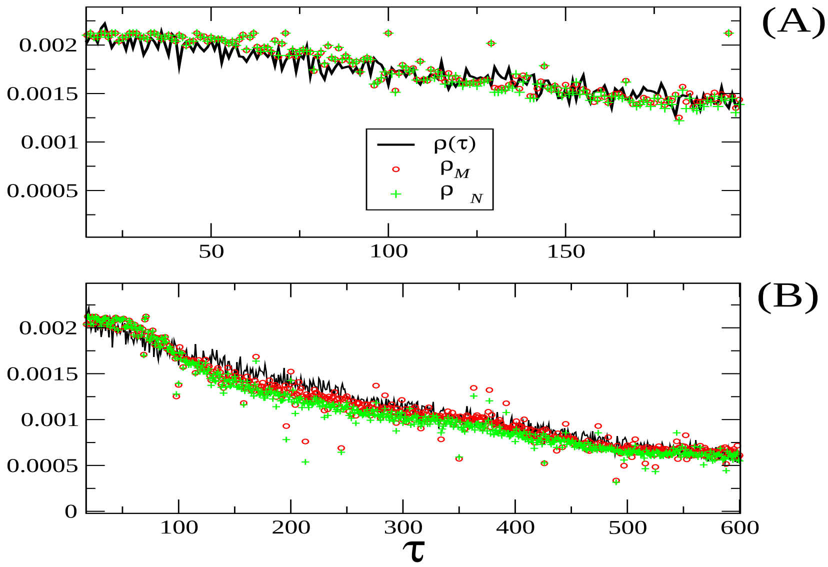


FIGURE 3. Red empty circles represent $\rho(\tau, S)$ estimated by Eq. (12), green crosses estimated by Eq. (18) and the black line calculated by Eq. (27). Picture (B) is just a similar reproduction of (A) considering longer first return times. We consider 200 UPOs inside $S = [0.1 - 0.001, 0.1 + 0.001]$, for each τ.

must be known. Considering larger periods than 20 would be computationally demanding, even thought the proposed method to calculate UPOs is capable of finding them all.

It is also required that $\tau > 2\tau_{min}$, once that to calculate the quotient q_1 in Eq. (14) there has to exist at least one recurrent UPO within the set of t UPOs considered, i.e. $r \geq 1$. Therefore, we need to choose the size of the interval such that 20-2$\tau_{min} - 1$ is sufficiently large, meaning an interval for which τ_{min} is sufficiently smaller. We have chosen ϵ=0.02.

Since the error of our estimation is proportional to a quotient between two quantities that depend on the number r of recurrent UPOs, it is advisable that one consider intervals for which a reasonable number of recurrent UPOs are found, even when their period is short (smaller or equal than 20). Such interval is positioned in places were the natural measure is large. In the case of the logistic map, these intervals are positioned either close to x=0 or $x = 1$. Therefore, we consider an interval positioned at $x = 0.04$. From the previous considerations, we consider that the interval has a size of $\epsilon = 0.02$.

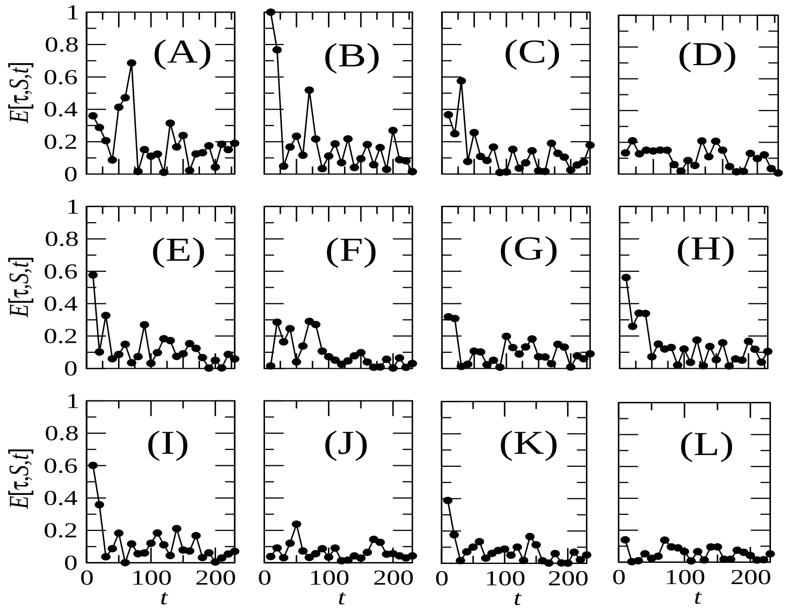
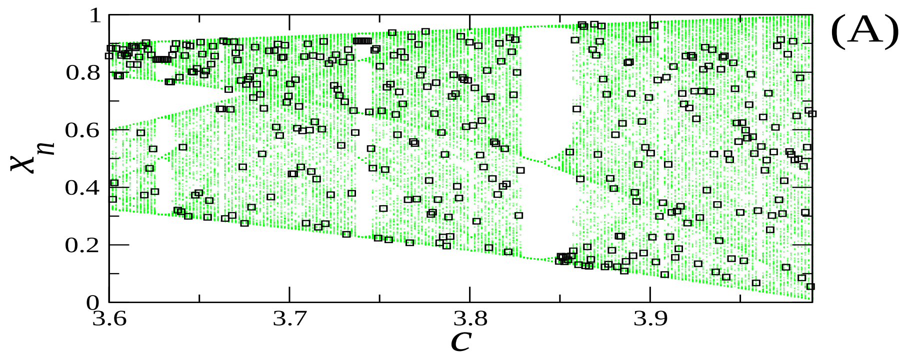


FIGURE 4. We show the quantity $E[\tau, S, t]$ with respect to a number t of UPOs randomly chosen, for $\tau = 9$ (A), $\tau = 10$ (B), $\tau = 11$ (C), $\tau = 12$ (D), $\tau = 13$ (E), $\tau = 14$ (F), $\tau = 15$ (G), $\tau = 16$ (H), $\tau = 17$ (I), $\tau = 18$ (J), $\tau = 19$ (K), and $\tau = 20$ (L). The quantity E gives the amount of deviation, in a scale from 0 to 1, of ρ_M [Eq. (13)] as when compared to the "exact" value of ρ [Eq. (10)]. We consider an interval positioned in $x = 0.04$ with size $\epsilon = 0.02$.

In Fig. 4(A-I), we show the quantity $E[\tau, S, t]$ with respect to the number t of UPOs randomly chosen, for $\tau = 9$ (A), $\tau = 10$ (B), $\tau = 11$ (C), $\tau = 12$ (D), $\tau = 13$ (E), $\tau = 14$ (F), $\tau = 15$ (G), $\tau = 16$ (H), $\tau = 17$ (I), $\tau = 18$ (J), $\tau = 19$ (K), and $\tau = 20$ (L).

The most important information from these figures is that as UPOs of longer periods are considered [going from Fig. (A) to (L)], the error E of our estimation decreases in an average sense considering all the values of t. Another relevant point is that the larger the number t of UPOs considered, the smaller the error. Notice that the total number of UPOs of period τ is given by 2^{τ}. Therefore, looking at Fig. 4(L), one can see that even considering only of about 0.0009% of all the UPOs (10 UPOs, out of a total of 2^{20}=1048576), the error of our estimation is smaller than 14% when compared to the "exact" value of ρ.



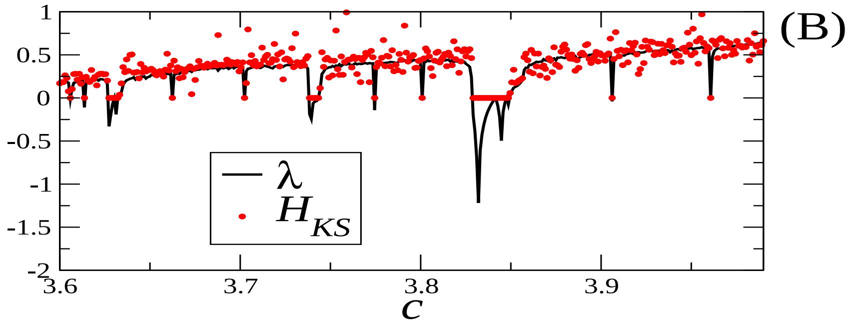


FIGURE 5. (A) A bifurcation diagram as points and the randomly chosen intervals as empty squares. (B) Lyapunov exponent as line and filled circles representing the H_{KS} entropy using Eq. (22), for the logistic family. We consider 400 values of c and for each c the size of the set S is $\epsilon = 0.002$.

6.4. **Estimating the KS entropy.** In order to know how good our estimation for H_{KS} is we use Pesin's equality which states that H_{KS} equals the sum of the positive Lyapunov exponents, here denoted by λ. For the logistic map there is at most one positive Lyapunov exponent.

Figure 5 shows the approximation for the quantity H_{KS} using Eq. (22). See that Eq. (22) only needs one subset S on the chaotic attractor to produce reasonable results. In this numerical simulation we vary the parameter c of the logistic family and for each c we use just one subset $S(c)$ randomly chosen [shown in Fig. 5 (A)] but satisfying $\tau_{min} \in [10, 14]$ so that τ considered in Eq. (22) is sufficiently large.

Finally, Fig. 6 shows the global estimation for H_{KS}, using the Eqs. (23) and (24), considering 40 intervals S_i for each value of c. Recall that if $\lambda < 0$, then $H_{KS} = 0$.

6.5. **Numerical work to find UPOs.** The analytical calculation of periodic orbits of a map is a difficult task. Even for the *logistic map* it is very difficult to calculate periodic orbits with a period as low as as

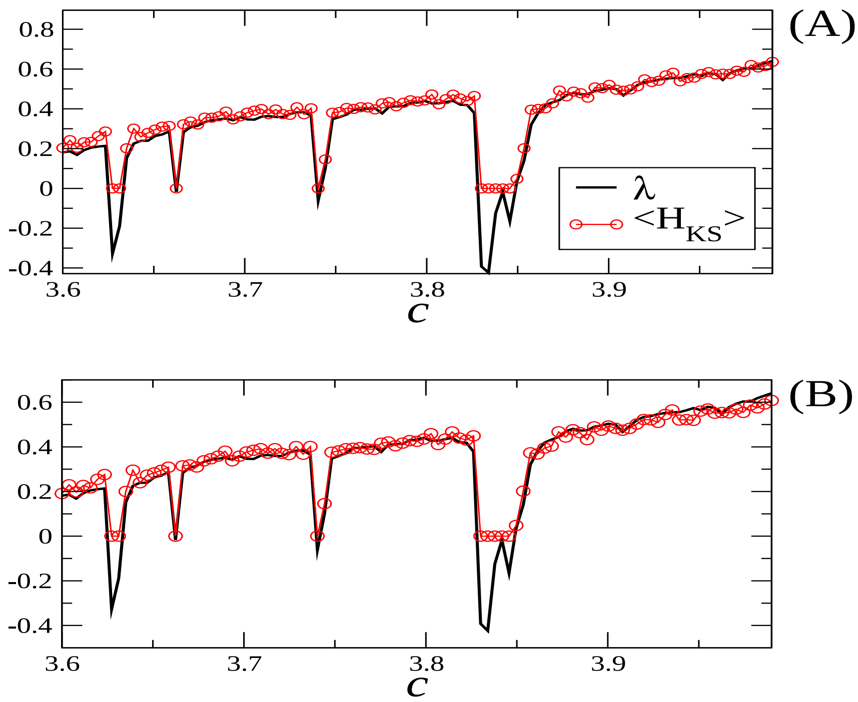


FIGURE 6. The Lyapunov exponent λ as line and the aproximation of H_{KS} entropy using Eqs. (23) and (24) as empty circles. (A), Eq. (23); (B), Eq. (24). In this simulation we consider 100 values of c and for each c we consider 40 subsets S_i each one with lenght $\epsilon = 0.002$. A subset S_i is picked only if $\tau_{min} \in [10, 14]$.

four or five. In our numerical work we need to find unstable periodic orbits and, in some cases, we need to find all different UPOs inside a subset of the phase space, for a sufficiently large period. For that, we use the method developed by Biham and Wenzel[19]. They suggest a way to obtain UPOs of a dynamical system with dimension D using a Hamiltonian, associated to the map, with dimension ND, where N is the number of UPOs with period p. The extremal configurations of this Hamiltonian are the UPOs of the map. The force $\partial H/\partial t$ directs trajectories of the Hamiltonian to the position of a UPO.

The Hamiltonian associated with the map gives a physical interpretation of the problem but in some cases it is impossible to know it. We propose a method with a similar interpretation that is simpler in the sense that we do not need to know the Hamiltonian associated with the map, just an array of N coupled systems where the linear coupling between nodes acts as the force directing the network to possible periodic solutions of the dynamical system concerned.

For this method we just need the force associated with the ith node, described by x^i, and satisfying the Euler-Lagrange (E-L) equations:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial x^i},$$

where L is the Lagrangian associated with the map. We are interested only in static extremum configurations of the Hamiltonian and therefore the kinetic term will be neglected[19]. This implies

$$\frac{\partial L}{\partial x^i} = 0$$

We illustrate the numerical calculation of UPOs with arbitrary length applying it to the logistic family. Because the static (E-L) equations reproduce the map, we have

$$\frac{\partial L}{\partial x_n^i} = x_n^{i+1} - cx_n^i(1 - x_n^i).$$

The force of the i node will be given by

$$F_i = -\frac{\partial L}{\partial x_n^i} = -x_n^{i+1} + cx_n^i(1 - x_n^i).$$

When the chain is in stable or unstable equilibrium (an extremum static configuration of the Hamiltonian), $F_i = 0$ for all i. To find a specific extremum configuration of order p of the Hamiltonian we introduce an artificial dynamical system defined by

$$\frac{\partial x_n^i}{\partial t} = s_i F_i,\ \ i = 1, ..., p, \tag{26}$$

where $s_i = \pm 1$ represents the direction of the force with respect to the ith node. This equation is solved subject to the periodic boundary condition $x^{p+1} = x^1$ and when the forces in all nodes decrease to zero the resulting structure x^i is simultaneously an extremum static configuration and an exact p-periodic orbit of the logistic map. For $c = 4$, if we take $s_i = -1\ \forall i$ then we obtain the trivial periodic point $x_i = 0\ \forall i$. The different ways to write s_i will give different UPOs. We may look at s_i as the representation of the orbit in a symbolic dynamics with $\Sigma = \{-1, 1\}$, taking the trivial partition on the logistic map, i.e., $s_i = -1$ if $x_i \in [0, 1/2]$ and $s_i = 1$ if $x_i \in [1/2, 1]$.

Equation (26) is in fact an equation for a network of coupled maps. The UPOs with period p embedded in the chaotic attractor can be calculated by finding the stable periodic orbits of the following array of maps constructed with $i = 1, ..., p$ nodes x_n^i, where every node is connected to its nearest neighbor as in

$$x_{n+1}^i = x_n^i - cs_i[x_n^{i+1} - F(x_n^i)],$$

with the periodic boundary condition $x_n^p = x_n^1$, where the term $cs_i[x_n^{i+1} - F(x_n^i)]$ represents the Lagrangian force.

7. Conclusions

In this work we propose two ways to compute the density function of the first Poincaré returns (DFP), using unstable periodic orbits (UPOs), where the first Poincaré return (FPR) is the sequence of time intervals that a trajectory takes to make two consecutive returns to a specific region. In the first way, the DFP can be exactly calculated considering all UPOs of a given low period. In the second way, the DFP is estimated considering only a few UPOs. We have numerically shown that the error of our estimation becomes smaller, the longer the period of the UPOs and the larger the number of UPOs considered.

The relation between DFP and UPOs allows us to compute easily an important invariant quantity, the Kolmogorov-Sinai entropy.

For non-uniformly hyperbolic systems there exists some particular subsets for which the UPOs that visit it are not sufficient to calculate their measure of the chaotic attractor inside it[20, 21]. For such cases our approach works in an approximate sense, but it still provides good estimates as we have shown in our simulations performed in the logistic map, a non-uniformly hyperbolic system. In addition, the approaches shown in here were applied in ref. [22] to estimate the value of the Lyapunov exponent in the experimental Chua's circuit and in the Hénon map, both systems being non-hyperbolic.

Our approach offers an easy way to estimate the KS entropy in experiments, since one does not need to calculate UPOs, but rather only to measure the DFP of trajectories that make shortest returns, i.e. the quantity $\rho(\tau_{min}, S)$. These trajectories are the most frequent trajectories, and as a consequence even if only a few returns are measured, one can obtain a good estimation of $\rho(\tau_{min}, S)$. More details of how to estimate the KS entropy from experimental data can be seen in Ref. [22].

8. Appendix

8.1. **Measure and density in terms of FPRs.** We calculate $\rho(\tau, S)$ also in terms of a finite set of FPRs by

$$\rho(\tau, S) = \frac{K(\tau, S)}{L(S)} \tag{27}$$

where $K(\tau, S)$ is the number of FPRs with a particular length τ that occurred in region S and $L(S)$ is the total number of FPRs measured in S with any possible length.

We calculate $\mu(S)$ also in terms of FPRs by

$$\mu(S) = \frac{L(S)}{n_L} \tag{28}$$

where n_L is the number of iterations considered to measure the $L(S)$ FPRs and so $n_L = \sum_{n=1}^{L} \tau_n$ (see definition 3.1).

We define the average of the returns by

$$< \tau >= \frac{n_L}{L(S)}. \tag{29}$$

Comparing Eqs. (28) and (29), we have that

$$\mu(S) = \frac{1}{< \tau >} \tag{30}$$

also known as Kac's lemma.

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Chapter 3

Analytical Results - Markov Transformations

This chapter is dedicated to the proof of the conjecture, presented in section 1.2, for a particular case of a well known class of dynamical systems called Markov transformations. Before showing the results, we construct special sets on the phase space of some particular expanding piecewise transformations and we show that for these special sets the conjecture is true. The way that these special subsets are constructed will be useful to understand the choice of the class of Markov transformations to prove the conjecture.

3.1 Special sets in expanding piecewise linear transformations

Class of dynamical systems

Denote by λ the Lebesgue measure and consider the class of expanding and piecewise (finite) linear maps

$$F : I \to I,$$

where I is an interval, $0 < \lambda(I) < \infty$ and our chaotic attractor is dense in I. We also suppose that the natural measure coincides with the Lebesgue measure. This class of dynamical systems will be called $PLC(I, I)$. Example of such a map in $[0, 1]$ is $f(x) = 2x \ (mod\, 1)$, we discuss this example in section 3.2, after the proof of Proposition 15.

Observe that in $PLC(I, I)$ all periodic points are unstable. Henceforth when we say periodic points it is implicit that is unstable.

Special subsets

Let's define $C_\alpha(\bar{x})$ as the interval $[\bar{x}-\alpha, \bar{x}+\alpha]$ where $\bar{x}$ is a periodic point of minimal period $p > 1$, non-recurrent with respect to this interval and there is no other periodic point of period less or equal to p inside $C_\alpha(\bar{x})$. For maps in $PLC(I, I)$ there is always $\alpha_0 > 0$, for each $\bar{x}$, such that $\forall\ \alpha \in (0, \alpha_0]$ $C_\alpha(\bar{x})$ has the previous property, i.e. there is no other periodic point of period less or equal to p inside it. Henceforth C_α is always an interval constructed around some $\bar{x}$ with the previous property and we omit the $\bar{x}$ in $C_\alpha(\bar{x})$ when no ambiguity arises.

Let B be the set of points where F fails to be of class C^1.

Results

Let $\bar{x}$ be a periodic point for $F \in PLC(I, I)$ with minimal period $p > 1$. Let $O(\bar{x}) = \{F^i(\bar{x}) :\ 0 \leq i < p\}$ be the orbit of $\bar{x}$, and suppose

$$O(\bar{x}) \cap B = \emptyset.$$

Then there exist $\alpha_* > 0$, $\beta_1 > 0$ and $\beta_2 > 0$ such that C_{α_*} and $C'' = [\bar{x}-\beta_1, \bar{x}+\beta_2]$ have the following properties:

1. $\frac{d}{dx}F^p(x)$ is constant in C'',
2. $F^p(C'') = C_{\alpha_*}$,
3. for $1 < j < p$, $F^j(C'') \cap C_{\alpha_*} = \emptyset$.

Lemma 9 *Fixing $p > 1$ and from properties 1., 2. and 3. it follows that $\forall \alpha \in (0, \alpha_*]$ we have*

$$\rho(p, C_\alpha) = \mu_{NR}(p, C_\alpha).$$

Proof. Since $\mu(C_\alpha) = \lambda(C_\alpha)$, using properties 2. and 3. we have

$$\rho(p, C_\alpha) = \frac{\lambda(C'')}{\lambda(C_\alpha)}.$$

Now, using properties 1. a 2. we conclude that

$$\lambda(C_\alpha) = L_{\bar{x}}\lambda(C'')$$

and we obtain

$$\rho(p, C_\alpha) = \frac{\lambda(C'')}{\lambda(C_\alpha)} = \frac{1}{L_{\bar{x}}} = \mu_{NR}(p, C_\alpha).$$

■

Proof of properties 1.,2. and 3.: We can choose $\alpha > 0$ such that

$$d(O(\bar{x}) - \{\bar{x}\}, C_\alpha) > \alpha.$$

This is always possible because $O(\bar{x})$ is a discrete set of points in I and then when $\alpha \to 0$ we have $d(O(\bar{x}) - \{\bar{x}\}, C_\alpha) \to \gamma > 0$. Also we have the same situation for period less than p and then it is always possible to find α such that property 3. holds. Now, observe that we can always choose $\alpha > 0$ such that the map $F|_{C_\alpha}$ is linear since $O(\bar{x}) \cap B = \emptyset$. Let α_* be such that all these hold. Also, for any $\beta_1, \beta_2 < \alpha_*$ we have $C'' \subset C_{\alpha_*}$ and property 1. holds. To see that C'' always exists, define $\phi_\pm(t) = F^p(\bar{x} \pm t) \mp \bar{x}$ and we conclude that there is $\beta_1 \leq \alpha_*$ such that $\phi_+(\beta_1) = \alpha_*$ (F is expanding). The same thing for $\phi_-(t)$ and we obtain $\beta_2 < \alpha_*$ and

$$C'' = [\bar{x} - \beta_1, \bar{x} + \beta_2].$$

Some remarks about lemma 9

The first relevant observation is about the special sets presented here. By construction, each one needs to have a non-recurrent periodic point inside it and the existence of such set depends on the existence of a non-recurrent periodic point. In general we want to choose any interval to observe the first returns and not be limited by conditions 1., 2. and 3. that are very restrictive. On the other hand, we can feel in lemma 9 that all the special sets, for some fixed period p, form a kind of partition on the phase space. The class of Markov transformations, as we will see it later, always have a well defined partition of the phase space for which we know if there exists or not periodic points of a particular period. Also the natural measure may coincide, under some assumptions, with the Lebesgue measure for Markov transformations.

The rest of the chapter is dedicated to the proof of the conjecture 1.2 for a particular case of Markov transformations.

3.2 Markov transformations

Elements of Markov partition and symbolic dynamics

Definition 10 *Denote with λ the Lebesgue measure. We say that $f : [0,1] \to [0,1]$ is a Markov transformation if there exists a finite or countable family $\{I_0, I_1, ...\}$ of open and disjoint intervals in $[0,1]$ such that:*

1. $\lambda([0,1] - \bigcup_j I_j) = 0$,

2. $\forall j$, $f(I_j)$ *is a union, except for a* λ*-measure* 0 *set, of some intervals of the family* $\{I_0, I_1, ...\}$ *and* $\lambda(f(I_j)) > 0$,

3. $\exists \alpha > 0$ *such that the derivative of* f *exist a.e.*(λ) *and satisfies* $|f'(x)| > \alpha \ \forall x \in \bigcup_j I_j$,

4. $\exists \beta > 1$ *and* $n_0 > 0$ *such that* $|(f^{n_0})'(x)| \geq \beta$ *for almost all* x *such that* $f^m(x) \in \bigcup_j I_j$ *for all* $0 \leq m \leq n_0 - 1$,

5. $\forall j, i \ \exists m > 0$ *such that* $\lambda(f^{-m}(I_j) \cap I_i) \neq 0$,

6. $\exists C > 0$ *and* $0 < \gamma < 1$ *such that* $|\frac{f'(x)}{f'(y)} - 1| \leq C|x - y|^\gamma$ *for all* x, y *on the same interval in the family.*

Condition 4. means that some iterate of f is uniformly expanding. We discuss this again in example 3.2.1 below. Condition 2. is usually called Markov condition and means that elements of the partition $\{I_0, I_1, ...\}$ are always mapped into unions of elements of the same partition.

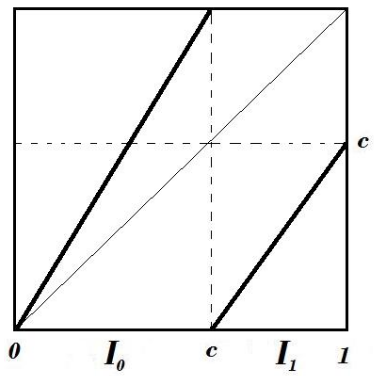
Definition 11 *We say that* $f : [0,1] \to [0,1]$ *is an N-linear Markov transformation if* f *is a Markov transformation,* f *has constant derivative in each* I_i *and the family of intervals* $\{I_0, I_1, ...\}$ *is finite with* N *elements.*

From now on when we refer to "Markov transformations" we always mean "N-linear Markov transformations".

Consider P_j as the closure of I_j for all $j \in \{0, ..., N-1\}$. We will abuse terminology and refer to $P = \{P_0, ..., P_{N-1}\}$ as a partition of $[0,1]$.

Example 3.2.1 *An example of an N-linear Markov transformation for* $N = 2$ *is:*

$$f(x) = \begin{cases} \frac{x}{c} & if \ x \in [0,c] \\ \frac{cx}{1-c} - \frac{c^2}{1-c} & if \ x \in (c,1] \end{cases}$$



Note that even though $||Df||_{I_1}$ may be less that 1, the second iterate of f is always expanding. Property 4. from the definition of Markov transformation holds with $n_0 = 2$ and $\beta = \min\{\frac{1}{1-c}, \frac{1}{c^2}\}$.

Because of the fact that each P_i is mapped, by f, into a union of some P_j's (as in example 3.2.1), we can study f using the subshift of finite type defined by the transition matrix

$$A = \begin{pmatrix} A_{0,0} & \cdots & A_{0,N-1} \\ A_{1,0} & \ddots & \vdots \\ \vdots & & \\ A_{N-1,0} & \cdots & A_{N-1,N-1} \end{pmatrix},$$

where $A_{i,j} \in \{0,1\}$ and $A_{i,j} = 1$ if and only if $f(P_i) \supset P_j$. The matrix A codes the allowed symbol sequences that represent the way f maps one interval into the others. In example 3.2.1 A is given by

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Define $X_A = \{(s_j) \in \Sigma_{\{0,\ldots,N-1\}} \ : \ A_{s_i,s_{i+1}} = 1 \ \forall i \geq 0\}$, where $\Sigma_{\{0,\ldots,N-1\}}$ is the space of all infinite sequences with the symbols $\{0, ..., N-1\}$. In example 3.2.1 X_A is the space of all infinite sequences of zeros and ones that do not contain the word '11'.

In X_A the topology is induced by the metric

$$m(s,t) = \sum_{i=0}^{\infty} \frac{1 - \delta_{s_i t_i}}{2^{i+1}}, \quad \forall s, t \in X_A,$$

where $s = (s_i)$ and $t = (t_i)$. In example 3.2.1, if $\overline{0}$ is the sequence $s_i = 0 \ \forall i$, then if for some sequence t we know that $m(\overline{0}, t) < 1/4$ then we conclude that the first two symbols of t needs to be zero. If, for some $s, t \in X_A$ we have $m(s,t) < 1/2^k$ then the first k symbols of the sequences s and t must coincide.

Defining $B = \{x \in [0,1] : \ x \in (\bigcup_{\{j,k:\ j \neq k\}} P_j \cap P_k)\}$, let $z \in [0,1]$ be a point such that $f^k(z) \notin B$ for any k. Then, for every k, $f^k(z)$ lies in some P_i. We associate with the point z an infinite sequence $(z_0, z_1, ...)$ by choosing z_i so that $f^i(z) \in P_{z_i}$. The sequence $(z_i) = (z_0, z_1, ...)$ is called the P-name of z and $\psi(z)$ is the coding map that sends a point in $[0,1]$ to its P-name. So, $\psi : [0,1] \to X_A$ with $\psi(z) = (z_i)$. In example 3.2.1, $\psi(0) = (0000...)$ because $\forall k\ f^k(0) \in I_0$ (0 is a fixed point for f).

The coding map ψ is not well defined at points z such that $f^k(z)$ lies at the boundary of the intervals in the partition P. These points do not have unique sequence of symbols. In example 3.2.1 these points are c and all its preimages by f. Thus the map ψ is well-defined in $[0,1] - \tilde{B}$, where $\tilde{B} = \{z \in [0,1] : \exists k : f^k(z) \in B\}$. It is a continuous map in this set and $\psi \circ f = \sigma \circ \psi$, where σ is the one-sided shift operator on X_A:

$$\sigma : X_A \to X_A$$

with $\sigma((s_0, s_1, s_2, ...)) = (s_1, s_2, s_3, ...)$.

The points with more than one image under ψ are the points inside $\tilde{B}$ but observe that $\lambda(\tilde{B}) = 0$ since $\tilde{B}$ is a countable set in $[0,1]$. The map ψ is the *semi-conjugacy* between $(f, [0,1] - \tilde{B})$ and (σ, X_A)[8, chapter 1.1]. We will show in lemma 12 that ψ is, in fact, a full topological conjugacy.

Elements of the Markov partition and measure

Define $Me(1) = P$. Now define $Me(p)$, $p \in \mathbf{N}$, $p > 1$ recursively as follows: if $Me(p-1) = \{D_0, ..., D_l\}$ then

$$Me(p) = \{f^{-p+1}(P_0) \cap D_0,\ f^{-p+1}(P_0) \cap D_1, ...,\ f^{-p+1}(P_0) \cap D_l,$$
$$f^{-p+1}(P_1) \cap D_0, ...,\ f^{-p+1}(P_{N-1}) \cap D_l\}.$$

For each $p > 0$, all these closed intervals define a finite partition of $[0,1]$ and all the elements of $Me(p)$ will be called the **elements of the Markov partition** of level p. If $D \in Me(p)$ then there is an index set $I \subset \{1, 2, \dots, \#Me(p)\}$ such that

$$f(D) = \bigcup_{j \in I} D_j$$

where $D_j \in Me(p)$ for $0 < j \leq \#Me(p)$. This property is called Markov condition. In example 3.2.1, $D = [0, c^2]$ is an element of $Me(2) = \{[0, c^2], [c^2, c], [c, 1]\}$ and its image by f is $[0, c]$ that is $[0, c^2] \cup [c^2, c]$.

Lemma 12 *Consider an N-linear Markov transformation f. If $p \geq kn_0$ and $D \in Me(p)$ then*

$$\lambda(D) \leq \frac{1}{\beta^k}$$

for n_0 and β given in property (4) of definition 10.

Proof. First of all observe that for any $m < p$, if $D \in Me(p)$ then $f^m(D) \in Me(p-m)$. Moreover, $(f^m)'(x)$ has the same value for all $x \in D$.

In particular, for $m = n_0$

$$f^{n_0}(x) = \alpha_D x + c \ \ \forall x \in D$$

with $|\alpha_D| \geq \beta$, by property (4), and

$$\lambda(D)|\alpha_D| = \lambda(f^{n_0}(D)).$$

Therefore, for all $D \in Me(p)$, with $p \geq n_0$ we have

$$\lambda(D) \leq \frac{\lambda(f^{n_0}(D))}{\beta}. \tag{3.1}$$

If $p > kn_0$ then applying (3.1) recursively we obtain

$$\lambda(D) \leq \frac{\lambda(f^{kn_0}(D))}{\beta^k}$$

and the result follows since $f^{kn_0}(D) \subset [0,1]$ and thus $\lambda(f^{kn_0}(D)) \leq 1$. ■

Lemma 12 can be proved in a more general than linear situation and if the number of elements of the partition is not finite then lemma 12 is not true anymore. A counter example can be found in [5].

Each $D \in Me(p)$ has a well defined and finite code $d_0, ..., d_{p-1}$, where $d_i \in \{0, ..., N-1\}$ $\forall i = 0, ..., p-1$, that we denote by $\psi_p(D)$. For some $D \in Me(p)$ with $\psi_p(D) = (d_0, ..., d_{p-1})$ we have $x \in D$ if and only if $\psi(x) = (x_i)$ with $x_0 = d_0,\ x_1 = d_1, \ldots$ and $x_{p-1} = d_{p-1}$. The number of elements in $Me(p)$ is the number of different words with length p that are contained in sequences of X_A.

It follows from lemma 12 that given a sequence $(z_i) \in X_A$ there is a unique point $z \in [0,1] - \tilde{B}$ having this sequence as its P-name since $(z_0, z_1, ..., z_{p-1})$ are, by definition, the first p symbols of the code of some point $z \in D \in Me(p)$ with $\psi_p(D) = (z_0, z_1, ..., z_{p-1})$ and $\lambda(D) \to 0$ when $p \to \infty$. We conclude that the map ψ is invertible. Let $\pi : X_A \to [0,1] - \tilde{B}$ be its inverse.

If we look at the probability of moving from P_i to P_j in one step, $i \neq j$, then we can write the *stochastic matrix*, associated to f, as

$$Z = \begin{pmatrix} Z_{0,0} & \cdots & Z_{0,N-1} \\ Z_{1,0} & \ddots & \vdots \\ \vdots & & \\ Z_{N-1,0} & \cdots & Z_{N-1,N-1} \end{pmatrix},$$

where

$$\sum_{i=0}^{N-1} Z_{k,i} = 1, \ \forall k = 0, ..., N-1.$$

This matrix describes all the probabilities of moving from one state to another one in one time step ($Z_{i,j}$ is the probability of moving from P_i to P_j by f) and so $0 \leq Z_{i,j} \leq 1 \ \forall i, j$. To calculate the elements $Z_{i,j}$ we just need to look for the portion of Lebesgue measure that is sent, by f, from some P_i to some P_j. For instance, in example 3.2.1, $Z_{0,0} = c$ because the portion, in terms of Lebesgue measure, of points in P_0 that stays in P_0 by f is c and so, the probability of choosing one point in P_0 that will stay in P_0 by f is c. In general, for a given Markov transformation f,

$$Z_{i,j} = \frac{\lambda(f^{-1}(P_j) \cap P_i)}{\lambda(P_i)}. \tag{3.2}$$

The *stationary probability vector* is a vector $v = (v_0, ..., v_{N-1})$ such that $vZ = v$ and $\sum_{k=0}^{N-1} v_k = 1$. The Perron Frobenius theorem[15, ch 0, sec. 9] ensures that every stochastic matrix has such an eigenvector, and that the largest absolute value of an eigenvalue is always 1. In general, there may be several such vectors. However, for a matrix with strictly positive entries, this vector is unique.

The stochastic matrix and the stationary probability vector will be useful to define an invariant measure in X_A.

To define an invariant measure in X_A let

$$C(j; a_0, ..., a_k) = \{(s_i) \in X_A \ : \ s_j = a_0, ..., s_{j+k} = a_k\}$$

be the cylinders in X_A. Observe that for each element $D \in Me(p)$ with $\psi_p(D) = (d_0, ..., d_{p-1})$ we have $\psi(D) = C(0; d_0, \ldots, d_{p-1})$. From now on we omit the p in $\psi_p(D)$. All these open subsets form a basis for the topology induced by the metric m in X_A and we define

$$\nu(C(j; a_0, ..., a_k)) = v_{a_0} Z_{a_0 a_1} \cdots Z_{a_{k-1} a_k}, \tag{3.3}$$

where v_{a_0} represents the probability of being in I_{a_0}.

It is easy to prove that ν is a well defined measure on X_A and does not depend on j, so it is σ-invariant.

We will finish this section showing that the measure induced by ψ of ν,

$$\nu_*(.) = \nu(\psi(.)),$$

is, in fact, the natural measure defined in $[0,1]$ for Markov transformations with finite partition. For this we need one more definition and a result.

Definition 13 *If μ and λ are two measures on the same measurable space then μ is said to be absolutely continuous with respect to λ, and we write $\mu << \lambda$, if $\mu(S) = 0$ for every set S for which $\lambda(S) = 0$*

Theorem 14 (Adler and Bowen[13]) *If f is an N-linear Markov transformation then there exists a unique probability measure on the borelians of $[0,1]$ that is f-invariant and absolutely continuous with respect to the Lebesgue measure.*

Proposition 15 *If f is a Markov transformation then $\mu = \nu_*$.*

Proof. From theorem 14, if we show that μ and ν_* are absolutely continuous with respect to the Lebesgue measure then they are the same measure. To verify that $\mu \ll \lambda$ we use directly the Lasota-Yorke theorem[10] that says, in particular, that there exists a constant c such that $\mu([a,b]) \leq c\lambda([a,b])$ for any $a, b \in [0,1]$. So, $\mu \ll \lambda$ and then, to finish the proof, we only need to show that

$$\nu_* \ll \lambda. \tag{3.4}$$

We have the following diagram:

$$\begin{array}{ccc} [0,1] - \tilde{B} & \xrightarrow{f} & [0,1] - \tilde{B} \\ \psi \downarrow & & \downarrow \psi \\ X_A & \xrightarrow{\sigma} & X_A \end{array}$$

and $\psi \circ f = \sigma \circ \psi$.

Let's consider some $S \subset [0,1]$ such that $\lambda(S) = 0$. We want to show that

$$\nu_*(S) = 0. \tag{3.5}$$

If S is a finite set of points then it is clear that $\nu_*(S) = 0$. If $\psi(S)$ contains a cylinder of the form $C = C(0, d_0, ..., d_{p-1})$ then $\nu(\psi(S)) > 0$ because all cylinders have positive measure in X_A. On the other hand, $\pi(C) \subset S$ represents an interval in $[0,1]$ because $\pi(C) \in Me(p)$. This contradicts the assumption $\lambda(S) = 0$. From now we assume $\psi(S)$ does not contain any cylinder.

Lemma 16 *The family of intervals* $\{\bigcup_p Me(p)\}$ *generates the Borelians in* $[0,1]$.

Proof. Consider $x \in [0,1] - \tilde{B}$. Then there exist $A_1 \in Me(n_0)$, $A_2 \in Me(2n_0), ...,$ $A_k \in Me(kn_0), ...$ such that $x \in A_i$ $\forall i$.

By lemma 12, we know that

$$\lambda(A_k) \rightarrow_{k\rightarrow\infty} 0.$$

Then, $x \in \cap_i^\infty A_i$ and $\lambda(A_i) \rightarrow_{i\rightarrow\infty} 0$.

We conclude that for any $x \in [0,1] - \tilde{B}$ we can construct a sequence of extremes of intervals in $\{\bigcup_p Me(p)\}$ such that the limit converges to x. This means that we can approximate any point in $[0,1]$ by a sequence of extremes of intervals in $\{\bigcup_p Me(p)\}$. So, we can approximate any rational number in $[0,1]$ and we conclude the proof because all open sets in $[0,1]$ with rational extremes generate the Borelians in $[0,1]$. ■

We are considering an N-linear Markov transformation with $Me(1) = \{P_1, ..., P_N\}$ then $\forall x \in P_j$, $f(x) = \alpha_j x + \gamma_j$. If $S = f^{-1}(P_j) \cap P_i$ then, by condition (2) of definition 10, $|\alpha_i|\lambda(S) = \lambda(P_j)$. So,

$$\lambda(S) = \frac{\lambda(P_j)}{|\alpha_i|} \tag{3.6}$$

For $D \in Me(p)$, $\nu_*(D) = v_{d_0} Z_{d_0 d_1} \cdots Z_{d_{p-2} d_{p-1}}$ where $\psi(D) = d_0, ..., d_{p-1}$, by Eq. (3.2) we obtain

$$\nu_*(D) = \frac{v_{d_0}}{\lambda(P_{d_0})} \frac{\lambda(P_{d_{p-1}})}{\prod_{i=0}^{p-2} |\alpha_{d_i}|}.$$

Also observe that $D \in Me(p)$, $\nu(\psi(D)) = v_{d_0} Z_{d_0 d_1} ... Z_{d_{p-2} d_{p-1}} \leq \gamma$ where, directly by the definition of Z, $\gamma < 1$.

Lemma 17 *With* $\psi(D) = \{d_0, ..., d_{p-1}\}$ *we have*

$$\lambda(D) = \frac{\lambda(P_{d_{p-1}})}{\prod_{i=0}^{p-2} |\alpha_{d_i}|} = \frac{\lambda(P_{d_0})}{v_{d_0}} \nu_*(D)$$

Proof. Because of the fact that $f^{p-1}(D) = P_{d_{p-1}}$ and using condition (2) of definition 10 we obtain

$$f^{p-1}(x) = \alpha_{d_{p-2}} \alpha_{d_{p-3}} \cdots \alpha_{d_0} x + constant$$

for all $x \in D$. Then using (3.6) we conclude that $\lambda(D) = \frac{\lambda(P_{d_{p-1}})}{|\alpha_{d_{p-2}} \alpha_{d_{p-3}} \cdots \alpha_{d_0}|}$ ■

Lemma 17 allows us to conclude that $\forall p > 1\ \exists m > 0$ such that $\forall D \in Me(p)$ we have
$$m\nu_*(D) \leq \lambda(D).$$
In fact, $m = \min_j \left\{\frac{\lambda(P_j)}{v_j}\right\}$.

Finally, to show that we have (3.5) if $\lambda(S) = 0$, we observe that $\forall \epsilon > 0$ we can chose, by lemma 12, a large p such that
$$\sum_j \lambda(B_j) < \epsilon$$
and $S \subset \cup_j B_j$ and $\forall j\ B_j \in Me(p)$. Therefore $\nu_*(S) < \frac{\epsilon}{m}$ and we conclude the proof of proposition 15. ■

Proposition 15 allows us to use symbolic dynamics to compute the natural measure. From now on we use the same symbol μ for both ν and ν_* measures.

Example 3.2.2 *Let* $f : [0,1] \to [0,1]$ *be given by*
$$f(x) = \begin{cases} \frac{x}{c} & if\ x \in [0, c[\\ \frac{x-c}{1-c} & if\ x \in [c, 1] \end{cases}$$
with $0 < c < 1$.

For $c = 1/2$, $f(x) = 2x\ (mod\ 1)$. *The stochastic matrix associated to* f *is*
$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$
with the stationary probability vector $v = (1/2, 1/2)$.

For any $D \in Me(p)$ *with* $\psi(D) = (d_0, ..., d_{p-1})$ *we have:*
$$\nu_*(D) = \nu(D) = \frac{v_{d_0}}{\lambda(P_{d_0})}\lambda(D).$$
For $c = 1/2$ *we have* $\lambda(P_0) = \lambda(P_1) = 1/2$ *and hence* $\nu_*(D) = \lambda(D)$. *For other values of c, the constant m above may be taken to be the minimum of* $2c$ *and* $2(1-c)$.

3.3 Density function of first returns for elements of a Markov partition

From now on we restrict our attention to 2-linear Markov transformation. In this section we prove the conjecture for first Poincaré returns of f^p, where

f is a 2-linear Markov transformation, to a set $D \in Me(p)$ with $p > 1$. We show that

$$\rho(p, D) = \mu_{NR}(p, D).$$

Some of the tools developed in this section will be useful in extending the result to other situations.

First Poincaré returns and symbolic dynamics

For $D \in Me(p)$, $\mu(D) = \mu(C(0; d_0, ..., d_{p-1}))$. Define

$$S_p(D) = \{x \in D \ : \ f^p(x) \in D\}$$

and

$$S^p_{NR}(D) = \{x \in S_p(D) \ : \ f^i(x) \notin D, \ 0 < i < p\}.$$

The density function of the first Poincaré returns can be written as

$$\rho(p, D) = \frac{\mu(S^p_{NR}(D))}{\mu(D)}. \tag{3.7}$$

Definition 18 *We say that the code $d_0, ..., d_{p-1}$ identifies $D \in Me(p)$ if and only if $\psi_p(D) = (d_0, ..., d_{p-1})$.*

Lemma 19 *If $D \in Me(p)$ then there exists at most one point of period p in D.*

Proof. Suppose that $d_0, ..., d_{p-1}$ is the code that identifies D. This means that $x \in D$ if and only if $\psi(x) = (d_0, ..., d_{p-1}, x_p, x_{p+1}, ...)$.

If $d_{p-1}d_0$ is an allowed word in X_A (i.e. if $A_{d_{p-1},d_0} = 1$) then the p-periodic sequence $(\overline{d_0, ..., d_{p-1}}) = (d_0, ..., d_{p-1}, d_0, ..., d_{p-1}, d_0, ...)$ is in X_A and $\pi(\overline{d_0, ..., d_{p-1}}) \in D$. Moreover, this is the only possible code for a p-periodic point in D. ■

Observe that for each $D \in Me(p)$ we have, at most, one periodic point of period p inside it, represented by $(\overline{d_0, ..., d_{p-1}})$. Here "at most" refers to cases for which we can not have periodic points like in example 3.2.1:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and consider D represented by the code $(1, 0, 1)$. In this case $(\overline{101}) \notin X_A$ and then there is no periodic point of period 3 inside D.

Lemma 20 *Consider some $D \in Me(p)$, $p > 1$. If $S_p(D) \neq \emptyset$ then either $S^p_{NR}(D) = S_p(D)$ or $S^p_{NR}(D) = \emptyset$. Moreover, $S^p_{NR}(D) \neq \emptyset$ if and only if the periodic point with period p in D is non-recurrent.*

Proof. Suppose that $d_0, ..., d_{p-1}$ identifies $D \in Me(p)$. For any $x \in D$, $\psi(x) = (d_0, ..., d_{p-1}, x_p, x_{p+1}, ...)$ where x_i $\forall i > p-1$ can be any symbol of the finite set $\{0, 1, ..., N-1\}$. If $f^p(x) \in D$ it means that

$$(x_i) = (d_0, ..., d_{p-1}, d_0, ...d_{p-1}, x_{2p}, ...).$$

So, this is the form of points in $S_p(D)$.

The last observation implies that there exists a periodic point of period p inside D, given by $(\overline{d_0, ..., d_{p-1}})$.

Now, if $S^p_{NR}(D) \neq \emptyset$ then

$$(d_j, ..., d_{p-1}, d_0, ..., d_{j-1}) \neq (d_0, ..., d_{p-1}) \quad \forall j \text{ with } 0 < j < p.$$

This last property of the code of D implies that $\forall x \in S_p(D)$, x is also a point of $S^p_{NR}(D)$. By definition, $S^p_{NR}(D) \subset S_p(D)$ and we conclude that $S^p_{NR}(D) = S_p(D)$.

Finally we observe that if $S^p_{NR}(D) \neq \emptyset$ then there is at least one non-recurrent point in D. The first p symbols of this non-recurrent point will define all the code of the periodic point with period p in D and, consequently, the periodic point needs to be non-recurrent. ■

Lemma 21 *Consider some $D \in Me(p-k)$, $p > 1$ and $0 < k < p$.*

If $S^p_{NR}(D) \neq \emptyset$ and if $S^p_{NR}(D) \neq S_p(D)$ then there exists at least one periodic point with period p in D that is non-recurrent in D and also there exists at least one periodic point with period p in D that is recurrent in D.

If $S^p_{NR}(D) = S_p(D) \neq \emptyset$ then there exists a p-periodic point in D and, moreover, all p-periodic points in D are non-recurrent.

If $S^p_{NR}(D) = \emptyset$ then any p-periodic points in D are recurrent.

Proof. It is a direct consequence of the existence of at least one point in D that is non-recurrent (or recurrent) in D as in lemmas 19 and 20. The first p symbols of the code of that non-recurrent (or recurrent) point will define all the code of the non-recurrent (or recurrent) periodic point with period p in D. ■

For $D \in Me(p)$, if $\psi(D) = C(0; d_0, \ldots, d_{p-1})$ i.e. if $d_0, \ldots, d_{p-1}$ is the code that identifies D, then $\psi(S_p(D)) = C(0; d_0, \ldots, d_{p-1}, d_0, \ldots, d_{p-1})$. By (3.3) it follows that

$$\mu(D) = v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}}$$

and

$$\mu(S_p(D)) = v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}} Z_{d_{p-1},d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}}$$
$$= \frac{Z_{d_{p-1},d_0} \mu(D)^2}{v_{d_0}}.$$

From this it follows that if $S_{NR}^p(D) \neq \emptyset$ and using Lemma 20 we have

$$\rho(p, D) = \frac{Z_{d_{p-1},d_0}}{v_{d_0}} \mu(D). \tag{3.8}$$

Some useful results about the space of the sequences with two symbols

In this section we will prove some useful results related with maps whose trajectories are encoded by two symbols, 2-linear Markov transformation.

If we assume that there is a chaotic attractor, C, on the phase space $[0, 1]$ such that $\overline{C} = [0, 1]$ then only some of all possible transition matrices, associated to the linear Markov transformation that can be represented by two symbols, are allowed since the transformation needs to have a dense orbit in $[0, 1]$. The allowed transition matrices are:

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.9}$$

The next result will be useful for the proof of the main result of the next section and, essentially, it gives us, for some fixed sequence $(x_j) \in \Sigma_{A_i}$, all the relations between the number of words with two symbols and the number of words with just one symbol, for some fixed length $p > 2$ of (x_j).

Let $(x_j) \in X_{A_i}$. Define $\hat{x_p} = (x_0, ..., x_{p-1})$ and $s_{word}^k = (s_0, ..., s_{k-1})$ where $s_l \in \{0, 1\}$ $\forall l = 0, ..., k-1$. Define $N(s_{word}^k, (x_j), p)$ as the number of occurrences of s_{word}^k in $\hat{x_p}$, for some fixed $p > 1$.

Lemma 22 *For some fixed $p > 1$ and $(x_j) \in X_{A_i}$, $i \in \{1, 2, 3\}$, we have*

1. $N(00, (x_j), p) + N(01, (x_j), p) = N(0, (x_j), p-1) = N(0, (x_j), p) - (1 - x_{p-1})$,
2. $N(00, (x_j), p) + N(10, (x_j), p) = N(0, \sigma((x_j)), p-1) = N(0, (x_j), p) - (1 - x_0)$,
3. $N(10, (x_j), p) + N(11, (x_j), p) = N(1, (x_j), p-1) = N(1, (x_j), p) - x_{p-1}$,
4. $N(01, (x_j), p) + N(11, (x_j), p) = N(1, \sigma((x_j)), p-1) = N(1, (x_j), p) - x_0$.

Proof. For equation (1) observe that every occurrence of a 0 in one of the terms $x_0, ..., x_{p-2}$ also corresponds to an occurrence of 00 or to one 01. For equation (2) observe that every occurrence of a 0 in one of the terms

$x_1, ..., x_{p-1}$ also corresponds to an occurrence of 00 or to one 10. The last two equations can be proved using similar arguments. Q.E.D.

For a more formal proof, let us use an induction argument on p: For $p = 2$, $\hat{x_p}$ can only have the form 00, 01, 10 or 11. In any of these cases equation (1), (2), (3) and (4) is valid. For $p = 2$ all equations are valid. Now, suppose equation (1), (2), (3) and (4) valid for some $p > 2$, let's prove for $p + 1$. First, observe that

$$N(00,(x_j),p+1) = N(00,(x_j),p) + \begin{cases} 1 & if\ x_{p-1}x_p = 00, \\ 0 & if\ x_{p-1}x_p \neq 00, \end{cases}$$

$$N(01,(x_j),p+1) = N(01,(x_j),p) + \begin{cases} 1 & if\ x_{p-1}x_p = 01, \\ 0 & if\ x_{p-1}x_p \neq 01, \end{cases}$$

$$N(10,(x_j),p+1) = N(10,(x_j),p) + \begin{cases} 1 & if\ x_{p-1}x_p = 10, \\ 0 & if\ x_{p-1}x_p \neq 10, \end{cases}$$

$$N(11,(x_j),p+1) = N(11,(x_j),p) + \begin{cases} 1 & if\ x_{p-1}x_p = 11, \\ 0 & if\ x_{p-1}x_p \neq 11, \end{cases}$$

$$N(0,(x_j),p+1) = N(0,(x_j),p) + \begin{cases} 1 & if\ x_p = 0, \\ 0 & if\ x_p = 1, \end{cases}$$

$$N(1,(x_j),p+1) = N(1,(x_j),p) + \begin{cases} 1 & if\ x_p = 1, \\ 0 & if\ x_p = 0. \end{cases}$$

As we can see from the last equations, the differences between all quantities only depends on the value of the terms x_{p-1} and x_p. If $(x_{p-1}, x_p) = (0, 0)$ then, using the induction condition and some of the last relations, we obtain

$$(1) \Leftrightarrow N(00,(x_j),p+1) - 1 + N(01,(x_j),p+1) = N(0,(x_j),p+1) - 1 - (1 - x_p)$$

$$\Leftrightarrow N(00,(x_j),p+1) + N(01,(x_j),p+1) = N(0,(x_j),p+1) - (1 - x_p),$$

$$(2) \Leftrightarrow N(00,(x_j),p+1) - 1 + N(10,(x_j),p+1) = N(0,(x_j),p+1) - 1 - (1 - x_0)$$

$$\Leftrightarrow N(00,(x_j),p+1) + N(10,(x_j),p+1) = N(0,(x_j),p+1) - (1 - x_0),$$

$$(3) \Leftrightarrow N(00,(x_j),p+1) - 1 + N(10,(x_j),p+1) = N(0,(x_j),p+1) - 1 - (1 - x_0)$$

$$\Leftrightarrow N(00,(x_j),p+1) + N(10,(x_j),p+1) = N(0,(x_j),p+1) - (1 - x_0),$$

$$(4) \Leftrightarrow N(00,(x_j),p+1) - 1 + N(10,(x_j),p+1) = N(0,(x_j),p+1) - 1 - (1 - x_0)$$

$$\Leftrightarrow N(00,(x_j),p+1) + N(10,(x_j),p+1) = N(0,(x_j),p+1) - (1 - x_0)\ Q.E.D.$$

The arguments to prove for the cases when $(x_{p-1}, x_p) = (0, 1)$, $(1, 0)$ or $(1, 1)$ are similar and we conclude that the equation (1), (2), (3) and (4) is valid for any $p > 1$. ■

Linear Markov transformations with two pieces

As we already described, some of the elements of $Me(p)$, for some fixed $p > 1$, do not contain a periodic point of period p. For a 2-linear Markov transformation and $D \in Me(p)$, the expression for $\mu_{NR}(p, D)$ in definition 8 takes the form

$$\mu_{NR}(p, D) = \frac{1}{(der_0)^{N(0)}(der_1)^{N(1)}}, \tag{3.10}$$

whenever $S_{NR}(D) \neq \emptyset$, where $N(0) = N(0, (x_j), p)$ and $N(1) = N(1, (x_j), p)$, for $(x_j) = (d_0(D), ..., d_{p-1}(D), x_p, ...)$, and der_0, der_1 are the absolute values of the derivative of the map in I_0, I_1, respectively.

The main result of this section is:

Theorem 23 *Consider f as a 2-linear Markov transformation. Also let's assume that there is a chaotic attractor, C, on the phase space $[0, 1]$ such that $\overline{C} = [0, 1]$. Then, fixing $p > 1$, for any element $D \in Me(p)$ we have*

$$\rho(p, D) = \mu_{NR}(p, D).$$

Before proving the theorem let us recall that, for a linear Markov transformation with 2 pieces and on the conditions of the theorem, the only allowed transition matrices are those in (3.9):

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 24 *On the conditions of the theorem 23, if either $S_p(D) = \emptyset$ or $S^p_{NR}(D) \neq S_p(D) \neq \emptyset$ then theorem 23 is a trivial observation.*

Proof. If $S_p(D) = \emptyset$ and if $d_0, ..., d_{p-1}$ is the code that identifies D then there are no points $x \in [0, 1]$ with the code $d_0, ..., d_{p-1}, d_0, ..., d_{p-1}, x_{2p-1}, ...$ and it means that there is no periodic point with period p inside D. This implies that $\mu_{NR}(p, D) = 0$. On the other hand, $S^p_{NR}(D) \subset S_p(D) = \emptyset$ and so $\rho(p, D) = 0$.

If $S_p(D) \neq \emptyset$ and $S^p_{NR}(D) \neq S_p(D)$ then, by Lemma 20, $S^p_{NR}(D) = \emptyset$ and it means that we do not have any non-recurrent periodic point of period p inside D and so $\mu_{NR}(p, D) = 0$. On the other hand, $S^p_{NR}(D) = \emptyset$ implies $\rho(p, D) = 0$. ■

The interesting and non trivial cases are those where the transition matrices are (3.9) and, at the same time, for a chosen $D \in Me(p)$ we have $S^p_{NR}(D) \neq \emptyset$. Then the proof of theorem 23 is divided in three lemmas, for the three possible transition matrices.

Lemma 25 *Let f be a linear Markov transformation with transition matrix A_1. Then, under the conditions of theorem 23, for any D such that $S^p_{NR}(D) \neq \emptyset$ we have*

$$\rho(p, D) = \mu_{NR}(p, D).$$

Proof. The stochastic matrix and the stationary vector, with $I_0 = (0, c)$ and $I_1 = (c, 1)$ for $0 < c < 1$, are given by

$$Z = \begin{pmatrix} c & 1-c \\ c & 1-c \end{pmatrix},$$

and $v = (c, 1-c)$. For a chosen $D \in Me(p)$ with code $d_0, ..., d_{p-1}$, we write its measure as

$$\mu(D) = \mu(C(0; d_0, ..., d_{p-1})) = v_{d_0} Z_{d_0 d_1} \cdots Z_{d_{p-2} d_{p-1}} =$$

$$= v_{d_0} c^{[N(00)+N(10)]}(1-c)^{[N(01)+N(11)]},$$

where $N(00) = N(00, (x_j), p)$, $N(01) = N(01, (x_j), p)$, $N(10) = N(10, (x_j), p)$ and $N(11) = N(11, (x_j), p)$, for $(x_j) = (d_0, ..., d_{p-1}, x_p, ...)$. We also write the measure for $S^p_{NR}(D)$ and using Eq. (3.8) we obtain

$$\rho(p, D) = c^{[N(00)+N(10)+1-d_0]}(1-c)^{[N(01)+N(11)+d_0]}.$$

Using the information about the derivatives of the map, we write Eq. (3.10) as

$$\mu_{NR}(p, D) = c^{N(0)}(1-c)^{N(1)},$$

where $N(0)$ and $N(1)$ are the numbers of occurrences of 0 and 1, respectively, in the code of D. We apply lemma 22 to the case $(x_j) = (d_0, ..., d_{p-1}, x_p, ...)$ and we conclude that $N(00)+N(10)+1-d_0 = N(0)$ and $N(01)+N(11)+d_0 = N(1)$. Then, it follows that

$$\rho(p, D) = c^{N(0)}(1-c)^{N(1)} = \mu_{NR}(p, D).$$

■

Lemma 26 *Let f be a linear Markov transformation with transition matrix A_2. Then, under the conditions of theorem 23, for any D such that $S^p_{NR}(D) \neq \emptyset$ we have*

$$\rho(p, D) = \mu_{NR}(p, D).$$

Proof. The stochastic matrix and the stationary vector, with $I_0 = (0, c)$ and $I_1 = (c, 1)$ for $0 < c < 1$, are given by

$$Z = \begin{pmatrix} 0 & 1 \\ c & 1-c \end{pmatrix},$$

and $v = (\frac{c}{1+c}, \frac{1}{1+c})$. For a chosen $D \in Me(p)$ with code $d_0, ..., d_{p-1}$, we write its measure as

$$\mu(D) = v_{d_0} c^{N(10)} (1-c)^{N(11)},$$

where $N(10)$ and $N(11)$ are the numbers of occurrences of the words 10 and 11, respectively, in the code of D. We also write the measure of $S^p_{NR}(D)$ and using Eq. (3.8) we obtain

$$\rho(p, D) = c^{[N(10)+1-d_0]} (1-c)^{[N(11)+d_0-1+d_{p-1}]}.$$

Using the information about the derivatives of the map, we write Eq. (3.10) as

$$\mu_{NR}(p, D) = c^{N(0)} (1-c)^{[N(1)-N(0)]},$$

where $N(0)$ and $N(1)$ are the numbers of occurrences of 0 and 1, respectively, in the code of D. We apply lemma 22 to the case $(x_j) = (d_0, ..., d_{p-1}, x_p, ...)$ and we conclude that $N(10) + 1 - d_0 = N(0)$ and $N(11) + d_0 + d_{p-1} - 1 = N(1) - N(0)$. Then, it follows that

$$\rho(p, D) = c^{N(0)} (1-c)^{[N(1)-N(0)]} = \mu_{NR}(p, D).$$

■

Lemma 27 *Let f be a linear Markov transformation with transition matrix A_3. Then, under the conditions of theorem 23, for any D such that $S^p_{NR}(D) \neq \emptyset$ we have*

$$\rho(p, D) = \mu_{NR}(p, D).$$

Proof. The stochastic matrix and the stationary vector, with $I_0 = (0, c)$ and $I_1 = (c, 1)$ for $0 < c < 1$, are given by

$$Z = \begin{pmatrix} c & 1-c \\ 1 & 0 \end{pmatrix},$$

and $v = (\frac{1}{2-c}, \frac{1-c}{2-c})$. For a chosen $D \in Me(p)$, we write its measure as

$$\mu(D) = v_{d_0} c^{N(00)} (1-c)^{N(01)},$$

where $N(00)$ and $N(01)$ are the numbers of occurrences of the words 00 and 01, respectively, in the code of D. We also write the measure of $S^p_{NR}(D)$ and using Eq. (3.8) we obtain

$$\rho(p, D) = c^{[N(00)+1-d_0-d_{p-1}]}(1-c)^{[N(01)+d_0]}.$$

Using the information about the derivatives of the map, we write Eq. (3.10) as

$$\mu_{NR}(p, D) = c^{[N(0)-N(1)]}(1-c)^{N(1)},$$

where $N(0)$ and $N(1)$ are the numbers of occurrences of 0 and 1, respectively, in the code of D. We apply lemma 22 to the case $(x_j) = (d_0, ..., d_{p-1}, x_p, ...)$ and we conclude that $N(01) + d_0 = N(1)$ and $N(00) + 1 - d_0 - d_{p-1} - 1 = N(0) - N(1)$. Then, it follows that

$$\rho(p, D) = c^{N(0)}(1-c)^{[N(1)-N(0)]} = \mu_{NR}(p, D).$$

■

3.4 Longer returns to elements of the Markov partition

In this section we generalize theorem 23 in terms of the subset that we want to observe the returns. If $S \in Me(p-k)$ then for $k = 0$ theorem 23 says that

$$\rho(p, S) = \mu_{NR}(p, S)$$

and in this section we will show that, in fact, is not only true for $k = 0$ but is also true for $0 \leq k < p$.

Subset as a perfect union of elements of the Markov partition

Let's take $A, B \in Me(p)$ such that $S = A \cup B \in Me(p-1)$. In these conditions we know that $\psi(A)$ and $\psi(B)$ differ only on the last digit. So, $\psi(S) = d_0, ..., d_{p-2}$, $\psi(A) = d_0, ..., d_{p-2}, d_{p-1}(A)$ and $\psi(B) = d_0, ..., d_{p-2}, d_{p-1}(B)$ where $d_{p-1}(A) \neq d_{p-1}(B)$.

Recall that

$$S_p(A \cup B) = \{x \in A \cup B \ : \ f^p(x) \in A \cup B\}$$

and

$$S^p_{NR}(A \cup B) = \{x \in S_p(A \cup B) \ : \ f^i(x) \notin A \cup B \ 0 < i < p\}.$$

Lemma 28 *Suppose $S = A \cup B \in Me(p-1)$, with $A, B \in Me(p)$ and $S^p_{NR}(S) \neq \emptyset$. If p is prime then $S^p_{NR}(S) = S_p(S)$. If p is not prime then in the cases where $S^p_{NR}(S) \neq S_p(S)$ there exists one and only one recurrent periodic point of period p in S and one and only one non-recurrent periodic point of period p in S.*

Proof. All $x \in S^p_{NR}(S)$ have the form

$$\psi(x) = (x_j) = (d_0, d_1, ..., d_{p-2}, x_{p-1}, d_0, d_1, ..., d_{p-2}, x_{2p-1}, ...)$$

and there is no $k = 1, ..., p-1$ such that $\sigma^k((x_j)) \in S$.

Let us take some $y \in S_p(S)$ and it has the form

$$\psi(y) = (y_j) = (d_0, d_1, ..., d_{p-2}, y_{p-1}, d_0, d_1, ..., d_{p-2}, y_{2p-1}, ...).$$

For $0 < k < p$, $\sigma^k((y_j)) = (d_k, d_{k+1}, ..., d_{p-2}, y_{p-1}, ...)$. So, if for some $0 < k < p$ we have $\sigma^k((y_j)) \in S \Leftrightarrow d_j = d_{j+k}$ and $d_{p-k-1} = y_{p-1}$ $\forall j = 0, ..., p-2$ $(mod\ p)$.

Case prime: See that $\{j : \ d_j = d_0\} = \{nk\ (mod\ p)\}$. By a classical result we know that k generates Z_n iff $(k, n) = 1$. As we are assuming p prime then we conclude that $\{nk\ (mod\ p)\} = Z_p$ and all d_j's has to have the same value. In this situation we only have the cases where either $S = \{11..11\}$ *or* $S = \{00..00\}$ but in these cases $S^p_{NR}(S) = \emptyset$ and we conclude that $S^p_{NR}(S) = S_p(S)$ whenever $S^p_{NR}(S) \neq \emptyset$ and p prime.

Case not prime: Let us consider some point in $S_p(S)$ that is not in $S^p_{NR}(S)$, we call it z. Now, z has the form

$$\psi(z) = (z_j) = (d_0, d_1, ..., d_{p-2}, z_{p-1}, d_0, d_1, ..., d_{p-2}, ...) \qquad (3.11)$$

and for some k, $0 < k < p$ we have

$$(d_k, ..., d_{p-2}, z_{p-1}, d_0, ..., d_{p-2,...}).$$

Moreover, every point y satisfying $\psi(y) = (d_0, d_1, ..., d_{p-2}, z_{p-1}, d_0, d_1, ..., d_{p-2}, ...)$ lies in $S_p(S) - S^p_{NR}(S)$. In particular, $\pi(d_0, d_1, ..., d_{p-2}, z_{p-1})$ is a recurrent periodic point in S.

To complete the proof, note that if $x \in S^p_{NR}(S)$ then $\psi(x)$ must be of the form

$$\psi(x) = (d_0, d_1, ..., d_{p-2}, 1 - z_{p-1}, d_0, d_1, ..., d_{p-2}, ...).$$

since we are assuming $S^p_{NR}(S) \neq \emptyset$ then $d_{p-2}, 1 - z_{p-1}, d_0$ is an allowed word in X_A and thus $\pi(\overline{d_0, d_1, ..., d_{p-2}, 1 - z_{p-1}}$ is a non recurrent periodic point in S. ■

Lemma 29 *Consider $S = A \cup B \in Me(p-1)$, with $A, B \in Me(p)$ and $S^p_{NR}(S) \neq \emptyset$. If $S^p_{NR}(S) = S_p(S)$ (for instance, if p is prime) we have*

$$\rho(p,S) = \frac{\mu(A) Z_{d_{p-1}(A),d_0}}{v_{d_0}} + \frac{\mu(B) Z_{d_{p-1}(B),d_0}}{v_{d_0}}$$

and if $S^p_{NR}(S) \neq S_p(S)$ then

$$\rho(p,S) = \frac{\mu(S^*) Z_{d_{p-1}(S^*),d_0}}{v_{d_0}}$$

where S^* represents the set, either A or B, that contains, by lemma 28, the non-recurrent periodic point of period p in S.

Proof. If $S^p_{NR}(A \cup B) = S_p(A \cup B) \neq \emptyset$ we have that

$$\rho(p,S) = \frac{\mu(S_p(A)) + \mu(S_p(B)) + \mu(A \to B) + \mu(B \to A)}{\mu(A \cup B)}, \tag{3.12}$$

where $A \to B$ represents the set of points in A that returns to B and $B \to A$ represents the set of points in B that returns to A after p iterations by F. Their measures are given by

$$\mu(A \to B) = v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}(A)} Z_{d_{p-1}(A),d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}(B)}$$

and

$$\mu(B \to A) = v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}(B)} Z_{d_{p-1}(B),d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}(A)}.$$

So, we write Eq.(3.12) only with the elements of the stochastic matrix Z and we obtain

$$\frac{v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}(A)} Z_{d_{p-1}(A),d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}(A)}}{v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-3},d_{p-2}}} +$$

$$+ \frac{v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}(B)} Z_{d_{p-1}(B),d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}(B)}}{v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-3},d_{p-2}}} +$$

$$+ \frac{v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}(A)} Z_{d_{p-1}(A),d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}(B)}}{v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-3},d_{p-2}}} +$$

$$+ \frac{v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}(B)} Z_{d_{p-1}(B),d_0} Z_{d_0,d_1} \cdots Z_{d_{p-2},d_{p-1}(A)}}{v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-3},d_{p-2}}}.$$

Now, the fact that Z is a stochastic matrix, in particular, implies that $Z_{d_{p-2},d_{p-1}(A)} + Z_{d_{p-2},d_{p-1}(B)} = 1$ and we obtain from the last expression

$$\rho(p,S) = \frac{\mu(A) Z_{d_{p-1}(A),d_0} + \mu(B) Z_{d_{p-1}(B),d_0}}{v_{d_0}}$$

and case $S^p_{NR}(A\cup B)=S_p(A\cup B)\neq\emptyset$ is done.

If $S^p_{NR}(A\cup B)\neq S_p(A\cup B)\neq\emptyset$, by lemma 28, let us suppose, W.L.G., that A is the set that contains the non-recurrent periodic point of period p. Then, by definition, we have

$$\rho(p,S)=\frac{\mu(S^p_{NR}(A))+\mu(A\to B)}{\mu(A\cup B)}=$$

by lemma 20

$$=\frac{\mu(S_p(A))+\mu(A\to B)}{\mu(A\cup B)},$$

and using the elements of the stochastic matrix Z, we obtain

$$\rho(p,S)=\frac{\mu(A)Z_{d_{p-1}(A),d_0}}{v_{d_0}}.$$

■

Lemma 30 *For any $A\in Me(p)$, $p>1$, we have*

$$\frac{\mu(A)Z_{d_{p-1}(A),d_0}}{v_{d_0}}=\mu_{NR}(p,A).$$

Proof. By theorem 23

$$\mu_{NR}(p,A)=\rho(p,A)=\frac{\mu(S^p_{NR}(A))}{\mu(A)}$$

and by lemma 20 if this is not zero then it equals

$$\frac{\mu(S_p(A))}{\mu(A)}.$$

Finally, we write the last expression only with the elements of the stochastic matrix Z and we obtain

$$\mu_{NR}(p,A)=\frac{\mu(A)Z_{d_{p-1}(A),d_0}}{v_{d_0}}.$$

■

Theorem 31 *Under the conditions of Theorem 23, let us consider $S=A\cup B\in Me(p-1)$, with $A,B\in Me(p)$ and $S^p_{NR}(S)\neq\emptyset$. Then*

$$\rho(p,S)=\mu_{NR}(p,S).$$

In particular, if $S^p_{NR}(S)\neq S_p(S)$(and therefore p is not prime) then either

$$\rho(p,S)=\mu_{NR}(p,A)$$

or

$$\rho(p,S)=\mu_{NR}(p,B).$$

Proof. First we consider the case $S^p_{NR}(A \cup B) = S_p(A \cup B) \neq \emptyset$. By Lemma 28 this is always true when p is prime. Then

$$\rho(p,S) = \frac{\mu(S_p(A \cup B))}{\mu(A \cup B)} = \tag{3.13}$$

by lemma 29

$$= \frac{\mu(A) Z_{d_{p-1}(A),d_0}}{v_{d_0}} + \frac{\mu(B) Z_{d_{p-1}(B),d_0}}{v_{d_0}} =$$

by lemma 30

$$= \mu_{NR}(p,A) + \mu_{NR}(p,B)$$

and the case p prime is done since both periodic points of period p (in A and in B) are non-recurrent in S and are not the same (the last symbol of the code of each point needs to be different). So,

$$\mu_{NR}(p,A) + \mu_{NR}(p,B) = \mu(p,S).$$

The only case that we still need to prove is the case when $S^p_{NR}(S) \neq S_p(S)$. By Lemma 28 we suppose, W.L.G., that A is the element that has the recurrent periodic point of period p, $A = \pi(d_0, ..., d_{p-2}, d_A)$ with $d_A = d_{p-1}(A)$ and $B = \pi(d_0, ..., d_{p-2}, d_B)$ with $d_B = d_{p-1}(B)$. Observe that

$$S^p_{NR}(A \cup B) = \pi(d_0, d_1, ..., d_{p-2}, d_B, d_0, d_1, ..., d_{p-2})$$

$$= S^p_{NR}(B) \dot{\cup} \pi(d_0, d_1, ..., d_{p-2}, d_B, d_0, d_1, ..., d_{p-2}, d_A)$$

$$= S^p_{NR}(B) \dot{\cup} (B \to A)$$

By definition we have

$$\rho(p,S) = \frac{\mu(S^p_{NR}(A \cup B))}{\mu(A \cup B)} =$$

$$= \frac{\mu(S^p_{NR}(B)) + \mu(\pi(d_0, d_1, ..., d_{p-2}, d_B, d_0, d_1, ..., d_{p-2}, d_A))}{\mu(A \cup B)} =$$

by lemma 20

$$= \frac{\mu(S_p(B)) + \mu(B \to A)}{\mu(A \cup B)}.$$

Using lemmas 29 and 30 we conclude that

$$\rho(p,S) = \mu_{NR}(p,B).$$

Because of our assumption on A (one and only one recurrent periodic point of period p) we have $\mu_{NR}(p,S) = \mu_{NR}(p,B)$ and we conclude the proof. ■

Theorem 32 *Under the conditions of Theorem 23, let us consider* $S = \bigcup_i A_i \in Me(p-k)$*, with* $A_i \in Me(p)$ $\forall i$ *with* $p > 0$ *and* $0 \leq k < p$*. Then*

$$\rho(p, S) = \mu_{NR}(p, S)$$

Proof. Suppose that we are in the case where $S^p_{NR}(S) = S_p(S) \neq \emptyset$. Observe that the number of $A_i's$ is $N \leq 2^k$ and if the code of S is $d_0, ..., d_{p-k-1}$ then

$$\psi(A_i) = d_0, ..., d_{p-k-1}, d_{p-k}(A_i), d_{p-k+1}(A_i), ..., d_{p-1}(A_i).$$

The density function of the first returns can be written as

$$\rho(p, S) = \frac{\mu(S^p_{NR}(S))}{\mu(S)} = \frac{\mu(S_p(S))}{\mu(S)} = \tag{3.14}$$

$$= \frac{\sum_{i=1}^N \mu(S_p(A_i)) + \sum_{i,j=1, i\neq j}^N \mu(A_i \to A_j)}{\mu(S)}$$

where

$$\mu(A_i \to A_j) =$$

$$= v_{d_0} Z_{d_0,d_1} \cdots Z_{d_{p-k-1},d_{p-k}(A_i)} Z_{d_{p-k}(A_i),d_{p-k+1}(A_i)} \cdots Z_{d_{p-1(A_i)},d_0} Z_{d_0,d_1} \cdots$$

$$\cdots Z_{d_{p-k-1}(A_j),d_{p-k}(A_j)} Z_{d_{p-k}(A_j),d_{p-k+1}(A_j)} \cdots Z_{d_{p-2(A_j)},d_{p-1(A_j)}}$$

$$= \mu(A_i) \frac{\mu(A_j)}{v_{d_0}} Z_{d_p-1}(A_i), d_0.$$

Finally we write Eq.(3.14) as

$$\rho(p, S) = \frac{\sum_{i=1}^N \frac{\mu(A_i)^2}{v_{d_0}} Z_{d_p-1}(A_i), d_0 + \sum_{i=1}^N \sum_{j=1, j\neq i}^N \frac{\mu(A_i)\mu(A_j)}{v_{d_0}} Z_{d_p-1}(A_i), d_0}{\mu(S)}$$

and using lemma 30 we obtain

$$\rho(p, S) = \mu_{NR}(p, A_1) \left(\frac{\mu(A_1) + ... + \mu(A_N)}{\mu(S)} \right) + ...$$

$$... + \mu_{NR}(p, A_N) \left(\frac{\mu(A_1) + ... + \mu(A_N)}{\mu(S)} \right) =$$

$$= \sum_{i=1}^N \mu_{NR}(p, A_i) = \mu_{NR}(p, S)$$

since we are assuming $S^p_{NR}(S) = S_p(S)$.

For the case where $S_p(S) \neq S_{NR}^p(S) \neq \emptyset$, we reorder the $A_i's$, using lemma 21, such that the first l A_i's will be the ones that contain a non-recurrent (in S) periodic point of period p. Then

$$\rho(p,S) = \frac{\sum_{i=1}^{l} \mu(S_{NR}^p(A_i)) + \sum_{i=1}^{l}\sum_{j=1}^{N} \mu(A_i \to A_j)}{\mu(S)} =$$

$$= \frac{\sum_{i=1}^{l} \mu(S_p(A_i)) + \sum_{i=1}^{l}\sum_{j=1}^{N} \mu(A_i \to A_j)}{\mu(S)} =$$

$$= \sum_{i=1}^{l} \mu_{NR}(p, A_i) = \mu_{NR}(p,S).$$

Finally if $S_{NR}(S) = \emptyset$ then $\rho(p,S) = 0$ and by lemma 21 $\mu_{NR}(p,S) = 0$.
■

Considering $k = 1$ in theorem 32 we have theorem 31 as expected but theorem 31 still has a little bit more information. Observe that, for the case $k = 1$, if p is prime then we automatically know that there are no recurrent periodic points of period p in S and then we just need to calculate all periodic points of period p inside D without verifying if they are or not recurrent points in D. We do not have anymore this property if $k > 1$.

3.5 Subset that is not a perfect union of elements of the Markov partition - numerical point of view

In this section we discuss the situation when, for some linear Markov transformation, our subset of the phase space is not anymore a perfect union of Markov elements. For simplicity, consider a 2-linear Markov transformation and a set $S \in [0,1]$ such that one of the boundaries of S is not a boundary of any element of the Markov partition, for some fixed level $p > 1$. More precisely, consider $A, B \in Me(p)$ with $A \cup B \in Me(p-1)$ and $S(\epsilon) = A \cup R(\epsilon)$ where $R(\epsilon) \subset B$ is an interval such that $S(0) = A$, $S(1) = A \cup B$ and, $\forall \epsilon_1, \epsilon_2 \in [0,1]$, if $\epsilon_1 > \epsilon_2$ then $S(\epsilon_2) \subset S(\epsilon_1)$. By theorem 32

$$\rho(p, S(0)) = \mu_{NR}(p, A)$$

and

$$\rho(p, S(1)) = \mu_{NR}(p, A \cup B).$$

For the sets A and B, we have the following possible situations:

With respect to S,

1. A contains a non-recurrent periodic point of period p and B contains a non-recurrent periodic point of period p that does not belong to the orbit of the non-recurrent periodic point of period p in A,

2. A contains a non-recurrent periodic point of period p and B contains a non-recurrent periodic point that belongs to the orbit of the non-recurrent periodic point of period p in A,

3. A contains a non-recurrent periodic point of period p and B does not contain a non-recurrent periodic point of period p,

4. A does not contain a non-recurrent periodic point of period p and B contains a non-recurrent periodic point of period p,

5. A does not contain a non-recurrent periodic point of period p and B also does not contain a non-recurrent periodic point of peroid p.

In each previous situation and $\forall \epsilon \in [0,1]$ we expect the following relations between $\rho(p, S(\epsilon))$ and the quantities $\mu_{NR}(p, A)$ and $\mu_{NR}(p, A \cup B)$:

1. $\mu_{NR}(p, A) \leq \rho(p, S(\epsilon)) \leq \mu_{NR}(p, A \cup B)$, $\forall \epsilon \in [0,1]$,

2. $\mu_{NR}(p, A \cup B) \leq \rho(p, S(\epsilon)) \leq \mu_{NR}(p, A)$, $\forall \epsilon \in [0,1]$,

3. $\mu_{NR}(p, A) = \mu_{NR}(p, A \cup B) \neq 0$,

4. $\mu_{NR}(p, A) \leq \rho(p, S(\epsilon)) \leq \mu_{NR}(p, A \cup B)$, $\forall \epsilon \in [0,1]$,

5. $\mu_{NR}(p, A) = \mu_{NR}(p, A \cup B) = 0$.

As a conclusion, we suggest that if $S \subset A \cup B \in Me(p-1)$ then $\rho(p, S)$ is always bounded by $\mu_{NR}(p, A)$ and $\mu_{NR}(p, A \cup B)$.

Numerical simulations have been done considering the Markov transformation from example 3.2.1 with $c = 0.6$. In Fig. 3.1 (A), are shown by pluses the order-5 Markov cell borders and in filled circles the 10 unstable periodic points of minimal period $p = 5$ and the fixed point $x = 0$.

In Fig. 3.1 (B) and (C) are shown the values of $\rho(p, S)$ and $\mu_{NR}(p, S)$ as we change the size of the subset S. We start with a subset being a Markov element $S_1 = [0.216, 0.3024]$ where $\psi(S_1) = (00100)$ and then we change it until it becomes a perfect union of two Markov elements $S_1 = [0.2160, 0.3024] \bigcup S_2 = [0.3024, 0.3600]$ where $\psi(S_2) = (00101)$. These two intervals are in the situation described in 1. In Fig. 3.1 (B) we consider the unstable periodic points of period $p = 5$ and in Fig. 3.1 (C) the unstable periodic points of period $p = 8$ for the same sets. In the horizontal axis of (B) and (C), $\lambda(S)$ represents the length of S when we increase S_1 in order to get $S_1 \cup S_2$.

According to theorems 23 and 31, $\rho(p, S) = \mu_{NR}(p, S)$ whenever $S \in Me(p)$ or $S \in Me(p-1)$. That happens in (B) when $\lambda(S) = 0.0864$ ($S = S_1$) and when $\lambda(S) = 0.144$ ($S = S_1 \cup S_2$).

According to theorem 32, as long as the subset is an element of a Markov partition of order p_1, $\rho(p, S) = \mu_{NR}(p, S)$ for any order $p > p_1$. In Fig. 3.1 (C) we observe that fact using $p = 8$.

When the subset $S \subset A \cup B \in Me(p-1)$ is not a perfect union of Markov elements, then it can also happen that $\rho(p, S) = \mu_{NR}(p, S)$. As an example, observe in Fig. 3.1 (C) when $\lambda(S)$ is close to 0.1 or when $\lambda(S)$ is close to 0.12.

When $\rho(p, S) \neq \mu_{NR}(p, S)$ notice that $\rho(p, S)$ is confined within the values of $\mu_{NR}(p, S_1 \cup S_2)$ and $\mu_{NR}(p, S_1)$. We strongly believe that this is always true when $S \subset A \cup B \in Me(p-1)$.

Fig. 3.2 shows similar numerical results for subsets $S \subset A \cup B$ where $A, B \in Me(5)$ but where $A \cup B$ are not necessarily in $Me(4)$. In some cases we still get either $\mu_{NR}(p, A) \leq \rho(p, S(\epsilon)) \leq \mu_{NR}(p, A \cup B)$, $\forall \epsilon \in [0, 1]$ or $\mu_{NR}(p, A \cup B) \leq \rho(p, S(\epsilon)) \leq \mu_{NR}(p, A)$, $\forall \epsilon \in [0, 1]$ but this is not true anymore in the case of $\psi(A) = (01010)$ and $\psi(B) = (10000)$ (Fig. 3.2 (D)). This shows that theorem 32 cannot be much further extended (with respect the subset of the phase space), and that from this point on we can only expect to find approximate results.

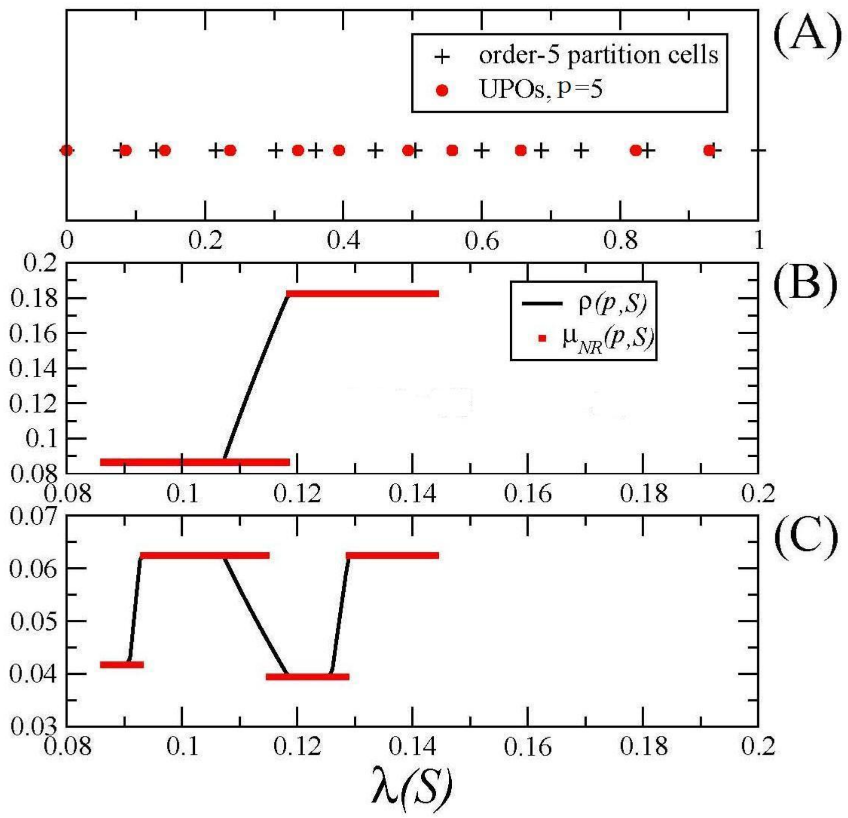


Figure 3.1: Comparing ρ and μ_{NR} using the map obtained by taking $c = 0.6$ in example 3.2.1.

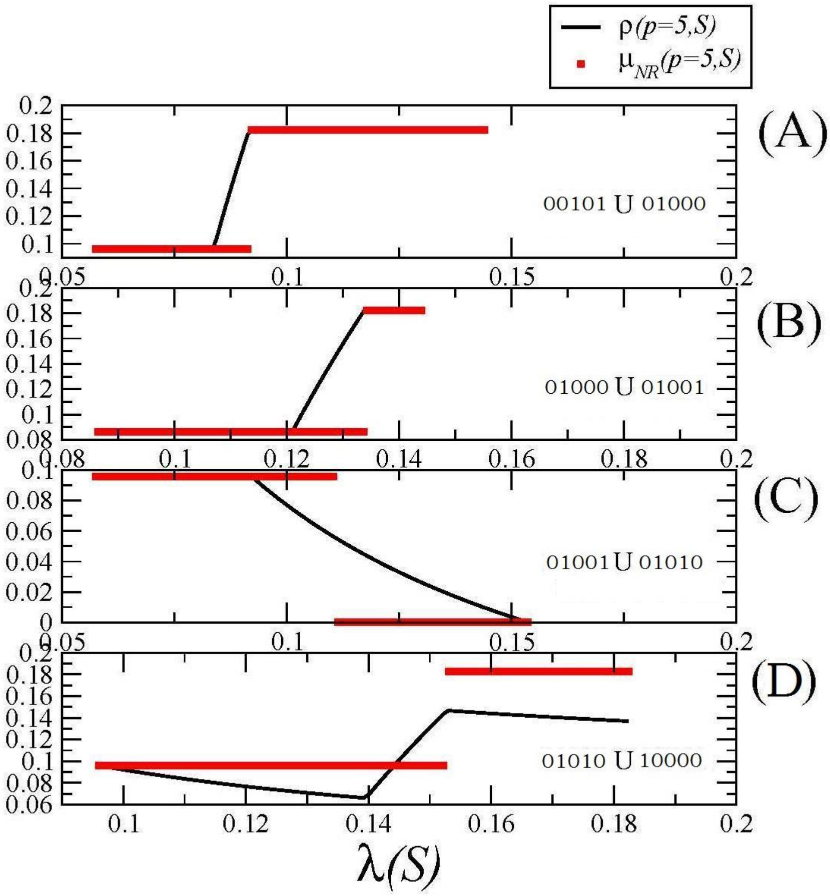


Figure 3.2: Comparing ρ and μ_{NR} using the map obtained by taking $c = 0.6$ in example 3.2.1.

Chapter 4

Conclusion and future work

This work is dedicated to the presentation and the proof of a conjecture for chaotic dynamical systems. The conjecture says, essentially, that the density function of the first Poincaré returns is completely determined by the unstable periodic points of a given chaotic map. The first Poincaré return is the time spent by a trajectory to make two consecutive returns to some specific region of the phase space. The relation between the density of such returns and the unstable periodic points allows us to compute easily important quantities as was done, for the Kolmogorov-Sinai entropy, in [14] with the logistic map. Even for nonuniformly hyperbolic systems, where there exist some particular subsets for which the unstable periodic orbits are not sufficient to calculate their measure[1], the simulations, in [14] with the logistic map and also in [2] with Chua's circuit and Henon map, suggest that the conjecture is still true but in an approximate sense. As a consequence of the conjecture and the fact that first Poincaré returns can be simply and quickly accessible in experiments, this work offers an easy way to obtain important quantities in dynamical systems by experiments.

Simulations suggest, in particular, that the conjecture presented in [14] can be proved in some particular classes of dynamical systems. In this work is provided a proof of such fact, in lemma 9, for particular case of expanding piecewise transformations and for special subsets of the phase space. Also here is provided a proof considering the class of all Markov transformations with a linear assumption. Theorem 23, 31 and 32 are the main results of this work where it is proved that in elements of the Markov partition (of any order) we can express the density of the first Poincaré returns in terms of the unstable non-recurrent periodic orbits.

There are some natural continuations of this work: first, to extend the results on piecewise linear Markov maps to some sets that are not elements of the Markov class $Me(p)$. Second, to reformulate the conjecture for the sets for which it may only hold in an approximate sense. Third, to extend the results to other maps, without the assumption of linearity.

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