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2015 Nonlinearity 28 1695

(http://iopscience.iop.org/0951-7715/28/6/1695)

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **28** (2015) 1695–1720 doi:10.1088/0951-7715/28/6/1695

Generic area-preserving reversible diffeomorphisms

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Received 17 December 2014, revised 17 March 2015
Accepted for publication 9 April 2015
Published 7 May 2015

CrossMark

Recommended by Leonid Bunimovich

Abstract

Let M be a surface and $R : M \to M$ an area-preserving C^∞ diffeomorphism which is an involution and whose set of fixed points is a submanifold with dimension one. We will prove that C^1 - generically either an area-preserving R-reversible diffeomorphism, is Anosov, or, for μ-almost every $x \in M$, the Lyapunov exponents at x vanish or else the orbit of x belongs to a compact hyperbolic set with an empty interior. We will also describe a nonempty C^1-open subset of area-preserving R-reversible diffeomorphisms where for C^1 - generically each map is either Anosov or its Lyapunov exponents vanish from almost everywhere.

Keywords: dominated splitting, Lyapunov exponent, reversibility
Mathematics Subject Classification: 37D25, 37C80, 37C05

1. Introduction

Let M be a C^∞ compact, connected, Riemannian two-dimensional manifold without boundary and μ its normalized Lebesgue measure. Denote by $\text{Diff}^r_\mu(M)$ the set of C^r-diffeomorphisms of M which preserve μ endowed with the C^r-topology ($r \in \mathbb{N} \cup \{\infty\}$). A diffeomorphism $f : M \to M$ is said to be Anosov if M is a hyperbolic set for f. In [45], it was proved that a generic μ-preserving diffeomorphism is either Anosov or the set of elliptic periodic points is dense in the surface. More recently [9,37,38], another C^1-generic dichotomy in this setting has been established. For $f \in \text{Diff}^1_\mu(M)$ and Lebesgue almost every $x \in M$, the upper *Lyapunov exponent* at x is given by

$$\lambda^+(f, x) = \lim_{n\to+\infty} \log \|Df^n_x\|^{1/n}.$$

The main theorem of [9] states that in a C^1-residual subset of $\mathrm{Diff}^1_\mu(M)$ each element is either Anosov or has zero upper Lyapunov exponent at Lebesgue almost every point.

In this paper we address a similar question within the subspace of $\mathrm{Diff}^1_\mu(M)$ which exhibits some symmetry. More precisely, given a diffeomorphism $R \in \mathrm{Diff}^\infty_\mu(M)$ such that $R = R^{-1}$, denote by $\mathrm{Diff}^1_{\mu,R}(M)$ the subset of maps $f \in \mathrm{Diff}^1_\mu(M)$, called R-reversible, such that R conjugates f and f^{-1}, that is,

$$R \circ f = f^{-1} \circ R.$$

The spaces $\mathrm{Diff}^{\,1}(M)$, $\mathrm{Diff}_\mu^{\,1}(M)$ and $\mathrm{Diff}_{\mu,R}^{\,1}(M)$ are Baire [19, 29] and an extension of [11, Theorem 7] allows one to deduce that there is a residual subset $\mathcal{C}$ of $\mathrm{Diff}^1_{\mu,R}(M)$ such that $f \in \mathcal{C}$ if and only if it is a continuity point of the (upper semi-continuous) map

$$h \in \mathrm{Diff}^1_{\mu,R}(M) \mapsto \int_M \lambda^+(h,x)\, \mathrm{d}\mu$$

and so, for almost every $x \in M$, either $\lambda^+(f,x) = 0$ or the orbit of x by f has a dominated splitting. Roughly speaking, the method used in [11] depends on the construction of Kakutani towers on regions far away from hyperbolicity; we will uncover the quite involved machinery behind this argument in section 8. To prevent the coexistence of both behaviours in substantial sets of M, that is, to ensure that either hyperbolicity occurs on the whole manifold (so the diffeomorphism is Anosov) or $\lambda^+(f,x) = 0$ Lebesgue almost everywhere, those towers had to be built at a full measure subset of M. This dichotomy was obtained in [9] for surfaces and area-preserving diffeomorphisms, assisted by the density of C^2 diffeomorphisms in $\mathrm{Diff}^1_\mu(M)$ proved in [55], and the fact that for a C^2 area-preserving diffeomorphism any uniformly hyperbolic set has zero Lebesgue measure, unless it coincides with M; see [11, Theorem 15] for details. However, for the time being, no such density is known in the setting of reversible conservative dynamics. To deal with positive Lebesgue measure hyperbolic sets, we adjusted the result of [55] and the reasoning of [9] to the presence of reversibility.

Given an involution $R \in \mathrm{Diff}^\infty_\mu(M)$ such that

$$\mathrm{Fix}\,(R) = \{x \in M : R(x) = x\}$$

is a submanifold of M with dimension equal to 1, consider the nonempty open set

$$\mathscr{W}_R = \left\{ f \in \mathrm{Diff}^{\,1}_{\mu,R}(M) : f(x) \neq R(x), \text{ for all } x \in M \right\}.$$

As section 6 will discuss, if $f \in \mathscr{W}_R$, then it may be C^1-approximated by a diffeomorphism $g \in \mathrm{Diff}^{\,2}_{\mu,R}(M)$, for which any compact hyperbolic set is M or has zero measure. With due regard to these constraints, we will prove the following generic characterization.

Theorem A. *There exists a C^1-residual $\mathscr{R}_R \subset \mathscr{W}_R$ whose diffeomorphisms are either Anosov or their Lyapunov exponents vanish Lebesgue almost everywhere.*

As the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the only surface that may support an Anosov diffeomorphism [22, 39], we may add that:

Corollary 1. *If $M \neq \mathbb{T}^2$, then a C^1-generic $f \in \mathscr{W}_R$ has zero Lyapunov exponents at Lebesgue almost every point.*

The set $\mathscr{W}_R$ deserves a further comment. If $f \in \mathscr{W}_R$, then the involution $R \circ f$ has no fixed points since

$$(R \circ f)(x) = x \;\Leftrightarrow\; R(R \circ f)(x) = R(x) \;\Leftrightarrow\; f(x) = R(x)$$

so $\mathscr{W}_R \neq \emptyset$ only on manifolds that support fixed point free involutions; [18, 27] contain more information on necessary and sufficient conditions on the manifold for the existence of those involutions. For instance, consider the 2-sphere $\mathbb{S}^2$, the antipodal map $A : \mathbb{S}^2 \to \mathbb{S}^2$ and the rotation $R : \mathbb{S}^2 \to \mathbb{S}^2$ of angle π around the north-south axis of $\mathbb{S}^2$. Then both A and R are involutions of $\mathrm{Diff}^1_\mu(\mathbb{S}^2)$, A is fixed point free and the diffeomorphism $f = R \circ A$ belongs to $\mathrm{Diff}^1_{\mu,R}(\mathbb{S}^2)$ since

$$R \circ f = R \circ (R \circ A) = A = f^{-1} \circ R.$$

Moreover, $R \circ f = A$ has no fixed points, so $f \in \mathscr{W}_R$.

As we will see in section 7, a compact hyperbolic set for $f \in \mathrm{Diff}^1_{\mu,R}(M)$ is equal to M or has empty interior. Thus, without the assumption on the common images of f and R that defines the set $\mathscr{W}_R$, we have:

Theorem B. *There exists a C^1-residual $\mathcal{R}_R \subset \mathrm{Diff}^1_{\mu,R}(M)$ such that, if $f \in \mathcal{R}_R$, then either f is Anosov or, for μ-almost every $x \in M$, the Lyapunov exponents at x vanish or else the orbit of x belongs to a compact hyperbolic set with empty interior.*

2. Framework

A substantial amount of information about the geometry of the stable/unstable manifolds may be obtained from the presence of nonzero Lyapunov exponents and the existence of a dominated splitting. Hence, it is of primary importance to understand when one can avoid vanishing exponents or to evaluate their prevalence. Several successful strategies to characterize the generic dynamical behavior are worth mentioning: [10, 12] for volume-preserving diffeomorphisms, sympletic maps and linear cocycles in any dimension; [5, 7] for volume-preserving flows; [6] for Hamiltonians with two degrees of freedom; [28] for diffeomorphisms acting in a three-dimensional manifold. In what follows, we will borrow ideas and techniques from these articles. To extend them to area-preserving reversible dynamical systems, the main difficulties are to handle with hyperbolic pieces which are not the entire manifold (see sections 6 and 7), and to set up a program of C^1 small perturbations which keep invariant both the area-preserving character and the reversibility (details in section 5), and collapse the expanding directions into the contracting ones to such an extent that the upper Lyapunov exponent diminishes (done in section 9).

A dynamical symmetry is a geometric invariant which plays an important role in several applications in physics, from classical [8] and quantum mechanics [48] to thermodynamics [31]. In this context we may essentially distinguish two types of natural symmetries: those which preserve orbits and those which invert them. Much attention has been paid to the former (see, for instance, [20, 24] and references therein); the latter, called reversibility [2–4], is a feature that most prominently arises in Hamiltonian systems and became a useful tool for the analysis of periodic orbits and homoclinic or heteroclinic cycles [19]. The references [32] and [51] present a thorough survey of reversible dynamical systems.

Another, more studied, dynamical invariant by smooth maps is a symplectic form [43]. Our research will be focused on surfaces, where symplectic maps are the area-preserving ones. Some dynamical systems are twofold invariant, both reversible and symplectic, like, for instance, the Chirikov–Taylor standard map [40], the Hénon conservative map [51] and the Arnold cat map [4]. To the best of our knowledge, only a few systematic comparisons between these two settings have been investigated, as in [19] and [53].

3. Preliminaries

In this section, we will discuss some of the consequences of reversibility and summarize a few properties of Lyapunov exponents and dominated splittings.

3.1. Reversibility

Let $R \in \text{Diff}^\infty_\mu(M)$ such that $R = R^{-1}$ and consider $f \in \text{Diff}^1_{\mu,R}(M)$. Geometrically, reversibility means that, applying R to an orbit of f, we get an orbit of f^{-1}. The f-orbit of a point $x \in M$, say $\mathcal{O}(x) = \{f^n(x), x \in \mathbb{Z}\}$, is said to be R-symmetric if $R(\mathcal{O}(x)) = \mathcal{O}(x)$. If x is a fixed point by f and its orbit is R-symmetric, then obviously x is a fixed point by R as well. Yet, in general, the fixed point set of f, say Fix (f), is not preserved by R. Each element of the set Fix $(f) \setminus$ Fix (R) is called asymmetric.

Consider $f, g \in \text{Diff}^1_{\mu,R}(M)$. Then $R \circ f^{-1} = f \circ R$, but $R \circ (f \circ g) = (f^{-1} \circ R) \circ g = (f^{-1} \circ g^{-1}) \circ R = (g \circ f)^{-1} \circ R$, so the set $\text{Diff}^1_{\mu,R}(M)$ endowed with the composition of maps is, in general, not a group. Moreover, if $f \in \text{Diff}^1_{\mu,R}(M)$ is conjugate through h to $g \in \text{Diff}^1_\mu(M)$, then, although $(R \circ h) \circ g = f^{-1}(R \circ h)$, g may be not R-reversible.

Remark 3.1. The space $\text{Diff}^\infty_\mu(M)$ is a Lie group whose differential structure is locally Fréchet [17, 34]. Its subset of involutions

$$\text{Inv}^\infty_\mu = \{R \in \text{Diff}_\mu^\infty(M) : R^2 = Id_M\}$$

is the fixed point set of the continuous group transformation

$$R \in \text{Diff}^\infty_\mu(M) \ \mapsto \ R^{-1},$$

and so Inv^∞_μ is a closed subgroup; therefore, it is also a Lie group (see [17, section 26] or [42]). Moreover, given $R \in \text{Inv}^\infty_\mu$,

$$\text{Diff}^\infty_{\mu,R}(M) = \{f \in \text{Diff}_\mu^\infty(M) : \exists \, U \in \text{Inv}^\infty_\mu : f = R \circ U\}$$

since

- if $f = R \circ U$, for some $U \in \text{Inv}^\infty_\mu$, then $R \circ f = U = (U \circ R) \circ R = f^{-1} \circ R$;
- if $R \circ f = f^{-1} \circ R$, then $U = R \circ f \in \text{Inv}^\infty_\mu$ and $f = R \circ U$.

So, $\text{Diff}^\infty_{\mu,R}(M) = R\,(\text{Inv}^\infty_\mu)$ and $\text{Diff}^\infty_{\mu,R}(M)$ is a Lie pseudogroup.

3.2. Dominated splitting

In the following we will use the canonical norm of a bounded linear map A given by $\|A\| = \sup_{\|v\|=1} \|A\,v\|$. For $f \in \text{Diff}^1(M)$, a compact f-invariant set $\Lambda \subseteq M$ is said to be *uniformly hyperbolic* if there is $m \in \mathbb{N}$ such that, for every $x \in \Lambda$, there is a Df-invariant continuous splitting $T_x M = E^u_x \oplus E^s_x$ such that

$$\|Df^m_x|_{E^s_x}\| \leqslant \frac{1}{2} \ \text{ and } \ \|(Df^m_x)^{-1}|_{E^u_x}\| \leqslant \frac{1}{2}.$$

There are several interesting ways to weaken the definition of uniform hyperbolicity. Here we use the one introduced in [35, 36, 47]. Given $m \in \mathbb{N}$, a compact f-invariant set $\Lambda \subseteq M$ is said to have an *m-dominated splitting* if, for every $x \in \Lambda$, there exists a Df-invariant continuous splitting $T_x\Lambda = E^u_x \oplus E^s_x$ satisfying

$$\|Df^m_x|_{E^s_x}\| \ \|(Df^m_x)^{-1}|_{E^u_x}\|^{-1} \leqslant \frac{1}{2}. \tag{3.1}$$

Observe that, if Λ displays an m-dominated splitting for f, then the same splitting is dominated for f^{-1}. Under a dominating splitting, both sub-bundles may expand or contract, although E^u expands more efficiently than E^s and, if both sub-bundles contract, E^u is less contracting than E^s. Moreover, as happens in the uniform hyperbolicity setting, the angle between these sub-bundles is uniformly bounded away from zero, because the splitting varies continuously with the point and Λ is compact, and the dominated splitting extends to the closure of Λ (see [14] for full details). Within two-dimensional area-preserving diffeomorphisms, hyperbolicity is in fact equivalent to the existence of a dominated splitting [9, Lemma 3.11].

Lemma 3.2. *Consider $f \in \text{Diff}^1_{\mu,R}(M)$ and a closed f-invariant set $\Lambda \subseteq M$ with an m-dominated splitting. Then $R(\Lambda)$ is closed, f-invariant and has an m-dominated splitting as well.*

Proof. Clearly, as R is an involution, $fR(\Lambda) = Rf^{-1}(\Lambda) = R(\Lambda)$. Let $x \in \Lambda$ whose orbit exhibits the decomposition $T_{f^i(x)}M = E^u_{f^i(x)} \oplus E^s_{f^i(x)}$, for $i \in \mathbb{Z}$. Then we also have a Df-invariant decomposition for $R(x)$, namely $T_{f^i(R(x))}M = E^u_{f^i(R(x))} \oplus E^s_{f^i(R(x))}$, for $i \in \mathbb{Z}$, where

$$E^u_{f^i(R(x))} = DR_{f^{-i}(x)}(E^s_{f^{-i}(x)})$$

and

$$E^s_{f^i(R(x))} = DR_{f^{-i}(x)}(E^u_{f^{-i}(x)}).$$

Indeed, for $x \in \Lambda$ and $i \in \mathbb{Z}$, we have

$$\begin{aligned} Df_{f^i(R(x))}(E^s_{f^i(R(x))}) &= Df_{R(f^{-i}(x))}(E^s_{f^i(R(x))}) = Df_{R(f^{-i}(x))}DR_{f^{-i}(x)}(E^u_{f^{-i}(x)}) \\ &= DR_{f^{-i-1}(x)}Df^{-1}_{f^{-i}(x)}(E^u_{f^{-i}(x)}) = DR_{f^{-i-1}(x)}(E^u_{f^{-i-1}(x)}) = E^s_{f^{i+1}(R(x))}, \end{aligned}$$

and a similar invariance holds for the sub-bundle E^u. Therefore, since R is a diffeomorphism in the compact Λ, we deduce that the angle between the sub-bundles at $R(x)$ is bounded away from zero. Finally, notice that

$$\begin{aligned} \|Df^m_{R(x)}|_{E^s_{R(x)}}\| \; \|(Df^m_{R(x)})^{-1}|_{E^u_{R(x)}}\|^{-1} &= \|Df^m_{R(x)}|_{DR_x(E^u_x)}\| \; \|(Df^m_{R(x)})^{-1}|_{DR_x(E^s_x)}\|^{-1} \\ &= \|R(Df^m_x)^{-1}|_{E^u_x}\| \; \|R(Df^m_x|_{E^s_x})\|^{-1} \\ &= \|(Df^m_x)^{-1}|_{E^u_x}\| \; \|Df^m_x|_{E^s_x}\|^{-1} \overset{(3.1)}{\leqslant} \frac{1}{2}. \end{aligned}$$

□

3.3. Lyapunov exponents

By Oseledets' theorem [46], for μ-a.e. point $x \in M$, there is a splitting $T_xM = E^1_x \oplus ... \oplus E^{k(x)}_x$ (called *Oseledets' splitting*) and real numbers $\lambda_1(x) > ... > \lambda_{k(x)}(x)$ (called *Lyapunov exponents*) such that $Df_x(E^i_x) = E^i_{f(x)}$ and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n_x(v^j)\| = \lambda_j(f, x)$$

for any $v^j \in E^j_x \backslash \{\vec{0}\}$ and $j = 1, ..., k(x)$. This allows us to conclude that, for μ-a.e. x,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\det(Df^n_x)| = \sum_{j=1}^{k(x)} \lambda_j(x)\dim(E^j_x), \tag{3.2}$$

which is related to the subexponential decrease of the angle between any two subspaces of the Oseledets splitting along μ-a.e. orbit. Since, in the area-preserving case, we have

$|\det(Df_x^n)| = 1$ for any $x \in M$, by (3.2) we get $\lambda_1(x) + \lambda_2(x) = 0$. Hence either $\lambda_1(x) = -\lambda_2(x) > 0$ or they are both equal to zero. If the former holds for μ-a.e. x, then there are two one-dimensional subspaces E_x^u and E_x^s, associated to the positive Lyapunov exponent $\lambda_1(x) = \lambda_u(x)$ and the negative $\lambda_2(x) = \lambda_s(x)$, respectively. We denote by $\mathscr{O}(f)$ the set of regular points, that is,

$$\mathscr{O}(f) = \{x \in M : \lambda_1(x), \lambda_2(x) \text{ exist}\}$$

by $\mathscr{O}^+(f) \subseteq \mathscr{O}(f)$ the subset of points with one positive Lyapunov exponent

$$\mathscr{O}^+(f) = \{x \in \mathscr{O}(f) : \lambda_1(x) > 0\}$$

and by $\mathscr{O}^0(f) \subseteq \mathscr{O}(f)$ the set of those points with both Lyapunov exponents equal to zero

$$\mathscr{O}^0(f) = \{x \in \mathscr{O}(f) : \lambda_1(x) = \lambda_2(x) = 0\}.$$

So $\mathscr{O}^+(f) = \mathscr{O}(f) \backslash \mathscr{O}^0(f)$. With this notation, we may summarize Oseledets' theorem in the area-preserving reversible setting as:

Theorem 3.3 ([46]). *Let $f \in \mathrm{Diff}^1_{\mu,R}(M)$. For Lebesgue almost every $x \in M$, the limit*

$$\lambda^+(f, x) = \lim_{n\to+\infty} \frac{1}{n} \log \|Df_x^n\|$$

exists and defines a non-negative measurable function of x. For almost any $x \in \mathscr{O}^+$, there is a splitting $E_x = E_x^u \oplus E_x^s$ which varies measurably with x and satisfies:

$$v \in E_x^u \backslash \{\vec{0}\} \ \Rightarrow \ \lim_{n\to\pm\infty} \frac{1}{n} \log \|Df_x^n(v)\| = \lambda^+(f, x).$$

$$v \in E_x^s \backslash \{\vec{0}\} \ \Rightarrow \ \lim_{n\to\pm\infty} \frac{1}{n} \log \|Df_x^n(v)\| = -\lambda^+(f, x).$$

$$\vec{0} \neq v \notin E_x^u \cup E_x^s \ \Rightarrow \ \lim_{n\to+\infty} \frac{1}{n} \log \|Df_x^n(v)\| = \lambda^+(f, x) \text{ and } \lim_{n\to-\infty} \frac{1}{n} \log \|Df_x^n(v)\| = -\lambda^+(f, x).$$

The next result shows the natural rigidity on the Lyapunov exponents of reversible diffeomorphisms.

Lemma 3.4. *Let $f \in \mathrm{Diff}^1_{\mu,R}(M)$. If $x \in \mathscr{O}^+$ has a decomposition $E_x^u \oplus E_x^s$, then*

(a) $R(x) \in \mathscr{O}^+$.
(b) The Oseledets splitting at $R(x)$ is $E^u_{R(x)} \oplus E^s_{R(x)}$ with $E^u_{R(x)} = DR_x(E_x^s)$, $E^s_{R(x)} = DR_x(E_x^u)$.
(c) $\lambda^+(f, R(x)) = \lambda^+(f, x)$ and $\lambda^-(f, R(x)) = \lambda^-(f, x) = -\lambda^+(f, x)$.
(d) If $x \in \mathscr{O}^0$, then $R(x) \in \mathscr{O}^0$.

Proof. Assume that $x \in \mathscr{O}^+$ and let $v \in E_x^u \backslash \{\vec{0}\}$. Consider the direction $v' = DR_x(v) \in T_{R(x)}M$ and let us compute the Lyapunov exponent at $R(x)$ along this direction:

$$\begin{aligned}
\lambda(f, R(x), v') &= \lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n_{R(x)}(v')\| = \lim_{n\to\pm\infty} \frac{1}{n} \log \|D(R \circ f^{-n} \circ R)_{R(x)}(v')\| \\
&= \lim_{n\to\pm\infty} \frac{1}{n} \log \|DR_{f^{-n}(R^2(x))} Df^{-n}_{R^2(x)} DR_{R(x)} DR_x(v)\| \\
&= \lim_{n\to\pm\infty} \frac{1}{n} \log \|DR_{f^{-n}(x)} Df_x^{-n}(v)\| \\
&= \lim_{n\to\pm\infty} \frac{1}{n} \log \|Df_x^{-n}(v)\| = -\lim_{n\to\pm\infty} \frac{1}{-n} \log \|Df_x^{-n}(v)\| = -\lambda^+(f, x, v).
\end{aligned}$$

Thus $R(x) \in \mathscr{O}^+$. The other properties are deduced similarly. □

3.4. Integrated Lyapunov exponent

It was proved in [9] that, when $\mathrm{Diff}^1_\mu(M)$ is endowed the C^1-topology and $[0, +\infty[$ has the usual distance, then the function

$$\begin{array}{rccc} \mathscr{L}\colon & \mathrm{Diff}^1_\mu(M) & \longrightarrow & [0, +\infty[\\ & f & \longrightarrow & \int_M \lambda^+(f, x)\, \mathrm{d}\mu \end{array}$$

is upper semicontinuous. This is due to the fact that $\mathscr{L}$ is the infimum of continuous functions, namely

$$\mathscr{L}(f) = \inf_{n\in\mathbb{N}} \frac{1}{n} \int_M \log \|Df^n_x\| \mathrm{d}\mu. \tag{3.3}$$

Clearly, the same holds for the restriction of $\mathscr{L}$ to $\mathrm{Diff}^1_{\mu,R}(M)$. Therefore, there exists a residual set in $\mathrm{Diff}^1_{\mu,R}(M)$ for which the map $\mathscr{L}$ is continuous. Now, the upper semicontinuity of $\mathscr{L}$ implies that $\mathscr{L}^{-1}([0, \tau[)$ is C^1-open for any $\tau > 0$; hence

$$\mathscr{A}_\tau = \left\{ f \in \mathrm{Diff}^1_{\mu,R}(M)\colon \mathscr{L}(f) < \tau \right\}$$

is C^1-open.

3.5. (R, f)-free orbits

Given a subset X of M, we say that X is (R, f)*-free* if $f(x) \neq R(y)$ for all $x, y \in X$.

Lemma 3.5. *Let $f \in \mathrm{Diff}^1_{\mu,R}(M)$. If $x \in M$ and $R(x)$ does not belong to the f-orbit of x, then this orbit is (R, f)-free.*

Proof. Let us assume that there exist $i, j \in \mathbb{Z}$ such that $f^i(x) = R(f^j(x))$. Then $f^i(x) = f^{-j}(R(x))$ and $f^{j+i}(x) = R(x)$, which contradicts the assumption. □

Proposition 3.6. *There is a residual $\mathscr{D} \subset \mathrm{Diff}^1_{\mu,R}(M)$ such that, for any $f \in \mathscr{D}$, the set of orbits outside* Fix (R) *which are not (R, f)-free is countable.*

Proof. Since f and R are smooth maps defined on M, by Thom's transversality theorem [25] there exists an open and dense set $\mathscr{D}_1 \subset \mathrm{Diff}^1_{\mu,R}(M)$ such that, if $f \in \mathscr{D}_1$, the graphs of f and R are transverse submanifolds of $M \times M$, intersecting only at isolated points. Therefore, we may find a neighborhood of each intersection point where it is unique. By compactness of M, we conclude that generically the graphs of f and R intersect at a finite number of points (and this is an open property). Denote by $\mathcal{F}_1 = \{x_{1,j}\}_{j=1}^{k_1}$ the set of points such that $f(x_{1,j}) = R(x_{1,j})$.

Analogously, for $n \in \mathbb{N}$, let $\mathscr{D}_n \subset \mathrm{Diff}^1_{\mu,R}(M)$ be the open and dense set of diffeomorphisms $f \in \mathrm{Diff}^1_{\mu,R}(M)$ such that the graphs of $\{f^{-n}, ..., f^{-1}, f, f^2, ..., f^n\}$ and R are transverse, and denote by $\mathcal{F}_n = \{x_{n,j}\}_{j=1}^{k_n}$ the finite set of f-orbits satisfying $f^i(x_{n,j}) = R(x_{n,j})$ for some $j \in \{1, ..., k_n\}$ and $i \in \{-n, ..., -1, 1, ..., n\}$. Finally, define

$$\mathscr{D} = \bigcap_{n\in\mathbb{N}} \mathscr{D}_n$$

and the countable set of (R, f)-not-free orbits by

$$\mathcal{F} = \bigcup_{n\in\mathbb{N},\, m\in\mathbb{Z}} f^m(\mathcal{F}_n).$$

We are left to show that, if $f \in \mathscr{D}$ and $x \in M \backslash [\mathcal{F} \cup \text{Fix}\,(R)]$, then the orbit of x is an (R, f)-free set. Indeed, by construction, for such an x, the iterate $R(x)$ does not belong to the f-orbit of x; thus, by lemma 3.5, this orbit is (R, f)-free. □

Remark 3.7. The previous argument may be performed in $\text{Diff}^{\,r}_{\mu,R}(M)$, for any $r \in \mathbb{N}$.

From this result and the fact that $\dim \text{Fix}\,(R) = 1$, we easily get:

Corollary 3.8. *Generically in* $\text{Diff}^1_{\mu,R}(M)$*, the set of* (R, f)*-free orbits has full Lebesgue measure.*

4. Stability of periodic orbits

Let $R \in \text{Diff}^{\infty}_{\mu,R}(M)$ be an involution such that Fix (R) is a submanifold of M with dimension equal to 1. Consider $f \in \text{Diff}^1_{\mu,R}(M)$. For area-preserving diffeomorphisms, hyperbolicity is an open but not dense property. Indeed, the C^1-stable periodic points are hyperbolic or elliptic; furthermore, in addition to openness, the area-preserving diffeomorphisms whose periodic points are either elliptic or hyperbolic are generic [52]. A version of Kupka–Smale's theorem for reversible area-preserving diffeomorphisms has been established in [19]. It certifies that, for a generic f in $\text{Diff}^1_{\mu,R}(M)$, all the periodic orbits of f with given period are isolated.

Theorem 4.1 ([19]). *Let* $\mathscr{S}_k = \{f \in \text{Diff}^r_{\mu,R}(M)$: every periodic point of period less or equal to k is elementary$\}$ *and*

$$\mathscr{S} = \bigcap_{k \in \mathbb{N}} \mathscr{S}_k.$$

Then, for each $k,\ r \in \mathbb{N}$*, the set* $\mathscr{S}_k$ *is residual in* $\text{Diff}^r_{\mu,R}(M)$*. Thus,* $\mathscr{S}$ *is* C^r*-residual as well.*

Therefore a generic $f \in \text{Diff}^r_{\mu,R}(M)$ has countably many periodic points, a finite number for each possible period.

Corollary 4.2. *There is a residual* $\mathscr{E}_r \subset \text{Diff}^r_{\mu,R}(M)$ *such that, for any* $f \in \mathscr{E}_r$*, the set of periodic points of* f *has Lebesgue measure zero.*

In [54], the author states generic properties of reversible vector fields on three-dimensional manifolds. To convey those features to diffeomorphisms on surfaces, we take the vector field defined by suspension of a reversible diffeomorphism $f : M \to M$, without losing differentiability [50], acting on a quotient manifold $\bar{M} = M \times \mathbb{R}/\sim$ where it is transversal to the section $M \times 0/\sim$. This vector field is reversible with respect to the involution obtained by projecting $R \times (-Id)$, whose fixed point set is still a submanifold of dimension 1 of $\bar{M}$. This way, we deduce from [54] that:

Proposition 4.3. *A generic* $f \in \text{Diff}^r_{\mu,R}(M)$ *has only asymmetric fixed points and all its periodic orbits are hyperbolic or elliptic.*

5. Local perturbations

Let $R \in \text{Diff}^{\infty}_{\mu,R}(M)$ be an involution such that Fix (R) is a submanifold of M with dimension equal to 1. Consider $f \in \text{Diff}^1_{\mu,R}(M)$. Given $p \in M$, if we differentiate the equality $R \circ f = f^{-1} \circ R$ at p, then we get $DR_{f(p)} \circ Df_p = Df^{-1}_{R(p)} \circ DR_p$, a linear constraint between

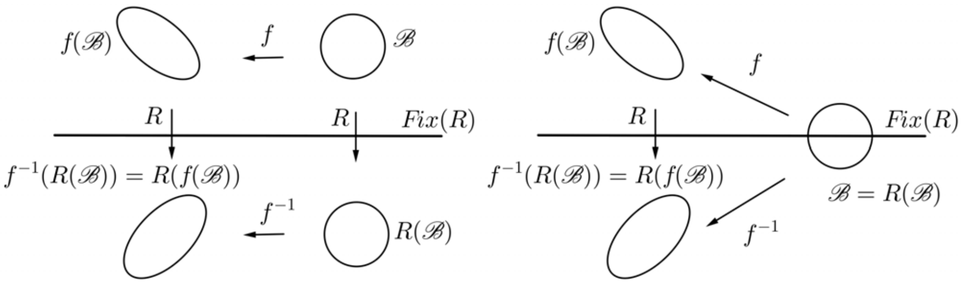

Figure 1. Illustration of the first perturbation lemma: $\mathscr{B}$ is the ball $B(x, r)$.

four matrices of SL(2, $\mathbb{R}$), two of which are also linked through the equality $R^2 = Id$. As the dimension SL(2, $\mathbb{R}$) is 3, there is some room to perform nontrivial perturbations.

In this section, we set two perturbation schemes that are the basis of the following sections. The first one describes a local small C^1 perturbation within reversible area-preserving diffeomorphisms in order to change a map and its derivative at a point, provided x has an (R, f)-free nonperiodic orbit of f. The second one is inspired by Franks' lemma ([21]), proved for dissipative diffeomorphisms, and allows locally small abstract perturbations to be performed, within the reversible setting, on the derivative along a segment of an orbit of an area-preserving diffeomorphism. These perturbation lemmas have been proved in the C^1 topology only, for reasons appositely illustrated in [14, 49].

5.1. First perturbation lemma

Consider $f \in \text{Diff}^{\,1}_{\mu,R}(M)$ and take a point $x \in M$ whose orbit by f is not periodic and $f(x) \neq R(x)$. Notice that those points exist if $f \in \mathscr{D}_1 \cap \mathscr{E}_1$, as described in Proposition 3.6 and Corollary 4.2. We will see how to slightly change f and Df at a small neighborhood of x without losing reversibility.

Denote by $B(x, \rho)$ the open ball centered at x with radius ρ and by C the union $B(x, \rho) \cup R(f(B(x, \rho)))$.

Lemma 5.1. *Given* $f \in \text{Diff}^1_{\mu,R}(M)$ *and* $\eta > 0$*, there exist* $\rho > 0$ *and* $\zeta > 0$ *such that, for any* $x \in M$ *satisfying* $f(x) \neq R(x)$ *and for every* C^1 *area-preserving diffeomorphism* $h: M \to M$ *coinciding with the identity in* $M \backslash B(x, \rho)$ *and* ζ*-*C^1*-close to the identity, there exists* $g \in \text{Diff}^1_{\mu,R}(M)$ *which is* η*-*C^1*-close to* f *and such that* $g = f$ *outside* C *and* $g = f \circ h$ *in* $B(x, \rho)$.

Proof. Using the uniform continuity of f on the compact M and the fact that f is C^1, we may choose $\tau > 0$ such that, each time the distance between two points z and w of M is smaller than τ, then the distance between their images by f, the norm of the difference of the linear maps Df_z and Df_w and the norm of the difference of the linear maps DR_z and DR_w are smaller than $\min\left\{\frac{\eta}{2}, \frac{\eta}{2\,\|f\|_{C^1}\,\|R\|_{C^1}}\right\}$.

As $f(x) \neq R(x)$, calling on the continuity of both f and R we may find $0 < \rho < \tau$ such that the open ball $B(x, \rho)$ satisfies $f(B(x, \rho)) \cap R(B(x, \rho)) = \emptyset$ (or, equivalently, $B(x, \rho) \cap R(f(B(x, \rho))) = \emptyset$). Moreover, if x is not a fixed point of f, we may choose ρ so that $B(x, \rho) \cap f(B(x, \rho)) = \emptyset$ (see figure 1).

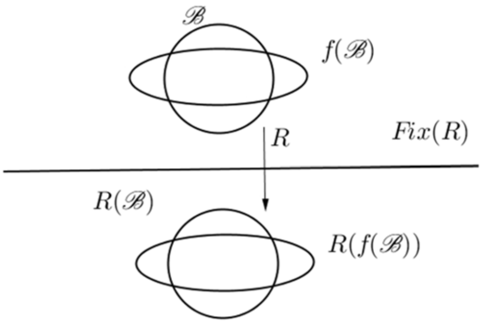

Figure 2. Illustration of the 1st perturbation lemma: x is fixed by f.

Consider the estimate

$$\zeta = \frac{1}{2}\min\left\{\tau, \frac{\eta}{2\max\{\ \|f\|_{C^1}(\|R\|_{C^1})^2, \|f\|_{C^1}\ \}}\right\}$$

and take a C^1 area-preserving diffeomorphism $h: M \to M$ equal to the identity in $M \setminus B(x, \rho)$ and ζ-C^1-close to the identity. If $x \notin \text{Fix}\,(f)$, define $g : M \to M$ by

$$g(t) = \begin{cases} f(t) & \text{if } t \notin C; \\ f \circ h\,(t) & \text{if } t \in B(x, \rho); \\ R \circ h^{-1} \circ f^{-1} \circ R\,(t) & \text{if } t \in R(f(B(x, \rho))); \\ f(t) & \text{if } t \in R(B(x, \rho)) \cup f(B(x, \rho)). \end{cases}$$

Otherwise, if $f(x) = x$ as illustrated in figure 2, let $g : M \to M$ be given by

$$g(t) = \begin{cases} f(t) & \text{if } t \notin C; \\ f \circ h\,(t) & \text{if } t \in B(x, \rho); \\ R \circ h^{-1} \circ f^{-1} \circ R\,(t) & \text{if } t \in R(f(B(x, \rho))). \end{cases}$$

We are left to confirm that $g \in \text{Diff}^1_{\mu,R}(M)$ and is η-C^1-close to f. We begin by showing that the equality $R \circ g = g^{-1} \circ R$ holds. If $y \notin B(x, \rho) \cup f(B(x, \rho)) \cup R(B(x, \rho)) \cup Rf(B(x, \rho))$, then $R(y)$ is also out of this union and, therefore, $g(y) = f(y)$ and $g^{-1}(R(y)) = f^{-1}(R(y))$. Hence $R(g(y)) = R(f(y)) = f^{-1}(R(y)) = g^{-1}(R(y))$. If $y \in B(x, \rho)$, then $R(y) \in R(B(x, \rho))$ and so

$$R(g(y)) = R(f \circ h)(y) = R(f \circ h)(R \circ R)(y) = (R \circ h^{-1} \circ f^{-1} \circ R)^{-1}(R(y)) = g^{-1}(R(y)).$$

Analogous computations prove the reversibility condition on $R(f(B(x, \rho)))$. Finally, if $y \in R(B(x, \rho))$, then $R(y) \in B(x, \rho)$ and $R(g(y)) = R(f(y)) = f^{-1}(R(y)) = g^{-1}(R(y))$. Similar reasoning works for $y \in f(B(x, \rho))$.

Let us now verify that g is η-C^1-close to f.

(a) C^0-approximation.

By definition, the differences between the values of g and f are bounded by the distortion the map h induces on the ball $B(x, \rho)$ plus the effect that deformation creates on the first iterate by f and the action of R (which preserves distances locally). Now, for $z \in B(x, \rho)$, the distance between $h(z)$ and z is small than ζ, which is smaller than τ. So, by the choice of τ, the distance between $g(z)$ and $f(z)$ is smaller than η.

(b) C^1-approximation.

We have to estimate, for $z \in B(x, \rho)$, the norm $\|Df_z - Dg_z\| = \|Df_z - Df_{h(z)}(Dh_z)\|$ and, for $z \in R(f(B(x, \rho)))$, $\|Df_z - D(R \circ h^{-1} \circ f^{-1} \circ R)_z\|$. Concerning the former, from the choices of τ and ζ, we have

$$\begin{aligned}\|Df_z - Df_{h(z)}Dh_z\| &\leqslant \|Df_z - Df_{h(z)}\| + \|Df_{h(z)} - Df_{h(z)}\,Dh_z\| \\ &\leqslant \frac{\eta}{2} + \|f\|_{C^1}\,\|Id_z - Dh_z\| \leqslant \frac{\eta}{2} + \|f\|_{C^1}\,\zeta < \eta.\end{aligned}$$

Regarding the latter,

$$\begin{aligned}\|Df_z - D(R \circ h^{-1} \circ R \circ f)_z\| &= \|Df_z - D(R \circ h^{-1} \circ R)_{f(z)}\,Df_z\| \\ &\leqslant \|Id_{f(z)} - D(R \circ h^{-1} \circ R)_{f(z)}\|\,\|f\|_{C^1} \\ &= \|DR_{R(f(z))}\,DR_{f(z)} - D(R \circ h^{-1})_{R(f(z))}\,DR_{f(z)}\|\,\|f\|_{C^1} \\ &\leqslant \|DR_{R(f(z))} - D(R \circ h^{-1})_{R(f(z))}\|\,\|f\|_{C^1}\,\|R\|_{C^1} \\ &\leqslant \|DR_{R(f(z))} - DR_{h^{-1}(R(f(z)))}\,Dh^{-1}_{R(f(z))}\|\,\|f\|_{C^1}\,\|R\|_{C^1} \\ &\leqslant \frac{\eta}{2} + \|Id_{R(f(z))} - Dh^{-1}_{R(f(z))}\|\,\|f\|_{C^1}\,(\|R\|_{C^1})^2 \\ &\leqslant \frac{\eta}{2} + \zeta\,\|f\|_{C^1}\,(\|R\|_{C^1})^2 < \eta.\end{aligned}$$

□

5.2. Second perturbation lemma

We will now consider an area-preserving reversible diffeomorphism, a finite set in M and an abstract tangent action that performs a small perturbation of the derivative along that set. Then we will search for an area-preserving reversible diffeomorphism, C^1 close to the initial one, whose derivative equals the perturbed cocycle on those iterates. To find such a perturbed diffeomorphism, we will benefit from the argument, suitable for area-preserving systems, presented in [13]. But before proceeding, let us analyze an example.

Example 5.2. Take the linear involution R induced on the torus by the linear matrix $A(x, y) = (x, -y)$, and consider the diffeomorphism $f = R$. Clearly, $R \circ f = f^{-1} \circ R$. The set of fixed points of f is the projection on the torus of $[0, 1] \times \{0\} \cup [0, 1] \times \{\frac{1}{2}\}$, and so it is made of two closed curves. All the other orbits of f are periodic with period 2. Given $p \notin \mathrm{Fix}\,(f)$, we have $Df_p = Df_{f(p)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Now, if $\eta > 0$ and

$$L(p) = \begin{pmatrix} 1+\eta & 0 \\ \eta & -\frac{1}{1+\eta} \end{pmatrix} = L(f(p))$$

we claim that no diffeomorphism g on the torus satisfying

$$Dg_p = L(p), \quad Dg_{f(p)} = L(f(p)) \quad \text{and} \quad g(p) = f(p)$$

can be R-reversible. Indeed, differentiating the equality $R \circ g = g^{-1} \circ R$ at p, we would get

$$A \circ Dg_p = Dg^{-1}_{R(p)} \circ A = Dg^{-1}_{f(p)} \circ A$$

that is,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1+\eta & 0 \\ \eta & -\frac{1}{1+\eta} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\eta} & 0 \\ \eta & -1+\eta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

thus

$$\begin{pmatrix} 1+\eta & 0 \\ -\eta & \frac{1}{1+\eta} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\eta} & 0 \\ \eta & 1+\eta \end{pmatrix}$$

which yields $\eta = 0$. This example evinces the need to impose some restrictions on the set where we wish to carry the perturbation.

Lemma 5.3. *Fix an involution R and $f \in \mathrm{Diff}^1_{\mu,R}(M)$. Let $X = \{x_1, x_2, \ldots, x_k\}$ be a finite (R, f)-free set of distinct points in M. Denote by $V = \oplus_{x\in X} T_x M$ and $V' = \oplus_{x\in X} T_{f(x)} M$ and let $P: V \to V'$ be a map such that, for each $x \in X$, $P(x) \in \mathrm{SL}(T_x M \to T_{f(x)} M)$. For every $\eta > 0$, there is $\zeta > 0$ such that, if $\|P - Df\| < \zeta$, then there exists $g \in \mathrm{Diff}^1_{\mu,R}(M)$ which is η-C^1-close to f and satisfies $Dg_x = P|_{T_x M}$ for every $x \in X$. Moreover, if $K \subset M$ is compact and $K \cap X = \emptyset$, then g can be found so that $g = f$ in K.*

Proof. Given $\eta > 0$, take the values of $\rho > 0$ and $\zeta > 0$ associated to $\frac{\eta}{k}$ by lemma 5.1, and note that each element of X satisfies the hypothesis of this lemma. Starting with x_1 and using Franks' lemma for area-preserving diffeomorphisms [13], we perform a perturbation of f supported in $B(x_1, \rho_1)$, where $0 < \rho_1 < \rho$ is sufficiently small, obtaining $G_1 \in \mathrm{Diff}^1_\mu(M)$ such that $DG_{1_{x_1}} = P(x_1)$ and G_1 is ζ-close to f.

Define $h_1 = f^{-1} \circ G_1$. The C^1 diffeomorphism h_1 is area-preserving, equal to the identity in $M \backslash B(x_1, \rho_1)$ and ζ-C^1-close to the identity. So, by lemma 5.1, there is $g_1 \in \mathrm{Diff}^1_{\mu,R}(M)$ which is $\frac{\eta}{k}$-C^1-close to f, $g_1 = f$ outside $C_1 = B(x_1, \rho_1) \cup R(f(B(x_1, \rho_1)))$ and $g_1 = f \circ h_1 = G_1$ inside $B(x_1, \rho_1)$.

We proceed repeating the above argument for x_2 and g_1 just constructed, taking care to choose an open ball centered at x_2, with radius $0 < \rho_2 < \rho$, such that $C_2 = B(x_2, \rho_2) \cup R(f(B(x_2, \rho_2)))$ does not intersect C_1: this is a legitimate step according to the constraints X has to fulfill. Applying again [13], we do a perturbation on g_1 supported in $B(x_2, \rho_2)$, which yields $G_2 \in \mathrm{Diff}^1_\mu(M)$ such that $DG_{2_{x_2}} = P(x_2)$ and G_2 is ζ-close to g_1. Therefore, the C^1 diffeomorphism $h_2 = g_1^{-1} \circ G_2$ is area-preserving, equal to the identity in $M \backslash B(x_2, \rho_2)$ and ζ-C^1-close to the identity. So, by lemma 5.1, there is $g_2 \in \mathrm{Diff}^1_{\mu,R}(M)$ which is $\frac{\eta}{k}$-C^1-close to g_1, thus $\frac{2\eta}{k}$-C^1-close to f, satisfies $g_2 = g_1$ outside C_2 and is such that $g_2 = g_1 \circ h_2 = G_2$ inside $B(x_2, \rho_2)$.

In a similar way we do the remaining $k - 2$ perturbations till we have taken into consideration all the elements of X. At the end of this process we obtain a diffeomorphism $g \in \mathrm{Diff}^1_{\mu,R}(M)$ which is η-C^1-close to f and differs from f only at $C = M \backslash \bigcup_{i=1}^k C_i$.

Surely, if K is compact and $K \cap X = \emptyset$, then C may be chosen inside the complement of K. □

6. Smoothing out a reversible diffeomorphism

In this section we will show that a C^1 reversible area-preserving diffeomorphism of the open and dense set $\mathscr{D}_1$ (see proposition 3.6) can be smoothed as an R-reversible area-preserving C^∞ diffeomorphism up to a set of arbitrarily small Lebesgue measure. The argument follows the guidelines of [55], although adapted to comply with reversibility.

Proposition 6.1. *Given $f \in \mathscr{D}_1$, a neighborhood $\mathcal{V}_f \subset \mathrm{Diff}^1_{\mu,R}(M)$ of f and $\epsilon > 0$, there exist $g \in \mathrm{Diff}^1_{\mu,R}(M)$, a compact $Z \subset M$ and an open neighborhood V_Z of Z such that:*

(a) $g \in \mathcal{V}_f$.
(b) g is C^∞ in V_Z.
(c) $\mu(M \setminus Z) < \epsilon$.
(d) $\mu(V_Z \setminus Z) < \epsilon/2$.
(e) $Z = M$, if $f(x) \neq R(x)$ for all $x \in M$.

Proof. For the sake of completeness, we will reconstruct the main steps of the proof of [55], highlighting the differences forced by the reversibility. Some details in the argument of [55] are easier in our context of compact manifolds (for instance, [55] addressed locally compact manifolds), while others demand full attention to reversibility.

6.1. Construction of Z

Assume that f is not C^2 and denote by $\mathcal{P} = \{x_1, \cdots, x_k\}$ the finite set $\{x \in M : f(x) \neq R(x)\}$. For arbitrary $\epsilon > 0$, take the open covering of $\mathcal{P}$ defined by $(B(x_i, r(\epsilon)))_{\{i=1,\cdots,k\}}$, where $r(\epsilon)$ is chosen small enough so that $B(x_i, r(\epsilon)) \cap R(B(x_i, r(\epsilon))) = \emptyset$. Then, let $\mathcal{B}$ be the set

$$\mathcal{B} = \bigcup_{i=1}^{k} B(x_i, r(\epsilon)) \cup R\left(\bigcup_{i=1}^{k} B(x_i, r(\epsilon))\right)$$

and consider the complement of $\mathcal{B}$ in M

$$Z = M \backslash \mathcal{B}.$$

Observe that we may select a value of $r(\epsilon)$ sufficiently small to guarantee that $\mu(M\backslash Z) < \epsilon/2$. Moreover, if $f(x) \neq R(x)$ for all $x \in M$, then $Z = M$.

6.2. Coverings and charts

As f is a C^1 area-preserving map, for each $x \in M$ there are symplectic charts, say $(U(x), \varphi_1)$ and $(V(x), \varphi_2)$, such that $x \in U(x)$, $f(x) \in V(x)$, $f(U(x)) \subset V(x)$, φ_1 and φ_2 are C^∞ local symplectomorphisms and

$$\psi = \varphi_2 \circ f \circ \varphi_1^{-1} \quad \in C^1\left[\varphi_1(U(x)), \varphi_2(V(x))\right]$$

where $C^1[A, B]$ stands for the set of C^1 maps from $A \subset \mathbb{R}^2$ to $B \subset \mathbb{R}^2$. In what follows, the map ψ will be also addressed as

$$(u, v) \mapsto \psi(u, v) := (\xi, \eta).$$

Due to the compactness of Z we may take a finite open covering of Z made of these symplectic charts

$$\mathcal{U}_1 = (U_i)_{\{i=1,\cdots,\ell\}}$$

such that $U_i \cap \mathcal{P} = \emptyset$, for all i, and the diameter of $\mathcal{U}_1$ is arbitrarily small. Denote by $\mathcal{U}$ the open covering

$$\mathcal{U}_f = \mathcal{U}_1 \cup R(f(\mathcal{U}_1)) \quad \text{where} \quad R(f(\mathcal{U}_1)) = (R(f(U_i)))_{\{i=1,\cdots,\ell\}}\,.$$

As $f(x) \neq R(x)$ for all $x \in U_i$ and every $i \in \{1, \cdots, \ell\}$, we may find U_i small enough so that U_i is disjoint from $R(f(U_i))$. In particular, denoting by dist_{C^0} the Hausdorff distance we have:

$$\min_{i \in \{1,\cdots,\ell\}} \{\mathrm{dist}_{C^0}\,(\overline{U_i}, \overline{R(f(U_i))})\} > 0.$$

Moreover, we may assume that the neighborhood $\mathcal{V}_f$ is small enough so that $\mathcal{V}_f \subset \mathscr{D}_1$ and

$$h \in \mathcal{V}_f \quad \Rightarrow \quad \min_{i \in \{1,\cdots,\ell\}} \{\mathrm{dist}_{C^0}(\overline{U_i}, \overline{R(h(U_i))}) > 0$$

which ensures, in particular, that

$$h \in \mathcal{V}_f \quad \Rightarrow \quad h(x) \neq R(x), \qquad \forall x \in U_i, \ \forall i \in \{1, \cdots, \ell\}.$$

6.3. Generating functions

The first part of Zehnder's argument (see [55, Lemma 1]) characterizes locally symplectic maps between symplectic manifolds of the same dimension by means of the so-called generating functions. In particular, it proves that a diffeomorphism $f : M \to M$ is symplectic if and only if, given $x \in M$, there exist a symplectic chart $(U(x), \varphi_1)$ at x and a C^2-function $S : \varphi_1(U(x)) \to \mathbb{R}$ such that, for all $(t, \eta) \in \varphi_1(U(x))$, we have

$$\xi = \frac{\partial S(t,\eta)}{\partial \eta}, \qquad y = \frac{\partial S(t,\eta)}{\partial t} \qquad \text{and} \qquad \det(d_t\, d_\eta\, S(t,\eta)) \neq 0.$$

In this case, we say that the map ψ is generated by S, or that S is a generating function for ψ.

6.4. Smoothing locally in $\mathbb{R}^2$

The second section of Zehnder's reasoning is devoted to smooth a generating function; see [55, Lemma 2]. Let W_1, W_2 be open sets of $\mathbb{R}^2$ and $\psi : W_1 \to W_2$ be a map generated by S. Given $\delta > 0$ and $z \in W_1$, take small open subsets $W_1^{(1)}(z)$, $W_1^{(2)}(z)$ and $W_1^{(3)}(z)$ containing z such that

$$W_1^{(3)}(z) \subset W_1^{(2)}(z) \subset W_1^{(1)}(z) \subset \overline{W_1^{(1)}(z)} \subset W_1$$

and so that inside $W_1^{(1)}(z)$ we can construct a C^2-map

$$S_1 = S - \omega_2\left(\omega_1\, S - X_\delta^\star(\omega_1\, S)\right),$$

where ω_1 and ω_2 are real smooth maps on $W_1^{(1)}(z)$ satisfying $\omega_1 \equiv 1$ on $\overline{W_1^{(3)}(z)}$, $\omega_2 \equiv 1$ on $W_1^{(2)}(z)$ and

$$\|\omega_2\,(\omega_1\, S - X_\delta^\star(\omega_1\, S))\|_{C^2} < \delta.$$

Here $X_\delta^\star(\omega_1\, S)$ stands for the δ-Friedrichs mollifier associated to $\omega_1\, S$; see [23] and references therein.

By construction, the map S_1 is C^1-δ-close to S and has the following properties:

(a) S_1 is C^∞ in $W_1^{(3)}(z)$;
(b) S_1 is C^2 in $W_1^{(2)}(z) \setminus W_1^{(3)}(z)$;
(c) $S_1 \equiv S$ in $W_1^{(1)}(z) \setminus W_1^{(2)}(z)$.

Let the map ψ_1, defined on $W_1^{(1)}(z)$, be generated by S_1. Then:

(a) ψ_1 is C^∞ on $W_1^{(3)}(z)$;
(b) ψ_1 is C^2 on $W_1^{(1)}(z)$;
(c) ψ_1 coincides with ψ on $W_1^{(1)}(z) \setminus \overline{W_1^{(2)}(z)}$.

6.5. Smoothing locally in M

The third step in Zehnder's proof (see [55, page 834]) is to smooth out the diffeomorphism f in U_i, for any $i \in \{1, \cdots, \ell\}$, by locally smoothing its generating function. As $f(x) \neq R(x)$ for all x in any element of the open covering $\mathcal{U}_f$ of Z, we can perform a balanced perturbation, as explained in lemma 5.1, in order to guarantee that the resulting diffeomorphism is still R-reversible: each time we smooth in U_i, we also induce smoothness in $R(f(U_i))$. Let us check this procedure in more detail.

Take the element U_1 of the covering $\mathcal{U}_f$ and fix $x \in U_1$. Consider $z = \varphi_1(x)$ and charts at x, say $U_1^{(1)}(x)$, $U_1^{(2)}(x)$ and $U_1^{(3)}(x)$, such that

$$U_1^{(3)}(x) \subset U_1^{(2)}(x) \subset U_1^{(1)}(x) \subset \overline{U_1^{(1)}(x)} \subset U_1,$$

$$\mu\left(U_1 \setminus U_1^{(3)}(x)\right) < \epsilon/(4\,\ell)$$

and so that, on $U_1^{(1)}(x)$, every symplectic diffeomorphism in $\mathcal{V}_f$ is given by a generating function as described before and the previous local construction can be carried out, for a given $\delta > 0$, with respect to

$$W_1^{(3)}(z) = \varphi_1(U_1^{(3)}(x)) \subset W_1^{(2)}(z) = \varphi_1(U_1^{(2)}(x)) \subset W_1^{(1)}(z) = \varphi_1(U_1^{(1)}(x)).$$

Define the C^1 area-preserving diffeomorphism $f_1 : M \to M$ as follows:

$$f_1(u) = \begin{cases} f(u) & \text{if } u \in M \setminus \overline{U_1^{(2)}(x)} \\ \varphi_2^{-1} \circ \psi_1 \circ \varphi_1(u) & \text{if } u \in U_1^{(1)}(x) \end{cases}$$

Notice that f_1 is C^2 on $U_1^{(2)}(x)$, is C^∞ in $U_1^{(3)}(x)$ and satisfies $f_1(U_1) = f(U_1)$. Besides, δ could have been chosen small enough so that $f_1 \in \mathcal{V}_f$.

Now, as $U_1 \cap Rf(U_1) = \emptyset$ and $R \in \text{Diff}^\infty_\mu(M)$, we use the method explained in lemma 5.1 to change f_1 into a diffeomorphism g_1, which is R-reversible, C^1, area-preserving, C^∞ in $U_1^{(3)}(x) \cup Rf(U_1^{(3)}(x))$, C^2 on $U_1^{(2)}(x) \cup Rf(U_1^{(2)}(x))$ and satisfies $g_1(U_1) = f(U_1)$. It is defined by

$$g_1(u) = \begin{cases} f(u) & \text{if } u \in M \setminus (U_1 \cup Rf(U_1)) \\ f_1(u) & \text{if } u \in U_1 \\ R \circ f_1^{-1} \circ R(u) & \text{if } u \in Rf(U_1) \end{cases}$$

Observe that, by definition, $g_1 \in \mathcal{V}_f$.

6.6. Construction of g and V_Z

Afterwards, we consider the open chart U_2 and find, in a similar way,

- three charts $U_2^{(1)}$, $U_2^{(2)}$ and $U_2^{(3)}$ such that

$$U_2^{(3)} \subset U_2^{(2)} \subset U_2^{(1)} \subset U_2 \quad \text{and} \quad \mu\left(U_2 \setminus U_2^{(3)}\right) < \epsilon/(4\,\ell);$$

- a C^1 area-preserving diffeomorphism $f_2 : M \to M$ such that
  - ∗ f_2 is C^∞ in $U_1^{(3)} \cup U_2^{(3)}$;
  - ∗ f_2 is C^2 in $U_1^{(2)} \cup U_2^{(2)}$;
  - ∗ $f_2 = g_1$ in $M \setminus \overline{U_2^{(2)}}$;
  - ∗ $f_2(U_2) = g_1(U_2)$.

Moreover, we have $f_2 \in \mathcal{V}_f$, so $U_2 \cap Rf_2(U_2) = \emptyset$. Therefore, as previously done, we get an R-reversible, C^1, area-preserving diffeomorphism g_2, which is C^∞ in $U_2^{(3)} \cup Rf_2(U_2^{(3)})$ and C^2 in $U_2^{(2)} \cup Rf_2(U_2^{(2)})$, by defining

$$g_2(u) = \begin{cases} g_1(u) & \text{if } u \in M \setminus (U_2 \cup Rf_2(U_2)) \\ f_2(u) & \text{if } u \in U_2 \\ R \circ f_2^{-1} \circ R(u) & \text{if } u \in Rf_2(U_2) \end{cases}$$

Notice that the definition of g_2 is compatible with the one of g_1 on intersecting charts and that $g_2 \in \mathcal{V}_f$.

Iterating this procedure, we reach $g = g_\ell$. This is a C^1 diffeomorphism which is R-reversible, area-preserving, C^∞ in $\bigcup_{i=1}^{\ell} U_i^{(3)} \cup Rf_{\ell-1}(U_i^{(3)})$ and C^2 in $\bigcup_{i=1}^{\ell} U_i^{(2)} \cup Rf_{\ell-1}(U_1^{(2)})$. Moreover, $g \in \mathcal{V}_f$.

The neighborhood V_Z of Z is given by

$$V_Z = \bigcup_{i=1}^{\ell} U_i^{(3)} \cup Rf_{\ell-1}(U_1^{(3)})$$

and we have

$$\mu(V_Z \setminus Z) \leqslant \sum_{i=1}^{\ell} \mu(U_i \setminus U_i^{(3)}) + \mu(Rf_{\ell-1}(U_i \setminus U_i^{(3)}))) = \sum_{i=1}^{\ell} 2\,\mu(U_i \setminus U_i^{(3)})) < \epsilon/2. \qquad \square$$

Remark 6.2. If Λ is a compact hyperbolic set for g such that $\mu(\Lambda) > 0$ but $\mu(\Lambda \cap Z) = 0$, then

$$\mu(\Lambda) = \mu(\Lambda \cap Z) + \mu(\Lambda \cap (M \setminus Z)) < \epsilon$$

and, as Λ and μ are g-invariant,

$$\begin{aligned} \mu\left((\Lambda \cap (M \backslash Z)) \cap \bigcup_{j \in \mathbb{Z}} g^{-j}(Z)\right) &= \mu\left((\Lambda \cap (M \backslash Z) \cap \bigcup_{j \in \mathbb{Z}} g^{-j}(\Lambda \cap Z)\right) \\ &= \mu\left(\bigcup_{j \in \mathbb{Z}} \Lambda \cap (M \backslash Z) \cap g^{-j}(\Lambda \cap Z)\right) \\ &\leqslant \sum_{j \in \mathbb{Z}} \mu\left(g^{-j}(\Lambda \cap Z)\right) = 0 \end{aligned}$$

which means that the iterates by g of μ almost every point in $\Lambda \cap (M \backslash Z)$ remain there.

7. Hyperbolic sets

It is known [16] that basic hyperbolic sets of C^2 non-Anosov diffeomorphisms have zero measure. In [11], it was proved that the same result holds for compact hyperbolic sets of C^2 symplectic diffeomorphisms without assuming that they are basic pieces. Using [55], which says that every C^1 symplectic diffeomorphism can be approximated by a C^2 one, it has been deduced in [12] that C^1 generically a symplectic diffeomorphism f is Anosov or every hyperbolic set of f has Lebesgue zero measure. Up to now, no such density of C^2 diffeomorphisms is known in the context of area-preserving reversible diffeomorphisms. Therefore, we can only ensure that:

Proposition 7.1. *If Λ_f is a compact hyperbolic set for $f \in \mathrm{Diff}^1_{\mu,R}(M)$, then either $\Lambda_f = M$ or Λ_f has empty interior.*

Proof. As Λ_f is hyperbolic, there exist a neighborhood V of Λ_f in M and a neighborhood W of f in $\mathrm{Diff}^1_\mu(M)$ such that, for each $g \in W$, there is a compact hyperbolic set $\Lambda_g \subset V$ and a (Hölder) homeomorphism $h : \Lambda_f \to \Lambda_g$ such that $h \circ f = g \circ h$; see [30, Theorem 19.1.1]. Now, if Λ_f has interior points, then, for each $g \in W$, the hyperbolic set Λ_g has nonempty interior as well. Moreover, by theorem 1.3 of [15], we may find a transitive diffeomorphism g_0 in W. A dense orbit of such a g_0 has to enter Λ_{g_0}, so, by invariance, this orbit is contained in Λ_{g_0}. By compactness of Λ_{g_0}, we finally conclude that $\Lambda_{g_0} = M$. Thus, $\Lambda_f = M$ as well. □

8. Proof of theorem A

Consider $f \in \mathrm{Diff}^1_{\mu,R}(M)$. If f is Anosov or its integrated Lyapunov exponent (see section 3.4) is zero, the proof ends. For instance, if $f = R$, then all orbits of f have zero Lyapunov exponents. Otherwise, we start approaching f by f_1 of the open and dense set $\mathscr{D}_1$. Then, given $\epsilon > 0$, by proposition 6.1 there exist a subset $Z \subset M$, whose complement in M has Lebesgue measure smaller than ϵ, and a diffeomorphism $f_2 \in \mathscr{D}_1$ which is C^1-close to f_1 (thus close to f) and is of class C^2 in an open neighborhood of Z. Using corollaries 4.2 and 3.8, we then find a diffeomorphism $F \in \mathscr{D}$ whose set of periodic points is countable (so it has null Lebesgue measure), is C^1 close to f_2 (hence close to f) and is still C^2 when restricted to Z.

According to the proposition 7.1, either F is Anosov or any compact hyperbolic set Λ of F has empty interior. Yet, in the latter case, $\Lambda \cap Z$ may have positive (although strictly smaller than one) Lebesgue measure. The next result, with which we will end the proof of theorem A, will be proved under the assumption that

$$\mu(\Lambda \cap Z) = 0, \quad \text{for any compact hyperbolic set } \Lambda \text{ of } F \tag{8.1}$$

in which case $\mu(\Lambda) < \epsilon$ (see remark 6.2). This happens, for instance, when $Z = M$, since then F is C^2 and $\mu(\Lambda) = 0$; or if μ is ergodic for F, because then $\mu(\Lambda) = 0$ as well, unless $\Lambda = M$.

Proposition 8.1. *Given $\delta > 0$, there is $g \in \mathrm{Diff}^1_{\mu,R}(M)$ which is C^1-close to F and satisfies $\mathscr{L}(g) < \epsilon + \delta$.*

Let $\mathscr{A}$ be the C^1-open subset of $\mathrm{Diff}^1_{\mu,R}(M)$ of the R-reversible Anosov diffeomorphisms and, for any $k, n \in \mathbb{N}$, denote by $\mathscr{A}_{k,n}$ the set

$$\mathscr{A}_{k,n} = \left\{ h \in \mathrm{Diff}^1_{\mu,R}(M) \colon \mathscr{L}(h) < \frac{1}{k} + \frac{1}{n} \right\}.$$

Clearly (see section 3.4), the union

$$\mathscr{A} \cup \mathscr{A}_{k,n}$$

is C^1-open in $\mathrm{Diff}^1_{\mu,R}(M)$. After proposition 8.1, we know that it is dense as well. Therefore, the set

$$\mathscr{A} \cup \left\{ h \in \mathrm{Diff}^1_{\mu,R}(M) \colon \mathscr{L}(h) = 0 \right\}$$

is a countable intersection of the C^1 open and dense sets

$$\mathscr{A} \cup \left\{ h \in \mathrm{Diff}^1_{\mu,R}(M) \colon \mathscr{L}(h) < \frac{1}{k} + \frac{1}{n} \right\}$$

and so it is residual.

9. Proof of proposition 8.1

Let $F \in \mathscr{D}$ be the diffeomorphism constructed on section 8 after fixing $f \in \mathrm{Diff}^1_{\mu,R}(M)$ and $\epsilon > 0$. Recall that F belongs to $\mathrm{Diff}^1_{\mu,R}(M)\backslash\mathscr{A}$, Lebesgue almost all its orbits are (R, F)-free and its set of periodic points has Lebesgue measure zero. In what follows we will assume that F satisfies the property (8.1).

9.1. Reducing locally the Lyapunov exponent

The prior ingredient to prove proposition 8.1 is the next lemma whose statement is the reversible version of the main lemma in [9].

Lemma 9.1. *Given η, $\delta > 0$ and $\kappa \in]0, 1[$, there exists a measurable function $\mathcal{N}\colon M \to \mathbb{N}$ such that, for x in a set $\hat{Z}$ with Lebesgue measure bigger than $1 - \epsilon$ and every $n \geqslant \mathcal{N}(x)$, there exists $\varrho = \varrho(x, n) > 0$ such that, for any ball $B(x, r)$, with $0 < r < \varrho$, we may find $G \in \mathrm{Diff}^1_{\mu,R}(M)$, which is η-C^1-close to F, and compact sets $K_1 \subset B(x, r)$ and $K_2 \subset R\,F^n(K_1) \subset R\,F^n(B(x, r))$ satisfying:*

(a) $F = G$ outside $\left(\bigcup_{j=0}^{n-1} F^j(\overline{B(x, r)})\right) \cup \left(\bigcup_{j=1}^{n} R\,F^j(\overline{B(x, r)})\right)$.

(b) For $j \in \{0, 1, ..., n-1\}$, the iterates $F^j(\overline{B(x, r)})$ and $R\,F^{j+1}(\overline{B(x, r)})$ are pairwise disjoint.

(c) $\mu(K_1) > \kappa\,\mu(B(x, r))$ and $\mu(K_2) > \kappa\,\mu(R\,F^n(B(x, r)))$.

(d) If $y_1 \in K_1$ and $y_2 \in K_2$, then $\frac{1}{n}\log\|DG^n_{y_i}\| < \delta$ for $i = 1, 2$.

Although the proof of this lemma follows closely the argument of [9], it is worth registering the fundamental differences between the previous result and [9, main lemma]. Firstly, each time we perturb the map F around $F^j(x)$, for $j \in \{0, ..., n-1\}$, we must balance with a perturbation around $R\,F^{j+1}(x)$ to prevent the perturbed diffeomorphism's exit from $\mathrm{Diff}^1_{\mu,R}(M)$. Thus the perturbations in $\bigcup_{j=0}^{n-1} F^j(\overline{B(x, r)})$ spread to a deformation of F in $\bigcup_{j=1}^{n} R\,F^j(\overline{B(x, r)})$. This is possible because $F \in \mathscr{D}$, but our choice of ϱ must be more judicious and, in general, smaller than the one in [9] to avoid inconvenient intersections. Secondly, we need an additional control on the function $\mathcal{N}$ and on $\mu(K_2)$ to localize the computation of the Lyapunov exponents along the orbits of elements of K_2.

Aside from this, we also have a loss in measure. As F is not globally C^2, instead of a function $\mathcal{N}\colon M \to \mathbb{N}$ with nice properties on a full measure set, during the proof [9] we have to take out a compact hyperbolic component with, perhaps, positive measure. Fortunately, that portion has measure smaller than ϵ, though its effect shows up in several computations and cannot be crossed off the final expression of the integrated Lyapunov exponent.

Regardless of these difficulties, reversibility also relieves our task here and there. For instance, the inequality for $y_2 \in K_2$ in the previous lemma, that is, $\|Dg^n_{y_2}\| < \mathrm{e}^{n\delta}$, follows from the corresponding one for y_1 due to the reversibility and the fact that $\|A\| = \|A^{-1}\|$ for any $A \in \mathrm{SL}(2, \mathbb{R})$. Indeed, given $y_2 \in K_2$, there exists $y_1 \in K_1$ such that $y_2 = R(F^n(y_1)) = F^{-n}(R(y_1))$. Then (see lemma 5.1)

$$\|DG^n_{y_2}\| = \|D(R\,G^{-n}\,R)(y_2)\| \leqslant \|DG^{-n}_{R(y_2)}\| = \|DG^n_{y_1}\| < \mathrm{e}^{n\delta}.$$

In what follows we will check where differences start emerging and summarize the essential lemmas where reversibility steps in.

9.1.1. Sending E^u to E^s

Definition 9.2 ([9, section 3.1]). Given $\eta > 0$, $\kappa \in]0, 1[$, $n \in \mathbb{N}$ and $x \in M$, a finite family of linear maps $L_j : T_{F^j(x)}M \to T_{F^{j+1}(x)}M$, for $j = 0, ..., n-1$, is an (η, κ)-realizable sequence of length n at x if, for all $\gamma > 0$, there is $\rho > 0$ such that, for $j \in \{0, 1, ..., n-1\}$, the iterates $F^j(B(x, \rho))$ and $R(F^j(B(x, \rho)))$ are pairwise disjoint and, for any open nonempty set $U \subseteq B(x, \rho)$, there exist

(a) a measurable set $K_1 \subseteq U$ such that $\mu(K_1) > \kappa\, \mu(U)$
(b) $h \in \mathrm{Diff}^1_{\mu,R}(M)$, η-C^1-close to F satisfying:

1 $h = F$ outside $\left(\bigcup_{j=0}^{n-1} F^j(\overline{U})\right) \bigcup \left(\bigcup_{j=1}^{n} R(F^j(\overline{U}))\right)$
2 if $y_1 \in K_1$, then $\|Dh_{h^j(y_1)} - L_j\| < \gamma$ for $j = 0, 1, ..., n-1$.

Notice that, if the orbit of x is (R, F)-free and not periodic (or periodic but with period greater than n) and we define $K_2 = R(F^n(K_1))$ and, for $j \in \{0, 1, ..., n-1\}$, the sequence

$$\begin{array}{rccc} \tilde{L}_j: & T_{R(F^{n-j}(x))}M & \longrightarrow & T_{R(F^{n-j-1}(x))}M \\ & v & \longmapsto & DR_{F^{n-j-1}(x)} L^{-1}_{n-j-1} DR_{R(F^{n-j}(x))}(v) \end{array}$$

then we obtain, for $y_2 \in K_2$ and $j = 0, 1, ..., n-1$, the inequality $\|Dh_{h^j(y_2)} - \tilde{L}_j\| < \gamma$.

The following lemma is an elementary tool to interchange bundles using rotations of the Oseledets directions, and thereby construct realizable sequences. If $x \in M$ and $\theta \in \mathbb{R}$, consider a local chart at x, $\varphi_x : V_x \to \mathbb{R}^2$ and the maps $D\varphi_x^{-1} \mathfrak{R}_\theta D\varphi_x : \mathbb{R}^2 \to \mathbb{R}^2$, where $\mathfrak{R}_\theta$ is the standard rotation of angle θ at $\varphi_x(x)$.

Denote by $\mathcal{Y}$ the full Lebesgue measure subset of M with countable complement, given by proposition 3.6 and Corollary 4.2, whose points have (R, F)-free and nonperiodic orbits by F.

Lemma 9.3 ([9, Lemma 3.3]). *Given $\eta > 0$ and $\kappa \in]0, 1[$, there is $\theta_0 > 0$ such that, if $x \in \mathcal{Y}$ and $|\theta| < \theta_0$, then $\{DF_x \mathfrak{R}_\theta\}$ and $\{\mathfrak{R}_\theta DF_x\}$ are (η, κ)-realizable sequence of length 1 at x.*

The next result enables us to construct realizable sequences with a purpose: to send expanding Oseledets directions into contracting ones. This will be done at a region of M without uniform hyperbolicity because there the Oseledets directions can be blended. More precisely, for $x \in \mathcal{O}^+(F)$ and $m \in \mathbb{N}$, let

$$\Delta_m(F, x) = \frac{\|DF_x^m|_{E^s(x)}\|}{\|DF_x^m|_{E^u(x)}\|}$$

and

$$\Gamma(F, m)^* = \left\{ x \in \mathcal{O}^+(F) \cap \mathcal{Y} : \Delta_m(F, x) \geqslant \frac{1}{2} \right\}.$$

Lemma 9.4 ([9, Lemma 3.8]). *Take $\eta > 0$ and $\kappa \in]0, 1[$. There is $m \in \mathbb{N}$ such that, for every $x \in \Gamma(F, m)^*$, there exists an (η, κ)-realizable sequence $\{L_0, L_1, ..., L_{m-1}\}$ at x with length m satisfying*

$$L_{m-1}(\ldots)L_1 L_0(E^u_x) = E^s_{F^m(x)}$$

and, consequently,

$$\tilde{L}_{m-1}(\ldots)\tilde{L}_1 \tilde{L}_0(E^u_{R(F^m(x))}) = E^s_{R(x)}.$$

The ensuing step is to verify that the above construction may be done in such a way that the composition of realizable sequences has small norm. Consider the F-invariant set

$$\Omega_m(F) = \bigcup_{n \in \mathbb{Z}} F^n(\Gamma(F, m)^*).$$

Then $\mathcal{H}_m = \mathcal{O}^+(F) - \Omega_m(F)$ is empty or its closure is a compact hyperbolic set [9, Lemma 3.11]. Under the hypothesis (8.1), we have $\mu(\mathcal{H}_m) < \epsilon$. Hence,

Lemma 9.5 ([9, Lemma 3.13]). *Consider $\eta > 0$, $\kappa \in]0, 1[$ and $\delta > 0$. There exists a measurable function $\mathcal{N}: M \to \mathbb{N}$ such that, for x in a subset with Lebesgue measure greater that $1 - \epsilon$ and all $n \geqslant N(x)$, we may find a (η, κ)-realizable sequence $\{L_j\}_{j=0}^{n-1}$ of length n such that*

$$\|L_{n-1}(\ldots)L_0\| < \mathrm{e}^{\frac{4}{5} n \delta}.$$

If γ is chosen small enough in the definition 9.2, lemma 9.1 is a direct consequence of the preceding one.

9.2. Reducing globally the Lyapunov exponent

After lemma 9.1 we know how to find large values of n such that, for some perturbation $G \in \mathrm{Diff}^1_{\mu,R}(M)$ of F, we get $\|DG^n_x\| < \mathrm{e}^{n\delta}$ for a considerable amount of points x inside a small ball and its image by RF. However, the Lyapunov exponent is an asymptotic concept and we need to evaluate, or find a good approximation of it on a set with full μ measure. In this section we will extend the local procedure to an almost global perturbation, which allows us to draw later on global conclusions. The classic ergodic theoretical construction of a Kakutani castle [1] is the bridge between these two approaches, as was discovered in [9, section 4]. The main novelties here are the possible presence of compact hyperbolic sets with positive measure and the fact that, when building some tower of the castle, we simultaneously build its mirror inverted reversible copy.

9.2.1. A reversible Kakutani castle Let $A \subseteq M$ be a Borelian subset of M with positive Lebesgue measure and $n \in \mathbb{N}$. The union of the mutually disjoint subsets $\bigcup_{i=0}^{n-1} F^i(A)$ is called a *tower*, n its *height* and A its *base*. The union of pairwise disjoint towers is called a *castle*. The *base of the castle* is the union of the bases of its towers. The first return map to A, say $\tau : A \to \mathbb{N} \cup \{\infty\}$, is defined as $\tau(x) = \inf\{n \in \mathbb{N} : F^n(x) \in A\}$. Since $\mu(A) > 0$ and F is measure-preserving, by Poincaré's recurrence theorem the orbit of Lebesgue-almost all points in A will come back to A. Thus, $\tau(x) \in \mathbb{N}$ for Lesbesgue almost every $x \in A$. If $A_n = \{x \in A : \tau(x) = n\}$, then $\mathcal{T}_{n,A} = A_n \cup F(A_n) \cup \ldots \cup F^{n-1}(A_n)$ is a tower and the F-invariant set $\bigcup_{n \in \mathbb{Z}} F^n(A)$ is the union of the towers $\mathcal{T}_{n,A}$, creating a castle with base A. Moreover,

Lemma 9.6 ([26, pp. 70 and 71]). *For every Borelian U such that $\mu(U) > 0$ and every $n \in \mathbb{N}$, there exists a positive measure set $V \subset U$ such that $V, F(V), \ldots, F^n(V)$ are pairwise disjoint. Besides, V can be chosen maximal, that is, no set containing V and with larger Lebesgue measure than V has this property.*

Fix η, $\delta > 0$ and take $0 < \kappa < 1$ such that $1 - \kappa < \delta^2$. Apply lemma 9.1 to get a function $\mathcal{N}$ as stated. For each $n \in \mathbb{N}$, consider $P_n = \{x \in M : \mathcal{N}(x) \leqslant n\}$. Clearly,

$$\lim_{n\to\infty} \mu(P_n) \geqslant 1 - \epsilon.$$

Hence, there is $\alpha \in \mathbb{N}$ such that

$$\mu(M\backslash P_\alpha) < \epsilon + \delta^2$$

and therefore

$$\mu\,(M\backslash(P_\alpha \cup R(P_\alpha))) < \epsilon + \delta^2.$$

For $U = P_\alpha \cup R(P_\alpha)$ and α, lemma 9.6 gives a maximal set $\mathcal{B} \subset P_\alpha \cup R(P_\alpha)$ with positive Lebesgue measure and such that $\mathcal{B}, F(\mathcal{B}), \ldots, F^\alpha(\mathcal{B})$ are mutually disjoint. Denote by $\hat{\mathcal{Q}}$ the Kakutani castle associated to the base $\mathcal{B}$, that is

$$\hat{\mathcal{Q}} = \cup_{n\in\mathbb{Z}} F^n(\mathcal{B}).$$

Observe that, by the maximality of $\mathcal{B}$, the set $\hat{\mathcal{Q}}$ contains $P_\alpha \cup R(P_\alpha)$, and so $\mu\left(\hat{\mathcal{Q}}\backslash(P_\alpha \cup R(P_\alpha))\right) < \epsilon + \delta^2$.

Consider now the castle $\mathcal{Q} \subset \hat{\mathcal{Q}}$ whose towers have heights less that 3α. Adapting the argument in [9, Lemma 4.2], we obtain:

Lemma 9.7. $\mu\left(\hat{\mathcal{Q}}\backslash\mathcal{Q}\right) < 3(\epsilon + \delta^2)$.

Furthermore,

Lemma 9.8. *(a)* $\mu\,(\mathcal{B}\triangle R(\mathcal{B})) = 0$.

(b) If $\mathcal{T}_{n,\mathcal{B}}$ is the tower of height n associated to $\mathcal{B}$, then also is $R(\mathcal{T}_{n,\mathcal{B}})$. Besides, $RF^n(\mathcal{T}_{n,\mathcal{B}} \cap \mathcal{B}) = R(\mathcal{T}_{n,\mathcal{B}}) \cap \mathcal{B}$.

Proof.

(a) We will show that $R(\mathcal{B}) \subset \mathcal{B}$ moduloμ. Assume that there exists a positive μ-measure subset $C \subset R(\mathcal{B})$ such that C is not contained in $\mathcal{B}$. Observe that $C \subset P_\alpha \cup R(P_\alpha)$ because $P_\alpha \cup R(P_\alpha)$ is R-invariant and $\mathcal{B} \subset P_\alpha \cup R(P_\alpha)$. As $\mathcal{B}$ is maximal and there are points of C out of $\mathcal{B}$, we have $F^i(C) \cap F^j(C) \neq \emptyset$ for some $i \neq j \in \{0, \ldots, \alpha\}$. However, $R(C) \subset \mathcal{B}$ and $\mu(R(C)) = \mu(C) > 0$, so $F^i(R(C)) \cap F^j(R(C)) = \emptyset$ which, using reversibility, is equivalent to $R(F^{-i}(C)) \cap R(F^{-j}(C)) = \emptyset$, that is, $F^{-i}(C) \cap F^{-j}(C) = \emptyset$, a contradiction.

(b) This is a direct consequence of (a). Since $\mathcal{T}_{n,\mathcal{B}}$ is a tower of height n with base $\mathcal{B}$, its first floor T_0 and its top floor T_n are in $\mathcal{B}$. By (a), $R(T_0)$ and $R(T_n)$ are in $\mathcal{B}$ as well, and so they are, respectively, the top and first floor of the tower $R(\mathcal{T}_{n,\mathcal{B}})$, and its height has to be n as well. □

Remark 9.9. At this stage, one may wonder about the effect of the existence of a hyperbolic set $\Lambda \cap (M\backslash Z)$ with positive, although small, Lebesgue measure. Could a typical orbit $x \in \mathcal{B}$ visit regions with hyperbolic-type behavior and positive measure? Yes, but only a null Lebesgue measure subset of points in $\mathcal{B}$ may visit $M\backslash Z$; see remark 6.2.

9.2.2. Regular families of sets. Following [33], we say that collection $\mathcal{V}$ of mensurable subsets of M is a *regular family* for the Lebesgue measure μ if there exists $\nu > 0$ such that $\mathrm{diam}(V)^2 \leqslant \nu\mu(V)$ for all $V \in \mathcal{V}$, where $\mathrm{diam}(A) = \sup\{\mathrm{d}(x, y), x, y \in A\}$. In what follows, we will prove that the family of all ellipses with controlled eccentricity constitutes a regular family for the Lebesgue measure.

An ellipse $E \subset M$ whose major and minor axes have lengths a and b, respectively, has eccentricity $e \geqslant 1$ if it is the image of the unitary disk $D \subset M$ under a diffeomorphism

$\Phi \in \mathrm{Diff}^1_\mu(O)$, defined on an open neighborhood O of D and satisfying $\|\Phi\|_{C^0} = e = \sqrt{a/b}$. Given $e_0 > 1$, the family of all ellipses whose eccentricity stays between 1 and e_0 is a regular family for the Lebesgue measure (just take $\nu = e_0^2$).

Let $\mathcal{B}$ be the base of the castle $\mathcal{Q}$ and let $n(x)$ be the height of the tower containing x. Recall that we have $\mathcal{N}(x) \leqslant \alpha \leqslant n(x)$.

Lemma 9.10. *Consider the castle $\mathcal{Q}$ and $x \in \mathcal{B}$. There exists $r(x) > 0$ and a ball $B(x, r(x))$ such that the set $B(x, r(x)) \cup R(F^{n(x)}(B(x, r(x))))$ is a regular family.*

Proof. Clearly, the sets $B(x, r(x))$ are regular (choose $\nu = 4/\pi$). Let us see that $R\,F^{n(x)}(B(x, r(x)))$ is also regular. Notice that, in general, this set is not an ellipse. However, if $B(x, r(x))$ is small, then $R\,F^n(B(x, r(x)))$ is close to its first order approximation, that is $DR\,DF^n(B(x, r(x)))$, which is an ellipse.

First observe that the height of a tower is constant in balls centered at points of $\mathcal{B}$ with sufficiently small radius [9, section 4.3]. Denote by $C_F = \max_{z \in M} \|DF_z\|$. Since μ is F and R invariant, if $r(x) < 1$ we have

$$\begin{aligned}
\left[\mathrm{diam}(R\,F^{n(x)}(B(x, r(x))))\right]^2 &= \left[\mathrm{diam}(F^{n(x)}(B(x, r(x))))\right]^2 \\
&\leqslant (2\,r(x)\,C_F)^{2\,n(x)} = \frac{(2\,C_F)^{2\,n(x)}\,r(x)^{2\,n(x)-2}}{\pi}\,\pi\,r(x)^2 \\
&\leqslant \frac{(2\,C_F)^{6\alpha}\,r(x)^{6\alpha-2}}{\pi}\,\pi\,r(x)^2 \leqslant \frac{(2\,C_F)^{6\alpha}}{\pi}\,\pi\,r(x)^2 \\
&= \frac{(2\,C_F)^{6\alpha}}{\pi}\,\mu(B(x, r(x)) \\
&= \nu\,\mu\left(R\,F^{n(x)}(B(x, r(x)))\right)
\end{aligned}$$

where $\nu = \frac{(2\,C_F)^{6\alpha}}{\pi}$. □

9.2.3. Construction of g. The last auxiliary result says that it is possible, using the Vitali covering lemma and lemma 9.10, to cover the base $\mathcal{B}$ essentially with balls and ellipses.

Lemma 9.11 ([9, section 4.3]). *Let $\gamma > 0$ satisfy $\gamma < \delta^2\alpha^{-1}$. Then:*

(a) There is a compact castle $\mathcal{Q}_1$ contained in $\mathcal{Q}$ and an open castle $\mathcal{Q}_2$ containing $\mathcal{Q}$ with the same shape[3] as $\mathcal{Q}$ and such that $\mu(\mathcal{Q}_2 \backslash \mathcal{Q}_1) < \gamma$.

(b) The base $\mathcal{B}_3$ of the castle $\mathcal{Q}_2 \cap \mathcal{Q}$ may be covered by a finite number of balls $B(x_i, r'(x_i))$ and their images $R\,F^{n_i}(B(x_i, r'(x_i))$, where $x_i \in \mathcal{B}_3$ and $r'(x_i)$ is small enough so that $n(x)|_{B(x_i, r'(x_i))} \equiv n_i$ and

$$\frac{\mu\left(\mathcal{B}_3 \setminus \dot{\bigcup} B(x_i, r(x_i)) \,\cup\, R\,F^{n_i}(B(x_i, r(x_i)))\right)}{\mu(\mathcal{B}_3)} < \gamma.$$

Once the covering $\bigcup B(x_i, r(x_i)) \cup R\,F^{n_i}(B(x_i, r(x_i)))$ is found, lemma 9.1 provides, for each i, a diffeomorphism $g_i \in \mathrm{Diff}^1_{\mu,R}(M)$ which is C^1-close to F and compact sets

$$K_1^i \subset B(x_i, r(x_i)) \quad \text{and} \quad K_2^i \subset R\,F^{n_i}(B(x_i, r(x_i))))$$

[3] This means that the castles have the same number of towers and the towers have the same heights.

such that:

(a) $g_i = F$ outside $[\bigcup_{j=0}^{n_i-1} F^j(\overline{B(x_i, r(x_i))})] \bigcup [\bigcup_{j=1}^{n_i} R(F^j(\overline{B(x_i, r(x_i))}))]$.

(b) For $j \in \{0, 1, ..., n_i - 1\}$, the iterates $F^j(\overline{B(x_i, r(x_i))})$ and $R(F^{j+1}(\overline{B(x_i, r(x_i))}))$ are pairwise disjoint.

(c) $\mu(K_1^i) > \kappa\, \mu(B(x_i, r(x_i)))$ and $\mu(K_2^i) > \kappa\, \mu(R\, F^{n_i}(B(x_i, r(x_i))))$.

(d) If $y_1 \in K_1^i$ and $y_2 \in K_2^i$, then $\log \|(Dg_i^{n_i})_{y_1}\| < n_i\, \delta$ and $\log \|(Dg_i^{n_i})_{y_2}\| < n_i\, \delta$.

Finally, we define the diffeomorphism $g \in \mathrm{Diff}^1_{\mu,R}(M)$ by $g = g_i$ in each component

$$\left[\bigcup_{j=0}^{n_i-1} F^j(\overline{B(x_i, r(x_i))})\right] \bigcup \left[\bigcup_{j=1}^{n_i} R\, F^j(\overline{B(x_i, r(x_i))})\right]$$

and $g = f$ elsewhere.

9.2.4. Estimation of $\mathscr{L}(g)$. For $\varphi \in \mathrm{Diff}^1(M)$, let $C_\varphi = \max\{\|D\varphi_z\| : z \in M\}$ and denote by C_1 the maximum of the set

$$\left\{ C(\varphi) : \varphi \in \mathrm{Diff}^1_{\mu,R}(M) \text{ and } \varphi \text{ is } \eta - C^1\text{–close to} F \right\}.$$

As in [9], despite the necessary adjustments, there are a constant $C_2 > 0$, a positive integer $N \geqslant \delta^{-1}\alpha$, a g-castle K of the same type as $\mathcal{Q}_2$ and a subset $\mathcal{G} = \bigcap_{j=1}^{N-1} g^{-j}(K)$ of M such that

$$\begin{aligned}
\mathscr{L}(g) &= \int_{\mathcal{G}} \lambda^+(g)\,\mathrm{d}\mu + \int_{\hat{Z}\setminus\mathcal{G}} \lambda^+(g)\,\mathrm{d}\mu + \int_{M\setminus\hat{Z}} \lambda^+(g)\,\mathrm{d}\mu \\
&\leqslant \int_{\mathcal{G}} \frac{1}{N} \log \|Dg^N\|\,\mathrm{d}\mu + \int_{\hat{Z}\setminus\mathcal{G}} \lambda^+(g)\,\mathrm{d}\mu + \int_{M\setminus\hat{Z}} \lambda^+(g)\,\mathrm{d}\mu \\
&\leqslant C_2\,\delta + \ln(C_1)(\delta+\epsilon) + \int_{M\setminus\hat{Z}} \lim_{n\to+\infty} \frac{1}{n} \ln \|Dg_x^n\|\,\mathrm{d}\mu \leqslant C_2\,\delta \\
&+ \ \ln(C_1)(\delta+\epsilon) + \ln(C_1)\,\epsilon \\
&= (C_2 + \ln(C_1))\,\delta + 2\ln(C_1)\,\epsilon.
\end{aligned}$$

10. Proof of theorem B

Given $m \in \mathbb{N}$ and $f \in \mathrm{Diff}^1_{\mu,R}(M)$, let $\Lambda(f, m) \subset M$ be the set of points with a m-dominated splitting, $\mathcal{Y}_f$ the set of points in M whose f-orbits are (R, f)-free and not periodic and

$$\begin{aligned}
\Gamma(f, m) &= M \setminus \Lambda(f, m) \\
\Gamma(f, m)^\sharp &= \mathscr{O}^+(f) \cap \Gamma(f, m) \\
\Gamma(f, m)^* &= \Gamma(f, m)^\sharp \cap \mathcal{Y}_f \\
\Gamma(f, \infty) &= \bigcap_{m\in\mathbb{N}} \Gamma(f, m) \\
\Gamma(f, \infty)^\sharp &= \bigcap_{m\in\mathbb{N}} \Gamma(f, m)^\sharp
\end{aligned}$$

The aim of this argument is to reduce the Lyapunov exponents in $\Gamma(f, m)^\sharp$, if they are positive, using a Kakutani castle in $\Gamma(f, m)^*$ (instead of in the whole M as we did on the previous section). The argument essentially follows three steps:

1st step . Mixing directions along an orbit segment and lowering the norm.

As proved in [11, Lemma 4.1], the set $\Gamma(f, \infty)^\sharp$ contains no periodic points for f. Moreover (see [11, Lemma 4.2]), given $\eta > 0$, $\kappa \in]0, 1[$, $\delta > 0$ and m sufficiently large, there exists a measurable function

$$\mathcal{N}: \Gamma(f, m)^* \to \mathbb{N}$$

such that, for $x \in \Gamma(f, m)^*$ and all $n \geqslant N(x)$, we may find a (η, κ)-realizable sequence $\{L_j\}_{j=0}^{n-1}$ of length n such that

$$\|L_{n-1}(\ldots)L_0\| < \mathrm{e}^{\frac{4}{5}n\delta}.$$

Moreover (see [11, Proposition 4.8]), given $f \in \mathrm{Diff}^1_{\mu,R}(M)$, $\epsilon_0 > 0$ and $\delta > 0$, then there exist $m \in \mathbb{N}$ and $g \in \mathrm{Diff}^1_{\mu,R}(M)$ which is ϵ_0-C^1-close to f, coincides with f outside the open set $\Gamma(f, m)$ and satisfies

$$\int_{\Gamma(f,m)} \lambda^+(g, x)\mathrm{d}\mu < \delta.$$

2nd step . Globalization.

Define

$$J_f = \int_{\Gamma(f,\infty)} \lambda^+(f, x)\mathrm{d}\mu.$$

Then [11, Lemma 4.17] proves that, given $f \in \mathrm{Diff}^1_{\mu,R}(M)$, $\epsilon_0 > 0$ and $\delta > 0$, then there exists $g \in \mathrm{Diff}^1_{\mu,R}(M)$ ϵ_0-C^1-close to f such that

$$\int_M \lambda^+(g, x)\mathrm{d}\mu < \int_M \lambda^+(f, x)\mathrm{d}\mu - J_f + \delta.$$

3rd step . Conclusion.

Let $f \in \mathrm{Diff}^1_{\mu,R}(M)$ be a point of continuity of $\mathscr{L}$ defined by

$$\mathscr{L} : f \mapsto \int_M \lambda^+(f, x)\mathrm{d}\mu.$$

Notice that J_f must be 0, which means that $\lambda^+(f, x) = 0$ for almost every $x \in \Gamma(f, \infty)$. If $x \in \mathscr{O}(f)$ and all Lyapunov exponents of f at x vanish, then there is nothing left to prove. Otherwise, if $\lambda^+(f, x) > 0$, then $x \notin \Gamma(f, \infty)$, that is, $x \in \Lambda(f, m)$ for some m; hence, there is a dominated splitting along the orbit of x. As we are dealing with surfaces, we conclude that the orbit of x is hyperbolic, and therefore its closure is a compact hyperbolic set which, according to proposition 7.1, is M or has empty interior.

Acknowledgments

We are grateful to the referee for the careful reading of our manuscript and the useful comments which much helped to improve this work. M Bessa was partially supported by National Funds through FCT (Fundação para a Ciência e a Tecnologia) project PEst-OE/MAT/UI0212/2011. M Carvalho and A Rodrigues have been funded by the European Regional Development Fund within the program COMPETE and by the Portuguese Government through FCT—Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0144/2013. A Rodrigues has benefited from the FCT grant SFRH/BPD/84709/2012.

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