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DISCRETE LAYER FINITE ELEMENT MODELING OF ANISOTROPIC LAMINATED SHELLS BASED ON A REFINED SEMI-INVERSE MIXED DISPLACEMENT FIELD FORMULATION

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SYNOPSIS

This paper concerns the finite element (FE) modeling of anisotropic laminated shells. A discrete layer approach is employed in this work and a single layer is first considered and isolated from the multilayer shell structure. The weak form of the governing equations of the anisotropic single layer of the multilayer shell is derived with Hamilton's principle using a "mixed" (stresses/displacements) definition of the displacement field, which is obtained through a *semi-inverse* (stresses/strains-displacements) approach. Results from 3-D elasticity solutions are used to postulate adequate definitions of the out-of-plane shear stress components, which, in conjunction with the Reissner-Mindlin theory (or first order shear deformation theory) definitions of the shell in-plane stresses, are utilized to derive the "mixed" displacement field. Afterward, the single layer shell FE is "regenerated" to a 3-D form, which allows interlayer displacements and out-of-plane stresses continuity between adjacent interfaces of different layers to be imposed, and a multilayer shell FE is obtained by assembling, at an elemental FE level, all the "regenerated" single layer FE contributions. A *fully refined* shell theory, where displacement and full out-of-plane stresses continuity and homogeneous stress conditions on the top and bottom surfaces are assured, is conceptually proposed, and a *partially refined* shell theory, where the out-of-plane normal stress continuity is relaxed and a plane stress state is considered, is developed and used to derive a FE solution for segmented multilayer doubly-curved anisotropic shells.

INTRODUCTION

In the last decades composite laminated structures have been vastly used in high-tech applications for the aerospace, aeronautical and automotive (among others) industries. In the mean time, a relative maturity in the adequate modeling and design of those complex structures has been achieved. However, with the ever increasing strong demands of lighter, stiffer, lower cost and more efficient and reliable structural composite components, the structural designers are faced nowadays with the requirement of having at their disposal refined and more accurate multiphysics models of those complex layered structures. That problem pushes the underlying complexity of these models to higher levels. Thus, allied with the ever increasing processing capabilities of modern computers, the 21st century has emerged with a "new" challenge for science, which demands the development of new, refined, more accurate and fully representative theories to solve the "classical" structural problems that have been tackled by scientists and engineers for a long time.

It is well known that modeling laminated shell structures is a very complicated subject involving a lot of thinking concerning kinematic assumptions, displacement-strain-stress relationships, different variational principles, constitutive relations, consistency of the resultant governing equations, etc., with the extra complication of having the problem formulated in curvilinear coordinates.

The finite element (FE) method is usually the preferable way of obtaining solutions for structures with more complicated geometries, boundary conditions and applied loads types. ”Perfect” FEs of general shell structures without any numerical pathologies are still an issue to be solved by the scientific community and a challenge to deal with. However, many models have been developed to design and simulate the structural system’s response and to assess their numerical stability, reliability, representativeness and performance. However, further refinement of these models and coming up with some new approaches should be the main emerging tendencies of this complex research issue which is far from being fully understood.

The derivation of shell theories has been one of the most prominent challenges in solid mechanics for many years. The idea is to develop appropriate models that can accurately simulate the effects of shear deformations and transverse normal strains in laminated shells with good trade-off between accuracy and complexity, which is a big mathematical difficulty. Physical 3-D shells are usually modeled recurring to approximated mathematical 2-D models. They are obtained by imposing some chosen kinematic and mechanical assumptions to the 3-D continuum, e.g., by explicitly assuming a through-the-thickness axiomatic displacement field definition and assuming a plane-stress state. When compared to 3-D solid FEs, 2-D shell FEs allow a significant reduction of the computational cost without losing much accuracy. However, to make matters worse, this sort of approximations lead to so-called *locking* effects (e.g., shear and membrane locking), which produces an overstiffening of the FE model which in turn yields erroneous results. Furthermore, for shell-type structures, with a more complex shear-membrane-bending coupling behavior, the locking effects are not yet fully understood yet, making these numerical pathologies difficult to remedy completely.

Over the years many authors have developed models for this type of problem. It’s very difficult to comprehensively review all the contributions because the literature is very vast and the technical developments and improvements appear dispersed in technical journals of different areas, being some of them very difficult of obtaining. However, the major works, considered relevant for this work, are shortly reviewed here. That will allow to introduce the subject of this paper and to justify the options and assumptions taken for developing the present modeling approaches. Furthermore, it will allow to identify the aspects of the work that are new and significant in a more founded way.

In the development of structural mathematical models, different theories have been considered to axiomatically define the kinematics of laminated structural systems, where the planar dimensions are one to two orders of magnitude larger than their thickness (e.g., beams, plates or shells). Usually, these structural systems are formed by stacking layers of different isotropic or orthotropic composite materials with arbitrary fiber orientation or then, in the case of vibration damping treatments, by arbitrary stacking sequences of active piezoelectric or passive viscoelastic damping layers. These theories, following Kraus (1967), were originally developed for single layer ”monocoque” thin structures made of traditional isotropic materials. Generally speaking, they can be grouped into two classes of alternate theories: one in which all of the Love’s original assumptions (see Love, 1944) are preserved, and other, following higher-order linear theories in which one or another of Love's assumptions are suspended. Many additional theories of thin and thick elastic shells have been proposed and the chronicle of these efforts are presented for example by Naghdi (1956) and Leissa (1993).

Two different approaches are often used. The first one, the so-called *Equivalent Single Layer* (ESL) theories, where the number of independent generalized variables doesn't depend of the number of layers, are derived from 3-D elasticity by making suitable assumptions concerning the kinematics of deformation or the stress state through the thickness of the laminate, allowing the reduction of a 3-D problem to a 2-D one. The second one, the so-called *Layerwise Theories* (LWT), where the number of generalized variables depends on the number of physical (or nonphysical) layers, rely on the basis that the kinematic assumptions are established for each individual layer, which might be modeled (or not) as a 3-D solid. The problem is then reduced to a 2-D problem, however, retaining the 3-D intralaminar and interlaminar effects.

As reported by Yang et al. (2000), plate and shell structures made of laminated composite materials have often been modeled as an ESL using the *Classical Laminate Theory* (CLT) [see for example the textbook of Reddy (2004)], in which the out-of-plane stress components are ignored. The CLT is a direct extension of the well-known Kirchhoff-Love kinematic hypothesis, i.e., plane sections before deformation remain plane and normal to the mid-plane after deformation and that normals to the middle surface suffer no extension (Kirchhoff contribution) and others (cf. Leissa, 1993), however applied to laminate composite structures. This theory is adequate when the ratio of the thickness to length (or other similar dimension) is small, the dynamic excitations are within the low-frequency range and the material anisotropy is not severe. The application of such theories to layered anisotropic composite shells could lead to 30% or more errors in deflections, stresses and frequencies (Reddy, 2004). In order to overcome the deficiencies in the CLT, new refined laminate theories have been proposed relaxing some of the Love's postulates according to Koiter's recommendations (Koiter, 1960), where it is stated that "... *a refinement of Love's approximation theory is indeed meaningless, in general, unless the effects of transverse shear and normal stresses are taken into account at the same time.*" However, as stated by Carrera (1999), for 2-D modelings of multilayered structures (such as laminated constructions, sandwich panels, layered structures used as thermal protection, intelligent structural systems embedding piezoelectric and/or viscoelastic layers) require amendments to Koiter's recommendation. Among these, the inclusion of continuity of displacements, *zig-zag* effects, and of transverse shear and normal stresses interlaminar continuity at the interface between two adjacent layers, are some of the amendments necessary. The role played by zig-zag effects and interlaminar continuity has been confirmed by many 3-D analysis of layered plates and shells (Srinivas et al., 1970; Srinivas, 1974; Pagano and Reddy, 1994; Ren, 1987; Varadan and Bhaskar, 1991; Bhaskar and Varadan, 1994; Soldatos, 1994). These amendments become more significant when complicating effects such as high in-plane and/or out-of-plane transverse anisotropy are present. Hence, as referred by Carrera (1999), Koiter's recommendation concerning isotropic shells could be re-written for the case of multilayered shells as "... *a refinement of ... unless the effects of interlaminar continuous transverse shear and normal stresses are taken into account at the same time.*" This enforces the need of also assuring the interlaminar continuity of the out-of-plane stresses, alternatively denoted by Carrera (1997) as the C_z^0 *requirements*.

A refinement of the CLT, in which the transverse shear stresses are taken into account, was achieved with the extension to laminates of the so-called Reissner-Mindlin theory, or *First-order transverse Shear Deformation Theory* (FSDT). It provides improved global response estimates for deflections, vibration frequencies and buckling loads of moderately thick composites when compared to the CLT (see Reddy, 2004). Both approaches (CLT and FSDT) consider all layers as one anisotropic ESL and, as a consequence, they cannot model the warping effect of cross-sections. Furthermore, the assumption of a non-deformable normal results in incompatible shearing stresses between adjacent layers. The latter approach, because it assumes constant transverse shear stress, also requires the introduction of an arbitrary shear correction factor which depends on the lamination parameters for obtaining accurate results. Such a theory is adequate to predict only the gross behavior of laminates. *Higher-Order Theories* (HOTs), overcoming some of these limitations, were presented for example by Reddy and Liu (1985) and Reddy (1990) for laminated plates and shells. However, because of the material mismatch at the intersection of the layers, the HOT also lead to transverse shear and normal stress mismatch at the intersection. In conclusion, ESL theories are found to be inadequate for detailed, accurate, local stress analysis of laminated structures.

If detailed response of individual layers is required, as is the case for example for piezoelectric layers, and if significant variations in displacements gradients between layers exist, as is the case of local phenomena usually in viscoelastic layers or sandwich structures with soft cores, LWT (discrete layer) become more suitable to model the intralaminar and interlaminar effects and the warping of the cross section. The LWT corresponds to the implementation of CLT, FSDT or HOT at a layer level. That is, each layer is seen as an individual plate or shell and compatibility (continuity) of displacement (and eventually out-of-plane stress) components with correspondence to each interface is imposed as a constraint. As can be seen in Garção et al. (2004) and Lage et al. (2004), high-order displacement-based or mixed LWT have been successfully used to accurately model the behavior of laminates taking into account the interlaminar and intralaminar effects.

Another alternative, with a reduced computational effort, in the framework of ESL theories, is the use of the so-called *Zig-Zag Theories* (ZZTs), which have their origins and most significant contributions coming from the Russian school. Refined ZZTs have therefore been motivated to fulfill *a priori* (in a complete or partial form) the C_z^0 requirements. The fundamental ideas in developing ZZTs consists to assume a certain displacement and/or stress model in each layer and then to use compatibility and equilibrium conditions at the interface to reduce the number of the unknown variables and keep the number of variables independent of the number of layers.

As stated by Carrera (2003b), the first contribution to the ZZTs was supposedly given by Lekhnitskii in the 1930s [see the brief treatment concerning a layered beam in the English translation of his book (Lekhnitskii, 1968, Section 18)]. Apart from the method proposed by Lekhnitskii, which was almost ignored, two other independent contributions, which received much more attention from the scientific community, have been proposed in the literature in the second half of last century. The first of these was originally given by Ambartsumian in the 1950s, motivated by the attempt to refine the CLT to include partially or completely the C_z^0 requirements, and was applied to anisotropic single and multilayer plates and shells [see his textbooks: Ambartsumyan (1991); Ambartsumian (1991)]. Several variations have been presented which consisted in direct or particular applications of the original Ambartsumian's idea. Whitney (see Ashton and Whitney, 1970, Chapter 7) introduced the theory in the Western community and applied it to non-symmetrical plates, whereas the extension to multilayer shells and dynamic problems was made by Rath and Das (1973). However, several useless works concerning particular applications of Ambartsumian's original theory were developed presenting progressive refinements towards the original idea. It was only in the 1990s that the original theory was re-obtained as can be found, among others, in the works of Cho and Parmerter (1993), Beakou and Touratier (1993) and Soldatos and Timarci (1993). Regarding the second independent contribution, it was given in the 1980s by Reissner (1984), who proposed a mixed variational theorem that allows both displacements and stress assumptions to be made. Significant contributions to the theory proposed by Reissner were made by Murakami (1986) that introduced a zig-zag form of displacement field and Carrera (2004) that presented a systematic generalized manner of using the Reissner mixed variational principle to develop FE applications of ESL theories and LWT of plates and shells. For further details, an historical review of ZZT was performed by Carrera (2003a,b). A discussion on the theories and FEs for multilayered structures, with numerical assessment and a benchmarks for plate and shell structures can also be found in the literature (Noor and Burton, 1989; Noor et al., 1996; Ghugal and Shimpi, 2001, 2002; Carrera, 2002, 2003a; Reddy and Arciniega, 2004).

This paper concerns the FE modeling of anisotropic laminated (or multilayer) shells. A discrete layer approach is employed in this work and a single layer is first considered and isolated from the multilayer shell structure. The weak form of the governing equations of the anisotropic single layer of the multilayer shell is derived with Hamilton's principle using a "mixed" (stresses/displacements) definition of the displacement field. A *semi-inverse* iterative procedure (stresses/strains-displacements) is used to derive the layer "mixed" non-linear displacement field, in terms of a blend of the generalized displacements of the Love-Kirchhoff and Reissner-Mindlin shell theories and the stress components at the generic layer interfaces, without any simplifying assumptions regarding the thinness of the shell being considered. Results from 3-D elasticity solutions are used to postulate adequate definitions of the out-of-plane shear stress components, which, in conjunction with the Reissner-Mindlin theory definitions of the shell in-plane stresses, are utilized to derive the "mixed" displacement field.

First, a *fully refined* shell theory, where displacement and full out-of-plane stresses continuity and homogeneous stress conditions on the top and bottom surfaces of the whole laminate might be considered, is conceptually proposed. Then, due the dramatic complexity of the *fully refined* shell theory, which for the physical problem to be treated in this work does not present an appellative trade-off between accuracy and complexity, its underlying refinements are not pursued here. Hence, some restrictions and simplifications are introduced and a *partially refined* shell theory, where the out-of-plane normal stress continuity is relaxed and a plane stress state is considered, is established and used to develop a FE solution for segmented multilayer doubly-curved orthotropic shells. Based on the weak forms a a *partially refined* shell FE solution is developed for the generic single shell layer. Afterward, the single layer 2-D four-noded shell FE is "regenerated" to an equivalent 3-D form, which allows interlayer displacements and out-of-plane stresses continuity between adjacent interfaces of different layers to be imposed, and a multilayer shell FE is obtained by assembling, at an elemental FE level, all the "regenerated" single layer FE contributions. A dynamic condensation technique is employed to eliminate the stress DoFs and to cast the problem in an equivalent displacement-based FE model form.

Regarding the deformation theory developed in this work, it is inspired in Ambartsumian's contributions for the deformation theory of single layer anisotropic plates and shells (Ambartsumian, 1958; Ambartsumyan, 1991; Ambartsumian, 1991). Ambartsumian basically used a *semi-inverse* method to develop refined shear deformation theories. They are based on assuming a refined distribution of the transverse shear stresses and in the use of the the equilibrium and constitutive equations to derive expressions for the in-plane displacements, which in turn become non-linear in the thickness coordinate. Improvements including the effect of transverse normal strain were also considered for plates and shells. In some of these refined theories transverse shear stress distributions are assumed to follow a parabolic law and to satisfy zero shear stress conditions at the top and bottom surfaces of the plate or shell. However, if out-of-plane stresses continuity is required, a "mixed" formulation should consider also the stresses on the interfaces of the single layer, which, for the sake of simplicity, were, in general, assumed to be nil in the single layer theories developed by Ambartsumian. That puts some limitations in the generalization of the theory to multilayer structures since neither the displacement nor the normal stresses are available on the interfaces and interlayer continuity can not be imposed. Furthermore, his multilayer approach doesn't allow to consider segmented layers, which is something usually required, for example, in the study of segmented hybrid damping treatments. When that is the case, individual refined theories must be considered for each individual discrete-layer.

When compared with Ambartsumian's *first* and *second improved theories* of anisotropic shells (cf. Ambartsumian, 1991), the proposed *fully* and *partially refined* shell theories, similar to Ambartsumian's *first* and *second improved theories*, respectively, assume as a first approximation of the in-plane stresses the ones obtained with the FSDT instead of the CLT. Additionally, all the surface shear stresses are retained in the formulation since they will be used afterward with the "regenerated" 3-D element to generalize the theory to segmented multilayered shells. Furthermore, the theory is extended to elastic multilayered shells and a FE solution is developed. Thus, however strongly inspired in Ambartsumian's work, the deformation theories of the present work have some important differences and novelties which represent a further step towards the demanded refinement of multilayer structural models. It is worthy to mention that it would be very complicated and cumbersome to fulfill *a priori* all the C_z^0 requirements for a multilayer anisotropic shell. Instead, a discrete layer approach is used, which allows interlayer displacement and out-of-plane stresses continuity to be imposed *a posteriori* in a more straightforward manner, by means of a through-the-thickness assemblage of the "regenerated" single layer FE.

Regarding the variational approach used to get the governing equations, similar analytical works, using theories denoted as *partial mixed* theories, where the displacement field is defined in terms of generalized displacements and generalized surface and transverse stresses, can be found for beams and plates in the open literature (Rao et al., 2001; Rao and Desai, 2004; Rao et al., 2004). In contrast to other *fully mixed* methods, as the ZZT based on Reissner's contribution, where mixed variational principles are used, Hamilton's principle has also been employed to derive the governing equations (i.e., no mixed-enhanced variational principles are considered). The present work extends a similar concept to multilayered shells and a FE solution is developed.

Despite the fact that this work is devoted to multilayer elastic composite laminated shells, one of the ultimate aims of the developed theories is to apply them to study multiphysics problems, as is the case when designing hybrid active-passive (piezoelectric and viscoelastic) damping treatments. Usually these damping treatments are discontinuous and composed of segmented damping layers which motivates the type of discrete layer approach presented in this work. Thus, works available in the open literature concerning structures with damping treatments are worthy to refer and to discuss with some detail since they have tackled a similar problem and dealt with the physical constraints that the present approaches, when extended for instance to coupled segmented multiphysics piezo-visco-elastic shell structures (Vasques and Dias Rodrigues, 2006), try to circumvent. A work regarding a three-layered coupled piezo-visco-elastic plate FE was developed by Chattopadhyay et al. (2001), where a HOT was used for the definition of the displacement field of each individual layer and the displacement and shear stress continuity were assured. However, the formulation is limited to the study of active constrained layer damping (ACLD) treatments on three-layered plates and doesn't allow the study of multilayer structures (arbitrary damping treatments).

As far as the ”regeneration” concept is concerned, the concept has been employed also by Cho and Averill (2000) in the the framework of the ZZTs, where a plate FE, avoiding the shortcomings of requiring C^1 continuity of the transverse displacement, has been developed using a first-order zig-zag sublaminate theory for laminated composite and sandwich panels. However, the formulation is based on the decomposition of the whole structure in sublaminates, a linear piecewise function is assumed for the displacement of each sublaminate and the interlaminar displacement and normal shear stress continuity is imposed between the layers of the same sublaminate but not between the sublaminates. In comparison, the present work uses a similar ”regeneration” concept to shell-type structures with a more refined deformation theory being employed which allows displacement and interlayer continuity between all layers to be imposed.

FULLY REFINED MATHEMATICAL MODEL OF GENERAL SHELLS

Physical Problem and Modeling Approach

The physical problem of this work concerns a general anisotropic multilayer shell composed of arbitrary stacking sequences of layers of different materials (e.g, elastic, piezoelectric or viscoelastic materials) which might represent either a composite laminated shell or an host shell-type structure to which damping layers are attached (Fig. 1). It is assumed that all the layers have constant thickness and that they might be continuous or segmented across the shell surface curvilinear directions.

A mathematical theory is established by treating the shell as a multilayer shell of arbitrary materials. First, a generic layer is isolated from the global laminate, and the week form of the governing equations is derived for the individual layer. Interlaminar (or interlayer) continuity conditions of displacements and stresses (surface stresses) and homogeneous (or not) stress boundary conditions at inner and outer boundary surfaces of the global multilayer shell are imposed later, at the multilayer FE level, by assembling the contributions of all the individual layers to be considered.

Shell Differential Geometry

Consider a generic anisotropic layer (Fig. 2) extracted from the multilayer shell in Fig. 1. The generic anisotropic (or rotated orthotropic) layer has a constant thickness of $2h$ and a plane of elastic symmetry parallel to the middle surface Ω_0. The latter surface is used as a reference surface, referred to a set of curvilinear orthogonal coordinates α and β which coincide with the lines of principal curvature of the middle surface. Let z denote the distance, comprised in the interval $[-h, h]$, measured along the normal of a point (α, β) of the middle surface Ω_0 and a point (α, β, z) of the shell layer and Ω denote a surface at a distance z and parallel to Ω_0. The square of an arbitrary differential element of arc length $\mathrm{d}s$, the infinitesimal area of a rectangle in Ω and the infinitesimal volume $\mathrm{d}V$ of the shell layer are given by

$$ds^2 = {H_\alpha}^2 d\alpha^2 + {H_\beta}^2 d\beta^2 + {H_z}^2 dz^2, \qquad d\Omega = H_\alpha H_\beta \, d\alpha \, d\beta, \qquad dV = H_\alpha H_\beta H_z \, d\alpha \, d\beta \, dz, \tag{1}$$

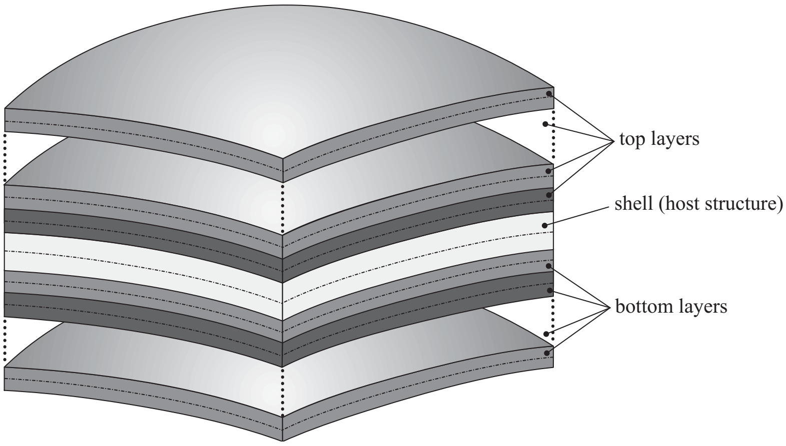

Figure 1: Multilayer shell with arbitrary stacking sequences of layers of different materials.

where $H_\alpha = H_\alpha(\alpha, \beta, z)$, $H_\beta = H_\beta(\alpha, \beta, z)$ and H_z are the so-called *Lamé parameters* given by

$$H_\alpha = A_\alpha \left(1 + \frac{z}{R_\alpha}\right), \qquad H_\beta = A_\beta \left(1 + \frac{z}{R_\beta}\right), \qquad H_z = 1. \tag{2}$$

$A_\alpha = A_\alpha\,(\alpha, \beta)$ and $A_\beta = A_\beta\,(\alpha, \beta)$ are the square root of the coefficients of the *first fundamental form* and $R_\alpha = R_\alpha\,(\alpha, \beta)$ and $R_\beta = R_\beta\,(\alpha, \beta)$ the principal radii of curvature of the middle surface Ω_0 (see, for example, Kraus, 1967).

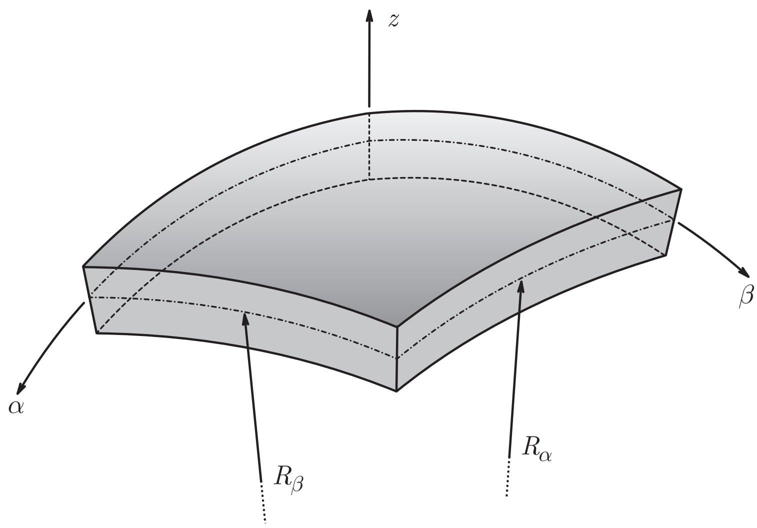

Figure 2: Generic anisotropic shell layer.

General Assumptions and Relationships

Different assumptions are made regarding the mechanical behavior of the generic shell layer:

(a) Strains and displacements are sufficiently small so that the quantities of the second- and higher-order magnitude in the strain-displacement relationships may be neglected in comparison with the first-order terms (i.e., infinitesimal strains and linear geometric elasticity conditions are assumed);

(b) The shear stresses $\sigma_{z\alpha}(\alpha,\beta,z)$ and $\sigma_{\beta z}(\alpha,\beta,z)$, or the corresponding strains $\varepsilon_{z\alpha}(\alpha,\beta,z)$ and $\varepsilon_{\beta z}(\alpha,\beta,z)$, vary in the generic shell layer thickness according to a specified law, which is defined by a "refining" even function $f(z)$ (e.g., parabolic, trigonometric), and the shear angle rotations $\psi_\alpha(\alpha,\beta)$ and $\psi_\beta(\alpha,\beta)$ of a normal to the reference middle surface obtained from the FSDT;

(c) The normal stresses $\sigma_{zz}(\alpha,\beta,z)$ at areas parallel to the middle surface are not negligible and are obtained through the out-of-plane equilibrium equation in orthogonal curvilinear coordinates;

(d) Non-homogeneous shear and normal stress conditions are assumed at the top and bottom surfaces of the generic shell layer;

(e) The strain $\varepsilon_{zz}(\alpha,\beta,z)$ is determined through the constitutive equation assuming as first approximations of the in-plane stresses the ones obtained with the FSDT for shells (defined in Appendix), $\sigma^*_{\alpha\alpha}(\alpha,\beta,z)$, $\sigma^*_{\beta\beta}(\alpha,\beta,z)$ and $\sigma^*_{\alpha\beta}(\alpha,\beta,z)$, without any simplification regarding the thinness of the shell being made (i.e., the terms $z/R_\alpha\,(\alpha,\beta)$ and $z/R_\beta\,(\alpha,\beta)$ are fully retained).

According to assumption (b) the out-of-plane shear stress components are postulated as

$$\begin{aligned}\sigma_{z\alpha}(\alpha,\beta,z) &= \frac{1}{H_\alpha}\left[\bar{\tau}_{z\alpha}(\alpha,\beta) + \frac{z}{2h}\tilde{\tau}_{z\alpha}(\alpha,\beta) + f(z)\psi_\alpha(\alpha,\beta)\right],\\ \sigma_{\beta z}(\alpha,\beta,z) &= \frac{1}{H_\beta}\left[\bar{\tau}_{\beta z}(\alpha,\beta) + \frac{z}{2h}\tilde{\tau}_{\beta z}(\alpha,\beta) + f(z)\psi_\beta(\alpha,\beta)\right],\end{aligned} \tag{3}$$

where $\psi_\alpha = \psi_\alpha(\alpha,\beta)$ and $\psi_\beta = \psi_\beta(\alpha,\beta)$ are the shear angles obtained with the FSDT (defined in Appendix) and the *bar* and *tilde* above $\tau_{z\alpha}$ and $\tau_{\beta z}$ are used to denote *mean* surface shear stresses, given by

$$\bar{\tau}_{z\alpha}(\alpha,\beta) = \frac{1}{2}\left[\sigma^t_{z\alpha}(\alpha,\beta) - \sigma^b_{z\alpha}(\alpha,\beta)\right], \qquad \bar{\tau}_{\beta z}(\alpha,\beta) = \frac{1}{2}\left[\sigma^t_{\beta z}(\alpha,\beta) - \sigma^b_{\beta z}(\alpha,\beta)\right], \tag{4}$$

and *relative* ones, given by

$$\tilde{\tau}_{z\alpha}(\alpha,\beta) = \sigma^t_{z\alpha}(\alpha,\beta) + \sigma^b_{z\alpha}(\alpha,\beta), \qquad \tilde{\tau}_{\beta z}(\alpha,\beta) = \sigma^t_{\beta z}(\alpha,\beta) + \sigma^b_{\beta z}(\alpha,\beta), \tag{5}$$

where $(\cdot)^t$ and $(\cdot)^b$ denote the shear stresses at the top and bottom surfaces (i.e., at $z = \pm h$) of the generic shell layer (see Fig. 3).

Substituting the values of the postulated out-of-plane shear stresses $\sigma_{z\alpha}(\alpha,\beta,z)$ and $\sigma_{\beta z}(\alpha,\beta,z)$ in Eqs. (3) into the static transverse equilibrium equation in orthogonal curvilinear coordinates in the third of Eqs. (A2) of the Appendix, in the absence of body forces, taking into account the definitions in (4) and (5) and integrating with respect to z, after some algebra, the normal stress component $\sigma_{zz}(\alpha,\beta,z)$ is expressed in terms of coefficients of powers of z (for the sake of simplicity, since they are at this point they are not important their detailed definitions will be given latter) as,

$$\sigma_{zz}(\alpha,\beta,z) = \sigma^{(0)}_{zz}(\alpha,\beta) + \sigma^{(1)}_{zz}(\alpha,\beta,z) + \sigma^{(2)}_{zz}(\alpha,\beta,z) + \sigma^{(f)}_{zz}(\alpha,\beta,z), \tag{6}$$

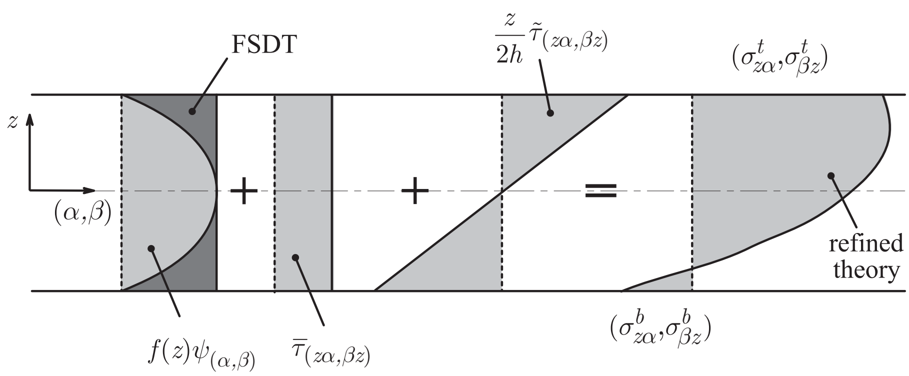

Figure 3: Postulated out-of-plane shear stress distributions of the refined and FSDT theories.

where $\sigma^{(0)}_{zz}(\alpha,\beta)$ is an integration function independent of z that is determined from the boundary conditions on the top and bottom surfaces of the shell layer, $\sigma^t_{zz}(\alpha,\beta)$ and $\sigma^b_{zz}(\alpha,\beta)$, respectively. Regarding the definitions of the higher order terms, the first-order one, $\sigma^{(1)}_{zz}(\alpha,\beta,z)$, depends of the mean shear stresses $\bar{\tau}_{z\alpha}(\alpha,\beta)$ and $\bar{\tau}_{\beta z}(\alpha,\beta)$, the second-order term, $\sigma^{(2)}_{zz}(\alpha,\beta,z)$, depends of the relative shear stresses $\tilde{\tau}_{z\alpha}(\alpha,\beta)$ and $\tilde{\tau}_{\beta z}(\alpha,\beta)$, and the term depending of $f(z)$, $\sigma^{(f)}_{zz}(\alpha,\beta,z)$, is related with the shear angles $\psi_\alpha(\alpha,\beta)$ and $\psi_\beta(\alpha,\beta)$.

By satisfying the stress boundary conditions $\sigma^t_{zz}(\alpha,\beta)$ and $\sigma^b_{zz}(\alpha,\beta)$ at the top and bottom surfaces, after some algebra, the function dependent only of (α,β) (independent of z) that results from the integration is given as

$$\begin{aligned}\sigma^{(0)}_{zz}(\alpha,\beta) &= \bar{\tau}_{zz}(\alpha,\beta) - h^2\sigma^{(2)}_{zz}(\alpha,\beta) - \frac{[F(h)+F(-h)]}{2F(z)}\sigma^{(f)}_{zz}(\alpha,\beta,z)\\ &= \bar{\tau}_{zz}(\alpha,\beta) - h^2\sigma^{(2)}_{zz}(\alpha,\beta),\end{aligned} \tag{7}$$

where

$$F(z) = \int f(z)dz. \tag{8}$$

Since $\sigma^{(0)}_{zz}(\alpha,\beta)$ can't depend on z, Eq. (7) is simplified, which is confirmed since $F(z)$ is and odd function, i.e., $F(h) = -F(-h)$. Additionally, an extra equation is obtained by imposing the top and bottom boundary conditions, which yields the term $\sigma^{(f)}_{zz}(\alpha,\beta,z)$ given as

$$\sigma^{(f)}_{zz}(\alpha,\beta,z) = \frac{F(z)}{F(h)-F(-h)}\left[\tilde{\tau}_{zz}(\alpha,\beta) - 2h\sigma^{(1)}_{zz}(\alpha,\beta,z)\right], \tag{9}$$

where similarly to Eqs. (4) and (5), the *bar* and *tilde* have been used to denote *mean* and *relative* transverse stresses,

$$\bar{\tau}_{zz}(\alpha,\beta) = \frac{1}{2}\left[\sigma^t_{zz}(\alpha,\beta) - \sigma^b_{zz}(\alpha,\beta)\right], \qquad \tilde{\tau}_{zz}(\alpha,\beta) = \sigma^t_{zz}(\alpha,\beta) + \sigma^b_{zz}(\alpha,\beta). \tag{10}$$

After some algebra, the remaining undefined terms of Eq. (6) are given by

$$\begin{aligned}\sigma^{(1)}_{zz}(\alpha,\beta,z) = \frac{z}{H_\alpha H_\beta}\Bigg[&H_\beta\frac{A_\alpha}{R_\alpha}\sigma^{*(0)}_{\alpha\alpha}(\alpha,\beta,z)\\ &+H_\alpha\frac{A_\beta}{R_\beta}\sigma^{*(0)}_{\beta\beta}(\alpha,\beta,z) - \frac{\partial\bar{\tau}_{z\alpha}(\alpha,\beta)}{\partial\alpha} - \frac{\partial\bar{\tau}_{\beta z}(\alpha,\beta)}{\partial\beta}\Bigg],\end{aligned} \tag{11}$$

$$\sigma_{zz}^{(2)}(\alpha,\beta,z)=\frac{z^2}{H_\alpha H_\beta}\left[H_\beta\frac{A_\alpha}{2R_\alpha}\sigma_{\alpha\alpha}^{*(1)}(\alpha,\beta,z)\right.$$
$$\left.+H_\alpha\frac{A_\beta}{2R_\beta}\sigma_{\beta\beta}^{*(1)}(\alpha,\beta,z)-\frac{1}{4h}\frac{\partial\tilde{\tau}_{z\alpha}(\alpha,\beta)}{\partial\alpha}-\frac{1}{4h}\frac{\partial\tilde{\tau}_{\beta z}(\alpha,\beta)}{\partial\beta}\right]. \quad (12)$$

At this point, it is worthy to mention that in order to keep the formulation of the transverse stress $\sigma_{zz}(\alpha,\beta,z)$ in Eq. (6) general, its last term depends of the integral of the shear "refining" function, $F(z)$. Depending of the type and/or order in z of the shear function $f(z)$, (e.g., polynomial, trigonometric function), different expansions in z can be obtained and different deformation theories can be considered.

Considering the strain-stress constitutive behavior of the out-of-plane shear strains $\varepsilon_{z\alpha}$ and $\varepsilon_{\beta z}$ expressed in Eq. (A13) of Appendix and the postulated shear stresses in Eqs. 3, the shear strains are given as

$$\begin{aligned}\varepsilon_{z\alpha}(\alpha,\beta,z)&=\frac{1}{H_\alpha}\left[\bar{\Sigma}_{z\alpha}(\alpha,\beta)+\tfrac{z}{2h}\tilde{\Sigma}_{z\alpha}(\alpha,\beta)+f(z)\Psi_\alpha(\alpha,\beta)\right],\\ \varepsilon_{\beta z}(\alpha,\beta,z)&=\frac{1}{H_\beta}\left[\bar{\Sigma}_{\beta z}(\alpha,\beta)+\tfrac{z}{2h}\tilde{\Sigma}_{\beta z}(\alpha,\beta)+f(z)\Psi_\beta(\alpha,\beta)\right],\end{aligned} \quad (13)$$

where the following notations regarding the *mean* surface stresses

$$\bar{\Sigma}_{z\alpha}(\alpha,\beta)=\bar{s}_{45}\bar{\tau}_{\beta z}(\alpha,\beta)+\bar{s}_{55}\bar{\tau}_{z\alpha}(\alpha,\beta),\quad \bar{\Sigma}_{\beta z}(\alpha,\beta)=\bar{s}_{44}\bar{\tau}_{\beta z}(\alpha,\beta)+\bar{s}_{45}\bar{\tau}_{z\alpha}(\alpha,\beta), \quad (14)$$

relative surface stresses

$$\tilde{\Sigma}_{z\alpha}(\alpha,\beta)=\bar{s}_{45}\tilde{\tau}_{\beta z}(\alpha,\beta)+\bar{s}_{55}\tilde{\tau}_{z\alpha}(\alpha,\beta),\quad \tilde{\Sigma}_{\beta z}(\alpha,\beta)=\bar{s}_{44}\tilde{\tau}_{\beta z}(\alpha,\beta)+\bar{s}_{45}\tilde{\tau}_{z\alpha}(\alpha,\beta), \quad (15)$$

and shear angles

$$\Psi_\alpha(\alpha,\beta)=\bar{s}_{45}\psi_\beta(\alpha,\beta)+\bar{s}_{55}\psi_\alpha(\alpha,\beta),\quad \Psi_\beta(\alpha,\beta)=\bar{s}_{44}\psi_\beta(\alpha,\beta)+\bar{s}_{45}\psi_\alpha(\alpha,\beta), \quad (16)$$

are used. Alternatively, for simplicity, the shear strains in Eqs. (13) can still be expressed in terms of coefficients of increasing powers and functions of z as

$$\begin{aligned}\varepsilon_{z\alpha}(\alpha,\beta,z)&=\frac{1}{H_\alpha}\left[\varepsilon_{z\alpha}^{(0)}(\alpha,\beta)+z\varepsilon_{z\alpha}^{(1)}(\alpha,\beta)+\varepsilon_{z\alpha}^{(f)}(\alpha,\beta,z)\right],\\ \varepsilon_{\beta z}(\alpha,\beta,z)&=\frac{1}{H_\beta}\left[\varepsilon_{\beta z}^{(0)}(\alpha,\beta)+z\varepsilon_{\beta z}^{(1)}(\alpha,\beta)+\varepsilon_{\beta z}^{(f)}(\alpha,\beta,z)\right],\end{aligned} \quad (17)$$

where the definitions of the terms in the previous equations are obvious from the analysis of Eqs. (13).

In a similar way, considering the transverse strain-stress constitutive behavior of $\varepsilon_{zz}(\alpha,\beta,z)$ expressed in Eq. (A13) of Appendix and taking as first approximations of the in-plane stress components the ones obtained with the FSDT for shells described in Eqs. (A7) of Appendix, i.e., $\sigma_{\alpha\alpha}^*(\alpha,\beta,z)$, $\sigma_{\beta\beta}^*(\alpha,\beta,z)$ and $\sigma_{\alpha\beta}^*(\alpha,\beta,z)$, and considering $\sigma_{zz}(\alpha,\beta,z)$ as defined in (6), the transverse strain component is given by

$$\varepsilon_{zz}(\alpha,\beta,z)\approx\bar{s}_{13}\sigma_{\alpha\alpha}^*(\alpha,\beta,z)+\bar{s}_{23}\sigma_{\beta\beta}^*(\alpha,\beta,z)+\bar{s}_{33}\sigma_{zz}(\alpha,\beta,z)+\bar{s}_{36}\sigma_{\alpha\beta}^*(\alpha,\beta,z). \quad (18)$$

Mixed Displacement Field and Strains

In this section the displacement field of the shell layer is derived by considering the out-of-plane strains $\varepsilon_{\beta z}$, $\varepsilon_{z\alpha}$ and ε_{zz} presented in the previous section in Eqs. (17) and (18), respectively, and the correspondent out-of-plane strain-displacement relations in Eqs. (A1) of the Appendix. By virtue of the assumptions previously considered,

$$\begin{aligned}\varepsilon_{zz} = \frac{\partial w}{\partial z} \approx \bar{s}_{13}\left[\sigma^{*(0)}_{\alpha\alpha}(\alpha,\beta,z) + \sigma^{*(1)}_{\alpha\alpha}(\alpha,\beta,z)\right] + \bar{s}_{23}\left[\sigma^{*(0)}_{\beta\beta}(\alpha,\beta,z) + \sigma^{*(1)}_{\beta\beta}(\alpha,\beta,z)\right] \\ + \bar{s}_{33}\left[\sigma^{(0)}_{zz}(\alpha,\beta,z) + \sigma^{(1)}_{zz}(\alpha,\beta,z) + \sigma^{(2)}_{zz}(\alpha,\beta,z) + \sigma^{(f)}_{zz}(\alpha,\beta,z)\right] \\ + \bar{s}_{36}\left[\sigma^{*(0)}_{\alpha\beta}(\alpha,\beta,z) + \sigma^{*(1)}_{\alpha\beta}(\alpha,\beta,z)\right]. \qquad (19)\end{aligned}$$

Thus, integrating the previous equation with respect to z over the limits from 0 to z, considering that when $z = 0$ we have $w(\alpha,\beta,z) = w_0(\alpha,\beta)$, and taking into account the time dependence of the strains and stresses definitions, the transverse displacement $w = w(\alpha,\beta,z,t)$ is given by

$$w(\alpha,\beta,z,t) = w^{(0)}(\alpha,\beta,t) + w^{(1)}(\alpha,\beta,z,t) + w^{(2)}(\alpha,\beta,z,t) + w^{(3)}(\alpha,\beta,z,t) + w^{(f)}(\alpha,\beta,z,t), \qquad (20)$$

where

$$\begin{aligned} w^{(0)}(\alpha,\beta,t) &= w_0(\alpha,\beta,t), \\ w^{(1)}(\alpha,\beta,z,t) &= \int_0^z \Big[\bar{s}_{13}\sigma^{*(0)}_{\alpha\alpha}(\alpha,\beta,z,t) + \bar{s}_{23}\sigma^{*(0)}_{\beta\beta}(\alpha,\beta,z,t) \\ &\quad + \bar{s}_{33}\sigma^{(0)}_{zz}(\alpha,\beta,z,t) + \bar{s}_{36}\sigma^{*(0)}_{\alpha\beta}(\alpha,\beta,z,t)\Big]\, dz, \\ w^{(2)}(\alpha,\beta,z,t) &= \int_0^z \Big[\bar{s}_{13}\sigma^{*(1)}_{\alpha\alpha}(\alpha,\beta,z,t) + \bar{s}_{23}\sigma^{*(1)}_{\beta\beta}(\alpha,\beta,z,t) \qquad (21) \\ &\quad + \bar{s}_{33}\sigma^{(1)}_{zz}(\alpha,\beta,z,t) + \bar{s}_{36}\sigma^{*(1)}_{\alpha\beta}(\alpha,\beta,z,t)\Big]\, dz, \\ w^{(3)}(\alpha,\beta,z,t) &= \int_0^z \bar{s}_{33}\sigma^{(2)}_{zz}(\alpha,\beta,z,t)dz, \\ w^{(f)}(\alpha,\beta,z,t) &= \int_0^z \bar{s}_{33}\sigma^{(f)}_{zz}(\alpha,\beta,z,t)dz. \end{aligned}$$

In a similar way, using the relations

$$\begin{aligned} \varepsilon_{\beta z}(\alpha,\beta,z) &= H_\beta \frac{\partial}{\partial z}\left(\frac{v(\alpha,\beta,z)}{H_\beta}\right) + \frac{1}{H_\beta}\frac{\partial w(\alpha,\beta,z)}{\partial \beta} \\ &\approx \frac{1}{H_\beta}\left[\varepsilon^{(0)}_{\beta z}(\alpha,\beta) + z\varepsilon^{(1)}_{\beta z}(\alpha,\beta) + \varepsilon^{(f)}_{\beta z}(\alpha,\beta,z)\right], \\ \varepsilon_{z\alpha}(\alpha,\beta,z) &= H_\alpha \frac{\partial}{\partial z}\left(\frac{u(\alpha,\beta,z)}{H_\alpha}\right) + \frac{1}{H_\alpha}\frac{\partial w(\alpha,\beta,z)}{\partial \alpha} \qquad (22) \\ &\approx \frac{1}{H_\alpha}\left[\varepsilon^{(0)}_{z\alpha}(\alpha,\beta) + z\varepsilon^{(1)}_{z\alpha}(\alpha,\beta) + \varepsilon^{(f)}_{z\alpha}(\alpha,\beta,z)\right], \end{aligned}$$

integrating in order to z over the limits from 0 to z and considering that for $z = 0$, the displacements on the middle surface are given by $u(\alpha,\beta,z) = u_0(\alpha,\beta)$ and $v(\alpha,\beta,z) = v_0(\alpha,\beta)$, the time dependent tangential displacements of any point of the shell are given by

$$\begin{aligned} u(\alpha,\beta,z,t) &= -H_\alpha \int_0^z \frac{1}{{H_\alpha}^2} \frac{\partial w(\alpha,\beta,z,t)}{\partial\alpha} dz \\ &+ H_\alpha \int_0^z \frac{1}{{H_\alpha}^2} \left[\varepsilon_{z\alpha}^{(0)}(\alpha,\beta,t) + z\varepsilon_{z\alpha}^{(1)}(\alpha,\beta,t) + \varepsilon_{z\alpha}^{(f)}(\alpha,\beta,z,t)\right] dz \\ &= u^{(0)}(\alpha,\beta,z,t) + u^{(1)}(\alpha,\beta,z,t) + u^{(2)}(\alpha,\beta,z,t) + u^{(3)}(\alpha,\beta,z,t) \\ &+ u^{(4)}(\alpha,\beta,z,t) + u^{(f)}(\alpha,\beta,z,t), \end{aligned} \tag{23}$$

$$\begin{aligned} v(\alpha,\beta,z,t) &= -H_\beta \int_0^z \frac{1}{{H_\beta}^2} \frac{\partial w(\alpha,\beta,z,t)}{\partial\beta} dz \\ &+ H_\beta \int_0^z \frac{1}{{H_\beta}^2} \left[\varepsilon_{\beta z}^{(0)}(\alpha,\beta,t) + z\varepsilon_{\beta z}^{(1)}(\alpha,\beta,t) + \varepsilon_{\beta z}^{(f)}(\alpha,\beta,z,t)\right] dz \\ &= v^{(0)}(\alpha,\beta,z,t) + v^{(1)}(\alpha,\beta,z,t) + v^{(2)}(\alpha,\beta,z,t) + v^{(3)}(\alpha,\beta,z,t) \\ &+ v^{(4)}(\alpha,\beta,z,t) + v^{(f)}(\alpha,\beta,z,t), \end{aligned} \tag{24}$$

where

$$\begin{aligned} u^{(0)}(\alpha,\beta,z,t) &= \frac{H_\alpha}{A_\alpha} u_0(\alpha,\beta,t), \\ u^{(1)}(\alpha,\beta,z,t) &= -H_\alpha \int_0^z \frac{1}{{H_\alpha}^2} \left[\frac{\partial w^{(0)}(\alpha,\beta,t)}{\partial\alpha} - \varepsilon_{z\alpha}^{(0)}(\alpha,\beta,t)\right] dz, \\ u^{(2)}(\alpha,\beta,z,t) &= -H_\alpha \int_0^z \frac{1}{{H_\alpha}^2} \left[\frac{\partial w^{(1)}(\alpha,\beta,z,t)}{\partial\alpha} - z\varepsilon_{z\alpha}^{(1)}(\alpha,\beta,t)\right] dz, \\ u^{(3)}(\alpha,\beta,z,t) &= -H_\alpha \int_0^z \frac{1}{{H_\alpha}^2} \frac{\partial w^{(2)}(\alpha,\beta,z,t)}{\partial\alpha} dz, \\ u^{(4)}(\alpha,\beta,z,t) &= -H_\alpha \int_0^z \frac{1}{{H_\alpha}^2} \frac{\partial w^{(3)}(\alpha,\beta,z,t)}{\partial\alpha} dz, \\ u^{(f)}(\alpha,\beta,z,t) &= -H_\alpha \int_0^z \frac{1}{{H_\alpha}^2} \left[\frac{\partial w^{(f)}(\alpha,\beta,z,t)}{\partial\alpha} - \varepsilon_{z\alpha}^{(f)}(\alpha,\beta,z,t)\right] dz, \end{aligned} \tag{25}$$

$$\begin{aligned} v^{(0)}(\alpha,\beta,z,t) &= \frac{H_\beta}{A_\beta} v_0(\alpha,\beta,t), \\ v^{(1)}(\alpha,\beta,z,t) &= -H_\beta \int_0^z \frac{1}{{H_\beta}^2} \left[\frac{\partial w^{(0)}(\alpha,\beta,t)}{\partial\beta} - \varepsilon_{\beta z}^{(0)}(\alpha,\beta,t)\right] dz, \\ v^{(2)}(\alpha,\beta,z,t) &= -H_\beta \int_0^z \frac{1}{{H_\beta}^2} \left[\frac{\partial w^{(1)}(\alpha,\beta,z,t)}{\partial\beta} - z\varepsilon_{\beta z}^{(1)}(\alpha,\beta,t)\right] dz, \\ v^{(3)}(\alpha,\beta,z,t) &= -H_\beta \int_0^z \frac{1}{{H_\beta}^2} \frac{\partial w^{(2)}(\alpha,\beta,z,t)}{\partial\beta} dz, \\ v^{(4)}(\alpha,\beta,z,t) &= -H_\beta \int_0^z \frac{1}{{H_\beta}^2} \frac{\partial w^{(3)}(\alpha,\beta,z,t)}{\partial\beta} dz, \\ v^{(f)}(\alpha,\beta,z,t) &= -H_\beta \int_0^z \frac{1}{{H_\beta}^2} \left[\frac{\partial w^{(f)}(\alpha,\beta,z,t)}{\partial\beta} - \varepsilon_{\beta z}^{(f)}(\alpha,\beta,z,t)\right] dz, \end{aligned} \tag{26}$$

In Eqs. (20), (23) and (24) it is shown that in comparison with the CLT of shells, following Love's first approximation, and the FSDT of shells discussed in Appendix, the in-plane and transverse displacements of any point of the shell are nonlinearly dependent on z. Additionally, the same 5 generalized displacements of the FSDT (see Appendix), $u_0 = u_0(\alpha, \beta, t)$, $v_0 = v_0(\alpha, \beta, t)$ and $w_0 = w_0(\alpha, \beta, t)$, which are are the tangential and transverse displacements referred to a point on the middle surface, respectively, and the shear angle rotations of a normal to the reference middle surface,

$$\begin{aligned} \psi_\alpha(\alpha, \beta, t) &= \frac{\partial w_0(\alpha, \beta, t)}{\partial \alpha} + A_\alpha(\alpha, \beta)\,\theta_\alpha(\alpha, \beta, t) - \frac{A_\alpha(\alpha, \beta)}{R_\alpha(\alpha, \beta)} u_0(\alpha, \beta, t), \\ \psi_\beta(\alpha, \beta, t) &= \frac{\partial w_0(\alpha, \beta, t)}{\partial \beta} + A_\beta(\alpha, \beta)\,\theta_\beta(\alpha, \beta, t) - \frac{A_\beta(\alpha, \beta)}{R_\beta(\alpha, \beta)} v_0(\alpha, \beta, t), \end{aligned} \tag{27}$$

are used to define the "generalized" displacements of the proposed refined theory which represent non-linear functions in z of the generalized displacements of FSDT. Thus, based on the assumptions (b)-(e) the 3-D problem of the theory of elasticity has been fully brought to a 2-D problem of the theory of the shell, with Eqs. (20), (23) and (24) establishing the geometrical model of the deformed state of the *fully refined* theory of the generic shell layer.

On the basis of the refined displacement field defined by Eqs. (20), (23) and (24) and the general strain-displacement relations in Eqs. (A1) in Appendix, the not yet defined in-plane strain components of the proposed theory may be determined. For the sake of brevity their definitions will not be given here since they are quite long equations, in terms of high-order derivatives of the generalized displacements of the FSDT, which can be derived from the previous definitions.

PARTIALLY REFINED MATHEMATICAL MODEL OF DOUBLY-CURVED SHELLS

Restrictions and Simplifications

The definitions of the displacement field presented in Eqs. (20), (23) and (24) are quite general and applicable to anisotropic shells of arbitrary curvature. They result from a *fully refined* interactive shell theory based, as a first approximation, on the in-plane stresses of the FSDT. Furthermore, all the terms regarding the thickness coordinate to radii or curvature ratios were retained and no simplifications were made regarding thin shell assumptions. Additionally, transverse shear strains and stresses were not considered negligible and as a result a non-linearly dependent on z transverse displacement was obtained by the iterative procedure. That theory was denoted as *fully refined* since all the strain and stress components of the 3-D elasticity are obtained directly from the "mixed" (in terms of surface stresses and generalized displacements) displacement field by using the strain-displacement relations and an anisotropic constitutive law. It can also be seen from the definitions of the terms of the in-plane displacements in Eqs. (25) and (26) that they involve high-order derivatives (which become even higher for the strain and stress components) of the generalized displacements of the FSDT, which complicates the formulation and FE solution dramatically. Additionally, the *fully refined* mathematical model allows full out-of-plane interlayer (or interlaminar) stress continuity (and, as obvious, displacements too) to be imposed when assembling all the layers contributions at the "regenerated" FE level (further details will be discussed later). This renders a 2-D theory representative of the *full* 3-D behavior of a shell with arbitrary geometry (curvature).

However, the *fully refined* mixed displacement definitions are quite tedious and, for the sake of making the calculations less cumbersome, the general "mixed" displacement field definitions will be restricted to orthotropic shells with constant curvatures, i.e., doubly-curved shells (cylindrical, spherical, toroidal geometries) for which

$$A_\alpha = A_\beta = 1, \qquad \frac{\partial A_\alpha}{\partial \beta} = \frac{\partial A_\beta}{\partial \alpha} = 0, \qquad R_\alpha(\alpha,\beta) = R_\alpha, \qquad R_\beta(\alpha,\beta) = R_\beta. \tag{28}$$

Additionally, the non-linear transverse displacement field definition will be discarded and only the zero-order term, i.e., $w(\alpha,\beta,z,t) = w^{(0)}(\alpha,\beta,t)$, will be retained. This simplification makes the theory less complicated and more suitable to be implemented, since it avoids the higher-order derivatives of the generalized variables, and is denoted as *partial refined* theory. It is well known, however, that in the major part of the problems, the transverse normal stress effects are small when compared with the other stress components (see Robbins and Chopra, 2006). An exception is, for example, in thermo-mechanical analysis where the transverse stress σ_{zz} plays an important role (see Carrera, 1999, 2005), which is not the case here. This simplification implies also the need to make the usual plane-stress assumption, which doesn't allow to impose interlayer transverse normal stress continuity. Regarding the shear stress "refining" function $f(z)$, several functions can be used. However, as stated by Ambartsumyan (1991, p. 37), some arbitrariness in the reasonable selection of $f(z)$ will not introduce inadmissible errors into the refined theory, which in this work will be assumed to follow the law of a quadratic parabola as

$$f(z) = 1 - \frac{z^2}{h^2}. \tag{29}$$

Displacements and Strains

For the sake of simplicity and compactness when writing the mathematical definitions, from this point henceforth, the spatial (α,β,z) and time t dependencies will be omitted from the equations when convenient and only written when necessary for the comprehension of the equations. Thus, under the doubly-curved shell restrictions to the general problem, and following the previously defined *partial refined* theory, the displacement definitions in Eqs. (20), (23) and (24), for an orthotropic shell layer, taking into account the definitions in (A5) in the Appendix, are given as

$$\begin{aligned}
u(\alpha,\beta,z,t) &= \frac{1}{z_\alpha^{(0)}}\left[u_0 + z_\alpha^{*(f)}\bar{s}_{55}^*\psi_\alpha - z_\alpha^{*(0)}\frac{\partial w_0}{\partial \alpha} + z_\alpha^{*(0)}\bar{s}_{55}^*\bar{\tau}_{z\alpha} + z_\alpha^{*(1)}\frac{\bar{s}_{55}^*}{2h}\tilde{\tau}_{z\alpha}\right],\\
v(\alpha,\beta,z,t) &= \frac{1}{z_\beta^{(0)}}\left[v_0 + z_\beta^{*(f)}\bar{s}_{44}^*\psi_\beta - z_\beta^{*(0)}\frac{\partial w_0}{\partial \beta} + z_\beta^{*(0)}\bar{s}_{44}^*\bar{\tau}_{\beta z} + z_\beta^{*(1)}\frac{\bar{s}_{44}^*}{2h}\tilde{\tau}_{\beta z}\right],\\
w(\alpha,\beta,z,t) &= w_0,
\end{aligned} \tag{30}$$

where $z_\alpha^{(0)}$ and $z_\beta^{(0)}$ are defined in Eqs. (A5) of the Appendix and

$$\begin{aligned}
z_\alpha^{*(i)} &= \int_0^z \frac{z^i}{(1+z/R_\alpha)^2}dz, \qquad & z_\beta^{*(i)} &= \int_0^z \frac{z^i}{(1+z/R_\beta)^2}dz,\\
z_\alpha^{*(f)} &= \int_0^z \frac{f(z)}{(1+z/R_\alpha)^2}dz, \qquad & z_\beta^{*(f)} &= \int_0^z \frac{f(z)}{(1+z/R_\beta)^2}dz,
\end{aligned} \tag{31}$$

with $i = 0, 1$. As can be seen in the "mixed" displacement field definition in Eqs. (30), the displacement field is defined in terms of the generalized displacements of the FSDT and mean and relative surface shear stresses. Additionally, as would be expected, if $f(z)$ is assumed equal to unity, which corresponds to case where no correction to the FSDT constant through-the-thickness shear stresses/strains is made, the displacement is the same as the one obtained with the FSDT, with the extra surface shear stress terms. In the limit case where $R_\alpha = R_\beta = \infty$, which corresponds to the case of planar structures such as plates, and considering the parabolic definition of $f(z)$, the displacements are consistent and are expanded in a power series of z up to z^3. In the present case, since the terms z/R_α and z/R_β were fully retained, the displacements are defined with more complex coefficients in terms of powers of z and $\ln(R_\alpha + z)$ and $\ln(R_\beta + z)$. For convenience, the "mixed" displacement field in Eqs. (30) can be expressed in matrix form as

$$\mathbf{u}(\alpha, \beta, z, t) = \mathbf{z}^u(z)\mathbf{u}_0(\alpha, \beta, t) + \mathbf{z}^\tau(z)\boldsymbol{\tau}(\alpha, \beta, t), \tag{32}$$

or, alternatively,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} z_{11}^u & 0 & z_{13}^u\partial_\alpha & z_{14}^u & 0 \\ 0 & z_{22}^u & z_{23}^u\partial_\beta & 0 & z_{25}^u \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{Bmatrix} + \begin{bmatrix} z_{11}^\tau & z_{12}^\tau & 0 & 0 \\ 0 & 0 & z_{23}^\tau & z_{24}^\tau \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \tilde{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \tilde{\tau}_{\beta z} \end{Bmatrix}, \tag{33}$$

where the coefficients of matrices $\mathbf{z}^u(z)$ and $\mathbf{z}^\tau(z)$ are derived from the displacements as explained in the Appendix. In the specific case of $z_{13}^u(z)$ and $z_{23}^u(z)$, which are related with $\partial w_0/\partial\alpha$ and $\partial w_0/\partial\beta$, the partial derivative operators ∂_α and ∂_β in order to α and β, respectively, were also included.

From the definition of the displacement field in Eq. (32), the out-of-plane shear strains are given by Eqs. (17), taking into account the restrictions in (28) and the transverse strain $\varepsilon_{zz} = 0$ [due to the fact that $w(\alpha, \beta, z, t)$ was assumed independent of z]. The in-plane strain components are obtained according to the displacement-strain relations in the first two and last equations of Eqs. (A1) by taking into account (28). Thus, the strains vector without the null component ε_{zz} may be expressed in matrix form as

$$\begin{aligned} \boldsymbol{\varepsilon}(\alpha, \beta, z, t) &= \partial_\varepsilon(z)\mathbf{z}^u(z)\mathbf{u}_0(\alpha, \beta, t) + \partial_\varepsilon(z)\mathbf{z}^\tau(z)\boldsymbol{\tau}(\alpha, \beta, t) \\ &= \mathbf{z}^{\varepsilon u}(z)\mathbf{u}_0(\alpha, \beta, t) + \mathbf{z}^{\varepsilon\tau}(z)\boldsymbol{\tau}(\alpha, \beta, t), \end{aligned} \tag{34}$$

where $\partial_\varepsilon(z)$ is a matrix differential operator given by

$$\partial_\varepsilon(z) = \begin{bmatrix} \partial_\alpha z_\alpha^{(0)} & 0 & z_\alpha^{(0)}/R_\alpha \\ 0 & \partial_\beta z_\beta^{(0)} & z_\beta^{(0)}/R_\beta \\ 0 & \partial_z - z_\beta^{(0)}/R_\beta & \partial_\beta z_\beta^{(0)} \\ \partial_z - z_\alpha^{(0)}/R_\alpha & 0 & \partial_\alpha z_\alpha^{(0)} \\ \partial_\beta z_\beta^{(0)} & \partial_\alpha z_\alpha^{(0)} & 0 \end{bmatrix}, \tag{35}$$

and ∂_z is an other partial differential operator, this time in order to z. Considering the previous operator matrix in Eq. (34), the strains vector is defined in terms of the matrices $\mathbf{z}^{\varepsilon u}(z)$ and

$\mathbf{z}^{\varepsilon\tau}(z)$ as

$$\begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} z_{11}^{\varepsilon u}\partial_\alpha & 0 & z_{13}^{\varepsilon u1} + z_{13}^{\varepsilon u2}\partial_{\alpha\alpha} & z_{14}^{\varepsilon u}\partial_\alpha & 0 \\ 0 & z_{22}^{\varepsilon u}\partial_\beta & z_{23}^{\varepsilon u1} + z_{23}^{\varepsilon u2}\partial_{\beta\beta} & 0 & z_{25}^{\varepsilon u}\partial_\beta \\ 0 & z_{32}^{\varepsilon u} & z_{33}^{\varepsilon u}\partial_\beta & 0 & z_{35}^{\varepsilon u} \\ z_{41}^{\varepsilon u} & 0 & z_{43}^{\varepsilon u}\partial_\alpha & z_{44}^{\varepsilon u} & 0 \\ z_{51}^{\varepsilon u}\partial_\beta & z_{52}^{\varepsilon u}\partial_\alpha & z_{53}^{\varepsilon u}\partial_{\alpha\beta} & z_{54}^{\varepsilon u}\partial_\beta & z_{55}^{\varepsilon u}\partial_\alpha \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{Bmatrix}$$
$$+ \begin{bmatrix} z_{11}^{\varepsilon\tau}\partial_\alpha & z_{12}^{\varepsilon\tau}\partial_\alpha & 0 & 0 \\ 0 & 0 & z_{23}^{\varepsilon\tau}\partial_\beta & z_{24}^{\varepsilon\tau}\partial_\beta \\ 0 & 0 & z_{33}^{\varepsilon\tau} & z_{34}^{\varepsilon\tau} \\ z_{41}^{\varepsilon\tau} & z_{42}^{\varepsilon\tau} & 0 & 0 \\ z_{51}^{\varepsilon\tau}\partial_\beta & z_{52}^{\varepsilon\tau}\partial_\beta & z_{53}^{\varepsilon\tau}\partial_\alpha & z_{54}^{\varepsilon\tau}\partial_\alpha \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \tilde{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \tilde{\tau}_{\beta z} \end{Bmatrix} \tag{36}$$

where $\partial_{\alpha\alpha}$, $\partial_{\beta\beta}$ and $\partial_{\alpha\beta}$ are double and crossed partial differential operators. The coefficients of $\mathbf{z}^{\varepsilon u}(z)$ and $\mathbf{z}^{\varepsilon\tau}(z)$ are given in Appendix. It is worthy to mention that the strain field is defined in terms of zero- and/or first-order derivatives of the generalized in-plane displacements, rotations and surface stresses, and cross derivatives and zero-, first- and second-order derivatives of the generalized transverse displacement w_0.

Variational Formulation

In order to derive the weak form of the equations governing the motion of the single layer shell, Hamilton's principle is used so that

$$\delta \int_{t_0}^{t_1} (T - U + W) dt = 0, \tag{37}$$

where t_0 and t_1 define the time interval, δ denotes the variation, T is the *kinetic energy*, U is the *potential strain energy* and W denotes the *work of the external non-conservative forces*.

Since the stresses have been replaced and considered by means of internal forces and moments due to the thickness integration it is appropriate to alter the definition of the fundamental element of the shell. Accordingly, it will be assumed henceforth that the element which was formerly defined to be dz thick, is replaced, on account of the integrations with respect to z, with an element of thickness h. Such an element is acted upon by the internal forces (stress resultants) and moments per unit arc length and by external effects such as the mechanical forces. The internal forces act upon the edges of the element while the mechanical forces act upon the inner and outer surfaces.

According to Eq. (32), the kinetic energy is given by

$$T = \frac{1}{2} \int_V \rho \dot{\mathbf{u}}^{\mathrm{T}} \dot{\mathbf{u}} \, dV, \tag{38}$$

where $\dot{\mathbf{u}} = \dot{\mathbf{u}}(\alpha, \beta, z, t)$ is the vector of generalized velocities taking into account the time differentiation of the three components of the displacement field expressed in the tri-orthogonal curvilinear coordinate system. The first variation of the kinetic energy yields the virtual work of the inertial forces, given by

$$\delta T = -\int_{\Omega_0} \left[\int_{-h}^{+h} \rho \delta \mathbf{u}^{\mathrm{T}} \ddot{\mathbf{u}} \, H_\alpha H_\beta \, H_z \, dz \right] d\alpha \, d\beta, \tag{39}$$

which is expanded with more detail in terms of the variations of the generalized displacements and stresses in Appendix .

The potential strain energy of the elastic medium is expressed in terms of mechanical stresses and strains by

$$U = \frac{1}{2} \int_V \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\varepsilon} \, dV. \tag{40}$$

Taking into account the plane-stress constitutive behavior in Eq. (A14) in the Appendix, the first variation of the previous equation yields

$$\delta U = \int_{\Omega_0} \left[\int_{-h}^{+h} \boldsymbol{\sigma}^{\mathrm{T}} \delta \boldsymbol{\varepsilon} \, H_\alpha H_\beta H_z \, dz \right] d\alpha \, d\beta, \tag{41}$$

which is expanded with more detail in terms of the variations of the generalized displacements and internal forces and moments in Appendix .

The last term of Eq. (37) considers the work done by the applied mechanical forces which are applied on the inner and outer surfaces and lateral edges of the shell. To write the expressions for the net external forces work recall that Ω denotes a surface at a distance z and parallel to the middle-surface, where Ω^t and Ω^b denote the top and bottom surfaces for which $z = \pm h$, and that Γ denotes the boundary of the shell element, with Γ_α and Γ_β being the boundary edges of constant β and α coordinates, respectively (with the circle on the integral implying that it includes the total boundary of the shell). Thus, the work of the non-conservative forces is given by

$$\begin{aligned} W = \int_{\Omega^t} F_z^t w \, d\Omega^t + \int_{\Omega^b} F_z^b w \, d\Omega^b + \oint_{\Gamma_\alpha} \left[\int_{-h}^{+h} (\hat{\sigma}_{\beta\beta} v + \hat{\sigma}_{\beta\alpha} u + \hat{\sigma}_{\beta z} w) \frac{1}{z_\alpha^{(0)}} dz \right] d\alpha \\ + \oint_{\Gamma_\beta} \left[\int_{-h}^{+h} (\hat{\sigma}_{\alpha\alpha} u + \hat{\sigma}_{\alpha\beta} v + \hat{\sigma}_{z\alpha} w) \frac{1}{z_\beta^{(0)}} dz \right] d\beta, \end{aligned} \tag{42}$$

where $F_z^t = F_z^t(\alpha, \beta, t)$ and $F_z^b = F_z^b(\alpha, \beta, t)$ are transverse normal forces applied on the top and bottom surfaces, and the *hat* over the stresses, $\hat{\sigma}_{\alpha\alpha} = \hat{\sigma}_{\alpha\alpha}(\beta, z, t)$, $\hat{\sigma}_{\alpha\beta} = \hat{\sigma}_{\alpha\beta}(\beta, z, t)$ and $\hat{\sigma}_{z\alpha} = \hat{\sigma}_{z\alpha}(\beta, z, t)$, for the edges normal to α, and $\hat{\sigma}_{\beta\beta} = \hat{\sigma}_{\beta\beta}(\alpha, z, t)$, $\hat{\sigma}_{\beta\alpha} = \hat{\sigma}_{\beta\alpha}(\alpha, z, t)$ and $\hat{\sigma}_{\beta z} = \hat{\sigma}_{\beta z}(\alpha, z, t)$, for the edges normal to β, denotes prescribed stresses on the boundary edges. Retaining only the normal mechanical load on the top surface F_z^t, the first variation of Eq. (42) yields the virtual work of the non-conservative external forces given by

$$\begin{aligned} \delta W = \int_{\Omega_0} Z \delta w_0 \, d\alpha \, d\beta + \oint_{\Gamma_\alpha} \left[\int_{-h}^{+h} (\hat{\sigma}_{\beta\beta} \delta v + \hat{\sigma}_{\beta\alpha} \delta u + \hat{\sigma}_{\beta z} \delta w) \frac{1}{z_\alpha^{(0)}} dz \right] d\alpha \\ + \oint_{\Gamma_\beta} \left[\int_{-h}^{+h} (\hat{\sigma}_{\alpha\alpha} \delta u + \hat{\sigma}_{\alpha\beta} \delta v + \hat{\sigma}_{z\alpha} \delta w) \frac{1}{z_\beta^{(0)}} dz \right] d\beta, \end{aligned} \tag{43}$$

where $Z = F_z^t \left(1 + h/R_\alpha\right) \left(1 + h/R_\beta\right)$. The virtual work of the non-conservative forces δW is expressed with more detail in terms of the variations of the generalized displacements and prescribed forces and moments in the Appendix.

Weak Forms of the Governing Equations in Terms of Internal Forces and Moments

The weak form of the governing mechanical equations in terms of internal forces and moments are obtained by substituting the variational terms in Eqs. (39), (41) and (43) into the Hamilton's principle in Eq. (37). The virtual generalized displacements and surface stresses are zero where the corresponding variables are specified. For time-dependent problems, the admissible virtual generalized variables must also vanish at time $t = t_0$ and $t = t_1$. Thus, using the fundamental lemma of variational calculus and collecting the coefficients of each variation of the different generalized variables (displacements and surface stresses) into independent equations yields for the generalized displacements δu_0, δv_0, δw_0, $\delta\theta_\alpha$ and $\delta\theta_\beta$:

$$\int_{\Omega_0}\Big[\delta u_0(I^{uu}_{11}\ddot{u}_0 + I^{uu}_{13}\frac{\partial \ddot{w}_0}{\partial\alpha} + I^{uu}_{14}\ddot{\theta}_\alpha + I^{\tau u}_{11}\ddot{\overline{\tau}}_{z\alpha} + I^{\tau u}_{21}\ddot{\tilde{\tau}}_{z\alpha}) + \delta\frac{\partial u_0}{\partial\alpha}(N^{\star 11}_{\alpha\alpha} + N^{\star 21}_{\beta\beta}) + \delta\frac{\partial u_0}{\partial\beta}(N^{51}_{\alpha\beta}) + \delta u_0(Q^{41}_{z\alpha})\Big]d\alpha\, d\beta - \oint_{\Gamma_\alpha}\delta u_0(\hat{N}^{11}_{\alpha\beta})d\alpha - \oint_{\Gamma_\beta}\delta u_0(\hat{N}^{11}_{\alpha\alpha})d\beta = 0, \quad (44)$$

$$\int_{\Omega_0}\Big[\delta v_0(I^{uu}_{22}\ddot{v}_0 + I^{uu}_{23}\frac{\partial \ddot{w}_0}{\partial\beta} + I^{uu}_{25}\ddot{\theta}_\beta + I^{\tau u}_{32}\ddot{\overline{\tau}}_{\beta z} + I^{\tau u}_{42}\ddot{\tilde{\tau}}_{\beta z}) + \delta\frac{\partial v_0}{\partial\beta}(N^{\star 12}_{\alpha\alpha} + N^{\star 22}_{\beta\beta}) + \delta\frac{\partial v_0}{\partial\alpha}(N^{52}_{\alpha\beta}) + \delta v_0(Q^{32}_{\beta z})\Big]d\alpha\, d\beta - \oint_{\Gamma_\alpha}\delta v_0(\hat{N}^{22}_{\beta\beta})d\alpha - \oint_{\Gamma_\beta}\delta v_0(\hat{N}^{22}_{\alpha\beta})d\beta = 0, \quad (45)$$

$$\int_{\Omega_0}\Big[\delta w_0(I^{uu}_{33}\ddot{w}) + \delta\frac{\partial w_0}{\partial\alpha}(I^{uu}_{13}\ddot{u}_0 + I^{u\alpha}_{33}\frac{\partial \ddot{w}_0}{\partial\alpha} + I^{uu}_{34}\ddot{\theta}_\alpha + I^{\tau u}_{13}\ddot{\overline{\tau}}_{z\alpha} + I^{\tau u}_{23}\ddot{\tilde{\tau}}_{z\alpha}) + \delta\frac{\partial w_0}{\partial\beta}(I^{uu}_{23}\ddot{v}_0 + I^{u\beta}_{33}\frac{\partial \ddot{w}_0}{\partial\beta} + I^{uu}_{35}\ddot{\theta}_\beta + I^{\tau u}_{33}\ddot{\overline{\tau}}_{\beta z} + I^{\tau u}_{43}\ddot{\tilde{\tau}}_{\beta z}) + \delta w_0(M^{\star 131}_{\alpha\alpha} + M^{\star 231}_{\beta\beta}) + \delta\frac{\partial^2 w_0}{\partial\alpha^2}(M^{\star 132}_{\alpha\alpha}) + \delta\frac{\partial^2 w_0}{\partial\beta^2}(M^{\star 232}_{\beta\beta}) + \delta\frac{\partial^2 w_0}{\partial\alpha\partial\beta}(M^{53}_{\alpha\beta}) + \delta\frac{\partial w_0}{\partial\alpha}(Q^{43}_{z\alpha}) + \delta\frac{\partial w_0}{\partial\beta}(Q^{33}_{\beta z}) - \delta w_0 Z\Big]d\alpha\, d\beta - \oint_{\Gamma_\alpha}\Big[\delta\frac{\partial w_0}{\partial\beta}(\hat{M}^{23}_{\beta\beta}) + \delta\frac{\partial w_0}{\partial\alpha}(\hat{M}^{13}_{\alpha\beta}) + \delta w_0(\hat{Q}^{33}_{\beta z})\Big]d\alpha - \oint_{\Gamma_\beta}\Big[\delta\frac{\partial w_0}{\partial\alpha}(\hat{M}^{13}_{\alpha\alpha}) + \delta\frac{\partial w_0}{\partial\beta}(\hat{M}^{23}_{\alpha\alpha}) + \delta w_0(\hat{Q}^{33}_{z\alpha})\Big]d\beta = 0, \quad (46)$$

$$\int_{\Omega_0}\Big[\delta\theta^\alpha_0(I^{uu}_{14}\ddot{u}_0 + I^{uu}_{43}\frac{\partial \ddot{w}_0}{\partial\alpha} + I^{uu}_{44}\ddot{\theta}^\alpha_0 + I^{\tau u}_{14}\ddot{\overline{\tau}}_{z\alpha} + I^{\tau u}_{24}\ddot{\tilde{\tau}}_{z\alpha}) + \delta\frac{\partial\theta_\alpha}{\partial\alpha}(M^{\star 14}_{\alpha\alpha} + M^{\star 24}_{\beta\beta}) + \delta\frac{\partial\theta_\alpha}{\partial\beta}(M^{54}_{\alpha\beta}) + \delta\theta_\alpha(Q^{44}_{z\alpha})\Big]d\alpha\, d\beta - \oint_{\Gamma_\alpha}\delta\theta_\alpha(\hat{M}^{14}_{\alpha\beta})d\alpha - \oint_{\Gamma_\beta}\delta\theta_\alpha(\hat{M}^{14}_{\alpha\alpha})d\beta = 0, \quad (47)$$

$$\int_{\Omega_0}\Big[\delta\theta^\beta_0(I^{uu}_{25}\ddot{v}_0 + I^{uu}_{35}\frac{\partial \ddot{w}_0}{\partial\beta} + I^{uu}_{55}\ddot{\theta}^\beta_0 + I^{\tau u}_{35}\ddot{\overline{\tau}}_{\beta z} + I^{\tau u}_{45}\ddot{\tilde{\tau}}_{\beta z}) + \delta\frac{\partial\theta_\beta}{\partial\beta}(M^{\star 15}_{\alpha\alpha} + M^{\star 25}_{\beta\beta}) + \delta\frac{\partial\theta_\beta}{\partial\alpha}(M^{55}_{\alpha\beta}) + \delta\theta_\beta(Q^{35}_{\beta z})\Big]d\alpha\, d\beta - \oint_{\Gamma_\alpha}\delta\theta_\beta(\hat{M}^{25}_{\beta\beta})d\alpha - \oint_{\Gamma_\beta}\delta\theta_\beta(\hat{M}^{25}_{\alpha\beta})d\beta = 0. \quad (48)$$

In a similar way, for the generalized surface stress variables, $\delta\bar{\tau}_{z\alpha}$, $\delta\tilde{\tau}_{z\alpha}$, $\delta\bar{\tau}_{\beta z}$ and $\delta\tilde{\tau}_{\beta z}$, one gets

$$\int_{\Omega_0}\Big[\delta\bar{\tau}_{z\alpha}(I^{\tau u}_{11}\ddot{u}_0 + I^{\tau u}_{13}\frac{\partial\ddot{w}_0}{\partial\alpha} + I^{\tau u}_{14}\ddot{\theta}_\alpha + I^{\tau\tau}_{11}\ddot{\bar{\tau}}_{z\alpha} + I^{\tau\tau}_{12}\ddot{\tilde{\tau}}_{z\alpha}) + \delta\frac{\partial\bar{\tau}_{z\alpha}}{\partial\alpha}(T^{\star 11}_{\alpha\alpha} + T^{\star 21}_{\beta\beta})$$
$$+ \delta\bar{\tau}_{z\alpha}(T^{41}_{z\alpha}) + \delta\frac{\partial\bar{\tau}_{z\alpha}}{\partial\beta}(T^{51}_{\alpha\beta})\Big]d\alpha\, d\beta - \oint_{\Gamma_\alpha}\delta\bar{\tau}_{z\alpha}(\hat{T}^{11}_{\beta\alpha})d\alpha - \oint_{\Gamma_\beta}\delta\bar{\tau}_{z\alpha}(\hat{T}^{11}_{\alpha\alpha})d\beta = 0, \quad (49)$$

$$\int_{\Omega_0}\Big[\delta\tilde{\tau}_{z\alpha}(I^{\tau u}_{21}\ddot{u}_0 + I^{\tau u}_{23}\frac{\partial\ddot{w}_0}{\partial\alpha} + I^{\tau u}_{24}\ddot{\theta}_\alpha + I^{\tau\tau}_{12}\ddot{\bar{\tau}}_{z\alpha} + I^{\tau\tau}_{22}\ddot{\tilde{\tau}}_{z\alpha}) + \delta\frac{\partial\tilde{\tau}_{z\alpha}}{\partial\alpha}(T^{\star 12}_{\alpha\alpha} + T^{\star 22}_{\beta\beta})$$
$$+ \delta\tilde{\tau}_{z\alpha}(T^{42}_{z\alpha}) + \delta\frac{\partial\tilde{\tau}_{z\alpha}}{\partial\beta}(T^{52}_{\alpha\beta})\Big]d\alpha\, d\beta - \oint_{\Gamma_\alpha}\delta\tilde{\tau}_{z\alpha}(\hat{T}^{12}_{\beta\alpha})d\alpha - \oint_{\Gamma_\beta}\delta\tilde{\tau}_{z\alpha}(\hat{T}^{12}_{\alpha\alpha})d\beta = 0, \quad (50)$$

$$\int_{\Omega_0}\Big[\delta\bar{\tau}_{\beta z}(I^{\tau u}_{32}\ddot{v}_0 + I^{\tau u}_{33}\frac{\partial\ddot{w}_0}{\partial\beta} + I^{\tau u}_{35}\ddot{\theta}_\beta + I^{\tau\tau}_{33}\ddot{\bar{\tau}}_{\beta z} + I^{\tau\tau}_{34}\ddot{\tilde{\tau}}_{\beta z}) + \delta\frac{\partial\bar{\tau}_{\beta z}}{\partial\beta}(T^{\star 13}_{\alpha\alpha} + T^{\star 23}_{\beta\beta})$$
$$+ \delta\bar{\tau}_{\beta z}(T^{33}_{\beta z}) + \delta\frac{\partial\bar{\tau}_{\beta z}}{\partial\alpha}(T^{53}_{\alpha\beta})\Big]d\alpha\, d\beta - \oint_{\Gamma_\alpha}\delta\bar{\tau}_{\beta z}(\hat{T}^{23}_{\beta\beta})d\alpha - \oint_{\Gamma_\beta}\delta\bar{\tau}_{\beta z}(\hat{T}^{23}_{\alpha\beta})d\beta = 0, \quad (51)$$

$$\int_{\Omega_0}\Big[\delta\tilde{\tau}_{\beta z}(I^{\tau u}_{42}\ddot{v}_0 + I^{\tau u}_{43}\frac{\partial\ddot{w}_0}{\partial\beta} + I^{\tau u}_{45}\ddot{\theta}_\beta + I^{\tau\tau}_{34}\ddot{\bar{\tau}}_{\beta z} + I^{\tau\tau}_{44}\ddot{\tilde{\tau}}_{z\alpha}) + \delta\frac{\partial\tilde{\tau}_{\beta z}}{\partial\beta}(T^{\star 14}_{\alpha\alpha} + T^{\star 24}_{\beta\beta})$$
$$+ \delta\tilde{\tau}_{\beta z}(T^{34}_{\beta z}) + \delta\frac{\partial\tilde{\tau}_{\beta z}}{\partial\alpha}(T^{54}_{\alpha\beta})\Big]d\alpha\, d\beta - \oint_{\Gamma_\alpha}\delta\tilde{\tau}_{\beta z}(\hat{T}^{24}_{\beta\beta})d\alpha - \oint_{\Gamma_\beta}\delta\tilde{\tau}_{\beta z}(\hat{T}^{24}_{\alpha\beta})d\beta = 0. \quad (52)$$

The previous equations are the weak forms of the governing equations of the doubly-curved orthotropic generic elastic shell layer. As can be seen, the 3-D problem has been brought to a 2-D form in function of the reference surface curvilinear coordinates α and β. Hence, the FE solution of the shell problem can be derived in a manner similar to that of plates with some additional terms regarding the curvatures. It is worthy to mention that in the present refined shell theory no assumptions regarding the thinness of the shell were considered and as a consequence the formulation fully accounts for the effects of the z/R_α and z/R_β terms. Additionally, the "mixed" *partially refined* theory also considers additional generalized variables concerning the shear stresses on the top and bottom surfaces of the shell layer which will be used at the elemental FE level to impose transverse interlaminar (interlayer) continuity of the shear stresses and homogeneous shear stress conditions on the top and bottom global surfaces of the multilayer shell.

Constitutive Equations of the Internal Forces and Moments

The strains, and there by the stresses, of the proposed theory where shown to be non-linearly dependent across the thickness of a thick anisotropic elastic shell. Thus, as far as the mathematical model is concerned, it is convenient to integrate the stress distributions through the thickness of the shell and to replace the usual consideration of stress by statically equivalent internal forces and moments. By performing such integration, the variations with respect to the thickness coordinate z are completely eliminated to yield a 2-D mathematical model of the

3-D physical problem. These integrations were carried out in Appendix, and the virtual work quantities of Hamilton's principle in Eq. (37) were expressed in terms of internal forces and moments.

Contrarily to what is often presented in the literature, and since the thickness terms z/R_α and z/R_β were fully retained in the formulation, in the definitions of force and moment resultants given in Eqs. (A26) and (A28) of the Appendix one may notice that the symmetry of the stress tensor (that is, $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$) doesn't necessarily implies that the correspondent force resultants or the moment resultants are equal, even if we consider the restriction of dealing with a doubly-curved shell as stated in Eqs. (28). That relation holds only for a spherical shell, flat plate or a thin shell of any type where the assumptions $1 + z/R_\alpha \approx 1$ and $1 + z/R_\beta \approx 1$ are taken into account. Vanishing of the moments about the normal to the differential element yields an additional relation among the twisting moments and twisting shear forces (cf. Reddy, 2004). In order to avoid inconsistency associated with rigid body rotations (i.e., rigid body rotation gives a nonvanishing torsion except for flat plates or spherical shells) the additional relation must be accounted for in the formulation [see the treatment of Sanders which is described for example by Kraus (1967, Sec. 3.2), Leissa (1993, Sec. 1.4.5) or Reddy (2004, Sec. 8.2.4)]. However, if the rotation is of the same order of magnitude as the strain components, which is actually the case in most problems, then, as noted by Koiter (1960), the torsion is negligible. Thus, for general engineering purposes the foregoing inconsistencies can generally be overlooked which will be the case in this work.

Next the constitutive equations that relate the internal forces and moments in Eqs. (A26) of Appendix with the strains of the layer and/or generalized displacements are derived. To this end it is recalled that an orthotropic elastic material is considered for the generic shell layer and that it obeys Hooke's law under plane-stress assumption as defined in Eq. (A14) of Appendix. Thus, the internal in-plane forces are collected as

$$\begin{Bmatrix} (N^{\star 11}_{\alpha\alpha}, N^{\star 12}_{\alpha\alpha}) \\ (N^{\star 21}_{\beta\beta}, N^{\star 22}_{\beta\beta}) \\ (N^{51}_{\alpha\beta}, N^{52}_{\alpha\beta}) \end{Bmatrix} = \left\langle \begin{Bmatrix} \sigma_{\alpha\alpha}(z^{\star\varepsilon u}_{11}, z^{\star\varepsilon u}_{12}) \\ \sigma_{\beta\beta}(z^{\star\varepsilon u}_{21}, z^{\star\varepsilon u}_{22}) \\ \sigma_{\alpha\beta}(z^{\varepsilon u}_{51}, z^{\varepsilon u}_{52}) \end{Bmatrix} \right\rangle, \tag{53}$$

where for convenience $\langle \ldots \rangle$ denotes thickness integration defined as

$$\langle \ldots \rangle = \int_{-h}^{+h} (\ldots) \frac{1}{z^{(0)}_\alpha z^{(0)}_\beta} dz. \tag{54}$$

Considering the constitutive behavior in Eq. (A14) yields

$$\begin{Bmatrix} (N^{\star 11}_{\alpha\alpha}, N^{\star 12}_{\alpha\alpha}) \\ (N^{\star 21}_{\beta\beta}, N^{\star 22}_{\beta\beta}) \\ (N^{51}_{\alpha\beta}, N^{52}_{\alpha\beta}) \end{Bmatrix} = \left\langle \begin{bmatrix} \bar{c}^*_{11} & \bar{c}^*_{12} & 0 \\ \bar{c}^*_{12} & \bar{c}^*_{22} & 0 \\ 0 & 0 & \bar{c}^*_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{\alpha\alpha}(z^{\star\varepsilon u}_{11}, z^{\star\varepsilon u}_{12}) \\ \varepsilon_{\beta\beta}(z^{\star\varepsilon u}_{21}, z^{\star\varepsilon u}_{22}) \\ \varepsilon_{\alpha\beta}(z^{\varepsilon u}_{51}, z^{\varepsilon u}_{52}) \end{Bmatrix} \right\rangle. \tag{55}$$

Similarly, the out-of-plane forces are collected as

$$\begin{Bmatrix} (Q^{32}_{\beta z}, Q^{33}_{\beta z}, Q^{35}_{\beta z}) \\ (Q^{41}_{z\alpha}, Q^{43}_{z\alpha}, Q^{44}_{z\alpha}) \end{Bmatrix} = \left\langle \begin{Bmatrix} \sigma_{\beta z}(z^{\varepsilon u}_{32}, z^{\varepsilon u}_{33}, z^{\varepsilon u}_{35}) \\ \sigma_{z\alpha}(z^{\varepsilon u}_{41}, z^{\varepsilon u}_{43}, z^{\varepsilon u}_{44}) \end{Bmatrix} \right\rangle, \tag{56}$$

which are expressed in terms of the shear strains as

$$\begin{Bmatrix} (Q^{32}_{\beta z}, Q^{33}_{\beta z}, Q^{35}_{\beta z}) \\ (Q^{41}_{z\alpha}, Q^{43}_{z\alpha}, Q^{44}_{z\alpha}) \end{Bmatrix} = \left\langle \begin{bmatrix} \bar{c}_{44} & 0 \\ 0 & \bar{c}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{\beta z}(z^{\varepsilon u}_{32}, z^{\varepsilon u}_{33}, z^{\varepsilon u}_{35}) \\ \varepsilon_{z\alpha}(z^{\varepsilon u}_{41}, z^{\varepsilon u}_{43}, z^{\varepsilon u}_{44}) \end{Bmatrix} \right\rangle. \tag{57}$$

The moment resultants of the in-plane stresses are collected and expressed by

$$\left\{\begin{matrix}(M_{\alpha\alpha}^{\star131}, M_{\alpha\alpha}^{\star132}, M_{\alpha\alpha}^{\star14}, M_{\alpha\alpha}^{\star15})\\(M_{\beta\beta}^{\star231}, M_{\beta\beta}^{\star232}, M_{\beta\beta}^{\star24}, M_{\beta\beta}^{\star25})\\(M_{\alpha\beta}^{53}, M_{\alpha\beta}^{54}, M_{\alpha\beta}^{55})\end{matrix}\right\} = \left\langle\left\{\begin{matrix}\sigma_{\alpha\alpha}(z_{13}^{\star\varepsilon u1}, z_{13}^{\star\varepsilon u2}, z_{14}^{\star\varepsilon u}, z_{15}^{\star\varepsilon u})\\\sigma_{\beta\beta}(z_{23}^{\star\varepsilon u1}, z_{23}^{\star\varepsilon u2}, z_{24}^{\star\varepsilon u}, z_{25}^{\star\varepsilon u})\\\sigma_{\alpha\beta}(z_{53}^{\varepsilon u}, z_{54}^{\varepsilon u}, z_{55}^{\varepsilon u})\end{matrix}\right\}\right\rangle. \tag{58}$$

Similarly to what has been considered for the in-plane forces in Eq. (55), the internal moments are re-written as

$$\left\{\begin{matrix}(M_{\alpha\alpha}^{\star131}, M_{\alpha\alpha}^{\star132}, M_{\alpha\alpha}^{\star14}, M_{\alpha\alpha}^{\star15})\\(M_{\beta\beta}^{\star231}, M_{\beta\beta}^{\star232}, M_{\beta\beta}^{\star24}, M_{\beta\beta}^{\star25})\\(M_{\alpha\beta}^{53}, M_{\alpha\beta}^{54}, M_{\alpha\beta}^{55})\end{matrix}\right\} = \left\langle\begin{bmatrix}\bar{c}_{11}^{*} & \bar{c}_{12}^{*} & 0\\\bar{c}_{12}^{*} & \bar{c}_{22}^{*} & 0\\0 & 0 & \bar{c}_{66}^{*}\end{bmatrix}\left\{\begin{matrix}\varepsilon_{\alpha\alpha}(z_{13}^{\star\varepsilon u1}, z_{13}^{\star\varepsilon u2}, z_{14}^{\star\varepsilon u}, z_{15}^{\star\varepsilon u})\\\varepsilon_{\beta\beta}(z_{23}^{\star\varepsilon u1}, z_{23}^{\star\varepsilon u2}, z_{24}^{\star\varepsilon u}, z_{25}^{\star\varepsilon u})\\\varepsilon_{\alpha\beta}(z_{53}^{\varepsilon u}, z_{54}^{\varepsilon u}, z_{55}^{\varepsilon u})\end{matrix}\right\}\right\rangle. \tag{59}$$

By last, in a similar way to what has been done in Eqs. (55) and (57), the resultants of the stresses related with the interlayer surface stresses are expressed by

$$\left\{\begin{matrix}(T_{\alpha\alpha}^{\star11}, T_{\alpha\alpha}^{\star12}, T_{\alpha\alpha}^{\star13}, T_{\alpha\alpha}^{\star14})\\(T_{\beta\beta}^{\star21}, T_{\beta\beta}^{\star22}, T_{\beta\beta}^{\star23}, T_{\beta\beta}^{\star24})\\(T_{\alpha\beta}^{51}, T_{\alpha\beta}^{52}, T_{\alpha\beta}^{53}, T_{\alpha\beta}^{54})\end{matrix}\right\} = \left\langle\begin{bmatrix}\bar{c}_{11}^{*} & \bar{c}_{12}^{*} & 0\\\bar{c}_{12}^{*} & \bar{c}_{22}^{*} & 0\\0 & 0 & \bar{c}_{66}^{*}\end{bmatrix}\left\{\begin{matrix}\varepsilon_{\alpha\alpha}(z_{11}^{\star\varepsilon\tau}, z_{12}^{\star\varepsilon\tau}, z_{13}^{\star\varepsilon\tau}, z_{14}^{\star\varepsilon\tau})\\\varepsilon_{\beta\beta}(z_{21}^{\star\varepsilon\tau}, z_{22}^{\star\varepsilon\tau}, z_{23}^{\star\varepsilon\tau}, z_{24}^{\star\varepsilon\tau})\\\varepsilon_{\alpha\beta}(z_{51}^{\varepsilon\tau}, z_{52}^{\varepsilon\tau}, z_{53}^{\varepsilon\tau}, z_{54}^{\varepsilon\tau})\end{matrix}\right\}\right\rangle, \tag{60}$$

and

$$\left\{\begin{matrix}(T_{\beta z}^{33}, T_{\beta z}^{34})\\(T_{z\alpha}^{41}, T_{z\alpha}^{42})\end{matrix}\right\} = \left\langle\begin{bmatrix}\bar{c}_{44} & 0\\0 & \bar{c}_{55}\end{bmatrix}\left\{\begin{matrix}\varepsilon_{\beta z}(z_{33}^{\varepsilon\tau}, z_{34}^{\varepsilon\tau})\\\varepsilon_{z\alpha}(z_{41}^{\varepsilon\tau}, z_{42}^{\varepsilon\tau})\end{matrix}\right\}\right\rangle. \tag{61}$$

For the sake of simplicity and easiness of reading of the internal forces and moments, the previous equations are not developed here in terms of the generalized displacements. The reader is referred to the Appendix for a more detailed derivation of these formulae.

FINITE ELEMENT SOLUTION

Preliminary Comments on the FE Solution of the *Fully* and *Partially Refined* Models

In this section the FE solution of the weak form of the governing equations of the *partially refined* mathematical model of doubly curved shells in Eqs. (44)-(52) is developed. Regarding the FE solution of the *fully refined* model, it will not be derived here for reasons related with the complexity of the formulation. As can be seen from the fully refined definitions of the displacement field presented in Eqs. (20), (23) and (24), the fully refined weak forms would involve high-order derivatives of the generalized displacements which would complicate the formulation and FE solution dramatically. That would require higher order continuity of the variables, which would be cumbersome for FE solutions, with the outcome of considering an equivalent 2-D theory fully representative of the 3-D elasticity problem. It is well known, however, that in the major part of the problems, the transverse stress is small when compared with the other stress components (see Robbins and Chopra, 2006). An exception is, for example, in thermomechanical analysis where the transverse stress σ_{zz} plays an important role (see Carrera, 1999, 2005), which is not the case here. That refinement is not pursued here since the trade-off between accuracy and complexity is not appellative for the physical problem to be treated in this work.

Spatial Approximation

For the sake of brevity the weak forms of the partially refined model in Eqs. (44)-(52) are expressed in terms of the internal forces and moments (stress resultants). However, if the constitutive equations of the internal forces and moments (detailed in the Appendix) are taken into account, the weak forms can still be expressed in terms of the generalized variables. That will not be made here explicitly, but those relations will be implicitly taken into account to derive the elemental matrices and vectors.

Thus, from the analysis of the weak forms in (44)-(52) and/or the internal forces and moments detailed in the Appendix, it can be seen that they contain at the most first-order derivatives of the generalized displacements, u_0, v_0, θ_α and θ_β, and surface stresses, $\bar{\tau}_{z\alpha}$, $\tilde{\tau}_{z\alpha}$, $\bar{\tau}_{\beta z}$ and $\tilde{\tau}_{\beta z}$, requiring C^0 continuity. Furthermore, and in contrast to what is traditionally obtained with the FSDT, the present partial refined model contain also at the most second-order derivatives of the transverse displacement w_0, requiring C^1 continuity, which is something that is typically obtained with the CLT. Thus, the partially refined model, at first sight, yields something that resembles a blend of the CLT and FSDT. Therefore, the displacement variables, u_0, v_0, θ_α, θ_β, w_0, $\partial w_0/\partial\alpha$, $\partial w_0/\partial\beta$ and (or not) $\partial^2 w_0/\partial\alpha\partial\beta$ (*nonconforming* or *conforming* elemental approaches), and shear stress variables, $\bar{\tau}_{z\alpha}$, $\tilde{\tau}_{z\alpha}$, $\bar{\tau}_{\beta z}$ and $\tilde{\tau}_{\beta z}$, must be carried as nodal variables in order to enforce their interelement continuity.

For the FE solution, linear Lagrange C^0 continuity rectangular interpolation functions might be used to approximate all the displacement and stress variables whereas the generalized transverse displacement w_0 should be approximated using Hermite C^1 continuity rectangular interpolation functions over a four-noded element Ω_0^e. The combined conforming or nonconforming elements have a total of 12 or 11 degrees of freedom (DoFs) per node, respectively. Therefore, let

$$\begin{aligned}
u_0(\alpha,\beta,t) &\approx \sum_{j=1}^{n} \bar{u}_0^j(t) L_j^e(\alpha,\beta), & v_0(\alpha,\beta,t) &\approx \sum_{j=1}^{n} \bar{v}_0^j(t) L_j^e(\alpha,\beta),\\
\theta_\alpha(\alpha,\beta,t) &\approx \sum_{j=1}^{n} \bar{\theta}_\alpha^j(t) L_j^e(\alpha,\beta), & \theta_\beta(\alpha,\beta,t) &\approx \sum_{j=1}^{n} \bar{\theta}_\beta^j(t) L_j^e(\alpha,\beta),\\
\bar{\tau}_{z\alpha}(\alpha,\beta,t) &\approx \sum_{j=1}^{n} \bar{\bar{\tau}}_{z\alpha}^j(t) L_j^e(\alpha,\beta), & \tilde{\tau}_{z\alpha}(\alpha,\beta,t) &\approx \sum_{j=1}^{n} \bar{\tilde{\tau}}_{z\alpha}^j(t) L_j^e(\alpha,\beta),\\
\bar{\tau}_{\beta z}(\alpha,\beta,t) &\approx \sum_{j=1}^{n} \bar{\bar{\tau}}_{\beta z}^j(t) L_j^e(\alpha,\beta), & \tilde{\tau}_{\beta z}(\alpha,\beta,t) &\approx \sum_{j=1}^{n} \bar{\tilde{\tau}}_{\beta z}^j(t) L_j^e(\alpha,\beta),\\
w_0(\alpha,\beta,t) &\approx \sum_{r=1}^{m} \bar{w}_0^r(t) H_r^e(\alpha,\beta),
\end{aligned} \tag{62}$$

where $(\bar{u}_0^j, \bar{v}_0^j, \bar{\theta}_\alpha^j, \bar{\theta}_\beta^j)$ and $(\bar{\bar{\tau}}_{z\alpha}^j, \bar{\tilde{\tau}}_{z\alpha}^j, \bar{\bar{\tau}}_{\beta z}^j, \bar{\tilde{\tau}}_{\beta z}^j)$ denote the values of the generalized in-plane displacements, rotations and surface shear stresses at the jth node of the Lagrange elements, $\bar{w}_0^r$ denote the values of w_0 and its derivatives with respect to α and β at the nodes of the Hermite elements, and L_j^e and H_r^e are the Lagrange and Hermite elemental interpolation functions, respectively. For the conforming four-noded rectangular element ($n = 4$ and $m = 12$) the total number of DoFs per element is 48 and for the nonconforming is 44.

Discrete Finite Element Equations of the Shell Layer

Substituting the spatial approximations of the generalized displacements and surface stresses in Eqs. (62) into the weak forms in Eqs. (44)-(52), the ith equation associated with each weak form is given as

$$\sum_{j=1}^{n}(M_{ij}^{11}\ddot{\bar{u}}_0^j + M_{ij}^{14}\ddot{\bar{\theta}}_\alpha^j + M_{ij}^{16}\ddot{\bar{\bar{\tau}}}_{z\alpha}^j + M_{ij}^{17}\ddot{\tilde{\bar{\tau}}}_{z\alpha}^j + K_{ij}^{11}\bar{u}_0^j + K_{ij}^{12}\bar{v}_0^j + K_{ij}^{14}\bar{\theta}_\alpha^j + K_{ij}^{15}\bar{\theta}_\beta^j$$
$$+ K_{ij}^{16}\bar{\bar{\tau}}_{z\alpha}^j + K_{ij}^{17}\tilde{\bar{\tau}}_{z\alpha}^j + K_{ij}^{18}\bar{\bar{\tau}}_{\beta z}^j + K_{ij}^{19}\tilde{\bar{\tau}}_{\beta z}^j) + \sum_{r=1}^{m}(M_{ir}^{13}\ddot{\bar{w}}_0^r + K_{ir}^{13}\bar{w}_0^r) - F_i^1 = 0, \quad (63)$$

$$\sum_{j=1}^{n}(M_{ij}^{22}\ddot{\bar{v}}_0^j + M_{ij}^{25}\ddot{\bar{\theta}}_\beta^j + M_{ij}^{28}\ddot{\bar{\bar{\tau}}}_{\beta z}^j + M_{ij}^{29}\ddot{\tilde{\bar{\tau}}}_{\beta z}^j + K_{ij}^{21}\bar{u}_0^j + K_{ij}^{22}\bar{v}_0^j + K_{ij}^{24}\bar{\theta}_\alpha^j + K_{ij}^{25}\bar{\theta}_\beta^j$$
$$+ K_{ij}^{26}\bar{\bar{\tau}}_{z\alpha}^j + K_{ij}^{27}\tilde{\bar{\tau}}_{z\alpha}^j + K_{ij}^{28}\bar{\bar{\tau}}_{\beta z}^j + K_{ij}^{29}\tilde{\bar{\tau}}_{\beta z}^j) + \sum_{r=1}^{m}(M_{ir}^{23}\ddot{\bar{w}}_0^r + K_{ir}^{23}\bar{w}_0^r) - F_i^2 = 0, \quad (64)$$

$$\sum_{j=1}^{n}(M_{rj}^{31}\ddot{\bar{u}}_0^j + M_{rj}^{32}\ddot{\bar{v}}_0^j + M_{rj}^{34}\ddot{\bar{\theta}}_\alpha^j + M_{rj}^{35}\ddot{\bar{\theta}}_\beta^j + M_{rj}^{36}\ddot{\bar{\bar{\tau}}}_{z\alpha}^j + M_{rj}^{37}\ddot{\tilde{\bar{\tau}}}_{z\alpha}^j + M_{rj}^{38}\ddot{\bar{\bar{\tau}}}_{\beta z}^j + M_{rj}^{39}\ddot{\tilde{\bar{\tau}}}_{\beta z}^j + K_{rj}^{31}\bar{u}_0^j + K_{rj}^{32}\bar{v}_0^j$$
$$+ K_{rj}^{34}\bar{\theta}_\alpha^j + K_{rj}^{35}\bar{\theta}_\beta^j + K_{rj}^{36}\bar{\bar{\tau}}_{z\alpha}^j + K_{rj}^{37}\tilde{\bar{\tau}}_{z\alpha}^j + K_{rj}^{38}\bar{\bar{\tau}}_{\beta z}^j + K_{rj}^{39}\tilde{\bar{\tau}}_{\beta z}^j) + \sum_{s=1}^{m}(M_{rs}^{33}\ddot{\bar{w}}_0^s + K_{rs}^{33}\bar{w}_0^s) - F_r^3 = 0, \quad (65)$$

$$\sum_{j=1}^{n}(M_{ij}^{41}\ddot{\bar{u}}_0^j + M_{ij}^{44}\ddot{\bar{\theta}}_\alpha^j + M_{ij}^{46}\ddot{\bar{\bar{\tau}}}_{z\alpha}^j + M_{ij}^{47}\ddot{\tilde{\bar{\tau}}}_{z\alpha}^j + K_{ij}^{41}\bar{u}_0^j + K_{ij}^{42}\bar{v}_0^j + K_{ij}^{44}\bar{\theta}_\alpha^j + K_{ij}^{45}\bar{\theta}_\beta^j$$
$$+ K_{ij}^{46}\bar{\bar{\tau}}_{z\alpha}^j + K_{ij}^{47}\tilde{\bar{\tau}}_{z\alpha}^j + K_{ij}^{48}\bar{\bar{\tau}}_{\beta z}^j + K_{ij}^{49}\tilde{\bar{\tau}}_{\beta z}^j) + \sum_{r=1}^{m}(M_{ir}^{43}\ddot{\bar{w}}_0^r + K_{ir}^{43}\bar{w}_0^r) - F_i^4 = 0, \quad (66)$$

$$\sum_{j=1}^{n}(M_{ij}^{52}\ddot{\bar{v}}_0^j + M_{ij}^{55}\ddot{\bar{\theta}}_\beta^j + M_{ij}^{58}\ddot{\bar{\bar{\tau}}}_{\beta z}^j + M_{ij}^{59}\ddot{\tilde{\bar{\tau}}}_{\beta z}^j + K_{ij}^{51}\bar{u}_0^j + K_{ij}^{52}\bar{v}_0^j + K_{ij}^{54}\bar{\theta}_\alpha^j + K_{ij}^{55}\bar{\theta}_\beta^j$$
$$+ K_{ij}^{56}\bar{\bar{\tau}}_{z\alpha}^j + K_{ij}^{57}\tilde{\bar{\tau}}_{z\alpha}^j + K_{ij}^{58}\bar{\bar{\tau}}_{\beta z}^j + K_{ij}^{59}\tilde{\bar{\tau}}_{\beta z}^j) + \sum_{r=1}^{m}(M_{ir}^{53}\ddot{\bar{w}}_0^r + K_{ir}^{53}\bar{w}_0^r) - F_i^5 = 0, \quad (67)$$

$$\sum_{j=1}^{n}(M_{ij}^{61}\ddot{\bar{u}}_0^j + M_{ij}^{64}\ddot{\bar{\theta}}_\alpha^j + M_{ij}^{66}\ddot{\bar{\bar{\tau}}}_{z\alpha}^j + M_{ij}^{67}\ddot{\tilde{\bar{\tau}}}_{z\alpha}^j + K_{ij}^{61}\bar{u}_0^j + K_{ij}^{62}\bar{v}_0^j + K_{ij}^{64}\bar{\theta}_\alpha^j + K_{ij}^{65}\bar{\theta}_\beta^j$$
$$+ K_{ij}^{66}\bar{\bar{\tau}}_{z\alpha}^j + K_{ij}^{67}\tilde{\bar{\tau}}_{z\alpha}^j + K_{ij}^{68}\bar{\bar{\tau}}_{\beta z}^j + K_{ij}^{69}\tilde{\bar{\tau}}_{\beta z}^j) + \sum_{r=1}^{m}(M_{ir}^{63}\ddot{\bar{w}}_0^r + K_{ir}^{63}\bar{w}_0^r) - F_i^6 = 0, \quad (68)$$

$$\sum_{j=1}^{n}(M_{ij}^{71}\ddot{\bar{u}}_0^j + M_{ij}^{74}\ddot{\bar{\theta}}_\alpha^j + M_{ij}^{76}\ddot{\bar{\bar{\tau}}}_{z\alpha}^j + M_{ij}^{77}\ddot{\tilde{\bar{\tau}}}_{z\alpha}^j + K_{ij}^{71}\bar{u}_0^j + K_{ij}^{72}\bar{v}_0^j + K_{ij}^{74}\bar{\theta}_\alpha^j + K_{ij}^{75}\bar{\theta}_\beta^j$$
$$+ K_{ij}^{76}\bar{\bar{\tau}}_{z\alpha}^j + K_{ij}^{77}\tilde{\bar{\tau}}_{z\alpha}^j + K_{ij}^{78}\bar{\bar{\tau}}_{\beta z}^j + K_{ij}^{79}\tilde{\bar{\tau}}_{\beta z}^j) + \sum_{r=1}^{m}(M_{ir}^{73}\ddot{\bar{w}}_0^r + K_{ir}^{73}\bar{w}_0^r) - F_i^7 = 0, \quad (69)$$

$$\sum_{j=1}^{n}(M_{ij}^{82}\ddot{\bar{v}}_0^j + M_{ij}^{85}\ddot{\bar{\theta}}_\beta^j + M_{ij}^{88}\ddot{\bar{\bar{\tau}}}_{\beta z}^j + M_{ij}^{89}\ddot{\bar{\bar{\tau}}}_{\beta z}^j + K_{ij}^{81}\bar{u}_0^j + K_{ij}^{82}\bar{v}_0^j + K_{ij}^{84}\bar{\theta}_\alpha^j + K_{ij}^{85}\bar{\theta}_\beta^j$$
$$+ K_{ij}^{86}\bar{\bar{\tau}}_{z\alpha}^j + K_{ij}^{87}\bar{\bar{\tau}}_{z\alpha}^j + K_{ij}^{88}\bar{\bar{\tau}}_{\beta z}^j + K_{ij}^{89}\bar{\bar{\tau}}_{\beta z}^j) + \sum_{r=1}^{m}(M_{ir}^{83}\ddot{\bar{w}}_0^r + K_{ir}^{83}\bar{w}_0^r) - F_i^8 = 0, \quad (70)$$

$$\sum_{j=1}^{n}(M_{ij}^{92}\ddot{\bar{v}}_0^j + M_{ij}^{95}\ddot{\bar{\theta}}_\beta^j + M_{ij}^{98}\ddot{\bar{\bar{\tau}}}_{\beta z}^j + M_{ij}^{99}\ddot{\bar{\bar{\tau}}}_{\beta z}^j + K_{ij}^{91}\bar{u}_0^j + K_{ij}^{92}\bar{v}_0^j + K_{ij}^{94}\bar{\theta}_\alpha^j + K_{ij}^{95}\bar{\theta}_\beta^j$$
$$+ K_{ij}^{96}\bar{\bar{\tau}}_{z\alpha}^j + K_{ij}^{97}\bar{\bar{\tau}}_{z\alpha}^j + K_{ij}^{98}\bar{\bar{\tau}}_{\beta z}^j + K_{ij}^{99}\bar{\bar{\tau}}_{\beta z}^j) + \sum_{r=1}^{m}(M_{ir}^{93}\ddot{\bar{w}}_0^r + K_{ir}^{93}\bar{w}_0^r) - F_i^9 = 0, \quad (71)$$

where $i = 1,\ldots,n$, and $r = 1,\ldots,m$. The coefficients of the mass matrix $M_{ij}^{xy} = M_{ji}^{yx}$, stiffness matrix $K_{ij}^{xy} = K_{ji}^{yx}$ and force vectors F_i^x are given in the Appendix.

In matrix notation Eqs. (63)-(71) can be expressed in terms of the elemental matrices and vectors of the generic layer l as

$$\begin{bmatrix} \mathbf{M}_{uu}^l & \mathbf{M}_{u\tau}^l \\ \mathbf{M}_{\tau u}^l & \mathbf{M}_{\tau\tau}^l \end{bmatrix} \begin{Bmatrix} \ddot{\bar{\mathbf{u}}}^l(t) \\ \ddot{\bar{\boldsymbol{\tau}}}^l(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{uu}^l & \mathbf{K}_{u\tau}^l \\ \mathbf{K}_{\tau u}^l & \mathbf{K}_{\tau\tau}^l \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{u}}^l(t) \\ \bar{\boldsymbol{\tau}}^l(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_u^l(t) \\ \mathbf{F}_\tau^l(t) \end{Bmatrix}, \quad (72)$$

where, since $\mathbf{K}^{yx} = (\mathbf{K}^{xy})^{\mathrm{T}}$ and $\mathbf{M}^{yx} = (\mathbf{M}^{xy})^{\mathrm{T}}$ one gets $\mathbf{M}_{\tau u}^l = (\mathbf{M}_{u\tau}^l)^{\mathrm{T}}$ and $\mathbf{K}_{\tau u}^l = (\mathbf{K}_{u\tau}^l)^{\mathrm{T}}$, and the matrices and vectors are defined by

$$\mathbf{M}_{uu}^l = \begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} & \mathbf{M}^{13} & \mathbf{M}^{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{M}^{23} & \mathbf{0} & \mathbf{M}^{25} \\ \mathbf{M}^{31} & \mathbf{M}^{32} & \mathbf{M}^{33} & \mathbf{M}^{34} & \mathbf{M}^{35} \\ \mathbf{M}^{41} & \mathbf{0} & \mathbf{M}^{43} & \mathbf{M}^{44} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{52} & \mathbf{M}^{53} & \mathbf{0} & \mathbf{M}^{55} \end{bmatrix}, \quad \mathbf{K}_{uu}^l = \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} & \mathbf{K}^{14} & \mathbf{K}^{15} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} & \mathbf{K}^{24} & \mathbf{K}^{25} \\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} & \mathbf{K}^{34} & \mathbf{K}^{35} \\ \mathbf{K}^{41} & \mathbf{K}^{42} & \mathbf{K}^{43} & \mathbf{K}^{44} & \mathbf{K}^{45} \\ \mathbf{K}^{51} & \mathbf{K}^{52} & \mathbf{K}^{53} & \mathbf{K}^{54} & \mathbf{K}^{55} \end{bmatrix}, \quad (73)$$

$$\mathbf{M}_{u\tau}^l = \begin{bmatrix} \mathbf{M}^{16} & \mathbf{M}^{17} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{28} & \mathbf{M}^{29} \\ \mathbf{M}^{36} & \mathbf{M}^{37} & \mathbf{M}^{38} & \mathbf{M}^{39} \\ \mathbf{M}^{46} & \mathbf{M}^{47} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{58} & \mathbf{M}^{59} \end{bmatrix}, \quad \mathbf{K}_{u\tau}^l = \begin{bmatrix} \mathbf{M}^{16} & \mathbf{M}^{17} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{28} & \mathbf{M}^{29} \\ \mathbf{M}^{36} & \mathbf{M}^{37} & \mathbf{M}^{38} & \mathbf{M}^{39} \\ \mathbf{M}^{46} & \mathbf{M}^{47} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{58} & \mathbf{M}^{59} \end{bmatrix}, \quad (74)$$

$$\mathbf{M}_{\tau\tau}^l = \begin{bmatrix} \mathbf{M}^{66} & \mathbf{M}^{67} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{76} & \mathbf{M}^{77} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{88} & \mathbf{M}^{89} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{98} & \mathbf{M}^{99} \end{bmatrix}, \quad \mathbf{K}_{\tau\tau}^l = \begin{bmatrix} \mathbf{K}^{66} & \mathbf{K}^{67} & \mathbf{K}^{68} & \mathbf{K}^{69} \\ \mathbf{K}^{76} & \mathbf{K}^{77} & \mathbf{K}^{78} & \mathbf{K}^{79} \\ \mathbf{K}^{86} & \mathbf{K}^{87} & \mathbf{K}^{88} & \mathbf{K}^{89} \\ \mathbf{K}^{96} & \mathbf{K}^{97} & \mathbf{K}^{98} & \mathbf{K}^{99} \end{bmatrix}, \quad (75)$$

$$\bar{\mathbf{u}}^l(t) = \begin{Bmatrix} \bar{\mathbf{u}}_0 \\ \bar{\mathbf{v}}_0 \\ \bar{\mathbf{w}}_0 \\ \bar{\boldsymbol{\theta}}_\alpha \\ \bar{\boldsymbol{\theta}}_\beta \end{Bmatrix}, \quad \bar{\boldsymbol{\tau}}^l(t) = \begin{Bmatrix} \bar{\bar{\boldsymbol{\tau}}}_{z\alpha} \\ \bar{\bar{\boldsymbol{\tau}}}_{z\alpha} \\ \bar{\bar{\boldsymbol{\tau}}}_{\beta z} \\ \bar{\bar{\boldsymbol{\tau}}}_{\beta z} \end{Bmatrix}, \quad \mathbf{F}_u^l(t) = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \mathbf{F}^3 \\ \mathbf{F}^4 \\ \mathbf{F}^5 \end{Bmatrix}, \quad \mathbf{F}_\tau^l(t) = \begin{Bmatrix} \mathbf{F}^6 \\ \mathbf{F}^7 \\ \mathbf{F}^8 \\ \mathbf{F}^9 \end{Bmatrix}. \quad (76)$$

The coupled "mixed" FE model in Eq. (72) is based on the weak forms of the equations of motion which where obtained through Hamilton's principle, and is expressed in terms of a coupled set of displacement (and the derivatives of the transverse displacement) and surface shear stresses DoFs. It should be noted that the contributions of the internal forces defined in vectors $\mathbf{F}_u^l$ and $\mathbf{F}_\tau^l$ to the force vector will cancel when element equations are assembled. They will remain in the force vector only when the element boundary coincides with the boundary of the domain being modeled. However, as is well known, the contributions of the distributed applied loads $Z(\alpha, \beta)$ to a node will add up from elements connected at the node and remain as a part of the force vector (see Reddy, 1993, pp. 313-318).

Assemblage of Matrices from Layer to Multilayer Level

In this section the elemental equations derived for the generic single shell layer are adapted in order to allow the generalization of the present theory to a multilayer, or discrete layer, type formulation. To that end, since the displacements DoFs of the elemental equations of the shell layer are defined in terms of in-plane generalized displacements in the middle surface and rotations of the normals to the middle surface, first the DoFs are transformed to equivalent in-plane displacements on the top and bottom surfaces of the generic shell layer element. Additionally, since the effects of the surface top and bottom shear stresses have been represented in terms of *mean* and *relative* quantities, another transformation is required to the stress DoFs to dispose of top and bottom shear stresses DoFs. The transverse displacement is assumed constant in the multilayer shell (i.e., is constant, and the same, for all layers). These transformations allow the displacement and stress DoFs of different layers to be assembled imposing not only displacement continuity but also shear stress continuity across the interfaces of the multilayer shell FE. Thus, the FE is "regenerated" (in opposition to the well-known "degeneration" approach) in the form of an equivalent eight-noded 3-D element with 2 in-plane displacement and 2 shear stress DoFs per node, and one transverse displacement (and its derivatives) per element. Therefore, the "regenerated" formulation is suitable for assemblage of elemental matrices from single layer to multilayer level.

The effects of the pairs of generalized variables (u_0, θ_α) and (v_0, θ_β) in the global displacement field are taken into account through new equivalent pairs of generalized variables (u_t, u_b) and (v_t, v_b), with each pair containing the in-plane translations at the top and bottom surfaces, respectively. Thus, rather than describing the in-plane displacement field by a translation and a rotation at one point, it can more conveniently be described here by the translation at two points on the top and bottom surfaces.

According to Eq. (33), and using the adequate coefficients of matrices $\mathbf{z}^u(z = h)$ and $\mathbf{z}^\tau(z = h)$, the displacement field $u(\alpha, \beta, z)$ on the top surface is given as

$$u_t = z_{11}^u(h) u_0 + z_{13}^u(h) \frac{\partial w_0}{\partial \alpha} + z_{14}^u(h) \theta_\alpha + z_{11}^\tau(h) \bar{\tau}_{z\alpha} + z_{12}^\tau(h) \tilde{\tau}_{z\alpha}. \tag{77}$$

Then, from the previous equation $u_0 = u_0(\alpha, \beta)$ is written as

$$u_0 = \frac{1}{z_{11}^u(h)} u_t - \frac{z_{13}^u(h)}{z_{11}^u(h)} \frac{\partial w_0}{\partial \alpha} - \frac{z_{14}^u(h)}{z_{11}^u(h)} \theta_\alpha - \frac{z_{11}^\tau(h)}{z_{11}^u(h)} \bar{\tau}_{z\alpha} - \frac{z_{12}^\tau(h)}{z_{11}^u(h)} \tilde{\tau}_{z\alpha}. \tag{78}$$

Substituting the definition of $u_0 = u_0(\alpha, \beta, z = 0)$ in terms of $u_t = u_t(\alpha, \beta, z = h)$ yields

$$u = \frac{z^u_{11}}{z^u_{11}(h)} u_t + \left[z^u_{13} - z^u_{11} \frac{z^u_{13}(h)}{z^u_{11}(h)} \right] \frac{\partial w_0}{\partial \alpha} + \left[z^u_{14} - z^u_{11} \frac{z^u_{14}(h)}{z^u_{11}(h)} \right] \theta_\alpha + \left[z^\tau_{11} - z^u_{11} \frac{z^\tau_{11}(h)}{z^u_{11}(h)} \right] \bar{\tau}_{z\alpha} + \left[z^\tau_{12} - z^u_{11} \frac{z^\tau_{12}(h)}{z^u_{11}(h)} \right] \tilde{\tau}_{z\alpha}. \quad (79)$$

From the previous equation it can be seen that some transformations to the first line of matrices $\mathbf{z}^u(z)$ and $\mathbf{z}^\tau(z)$ were performed in order to make a transformation of the generalized in-plane displacement u_0 on the middle surface to the translation on the top surface u_t. Performing a similar process to eliminate the rotation θ_α of Eq. (79) and express the displacement u also in terms of the translation on the bottom surface u_b, another transformation is performed considering also the terms of the first line of $\mathbf{z}^u(z = -h)$ and $\mathbf{z}^\tau(z = -h)$. Similar relations hold for the second pair of variables (v_0, θ_β). For the sake of brevity the algebra of these relations will not be be presented here but can easily be derived from the previous explanation.

Other required transformation to "regenerate" the 2-D element is performed according to Eqs. (4) and (5), where relationships between the *mean* and *relative* shear stresses and the shear stresses on the interfaces of the generic shell layer $\sigma^t_{z\alpha}$, $\sigma^b_{z\alpha}$, $\sigma^t_{\beta z}$ and $\sigma^b_{\beta z}$ can be easily established.

According to the previous discussion, the relationship between the original and "regenerated" set of generalized variables used to defined the in-plane displacement field can be established by means of a transformation matrices $\mathbf{T}_u$ and $\mathbf{T}_\tau$ as

$$\begin{Bmatrix} \bar{\mathbf{u}}_0 \\ \bar{\mathbf{v}}_0 \\ \bar{\mathbf{w}}_0 \\ \bar{\boldsymbol{\theta}}_\alpha \\ \bar{\boldsymbol{\theta}}_\beta \end{Bmatrix} = \mathbf{T}_u \begin{Bmatrix} \bar{\mathbf{u}}_t \\ \bar{\mathbf{u}}_b \\ \bar{\mathbf{v}}_t \\ \bar{\mathbf{v}}_b \\ \bar{\mathbf{w}} \end{Bmatrix}, \qquad \begin{Bmatrix} \bar{\bar{\boldsymbol{\tau}}}_{z\alpha} \\ \bar{\tilde{\boldsymbol{\tau}}}_{z\alpha} \\ \bar{\bar{\boldsymbol{\tau}}}_{\beta z} \\ \bar{\tilde{\boldsymbol{\tau}}}_{\beta z} \end{Bmatrix} = \mathbf{T}_\tau \begin{Bmatrix} \bar{\boldsymbol{\sigma}}^t_{z\alpha} \\ \bar{\boldsymbol{\sigma}}^b_{z\alpha} \\ \bar{\boldsymbol{\sigma}}^t_{\beta z} \\ \bar{\boldsymbol{\sigma}}^b_{\beta z} \end{Bmatrix}. \quad (80)$$

Performing the previous transformations into the FE elemental matrices in Eq. (72), where the elemental matrices and vectors are transformed according to (similar relations hold for the stiffness matrices)

$$\begin{gathered} \mathbf{M}^{*l}_{uu} = \mathbf{T}^{\mathrm{T}}_u \mathbf{M}^l_{uu} \mathbf{T}_u, \qquad \mathbf{M}^{*l}_{\tau\tau} = \mathbf{T}^{\mathrm{T}}_\tau \mathbf{M}^l_{\tau\tau} \mathbf{T}_\tau, \qquad \mathbf{M}^{*l}_{u\tau} = \mathbf{T}^{\mathrm{T}}_u \mathbf{M}^l_{u\tau} \mathbf{T}_\tau, \\ \mathbf{F}^{*l}_u = \mathbf{T}^{\mathrm{T}}_u \mathbf{F}^l_u, \qquad \mathbf{F}^{*l}_\tau = \mathbf{T}^{\mathrm{T}}_\tau \mathbf{F}^l_\tau, \end{gathered} \quad (81)$$

yields

$$\begin{bmatrix} \mathbf{M}^{*l}_{uu} & \mathbf{M}^{*l}_{u\tau} \\ \mathbf{M}^{*l}_{\tau u} & \mathbf{M}^{*l}_{\tau\tau} \end{bmatrix} \begin{Bmatrix} \ddot{\bar{\mathbf{u}}}^{*l}(t) \\ \ddot{\bar{\boldsymbol{\tau}}}^{*l}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{*l}_{uu} & \mathbf{K}^{*l}_{u\tau} \\ \mathbf{K}^{*l}_{\tau u} & \mathbf{K}^{*l}_{\tau\tau} \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{u}}^{*l}(t) \\ \bar{\boldsymbol{\tau}}^{*l}(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^{*l}_u(t) \\ \mathbf{F}^{*l}_\tau(t) \end{Bmatrix}, \quad (82)$$

From this point forward, the generic layer elemental matrices can be assembled in the thickness direction in order to create the desired multilayer FE according to the representative multilayer shell model to be generated. Displacement and shear stress continuity at the through-the-thickness interfaces of adjacent elements (discrete layers) is imposed in the assemblage process, as is usually done with the displacement DoF of 3-D elements, and it is assumed that no slippage occurs in the interfaces between adjacent layers. It is worthy to mention that the resultant multilayer elemental matrices are needed, for example, when there is the need to consider segmented layers, as is the case when dealing with arbitrary damping treatments (piezoelectric or viscoelastic patches) mounted on a host shell structure or stiffness reinforcements in a particular zone of a composite shell structure. After the through-the-thickness assemblage, the multilayer elemental matrices are written as

$$\begin{bmatrix} \mathbf{M}^e_{uu} & \mathbf{M}^e_{u\tau} \\ \mathbf{M}^e_{\tau u} & \mathbf{M}^e_{\tau\tau} \end{bmatrix} \begin{Bmatrix} \ddot{\bar{\mathbf{u}}}^e(t) \\ \ddot{\bar{\boldsymbol{\tau}}}^e(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^e_{uu} & \mathbf{K}^e_{u\tau} \\ \mathbf{K}^e_{\tau u} & \mathbf{K}^e_{\tau\tau} \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{u}}^e(t) \\ \bar{\boldsymbol{\tau}}^e(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^e_u(t) \\ \mathbf{F}^e_\tau(t) \end{Bmatrix}, \tag{83}$$

where the superscript $(\cdot)^e$ is used to denote multilayer elemental matrices and vectors.

Assuming homogeneous shear stress conditions on the free top and bottom surfaces of the multilayer shell element and performing a dynamic condensation to the shear stress DoFs, as suggested for a generic system, for example, by Kidder (1973), O'Callahan (1989) or Gordis (1994), one gets the multilayer FE elemental equations in terms of elemental reduced matrices and vectors in terms of only displacement variables as

$$\hat{\mathbf{M}}^e_{uu}\ddot{\bar{\mathbf{u}}}^e(t) + \hat{\mathbf{K}}^e_{uu}\bar{\mathbf{u}}^e(t) = \hat{\mathbf{F}}^e_u(t). \tag{84}$$

A generic fully discretized global electro-mechanical system is obtained by "in-plane" assembling the elemental multilayer FE matrices and vectors yielding

$$\mathbf{M}_{uu}\ddot{\bar{\mathbf{u}}}(t) + \mathbf{K}_{uu}\bar{\mathbf{u}}(t) = \mathbf{F}_u(t), \tag{85}$$

where the superscript $(\cdot)^e$ and the *hat* above the elemental matrices and vectors have been dropped to denote global matrices and vectors of the fully discretized FE model.

CONCLUSION

Based on a *fully refined* mathematical model of general anisotropic shells a "mixed" FE model has been conceptually proposed for multilayer shells. However, for practical reasons, related with the complexity of the formulation, simplifications regarding the through-the-thickness distribution of the transverse displacement were considered and a *partially refined* theory was derived with additional restrictions inherent to doubly-curved orthotropic shells physics. No simplifications regarding the thinness of the shell were considered and a plane stress state was considered for the partially refined theory.

It was shown that the refined assumptions and relaxation of some of Love's classical assumptions led to a "mixed" definition of the displacement field in terms of the same generalized displacements of the FSDT and CLT, and shear stresses on the top and bottom surfaces. The governing equations of a generic single layer of the multilayered shell were derived with Hamilton's principle in conjunction with the "mixed" displacement field definition. The DoFs of the resultant four-noded generic elastic single layer FE model were then "regenerated" into an equivalent eight-node 3-D formulation in terms of top and bottom translations and shear stresses, and a transverse displacement (and its derivatives) constant in the elemental volume. The through-the-thickness assemblage of the "regenerated" FE model of the single layer allowed the generation of a "refined" multilayer FE assuring displacement and shear stress interlayer continuity. The dynamic condensation of the stress DoFs allowed the reduction of the refined multilayer piezo-elastic FE to a an equivalent representation similar in structure to the one obtained with a first-order partial layerwise theory, but considering nonlinear in-plane displacements, quadratic shear stresses definitions and also interlayer continuity and homogeneous conditions of the shear stresses at the top and bottom surfaces of the multilayer FE.

The resultant *partially refined* ”mixed” FE model presents a good trade-off between accuracy and complexity and is therefore expected to yield a good representativeness of doubly curved orthotropic shells with segmented (discontinuous) layers in static and vibration analysis.

APPENDIX

Strain-Displacement and Equilibrium Equations in Orthogonal Curvilinear Coordinates

Taking into account that for shell structures $H_z = 1$, from the equations of 3-D theory of elasticity, the strain components of the shell layer are defined as a function of displacements by Sokolnikoff (1956, pp. 177-184) as

$$\begin{aligned}
\varepsilon_{\alpha\alpha} &= \frac{1}{H_\alpha}\frac{\partial u}{\partial \alpha} + \frac{1}{H_\alpha H_\beta}\frac{\partial H_\alpha}{\partial \beta}v + \frac{1}{H_\alpha}\frac{\partial H_\alpha}{\partial z}w, & \varepsilon_{z\alpha} &= H_\alpha \frac{\partial}{\partial z}\left(\frac{u}{H_\alpha}\right) + \frac{1}{H_\alpha}\frac{\partial w}{\partial \alpha}, \\
\varepsilon_{\beta\beta} &= \frac{1}{H_\alpha H_\beta}\frac{\partial H_\beta}{\partial \alpha}u + \frac{1}{H_\beta}\frac{\partial v}{\partial \beta} + \frac{1}{H_\beta}\frac{\partial H_\beta}{\partial z}w, & \varepsilon_{\beta z} &= H_\beta \frac{\partial}{\partial z}\left(\frac{v}{H_\beta}\right) + \frac{1}{H_\beta}\frac{\partial w}{\partial \beta}, \\
\varepsilon_{\alpha\beta} &= \frac{H_\alpha}{H_\beta}\frac{\partial}{\partial \beta}\left(\frac{u}{H_\alpha}\right) + \frac{H_\beta}{H_\alpha}\frac{\partial}{\partial \alpha}\left(\frac{v}{H_\beta}\right), & \varepsilon_{zz} &= \frac{\partial w}{\partial z},
\end{aligned} \tag{A1}$$

where $u = u(\alpha, \beta, z)$, $v = v(\alpha, \beta, z)$ and $w = w(\alpha, \beta, z)$ are the displacement components of an arbitrary point of the shell in the directions of the tangents to the coordinate lines (α, β, z), respectively.

The equilibrium equations of a differential element of the body of the shell layer in the tri-orthogonal system of curvilinear coordinates (Sokolnikoff, 1956) is represented by the partial differential equations

$$\begin{aligned}
&\frac{\partial}{\partial \alpha}(H_\beta \sigma_{\alpha\alpha}) + \frac{\partial}{\partial \beta}(H_\alpha \sigma_{\alpha\beta}) + \frac{\partial}{\partial z}(H_\alpha H_\beta \sigma_{z\alpha}) \\
&\qquad - \frac{\partial H_\beta}{\partial \alpha}\sigma_{\beta\beta} - \frac{\partial H_\alpha}{\partial \beta}\sigma_{\alpha\beta} + H_\beta \frac{\partial H_\alpha}{\partial z}\sigma_{z\alpha} + H_\alpha H_\beta P_\alpha = 0, \\
&\frac{\partial}{\partial \beta}(H_\alpha \sigma_{\beta\beta}) + \frac{\partial}{\partial \alpha}(H_\beta \sigma_{\alpha\beta}) + \frac{\partial}{\partial z}(H_\alpha H_\beta \sigma_{\beta z}) \\
&\qquad - \frac{\partial H_\alpha}{\partial \beta}\sigma_{\alpha\alpha} - \frac{\partial H_\beta}{\partial \alpha}\sigma_{\alpha\beta} + H_\alpha \frac{\partial H_\beta}{\partial z}\sigma_{z\beta} + H_\alpha H_\beta P_\beta = 0, \\
&\frac{\partial}{\partial z}(H_\alpha H_\beta \sigma_{zz}) + \frac{\partial}{\partial \alpha}(H_\beta \sigma_{z\alpha}) + \frac{\partial}{\partial \beta}(H_\alpha \sigma_{\beta z}) \\
&\qquad - H_\beta \frac{\partial H_\alpha}{\partial z}\sigma_{\alpha\alpha} - H_\alpha \frac{\partial H_\beta}{\partial z}\sigma_{\beta\beta} + H_\alpha H_\beta P_z = 0,
\end{aligned} \tag{A2}$$

where $P_\alpha = P_\alpha(\alpha, \beta, z)$, $P_\beta = P_\beta(\alpha, \beta, z)$ and $P_z = P_z(\alpha, \beta, z)$ are the corresponding projections of the volumetric force in the direction of the tangents to the shell curvilinear coordinate system.

First Order Shear Deformation Theory (FSDT) of Anisotropic Shells

According to *Love's first approximation* assumptions for thin shells (the so-called classical *Kirchhoff-Love theory* of shells), the following strain and stress definitions are derived for anisotropic shells, by relaxing the so-called Kirchhoff's hypothesis that *normals to the undeformed middle surface remain straight and normal to the deformed middle surface and suffer no extension* (see Leissa, 1993, Sec. 1.3). Instead, it is considered that *normals before deformation remain straight but not necessarily normal after deformation*, which basically relaxes the condition of nil out-of-plane shear strains. That theory is known as FSDT, or *Reissner-Mindlin theory* applied to shells, and over the years has been shown to be a more accurate approach for modeling moderately thick shells.

Consistent with the assumptions of a moderately thick shell theory, the displacement components were postulated as

$$\begin{aligned} u^*(\alpha,\beta,z) &= u_0(\alpha,\beta) + z\theta_\alpha(\alpha,\beta), \\ v^*(\alpha,\beta,z) &= v_0(\alpha,\beta) + z\theta_\beta(\alpha,\beta), \\ w^*(\alpha,\beta,z) &= w_0(\alpha,\beta), \end{aligned} \tag{A3}$$

where $u_0 = u_0(\alpha,\beta)$, $v_0 = v_0(\alpha,\beta)$ and $w_0 = w_0(\alpha,\beta)$ are the tangential and transverse displacements referred to a point on the middle surface, respectively, and $\theta_\alpha = \theta_\alpha(\alpha,\beta)$ and $\theta_\beta = \theta_\beta(\alpha,\beta)$ are the rotations of a normal to the reference middle surface.

Thus, taken into account the strain-displacement equations of the 3-D elasticity in orthogonal curvilinear coordinates in Eqs. (A1), as proposed by Byrne, Flügge, Goldenveizer, Lur'ye and Novozhilov between the 1940s and 1960s (cf. Leissa, 1993, Sec. 1.4), and in a similar form to what has been presented, for example, by Reddy (2004, Sec. 8.2.3) or Leissa (1993, Sec. 1.4.1), the in-plane strain definitions are given as

$$\begin{aligned} \varepsilon^*_{\alpha\alpha}(\alpha,\beta,z) &= z^{(0)}_\alpha \varepsilon^{*(0)}_{\alpha\alpha}(\alpha,\beta) + z^{(1)}_\alpha \varepsilon^{*(1)}_{\alpha\alpha}(\alpha,\beta), \\ \varepsilon^*_{\beta\beta}(\alpha,\beta,z) &= z^{(0)}_\beta \varepsilon^{*(0)}_{\beta\beta}(\alpha,\beta) + z^{(1)}_\beta \varepsilon^{*(1)}_{\beta\beta}(\alpha,\beta), \\ \varepsilon^*_{\alpha\beta}(\alpha,\beta,z) &= z^{(0)}_{\alpha\beta} \varepsilon^{*(0)}_{\alpha\beta}(\alpha,\beta) + z^{(1)}_{\alpha\beta} \varepsilon^{*(1)}_{\alpha\beta}(\alpha,\beta), \end{aligned} \tag{A4}$$

where

$$\begin{aligned} z^{(i)}_\alpha &= \frac{z^i}{(1+z/R_\alpha)}, & z^{(0)}_{\alpha\beta} &= z^{(0)}_\alpha z^{(0)}_\beta \left(1 - \frac{z^2}{R_\alpha R_\beta}\right), \\ z^{(i)}_\beta &= \frac{z^i}{(1+z/R_\beta)}, & z^{(1)}_{\alpha\beta} &= z^{(0)}_\alpha z^{(0)}_\beta z \left(1 + \frac{z}{2R_\alpha} + \frac{z}{2R_\beta}\right), \end{aligned} \tag{A5}$$

with $i = 0, 1$ and

$$\begin{aligned} \varepsilon^{*(0)}_{\alpha\alpha}(\alpha,\beta) &= \frac{1}{A_\alpha}\frac{\partial u_0}{\partial \alpha} + \frac{v_0}{A_\alpha A_\beta}\frac{\partial A_\alpha}{\partial \beta} + \frac{w_0}{R_\alpha}, \\ \varepsilon^{*(0)}_{\beta\beta}(\alpha,\beta) &= \frac{1}{A_\beta}\frac{\partial v_0}{\partial \beta} + \frac{u_0}{A_\alpha A_\beta}\frac{\partial A_\beta}{\partial \alpha} + \frac{w_0}{R_\beta}, \\ \varepsilon^{*(0)}_{\alpha\beta}(\alpha,\beta) &= \frac{A_\alpha}{A_\beta}\frac{\partial}{\partial \beta}\left(\frac{u_0}{A_\alpha}\right) + \frac{A_\beta}{A_\alpha}\frac{\partial}{\partial \alpha}\left(\frac{v_0}{A_\beta}\right), \\ \varepsilon^{*(1)}_{\alpha\alpha}(\alpha,\beta) &= \frac{1}{A_\alpha}\frac{\partial \theta_\alpha}{\partial \alpha} + \frac{1}{A_\alpha A_\beta}\frac{\partial A_\alpha}{\partial \beta}\theta_\beta, \end{aligned} \tag{A6}$$

$$\varepsilon_{\beta\beta}^{*(1)}(\alpha,\beta) = \frac{1}{A_\beta}\frac{\partial \theta_\beta}{\partial \beta} + \frac{1}{A_\alpha A_\beta}\frac{\partial A_\beta}{\partial \alpha}\theta_\alpha,$$
$$\varepsilon_{\alpha\beta}^{*(1)}(\alpha,\beta) = \frac{A_\alpha}{A_\beta}\frac{\partial}{\partial \beta}\left(\frac{\theta_\alpha}{A_\alpha}\right) + \frac{A_\beta}{A_\alpha}\frac{\partial}{\partial \alpha}\left(\frac{\theta_\beta}{A_\beta}\right)$$
$$+ \frac{1}{R_\alpha}\left(\frac{1}{A_\beta}\frac{\partial u_0}{\partial \beta} - \frac{1}{A_\alpha A_\beta}\frac{\partial A_\beta}{\partial \alpha}v_0\right) + \frac{1}{R_\beta}\left(\frac{1}{A_\alpha}\frac{\partial v_0}{\partial \alpha} - \frac{1}{A_\alpha A_\beta}\frac{\partial A_\alpha}{\partial \beta}u_0\right).$$

Regarding the previous strain definitions it is worthy to mention that the zero-order terms $\varepsilon_{\alpha\alpha}^{*(0)}$, $\varepsilon_{\beta\beta}^{*(0)}$ and $\varepsilon_{\alpha\beta}^{*(0)}$ represent the normal (membrane) and shearing strains of the reference surface, respectively, and the first order terms $\varepsilon_{\alpha\alpha}^{*(1)}$, $\varepsilon_{\beta\beta}^{*(1)}$ and $\varepsilon_{\alpha\beta}^{*(1)}$ represent the linearly distributed bending components of strain and the torsion of the reference surface during deformation.

Considering the anisotropic (rotated orthotropic) plane-stress constitutive behavior presented in Eq. (A14), the in-plane stresses are given as

$$\begin{aligned}
\sigma_{\alpha\alpha}^{*}(\alpha,\beta,z) &= \bar{c}_{11}^{*}\varepsilon_{\alpha\alpha}^{*}(\alpha,\beta,z) + \bar{c}_{12}^{*}\varepsilon_{\beta\beta}^{*}(\alpha,\beta,z) + \bar{c}_{16}^{*}\varepsilon_{\alpha\beta}^{*}(\alpha,\beta,z)\\
&= \sigma_{\alpha\alpha}^{*(0)}(\alpha,\beta,z) + \sigma_{\alpha\alpha}^{*(1)}(\alpha,\beta,z),\\
\sigma_{\beta\beta}^{*}(\alpha,\beta,z) &= \bar{c}_{12}^{*}\varepsilon_{\alpha\alpha}^{*}(\alpha,\beta,z) + \bar{c}_{22}^{*}\varepsilon_{\beta\beta}^{*}(\alpha,\beta,z) + \bar{c}_{26}^{*}\varepsilon_{\alpha\beta}^{*}(\alpha,\beta,z)\\
&= \sigma_{\beta\beta}^{*(0)}(\alpha,\beta,z) + \sigma_{\beta\beta}^{*(1)}(\alpha,\beta,z),\\
\sigma_{\alpha\beta}^{*}(\alpha,\beta,z) &= \bar{c}_{16}^{*}\varepsilon_{\alpha\alpha}^{*}(\alpha,\beta,z) + \bar{c}_{26}^{*}\varepsilon_{\beta\beta}^{*}(\alpha,\beta,z) + \bar{c}_{66}^{*}\varepsilon_{\alpha\beta}^{*}(\alpha,\beta,z)\\
&= \sigma_{\alpha\beta}^{*(0)}(\alpha,\beta,z) + \sigma_{\alpha\beta}^{*(1)}(\alpha,\beta,z),
\end{aligned} \tag{A7}$$

where in a similar way to the zero-order and first-order strain definitions, Eqs. (A7) are split into zero-order and first-order stress components, where, for example, for the first of Eqs. (A7),

$$\begin{aligned}
\sigma_{\alpha\alpha}^{*(0)}(\alpha,\beta,z) &= z_\alpha^{(0)}\bar{c}_{11}^{*}\varepsilon_{\alpha\alpha}^{*(0)}(\alpha,\beta) + z_\beta^{(0)}\bar{c}_{12}^{*}\varepsilon_{\beta\beta}^{*(0)}(\alpha,\beta) + z_{\alpha\beta}^{(0)}\bar{c}_{16}^{*}\varepsilon_{\alpha\beta}^{*(0)}(\alpha,\beta),\\
\sigma_{\alpha\alpha}^{*(1)}(\alpha,\beta,z) &= z_\alpha^{(1)}\bar{c}_{11}^{*}\varepsilon_{\alpha\alpha}^{*(1)}(\alpha,\beta) + z_\beta^{(1)}\bar{c}_{12}^{*}\varepsilon_{\beta\beta}^{*(1)}(\alpha,\beta) + z_{\alpha\beta}^{(1)}\bar{c}_{16}^{*}\varepsilon_{\alpha\beta}^{*(1)}(\alpha,\beta).
\end{aligned} \tag{A8}$$

Similar relations hold for the second and third Eqs. of (A7) which for the sake of brevity are not presented here.

Elastic Constitutive Behavior

The linear elastic constitutive equation in *engineering notation* is given by

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}, \tag{A9}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and strain vectors and $\mathbf{C}$ is the elasticity matrix appropriate for the material. The elastic material is assumed to be rotated orthotropic (anisotropic), with the axes of orthotropy not necessarily parallel (arbitrary orientation of the generic shell layer) to the axes of principal curvature of the shell layer (α,β,z). Representing Eq. (A9) with their full matrix and vector terms yields

$$\begin{Bmatrix}\sigma_{\alpha\alpha}\\ \sigma_{\beta\beta}\\ \sigma_{zz}\\ \sigma_{\beta z}\\ \sigma_{z\alpha}\\ \sigma_{\alpha\beta}\end{Bmatrix} = \begin{bmatrix}\bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} & 0 & 0 & \bar{c}_{16}\\ \bar{c}_{12} & \bar{c}_{22} & \bar{c}_{23} & 0 & 0 & \bar{c}_{26}\\ \bar{c}_{13} & \bar{c}_{23} & \bar{c}_{33} & 0 & 0 & \bar{c}_{36}\\ 0 & 0 & 0 & \bar{c}_{44} & \bar{c}_{45} & 0\\ 0 & 0 & 0 & \bar{c}_{45} & \bar{c}_{55} & 0\\ \bar{c}_{16} & \bar{c}_{26} & \bar{c}_{36} & 0 & 0 & \bar{c}_{66}\end{bmatrix} \begin{Bmatrix}\varepsilon_{\alpha\alpha}\\ \varepsilon_{\beta\beta}\\ \varepsilon_{zz}\\ \varepsilon_{\beta z}\\ \varepsilon_{z\alpha}\\ \varepsilon_{\alpha\beta}\end{Bmatrix} \tag{A10}$$

The relationships between the problem quantities $\bar{c}_{ij}$ $(i, j = 1, \ldots, 6)$ and the original material c_{ij} when the material system $\alpha\beta$-plane is rotated an angle $+\theta$ (rotating from α to β) around the z-axis are given by

$$\begin{aligned}
\bar{c}_{11} &= c_{11}\cos^4\theta + 2\left(c_{12} + 2c_{66}\right)\sin^2\theta\cos^2\theta + c_{22}\sin^4\theta,\\
\bar{c}_{12} &= c_{12}\left(\sin^4\theta + \cos^4\theta\right) + \left(c_{11} + c_{22} - 4c_{66}\right)\sin^2\theta\cos^2\theta,\\
\bar{c}_{13} &= c_{13}\cos^2\theta + c_{23}\sin^2\theta,\\
\bar{c}_{16} &= \left(c_{11} - c_{12} - 2c_{66}\right)\sin\theta\cos^3\theta + \left(c_{12} - c_{22} + 2c_{66}\right)\sin^3\theta\cos\theta,\\
\bar{c}_{22} &= c_{11}\sin^4\theta + 2\left(c_{12} + 2c_{66}\right)\sin^2\theta\cos^2\theta + c_{22}\cos^4\theta,\\
\bar{c}_{23} &= c_{13}\sin^2\theta + c_{23}\cos^2\theta,\\
\bar{c}_{26} &= \left(c_{11} - c_{12} - 2c_{66}\right)\sin^3\theta\cos\theta + \left(c_{12} - c_{22} + 2c_{66}\right)\sin\theta\cos^3\theta,\\
\bar{c}_{33} &= c_{33},\\
\bar{c}_{36} &= \left(c_{13} - c_{23}\right)\sin\theta\cos\theta,\\
\bar{c}_{44} &= c_{44}\cos^2\theta + c_{55}\sin^2\theta,\\
\bar{c}_{45} &= \left(c_{55} - c_{44}\right)\sin\theta\cos\theta,\\
\bar{c}_{55} &= c_{55}\cos^2\theta + c_{44}\sin^2\theta,\\
\bar{c}_{66} &= 2(c_{11} + c_{22} - 2c_{12})\sin^2\theta\cos^2\theta + c_{66}(\sin^2\theta - \cos^2\theta)^2.
\end{aligned} \tag{A11}$$

Similar relations to Eq. (A9) can be expressed for the full strain-stress relationship,

$$\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma}, \tag{A12}$$

where $\mathbf{S}$ is the compliance matrix, which in matrix form is written as

$$\begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{zz} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} \bar{s}_{11} & \bar{s}_{12} & \bar{s}_{13} & 0 & 0 & \bar{s}_{16} \\ \bar{s}_{12} & \bar{s}_{22} & \bar{s}_{23} & 0 & 0 & \bar{s}_{26} \\ \bar{s}_{13} & \bar{s}_{23} & \bar{s}_{33} & 0 & 0 & \bar{s}_{36} \\ 0 & 0 & 0 & \bar{s}_{44} & \bar{s}_{45} & 0 \\ 0 & 0 & 0 & \bar{s}_{45} & \bar{s}_{55} & 0 \\ \bar{s}_{16} & \bar{s}_{26} & \bar{s}_{36} & 0 & 0 & \bar{s}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{zz} \\ \sigma_{\beta z} \\ \sigma_{z\alpha} \\ \sigma_{\alpha\beta} \end{Bmatrix}, \tag{A13}$$

where the problem quantities $\bar{s}_{ij}$ might be obtained from the relationships $\mathbf{S} = \mathbf{C}^{-1}$ by taking the matrix $\mathbf{C}$ defined in Eq. (A10).

If one now reduces Eq. (A10) to the plane-stress constitutive behavior, where $\sigma_{zz} \approx 0$, yields

$$\begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\beta z} \\ \sigma_{z\alpha} \\ \sigma_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} \bar{c}^*_{11} & \bar{c}^*_{12} & 0 & 0 & \bar{c}^*_{16} \\ \bar{c}^*_{12} & \bar{c}^*_{22} & 0 & 0 & \bar{c}^*_{26} \\ 0 & 0 & \bar{c}_{44} & \bar{c}_{45} & 0 \\ 0 & 0 & \bar{c}_{45} & \bar{c}_{55} & 0 \\ \bar{c}^*_{16} & \bar{c}^*_{26} & 0 & 0 & \bar{c}^*_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix}, \tag{A14}$$

where the elastic stiffness constants have been modified to take the plane-stress assumption into account and are defined as

$$\begin{aligned}
\bar{c}^*_{11} &= \bar{c}_{11} - \frac{\left(\bar{c}_{13}\right)^2}{\bar{c}_{33}}, & \bar{c}^*_{22} &= \bar{c}_{22} - \frac{\left(\bar{c}_{23}\right)^2}{\bar{c}_{33}}, & \bar{c}^*_{66} &= \bar{c}_{66} - \frac{\left(\bar{c}_{36}\right)^2}{\bar{c}_{33}},\\
\bar{c}^*_{12} &= \bar{c}_{12} - \frac{\bar{c}_{13}\bar{c}_{23}}{\bar{c}_{33}}, & \bar{c}^*_{16} &= \bar{c}_{16} - \frac{\bar{c}_{13}\bar{c}_{36}}{\bar{c}_{33}}, & \bar{c}^*_{26} &= \bar{c}_{26} - \frac{\bar{c}_{23}\bar{c}_{36}}{\bar{c}_{33}}.
\end{aligned} \tag{A15}$$

Similar relations to the previous equations hold for the reduced compliances when the plane-stress is considered.

”Mixed” Displacement Field and Strains

According to the in-plane ”mixed” displacements definition in Eqs. (30) and taking into account the shear angles definition in Eqs. (27), the displacement field can be expresses as

$$\begin{aligned} u(\alpha,\beta,z,t) &= \frac{u_0}{z_\alpha^{(0)}} + \frac{z_\alpha^{*(f)}\bar{s}_{55}}{z_\alpha^{(0)}}\left(\frac{\partial w_0}{\partial\alpha} + \theta_\alpha - \frac{u_0}{R_\alpha}\right) - \frac{z_\alpha^{*(0)}}{z_\alpha^{(0)}}\frac{\partial w_0}{\partial\alpha} + \frac{z_\alpha^{*(0)}\bar{s}_{55}}{z_\alpha^{(0)}}\bar{\tau}_{z\alpha} + \frac{z_\alpha^{*(1)}\bar{s}_{55}}{z_\alpha^{(0)}2h}\tilde{\tau}_{z\alpha}, \\ v(\alpha,\beta,z,t) &= \frac{v_0}{z_\beta^{(0)}} + \frac{z_\beta^{*(f)}\bar{s}_{44}}{z_\beta^{(0)}}\left(\frac{\partial w_0}{\partial\beta} + \theta_\beta - \frac{v_0}{R_\beta}\right) - \frac{z_\beta^{*(0)}}{z_\beta^{(0)}}\frac{\partial w_0}{\partial\beta} + \frac{z_\beta^{*(0)}\bar{s}_{44}}{z_\beta^{(0)}}\bar{\tau}_{\beta z} + \frac{z_\beta^{*(1)}\bar{s}_{44}}{z_\beta^{(0)}2h}\tilde{\tau}_{\beta z}, \end{aligned} \tag{A16}$$

or, alternatively,

$$\begin{aligned} u(\alpha,\beta,z,t) &= z_{11}^u u_0 + z_{13}^u \frac{\partial w_0}{\partial\alpha} + z_{14}^u\theta_\alpha + z_{11}^\tau\bar{\tau}_{z\alpha} + z_{12}^\tau\tilde{\tau}_{z\alpha}, \\ v(\alpha,\beta,z,t) &= z_{22}^u v_0 + z_{23}^u \frac{\partial w_0}{\partial\beta} + z_{25}^u\theta_\beta + z_{23}^\tau\bar{\tau}_{\beta z} + z_{24}^\tau\tilde{\tau}_{\beta z}, \end{aligned} \tag{A17}$$

where the $z_{ij}^u = z_{ij}^u(z)$ and $z_{ir}^\tau = z_{ir}^\tau(z)$ coefficients are given by

$$\begin{aligned} z_{11}^u &= \frac{1}{z_\alpha^{(0)}} - \frac{z_\alpha^{*(f)}\bar{s}_{55}^*}{z_\alpha^{(0)}R_\alpha}, \qquad z_{13}^u = \frac{z_\alpha^{*(f)}\bar{s}_{55}^*}{z_\alpha^{(0)}} - \frac{z_\alpha^{*(0)}}{z_\alpha^{(0)}}, \qquad z_{14}^u = \frac{z_\alpha^{*(f)}\bar{s}_{55}^*}{z_\alpha^{(0)}}, \\ z_{22}^u &= \frac{1}{z_\beta^{(0)}} - \frac{z_\beta^{*(f)}\bar{s}_{44}^*}{z_\beta^{(0)}R_\beta}, \qquad z_{23}^u = \frac{z_\beta^{*(f)}\bar{s}_{44}^*}{z_\beta^{(0)}} - \frac{z_\beta^{*(0)}}{z_\beta^{(0)}}, \qquad z_{25}^u = \frac{z_\beta^{*(f)}\bar{s}_{44}^*}{z_\beta^{(0)}}, \end{aligned} \tag{A18}$$

$$z_{11}^\tau = \frac{z_\alpha^{*(0)}\bar{s}_{55}^*}{z_\alpha^{(0)}}, \qquad z_{12}^\tau = \frac{z_\alpha^{*(1)}}{z_\alpha^{(0)}}\frac{\bar{s}_{55}^*}{2h}, \qquad z_{23}^\tau = \frac{z_\beta^{*(0)}\bar{s}_{44}^*}{z_\beta^{(0)}}, \qquad z_{24}^\tau = \frac{z_\beta^{*(1)}}{z_\beta^{(0)}}\frac{\bar{s}_{44}^*}{2h}. \tag{A19}$$

It is worthy to mention that these terms are functions of z and incorporate elastic constants of the material and geometric variables of the shell layer.

The non-zero coefficients of matrices $\mathbf{z}^{\varepsilon u}(z)$ and $\mathbf{z}^{\varepsilon\tau}(z)$ used to define the strain field in Eq. (36) are given by

$$\begin{aligned} & z_{11}^{\varepsilon u} = z_\alpha^{(0)} z_{11}^u, \qquad z_{13}^{\varepsilon u1} = \frac{z_\alpha^{(0)}}{R_\alpha}, \qquad z_{13}^{\varepsilon u2} = z_\alpha^{*(f)}, \\ & z_{14}^{\varepsilon u} = z_{14}^u z_\alpha^{(0)}, \qquad z_{22}^{\varepsilon u} = z_{22}^{(0)} z_\beta^u, \qquad z_{23}^{\varepsilon u1} = \frac{z_\beta^{(0)}}{R_\beta}, \\ & z_{23}^{\varepsilon u2} = z_\beta^{*(f)}, \qquad z_{25}^{\varepsilon u} = z_{25}^u z_\beta^{(0)}, \qquad z_{32}^{\varepsilon u} = \frac{\partial z_{22}^u}{\partial z} - \frac{z_{22}^u z_\beta^{(0)}}{R_\beta}, \\ & z_{33}^{\varepsilon u} = z_\beta^{(0)} + \frac{\partial z_{23}^u}{\partial z} - \frac{z_{23}^u z_\beta^{(0)}}{R_\beta}, \qquad z_{35}^{\varepsilon u} = \frac{\partial z_{25}^u}{\partial z} - \frac{z_{25}^u z_\beta^{(0)}}{R_\beta}, \qquad z_{41}^{\varepsilon u} = \frac{\partial z_{11}^u}{\partial z} - \frac{z_{11}^u z_\alpha^{(0)}}{R_\alpha}, \\ & z_{43}^{\varepsilon u} = z_\alpha^{(0)} + \frac{\partial z_{13}^u}{\partial z} - \frac{z_{13}^u z_\alpha^{(0)}}{R_\alpha}, \qquad z_{44}^{\varepsilon u} = \frac{\partial z_{14}^u}{\partial z} - \frac{z_{14}^u z_\alpha^{(0)}}{R_\alpha}, \qquad z_{51}^{\varepsilon u} = z_{11}^u z_\beta^{(0)}, \\ & z_{52}^{\varepsilon u} = z_{22}^u z_\alpha^{(0)}, \qquad z_{53}^{\varepsilon u} = z_{13}^u z_\beta^{(0)} + z_{23}^u z_\alpha^{(0)}, \qquad z_{54}^{\varepsilon u} = z_{14}^u z_\beta^{(0)}, \\ & z_{55}^{\varepsilon u} = z_{25}^u z_\alpha^{(0)}, \end{aligned} \tag{A20}$$

and

$$\begin{aligned}
&z_{11}^{\varepsilon\tau} = z_\alpha^{(0)} z_{11}^\tau, && z_{12}^{\varepsilon\tau} = z_\alpha^{(0)} z_{12}^\tau, && z_{23}^{\varepsilon\tau} = z_\beta^{(0)} z_{23}^\tau,\\
&z_{24}^{\varepsilon\tau} = z_\beta^{(0)} z_{24}^\tau, && z_{33}^{\varepsilon\tau} = \frac{\partial z_{23}^\tau}{\partial z} - \frac{z_{23}^\tau}{R_\beta} z_\beta^{(0)}, && z_{34}^{\varepsilon\tau} = \frac{\partial z_{24}^\tau}{\partial z} - \frac{z_{24}^\tau}{R_\beta} z_\beta^{(0)},\\
&z_{41}^{\varepsilon\tau} = \frac{\partial z_{11}^\tau}{\partial z} - \frac{z_{11}^\tau}{R_\alpha} z_\alpha^{(0)}, && z_{42}^{\varepsilon\tau} = \frac{\partial z_{12}^\tau}{\partial z} - \frac{z_{12}^\tau}{R_\alpha} z_\alpha^{(0)}, && z_{51}^{\varepsilon\tau} = z_\beta^{(0)} z_{11}^\tau,\\
&z_{52}^{\varepsilon\tau} = z_\beta^{(0)} z_{12}^\tau, && z_{53}^{\varepsilon\tau} = z_\alpha^{(0)} z_{23}^\tau, && z_{54}^{\varepsilon\tau} = z_\alpha^{(0)} z_{24}^\tau.
\end{aligned} \tag{A21}$$

Virtual Work Terms

Virtual Work of the Inertial Forces

Taking into account Eqs. (2) and the doubly-curved shells restrictions in Eqs. (28) into Eq. (39) yields

$$\begin{aligned}
\delta T = &-\int_{\Omega_0} \left[\int_{-h}^{+h} \rho\left(\delta u \ddot{u} + \delta v \ddot{v} + \delta w \ddot{w}\right) \frac{1}{z_\alpha^{(0)} z_\beta^{(0)}} dz \right] d\alpha\, d\beta \\
= &-\int_{\Omega_0} \Big[\delta u_0 (I_{11}^{uu} \ddot{u}_0 + I_{13}^{uu} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{14}^{uu} \ddot{\theta}_0^\alpha + I_{11}^{\tau u} \ddot{\bar{\tau}}_{z\alpha} + I_{21}^{\tau u} \ddot{\tilde{\tau}}_{z\alpha}) + \delta v_0 (I_{22}^{uu} \ddot{v}_0 + I_{23}^{uu} \frac{\partial \ddot{w}_0}{\partial \beta} \\
&+ I_{25}^{uu} \ddot{\theta}_0^\beta + I_{32}^{\tau u} \ddot{\bar{\tau}}_{\beta z} + I_{42}^{\tau u} \ddot{\tilde{\tau}}_{\beta z}) + \delta w (I_{33}^{uu} \ddot{w}) + \delta \frac{\partial w_0}{\partial \alpha} (I_{13}^{uu} \ddot{u}_0 + I_{33}^{u\alpha} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{34}^{uu} \ddot{\theta}_0^\alpha + I_{13}^{\tau u} \ddot{\bar{\tau}}_{z\alpha} \\
&+ I_{23}^{\tau u} \ddot{\tilde{\tau}}_{z\alpha}) + \delta \frac{\partial w_0}{\partial \beta} (I_{23}^{uu} \ddot{v}_0 + I_{33}^{u\beta} \frac{\partial \ddot{w}_0}{\partial \beta} + I_{35}^{uu} \ddot{\theta}_0^\beta + I_{33}^{\tau u} \ddot{\bar{\tau}}_{\beta z} + I_{43}^{\tau u} \ddot{\tilde{\tau}}_{\beta z}) + \delta \theta_0^\alpha (I_{14}^{uu} \ddot{u}_0 \\
&+ I_{43}^{uu} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{44}^{uu} \ddot{\theta}_0^\alpha + I_{14}^{\tau u} \ddot{\bar{\tau}}_{z\alpha} + I_{24}^{\tau u} \ddot{\tilde{\tau}}_{z\alpha}) + \delta \theta_0^\beta (I_{25}^{uu} \ddot{v}_0 + I_{35}^{uu} \frac{\partial \ddot{w}_0}{\partial \beta} + I_{55}^{uu} \ddot{\theta}_0^\beta + I_{35}^{\tau u} \ddot{\bar{\tau}}_{\beta z} \\
&+ I_{45}^{\tau u} \ddot{\tilde{\tau}}_{\beta z}) + \delta \bar{\tau}_{z\alpha} (I_{11}^{\tau u} \ddot{u}_0 + I_{13}^{\tau u} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{14}^{\tau u} \ddot{\theta}_0^\alpha + I_{11}^{\tau\tau} \ddot{\bar{\tau}}_{z\alpha} + I_{12}^{\tau\tau} \ddot{\tilde{\tau}}_{z\alpha}) + \delta \tilde{\tau}_{z\alpha} (I_{21}^{\tau u} \ddot{u}_0 \\
&+ I_{23}^{\tau u} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{24}^{\tau u} \ddot{\theta}_0^\alpha + I_{12}^{\tau\tau} \ddot{\bar{\tau}}_{z\alpha} + I_{22}^{\tau\tau} \ddot{\tilde{\tau}}_{z\alpha}) + \delta \bar{\tau}_{\beta z} (I_{32}^{\tau u} \ddot{v}_0 + I_{33}^{\tau u} \frac{\partial \ddot{w}_0}{\partial \beta} + I_{35}^{\tau u} \ddot{\theta}_0^\beta + I_{33}^{\tau\tau} \ddot{\bar{\tau}}_{\beta z} \\
&+ I_{34}^{\tau\tau} \ddot{\tilde{\tau}}_{\beta z}) + \delta \tilde{\tau}_{\beta z} (I_{42}^{\tau u} \ddot{v}_0 + I_{43}^{\tau u} \frac{\partial \ddot{w}_0}{\partial \beta} + I_{45}^{\tau u} \ddot{\theta}_0^\beta + I_{34}^{\tau\tau} \ddot{\bar{\tau}}_{\beta z} + I_{44}^{\tau\tau} \ddot{\tilde{\tau}}_{z\alpha}) \Big] d\alpha\, d\beta,
\end{aligned} \tag{A22}$$

where

$$\begin{aligned}
(I_{11}^{uu}, I_{13}^{uu}, I_{14}^{uu}) &= \rho \left\langle z_{11}^u (z_{11}^u, z_{13}^u, z_{14}^u) \right\rangle,\\
(I_{22}^{uu}, I_{23}^{uu}, I_{25}^{uu}) &= \rho \left\langle z_{22}^u (z_{22}^u, z_{23}^u, z_{25}^u) \right\rangle,\\
(I_{33}^{uu}) &= \rho \left\langle 1 \right\rangle,\\
(I_{33}^{u\alpha}, I_{34}^{uu}) &= \rho \left\langle z_{13}^u (z_{13}^u, z_{14}^u) \right\rangle,\\
(I_{33}^{u\beta}, I_{35}^{uu}) &= \rho \left\langle z_{23}^u (z_{23}^u, z_{25}^u) \right\rangle,\\
(I_{44}^{uu}, I_{55}^{uu}) &= \rho \left\langle (z_{14}^u z_{14}^u, z_{25}^u z_{25}^u) \right\rangle,\\
(I_{11}^{\tau u}, I_{13}^{\tau u}, I_{14}^{\tau u}, I_{11}^{\tau\tau}, I_{12}^{\tau\tau}) &= \rho \left\langle z_{11}^\tau (z_{11}^u, z_{13}^u, z_{14}^u, z_{11}^\tau, z_{12}^\tau) \right\rangle,\\
(I_{21}^{\tau u}, I_{23}^{\tau u}, I_{24}^{\tau u}, I_{22}^{\tau\tau}) &= \rho \left\langle z_{12}^\tau (z_{11}^u, z_{13}^u, z_{14}^u, z_{12}^\tau) \right\rangle,\\
(I_{32}^{\tau u}, I_{33}^{\tau u}, I_{35}^{\tau u}, I_{33}^{\tau\tau}, I_{34}^{\tau\tau}) &= \rho \left\langle z_{23}^\tau (z_{22}^u, z_{23}^u, z_{25}^u, z_{23}^\tau, z_{34}^\tau) \right\rangle,\\
(I_{42}^{\tau u}, I_{43}^{\tau u}, I_{45}^{\tau u}, I_{44}^{\tau\tau}) &= \rho \left\langle z_{24}^\tau (z_{22}^u, z_{23}^u, z_{25}^u, z_{24}^\tau) \right\rangle.
\end{aligned} \tag{A23}$$

For convenience $\langle \ldots \rangle$ denotes thickness integration and it is defined by

$$\langle \ldots \rangle = \int_{-h}^{+h} (\ldots) \frac{1}{z_\alpha^{(0)} z_\beta^{(0)}} dz. \tag{A24}$$

Virtual Work of the Internal Mechanical Forces

Considering the strain definitions in Eq. (34), the term δU of Eq. (41) is given by

$$\begin{aligned}
\delta U &= \int_{\Omega_0} \left[\int_{-h}^{h} (\sigma_{\alpha\alpha}\delta\varepsilon_{\alpha\alpha} + \sigma_{\beta\beta}\delta\varepsilon_{\beta\beta} + \sigma_{\beta z}\delta\varepsilon_{\beta z} + \sigma_{z\alpha}\delta\varepsilon_{z\alpha} + \sigma_{\alpha\beta}\delta\varepsilon_{\alpha\beta}) \frac{1}{z_\alpha^{(0)} z_\beta^{(0)}} dz \right] d\alpha\, d\beta \\
&= \int_{\Omega_0} \Big[\delta\frac{\partial u_0}{\partial\alpha}(N_{\alpha\alpha}^{\star 11} + N_{\beta\beta}^{\star 21}) + \delta\frac{\partial u_0}{\partial\beta}(N_{\alpha\beta}^{51}) + \delta u_0(Q_{z\alpha}^{41}) + \delta\frac{\partial v_0}{\partial\beta}(N_{\alpha\alpha}^{\star 12} + N_{\beta\beta}^{\star 22}) \\
&+ \delta\frac{\partial v_0}{\partial\alpha}(N_{\alpha\beta}^{52}) + \delta v_0(Q_{\beta z}^{32}) + \delta w_0(M_{\alpha\alpha}^{\star 131} + M_{\beta\beta}^{\star 231}) + \delta\frac{\partial^2 w_0}{\partial\alpha^2}(M_{\alpha\alpha}^{\star 132}) + \delta\frac{\partial^2 w_0}{\partial\beta^2}(M_{\beta\beta}^{\star 232}) \\
&+ \delta\frac{\partial^2 w_0}{\partial\alpha\partial\beta}(M_{\alpha\beta}^{53}) + \delta\frac{\partial w_0}{\partial\alpha}(Q_{z\alpha}^{43}) + \delta\frac{\partial w_0}{\partial\beta}(Q_{\beta z}^{33}) + \delta\frac{\partial\theta_\alpha}{\partial\alpha}(M_{\alpha\alpha}^{\star 14} + M_{\beta\beta}^{\star 24}) + \delta\frac{\partial\theta_\alpha}{\partial\beta}(M_{\alpha\beta}^{54}) \\
&+ \delta\theta_\alpha(Q_{z\alpha}^{44}) + \delta\frac{\partial\theta_\beta}{\partial\beta}(M_{\alpha\alpha}^{\star 15} + M_{\beta\beta}^{\star 25}) + \delta\frac{\partial\theta_\beta}{\partial\alpha}(M_{\alpha\beta}^{55}) + \delta\theta_\beta(Q_{\beta z}^{35}) + \delta\frac{\partial\bar{\tau}_{z\alpha}}{\partial\alpha}(T_{\alpha\alpha}^{\star 11} + T_{\beta\beta}^{\star 21}) \\
&+ \delta\bar{\tau}_{z\alpha}(T_{z\alpha}^{41}) + \delta\frac{\partial\bar{\tau}_{z\alpha}}{\partial\beta}(T_{\alpha\beta}^{51}) + \delta\frac{\partial\tilde{\tau}_{z\alpha}}{\partial\alpha}(T_{\alpha\alpha}^{\star 12} + T_{\beta\beta}^{\star 22}) + \delta\tilde{\tau}_{z\alpha}(T_{z\alpha}^{42}) + \delta\frac{\partial\tilde{\tau}_{z\alpha}}{\partial\beta}(T_{\alpha\beta}^{52}) \\
&+ \delta\frac{\partial\bar{\tau}_{\beta z}}{\partial\beta}(T_{\alpha\alpha}^{\star 13} + T_{\beta\beta}^{\star 23}) + \delta\bar{\tau}_{\beta z}(T_{\beta z}^{33}) + \delta\frac{\partial\bar{\tau}_{\beta z}}{\partial\alpha}(T_{\alpha\beta}^{53}) + \delta\frac{\partial\tilde{\tau}_{\beta z}}{\partial\beta}(T_{\alpha\alpha}^{\star 14} + T_{\beta\beta}^{\star 24}) \\
&+ \delta\tilde{\tau}_{\beta z}(T_{\beta z}^{34}) + \delta\frac{\partial\tilde{\tau}_{\beta z}}{\partial\alpha}(T_{\alpha\beta}^{54}) \Big] d\alpha\, d\beta,
\end{aligned} \tag{A25}$$

where

$$\begin{aligned}
\left(N_{\alpha\alpha}^{\star 11}, N_{\alpha\alpha}^{\star 12}, M_{\alpha\alpha}^{\star 131}, M_{\alpha\alpha}^{\star 132}, M_{\alpha\alpha}^{\star 14}, M_{\alpha\alpha}^{\star 15}\right) &= \left\langle \sigma_{\alpha\alpha}(z_{11}^{\star\varepsilon u}, z_{12}^{\star\varepsilon u}, z_{13}^{\star\varepsilon u1}, z_{13}^{\star\varepsilon u2}, z_{14}^{\star\varepsilon u}, z_{15}^{\star\varepsilon u}) \right\rangle, \\
\left(N_{\beta\beta}^{\star 21}, N_{\beta\beta}^{\star 22}, M_{\beta\beta}^{\star 231}, M_{\beta\beta}^{\star 232}, M_{\beta\beta}^{\star 24}, M_{\beta\beta}^{\star 25}\right) &= \left\langle \sigma_{\beta\beta}(z_{21}^{\star\varepsilon u}, z_{22}^{\star\varepsilon u}, z_{23}^{\star\varepsilon u1}, z_{23}^{\star\varepsilon u2}, z_{24}^{\star\varepsilon u}, z_{25}^{\star\varepsilon u}) \right\rangle, \\
\left(Q_{\beta z}^{32}, Q_{\beta z}^{33}, Q_{\beta z}^{35}\right) &= \left\langle \sigma_{\beta z}(z_{32}^{\varepsilon u}, z_{33}^{\varepsilon u}, z_{35}^{\varepsilon u}) \right\rangle, \\
\left(Q_{z\alpha}^{41}, Q_{z\alpha}^{43}, Q_{z\alpha}^{44}\right) &= \left\langle \sigma_{z\alpha}(z_{41}^{\varepsilon u}, z_{43}^{\varepsilon u}, z_{44}^{\varepsilon u}) \right\rangle, \\
\left(N_{\alpha\beta}^{51}, N_{\alpha\beta}^{52}, M_{\alpha\beta}^{53}, M_{\alpha\beta}^{54}, M_{\alpha\beta}^{55}\right) &= \left\langle \sigma_{\alpha\beta}(z_{51}^{\varepsilon u}, z_{52}^{\varepsilon u}, z_{53}^{\varepsilon u}, z_{54}^{\varepsilon u}, z_{55}^{\varepsilon u}) \right\rangle, \\
\left(T_{\alpha\alpha}^{\star 11}, T_{\alpha\alpha}^{\star 12}, T_{\alpha\alpha}^{\star 13}, T_{\alpha\alpha}^{\star 14}\right) &= \left\langle \sigma_{\alpha\alpha}(z_{11}^{\star\varepsilon\tau}, z_{12}^{\star\varepsilon\tau}, z_{13}^{\star\varepsilon\tau}, z_{14}^{\star\varepsilon\tau}) \right\rangle, \\
\left(T_{\beta\beta}^{\star 21}, T_{\beta\beta}^{\star 22}, T_{\beta\beta}^{\star 23}, T_{\beta\beta}^{\star 24}\right) &= \left\langle \sigma_{\beta\beta}(z_{21}^{\star\varepsilon\tau}, z_{22}^{\star\varepsilon\tau}, z_{23}^{\star\varepsilon\tau}, z_{24}^{\star\varepsilon\tau}) \right\rangle, \\
\left(T_{\beta z}^{33}, T_{\beta z}^{34}\right) &= \left\langle \sigma_{\beta z}(z_{33}^{\varepsilon\tau}, z_{34}^{\varepsilon\tau}) \right\rangle, \\
\left(T_{z\alpha}^{41}, T_{z\alpha}^{42}\right) &= \left\langle \sigma_{z\alpha}(z_{41}^{\varepsilon\tau}, z_{42}^{\varepsilon\tau}) \right\rangle, \\
\left(T_{\alpha\beta}^{51}, T_{\alpha\beta}^{52}, T_{\alpha\beta}^{53}, T_{\alpha\beta}^{54}\right) &= \left\langle \sigma_{\alpha\beta}(z_{51}^{\varepsilon\tau}, z_{52}^{\varepsilon\tau}, z_{53}^{\varepsilon\tau}, z_{54}^{\varepsilon\tau}) \right\rangle.
\end{aligned} \tag{A26}$$

Virtual Work of the Non-Conservative Forces

In a similar way to what has been done before, the integration with respect to z is conveniently carrying out, and the virtual work of the non-conservative forces δW can be expressed in terms of prescribed forces and moments as

$$\delta W = \int_{\Omega_0} \delta w_0(Z) d\alpha\, d\beta + \oint_{\Gamma_\alpha} \Big[\delta u_0(\hat{N}^{11}_{\beta\alpha}) + \delta v_0(\hat{N}^{22}_{\beta\beta}) + \delta\frac{\partial w_0}{\partial\beta}(\hat{M}^{23}_{\beta\beta}) + \delta\frac{\partial w_0}{\partial\alpha}(\hat{M}^{13}_{\beta\alpha}) + \delta w_0(\hat{Q}^{33}_{\beta z})$$
$$+ \delta\theta_\alpha(\hat{M}^{14}_{\beta\alpha}) + \delta\theta_\beta(\hat{M}^{25}_{\beta\beta}) + \delta\bar{\tau}_{z\alpha}(\hat{T}^{11}_{\beta\alpha}) + \delta\tilde{\tau}_{z\alpha}(\hat{T}^{12}_{\beta\alpha}) + \delta\bar{\tau}_{\beta z}(\hat{T}^{23}_{\beta\beta}) + \delta\tilde{\tau}_{\beta z}(\hat{T}^{24}_{\beta\beta})\Big] d\alpha$$
$$+ \oint_{\Gamma_\beta} \Big[\delta u_0(\hat{N}^{11}_{\alpha\alpha}) + \delta v_0(\hat{N}^{22}_{\alpha\beta}) + \delta\frac{\partial w_0}{\partial\alpha}(\hat{M}^{13}_{\alpha\alpha}) + \delta\frac{\partial w_0}{\partial\beta}(\hat{M}^{23}_{\alpha\alpha}) + \delta w_0(\hat{Q}^{33}_{z\alpha}) + \delta\theta_\alpha(\hat{M}^{14}_{\alpha\alpha})$$
$$+ \delta\theta_\beta(\hat{M}^{25}_{\alpha\beta}) + \delta\bar{\tau}_{z\alpha}(\hat{T}^{11}_{\alpha\alpha}) + \delta\tilde{\tau}_{z\alpha}(\hat{T}^{12}_{\alpha\alpha}) + \delta\bar{\tau}_{\beta z}(\hat{T}^{23}_{\alpha\beta}) + \delta\tilde{\tau}_{\beta z}(\hat{T}^{24}_{\alpha\beta})\Big] d\beta, \quad \text{(A27)}$$

where

$$\begin{aligned}
(\hat{N}^{11}_{\alpha\alpha}, \hat{M}^{13}_{\alpha\alpha}, \hat{M}^{14}_{\alpha\alpha}) &= \langle \hat{\sigma}_{\alpha\alpha}\,(z^u_{11}, z^u_{13}, z^u_{14})\rangle_\beta\,, \\
(\hat{N}^{22}_{\alpha\beta}, \hat{M}^{23}_{\alpha\beta}, \hat{M}^{25}_{\alpha\beta}) &= \langle \hat{\sigma}_{\alpha\beta}\,(z^u_{22}, z^u_{23}, z^u_{25})\rangle_\beta\,, \\
(\hat{Q}^{33}_{z\alpha}) &= \langle \hat{\sigma}_{z\alpha}\rangle_\beta\,, \\
(\hat{T}^{11}_{\alpha\alpha}, \hat{T}^{12}_{\alpha\alpha}) &= \langle \hat{\sigma}_{\alpha\alpha}\,(z^\tau_{11}, z^\tau_{12})\rangle_\beta\,, \\
(\hat{T}^{23}_{\alpha\beta}, \hat{T}^{24}_{\alpha\beta}) &= \langle \hat{\sigma}_{\alpha\beta}\,(z^\tau_{23}, z^\tau_{24})\rangle_\beta\,, \\
(\hat{N}^{22}_{\beta\beta}, \hat{M}^{23}_{\beta\beta}, \hat{M}^{25}_{\beta\beta}) &= \langle \hat{\sigma}_{\beta\beta}\,(z^u_{22}, z^u_{23}, z^u_{25})\rangle_\alpha\,, \\
(\hat{N}^{11}_{\beta\alpha}, \hat{M}^{13}_{\beta\alpha}, \hat{M}^{14}_{\beta\alpha}) &= \langle \hat{\sigma}_{\beta\alpha}\,(z^u_{11}, z^u_{13}, z^u_{14})\rangle_\alpha\,, \\
(\hat{Q}^{33}_{\beta z}) &= \langle \hat{\sigma}_{\beta z}\rangle_\alpha\,, \\
(\hat{T}^{23}_{\beta\beta}, \hat{T}^{24}_{\beta\beta}) &= \langle \hat{\sigma}_{\beta\beta}\,(z^\tau_{23}, z^\tau_{24})\rangle_\alpha\,, \\
(\hat{T}^{11}_{\beta\alpha}, \hat{T}^{12}_{\beta\alpha}) &= \langle \hat{\sigma}_{\beta\alpha}\,(z^\tau_{11}, z^\tau_{12})\rangle_\alpha\,.
\end{aligned} \quad \text{(A28)}$$

Once again, for convenience, $\langle \ldots \rangle_\alpha$ and $\langle \ldots \rangle_\beta$ denote thickness integration and are defined as

$$\langle \ldots \rangle_\alpha = \int_{-h}^{+h} (\ldots)\,\frac{1}{z^{(0)}_\alpha} dz, \qquad \langle \ldots \rangle_\beta = \int_{-h}^{+h} (\ldots)\,\frac{1}{z^{(0)}_\beta} dz. \quad \text{(A29)}$$

Internal Forces and Moments in Terms of Generalized Variables

$$\left\{\begin{matrix} (N^{\star 11}_{\alpha\alpha}, N^{\star 12}_{\alpha\alpha}) \\ (N^{\star 21}_{\beta\beta}, N^{\star 22}_{\beta\beta}) \\ (N^{51}_{\alpha\beta}, N^{52}_{\alpha\beta}) \end{matrix}\right\} = \begin{bmatrix} A_{11}\partial_\alpha & A_{12}\partial_\beta & A^1_{13} + A^2_{13}\partial_{\alpha\alpha} + A^3_{13}\partial_{\beta\beta} & A_{14}\partial_\alpha & A_{15}\partial_\beta \\ A_{21}\partial_\alpha & A_{22}\partial_\beta & A^1_{23} + A^2_{23}\partial_{\alpha\alpha} + A^3_{23}\partial_{\beta\beta} & A_{24}\partial_\alpha & A_{25}\partial_\beta \\ A_{51}\partial_\beta & A_{52}\partial_\alpha & A_{53}\partial_{\alpha\beta} & A_{54}\partial_\beta & A_{55}\partial_\alpha \end{bmatrix} \left\{\begin{matrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{matrix}\right\}$$
$$+ \begin{bmatrix} A_{16}\partial_\alpha & A_{17}\partial_\alpha & A_{18}\partial_\beta & A_{19}\partial_\beta \\ A_{26}\partial_\alpha & A_{27}\partial_\alpha & A_{28}\partial_\beta & A_{29}\partial_\beta \\ A_{56}\partial_\beta & A_{57}\partial_\beta & A_{58}\partial_\alpha & A_{59}\partial_\alpha \end{bmatrix} \left\{\begin{matrix} \bar{\tau}_{z\alpha} \\ \tilde{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \tilde{\tau}_{\beta z} \end{matrix}\right\}, \quad \text{(A30)}$$

$$\begin{Bmatrix} \left(Q^{32}_{\beta z}, Q^{33}_{\beta z}, Q^{35}_{\beta z}\right) \\ \left(Q^{41}_{z\alpha}, Q^{43}_{z\alpha}, Q^{44}_{z\alpha}\right) \end{Bmatrix} = \begin{bmatrix} 0 & A_{32} & A_{33}\partial_\beta & 0 & A_{35} \\ A_{41} & 0 & A_{43}\partial_\alpha & A_{44} & 0 \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{Bmatrix} + \begin{bmatrix} 0 & 0 & A_{38} & A_{39} \\ A_{46} & A_{47} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \tilde{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \tilde{\tau}_{\beta z} \end{Bmatrix}, \quad \text{(A31)}$$

$$\begin{Bmatrix} \left(M^{\star 131}_{\alpha\alpha}, M^{\star 132}_{\alpha\alpha}, M^{\star 14}_{\alpha\alpha}, M^{\star 15}_{\alpha\alpha}\right) \\ \left(M^{\star 231}_{\beta\beta}, M^{\star 232}_{\beta\beta}, M^{\star 24}_{\beta\beta}, M^{\star 25}_{\beta\beta}\right) \\ \left(M^{53}_{\alpha\beta}, M^{54}_{\alpha\beta}, M^{55}_{\alpha\beta}\right) \end{Bmatrix} = \begin{bmatrix} B_{11}\partial_\alpha & B_{12}\partial_\beta & B^1_{13} + B^2_{13}\partial_{\alpha\alpha} + B^3_{13}\partial_{\beta\beta} & B_{14}\partial_\alpha & B_{15}\partial_\beta \\ B_{21}\partial_\alpha & B_{22}\partial_\beta & B^1_{23} + B^2_{23}\partial_{\alpha\alpha} + B^3_{23}\partial_{\beta\beta} & B_{24}\partial_\alpha & B_{25}\partial_\beta \\ B_{51}\partial_\beta & B_{52}\partial_\alpha & B_{53}\partial_{\alpha\beta} & B_{54}\partial_\beta & B_{55}\partial_\alpha \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{Bmatrix} + \begin{bmatrix} B_{16}\partial_\alpha & B_{17}\partial_\alpha & B_{18}\partial_\beta & B_{19}\partial_\beta \\ B_{26}\partial_\alpha & B_{27}\partial_\alpha & B_{28}\partial_\beta & B_{29}\partial_\beta \\ B_{56}\partial_\beta & B_{57}\partial_\beta & B_{58}\partial_\alpha & B_{59}\partial_\alpha \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \tilde{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \tilde{\tau}_{\beta z} \end{Bmatrix}, \quad \text{(A32)}$$

$$\begin{Bmatrix} \left(T^{\star 11}_{\alpha\alpha}, T^{\star 12}_{\alpha\alpha}, T^{\star 13}_{\alpha\alpha}, T^{\star 14}_{\alpha\alpha}\right) \\ \left(T^{\star 21}_{\beta\beta}, T^{\star 22}_{\beta\beta}, T^{\star 23}_{\beta\beta}, T^{\star 24}_{\beta\beta}\right) \\ \left(T^{51}_{\alpha\beta}, T^{52}_{\alpha\beta}, T^{53}_{\alpha\beta}, T^{54}_{\alpha\beta}\right) \end{Bmatrix} = \begin{bmatrix} C_{11}\partial_\alpha & C_{12}\partial_\beta & C^1_{13} + C^2_{13}\partial_{\alpha\alpha} + C^3_{13}\partial_{\beta\beta} & C_{14}\partial_\alpha & C_{15}\partial_\beta \\ C_{21}\partial_\alpha & C_{22}\partial_\beta & C^1_{23} + C^2_{23}\partial_{\alpha\alpha} + C^3_{23}\partial_{\beta\beta} & C_{24}\partial_\alpha & C_{25}\partial_\beta \\ C_{51}\partial_\beta & C_{52}\partial_\alpha & C_{53}\partial_{\alpha\beta} & C_{54}\partial_\beta & C_{55}\partial_\alpha \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{Bmatrix} + \begin{bmatrix} C_{16}\partial_\alpha & C_{17}\partial_\alpha & C_{18}\partial_\beta & C_{19}\partial_\beta \\ C_{26}\partial_\alpha & C_{27}\partial_\alpha & C_{28}\partial_\beta & C_{29}\partial_\beta \\ C_{56}\partial_\beta & C_{57}\partial_\beta & C_{58}\partial_\alpha & C_{59}\partial_\alpha \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \tilde{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \tilde{\tau}_{\beta z} \end{Bmatrix}, \quad \text{(A33)}$$

$$\begin{Bmatrix} \left(T^{33}_{\beta z}, T^{34}_{\beta z}\right) \\ \left(T^{41}_{z\alpha}, T^{42}_{z\alpha}\right) \end{Bmatrix} = \begin{bmatrix} 0 & C_{32} & C_{33}\partial_\beta & 0 & C_{35} \\ C_{41} & 0 & C_{43}\partial_\alpha & C_{44} & 0 \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{Bmatrix} + \begin{bmatrix} 0 & 0 & C_{38} & C_{39} \\ C_{46} & C_{47} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \tilde{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \tilde{\tau}_{\beta z} \end{Bmatrix}, \quad \text{(A34)}$$

Considering the generic internal force or moment terms in Eqs. (55),(59) and (60) as

$$\left\langle \begin{bmatrix} \bar{c}^*_{11} & \bar{c}^*_{12} & 0 \\ \bar{c}^*_{12} & \bar{c}^*_{22} & 0 \\ 0 & 0 & \bar{c}^*_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{\alpha\alpha}(g_1) \\ \varepsilon_{\beta\beta}(g_2) \\ \varepsilon_{\alpha\beta}(g_5) \end{Bmatrix} \right\rangle , \tag{A35}$$

where g_i is used to denote a series of coefficients $(g_{i1}, g_{i2}, \ldots)$ of the correspondent z's used to define the internal forces and moments, the following rule holds to determine the A_{ij}, B_{ij} and C_{ij} coefficients (with $i = 1, 2, 5$ and $j = 1, \ldots, 9$), denoted by the generic G_{ij},

$$\begin{aligned}
G_{11} &= \langle \bar{c}^*_{11} z^{\star\varepsilon u}_{11} g_1 + \bar{c}^*_{12} z^{u\star\varepsilon}_{21} g_2 \rangle, & G_{53} &= \langle \bar{c}^*_{66} z^{\varepsilon u}_{53} g_5 \rangle , \\
G_{12} &= \langle \bar{c}^*_{11} z^{\star\varepsilon u}_{12} g_1 + \bar{c}^*_{12} z^{\star\varepsilon u}_{22} g_2 \rangle , & G_{54} &= \langle \bar{c}^*_{66} z^{\varepsilon u}_{54} g_5 \rangle , \\
G^1_{13} &= \langle \bar{c}^*_{11} z^{\star\varepsilon u1}_{13} g_1 + \bar{c}^*_{12} z^{\star\varepsilon u1}_{23} g_2 \rangle , & G_{55} &= \langle \bar{c}^*_{66} z^{\varepsilon u}_{55} g_5 \rangle, \\
G^2_{13} &= \langle \bar{c}^*_{11} z^{\star\varepsilon u2}_{13} g_1 \rangle , & G_{16} &= \langle \bar{c}^*_{11} z^{\star\varepsilon\tau}_{11} g_1 + \bar{c}^*_{12} z^{\star\varepsilon\tau}_{21} g_2 \rangle , \\
G^3_{13} &= \langle \bar{c}^*_{12} z^{\star\varepsilon u2}_{23} g_1 \rangle , & G_{17} &= \langle \bar{c}^*_{11} z^{\star\varepsilon\tau}_{12} g_1 + \bar{c}^*_{12} z^{\star\varepsilon\tau}_{22} g_2 \rangle , \\
G_{14} &= \langle \bar{c}^*_{11} z^{\star\varepsilon u}_{14} g_1 + \bar{c}^*_{12} z^{\star\varepsilon u}_{24} g_2 \rangle , & G_{18} &= \langle \bar{c}^*_{11} z^{\star\varepsilon\tau}_{13} g_1 + \bar{c}^*_{12} z^{\star\varepsilon\tau}_{23} g_2 \rangle , \\
G_{15} &= \langle \bar{c}^*_{11} z^{\star\varepsilon u}_{15} g_1 + \bar{c}^*_{12} z^{\star\varepsilon u}_{25} g_2 \rangle , & G_{19} &= \langle \bar{c}^*_{11} z^{\star\varepsilon\tau}_{14} g_1 + \bar{c}^*_{12} z^{\star\varepsilon\tau}_{24} g_2 \rangle , \\
G_{21} &= \langle \bar{c}^*_{12} z^{\star\varepsilon u}_{11} g_1 + \bar{c}^*_{22} z^{\star\varepsilon u}_{21} g_2 \rangle , & G_{26} &= \langle \bar{c}^*_{12} z^{\star\varepsilon\tau}_{11} g_1 + \bar{c}^*_{22} z^{\star\varepsilon\tau}_{21} g_2 \rangle , \\
G_{22} &= \langle \bar{c}^*_{12} z^{\star\varepsilon u}_{12} g_1 + \bar{c}^*_{22} z^{\star\varepsilon u}_{22} g_2 \rangle , & G_{27} &= \langle \bar{c}^*_{12} z^{\star\varepsilon\tau}_{12} g_1 + \bar{c}^*_{22} z^{\star\varepsilon\tau}_{22} g_2 \rangle , \\
G^1_{23} &= \langle \bar{c}^*_{12} z^{\star\varepsilon u1}_{13} g_1 + \bar{c}^*_{22} z^{\star\varepsilon u1}_{23} g_2 \rangle , & G_{28} &= \langle \bar{c}^*_{12} z^{\star\varepsilon\tau}_{13} g_1 + \bar{c}^*_{22} z^{\star\varepsilon\tau}_{23} g_2 \rangle , \\
G^2_{23} &= \langle \bar{c}^*_{12} z^{\star\varepsilon u2}_{13} g_1 \rangle, & G_{29} &= \langle \bar{c}^*_{12} z^{\star\varepsilon\tau}_{14} g_1 + \bar{c}^*_{22} z^{\star\varepsilon\tau}_{24} g_2 \rangle , \\
G^3_{23} &= \langle \bar{c}^*_{22} z^{\star\varepsilon u2}_{23} g_1 \rangle , & G_{56} &= \langle \bar{c}^*_{66} z^{\varepsilon\tau}_{51} g_5 \rangle , \\
G_{24} &= \langle \bar{c}^*_{12} z^{\star\varepsilon u}_{14} g_1 + \bar{c}^*_{22} z^{\star\varepsilon u}_{24} g_2 \rangle , & G_{57} &= \langle \bar{c}^*_{66} z^{\varepsilon\tau}_{52} g_5 \rangle , \\
G_{25} &= \langle \bar{c}^*_{12} z^{\star\varepsilon u}_{15} g_1 + \bar{c}^*_{22} z^{\star\varepsilon u}_{25} g_2 \rangle , & G_{58} &= \langle \bar{c}^*_{66} z^{\varepsilon\tau}_{53} g_5 \rangle , \\
G_{51} &= \langle \bar{c}^*_{66} z^{\varepsilon u}_{51} g_5 \rangle , & G_{59} &= \langle \bar{c}^*_{66} z^{\varepsilon\tau}_{54} g_5 \rangle, \\
G_{52} &= \langle \bar{c}^*_{66} z^{\varepsilon u}_{52} g_5 \rangle .
\end{aligned} \tag{A36}$$

The shear coefficients A_{ij} (with $i = 3, 4$ and $l = 1, \ldots, 9$) are given by

$$\begin{aligned}
A_{32} &= \langle \bar{c}^\star_{44} z^{\varepsilon u}_{32} (z^{\varepsilon u}_{32}, z^{\varepsilon u}_{33}, z^{\varepsilon u}_{35}) \rangle , & A_{33} &= \langle \bar{c}^\star_{44} z^{\varepsilon u}_{33} (z^{\varepsilon u}_{32}, z^{\varepsilon u}_{33}, z^{\varepsilon u}_{35}) \rangle , \\
A_{35} &= \langle \bar{c}^\star_{44} z^{\varepsilon u}_{35} (z^{\varepsilon u}_{32}, z^{\varepsilon u}_{33}, z^{\varepsilon u}_{35}) \rangle , & A_{41} &= \langle \bar{c}^\star_{55} z^{\varepsilon u}_{41} (z^{\varepsilon u}_{41}, z^{\varepsilon u}_{43}, z^{\varepsilon u}_{44}) \rangle , \\
A_{43} &= \langle \bar{c}^\star_{55} z^{\varepsilon u}_{43} (z^{\varepsilon u}_{41}, z^{\varepsilon u}_{43}, z^{\varepsilon u}_{44}) \rangle , & A_{44} &= \langle \bar{c}^\star_{55} z^{\varepsilon u}_{44} (z^{\varepsilon u}_{41}, z^{\varepsilon u}_{43}, z^{\varepsilon u}_{44}) \rangle , \\
A_{38} &= \langle c^\star_{44} z^{\varepsilon\tau}_{33} (z^{\varepsilon u}_{32}, z^{\varepsilon u}_{33}, z^{\varepsilon u}_{35}) \rangle , & A_{39} &= \langle c^\star_{44} z^{\varepsilon\tau}_{34} (z^{\varepsilon u}_{32}, z^{\varepsilon u}_{33}, z^{\varepsilon u}_{35}) \rangle , \\
A_{46} &= \langle c^\star_{55} z^{\varepsilon\tau}_{41} (z^{\varepsilon u}_{41}, z^{\varepsilon u}_{43}, z^{\varepsilon u}_{44}) \rangle , & A_{47} &= \langle c^\star_{55} z^{\varepsilon\tau}_{42} (z^{\varepsilon u}_{41}, z^{\varepsilon u}_{43}, z^{\varepsilon u}_{44}) \rangle .
\end{aligned} \tag{A37}$$

Coefficients of the Finite Element Matrices

$$\begin{aligned}
&(M^{11}_{ij}, M^{14}_{ij}, M^{16}_{ij}, M^{17}_{ij}) = (I^{uu}_{11}, I^{uu}_{14}, I^{\tau u}_{11}, I^{\tau u}_{21}) S^{LL}_{11}, & &(M^{44}_{ij}, M^{46}_{ij}, M^{47}_{ij}) = (I^{uu}_{44}, I^{\tau u}_{14}, I^{\tau u}_{24}) S^{LL}_{11}, \\
&M^{13}_{ir} = I^{uu}_{13} S^{LH}_{1\alpha}, & &(M^{55}_{ij}, M^{58}_{ij}, M^{59}_{ij}) = (I^{uu}_{55}, I^{\tau u}_{35}, I^{\tau u}_{45}) S^{LL}_{11}, \\
&(M^{22}_{ij}, M^{25}_{ij}, M^{28}_{ij}, M^{29}_{ij}) = (I^{uu}_{22}, I^{uu}_{25}, I^{\tau u}_{32}, I^{\tau u}_{42}) S^{LL}_{11}, & &(M^{66}_{ij}, M^{67}_{ij}) = (I^{\tau\tau}_{11}, I^{\tau\tau}_{12}) S^{LL}_{11},
\end{aligned}$$

$$\begin{aligned}
&M_{ir}^{23} = I_{23}^{uu} S_{1\beta}^{LH}, && M_{ij}^{77} = I_{22}^{\tau\tau} S_{11}^{LL},\\
&M_{rs}^{33} = I_{33}^{uu} S_{11}^{HH} + I_{33}^{u\alpha} S_{\alpha\alpha}^{HH} + I_{33}^{u\beta} S_{\beta\beta}^{HH}, && (M_{ij}^{88}, M_{ij}^{89}) = (I_{33}^{\tau\tau}, I_{34}^{\tau\tau}) S_{11}^{LL},\\
&(M_{rj}^{34}, M_{rj}^{36}, M_{rj}^{37}) = (I_{34}^{uu}, I_{13}^{\tau u}, I_{23}^{\tau u}) S_{\alpha 1}^{HL}, && M_{ij}^{99} = I_{44}^{\tau\tau} S_{11}^{LL}.\\
&(M_{rj}^{35}, M_{rj}^{38}, M_{rj}^{39}) = (I_{35}^{uu}, I_{33}^{\tau u}, I_{43}^{\tau u}) S_{\beta 1}^{HL}, &&
\end{aligned} \tag{A38}$$

$$\begin{aligned}
K_{ij}^{11} &= \left[A_{11}^{(11)} + A_{21}^{(21)}\right] S_{\alpha\alpha}^{LL} + A_{51}^{(51)} S_{\beta\beta}^{LL} + A_{41}^{(41)} S_{11}^{LL},\\
K_{ij}^{12} &= \left[A_{12}^{(11)} + A_{22}^{(21)}\right] S_{\alpha\beta}^{LL} + A_{52}^{(51)} S_{\beta\alpha}^{LL},\\
K_{ir}^{13} &= \left[A_{13}^{1(11)} + A_{23}^{1(21)}\right] S_{\alpha 1}^{LH} + \left[A_{13}^{2(11)} + A_{23}^{2(21)}\right] S_{\alpha\alpha\alpha}^{LHH}\\
&\quad + \left[A_{13}^{3(11)} + A_{23}^{3(21)}\right] S_{\alpha\beta\beta}^{LHH} + A_{53}^{(51)} S_{\beta\alpha\beta}^{LHH} + A_{43}^{(41)} S_{1\alpha}^{LH},\\
K_{ij}^{14} &= \left[A_{14}^{(11)} + A_{24}^{(21)}\right] S_{\alpha\alpha}^{LL} + A_{54}^{(51)} S_{\beta\beta}^{LL} + A_{44}^{(41)} S_{11}^{LL},\\
K_{ij}^{15} &= \left[A_{15}^{(11)} + A_{25}^{(21)}\right] S_{\alpha\beta}^{LL} + A_{55}^{(51)} S_{\beta\alpha}^{LL},\\
K_{ij}^{16} &= \left[A_{16}^{(11)} + A_{26}^{(21)}\right] S_{\alpha\alpha}^{LL} + A_{56}^{(51)} S_{\beta\beta}^{LL} + A_{46}^{(41)} S_{11}^{LL},\\
K_{ij}^{17} &= \left[A_{17}^{(11)} + A_{27}^{(21)}\right] S_{\alpha\alpha}^{LL} + A_{57}^{(51)} S_{\beta\beta}^{LL} + A_{47}^{(41)} S_{11}^{LL},\\
K_{ij}^{18} &= \left[A_{18}^{(11)} + A_{28}^{(21)}\right] S_{\alpha\beta}^{LL} + A_{58}^{(51)} S_{\beta\alpha}^{LL},\\
K_{ij}^{19} &= \left[A_{19}^{(11)} + A_{29}^{(21)}\right] S_{\alpha\beta}^{LL} + A_{59}^{(51)} S_{\beta\alpha}^{LL},\\
K_{ij}^{22} &= \left[A_{12}^{(12)} + A_{22}^{(22)}\right] S_{\beta\beta}^{LL} + A_{52}^{(52)} S_{\alpha\alpha}^{LL} + A_{32}^{(32)} S_{11}^{LL},\\
K_{ir}^{23} &= \left[A_{13}^{1(12)} + A_{23}^{1(22)}\right] S_{\beta 1}^{LH} + \left[A_{13}^{2(12)} + A_{23}^{2(22)}\right] S_{\beta\alpha\alpha}^{LHH}\\
&\quad + \left[A_{13}^{3(12)} + A_{23}^{3(22)}\right] S_{\beta\beta\beta}^{LHH} + A_{53}^{(52)} S_{\alpha\alpha\beta}^{LHH} + A_{33}^{(32)} S_{1\beta}^{LH},\\
K_{ij}^{24} &= \left[A_{14}^{(12)} + A_{24}^{(22)}\right] S_{\beta\alpha}^{LL} + A_{54}^{(52)} S_{\alpha\beta}^{LL},\\
K_{ij}^{25} &= \left[A_{15}^{(12)} + A_{25}^{(22)}\right] S_{\beta\beta}^{LL} + A_{55}^{(52)} S_{\alpha\alpha}^{LL} + A_{35}^{(32)} S_{11}^{LL},\\
K_{ij}^{26} &= \left[A_{16}^{(12)} + A_{26}^{(22)}\right] S_{\beta\alpha}^{LL} + A_{56}^{(52)} S_{\alpha\beta}^{LL},\\
K_{ij}^{27} &= \left[A_{17}^{(12)} + A_{27}^{(22)}\right] S_{\beta\alpha}^{LL} + A_{57}^{(52)} S_{\alpha\beta}^{LL},\\
K_{ij}^{28} &= \left[A_{18}^{(12)} + A_{28}^{(22)}\right] S_{\beta\beta}^{LL} + A_{58}^{(52)} S_{\alpha\alpha}^{LL} + A_{38}^{(32)} S_{11}^{LL},\\
K_{ij}^{29} &= \left[A_{19}^{(12)} + A_{29}^{(22)}\right] S_{\beta\beta}^{LL} + A_{59}^{(52)} S_{\alpha\alpha}^{LL} + A_{39}^{(32)} S_{11}^{LL},\\
K_{rs}^{33} &= \left[B_{13}^{1(131)} + B_{23}^{1(231)}\right] S_{11}^{HH} + \left[B_{13}^{2(131)} + B_{23}^{2(231)}\right] S_{1\alpha\alpha}^{HHH} + \left[B_{13}^{3(131)} + B_{23}^{3(231)}\right] S_{1\beta\beta}^{HHH}\\
&\quad + B_{13}^{1(132)} S_{\alpha\alpha 1}^{HHH} + B_{13}^{2(132)} S_{\alpha\alpha\alpha\alpha}^{HHHH} + B_{13}^{3(132)} S_{\alpha\alpha\beta\beta}^{HHHH} + B_{23}^{1(232)} S_{\beta\beta 1}^{HHH} + B_{23}^{2(232)} S_{\beta\beta\alpha\alpha}^{HHHH}\\
&\quad + B_{23}^{3(232)} S_{\beta\beta\beta\beta}^{HHHH} + B_{53}^{(53)} S_{\alpha\beta\alpha\beta}^{HHHH} + A_{43}^{(43)} S_{\alpha\alpha}^{HH} + A_{33}^{(33)} S_{\beta\beta}^{HH},\\
K_{rj}^{34} &= \left[B_{14}^{(131)} + B_{24}^{(231)}\right] S_{1\alpha}^{HL} + B_{14}^{(132)} S_{\alpha\alpha\alpha}^{HHL} + B_{24}^{(232)} S_{\beta\beta\alpha}^{HHL} + B_{54}^{(53)} S_{\alpha\beta\beta}^{HHL} + A_{44}^{(43)} S_{\alpha 1}^{HL},
\end{aligned}$$

$$
\begin{aligned}
K_{rj}^{35} &= \left[B_{15}^{(131)} + B_{25}^{(231)}\right] S_{1\beta}^{HL} + B_{15}^{(132)} S_{\alpha\alpha\beta}^{HHL} + B_{25}^{(232)} S_{\beta\beta\beta}^{HHL} + B_{55}^{(53)} S_{\alpha\beta\alpha}^{HHL} + A_{35}^{(33)} S_{\beta 1}^{HL}, \\
K_{rj}^{36} &= \left[B_{16}^{(131)} + B_{26}^{(231)}\right] S_{1\alpha}^{HL} + B_{16}^{(132)} S_{\alpha\alpha\alpha}^{HHL} + B_{26}^{(232)} S_{\beta\beta\alpha}^{HHL} + B_{56}^{(53)} S_{\alpha\beta\beta}^{HHL} + A_{46}^{(43)} S_{\alpha 1}^{HL}, \\
K_{rj}^{37} &= \left[B_{17}^{(131)} + B_{27}^{(231)}\right] S_{1\alpha}^{HL} + B_{17}^{(132)} S_{\alpha\alpha\alpha}^{HHL} + B_{27}^{(232)} S_{\beta\beta\alpha}^{HHL} + B_{57}^{(53)} S_{\alpha\beta\beta}^{HHL} + A_{47}^{(43)} S_{\alpha 1}^{HL}, \\
K_{rj}^{38} &= \left[B_{18}^{(131)} + B_{28}^{(231)}\right] S_{1\beta}^{HL} + B_{18}^{(132)} S_{\alpha\alpha\beta}^{HHL} + B_{28}^{(232)} S_{\beta\beta\beta}^{HHL} + B_{58}^{(53)} S_{\alpha\beta\alpha}^{HHL} + A_{38}^{(33)} S_{\beta 1}^{HL}, \\
K_{rj}^{39} &= \left[B_{19}^{(131)} + B_{29}^{(231)}\right] S_{1\beta}^{HL} + B_{19}^{(132)} S_{\alpha\alpha\beta}^{HHL} + B_{29}^{(232)} S_{\beta\beta\beta}^{HHL} + B_{59}^{(53)} S_{\alpha\beta\alpha}^{HHL} + A_{39}^{(33)} S_{\beta 1}^{HL}, \\
K_{ij}^{44} &= \left[B_{14}^{(14)} + B_{24}^{(24)}\right] S_{\alpha\alpha}^{LL} + B_{54}^{(54)} S_{\beta\beta}^{LL} + A_{44}^{(44)} S_{11}^{LL}, \\
K_{ij}^{45} &= \left[B_{15}^{(14)} + B_{25}^{(24)}\right] S_{\alpha\beta}^{LL} + B_{55}^{(54)} S_{\beta\alpha}^{LL}, \\
K_{ij}^{46} &= \left[B_{16}^{(14)} + B_{26}^{(24)}\right] S_{\alpha\alpha}^{LL} + B_{56}^{(54)} S_{\beta\beta}^{LL} + A_{46}^{(44)} S_{11}^{LL}, \\
K_{ij}^{47} &= \left[B_{17}^{(14)} + B_{27}^{(24)}\right] S_{\alpha\alpha}^{LL} + B_{57}^{(54)} S_{\beta\beta}^{LL} + A_{47}^{(44)} S_{11}^{LL}, \\
K_{ij}^{48} &= \left[B_{18}^{(14)} + B_{28}^{(24)}\right] S_{\alpha\beta}^{LL} + B_{58}^{(54)} S_{\beta\alpha}^{LL}, \\
K_{ij}^{49} &= \left[B_{19}^{(14)} + B_{29}^{(24)}\right] S_{\alpha\beta}^{LL} + B_{59}^{(54)} S_{\beta\alpha}^{LL}, \\
K_{ij}^{55} &= \left[B_{15}^{(15)} + B_{25}^{(25)}\right] S_{\beta\beta}^{LL} + B_{55}^{(55)} S_{\alpha\alpha}^{LL} + A_{35}^{(35)} S_{11}^{LL}, \\
K_{ij}^{56} &= \left[B_{16}^{(15)} + B_{26}^{(25)}\right] S_{\beta\alpha}^{LL} + B_{56}^{(55)} S_{\alpha\beta}^{LL}, \\
K_{ij}^{57} &= \left[B_{17}^{(15)} + B_{27}^{(25)}\right] S_{\beta\alpha}^{LL} + B_{57}^{(55)} S_{\alpha\beta}^{LL}, \\
K_{ij}^{58} &= \left[B_{18}^{(15)} + B_{28}^{(25)}\right] S_{\beta\beta}^{LL} + B_{58}^{(55)} S_{\alpha\alpha}^{LL} + A_{38}^{(35)} S_{11}^{LL}, \\
K_{ij}^{59} &= \left[B_{19}^{(15)} + B_{29}^{(25)}\right] S_{\beta\beta}^{LL} + B_{59}^{(55)} S_{\alpha\alpha}^{LL} + A_{39}^{(35)} S_{11}^{LL}, \\
K_{ij}^{66} &= \left[C_{16}^{(11)} + C_{26}^{(21)}\right] S_{\alpha\alpha}^{LL} + C_{56}^{(51)} S_{\beta\beta}^{LL} + C_{46}^{(41)} S_{11}^{LL}, \\
K_{ij}^{67} &= \left[C_{17}^{(11)} + C_{27}^{(21)}\right] S_{\alpha\alpha}^{LL} + C_{57}^{(51)} S_{\beta\beta}^{LL} + C_{47}^{(41)} S_{11}^{LL}, \\
K_{ij}^{68} &= \left[C_{18}^{(11)} + C_{28}^{(21)}\right] S_{\alpha\beta}^{LL} + C_{58}^{(51)} S_{\beta\alpha}^{LL}, \\
K_{ij}^{69} &= \left[C_{19}^{(11)} + C_{29}^{(21)}\right] S_{\alpha\beta}^{LL} + C_{59}^{(51)} S_{\beta\alpha}^{LL}, \\
K_{ij}^{77} &= \left[C_{17}^{(12)} + C_{27}^{(22)}\right] S_{\alpha\alpha}^{LL} + C_{57}^{(52)} S_{\beta\beta}^{LL} + C_{47}^{(42)} S_{11}^{LL}, \\
K_{ij}^{78} &= \left[C_{18}^{(12)} + C_{28}^{(22)}\right] S_{\alpha\beta}^{LL} + C_{58}^{(52)} S_{\beta\alpha}^{LL}, \\
K_{ij}^{79} &= \left[C_{19}^{(12)} + C_{29}^{(22)}\right] S_{\alpha\beta}^{LL} + C_{59}^{(52)} S_{\beta\alpha}^{LL}, \\
K_{ij}^{88} &= \left[C_{18}^{(13)} + C_{28}^{(23)}\right] S_{\beta\beta}^{LL} + C_{58}^{(53)} S_{\alpha\alpha}^{LL} + C_{38}^{(33)} S_{11}^{LL}, \\
K_{ij}^{89} &= \left[C_{19}^{(13)} + C_{29}^{(23)}\right] S_{\beta\beta}^{LL} + C_{59}^{(53)} S_{\alpha\alpha}^{LL} + C_{39}^{(33)} S_{11}^{LL}, \\
K_{ij}^{99} &= \left[C_{19}^{(14)} + C_{29}^{(24)}\right] S_{\beta\beta}^{LL} + C_{59}^{(54)} S_{\alpha\alpha}^{LL} + C_{39}^{(34)} S_{11}^{LL},
\end{aligned}
\tag{A39}
$$

$$F_i^1 = \oint_{\Gamma_\alpha^e} L_i^e \hat{N}_{\alpha\beta}^{11}\, d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{N}_{\alpha\alpha}^{11}\, d\beta, \qquad F_i^2 = \oint_{\Gamma_\alpha^e} L_i^e \hat{N}_{\beta\beta}^{22}\, d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{N}_{\alpha\beta}^{22}\, d\beta,$$

$$F_i^4 = \oint_{\Gamma_\alpha^e} L_i^e \hat{M}_{\alpha\beta}^{14}\, d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{M}_{\alpha\alpha}^{14}\, d\beta, \qquad F_i^5 = \oint_{\Gamma_\alpha^e} L_i^e \hat{M}_{\beta\beta}^{25}\, d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{M}_{\alpha\beta}^{25}\, d\beta,$$

$$F_i^6 = \oint_{\Gamma_\alpha^e} L_i^e \hat{T}_{\beta\alpha}^{11}\, d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{T}_{\alpha\alpha}^{11}\, d\beta, \qquad F_i^7 = \oint_{\Gamma_\alpha^e} L_i^e \hat{T}_{\beta\alpha}^{12}\, d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{T}_{\alpha\alpha}^{12}\, d\beta,$$

$$F_i^8 = \oint_{\Gamma_\alpha^e} L_i^e \hat{T}_{\beta\beta}^{23}\, d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{T}_{\alpha\beta}^{23}\, d\beta, \qquad F_i^9 = \oint_{\Gamma_\alpha^e} L_i^e \hat{T}_{\beta\beta}^{24}\, d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{T}_{\alpha\beta}^{24}\, d\beta,$$

$$F_i^3 = \int_{\Omega_0^e} H_r^e Z\, d\alpha\, d\beta + \oint_{\Gamma_\alpha^e} \Big(\frac{\partial H_r^e}{\partial \beta} \hat{M}_{\beta\beta}^{23} + \frac{\partial H_r^e}{\partial \alpha} \hat{M}_{\alpha\beta}^{13} + H_r^e \hat{Q}_{\beta z}^{33} \Big) d\alpha$$
$$+ \oint_{\Gamma_\beta^e} \Big(\frac{\partial H_r^e}{\partial \alpha} \hat{M}_{\alpha\alpha}^{13} + \frac{\partial H_r^e}{\partial \beta} \hat{M}_{\alpha\alpha}^{23} + H_r^e \hat{Q}_{z\alpha}^{33} \Big) d\beta, \tag{A40}$$

In the previous equations a notation was used where the zero- (1), first- (α or β) and second-order ($\alpha\alpha$, $\beta\beta$ or crossed $\alpha\beta$) derivatives, taken as subscripts of S, of the Lagrange (L) and Hermite (H) interpolation functions, taken as superscripts of S, define the following terms (for the sake of brevity only a few terms are presented since the rest are obvious from the following relations),

$$S_{11}^{LL} = \int_{\Omega_0^e} L_i^e L_j^e\, d\alpha\, d\beta, \qquad S_{1\alpha}^{LH} = \int_{\Omega_0^e} L_i^e \frac{\partial H_r^e}{\partial \alpha}\, d\alpha\, d\beta, \qquad S_{\beta\beta}^{LL} = \int_{\Omega_0^e} \frac{\partial L_i^e}{\partial \beta} \frac{\partial L_j^e}{\partial \beta}\, d\alpha\, d\beta,$$
$$S_{\beta\alpha\beta}^{LHH} = \int_{\Omega_0^e} \frac{\partial L_i^e}{\partial \beta} \frac{\partial^2 H_r^e}{\partial \alpha \partial \beta}\, d\alpha\, d\beta, \qquad S_{\alpha\beta\alpha\beta}^{HHHH} = \int_{\Omega_0^e} \frac{\partial^2 H_r^e}{\partial \alpha \partial \beta} \frac{\partial^2 H_r^e}{\partial \alpha \partial \beta}\, d\alpha\, d\beta. \tag{A41}$$

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