INVARIANCE PROPERTIES OF MEASURE DRIVEN DIFFERENTIAL INCLUSIONS

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Abstract. The conventional concepts of invariance are extended in this article to include impulsive control systems represented by measure driven differential inclusions. Invariance conditions and some of their main features are derived. The solution concept plays a critical role in the extension of the conventional conditions to the impulsive control context.

Keywords: Differential inclusion, impulsive control, dynamical systems, invariance

1. INTRODUCTION

We consider impulsive control systems of the form

$$\begin{cases} dx(t) \in F(t,x(t))dt + \mathbf{G}(t,x(t))\mu(dt), \ t \in [0,\infty) \\ x(0) \in C_0, \end{cases} \quad (1)$$

where $F : [0,\infty) \times I\!R^n \hookrightarrow \mathcal{P}(I\!R^n)$, $\mathbf{G} : [0,\infty) \times I\!R^n \hookrightarrow \mathcal{P}(I\!R^{n \times q})$ are given multi-functions and $\mu \in C^*([0,\infty); K)$ is the set in the dual space of continuous functions from $[0,\infty)$ into $I\!R^q$ with values in K. $\mathcal{P}(I\!R^n)$ denotes the collection of subsets of $I\!R^n$ and the set K is the positive pointed cone in $I\!R^q$.

In what follows, the space of absolutely continuous functions and the space of functions of bounded variation from $[0,\infty)$ to $I\!R^n$ are, respectively, denoted by $AC([0,\infty); I\!R^n)$ and $BV^+([0,\infty); I\!R^n)$. $\bar{\mu}$ denotes the total variation of the measure μ, i.e., $\bar{\mu}(dt) := \sum_{i=1}^{q} \mu_i(dt)$. $\mathcal{L} \times \mathcal{B}$ is the product σ-field, where $\mathcal{L}$ denotes the Lebesgue subsets of $[0,\infty)$ and $\mathcal{B}$ denotes the Borel subsets of $I\!R^q$. B is the open unit ball in the Euclidean space.

In several application areas, such as finance, management of renewable resources, and aerospace navigation, (consider (Clark *et al.*, 1979; Baumeister, 2001; Gurman, 1981; Brogliato, 1996; Marec, 1979), to name just a few references), the addressed class of problems play an important role in modelling and analysis process. So, a considerable body of theory for this class of systems (see, for example, (Arutyunov *et al.*, 2005; Bressan and Rampazzo, 1991; Bressan and Rampazzo, 1994; Dykhta, 1990; Dykhta and Samsonyuk, 2000; Gurman, 1972; Kolokolnikova, 1996; Miller, 1996; Motta and Rampazzo, 1995; Pereira and Silva, 2000; Rishel, 1965; Silva and Vinter, 1996; Vinter and Pereira, 1988; Silva and Vinter, 1997), and references therein) and supporting control strategies computation schemes, (Dykhta and Derenko,

1994; Baturin and Goncharova, 1999; Baturin and Ourbanovich, 1997; Krotov and Gurman, 1987) have been developed so far.

The research reported in this paper is part of a larger effort to understand the features of impulsive dynamical systems governed by measure differential inclusions. In a previous work (Pereira and Silva, 2002), an extension of the Lyapounov stability theory to this class systems is presented. The basic condition is that the Lyapounov function composed with any trajectory of the system has to be strictly decreasing to zero as time goes to infinity. This Lyapounov condition is relaxed to allow for Lyapounov functions that are decreasing in "average" (Pereira and Silva, 2004). That roughly means that the composition of the Lyapounov function with the state trajectory might jump to higher values, but tends to zero as time goes to infinity. In other words, there is no need for strict monotony. Here, we set out to extend invariance results, i.e., conditions which ensure that a trajectory starting within a specified set remains there forever, to the impulsive context.

This article is organized as follows: in the next section we introduce the solution concept and some background results. Then, together with the presentation of relevant preliminary conventional definitions and results, we present, in Section 3, both weak and strong invariance conditions as well as the proofs of the main results.

2. SOLUTION CONCEPT AND BASIC RESULTS

For the concept of solution we use that introduced in (Pereira and Silva, 2000; Pereira and Silva, 2002), making the necessary changes to encompass the unbounded interval $[0, \infty)$. This concept has some important robustness properties. For this we need to describe a change of variables technique. Now, in order to define a required change of variable, let, for $i = 1, \ldots, q$,

$$M_i(t) = \int_{[0,t]} \mu_i(ds), \text{ for } t > 0 \text{ with } M_i(0) = 0, \text{ and}$$

consider:

- $\eta(t) := t + \sum_{i=1}^{m} M_i(t)$,

- $\bar{\eta}(t) := \begin{cases} \{\eta(t)\} & \text{if } \bar{\mu}(\{t\}) = 0, \\ [\eta(t^-), \eta(t)] & \text{if } \bar{\mu}(\{t\}) > 0. \end{cases}$

The above defined function η is a *reparameterization* of the time variable t. Now, we introduce the notion of graph completion for the set-valued measure μ.

Definition 2.1. A family of *graph completions* associated to the set-valued measure μ is the set of the pairs $(\theta, \gamma_\mu) : [0, \infty) \rightarrow [0, \infty) \times K$, where

$\theta : [0, \infty) \rightarrow [0, \infty)$ is the "inverse" of $\bar{\eta} : [0, \infty) \hookrightarrow \mathcal{P}([0, \infty))$ in the sense that $\theta(s) = t$, $\forall s \in \bar{\eta}(t)$ and $\gamma_\mu : [0, \infty) \rightarrow I\!\!R^q$ is defined $\forall s \in \bar{\eta}(t)$, $\forall t \in [0, \infty)$, by

$$\gamma(s) := \begin{cases} M(\theta(s)) & \text{if } \bar{\mu}(\{t\}) = 0 \\ M(t^-) + \int_{\eta(t^-)}^{s} v(\sigma)d\sigma & \text{if } \bar{\mu}(\{t\}) > 0, \end{cases}$$

for some $v(\cdot) \in V^t$, where $t = \theta(s)$. Here, $M(\cdot) := \text{col}(M_1(\cdot), \ldots, M_q(\cdot))$ and

$$V^t := \Big\{ v : \bar{\eta}(t) \rightarrow K \mid \dot{\theta}(s) + \sum_{i=1}^{m} v_i(s) = 1 \ \forall s \in \bar{\eta}(t),$$

$$\int_{\bar{\eta}(t)} v(s)ds = \mu(\{t\}) \Big\}.$$

Finally, we introduce the concept of robust solution.

Definition 2.2. The trajectory x, with $x(0) = x_0$, is admissible for (1) if $x(t) = x_{ac}(t) + x_s(t)$, $\forall t \in [0, \infty)$, where

$$\begin{cases} \dot{x}_{ac}(t) \in F(t, x(t)) + \mathbf{G}(t, x(t)) \cdot w_{ac}(t) \text{ a.e.} \\ x_s(t) = \int_{[0,t]} g_c(\tau) w_c(\tau) \bar{d}\mu_{sc}(d\tau) + \int_{[0,t]} g_a(\tau) \bar{\mu}_{sa}(d\tau). \end{cases}$$

Here, $\bar{\mu}$ is the total variation measure associated with μ, μ_{sc}, μ_{sa} and μ_{ac} are, respectively, the singular continuous, the singular atomic, and the absolutely continuous components of μ, w_{ac} is the time derivative of μ_{ac}, w_{sc} is the Radon-Nicodym derivative of μ_{sc} with respect to its total variation, $g_c(\cdot)$ is a $\bar{\mu}_{sc}$ measurable selection of $\mathbf{G}(\cdot, x(\cdot))$ and $g_a(\cdot)$ is a $\bar{\mu}_{sa}$ measurable selection of the multifunction

$$\tilde{G}(t, x(t^-); \mu(\{t\})) : [0, \infty) \times I\!\!R^n \times K \hookrightarrow \mathcal{P}(I\!\!R^n)$$

that takes, as values, the set of all $\bar{\xi}(\eta(t))$ where $(\xi(\cdot), \gamma_\mu(\cdot))$ satisfies:

$$\xi(\eta(t^-)) = x(t^-), \qquad (2)$$
$$\dot{\xi}(s) = G(t, \xi(s))\dot{\gamma}_\mu(s) \quad \text{a.e. in } \bar{\eta}(t), \quad (3)$$
$$\mu(\{t\}) = \gamma_\mu(\eta(t)) - \gamma_\mu(\eta(t^-)), \qquad (4)$$

for some function $G \in \mathbf{G}$ continuous in t and Lipschitz in x. Here, $(\xi, \gamma_\mu) \in AC([0, \infty); I\!\!R^n \times I\!\!R^q)$, and the pair (θ, γ_μ) is a graph completion of μ.

Remark. We treat the trajectories of (1) as path-valued functions. That means that their images are curves in ^{n}, for each time t. We denote such a trajectory by $x_t(\cdot)$. When t is a continuity point of the control measure μ, $x_t(\cdot)$ is a singleton $(x(t^-) = x(t^+) = x(t))$, while if t is an atom of the control measure μ, $x_t(\cdot)$ is regarded as a set of curves:

$$x_t(\cdot) := \{\xi : \bar{\eta}(t) \rightarrow I\!\!R^n : \xi \text{ satisfies (2)-(4)}\}.$$

Let $\{t_i\}_{i=1}^{\infty}$ be a sequence of atoms of μ and (x,μ) be a feasible process. Then, $x_t(\cdot) \in S$ means

$$x(t) \in S, \ \forall t \in [0,\infty) \text{ and } x_{t_i}(s) \subset S, \ \forall s \in \bar{\eta}(t_i).$$

We say that (x,μ) is a *feasible process* for (1) if $\mu \in C^*([0,\infty); K)$ and x is robust solution to (1).

By using a change of variables technique, we can define conventional differential inclusions associated to the impulsive differential inclusion. In the first case, Theorem 2.3 below, the control measure is fixed and then we have a particular conventional differential inclusion, stated with dependence on the graph completions of the measure. A more elaborated result that allows for the measure being a choice variable (a desired feature) is stated as Theorem 2.4 soon after.

Theorem 2.3. Suppose that the multi-functions F and $\mathbf{G}$ satisfy:

- F takes values in closed sets and is $\mathcal{L} \times \mathcal{B}$-measurable.
- $\mathbf{G}$ takes values in closed sets and is Borel-measurable.

Fix a measure $\mu \in C^*([0,\infty); K)$ and a initial value x_0. Let (θ, γ_μ) be the graph completion of μ and η the reparameterization function.

(i) Suppose that $x(\cdot) \in BV^+([0,\infty); \mathbb{R}^n)$ is a robust solution to (1) (with respect to μ and x_0). Then, there is a solution $y(\cdot) \in AC([0,\infty); \mathbb{R}^n)$ to

$$\begin{cases} \dot{y}(s) \in F(\theta(s), y(s))\dot{\theta}(s) + \mathbf{G}(\theta(s), y(s))\dot{\gamma}_\mu(s) \\ y(0) = x_0 \end{cases} \quad (5)$$

for which

$$x(t) = y(\eta(t)) \quad \text{for all } t \in (0,\infty). \quad (6)$$

Conversely,

(ii) Suppose that $y(\cdot) \in AC([0,\infty); \mathbb{R}^n)$ is a solution to (5). Then there exists a solution $x(\cdot) \in BV^+([0,\infty); ^n)$ to (1) for which (6) is satisfied.

(iii) Take a solution x to (1). Let y be a solution to (5) such that (6) is satisfied. Then

$$\|x\|_{T.V.} \le \|y\|_{T.V.}.$$

Proof. The proof is similar to of a similar result (see Theorem 4.1 of (Silva and Vinter, 1996)) in which the control measure is scalar-valued and, therefore, we omit it. □

In the sequel, we will denote functions and variables of the extended reparameterized system by $\tilde{\ }$, i.e., we have $\dot{\tilde{x}} \in \tilde{F}(y)$ where $\tilde{x} = \text{col}(x^0, y)$ and

$$\tilde{F} := \{\text{col}(v_0, Fv_0 + \mathbf{G}v) : \text{col}(v_0, v) \in \bar{V}\}.$$

In this context $\tilde{S} = [0,\infty) \times S$.

Theorem 2.4. Assume that F and $\mathbf{G}$ are Borel measurable.

If (x,μ) is a feasible control process for (1) then, there exists a trajectory $\tilde{x}$ to

$$\begin{cases} \dot{\tilde{x}}(s) \in \tilde{F}(\tilde{x}) \\ x^0(0) = 0, \quad \text{with } x^0(s) \to \infty \text{ as } s \to \infty \\ y(0) = x_0. \end{cases} \quad (7)$$

Conversely, for each trajectory $\tilde{x}$ of (7), there exists an admissible process (x,μ) for (1) such that $x(t) = \tilde{x}(\eta(t))$.

Proof. The proof of this result is found in (Pereira and Silva, 2004). □

3. INVARIANCE RESULTS

In this section, we state the invariance results for impulsive control systems. Let us first introduce the notion of invariance informally.

Let $S \subset \mathbb{R}^n$ be a closed set and F be a set valued map on the (t,x) space specifying the dynamics in a differential inclusion form. We say that a certain system (F,S) is invariant when all or some of the trajectories of F that start in S remain in this set for all future times.

Definition 3.1. Let $S \subset \mathbb{R}^n$. We say that the system $((F, \mathbf{G}), S)$ is *weakly invariant* if $\forall x_0 \in S$ there exists a feasible process $(x(\cdot), \mu(\cdot))$ of (1) with $x(0) = x_0$ and $x(t) \in S, \ \forall t \ge 0$. Also, $\forall i \in \mathbb{N}$, $\exists \xi_i(\cdot) \in x_{t_i}(\cdot)$ such that $\xi_i(s) \in S, \ \forall s \in \bar{\eta}(t_i)$.

Definition 3.2. If $x_t(\cdot) \in S, \forall t \ge 0$, for all feasible process $(x(\cdot), \mu(\cdot))$ of (1) such that $x(0) \in S$, we say that the system $((F, \mathbf{G}), S)$ is *strongly invariant*.

Definition 3.3. The *attainable set* $\mathcal{A}(x_0; T)$ from x_0 at the time T is given by:

$$\mathcal{A}(x_0; T) := \Big\{ x(T) : (x,\mu) \text{ is a feasible process}$$
$$\text{of (1) with } x(0) = x_0 \Big\}.$$

The results presented in this section require a set of assumptions on the multi-functions F and $\mathbf{G}$ that we call by *Standing Hypotheses*. These assumptions will hold hereafter.

Standing Hypotheses

(H1) For every $x \in \mathbb{R}^n$, $F(x)$ and $\mathbf{G}(x)$ are convex, compact and non-empty sets.

(H2) F and $\mathbf{G}$ are upper semicontinuous.

(H3) There are constants a and b such that, for every $x \in \mathbb{R}^n$,
if $v \in F(x)$ then $\|v\| \le a\|x\| + b$, and
if $V \in \mathbf{G}(x)$, then $\|V\| \le a\|x\| + b$.

The hypothesis **(H3)** is known as linear growth condition.

We recall that F is upper semicontinuous at x if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|y - x\| < \delta \implies F(y) \subset F(x) + \varepsilon B.$$

Before stating some equivalent forms to the weak invariance of the system $((F, \mathbf{G}), S)$, a result that generalizes the one for conventional control problems to the impulsive context, we will introduce the following assumption on the first component of the extended trajectory $\tilde{x}$ of the extended system.

$$x^0(0) = 0 \text{ and } \lim_{s \to \infty} x^0(s) = \infty. \tag{8}$$

Notice that this condition is naturally satisfied if the total variation measure $\bar{\mu}$ of the control measure μ satisfies

$$\forall T > 0, \ \lim_{t \to \infty} \bar{\mu}([t, t + T]) = 0.$$

Proposition 3.4. The system $(\tilde{F}, \tilde{S})$ is weakly invariant if and only if the system $((F, \mathbf{G}), S)$ is also weakly invariant.

Proof. $[\Rightarrow]$ Suppose that the system $(\tilde{F}, \tilde{S})$ is weakly invariant. Let $x_0 \in S$. Then, there exists a trajectory $\tilde{x} = \mathrm{col}(x^0, y)$ of $\tilde{F}$ such that $x^0(0) = 0$, $y(0) = x_0 \in S$ and $\tilde{x}(s) \in \tilde{S}$, $\forall s \geq 0$. By Theorem 2.4, there exists a process (x, μ) of $(F, \mathbf{G})$ such that $x(t) = y(\eta(t))$, $\forall t \geq 0$. That implies that $x(t) \in S$, $\forall t \geq 0$. Let $\{t_i\}$ be a sequence of the atoms of μ. By construction of x (see (Pereira and Silva, 2004)) we have that $y(s) \in x_{t_i}(s)$, $\forall s \in \eta(t_i)$. Thus, we may conclude that $((F, \mathbf{G}), S)$ is weakly invariant.

$[\Leftarrow]$ Let $x_0 \in S$. If $((F, \mathbf{G}), S)$ is weakly invariant, then there exists a feasible process (x, μ) of (1) with $x(0) = x_0$, $x(t) \in S$ $\forall t \geq 0$, and, $\forall i$, $\exists \xi_i(\cdot) \in x_{t_i}(\cdot)$ such that $\xi_i(s) \in S$, for all $s \in \bar{\eta}(t_i)$. Here, $\{t_i\}$ be a sequence of the atoms of μ. By Theorem 2.3, there exists $\mathrm{col}(x^0, y)$, a solution of (7), satisfying

$$y(s) = \begin{cases} x(\theta(s)), \text{ if } s \in [0, \infty) \setminus \bigcup_{i=1}^{\infty} \bar{\eta}(t_i) \\ \xi_i(s), \quad \text{if } s \in \bigcup_{i=1}^{\infty} \bar{\eta}(t_i). \end{cases}$$

Hence, we have $\tilde{x}(x) \in \tilde{S}$, $\forall s \geq 0$, and, therefore, $(\tilde{F}, \tilde{S})$ is weakly invariant. $\square$

Before pursuing, let us introduce the definition of proximal normal cone (see (Clarke *et al.*, 1998)) used in the next result.

Let $S \subset \mathbb{R}^n$ be a closed set and take $x_0 \in S$. Then $\zeta \in \mathbb{R}^n$ is a proximal normal vector to S at x_0 if $\exists \alpha > 0$ such that

$$d_S(x_0 + \alpha\zeta) = \alpha\|\zeta\|$$

where $d_S(\cdot)$ is the *distance function* given by $d_S(y) := \inf\{\|y - s\| : s \in S\}$.

The proximal normal cone to S at x_0, $N_S^p(x_0)$, is the set of all proximal normals of S at x_0.

Theorem 3.5. Suppose that the condition (8) holds. Then,

$$\begin{cases} \text{For each } \tilde{x} \in \tilde{S}, \text{ there exist } (v_0, v) \in \bar{V}, \\ \quad f \in F(x) \text{ and } G \in \mathbf{G}(x) \text{ such that} \\ \left\langle (v_0, v_0 f + \sum_{i=1}^{q} v_i g_i), \tilde{\zeta} \right\rangle \leq 0, \ \ \forall \tilde{\zeta} \in N_{\tilde{S}}^p(\tilde{x}) \end{cases} \tag{9}$$

if and only if

The system $((F, \mathbf{G}), S)$ is weakly invariant. (10)

Proof. The proof of this result follows the structure of the corresponding result for conventional systems in (Clarke *et al.*, 1998).

From this reference, it can be shown that the following sufficient condition for (9) in terms of the *Bouligand tangent cone* $T_{\tilde{S}}^B(\tilde{x})$ [3] holds

$$\tilde{F}(\tilde{x}) \cap \mathrm{co} T_{\tilde{S}}^B(\tilde{x}) \neq \emptyset, \ \forall \tilde{x} \in \tilde{S} \tag{11}$$

which, in turn, is implied by

$$\tilde{F}(x) \cap T_{\tilde{S}}^B(\tilde{x}) \neq \emptyset, \ \forall \tilde{x} \in \tilde{S}. \tag{12}$$

Furthermore, it can be easily concluded that (10) is equivalent to

$$\forall x_0 \in S, \ \forall \varepsilon > 0, \ \exists \delta \in (0, \varepsilon) \text{ s.t.} \\ \mathcal{A}(x_0; \delta) \cap S \neq \emptyset. \tag{13}$$

Therefore, in order to complete the proof, we only need to show that (9) implies (10) and that (13) implies (12).

Observe that $T_{\tilde{S}}^B(\tilde{x}) = \mathbb{R} \times T_S^B(x)$ and, as a consequence, (12) is equivalent to

$$\bar{F}(x) \cap T_S^B(x) \neq \emptyset, \ \forall x \in S, \tag{14}$$

being

$$\bar{F} := \{Fv_0 + Gv : \mathrm{col}(v_0, v) \in \bar{V}\}. \tag{15}$$

It can also be easy concluded that $N_{\tilde{S}}^p(\tilde{x}) = \{0\} \times N_S^p(x)$ and, therefore, (9) is equivalent to

$$\exists (v_0, v) \in \bar{V}, \ f(x) \in F(x), \ G(x) \in \mathbf{G}(x) \text{ s.t.} \\ \left\langle v_0 f + \sum_{i=1}^{q} v_i g_i, \zeta \right\rangle \leq 0, \ \forall \zeta \in N_S^p(x), \ \forall x \in S. \tag{16}$$

Now let us start with the first implication. Consider the system $(\bar{F}(x), S)$, and let $x_0 \in S$. From (9), and, obviously (16), it follows that

$$h(x, \zeta) \leq 0, \quad \forall \zeta \in N_S^p(x),$$

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[3] The *Bouligand tangent cone* to S at x defined as

$$T_S^B(x) := \left\{ \lim_{i \to \infty} \frac{x_i - x}{\lambda_i} : \ x_i \xrightarrow{S} x, \ \lambda_i \downarrow 0 \right\},$$

where $x_i \xrightarrow{S} x$ means $x_i \in S$ and $x_i \to x$.

where $h(x,p) := \min\{\langle w,p\rangle : w \in \bar{F}\}$. This allows us to use Theorem 4.2.4 of (Clarke *et al.*, 1998) to find a trajectory $y(\cdot)$ of $\bar{F}$ in $[0,\infty)$ such that $y(0) = x_0$ and $y(s) \in S$, $\forall s \geq 0$. Now, we only have to construct a process (x,μ) of (1) with $x(t) = y(\eta(t))$ for a suitable function η such that $x(0) = x_0$, $x(t) \in S$, $\forall t \geq 0$, and $\xi_{t_i}(s) \in S$, $\forall s \in \bar{\eta}(t_i)$, for all atoms t_i of μ and some function $\xi_i(\cdot) \in x_{t_i}(\cdot)$. This can be done as in Proposition 3.4 and, therefore the system $((F,\mathbf{G}),S)$ is weakly invariant.

Now, we show that (13) $\Rightarrow$ (12) or, after the remark earlier in this proof, its lower dimensional equivalent (14). First, notice that (10) is equivalent to (13). So, by Proposition 3.4, we can conclude that the claim (13) holds also to the attainable set for the absolutely continuous case

$$\dot{y}(s) \in F(y(s))\dot{\theta}(s) + \mathbf{G}(y(s))\dot{\gamma}_\mu(s).$$

This allow us adapt the proof of the similar result for the conventional control problem in (Clarke *et al.*, 1998). Suppose that (13) holds for the absolutely continuous version of the attainable set, which we denote here by $\mathcal{A}^y(x_0,t)$. Then, for all $n \in$, $\exists \delta_n \in (0, \frac{1}{n})$ with $\mathcal{A}^y(x_0,\delta_n) \cap S \neq \emptyset$. Therefore, for every n,

$$\begin{aligned}
\dot{y}_n(s) &\in F(y_n(s))\dot{\theta}_n(s) + \mathbf{G}(y_n(s))\dot{\gamma}_n(s) \\
&\subset \{F(y_n(s))v_0 + \mathbf{G}(y_n(s))v : (v_0,v) \in \bar{V}\} \\
&= \bar{F}(y_n(s)).
\end{aligned}$$

Here (θ_n,γ_n) are graph completion of measures $\mu_n \in C^*([0,\infty);K)$.

The functions y_n have the same Lipschitz constant K, so that

$$\frac{\|y_n(\delta_n) - x_0\|}{\delta_n} \leq K, \quad \forall n.$$

Thus, by taking a subsequence (no relabelling), there exists $v \in^n$ such that $v := \lim_{n\to\infty} \frac{y_n(\delta_n) - x_0}{\delta_n}$. That is, $v \in T_S^B(x_0)$. Then, we need only to show that $v \in \bar{F}(x_0)$ in order to deduce (12). We can write

$$y_n(\delta_n) - x_0 = \int_0^{\delta_n} \dot{y}_n(s)ds.$$

We have that $\bar{F}$ is upper semi-continuous. Let $\varepsilon \geq 0$. Then, for n sufficiently large, follows

$$y_n(\delta_n) - x_0 \in \int_0^{\delta_n} [\tilde{F}(x_0) + \varepsilon B]ds.$$

By dividing by δ_n and passing to the limit when $n \to \infty$ we obtain

$$v \in \bar{F}(x_0) + \varepsilon B.$$

The result is obtained by taking into account that ε is arbitrary. □

Proposition 3.6. The system $(\tilde{F},\tilde{S})$ is strongly invariant if and only if the system $((F,\mathbf{G}),S)$ is also strongly invariant.

Proof. [$\Rightarrow$] Let (x,μ) a feasible process for (1) such that $x(0) \in S$. Then, by Theorem 2.3, there exists a trajectory y for $F(y)\dot{\theta}(s) + \mathbf{G}(y)\dot{\gamma}(s) \subset \bar{F}(y)$ [4] such that $y(0) = x(0) \in S$, $y(s) = x(\theta(s))$, $\forall s \in [0,\infty) \setminus \cup_{i=1}^\infty \bar{\eta}(t_i)$ and $y(s) \in x_{t_i}(s)$, $\forall i \in \mathbb{N}$.

Let $x^0(s) = \theta(s)$ and notice that $\tilde{x} = \text{col}(x^0,y)$ is a trajectory for $\tilde{F}$ satisfying $\tilde{y}(0) \in \tilde{S}$. Since $(\tilde{F},\tilde{S})$ is strongly invariant, we have that $y(s) \in S$ for all $s \geq 0$. Let $\{t_i\}$ be an ordering of the atoms of μ. Since, at the interval $\bar{\eta}(t_i)$, $y(s)$ can be any curve in x_{t_i} that satisfies (2)-(4), we conclude that $x_{t_i}(\cdot) \subset S$ $\forall i \in \mathbb{N}$. Therefore, it follows from $x(t) = y(\eta(t))$ that $x_t(\cdot) \subset S$ for all $t \geq 0$, i.e., $((F,\mathbf{G}),S)$ is strongly invariant.

[$\Leftarrow$] Let $\tilde{x}$ be an arbitrary trajectory of $\tilde{F}$ with $\tilde{x}(0) \in \tilde{S}$. We will show that, for any arbitrary $T > 0$, $\tilde{x}(T) \in \tilde{S}$.

We can construct an admissible process (x,μ) of $(F,\mathbf{G})$ in $[0,T]$, with $x(0) = y(0) \in S$ by Theorem 2.4.

Let $T^* := \eta(T)$. By assumption $((F,\mathbf{G}),S)$ is strongly invariant, and, therefore, $x_t(\cdot) \subset S$ for all $t \in [0,T]$. By construction, we have $y(s) = x(\theta(s))$ for all $s \in [0,T^*] \setminus \cup_{i=1}^\infty \bar{\eta}(t_i)$ and $y(s) \in x_{t_i}(s)$ for all $s \in \bar{\eta}(t_i)$. Then, $y(s) \in S$ for all $s \in [0,T^*]$. But $T^* \geq T$ and this implies that $y(T) \in S$. Thus $(\tilde{F},\tilde{S})$ is strongly invariant, since $\tilde{x}$ is an arbitrary solution of (7). □

In the next result, that is a generalization of the similar result for the regular case (see e.g. (Clarke *et al.*, 1998)), we need the Lipschitz condition for multi-functions. We say that a multi-function $\Gamma : \mathbb{R}^n \hookrightarrow \mathbb{R}^m$ is locally Lipschitz if for each $x_0 \in \mathbb{R}^n$, there exists δ, $K > 0$ such that

$$\Gamma(x) \subset \Gamma(y) + K\|x-y\|B, \quad \forall x, y \in x_0 + \delta B.$$

Theorem 3.7. Suppose that F and $\mathbf{G}$ are locally Lipschitz. Then,

$$\begin{cases} \forall \tilde{x} \in \tilde{S}, \ \forall (v_0,v) \in \bar{V}, \ \text{we have}, \ \forall \tilde{\zeta} \in N_{\tilde{S}}^P(\tilde{x}), \\ \max_{f \in F(x), \ G \in \mathbf{G}(x)} \left\langle v_0 f + \sum_{i=1}^q v_i g_i, \tilde{\zeta} \right\rangle \leq 0 \end{cases} \quad (17)$$

if and only if

The system $((F,\mathbf{G}),S)$ is strongly invariant. (18)

Remark. Following the arguments in (Clarke *et al.*, 1998) it is straightforward to show that alternative equivalent characterizations of strong invariance are:

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[4] The multifunction defined in (15)

(a) $\tilde{F}(x) \subset T_{\tilde{S}}^C(\tilde{x})$ [5] , $\forall \tilde{x} \in \tilde{S}$.

(b) $\tilde{F}(\tilde{x}) \subset T_{\tilde{S}}^B(\tilde{x}) \ \forall \tilde{x} \in \tilde{S}$.

(c) $\tilde{F}(\tilde{x}) \subset \mathrm{co} T_{\tilde{S}}^B(\tilde{x}) \ \forall \tilde{x} \in \tilde{S}$.

(d) $\forall x_0 \in S$, $\exists \varepsilon > 0$, such that $\mathcal{A}(x_0; t) \subset S \ \forall t \in [0, \varepsilon]$.

Proof. It follows from Theorem 4.3.8 of (Clarke *et al.*, 1998) that (17) is a necessary and sufficient condition for strong invariance of the system $(\tilde{F}, \tilde{S})$ (as well as conditions $(a) - (d)$ in the above remark). Then, the conclusion follows immediately from Proposition 3.6. □

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[5] $T_S^C(x)$, denotes the *Clarke tangent cone* to S at x, which is given by:

$$T_S^C(x) := \{v \in \Re^n : d_S^\circ(x; v) \le 0\},$$

where $d_S^\circ(x; v)$ is the generalized directional derivative of $d_S(\cdot)$ at x, in the direction v. This is defined by

$$d^\circ(x; v) := \limsup_{y \to x; \ t \downarrow 0} \frac{d_S(y + tv) - d_S(y)}{t}.$$