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Mapping Spaces of Gray-Categories

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Summary

Gray-categories are semistrict version of tricategories, which however (unlike 3-categories) fully retain the richness of the theory, in the sense that any tricategory can be strictified to a triequivalent Gray category. They can be defined as categories enriched in the monoidal category Gray of 2-categories, with the Gray-tensor product, which in turn can be defined as the left adjoint to the internal hom functor of 2-functors, pseudo-transformations and modifications between a given pair of 2-categories. Gray-categories are similar to 3-categories, the crucial difference being that the horizontal composites of 2-cells coinciding on 0-cells are not unique; the two such composites are however connected by an invertible 3-cell called the tensor of the respective 2-cells, satisfying coherence conditions.

In this work we define a Gray-category of functors, lax transformations, modifications and perturbations between a given pair of Gray-categories, thereby providing a partial generalization of the internal hom functor for 2-categories. The principal obstacle here is that when the composite of two composable strict transformations is defined as the obvious pasting of diagrams, not all such composites exist. This is due to the lack of unique horizontal composites of 2-cells in the codomain Gray-category.

We solve this problem by introducing a minimally extended notion of transformation, avoiding the full generality of tricategories. There are two essential technical ingredients that make this possible: First, we construct a Gray-category, called path space, for every given Gray-category, and this pair constitutes an internal reflexive graph in the category of Gray-categories. The second tool we introduce is a resolution of the 1-dimensional structure of a given Gray-category, which is given by a co-monad derived from the canonical fibration of Gray-categories over categories and the free category co-monad.

Taking the co-Kleisli category for this co-monad gives us a suitably weakened kind of functor between Gray-categories. This provides just enough freedom to define a composition operation for the path space described above, turning it into an internal category in the category of Gray-categories and weak functors.

Given this internal category we can define lax transformations between weak Gray-functors as weak Gray-functors into the path space of the co-domain, satisfying the obvious incidence conditions. Now, given the composition operation of the path space, composable lax transformations have an obvious, well defined composition.

In turn, modifications and perturbations can be defined by iterating this idea to the second and third degree: For every Gray-category we define an internal Gray-category in the category of Gray-categories and weak functors, extending the path space. Modifications and perturbations are now describable as pseudo-functors into the second and third degree part of this internal Gray-category, called the 2-path and 3-path spaces, respectively; again, the various compositions of modifications and perturbations are defined using the operations of the extended path space.

By virtue of this construction, taking all weak functors from one Gray-category into the various degrees of the extended path space of another gives us a Gray-category of functors, transformations, modifications and perturbation as 0-, 1-, 2- and 3-cells respectively. We provide detailed explications of the objects thus obtained.

Resumo

Uma Gray-categoria é um caso particular, semi-estrito, do conceito de tricategoria. Não obstante (ao contrário das 3-categorias) as Gray-categorias retêm completamente a riqueza da teoria, no sentido que qualquer tricategoria pode ser estritificada numa Gray-categoria tri-equivalente.

As Gray-categorias podem ser definidas como categorias enriquecidas sobre a categoria monoidal Gray das 2-categorias, munidas do produto tensorial Gray, que por sua vez pode ser definido como o adjunto à esquerda do objecto exponencial de 2-functores, pseudo-transformações naturais e modificações entre um dado par de 2-categorias. As Gray-categorias são semelhantes às 3-categorias, sendo a diferença crucial o facto que as duas composições horizontais possíveis de 2-morfismos, adjacentes a um dado objecto, não coincidem; pese embora estejam ligadas por um 3-morfismo invertível (chamado produto tensorial dos respectivos 2-morfismos) satisfazendo este condições de coerência.

Neste trabalho, definimos uma Gray-categoria de functores fracos, transformações maleáveis, modificações e perturbações entre um determinado par de Gray-categorias, proporcionando assim uma generalização parcial do objecto exponencial para as 2-categorias. O principal obstáculo aqui é que, quando a composição de duas transformações rígidas é definido como sendo a colagem óbvia de diagramas, nem todas as composições fazem sentido. Isto acontece devido à falha na unicidade das composições horizontais de 2-morfismos na Gray-categoria alvo.

Superámos este problema introduzindo uma noção, minimamente estendida, de transformação (transformação maleável) entre functores, evitando assim a generalidade completa das tricategorias. Existem dois ingredientes técnicos essenciais na definição de transformação maleável: Em primeiro lugar, construímos uma Gray-categoria, chamada categoria dos caminhos numa Gray-categoria, dada uma certa Gray-categoria, sendo que o par constituído por uma Gray-categoria e o seu espaço dos caminhos define um grafo reflexivo interno à categoria das Gray-categorias. A segunda ferramenta que nós apresentamos é uma resolução da estrutura uni-dimensional de uma dada Gray-categoria, que é dada por uma co-mónade derivada da fibração canónica das Gray-categorias sobre as categorias, e da co-mónade da categoria livre numa categoria.

Considerando a categoria co-Kleisli desta co-mónade fornece-nos uma noção adequadamente fraca de functor entre duas Gray-categorias. Isto proporciona-nos exactamente a liberdade necessária para definir uma operação de composição dentro da Gray-categoria dos caminhos numa Gray-categoria, descrita acima, tornando-a numa categoria interna à categoria das Gray-categorias e functores fracos.

Dada esta categoria interna podemos definir transformações maleáveis entre Gray-functores fracos como sendo Gray-functores fracos para o espaço dos caminhos na Gray-categoria alvo, satisfazendo estes as condições óbvias de incidência. Devido à operação de composição no espaço dos caminhos, as transformações maleáveis têm agora uma composição óbvia e bem definida.

Por sua vez, as modificações e as perturbações podem ser definidas por iteração dessa ideia, para o segundo e terceiro grau: Dada uma Gray-categoria, definimos uma Gray-categoria interna à categoria das Gray-categorias e dos functores fracos, estendendo o espaço dos caminhos na Gray-categoria. Modificações e perturbações podem agora ser descritas como sendo functores para as

Gray-categorias dos 2- e 3-morfismos desta Gray-categoria interna, sendo estas últimas chamadas os espaços dos 2-caminhos e dos 3-caminhos na Gray categoria alvo, respectivamente. Mais uma vez, as várias composições de modificações e perturbações são definidas utilizando as operações na extensão do espaço dos caminhos numa Gray-categoria.

Devido a essa construção, considerando todos os functores fracos de uma Gray-categoria para os vários graus do espaço caminho estendido de uma outra dá-nós uma Gray-categoria de functores fracos, transformações maleáveis, modificações e perturbações como 0-, 1-, 2- e 3-morfismos respectivamente. Faremos explicações detalhadas dos objectos assim obtidos.

Contents

Chapter 1

Introduction

Folk knowledge of yore, among algebraic models for homotopy n-types Gray-groupoids model 3-types; Lack [2011] gives us a proof using a model category methods. Wanting to study the homotopy 3-type of the moduli space of 3-connections on a manifold, we thought it apt to define a mapping space $[\mathcal{S}_3(M), \mathcal{C}(\mathcal{H})]$ of Gray-groupoids that could model that moduli space, where $\mathcal{S}_3(M)$ is the fundamental Gray-groupoid and $\mathcal{C}(\mathcal{H})$ is the Gray-groupoid ultimately derived from a 2-crossed Lie-algebra where the triconnections take their values; this is the obvious next step after 2-connections, see for example Schreiber and Waldorf [2011]. See [Martins and Picken 2011] for the background on the fundamental Gray-groupoid and triconnections.

In 1999 Crans gave a partial solution the mapping space problem; however, the absence of an interchange law in Gray-categories prevents lax transformations between Gray-functors from being composable in general. The slightly unsatisfactory solution is to restrict to those transformations and higher cells that can in fact be composed; this does give mapping space Gray-category, but a mere stopgap not sufficient for our purposes.

Instead we enlarge the repertoire of maps, and thereby transformations, in a way that will permit forming all composites of transformations; specifically we introduce a 2-cocycle that intermediates coherently between the two possible evaluations of arrangements of squares shown in (5.5) and (5.6). In analogy with Garner [2010] we introduce a co-monadic weakening of strict Gray-functors in section 3. The comonad Q^1 then yields a co-Kleisli category $\mathsf{GrayCat}_{\mathrm{Q}^1}$. We use in an essential way that GrayCat is fibered over Cat.

Inspired by [Bénabou 1967] we axiomatise lax transformations by maps into a path-space. In section 4 we introduce a functorial path-space construction for Gray-categories; subsequently in section 5 it is shown that this yields an internal category $\overrightarrow{\mathbb{H}} \rightrightarrows \mathbb{H}$ in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ for a given $\mathbb{H}$ in GrayCat.

The n-th iterate of $\overrightarrow{(\_)}$ yields an n-truncated internal cubical object in GrayCat. In section 6 we construct an internal Gray-category

$$\overline{\overline{\overline{\mathbb{H}}}} \rightrightarrows \overline{\overline{\mathbb{H}}} \rightrightarrows \overrightarrow{\mathbb{H}} \rightrightarrows \mathbb{H}$$

in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ as a subobject of the third iterated path-space. It is then a trivial consequence that we obtain a mapping space Gray-category by applying the hom functor

$$[\mathbb{G}, \mathbb{H}] := \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overline{\overline{\overline{\mathbb{H}}}} \rightrightarrows \overline{\overline{\mathbb{H}}} \rightrightarrows \overrightarrow{\mathbb{H}} \rightrightarrows \mathbb{H}).$$

We hope to be able to prove in a later paper that this internal hom is part of a monoidal closed structure on $\mathsf{GrayCat}_{Q^1}$ involving a suitable extension of Crans' tensor product.

Lastly, we remark that if $\mathbb{H}$ is a Gray-groupoid then $\overrightarrow{\mathbb{H}}$ as well as $[\mathbb{G}, \mathbb{H}]$ will be Gray-groupoids.

Chapter 2

Gray-Categories

We shall give an overview of 2- and 3-dimensional categories before giving the precise definition internal to a category.

Sesquicategories

Ordinary categories have objects and arrows

$$x \xrightarrow{f} y \ . \tag{2.1}$$

We shall often talk about 0- and 1-cells instead when the category in question is the structure being investigated rather than the context in which the investigation is carried out. Objects and arrows may also bear upper case names. There are units and composition

$$x \xrightarrow{\mathrm{id}_x} x \qquad x \xrightarrow{f} y \xrightarrow{g} z, \quad x \xrightarrow{gf} z \tag{2.2}$$

that obey the obvious unit and associativity laws. In the presence of higher cells it will be convenient to denote the composition by $g\#_0 f$, that is, we note down the dimension of the incidence cell.

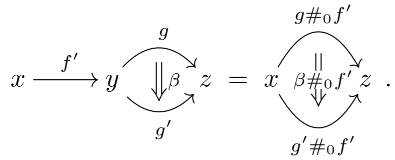
If we add 2-cells

$$x \underset{f'}{\overset{f}{\rightrightarrows}} y \ (\alpha: f \Rightarrow f') \qquad y \underset{g'}{\overset{g}{\rightrightarrows}} z \ (\beta: g \Rightarrow g') \tag{2.3}$$

into the mix we can define a sesquicategory by defining an action of the 1-cells on the 2-cells when they coincide on a 0-cell. We call this the «right whiskering» when the 1-cell appears on the right hand side in the diagram, and «left whiskering» in the opposite case. For example

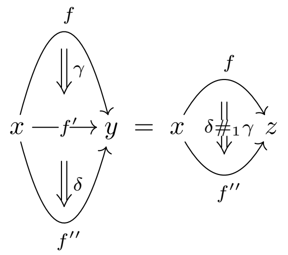
$$\left(x \underset{f'}{\overset{f}{\rightrightarrows}} y \ (\Downarrow \alpha) \right) \xrightarrow{g} z \;=\; x \underset{g\#_0 f'}{\overset{g\#_0 f}{\rightrightarrows}} z \ (\Downarrow g\#_0\alpha) \tag{2.4}$$

and



(2.5)

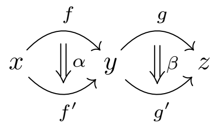
Also, the 2-cells can be composed along 1-cells



(2.6)

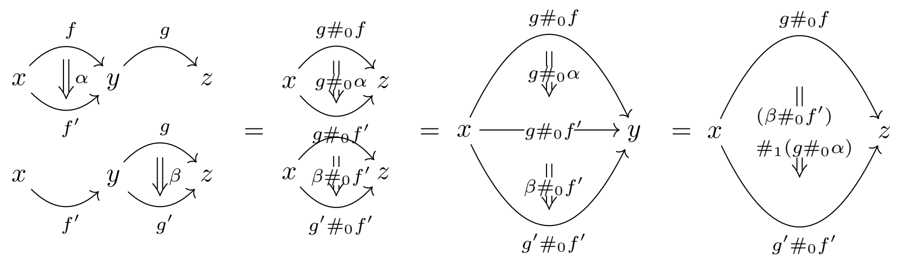
We assume units and and associativity for the 2-cells well.

Now, we can define derived operations called left and right horizontal composition. Given a diagram



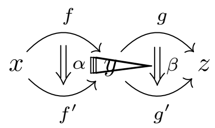
(2.7)

there are two ways to evaluate it in terms of the operations $\#_0$ and $\#_1$ as follows:



(2.8)

We shall call this the left horizontal composite, for no other reason than that «the left hand cell goes on top of the right hand cell». We denote it diagrammatically by



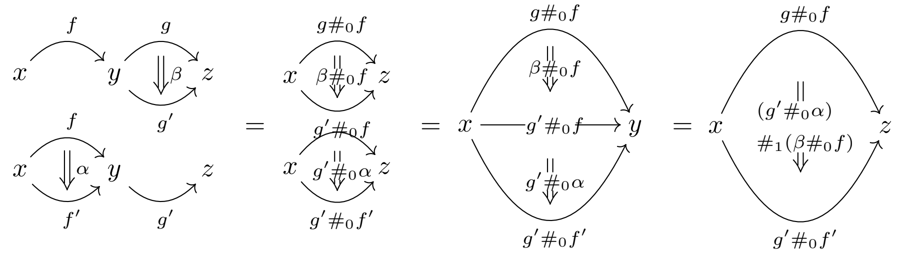
(2.9)

and define

$$\beta \triangleleft \alpha = (\beta \#_0 f') \#_1 (g \#_0 \alpha) . \tag{2.10}$$

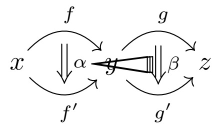
Note how when reading this expression from left to right one traverses the corresponding diagram from bottom to top and from right to left.

The other way to evalute (2.7) is



(2.11)

We draw this as



(2.12)

and define the right horizontal composite as

$$\beta \triangleright \alpha = (g' \#_0 \alpha) \#_1 (\beta \#_0 f)\,. \tag{2.13}$$

Of course we assume that whiskering distributes over the vertical composition of 2-cells.

In addition one might insist on the interchange condition

$$\beta \triangleleft \alpha = \beta \triangleright \alpha \tag{2.14}$$

making (2.7) a well defined composite. This is of course what turns the sesquicategory into a 2-category.

One can take a slightly more abstract view describing sesquicategories as categories enriched in the category of categories with a peculiar symmetric monoidal structure. First, for two categories B, C we can consider $\mathrm{Un}(\mathsf{B}, \mathsf{C})$ with ordinary functors as objects, and unnatural transformations, that is, families of C-morphisms indexed by B-objects as morphisms. So, unnatural transformations are like natural transformations, except that we do not impose naturality.

One can easily check that there is a symmeric tensor product $\mathsf{A} \square \mathsf{B}$ having $\mathsf{A}_0 \times \mathsf{B}_0$ as the set of vertices and as arrows sequences generated from expressions (f, y) and (x, g), where $f \in \mathsf{A}_1$, $y \in \mathsf{B}_0$, $x \in \mathsf{A}_0$ and $g \in \mathsf{B}_1$, subject to the relations

$$(f', y)(f, y) = (f'f, y) \tag{2.15}$$

$$(x, g')(x, g) = (x, g'g)\,. \tag{2.16}$$

Furthermore, one checks that there is an adjunction

$$\_\square \mathsf{B} \dashv \mathrm{Un}(\mathsf{B}, \_) \tag{2.17}$$

for all categories B.

Definition 1 Sesquicategories *are categories enriched in* $(\mathsf{Cat}, \square)$[1].

[1] Perhaps this monoidal category should be called Sesqui, so we can call sesquicategories Sesqui-categories.

For the definition of enriched categories see Kelly [2005].

Remark 2 *The failure of the interchange condition (2.14) is reflected in the fact that in* $\mathsf{A}\Box\mathsf{B}$ *the square*

$$\begin{array}{ccc} (x,y) & \xrightarrow{(x,g)} & (x,y') \\ {\scriptstyle (f,y)}\big\downarrow & & \big\downarrow{\scriptstyle (f,y')} \\ (x',y) & \xrightarrow[(x',g)]{} & (x',y') \end{array} \tag{2.18}$$

is in general not commutative; as opposed to the situation in $\mathsf{A} \times \mathsf{B}$.

We now unravel definition 1 in terms of internal structures in a cateogory with the necessary limits. An internal sesquicategory is given by the following data:

- a reflexive 2-globular object

$$C_2 \;\begin{array}{c}\xrightarrow{s_1}\\ \xleftarrow{\mathrm{id}_1}\\ \xrightarrow[t_1]{}\end{array}\; C_1 \;\begin{array}{c}\xrightarrow{s_0}\\ \xleftarrow{\mathrm{id}_0}\\ \xrightarrow[t_0]{}\end{array}\; C_0 \tag{2.19}$$

  globularity means

$$s_n s_{n+1} = s_n t_{n+1} \tag{2.20}$$
$$t_n s_{n+1} = t_n t_{n+1} \tag{2.21}$$

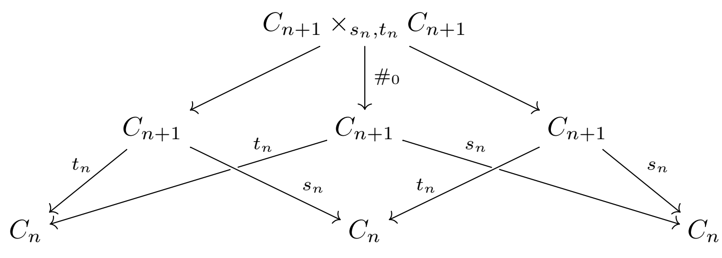
  so by abuse of notation we shall write

$$s_n = s_n s_{n+1} = s_n t_{n+1} \tag{2.22}$$
$$t_n = t_n s_{n+1} = t_n t_{n+1}\,. \tag{2.23}$$

  Reflexive means

$$C_n = s_n \mathrm{id}_n = t_n \mathrm{id}_n\,. \tag{2.24}$$

- composition operations:



(2.25)

  such that each

$$(C_n, C_{n+1}, \#_n, s_n, t_n, \mathrm{id}_n) \tag{2.26}$$

  is a category for $n = 0, 1$.

- Functorial, compatible, unital, associative left and right actions of C_1 on the category $C_2 \rightrightarrows C_1$, given by maps $\#_0 \colon C_1 \times_{s_0,t_0} C_2 \longrightarrow C_2$ and $\#_0 \colon C_2 \times_{s_0,t_0} C_1 \longrightarrow C_2$. In detail this means, left and right functoriality with respect to 2-cells

$$\begin{array}{ccc}
(C_1 \times_{s_0,t_0} C_2) \times_{C_1 \times s_1, C_1 \times t_1} (C_1 \times_{s_0,t_0} C_2) & \xrightarrow{\#_0 \times \#_0} & C_2 \times_{s_1,t_1} C_2 \\
\downarrow{\scriptstyle C_1 \times \#_1} & & \downarrow{\scriptstyle \#_1} \\
C_1 \times_{s_0,t_0} C_2 & \xrightarrow{\#_0} & C_2 \\
{\scriptstyle C_1 \times s_1}\downarrow \; \uparrow{\scriptstyle C_1 \times \mathrm{id}_1} \; \downarrow{\scriptstyle C_1 \times t_1} & & {\scriptstyle s_1}\downarrow \; \uparrow{\scriptstyle \mathrm{id}_1} \; \downarrow{\scriptstyle t_1} \\
C_1 \times_{s_0,t_0} C_1 & \xrightarrow[\#_0]{} & C_1
\end{array} , \tag{2.27}$$

$$\begin{array}{ccc}
(C_2 \times_{s_0,t_0} C_1) \times_{s_1 \times C_1, t_1 \times C_1} (C_2 \times_{s_0,t_0} C_1) & \xrightarrow{\#_0 \times \#_0} & C_2 \times_{s_1,t_1} C_2 \\
\downarrow{\scriptstyle \#_1 \times C_1} & & \downarrow{\scriptstyle \#_1} \\
C_2 \times_{s_0,t_0} C_1 & \xrightarrow{\#_0} & C_2 \\
{\scriptstyle s_1 \times C_1}\downarrow \; \uparrow{\scriptstyle \mathrm{id}_1 \times C_1} \; \downarrow{\scriptstyle t_1 \times C_1} & & {\scriptstyle s_1}\downarrow \; \uparrow{\scriptstyle \mathrm{id}_1} \; \downarrow{\scriptstyle t_1} \\
C_1 \times_{s_0,t_0} C_1 & \xrightarrow[\#_0]{} & C_1
\end{array} ; \tag{2.28}$$

Unitality of the $\#_0$ whiskering actions means

$$\begin{array}{ccc}
C_2 & \xrightarrow{\langle \mathrm{id}_0 t_0, C_2 \rangle} & C_1 \times_{s_0,t_0} C_2 \\
& {\scriptstyle C_2}\searrow & \downarrow{\scriptstyle \#_0} \\
& & C_2
\end{array} , \tag{2.29}$$

$$\begin{array}{ccc}
C_2 & \xrightarrow{\langle C_2, \mathrm{id}_0 s_0 \rangle} & C_2 \times_{s_0,t_0} C_1 \\
& {\scriptstyle C_2}\searrow & \downarrow{\scriptstyle \#_0} \\
& & C_2
\end{array} . \tag{2.30}$$

Left and right associativity as well as compatibility mean

$$
\begin{array}{ccc}
C_1 \times_{s_0,t_0} C_1 \times_{s_0,t_0} C_2 & \xrightarrow{(\#_0)\times C_2} & C_1 \times_{s_0,t_0} C_2 \\
{\scriptstyle C_1\times(\#_0)}\downarrow & & \downarrow{\scriptstyle \#_0} \\
C_1 \times_{s_1,t_1} C_2 & \xrightarrow[\#_0]{} & C_2
\end{array}
\qquad
\begin{array}{ccc}
C_2 \times_{s_0,t_0} C_1 \times_{s_0,t_0} C_1 & \xrightarrow{(\#_0)\times C_1} & C_2 \times_{s_0,t_0} C_1 \\
{\scriptstyle C_2\times(\#_0)}\downarrow & & \downarrow{\scriptstyle \#_0} \\
C_2 \times_{s_0,t_0} C_2 & \xrightarrow[\#_0]{} & C_2
\end{array}
,$$

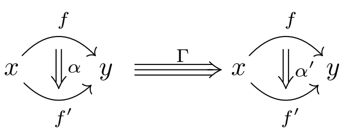
(2.31) (2.32)

$$
\begin{array}{ccc}
C_1 \times_{s_0,t_0} C_2 \times_{s_0,t_0} C_1 & \xrightarrow{(\#_0)\times C_1} & C_2 \times_{s_0,t_0} C_1 \\
{\scriptstyle C_1\times(\#_0)}\downarrow & & \downarrow{\scriptstyle \#_0} \\
C_1 \times_{s_0,t_0} C_2 & \xrightarrow[\#_0]{} & C_2
\end{array}
\quad . \tag{2.33}
$$

Gray-categories

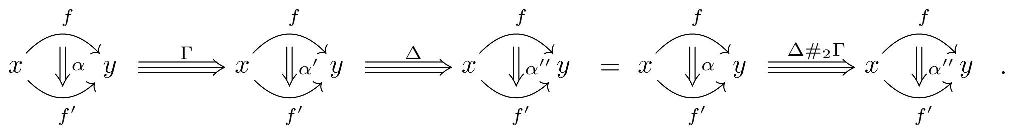
Having an idea about sesquicategories we can now go one dimension higher, introducing Gray-categories. They are the principal objects of study in this paper. For a more algebraic but similarly explicit exposition of them, see [Crans 1999].

We add 3-cells to sesquicategories



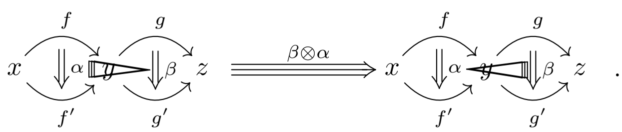
(2.34)

and of course we demand that 3-cells coinciding on a 2-cell compose associatively and that there are unit 3-cells



(2.35)

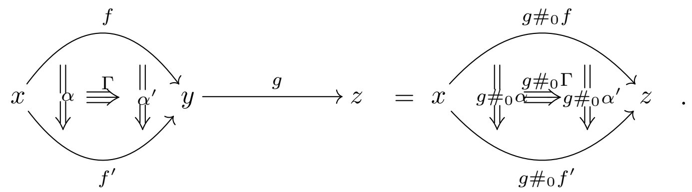
Moreover, we can somewhat mend the oddity of two horizontal composites ◁ and ▷ in the diagram (2.7) in a sesquicategory by inserting an invertible 3-cell between them



(2.36)

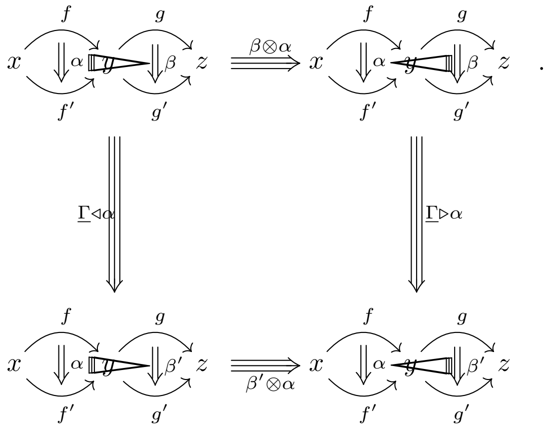
called the tensor of the respective 2-cells. Of course there are also actions of 1-

and 2-cells on 3-cells. For example



(2.37)

Of course, the tensor has to fulfill certain compatibilities, for example



(2.38)

commutes. Here we have extended $\triangleleft$ and $\triangleright$ to pairs of 3-cells coinciding with a 2-cell along a 0-cell, that is

$$\Gamma \triangleright \alpha = (g'\#_0\alpha)\#_1(\Gamma\#_0 f) \tag{2.39}$$
$$\Gamma \triangleleft \alpha = (\Gamma\#_0 f')\#_1(g\#_0\alpha)\,. \tag{2.40}$$

We will sometimes use underlining to emphasise the top-dimensional operands in an expression.

But beyond the tensor there are no further pathologies, meaning that the 1-, 2- and 3-cells between any given pair of 0-cells actually are the 0-, 1- and 2-cells of a 2-category. By 2.14 this means in particular that two 3-cells incident on a 1-cell have a unique composite $\#_1$.

Remember how in definition 1, the enrichement was in $(\mathsf{Cat}, \square)$ meaning that locally a sesquicategory is a category. Now, a Gray-category is locally a 2-category, so we have to replace Cat with 2Cat and extend the tensor $\square$ to something that allows us to fill in the square 2.18 with an invertible local 2-cell, that will yield the invertible 3-cell in 2.36. This extension is called the Gray-tensor product for 2-categories, also denoted by $\otimes$, see Gray [1974]. It can be defined as a left adjoint analogous to (2.17)

$$\_\otimes \mathsf{B} \dashv \mathrm{Ps}(\mathsf{B}, \_) \tag{2.41}$$

for all 2-categories B where $\mathrm{Ps}(\mathsf{B}, \_)$ is the 2-categoy of 2-functors, pseudotransformations and modifications.

For the moment we make the following observation

Remark 3 *A* Gray*-category is a reflexive 3-globular set $\mathbb{G}_{0,\ldots,3}$, with composition operations $\#_k$, where k denotes the dimension of the incidence cell. In general we can say that composing an i-cell with a j-cell along a k-cell yields an $i+j-(k+1)$-cell. The ones where $i=j$ and $k=i-1$ are called vertical. The ones where $i+j-(k+1)=\max\{i,j\}$ are called whiskers. This seems to suggest a certain relationship with graded Lie algebras. For considerations of dimension raising see also [Crans 1999, section 1].*

Definition 4 *A* **Gray-category** *is a category enriched in the category* Gray $=$ $(\mathsf{2Cat},\otimes)$ *of 2-categories with the Gray-tensor product.*

We summarize here the axioms of Gray-categories in an internal fashion, that is, using diagrams in a category with pullbacks. We crossreference the definition given in [Crans 1999, section 2].

Explicitly, if Gray was internal to a category with limits C, then we get a notion of Gray-categories internal to C, which is given by the following data, which is a translation of Crans' definition:

- a reflexive 3-globular object

$$C_3 \underset{t_2}{\overset{s_2}{\rightrightarrows}} \xleftarrow{\mathrm{id}_2} C_2 \underset{t_1}{\overset{s_1}{\rightrightarrows}} \xleftarrow{\mathrm{id}_1} C_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} \xleftarrow{\mathrm{id}_0} C_0 \tag{2.42}$$

  globularity means

$$s_n s_{n+1} = s_n t_{n+1} \tag{2.43}$$
$$t_n s_{n+1} = t_n t_{n+1} \tag{2.44}$$

  so by abuse of notation we shall write

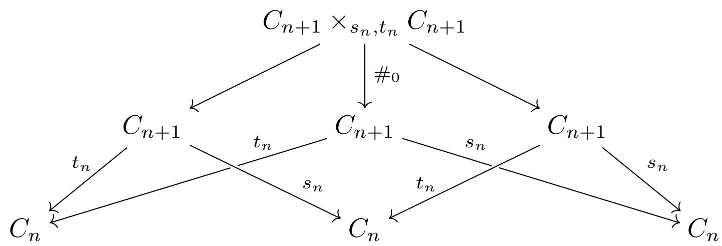
$$s_n = s_n s_{n+1} = s_n t_{n+1} \tag{2.45}$$
$$t_n = t_n s_{n+1} = t_n t_{n+1}\,. \tag{2.46}$$

  Reflexive means

$$C_n = s_n \mathrm{id}_n = t_n \mathrm{id}_n\,. \tag{2.47}$$

  This already captures condition [Crans 1999, 2.3(i)].

- vertical composition operations:



(2.48)

  such that each

$$(C_n, C_{n+1}, \#_n, s_n, t_n, \mathrm{id}_n) \tag{2.49}$$

  is a category.

- compatible, unital, associative left and right actions of C_2 on $C_3 \rightrightarrows C_2$, that is, maps $\#_1 \colon C_2 \times_{s_1,t_1} C_3 \longrightarrow C_3$ and $\#_1 \colon C_3 \times_{s_1,t_1} C_2 \longrightarrow C_3$, that form internal functors as follows:

$$\begin{array}{ccc}
(C_2 \times_{s_1,t_1} C_3) \times_{C_2 \times s_2, C_2 \times t_2} (C_2 \times_{s_1,t_1} C_3) & \xrightarrow{\#_1 \times \#_1} & C_3 \times_{s_2,t_2} C_3 \\
\Big\downarrow{\scriptstyle C_2 \times \#_2} & & \Big\downarrow{\scriptstyle \#_2} \\
C_2 \times_{s_1,t_1} C_3 & \xrightarrow{\#_1} & C_3 \\
{\scriptstyle C_2 \times s_2}\Big\downarrow\Big\uparrow{\scriptstyle C_2 \times \mathrm{id}_2}\Big\downarrow{\scriptstyle C_2 \times t_2} & & {\scriptstyle s_2}\Big\downarrow\Big\uparrow{\scriptstyle \mathrm{id}_2}\Big\downarrow{\scriptstyle t_2} \\
C_2 \times_{s_1,t_1} C_2 & \xrightarrow[\#_1]{} & C_2
\end{array} \tag{2.50}$$

and

$$\begin{array}{ccc}
(C_3 \times_{s_1,t_1} C_2) \times_{s_2 \times C_2, t_2 \times C_2} (C_3 \times_{s_1,t_1} C_2) & \xrightarrow{\#_1 \times \#_1} & C_3 \times_{s_2,t_2} C_3 \\
\Big\downarrow{\scriptstyle \#_2 \times C_2} & & \Big\downarrow{\scriptstyle \#_2} \\
C_3 \times_{s_1,t_1} C_2 & \xrightarrow{\#_1} & C_3 \\
{\scriptstyle s_2 \times C_2}\Big\downarrow\Big\uparrow{\scriptstyle \mathrm{id}_2 \times C_2}\Big\downarrow{\scriptstyle t_2 \times C_2} & & {\scriptstyle s_2}\Big\downarrow\Big\uparrow{\scriptstyle \mathrm{id}_2}\Big\downarrow{\scriptstyle t_2} \\
C_2 \times_{s_1,t_1} C_2 & \xrightarrow[\#_1]{} & C_2
\end{array} \quad . \tag{2.51}$$

Unital means

$$\begin{array}{ccc}
C_3 & \xrightarrow{\langle \mathrm{id}_1 t_1, C_3 \rangle} & C_2 \times_{s_1,t_1} C_3 \\
& {\scriptstyle C_3} \searrow & \Big\downarrow{\scriptstyle \#_1} \\
& & C_3
\end{array} \tag{2.52}$$

and

$$\begin{array}{ccc}
C_3 & \xrightarrow{\langle C_3, \mathrm{id}_1 s_1 \rangle} & C_3 \times_{s_1,t_1} C_2 \\
& {\scriptstyle C_3} \searrow & \Big\downarrow{\scriptstyle \#_1} \\
& & C_3
\end{array} \quad , \tag{2.53}$$

and left and right associativity means

$$\begin{array}{ccc} C_2 \times_{s_1,t_1} C_2 \times_{s_1,t_1} C_3 & \xrightarrow{(\#_1)\times C_3} & C_2 \times_{s_1,t_1} C_3 \\ {\scriptstyle C_2\times(\#_1)}\downarrow & & \downarrow{\scriptstyle \#_1} \\ C_2 \times_{s_1,t_1} C_3 & \xrightarrow[\#_1]{} & C_3 \end{array} \quad (2.54) \quad \text{and} \quad \begin{array}{ccc} C_3 \times_{s_1,t_1} C_2 \times_{s_1,t_1} C_2 & \xrightarrow{(\#_1)\times C_2} & C_3 \times_{s_1,t_1} C_2 \\ {\scriptstyle C_3\times(\#_1)}\downarrow & & \downarrow{\scriptstyle \#_1} \\ C_3 \times_{s_1,t_1} C_2 & \xrightarrow[\#_1]{} & C_3 \end{array} \ . \quad (2.55)$$

Compatibility means

$$\begin{array}{ccc} C_2 \times_{s_1,t_1} C_3 \times_{s_1,t_1} C_2 & \xrightarrow{(\#_1)\times C_2} & C_3 \times_{s_1,t_1} C_2 \\ {\scriptstyle C_2\times(\#_1)}\downarrow & & \downarrow{\scriptstyle \#_1} \\ C_3 \times_{s_1,t_1} C_2 & \xrightarrow[\#_1]{} & C_3 \end{array} \ . \qquad (2.56)$$

Furthermore we demand that the horizontal whiskers $\#_1$ of 3-cells by 2-cells along 1-cells, and vertical composition $\#_2$ of 3-cells along 2-cells define a unique horizontal composition of 3-cells along a 1-cell, that is,

$$\begin{array}{ccc} C_3 \times_{s_1,t_1} C_3 & \xrightarrow{\langle(\#_1)(t_2\times C_3),(\#_1)(C_3\times s_2)\rangle} & C_3 \times_{s_2,t_2} C_3 \\ {\scriptstyle \langle(\#_1)(C_3\times t_2),(\#_1)(s_2\times C_3)\rangle}\downarrow & & \downarrow{\scriptstyle \#_2} \\ C_3 \times_{s_2,t_2} C_3 & \xrightarrow[\#_2]{} & C_3 \end{array} \ . \qquad (2.57)$$

This point together with the previous one captures [Crans 1999, 2.4(ii)].

- Furthermore, 2-functorial, compatible, unital, associative left and right actions of C_1 on the 2-category $C_3 \rightrightarrows C_2 \rightrightarrows C_1$, given by maps $\#_0\colon C_1 \times_{s_0,t_0} C_2 \longrightarrow C_2$, $\#_0\colon C_2 \times_{s_0,t_0} C_1 \longrightarrow C_2$, $\#_0\colon C_1 \times_{s_0,t_0} C_3 \longrightarrow C_3$, and $\#_0\colon C_3 \times_{s_0,t_0} C_1 \longrightarrow C_3$. In detail this means, left and right functoriality with respect to 2-cells

$$\begin{array}{ccc} (C_1 \times_{s_0,t_0} C_2) \times_{C_1\times s_1, C_1\times t_1} (C_1 \times_{s_0,t_0} C_2) & \xrightarrow{\#_0\times\#_0} & C_2 \times_{s_1,t_1} C_2 \\ {\scriptstyle C_1\times\#_1}\downarrow & & \downarrow{\scriptstyle \#_1} \\ C_1 \times_{s_0,t_0} C_2 & \xrightarrow{\#_0} & C_2 \\ {\scriptstyle C_1\times s_1}\downarrow\ \uparrow{\scriptstyle C_1\times \mathrm{id}_1}\ \downarrow{\scriptstyle C_1\times t_1} & & {\scriptstyle s_1}\downarrow\ \uparrow{\scriptstyle \mathrm{id}_1}\ \downarrow{\scriptstyle t_1} \\ C_1 \times_{s_0,t_0} C_1 & \xrightarrow[\#_0]{} & C_1 \end{array} \ , \qquad (2.58)$$

$$
\begin{array}{ccc}
(C_2 \times_{s_0,t_0} C_1) \times_{s_1 \times C_1, t_1 \times C_1} (C_2 \times_{s_0,t_0} C_1) & \xrightarrow{\#_0 \times \#_0} & C_2 \times_{s_1,t_1} C_2 \\
\Big\downarrow{\scriptstyle \#_1 \times C_1} & & \Big\downarrow{\scriptstyle \#_1} \\
C_2 \times_{s_0,t_0} C_1 & \xrightarrow{\qquad \#_0 \qquad} & C_2 \\
{\scriptstyle s_1 \times C_1}\Big\downarrow \Big\uparrow{\scriptstyle \mathrm{id}_1 \times C_1} \Big\downarrow{\scriptstyle t_1 \times C_1} & & {\scriptstyle s_1}\Big\downarrow \Big\uparrow{\scriptstyle \mathrm{id}_1} \Big\downarrow{\scriptstyle t_1} \\
C_1 \times_{s_0,t_0} C_1 & \xrightarrow[\#_0]{\qquad\qquad} & C_1
\end{array}
\qquad ; \tag{2.59}
$$

left and right functoriality with respect to 3-cells

$$
\begin{array}{ccc}
(C_1 \times_{s_0,t_0} C_3) \times_{C_1 \times s_2, C_1 \times t_2} (C_1 \times_{s_0,t_0} C_3) & \xrightarrow{\#_0 \times \#_0} & C_3 \times_{s_2,t_2} C_3 \\
\Big\downarrow{\scriptstyle C_1 \times \#_2} & & \Big\downarrow{\scriptstyle \#_2} \\
C_1 \times_{s_0,t_0} C_3 & \xrightarrow{\qquad \#_0 \qquad} & C_3 \\
{\scriptstyle C_1 \times s_2}\Big\downarrow \Big\uparrow{\scriptstyle C_1 \times \mathrm{id}_2} \Big\downarrow{\scriptstyle C_1 \times t_2} & & {\scriptstyle s_2}\Big\downarrow \Big\uparrow{\scriptstyle \mathrm{id}_2} \Big\downarrow{\scriptstyle t_2} \\
C_1 \times_{s_0,t_0} C_2 & \xrightarrow[\#_0]{\qquad\qquad} & C_2
\end{array}
\qquad , \tag{2.60}
$$

$$
\begin{array}{ccc}
(C_3 \times_{s_0,t_0} C_1) \times_{s_2 \times C_1, t_2 \times C_1} (C_3 \times_{s_0,t_0} C_1) & \xrightarrow{\#_0 \times \#_0} & C_3 \times_{s_2,t_2} C_3 \\
\Big\downarrow{\scriptstyle \#_2 \times C_1} & & \Big\downarrow{\scriptstyle \#_2} \\
C_3 \times_{s_0,t_0} C_1 & \xrightarrow{\qquad \#_0 \qquad} & C_2 \\
{\scriptstyle s_1 \times C_1}\Big\downarrow \Big\uparrow{\scriptstyle \mathrm{id}_1 \times C_1} \Big\downarrow{\scriptstyle t_1 \times C_1} & & {\scriptstyle s_1}\Big\downarrow \Big\uparrow{\scriptstyle \mathrm{id}_2} \Big\downarrow{\scriptstyle t_1} \\
C_2 \times_{s_0,t_0} C_1 & \xrightarrow[\#_0]{\qquad\qquad} & C_2
\end{array}
\qquad . \tag{2.61}
$$

Unitality of the $\#_0$ whiskering actions means

$$\begin{array}{ccc} C_2 & \xrightarrow{\langle \mathrm{id}_0 t_0, C_2 \rangle} & C_1 \times_{s_0,t_0} C_2 \\ & {\scriptstyle C_2} \searrow & \downarrow {\scriptstyle \#_0} \\ & & C_2 \end{array} , \tag{2.62}$$

$$\begin{array}{ccc} C_2 & \xrightarrow{\langle C_2, \mathrm{id}_0 s_0 \rangle} & C_2 \times_{s_0,t_0} C_1 \\ & {\scriptstyle C_2} \searrow & \downarrow {\scriptstyle \#_0} \\ & & C_2 \end{array} , \tag{2.63}$$

similarly for the action of 1-cells on 3-cells,

$$\begin{array}{ccc} C_3 & \xrightarrow{\langle \mathrm{id}_0 t_0, C_3 \rangle} & C_1 \times_{s_0,t_0} C_3 \\ & {\scriptstyle C_3} \searrow & \downarrow {\scriptstyle \#_0} \\ & & C_3 \end{array} , \tag{2.64}$$

$$\begin{array}{ccc} C_3 & \xrightarrow{\langle C_3, \mathrm{id}_0 s_0 \rangle} & C_3 \times_{s_0,t_0} C_1 \\ & {\scriptstyle C_3} \searrow & \downarrow {\scriptstyle \#_0} \\ & & C_3 \end{array} . \tag{2.65}$$

Left and right associativity as well as compatibility mean

$$\begin{array}{ccc} C_1 \times_{s_0,t_0} C_1 \times_{s_0,t_0} C_2 & \xrightarrow{(\#_0) \times C_2} & C_1 \times_{s_0,t_0} C_2 \\ {\scriptstyle C_1 \times (\#_0)} \downarrow & & \downarrow {\scriptstyle \#_0} \\ C_1 \times_{s_1,t_1} C_2 & \xrightarrow[\#_0]{} & C_2 \end{array} , \tag{2.66}$$

$$\begin{array}{ccc} C_2 \times_{s_0,t_0} C_1 \times_{s_0,t_0} C_1 & \xrightarrow{(\#_0) \times C_1} & C_2 \times_{s_0,t_0} C_1 \\ {\scriptstyle C_2 \times (\#_0)} \downarrow & & \downarrow {\scriptstyle \#_0} \\ C_2 \times_{s_0,t_0} C_2 & \xrightarrow[\#_0]{} & C_2 \end{array} , \tag{2.67}$$

$$\begin{array}{ccc} C_1 \times_{s_0,t_0} C_2 \times_{s_0,t_0} C_1 & \xrightarrow{(\#_0) \times C_1} & C_2 \times_{s_0,t_0} C_1 \\ {\scriptstyle C_1 \times (\#_0)} \downarrow & & \downarrow {\scriptstyle \#_0} \\ C_1 \times_{s_0,t_0} C_2 & \xrightarrow[\#_0]{} & C_2 \end{array} , \tag{2.68}$$

$$\begin{array}{ccc} C_1 \times_{s_0,t_0} C_1 \times_{s_0,t_0} C_3 & \xrightarrow{(\#_0)\times C_3} & C_1 \times_{s_0,t_0} C_3 \\ {\scriptstyle C_1\times(\#_0)}\downarrow & & \downarrow{\scriptstyle \#_0} \\ C_1 \times_{s_1,t_1} C_3 & \xrightarrow[\#_0]{} & C_3 \end{array} , \tag{2.69}$$

$$\begin{array}{ccc} C_3 \times_{s_0,t_0} C_1 \times_{s_0,t_0} C_1 & \xrightarrow{(\#_0)\times C_1} & C_3 \times_{s_0,t_0} C_1 \\ {\scriptstyle C_3\times(\#_0)}\downarrow & & \downarrow{\scriptstyle \#_0} \\ C_3 \times_{s_0,t_0} C_1 & \xrightarrow[\#_0]{} & C_3 \end{array} , \tag{2.70}$$

$$\begin{array}{ccc} C_1 \times_{s_0,t_0} C_3 \times_{s_0,t_0} C_1 & \xrightarrow{(\#_0)\times C_1} & C_3 \times_{s_0,t_0} C_1 \\ {\scriptstyle C_1\times(\#_0)}\downarrow & & \downarrow{\scriptstyle \#_0} \\ C_1 \times_{s_0,t_0} C_3 & \xrightarrow[\#_0]{} & C_3 \end{array} . \tag{2.71}$$

This covers conditions Crans [1999, 2.4(iii)&(iv)].

- a map $\otimes\colon C_2 \times_{s_0,t_0} C_2 \longrightarrow C_3$ such that

$$\begin{array}{ccc} C_2 \times_{s_0,t_0} C_2 & \xrightarrow{\langle(\#_0)(t_1\times C_2),(\#_0)(C_2\times s_1)\rangle} & C_2 \times_{s_1,t_1} C_2 \\ {\scriptstyle \langle(\#_0)(C_2\times t_1),(\#_0)(s_1\times C_2)\rangle}\downarrow & \searrow^{\otimes}\ \dot{C}_3 \ \overset{t_2}{\underset{s_2}{\rightrightarrows}} & \downarrow{\scriptstyle \#_1} \\ C_2 \times_{s_1,t_1} C_2 & \xrightarrow[\#_1]{} & C_2 \end{array} , \tag{2.72}$$

where $\dot{C}_3$ is the object of invertible 3-cells. This means that $\dot{C}_3 \rightrightarrows C_2$ is an internal groupoid with an inversion $\overline{(\_)}\colon \dot{C}_3 \longrightarrow \dot{C}_3$ such that

$$\begin{array}{ccc} \dot{C}_3 & \xrightarrow{\overline{(\_)}} & \dot{C}_3 \\ & {\scriptstyle s_2}\searrow\!\!\searrow{\scriptstyle t_2} \quad {\scriptstyle t_2}\swarrow\!\!\swarrow{\scriptstyle s_2} & \\ & C_2 & \end{array} , \tag{2.73}$$

$$\begin{array}{ccc} \dot{C}_3 & \xrightarrow{\langle\overline{(\_)},\dot{C}_3\rangle} & \dot{C}_3 \times_{t_2,t_2} \dot{C}_3 \\ {\scriptstyle s_2}\downarrow & & \downarrow{\scriptstyle \#_2} \\ C_2 & \xrightarrow[\mathrm{id}_2]{} & \dot{C}_3 \end{array} \tag{2.74}$$

and

$$\begin{array}{ccc} \dot{C}_3 & \xrightarrow{\langle\dot{C}_3,\overline{(\_)}\rangle} & \dot{C}_3 \times_{s_2,s_2} \dot{C}_3 \\ {\scriptstyle t_2}\downarrow & & \downarrow{\scriptstyle \#_2} \\ C_2 & \xrightarrow[\mathrm{id}_2]{} & \dot{C}_3 \end{array} \quad . \tag{2.75}$$

This expresses condition [Crans 1999, 2.4(v)].

- Abbreviating

$$\triangleright = (\#_1)\langle(\#_0)(t_1 \times C_2), (\#_0)(C_2 \times s_1)\rangle \tag{2.76}$$
$$\triangleleft = (\#_1)\langle(\#_0)(C_2 \times t_1), (\#_0)(s_1 \times C_2)\rangle \tag{2.77}$$
$$\triangleright_\ell = (\#_1)\langle(\#_0)(t_1 \times C_3), (\#_0)(C_2 \times s_1)\rangle \tag{2.78}$$
$$\triangleleft_\ell = (\#_1)\langle(\#_0)(C_2 \times t_1), (\#_0)(s_1 \times C_3)\rangle \tag{2.79}$$
$$\triangleright_r = (\#_1)\langle(\#_0)(t_1 \times C_2), (\#_0)(C_3 \times s_1)\rangle \tag{2.80}$$
$$\triangleleft_r = (\#_1)\langle(\#_0)(C_3 \times t_1), (\#_0)(s_1 \times C_2)\rangle \tag{2.81}$$

we require $\otimes$ to have the following naturality properties

$$\begin{array}{ccc} C_3 \times_{s_0,t_0} C_2 & \xrightarrow{\langle(\triangleright_r),\otimes(s_2\times C_2)\rangle} & \dot{C}_3 \times_{s_2,t_2} \dot{C}_3 \\ {\scriptstyle \langle\otimes(t_2\times C_2),(\triangleleft_r)\rangle}\downarrow & & \downarrow{\scriptstyle \#_2} \\ \dot{C}_3 \times_{s_2,t_2} C_3 & \xrightarrow[\#_2]{} & C_3 \end{array} \tag{2.82}$$

and

$$\begin{array}{ccc} C_2 \times_{s_0,t_0} C_3 & \xrightarrow{\langle(\triangleright_\ell),\otimes(C_2\times s_2)\rangle} & \dot{C}_3 \times_{s_2,t_2} \dot{C}_3 \\ {\scriptstyle \langle\otimes(C_2\times t_2),(\triangleleft_\ell)\rangle}\downarrow & & \downarrow{\scriptstyle \#_2} \\ \dot{C}_3 \times_{s_2,t_2} C_3 & \xrightarrow[\#_2]{} & C_3 \end{array} \quad . \tag{2.83}$$

This expresses condition [Crans 1999, 2.4(vi)].

- Functoriality of the tensor. [Crans 1999, (vii)]

$$\begin{array}{ccc} C_2 \times_{s_0,t_0} (C_2 \times_{s_1,t_1} C_2) & \xrightarrow{C_2 \times (\#_1)} & C_2 \times_{s_0,t_0} C_2 \\ {\scriptstyle \langle \#_1 \langle \#_0 (t_1 \times p_0), \otimes (C_2 \times p_1) \rangle, \#_1 \langle \otimes (C_2 \times p_0), \#_0 (s_1 \times p_1) \rangle \rangle} \Big\downarrow & & \Big\downarrow {\scriptstyle \otimes} \\ \dot{C}_3 \times_{s_2,t_2} \dot{C}_3 & \xrightarrow[\#_2]{} & \dot{C}_3 \end{array} \tag{2.84}$$

$$\begin{array}{ccc} (C_2 \times_{s_1,t_1} C_2) \times_{s_0,t_0} C_2 & \xrightarrow{(\#_1) \times C_2} & C_2 \times_{s_0,t_0} C_2 \\ {\scriptstyle \langle \#_1 \langle \otimes (p_0 \times C_2), \#_0 (p_1 \times C_2) \rangle \rangle, \#_1 \langle \#_0 (p_0 \times t_1), \otimes (p_1 \times C_2) \rangle \rangle} \Big\downarrow & & \Big\downarrow {\scriptstyle \otimes} \\ \dot{C}_3 \times_{s_2,t_2} \dot{C}_3 & \xrightarrow[\#_2]{} & \dot{C}_3 \end{array} \tag{2.85}$$

- Associativity of the $\#_0$ compositions [Crans 1999, (ix)]

$$\begin{array}{ccc} C_2 \times_{s_0,t_0} C_2 \times_{s_0,t_0} C_1 & \xrightarrow{\otimes \times C_1} & \dot{C}_3 \times_{s_0,t_0} C_1 \\ {\scriptstyle C_2 \times \#_0} \Big\downarrow & & \Big\downarrow {\scriptstyle \#_0} \\ C_2 \times_{s_0,t_0} C_2 & \xrightarrow[\otimes]{} & \dot{C}_3 \end{array} \tag{2.86}$$

$$\begin{array}{ccc} C_2 \times_{s_0,t_0} C_1 \times_{s_0,t_0} C_2 & \xrightarrow{\#_0 \times C_2} & C_2 \times_{s_0,t_0} C_2 \\ {\scriptstyle C_2 \times \#_0} \Big\downarrow & & \Big\downarrow {\scriptstyle \otimes} \\ C_2 \times_{s_0,t_0} C_2 & \xrightarrow[\otimes]{} & \dot{C}_3 \end{array} \tag{2.87}$$

$$\begin{array}{ccc} C_1 \times_{s_0,t_0} C_2 \times_{s_0,t_0} C_2 & \xrightarrow{C_1 \times \otimes} & C_1 \times_{s_0,t_0} \dot{C}_3 \\ {\scriptstyle \#_0 \times C_2} \Big\downarrow & & \Big\downarrow {\scriptstyle \#_0} \\ C_2 \times_{s_0,t_0} C_2 & \xrightarrow[\otimes]{} & \dot{C}_3 \end{array} \tag{2.88}$$

- Tensoring is unital

$$\begin{array}{ccc} C_2 \times_{s_0,t_0} C_1 & \xrightarrow{\mathrm{id}_2 \times C_2} & C_2 \times_{s_0,t_0} C_2 \\ {\scriptstyle \#_0} \Big\downarrow & & \Big\downarrow {\scriptstyle \otimes} \\ C_2 & \xrightarrow[\mathrm{id}_2]{} & C_3 \end{array} \tag{2.89}$$

$$\begin{array}{ccc} C_1 \times_{s_0,t_0} C_2 & \xrightarrow{C_2 \times \mathrm{id}_2} & C_2 \times_{s_0,t_0} C_2 \\ {\scriptstyle \#_0} \Big\downarrow & & \Big\downarrow {\scriptstyle \otimes} \\ C_2 & \xrightarrow[\mathrm{id}_2]{} & C_3 \end{array} \tag{2.90}$$

This encodes [Crans 1999, (viii)].

Definition 5 *A* Gray**-functor** *is a* Gray*-enriched functor.*

Internally this means of course that a Gray-functor between Gray-categories is a map of globular sets, that preserves all the above operations.

Chapter 3

Resolution in Dimension One

We define a resolution of the 1-dimensional structure of a Gray-category using a comonad, by lifting the free category comonad called "path" in [Dawson et al. 2006] to Gray-categories; but note that we use the term in a different way in this paper.

The resulting co-Kleisli category can be seen as the category of Gray-categories with an enlarged repertoire of maps, that is flexible enough to carry out our path space construction. After giving an abstract construction of this category of pseudo maps we proceed to characterize them explicitly.

3.1 Basic Fibrations

There are obvious functors

$$\mathsf{GrayCat} \xrightarrow{(\_)_2} \mathsf{SesquiCat} \xrightarrow{(\_)_1} \mathsf{Cat} \xrightarrow{(\_)_0} \mathsf{Set} \tag{3.1}$$

that forget the 3-cells, the 2-cells and 1-cells respectively. The last one will not play an explicit role here.

Let $\mathfrak{S}$ be a sesquicategory, $\mathbb{G}$ a Gray-category, and $F\colon \mathfrak{S} \longrightarrow \mathbb{G}_2$ a sesquifunctor. We define $\overline{F}\colon F^*\mathfrak{S} \longrightarrow \mathbb{G}$ as follows:

$$(F^*\mathfrak{S})_0 = \mathfrak{S}_0 \tag{3.2}$$
$$(F^*\mathfrak{S})_1 = \mathfrak{S}_1 \tag{3.3}$$
$$(F^*\mathfrak{S})_2 = \mathfrak{S}_2 \tag{3.4}$$
$$(F^*\mathfrak{S})_3 = \{(\Gamma; \alpha, \beta) | \Gamma\colon F\alpha \longrightarrow F\beta\} \tag{3.5}$$

Note that the interchange of two 2-cells α, β in $F^*\mathfrak{S}$ incident on a 0-cell is given essentially by the interchange of their images under F:

$$\beta \otimes \alpha = (F\beta \otimes F\beta; \beta \triangleright \alpha, \beta \triangleleft \alpha)\,. \tag{3.6}$$

Let us take note of the following useful fact that helps to characterize the Cartesian maps:

Remark 6 *For a functor $p\colon \mathsf{E} \longrightarrow \mathsf{B}$ that preserves co-limits, let $D\colon \mathsf{D} \longrightarrow \mathsf{E}$ a diagram in E with co-limit (C, k_i)*

$$\begin{array}{ccccc} D_i & \xrightarrow{k_i} & C & & \\ & & & \searrow^{g} & \\ & & A & \xrightarrow[f]{} & B \end{array} , \tag{3.7}$$

assume $p(g)$ factors below as $p(f)u = p(g)$. Furthermore, assume that the induced sink $(u_i) = up(k_i)$ has fillers $\langle u_i \rangle$ above with $f\langle u_i \rangle = gk_i$, then the co-universally induced map $\langle u \rangle : C \longrightarrow A$ is a filler over u.

This means that to check whether a map f is Cartesian we don't need to give the filler u directly, but we can define it on presumably simpler parts of C. These then combine into a valid filler.

Remark 7 *Maps Cartesian with respect to $(\_)_2$ are exactly the* Gray*-functors, that are 2-locally isomorphisms of sets. That is, given two parallel 2-cells on the intervening 3-cells the map is bijective.*

Lemma 8 *$F^*\mathfrak{G}$ is a* Gray*-category, $\overline{F}$ is a* Gray*-functor and Cartesian with respect to $(\_)_2$.* □

Similarly, let $\mathfrak{S}$ a sesquicategory and C a category, $F\colon \mathsf{C} \longrightarrow \mathfrak{S}_1$ a functor, then we define a sesquicategory:

$$(F^*\mathsf{C})_0 = \mathsf{C}_0 \tag{3.8}$$
$$(F^*\mathsf{C})_1 = \mathsf{C}_1 \tag{3.9}$$
$$(F^*\mathsf{C})_2 = \{(\alpha; f, g) | \alpha\colon Ff \longrightarrow Fg\} \tag{3.10}$$

Lemma 9 *$F^*\mathsf{C}$ is a sesquicategory, $\overline{F}$ is a sesquifunctor, and Cartesian with respect to $(\_)_1$.* □

Remark 10 *Maps Cartesian with respect to $(\_)_1$ are exactly the sesquifunctors, that are 1-locally isomorphisms of sets. That is, given two parallel 1-cells on the intervening 2-cells the map is bijective.*

We will denote the composite $(\_)_1(\_)_2$ also by $(\_)_1$, it is of course a fibration as well. For later reference we describe its Cartesian liftings explicitly as well. Let $\mathbb{G}$ be a Gray-category, $\mathbb{G}_1$ its underlying category. Let C be an ordinary category and $F\colon \mathsf{C} \longrightarrow \mathbb{G}_1$ a functor. Then $F^*\mathbb{G}$ is given by:

$$(F^*\mathbb{G})_0 = \mathsf{C}_0 \tag{3.11}$$
$$(F^*\mathbb{G})_1 = \mathsf{C}_1 \tag{3.12}$$
$$(F^*\mathbb{G})_2 = \{(\alpha; f, g) | f, g\colon x \longrightarrow y,\, \alpha\colon Ff \longrightarrow Fg\} \tag{3.13}$$
$$(F^*\mathbb{G})_3 = \{(\Gamma; \alpha, \beta; f, g) | f, g\colon x \longrightarrow y,\, \Gamma\colon F\alpha \longrightarrow F\beta\} \tag{3.14}$$

Source and target maps are as follows:

$$s_2(\Gamma; \alpha, \beta; f, g) = (\alpha; f, g) \qquad t_2(\Gamma; \alpha, \beta; f, g) = (\beta; f, g) \tag{3.15}$$

$$s_1(\alpha; f, g) = f \qquad t_1(\alpha; f, g) = g\,. \tag{3.16}$$

and s_0, t_0 are as given by C. As identities we take:

$$i_1(f) = (\mathrm{id}_{Ff}; f, f) \quad i_2(\alpha; f, g) = (\mathrm{id}_\alpha; \alpha, \alpha, f, g)\,. \tag{3.17}$$

The tensor in $F^*\mathbb{G}$ of two 2-cells is

$$(\beta; g, g')\otimes(\alpha; f, f') = (\beta\otimes\alpha; \beta \triangleleft \alpha, \beta \triangleright \alpha; g\#_0 f, g'\#_0 f') \tag{3.18}$$

where

$$\beta \triangleleft \alpha = (\beta\#_0 Ff')\#_1(Fg\#_0\alpha), \quad \beta \triangleright \alpha = (Fg'\#_0\alpha)\#_1(\beta\#_1 Ff)\,. \tag{3.19}$$

There is an obvious map $\overline{F}\colon F^*\mathbb{G} \longrightarrow \mathbb{G}$ over F that acts like F on 0- and 1-cells, and on 2- and 3-cells as a projection to $\mathbb{G}$.

Remark 11 *The globular set $F^*\mathbb{G}$ is a* Gray*-category. The composition operations of $F^*\mathbb{G}$ are given by those of* C *and $\mathbb{G}$ and it is easy to see that they fulfill the axioms of a* Gray*-category.*

Obviously $G^*F^*\mathbb{G} \cong (FG)^*\mathbb{G}$ and $\mathrm{id}^*_{\mathsf{C}} \cong \mathrm{id}_{\mathsf{GrayCat_C}}$ coherently. Also, we can always choose $\mathrm{id}^*_{\mathsf{C}} = \mathrm{id}_{\mathsf{GrayCat_C}}$, but this is not necessary in what follows.

Lemma 12 *A map of* Gray*-categories is Cartesian with respect to $\mathbb{G} \mapsto \mathbb{G}_1$ iff it is 1-locally an isomorphism of categories, i.e. given two parallel 1-cells the map is bijective on the intervening 2-cells and in turn bijective on the 3-cells between parallel such.* □

Definition 13 *We define a map of* Gray*-categories to be an* n**-isomorphism** *if it is Cartesian with respect to $(\_)_n$. It is* n**-faithful** *if fillers of factorizations under $(\_)_n$ are unique, and* n**-full** *is there (not necessarily unique) fillers for all factorizations under $(\_)_n$.*

With this definition 0-fidelity is ordinary fidelity of functors, 1-fidelity is local fidelity, and so on.

Remark 14 *One property of Cartesian maps in a fibration p that we are going to exploit in the proof of the following theorem is that for three arrows upstairs,*

$$\underset{s}{\overset{r}{\rightrightarrows}} \xrightarrow{\;f\;} \tag{3.20}$$

with f Cartesian, $p(r) = p(s)$ downstairs and $fr = fs$ upstairs imply $r = s$, on account of f being p-faithful.

Lemma 15 *If fg is Cartesian with respect to a given fibration p and f is p-faithful, then g is p-Cartesian.*

Proof Assume k and u such that $p(g)u = p(k)$, then $p(fg)u = p(fk)$ and hence by fg being p-full there is a filler $\langle u\rangle$ such that $fg\,\langle u\rangle = fk$. Then by f being p-faithful $g\,\langle u\rangle = k$.

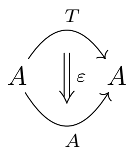
By fg being p-faithful $\langle u\rangle$ is the unique such filler. □

3.2 Comonad Liftings

Definition 16 *In a arbitrary 2-category a* **comonad** *on an object A is given by an endomorphism*

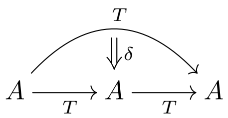
$$A \xrightarrow{T} A \tag{3.21}$$

and 2-cells



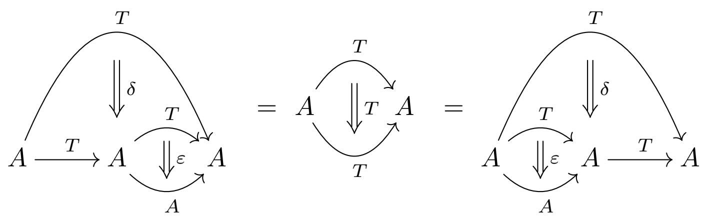
(3.22)

and



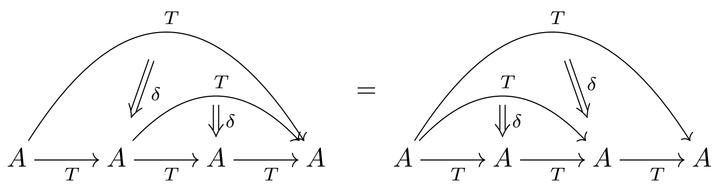
(3.23)

such that



(3.24)

and



. (3.25)

See, for example, Mac Lane [1998].

If A is a category, T a functor and ε and δ natural transformations these equations of course amount to the usual equations objectwise:

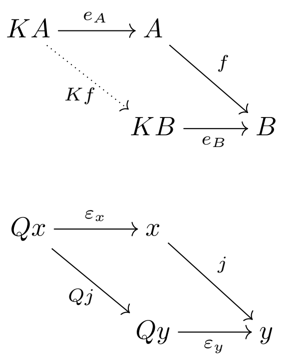
$$\begin{array}{ccccc} & & Tx & & \\ & \swarrow^{Tx} & \downarrow{\scriptstyle \delta_x} & \searrow^{Tx} & \\ Tx & \xleftarrow[\varepsilon_{Tx}]{} & TTx & \xrightarrow[T\varepsilon_x]{} & Tx \end{array} \tag{3.26}$$

and

$$\begin{array}{ccc} Tx & \xrightarrow{\delta_x} & TTx \\ {\scriptstyle \delta_x}\downarrow & & \downarrow{\scriptstyle T\delta_x} \\ TTx & \xrightarrow[\delta_{Tx}]{} & TTTx \end{array} . \tag{3.27}$$

Theorem 17 *Given a fibration of categories $p\colon \mathsf{E} \longrightarrow \mathsf{B}$, a comonad (Q, δ, ε) on B can be lifted to a comonad (K, d, e) on E such that $(K, Q)\colon p \longrightarrow p$ is a comonad in the 2-category of all fibrations.*

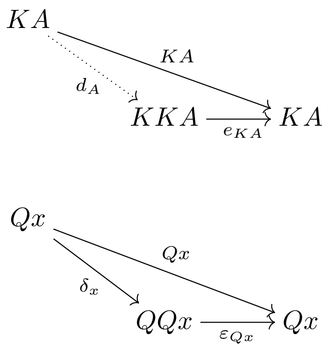
Proof Let $(\_)^*\colon \mathsf{B}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ be a chosen cleavage. For every $A \in \mathsf{E}_x$ we let $e_A\colon (KA = \varepsilon_x^* A) \longrightarrow A$ be the chosen Cartesian lift of $\varepsilon_x\colon Qx \longrightarrow x$. For a morphism f over j in



(3.28)

the dotted arrow is the unique filler induced by the factorization below. This makes K a functor and $e\colon K \longrightarrow \mathrm{id}_{\mathsf{E}}$ a natural transformation.

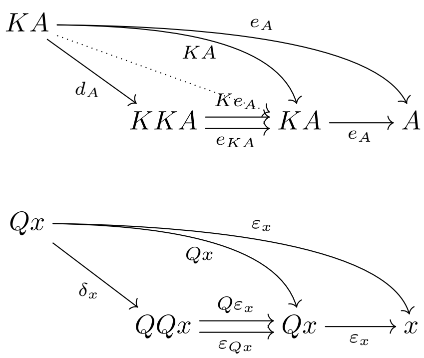
We define a family of co-multiplication maps d_A as the unique fillers in



(3.29)

where the triangle below commutes because is Q co-unital.

In the diagram



(3.30)

we see that $e_A e_{KA} d_A = e_A K e_A d_A$ by the naturality of e, and $p(e_{KA} d_A) = p(K e_A d_A)$ by Q being a monad. Hence by 14 the three endomorphisms of KA above have to coincide, meaning d is co-unital component wise.

The naturality of d, that is, that $d_B Kf = KKf d_A$ is the unique filler making the left-hand upstairs square commute

$$\begin{array}{cccccc} KA & \xrightarrow{d_A} & KKA & & & \\ & \searrow^{Kf} & \cdots & \searrow^{KKf} & & \\ & & KB & \xrightarrow[d_B]{} & KKB & \xrightarrow[e_{KB}]{} KB \end{array} \tag{3.31}$$

$$\begin{array}{cccccc} Qx & \xrightarrow{\delta_x} & QQx & & & \\ & \searrow^{Qj} & & \searrow^{QQj} & & \\ & & Qy & \xrightarrow[\delta_y]{} & QQy & \xrightarrow[\varepsilon_{Qy}]{} Qy \end{array}$$

is obtained by observing that $e_{KB} d_B Kf = KF = Kf e_{KA} d_A = e_{KB} KKf d_A$, from e being natural and a retraction. Also, $p(d_B Kf) = p(KKf d_a)$ by naturality of δ. We apply 14 again.

Finally, we show that d is co-associative:

$$\begin{array}{cccccc} KA & \xrightarrow{d_A} & KKA & & & \\ & \searrow^{d_A} & \cdots & \searrow^{d_{KA}} & & \\ & & KKA & \xrightarrow[Kd_A]{} & KKKA & \xrightarrow[e_{KKA}]{} KKA \end{array} \tag{3.32}$$

$$\begin{array}{cccccc} Qx & \xrightarrow{\delta_x} & QQx & & & \\ & \searrow^{\delta_x} & & \searrow^{\delta_{Qx}} & & \\ & & QQx & \xrightarrow[Q\delta_x]{} & QQQx & \xrightarrow[\varepsilon_{QQx}]{} QQx \end{array}$$

we calculate that $e_{KKA} K d_A d_A = d_A e_{KA} d_A = d_A = e_{KKA} d_{KA} d_A$, again by naturality of e and its retractiveness. Moreover, δ is co-associative, hence we can apply 14 once more. □

We observe that K preserves Cartesianness of maps, hence in particular Ke is Cartesian component wise.

Finally we can define our resolution comonad. Let $(Q, \delta, \varepsilon) = (FU, F\eta U, \varepsilon)$ be the comonad that arises from the adjunction

$$\mathsf{RGrph} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{Cat} \quad (\bot) \quad . \tag{3.33}$$

Then, according to theorem 17 we obtain the comonad (Q^1, d, e) on $\mathsf{GrayCat}$ induced by lifting Q along $(\_)_1$. The exponent reminds us that this provides a resolution of the 1-dimensional structure of Gray-categories. See 8 for a more abstract point of view on this construction. In section 3.3 we will show explicitly how this comonad acts.

Corollary 18 *By the above theorem there is a comonad* Q^1 *on* GrayCat *that pulls back the* Gray*-structure onto the free category on the underlying 1-graph.*

Definition 19 *The category of* Gray*-categories and* **pseudo Gray-maps** *is the co-Kleisli-category* $\mathsf{GrayCat}_{\mathrm{Q}^1}$ *of the comonad* Q^1*.*

This category has Gray-categories as objects, and morphisms

$$\mathbb{G} \overset{f}{\not\longrightarrow} \mathbb{H} \qquad \text{are morphisms} \qquad \mathrm{Q}^1\mathbb{G} \xrightarrow{f} \mathbb{H} \tag{3.34}$$

in GrayCat. Composition of two maps

$$\mathbb{G} \overset{f}{\not\longrightarrow} \mathbb{H} \overset{g}{\not\longrightarrow} \mathbb{K} \tag{3.35}$$

is defined by

$$\mathrm{Q}^1\mathbb{G} \xrightarrow{d_{\mathbb{G}}} \mathrm{Q}^1\mathrm{Q}^1\mathbb{G} \xrightarrow{\mathrm{Q}^1 f} \mathrm{Q}^1\mathbb{H} \xrightarrow{g} \mathbb{K}\,. \tag{3.36}$$

Identities are of the form

$$\mathbb{G} \overset{\mathrm{id}_{\mathbb{G}}}{\not\longrightarrow} \mathbb{G} \;=\; \mathrm{Q}^1\mathbb{G} \xrightarrow{e_{\mathbb{G}}} \mathbb{G}\,. \tag{3.37}$$

By way of notational convenience in diagrams in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ we use unslashed arrows $f\colon \mathbb{G} \longrightarrow \mathbb{H}$ to denote a strict arrow that is included in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ as $fe\colon \mathbb{G} \not\rightarrow \mathbb{H}$.

The comonad axioms make sure this is a category; c.f. e.g. [Mac Lane 1998].

There is an adjunction

$$\mathsf{GrayCat} \underset{L}{\overset{R}{\rightleftarrows}} \mathsf{GrayCat}_{\mathrm{Q}^1} \quad (L \dashv R) \tag{3.38}$$

The functor R takes a strict map $f\colon \mathbb{G} \longrightarrow \mathbb{H}$ to a pseudo map $fe\colon \mathbb{G} \not\rightarrow \mathbb{H}$ where e is the co-unit of Q^1. Moreover, since e is an epimorphism, R is faithful, and it is bijective on objects, hence R is actually an inclusion.

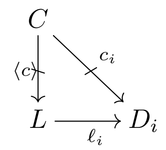
We note that the composite of a strict map after a pseudo map is particularly simple:

$$\mathbb{G} \overset{f}{\not\longrightarrow} \mathbb{H} \overset{ge}{\not\longrightarrow} \mathbb{K} \;=\; \begin{array}{ccccccc} \mathrm{Q}^1\mathbb{G} & \xrightarrow{d_{\mathrm{Q}^1\mathbb{G}}} & \mathrm{Q}^1\mathrm{Q}^1\mathbb{G} & \xrightarrow{\mathrm{Q}^1 f} & \mathrm{Q}^1\mathbb{H} & \xrightarrow{ge} & \mathbb{K} \\ & \searrow\!\!\searrow & \downarrow{\scriptstyle e_{\mathrm{Q}^1\mathbb{G}}} & & \downarrow{\scriptstyle e_{\mathbb{H}}} & \nearrow{\scriptstyle g} & \\ & & \mathrm{Q}^1\mathbb{G} & \xrightarrow[f]{} & \mathbb{H} & & \end{array}\,. \tag{3.39}$$

Lemma 20 *The category* $\mathsf{GrayCat}_{\mathrm{Q}^1}$ *has all limits of diagrams of strict maps, that is, those in the subcategory* GrayCat*, that is,* GrayCat *is complete and the inclusion* $\mathsf{GrayCat} \longrightarrow \mathsf{GrayCat}_{\mathrm{Q}^1}$ *preserves all limits.*

Proof Let D be a diagram in GrayCat, let $(\ell_i\colon L \longrightarrow D_i)_i$ be a limiting source in GrayCat, we claim its embedding into $\mathsf{GrayCat}_{\mathrm{Q}^1}$ is a limiting source there as well.

Let $(c_i\colon C \twoheadrightarrow D_i)_i$ be a source over D in $\mathsf{GrayCat}_{\mathrm{Q}^1}$. Thus there is a source $(c_i\colon \mathrm{Q}^1 C \longrightarrow D_i)_i$ in GrayCat, which induces a map $\langle c\rangle\colon \mathrm{Q}^1 C \longrightarrow L$ and this is of course a map $\langle c\rangle\colon C \twoheadrightarrow L$. The diagram

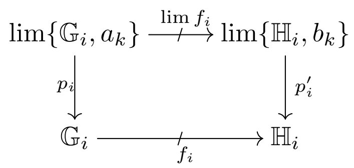


(3.40)

commutes for all i by the co-unit axiom of Q^1 and the naturality of e; c. f. also (3.39). Because e is an epimorphism $\langle c\rangle$ is the unique filler. □

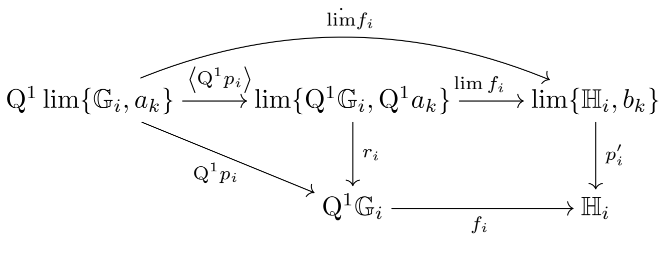
In particular, the pullback of two strict maps in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ is the same as its pullback in GrayCat. Products are obviously simply the same in both categories since their diagrams do not include any nontrivial morphisms.

Remark 21 *For two diagrams* $\{a_k\colon \mathbb{G}_i \longrightarrow \mathbb{G}_j\}$, $\{b_k\colon \mathbb{H}_i \longrightarrow \mathbb{H}_j\}$ *of strict maps of the same type in* $\mathsf{GrayCat}_{\mathrm{Q}^1}$ *and a natural transformation* $f_i\colon \mathbb{G}_i \twoheadrightarrow \mathbb{H}_i$ *between them there is an induced map* $\dot{\lim}\{f_i\}$ *such that:*



(3.41)

We unravel this diagram in terms of maps in GrayCat *and obtain*



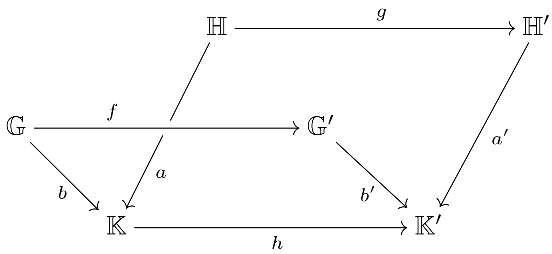
(3.42)

where the map $\dot{\lim} f_i$ *is induced by the universal property of the source* $\{f_i \mathrm{Q}^1 p_i\}$ *in* GrayCat, *that is,* $\dot{\lim}\{f_i\} = \langle f_i \mathrm{Q}^1 p_i\rangle$, *which then is the appropriate map in* $\mathsf{GrayCat}_{\mathrm{Q}^1}$. *On the other hand,* $\lim f_i$ *is induced by the cone* $f_i r_i$. *By universality* $\dot{\lim} f_i = \lim f_i \langle \mathrm{Q}^1 p_i\rangle$.

In particular this applies to pullbacks, that is, there is a canonical map

$$f \dot{\times} g\colon \mathbb{G} \times_{\mathbb{K}} \mathbb{H} \twoheadrightarrow \mathbb{G}' \times_{\mathbb{K}'} \mathbb{H}' \tag{3.43}$$

determined by f, g, h in



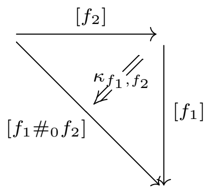
. (3.44)

3.3 Special Cells in the Resolved Space

We now take a closer look at the structure of $\mathrm{Q}^1\mathbb{G}$. By definition 1-cells here are non-empty lists $[f_1, \dots, f_n]$ of composable $\mathbb{G}$-1-cells modulo insertion or removal of identity 1-cells of $\mathbb{G}$; composition is concatenation. For composable 1-cells in $\mathbb{G}$, say, $f_1, \dots, f_n$ we have several 1-cells in $\mathrm{Q}^1\mathbb{G}$, in particular $[f_1, \dots, f_n] = [f_1]\#_0 \cdots \#_0[f_n]$ and $[f_1\#_0 \cdots \#_0 f_n]$ and $e_{\mathbb{G}}$ maps all of these to $f_1\#_0 \cdots \#_0 f_n$. Between $[f_1, \dots, f_n]$ and $[f_1\#_0 \cdots \#_0 f_n]$ we have a 2-cell

$$\kappa_{f_1,\dots,f_n} = (\mathrm{id}_{f_1\#_0\cdots\#_0 f_n}; [f_1, \dots, f_n], [f_1\#_0 \cdots \#_0 f_n]) \tag{3.45}$$

that is the pulled back identity 2-cell of $f_1\#_0 \cdots \#_0 f_n$. In particular we have



(3.46)

for all for all pairs f_1, f_2 of 1-cells of $\mathbb{G}$. Whiskers and composites of higher cells in $\mathrm{Q}^1\mathbb{G}$ are simply carried out in $\mathbb{G}$, hence for example

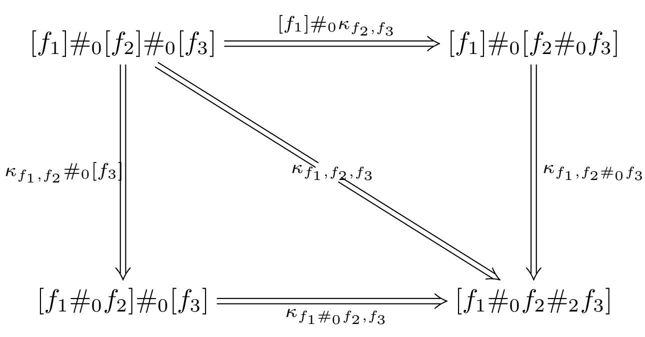
$$\kappa_{f_1,f_2}\#_0[f_3] = (\mathrm{id}_{f_1\#_0 f_2}\#_0 f_3; [f_1, f_2]\#_0[f_3], [f_1\#_0 f_2]\#_0[f_3]) \tag{3.47}$$

$$= (\mathrm{id}_{f_1\#_0 f_2\#_0 f_3}; [f_1, f_2, f_3], [f_1\#_0 f_2, f_3]) \tag{3.48}$$

and

$$\kappa_{f_1\#_0 f_2,f_3}\#_1\,(\kappa_{f_1,f_2}\#_0[f_3]) = (\mathrm{id}_{f_1\#_0 f_2\#_0 f_3}; [f_1, f_2, f_3], [f_1\#_0 f_2\#_0 f_3]) = \kappa_{f_1,f_2,f_3}\,. \tag{3.49}$$

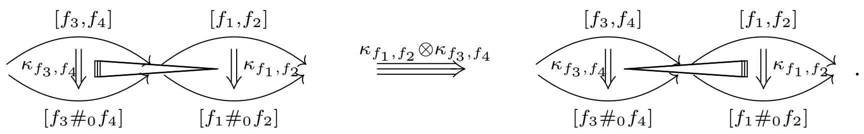
Hence we obtain that



(3.50)

commutes.

We consider the possible horizontal composites of κ_{f_1,f_2} and κ_{f_3,f_4} and their tensor:



(3.51)

By (3.18) we obtain

$$\begin{aligned}\kappa_{f_1,f_2}\otimes\kappa_{f_3,f_4} &= (\mathrm{id}_{f_1\#_0f_2};[f_1,f_2],[f_1\#_0f_2])\otimes(\mathrm{id}_{f_3\#_0f_4};[f_3,f_4],[f_3\#_0f_4])\\ &= \begin{pmatrix}\mathrm{id}_{f_1\#_0f_2}\otimes\mathrm{id}_{f_3\#_0f_4};\\ (\mathrm{id}_{f_1\#_0f_2}\#_0e[f_3\#_0f_4])\#_1(e[f_1,f_2]\#_0\mathrm{id}_{f_3\#_0f_4}),\\ (e[f_1\#_0f_2]\#_0\mathrm{id}_{f_3\#_0f_4})\#_1(\mathrm{id}_{f_1\#_0f_2}\#_0e[f_3,f_4]);\\ [f_1,f_2,f_3,f_4],[f_1\#_0f_2,f_3\#_0f_4]\end{pmatrix}\\ &= \begin{pmatrix}\mathrm{id}_{\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}};\\ (\mathrm{id}_{f_1\#_0f_2}\#_0f_3\#_0f_4)\#_1(f_1\#_0f_2\#_0\mathrm{id}_{f_3\#_0f_4}),\\ (f_1\#_0f_2\#_0\mathrm{id}_{f_3\#_0f_4})\#_1(\mathrm{id}_{f_1\#_0f_2}\#_0f_3\#_0f_4);\\ [f_1,f_2,f_3,f_4],[f_1\#_0f_2,f_3\#_0f_4]\end{pmatrix}\\ &= \begin{pmatrix}\mathrm{id}_{\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}};\\ (\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4})\#_1(\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}),\\ (\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4})\#_1(\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4});\\ [f_1,f_2,f_3,f_4],[f_1\#_0f_2,f_3\#_0f_4]\end{pmatrix}\\ &= \begin{pmatrix}\mathrm{id}_{\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}};\\ \mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4},\\ \mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4};\\ [f_1,f_2,f_3,f_4],[f_1\#_0f_2,f_3\#_0f_4]\end{pmatrix},\end{aligned} \tag{3.52}$$

meaning that this tensor is the identity of the two possible horizontal composites of κ_{f_1,f_2} and κ_{f_3,f_4}.

Finally, note that by construction the $\kappa_{f_1,\dots,f_n}$ are all invertible.

3.4 Pseudo Maps Explicitly

We provide an elementary characterization of pseudo Gray-functors.

Definition 22 *A* **pseudo** Q^1 **graph map** $F\colon \mathbb{G}\longrightarrow\mathbb{H}$ *between* Gray*-categories is a map of 3-globular sets, together with a function* $F^2\colon \mathbb{G}_1\times_{\mathbb{G}_0}\mathbb{G}_1\longrightarrow\mathbb{H}_2$*, such that the following conditions hold:*

1. *the restriction of* F *to* $\mathbb{G}(x,y)$ *is a sesquifunctor for all 0-cells* x,y *of* $\mathbb{G}$*,*
2. F^2 *is a normalized 2-cocycle, that is, the* $F^2_{f_1,f_2}$ *are invertible 2-cells* $F^2_{f_1,f_2}\colon F(f_1)\#_0F(f_2)\Longrightarrow F(f_1\#_0f_2)$ *with*

$$F^2_{f_1,f_2\#_0f_3}\#_1(F(f_1)\#_0F^2_{f_2,f_3}) = F^2_{f_1\#_0f_2,f_3}\#_1(F^2_{f_1,f_2}\#_0F(f_3)), \tag{3.53}$$

and for f_1 *or* f_2 *an identity 1-cell we have*

$$F^2_{f_1,f_2} = \mathrm{id}_{f_1\#_0f_2}, \tag{3.54}$$

3. *left and right whiskers of 2-cells by 1-cells along 0-cells are coherently preserved:*

$$F(\alpha\#_0 f)\#_1 F^2_{g,f} = F^2_{g',f}\#_1(F\alpha\#_0 Ff) \tag{3.55}$$
$$F(g\#_0\beta)\#_1 F^2_{g,f} = F^2_{g,f'}\#_1(Fg\#_0 F\beta)$$

4. *left and right whiskers of 3-cells by 1-cells along 0-cells are coherently preserved:*

$$F(\Gamma\#_0 f)\#_1 F^2_{g,f} = F^2_{g',f}\#_1(F\Gamma\#_0 Ff) \tag{3.56}$$
$$F(g\#_0\Delta)\#_1 F^2_{g,f} = F^2_{g,f'}\#_1(Fg\#_0 F\Delta)$$

5. *the tensor is coherently preserved:*

$$F(\beta\otimes\alpha)\#_1 F^2_{g,f} = F^2_{g',f'}\#_1(F\beta\otimes F\alpha) \tag{3.57}$$

6. *the tensors of compositors are trivial:*

$$\left(F^2_{f_1,f_2} \lhd F^2_{f_3,f_4} \xRightarrow{F^2_{f_1,f_2}\otimes F^2_{f_3,f_4}} F^2_{f_1,f_2} \rhd F^2_{fe_3,f_4} \right) = \mathrm{id} \tag{3.58}$$

7. *tensors of 2-co-cycle elements with images of 2-cells vanish:*

$$\left(F\alpha \lhd F^2_{g,f} \xRightarrow{F\alpha\otimes F^2_{g,f}} F\alpha \rhd F^2_{g,f} \right) = \mathrm{id} \tag{3.59}$$
$$\left(F^2_{h,g} \lhd F\beta \xRightarrow{F^2_{h,g}\otimes F\beta} F^2_{h,g} \rhd F\beta \right) = \mathrm{id} \tag{3.60}$$

for all suitably incident cells.

Note how this definition implies that the horizontal composites are also coherently preserved as a consequence of (3.55):

$$F(\alpha\lhd\beta)\#_1 F^2_{g,f} = F^2_{g',f'}\#_1(F\alpha\lhd F\beta) \tag{3.61}$$
$$F(\alpha\rhd\beta)\#_1 F^2_{g,f} = F^2_{g',f'}\#_1(F\alpha\rhd F\beta)\,.$$

Lemma 23 *There is a canonical correspondence between the set of pseudo* Q^1 *graph maps* $\mathbb{G}\longrightarrow\mathbb{H}$ *and* $\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},\mathbb{H})$*.*

PROOF Given a Q^1 graph map $F\colon\mathbb{G}\longrightarrow\mathbb{H}$ we define a Gray-functor $\tilde F\colon \mathrm{Q}^1\mathbb{G}\longrightarrow\mathbb{H}$ as follows

1. 0-cells:
$$\tilde F(x) = F(x), \tag{3.62}$$

2. 1-cells:
$$\tilde F[f_1,\ldots,f_n] = Ff_1\#_0\cdots\#_0 Ff_n, \tag{3.63}$$

3. 2-cells:

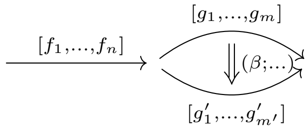
$$\tilde{F}(\alpha;[f_1,\dots,f_n],[g_1,\dots,g_m]) = \overline{\tilde{F}\kappa_{g_1,\dots,g_m}}\#_1 F\alpha \#_1 \tilde{F}\kappa_{f_1,\dots,f_n} \quad (3.64)$$

   where for $n=2$ the 2-cell $\tilde{F}\kappa_{f_1,\dots,f_n}$ is defined as $F^2_{f_1,f_2}$ and for $n \geq 3$ as the unique extension due to (3.53), (3.58),

4. 3-cells:

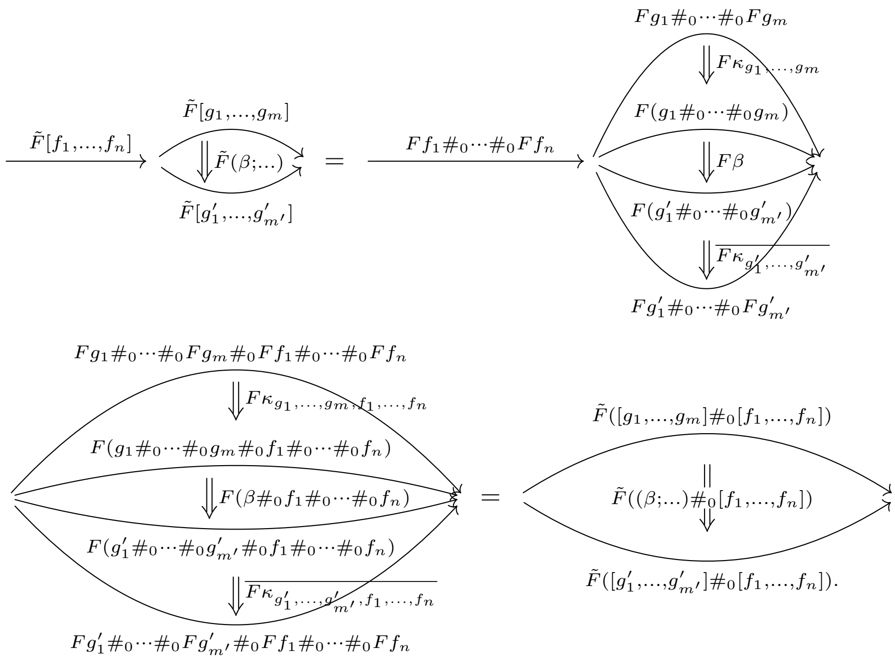
$$\tilde{F}(\Gamma;\alpha,\beta;[f_1,\dots,f_n],[g_1,\dots,g_m]) = \overline{\tilde{F}\kappa_{g_1,\dots,g_m}}\#_1 F\Gamma \#_1 \tilde{F}\kappa_{f_1,\dots,f_n}\,. \quad (3.65)$$

To elucidate, we show that 1-2-whiskers are preserved by $\tilde{F}$. For whiskerable cells



(3.66)

the equation



(3.67)

is a consequence of (3.64).

Similarly, we can verify that $\tilde{F}$ preserves tensors:

$$\begin{aligned}
&\tilde{F}(\beta;[g_1,\dots,g_m],[g'_1,\dots,g'_{m'}])\otimes(\alpha;[f_1,\dots,f_n],[f'_1,\dots,f'_{n'}])\\
&\quad = \tilde{F}\left(\beta\otimes\alpha;\beta\triangleleft\alpha,\beta\triangleright\alpha;[g_1,\dots,g_m,f_1,\dots,f_n],[g'_1,\dots,g'_{m'},f'_1,\dots,f'_{n'}]\right)\\
&\quad = \overline{\tilde{F}\kappa_{g'_1,\dots,g'_{m'},f'_1,\dots,f'_{n'}}}\#_1 F(\beta\otimes\alpha)\#_1\tilde{F}_{g_1,\dots,g_m,f_1,\dots,f_n}\\
&\quad = (\overline{\tilde{F}\kappa_{g'_1,\dots,g'_{m'}}}\otimes\overline{\tilde{F}\kappa_{f'_1,\dots,f'_{n'}}})\#_1(F\beta\otimes F\alpha)\#_1(\tilde{F}_{g_1,\dots,g_m}\otimes\tilde{F}_{f_1,\dots,f_n})
\end{aligned}$$

$$= (\overline{\tilde{F}\kappa_{g'_1,\dots,g'_{m'}}}\#_1 F\beta\#_1\tilde{F}_{g_1,\dots,g_m})\otimes(\overline{\tilde{F}\kappa_{f'_1,\dots,f'_{n'}}}\#_1 F\alpha\#_1\tilde{F}_{f_1,\dots,f_n})$$
$$\tilde{F}(\beta;[g_1,\dots,g_m],[g'_1,\dots,g'_{m'}])\otimes\tilde{F}(\alpha;[f_1,\dots,f_n],[f'_1,\dots,f'_{n'}]) \quad (3.68)$$

using (3.57) and (3.58). Preservation of the remaining operations is equally simple to verify.

Conversely, given a Gray-functor $G\colon \mathrm{Q}^1\mathbb{G} \longrightarrow \mathbb{H}$ we define a pseudo Q^1 graph map $\check{G}\colon \mathbb{G} \longrightarrow \mathbb{H}$ as follows:

1. 0-cells: $\check{G}(x) = G(x)$
2. 1-cells: $\check{G}(f) = G[f]$
3. 2-cells: $\check{G}(\alpha) = G(\alpha; [f], [f'])$
4. 3-cells: $\check{G}(\Gamma) = G(\Gamma; \alpha, \beta; [f], [f'])$
5. 2-co-cycle: $\check{G}^2_{f_1,f_2} = G\kappa_{f_1,f_2} = G(\mathrm{id}_{f_1\#_0 f_2}; [f_1\#_0 f_2], [f_1, f_2])$

This is obviously locally a sesquifunctor. We check the co-cycle condition:

$$\begin{aligned}
&\check{G}^2_{f_1,f_2\#_0 f_3}\#_1(\check{G}f_1\#_0\check{G}^2_{f_2,f_3})\\
&= G(\mathrm{id}_{f_1\#_0 f_2\#_0 f_3};[f_1,f_2\#_0 f_3],[f_1\#_0 f_2\#_0 f_3])\#_1(G[f_1]\#_0 G(\mathrm{id}_{f_2\#_0 f_3};[f_2,f_3],[f_2\#_0 f_3]))\\
&= G(\mathrm{id}_{f_1\#_0 f_2\#_0 f_3};[f_1,f_2\#_0 f_3],[f_1\#_0 f_2\#_0 f_3])\#_1 G(\mathrm{id}_{f_1\#_0 f_2\#_0 f_3};[f_1,f_2,f_3],[f_1,f_2\#_0 f_3])\\
&= G(id_{f_1\#_0 f_2\#_0 f_3};[f_1,f_2,f_3],[f_1\#_0 f_2\#_0 f_3])\\
&= G(\mathrm{id}_{f_1\#_0 f_2\#_0 f_3};[f_1\#_0 f_2,f_3],[f_1\#_0 f_2\#_0 f_3])\#_1 G(\mathrm{id}_{f_1\#_0 f_2\#_0 f_3};[f_1,f_2,f_3],[f_1\#_0 f_2,f_3])\\
&= G(\mathrm{id}_{f_1\#_0 f_2\#_0 f_3};[f_1\#_0 f_2,f_3],[f_1\#_0 f_2\#_0 f_3])\#_1(G(\mathrm{id}_{f_1\#_0 f_2};[f_1,f_2],[f_1\#_0 f_2])\#_0 G[f_3])\\
&= \check{G}^2_{f_1\#_0 f_2,f_3}\#_1(\check{G}^2_{f_1,f_2}\#_0\check{G}f_3) \quad (3.69)
\end{aligned}$$

Furthermore, we check the coherent preservation of whiskers:

$$\begin{aligned}
&\check{G}(\alpha\#_0 f)\#_1\check{G}^2_{g,f}\\
&= G(\alpha\#_0 f;[g\#_0 f],[g'\#_0 f])\#_1 G(\mathrm{id}_{g\#_0 f};[g,f],[g\#_0 f])\\
&= G(\alpha\#_0 f;[g,f],[g'\#_0 f])\\
&= G(\mathrm{id}_{g'\#_0 f};[g',f],[g'\#_0 f])\#_1 G(\alpha\#_0;[g,f],[g',f])\\
&= G(\mathrm{id}_{g'\#_0 f};[g',f],[g'\#_0 f])\#_1(G(\alpha;[g],[g'])\#_0 G[f])\\
&= \check{G}^2_{g',f}\#_1(\check{G}\alpha\#_0\check{G}f) \quad (3.70)
\end{aligned}$$

The remaining axioms are verified just as easily.

We verify briefly that $\tilde{\check{G}} = G$, for 1-cells we have

$$\tilde{\check{G}}[f_1,\dots,f_n] = \check{G}f_1\#_0\dots\#_0\check{G}f_n = G[f_1]\#_0\dots\#_0 G[f_n] = G[f_1,\dots,f_n] \quad (3.71)$$

and for 2-cells:

$$\begin{aligned}
\tilde{\check{G}}(\alpha;[f_1,\dots,f_n],[f'_1,\dots,f'_{n'}]) &= \overline{\check{G}\kappa_{f'_1,\dots,f'_{n'}}}\#_1\check{G}\#_1\check{G}\kappa_{f_1,\dots,f_n}\\
&= \begin{pmatrix} G(\mathrm{id}_{f'_1\#_0\cdots\#_0 f'_{n'}};[f'_1\#_0\cdots\#_0 f'_{n'}],[f'_1,\dots,f'_{n'}])\\ \#_1 G(\alpha;[f'_1\#_0\cdots\#_0 f'_{n'}],[f_1\#_0\cdots\#_0 f_n])\\ \#_1 G(\mathrm{id}_{f_1\#_0\cdots\#_0 f_n};[f_1,\dots,f_n],[f_1\#_0\cdots\#_0 f_n])\end{pmatrix}
\end{aligned}$$

$$G(\alpha; [f_1, \ldots, f_n], [f'_1, \ldots, f'_{n'}]) \quad (3.72)$$

Finally, $\check{\check{F}} = F$. □

Remark 24 *Given two pseudo* Q^1 *graph maps* $F \colon \mathbb{G} \longrightarrow \mathbb{H}$ *and* $G \colon \mathbb{H} \longrightarrow \mathbb{K}$ *their composite* GF *is simply the composite of the underlying globular maps with cocycle*

$$(GF)^2_{f_1,f_2} = GF^2_{f_1,f_2} \#_1 G^2_{Ff_1,Ff_2} \,. \quad (3.73)$$

Chapter 4

Path Spaces

We construct a path space for Gray-categories and prove some essential properties. We derived the idea for this construction from Bénabou [1967].

Definition 25 *Given a* Gray*-groupoid* $\mathbb{H}$ *we define the* **path space** $\overrightarrow{\mathbb{H}}$ *where the cells in each dimension are diagrams in* $\mathbb{H}$*:*

$$\overrightarrow{\mathbb{H}}_0 = \{ \xrightarrow{\;f\;} \} \tag{4.1}$$

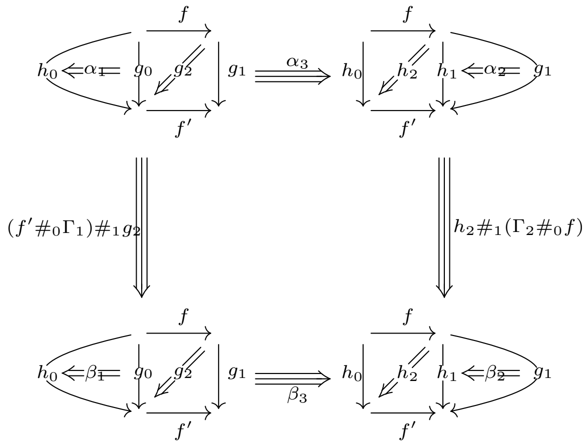
$$\overrightarrow{\mathbb{H}}_1 = \left\{ (g_2; g_0, g_1, f, f') \;\middle|\; \begin{array}{ccc} \cdot & \xrightarrow{\;f\;} & \cdot \\ {\scriptstyle g_0}\big\downarrow & \Swarrow_{g_2} & \big\downarrow{\scriptstyle g_1} \\ \cdot & \xrightarrow[\;f'\;]{} & \cdot \end{array} \right\} \tag{4.2}$$

$$\overrightarrow{\mathbb{H}}_2 = \left\{ \begin{pmatrix} \alpha_3; \alpha_1, \alpha_2, g_2, h_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix} \;\middle|\; \begin{array}{cccc} & \cdot & \xrightarrow{\;f\;} & \cdot \\ h_0 \; \overset{\Leftarrow}{\scriptstyle \alpha_1} & {\scriptstyle g_0}\big\downarrow & \Swarrow_{g_2} & \big\downarrow{\scriptstyle g_1} \\ & \cdot & \xrightarrow[\;f'\;]{} & \cdot \end{array} \;\overset{\alpha_3}{\Rrightarrow}\; \begin{array}{cccc} \cdot & \xrightarrow{\;f\;} & \cdot & \\ {\scriptstyle h_0}\big\downarrow & \Swarrow_{h_2} & \big\downarrow{\scriptstyle h_1} & \overset{\Leftarrow}{\scriptstyle \alpha_2} \; g_1 \\ \cdot & \xrightarrow[\;f'\;]{} & \cdot & \end{array} \right\} \tag{4.3}$$

$$\overrightarrow{\mathbb{H}}_3 = \left\{ \begin{pmatrix} \Gamma_1, \Gamma_2, \alpha_3, \beta_3; g_2, h_2, \\ \alpha_1, \alpha_2, \beta_1, \beta_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix} \;\middle|\; \begin{pmatrix} \Gamma_1 \colon \alpha_1 \Rrightarrow \beta_1, \\ \Gamma_2 \colon \alpha_2 \Rrightarrow \beta_2 \end{pmatrix} \; \textit{such that} \; \begin{array}{c} \beta_3 \#_2 ((f' \#_0 \Gamma_1) \#_1 g_2) \\ = (h_2' \#_1 (\Gamma_2 \#_0 f)) \#_2 \alpha_3 \end{array} \right\} \tag{4.4}$$

Compositions and identities arise canonically from pasting of diagrams in $\mathbb{H}$*, as detailed below.*

The condition in (4.4) on the 3-cells is the commutativity of the following diagram



(4.5)

The identities in each dimension are obviously the ones consisting of identity cells.

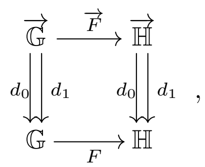
Remark 26 *By construction the map* $(d_0, d_1)\colon \overrightarrow{\mathbb{H}} \longrightarrow \mathbb{H} \times \mathbb{H}$ *is 2-faithful in the sense of definition 13, but in general not full.*

Remark 27 *The map* $i\colon \mathbb{H} \longrightarrow \overrightarrow{\mathbb{H}}$ *is 2-Cartesian and 1-faithful, but not in general 1-full.*

4.1 Path Spaces and Cartesian Maps

Lemma 28 *The path space construction* $\overrightarrow{(\_)}$ *of* Gray*-categories preserves 1-Cartesianness of maps.*

Proof Assume a situation



(4.6)

assume a pair of parallel 1-cells in $\overrightarrow{\mathbb{G}}$

(4.7)

we need to show that $\overrightarrow{F}$ is bijective on the intervening 2-cells. That means given

$$\beta_1\colon F(g_0) \Longrightarrow f(h_0) \quad \beta_2\colon F(g_1) \Longrightarrow F(h_1) \quad \beta_3\colon F(g_2\#_1(\beta_2\#_0 f)) \Rrightarrow F((f'\#_0\beta_1)\#_1 g_2) \tag{4.8}$$

there are unique

$$\alpha_1 \colon g_0 \Longrightarrow h_0 \quad \alpha_2 \colon g_1 \Longrightarrow h_1 \quad \alpha_3 \colon g_2 \#_1 (\alpha_2 \#_0 f) \Rrightarrow (f' \#_0 \alpha_1) \#_1 g_2 \tag{4.9}$$

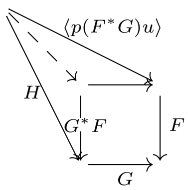
with $F(\alpha_i) = \beta_i$. But these exist uniquely by the 1-Cartesianness of F.

The same kind of argument can be applied to parallel 2-cells in $\overrightarrow{\mathbb{G}}$. □

Remark 29 *The functor $\overrightarrow{(\_)}$ preserves 2-Cartesian maps.*

Lemma 30 *A pullback of a Cartesian map is Cartesian if p preserves pullbacks.*

Proof Let F be p-Cartesian, and G^*F the pullback of F along G.



(4.10)

Let H factor through G below as $p(H) = p(G^*F)u$, then GH factors through F below as $p(GH) = p(GG^*F)u = p(F)p(F^*G)u$, hence there is a unique lift $\langle p(F^*G)u \rangle$. Hence there is a universally induced $\langle u \rangle$ with $G^*F\langle u \rangle = H$.

The functor p preserving pullbacks ensures that $p\langle u \rangle = u$. □

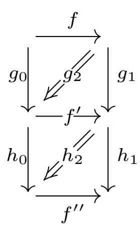
4.2 Vertical Composition Operations in the Path Space

We need to describe the vertical composition of 1-, 2-, 3-cells along 0-, 1-, 2-cells respectively.

We designate the composition in $\mathbb{H}$ by $\#_i$ and the interchange by $\otimes$, in $\overrightarrow{\mathbb{H}}$ we define the respective operations $\Box_i$ and $\boxtimes$ as follows:

$$h \Box_0 g = (h_2; h_0, h_1, f'', f') \Box_0 (g_2; g_0, g_1, f, f') = \begin{pmatrix} (h_2 \#_0 g_0) \#_1 (h_1 \#_0 g_2); \\ h_0 \#_0 g_0, h_1 \#_0 g_1, f, f'' \end{pmatrix} \tag{4.11}$$

This is just the vertical pasting



. (4.12)

Obviously this composition is associative and unital.

Remark 31 *Considering (4.12) we note that if the 1-cells in $\mathbb{H}$ are invertible, with inverse $\overline{(\_)}$, then the 2-cell*

$$(h_2 \#_0 g_0) \#_1 (h_1 \#_0 g_2) \tag{4.13}$$

in (4.12) *can also be written as a horizontal composite in two different ways:*

$$(h_2 \#_0 \overline{f'}) \lhd g_2 = h_2 \lhd (\overline{f'} \#_0 g_2) \tag{4.14}$$

There is of course also the opposite horizontal composite

$$(h_2 \#_0 \overline{f'}) \rhd g_2 = h_2 \rhd (\overline{f'} \#_0 g_2) \tag{4.15}$$

and a 3-cell

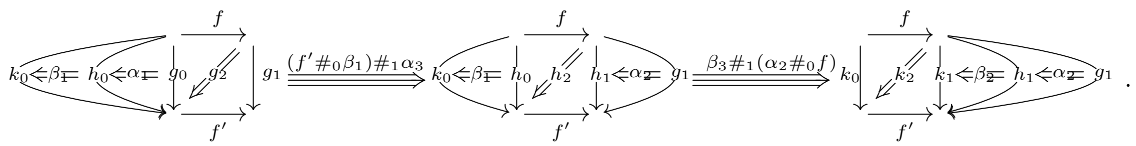
$$(h_2 \#_0 \overline{f'}) \otimes g_2 = h_2 \otimes (\overline{f'} \#_0 g_2) \tag{4.16}$$

going from (4.14) *to* (4.15)*. The picture* (4.12)*, however, always means* (4.14)*.*

The vertical composite of two 2-cells is

$$\begin{aligned} \beta \square_1 \alpha &= \begin{pmatrix} \beta_3; \beta_1, \beta_2, h_2, k_2; \\ h_0, h_1, k_0, k_1, f, f' \end{pmatrix} \square_1 \begin{pmatrix} \alpha_3; \alpha_1, \alpha_2, g_2, h_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix} \\ &= \begin{pmatrix} (\beta_3 \#_1 (\alpha_2 \#_0 f)) \#_2 ((f' \#_0 \beta_1) \#_1 \alpha_3); \\ \beta_1 \#_1 \alpha_1, \beta_2 \#_1 \alpha_2, g_2, h_2; g_0, g_1, k_0, k_1, f, f' \end{pmatrix} \end{aligned} \tag{4.17}$$

which has as its first component the following composite of $\mathbb{H}$-3-cells



(4.18)

We shall henceforth argue mostly diagrammatically in terms of such 3-cell diagrams, as it is fairly obvious what the lower dimensional components are.

Vertical composition of $\overrightarrow{\mathbb{H}}$-3-cells is particularly simple:

$$\Delta \square_2 \Gamma = \begin{pmatrix} \Delta_1 \colon \beta_1 \Rrightarrow \gamma_1, \\ \Delta_2 \colon \beta_2 \Rrightarrow \gamma_2 \end{pmatrix} \square_2 \begin{pmatrix} \Gamma_1 \colon \alpha_1 \Rrightarrow \beta_1, \\ \Gamma_2 \colon \alpha_2 \Rrightarrow \beta_2 \end{pmatrix} = \begin{pmatrix} \Delta_1 \#_2 \Gamma_1 \colon \alpha_1 \Rrightarrow \gamma_1, \\ \Delta_2 \#_2 \Gamma_2 \colon \alpha_2 \Rrightarrow \gamma_2 \end{pmatrix} \tag{4.19}$$

the condition 4.5 is obviously satisfied, since we just paste two instances of the commuting square vertically.

4.3 Whiskers

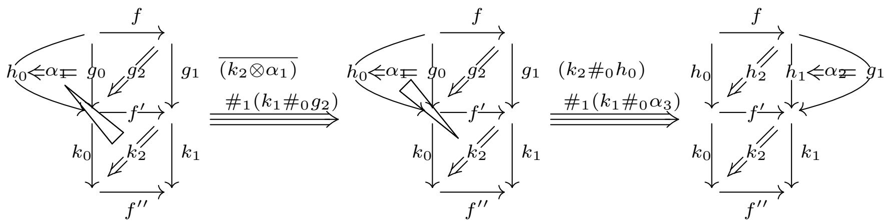
We need to define three whiskering operations, ${}^1\square_0^2$, ${}^1\square_0^3$, ${}^2\square_1^3$, where the raised indices indicate the dimension of the operands, the lower one the dimension of the incidence cell. Their symmetry partners are then obvious.

We define right whiskering of a 2-cell by a 1-cell as:

$$k \, {}^1\square_0^2 \alpha = (k_2; k_0, k_1, f', f'') \, {}^1\square_0^2 \begin{pmatrix} \alpha_3; \alpha_1, \alpha_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix}$$

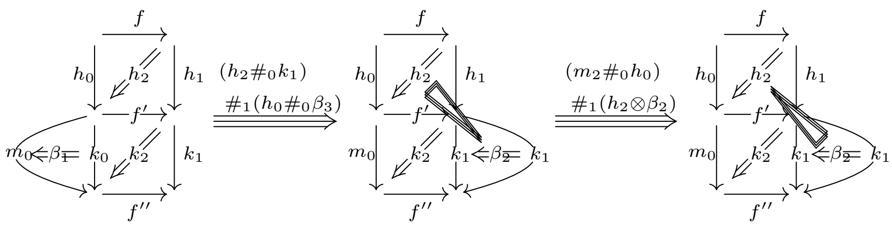
$$= \begin{pmatrix} ((k_2\#_0 h_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)); \\ k_0\#_0\alpha_1, k_1\#_0\alpha_2; \\ k_0\#_0 g_0, k_1\#_1 g_1, k_0\#_0 h_0, k_1\#_0 h_1, f, f'' \end{pmatrix}. \tag{4.20}$$

Diagrammatically this is the following composite:



(4.21)

For reference $(\beta_1, \beta_2, \beta_3)\square_0(h_0, h_1, h_2)$ is



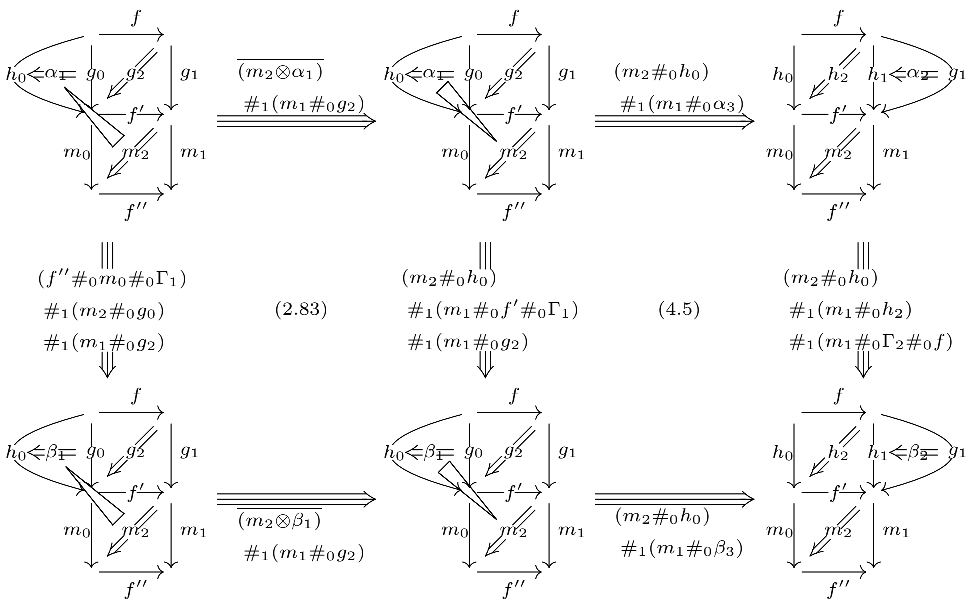
(4.22)

The action of 1-cells on 3-cells is as follows:

$$m^1\square_0^3\Gamma = (m_2; m_1, m_2, f', f'')^1\square_0^3 \begin{pmatrix} \Gamma_1, \Gamma_2, \alpha_3, \beta_3; \\ \alpha_1, \alpha_2, \beta_1\beta_2, g_2, h_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix}$$

$$= \begin{pmatrix} m_0\#_0\Gamma_1, m_1\#_0\Gamma_2, \\ ((m_2\#_0 h_0)\#_1(m_1\#_0\alpha_3))\#_2(\overline{(m_2\otimes\alpha_1)}\#_1(m_1\#_0 g_2)), \\ ((m_2\#_0 h_0)\#_1(m_1\#_0\beta_3))\#_2(\overline{((m_2\otimes\beta_1))}\#_1(m_1\#_0 g_2)); \\ m_0\#_0\alpha_1, m_0\#_1\alpha_2, m_0\#_0\beta_1, m_1\#_0\beta_2, \\ (m_2\#_0 g_0)\#_1(m_1\#_0 g_2), (m_2\#_0 h_0)\#_1(m_1\#_0 h_2); \\ m_0\#_0 g_0, m_1\#_0 g_1, m_0\#_0 h_0, m_1\#_0 h_1, f, f'' \end{pmatrix} \tag{4.23}$$

We claim this is again a proper 3-cell in $\overrightarrow{\mathbb{H}}$, that is, the whisker satisfies (4.5), as can be easily seen:

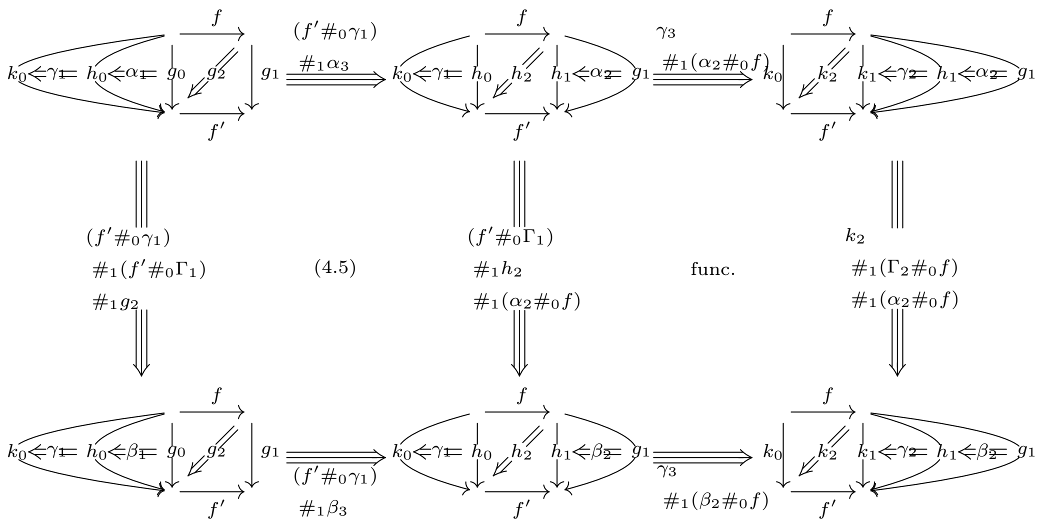


(4.24)

Finally, we define 3-2-whiskering:

$$\gamma^2\square_1^3\Gamma = \begin{pmatrix} \gamma_3;\gamma_1,\gamma_2,h_2,k_2; \\ h_0,h_1,k_0,k_1,f,f' \end{pmatrix} {}^2\square_1^3 \begin{pmatrix} \Gamma_1,\Gamma_2,\alpha_3,\beta_3;g_2,h_2, \\ \alpha_1,\alpha_2,\beta_1,\beta_2; \\ g_0,g_1,h_0,h_1,f,f' \end{pmatrix}$$
$$= \begin{pmatrix} \gamma_1\#_1\Gamma_1,\gamma_2\#_1\Gamma_2, \\ (\gamma_3\#_1(\alpha_2\#_0 f))\#_2((f'\#_0\gamma_1)\#_1\alpha_3), \\ (\gamma_3\#_1(\beta_2\#_0 f))\#_2((f'\#_0\gamma_1)\#_1\beta_3); \\ g_2,k_2,\gamma_1\#_1\alpha_1,\gamma_2\#_1\alpha_2,\gamma_1\beta_1,\gamma_2\beta_2; \\ g_0,g_1,k_0,k_1,f,f' \end{pmatrix} \quad (4.25)$$

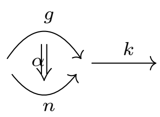
It gives a 3-cell in $\overrightarrow{\mathbb{H}}$ again.



(4.26)

4.4 Horizontal Composition of 2-Cells

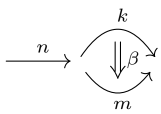
We shall use the following slightly abbreviated notation for the higher cells of the mapping space, for example writing (4.20) as:



$$= k^1 \square_0^2 \alpha = (k_2; k_0, k_1, f', f'')^1 \square_0^2 \left(\alpha_3; \alpha_1, \alpha_2 \middle| g, n\right)$$

$$= \begin{pmatrix} ((k_2 \#_0 n_0) \#_1 (k_1 \#_0 \alpha_3)) \#_2 (\overline{(k_2 \otimes \alpha_1)} \#_1 (k_1 \#_0 g_2)); \\ k_0 \#_0 \alpha_1, k_1 \#_0 \alpha_2 \middle| k \square_0 g, k \square_0 n \end{pmatrix}. \quad (4.27)$$

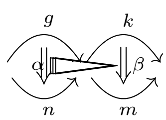
In the same spirit we write the opposite whiskering:



$$= \beta^2 \square_0^1 n = \left(\beta_3; \beta_1, \beta_2 \middle| k, m\right)$$

$$= \begin{pmatrix} ((m_2 \#_0 n_0) \#_1 (\beta_2 \otimes n_2)) \#_2 (\beta_3 \#_1 (k_1 \#_0 n_2)); \\ \beta_1 \#_0 n_0, \beta_2 \#_0 n_1 \middle| k \square_0 n, m \square_0 n \end{pmatrix}. \quad (4.28)$$

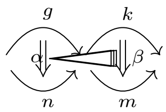
So now we can define the left horizontal composite:



$$= \beta \boxtimes \alpha = \begin{pmatrix} ((m_2 \#_0 n_0) \#_1 (\beta_2 \otimes n_2)) \\ \#_2 (\beta_3 \#_1 (k_1 \#_0 n_2)); \\ \beta_1 \#_0 n_0, \beta_2 \#_0 n_1 \middle| k \square_0 n, m \square_0 n \end{pmatrix} \square_1 \begin{pmatrix} ((k_2 \#_0 n_0) \#_1 (k_1 \#_0 \alpha_3)) \\ \#_2 (\overline{(k_2 \otimes \alpha_1)} \#_1 (k_1 \#_0 g_2)); \\ k_0 \#_0 \alpha_1, k_1 \#_0 \alpha_2 \middle| k \square_0 g, k \square_0 n \end{pmatrix}$$

$$= \left(\begin{array}{c} \left(\begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2)) \\ \#_2(\beta_3\#_1(k_1\#_0 n_2)) \end{pmatrix} \#_1(k_1\#_0\alpha_2\#_0 f) \right) \\ \#_2 \left((f''\#_0\beta_1\#_0 n_0)\#_1 \begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)) \end{pmatrix} \right); \\ \alpha_1 \triangleleft \beta_1, \alpha_2\triangleleft\beta_2 \,\big|\, k\square_0 g, m\square_0 n \end{array} \right) \tag{4.29}$$

Conversely

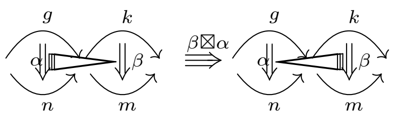


$$= \beta \boxbslash \alpha = \begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2)) \\ \#_2(\beta_3\#_1(k_1\#_0 n_2)); \\ \beta_1\#_0 n_0, \beta_2\#_0 n_1 \,\big|\, k\square_0 n, m\square_0 n \end{pmatrix} \square_1 \begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)); \\ k_0\#_0\alpha_1, k_1\#_0\alpha_2 \,\big|\, k\square_0 g, k\square_0 n \end{pmatrix}$$

$$= \left(\begin{array}{c} \left(\begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2)) \\ \#_2(\beta_3\#_1(k_1\#_0 n_2)) \end{pmatrix} \#_1(k_1\#_0\alpha_2\#_0 f) \right) \\ \#_2 \left((f''\#_0\beta_1\#_0 n_0)\#_1 \begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)) \end{pmatrix} \right); \\ \alpha_1 \triangleleft \beta_1, \alpha_2\triangleleft\beta_2 \,\big|\, k\square_0 g, m\square_0 n \end{array} \right) \tag{4.30}$$

4.5 Tensors

Finally, in



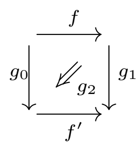
(4.31)

letting $\beta \boxtimes \alpha = (\beta_1\otimes\alpha_1, \beta_2\otimes\alpha_2)$ makes $\overrightarrow{\mathbb{H}}$ a Gray-category. This is a well defined 3-cell.

4.6 Identities

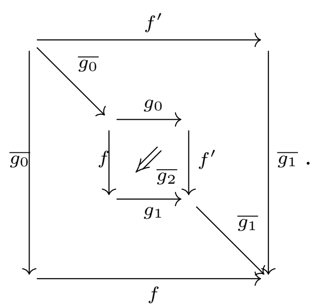
4.7 Inverses

If $\mathbb{H}$ has invertible 1- and 2-cells the inverse of of a 1-cell



(4.32)

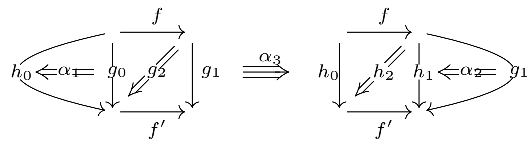
in $\overrightarrow{\mathbb{H}}$ is given by

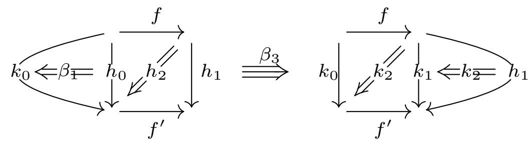


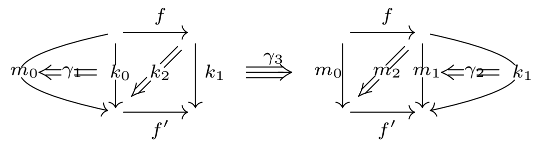
(4.33)

4.8 Axioms

This composition of $\overrightarrow{\mathbb{H}}$-2-cells is associative: Given three 2-cells

$\alpha =$  (4.34)

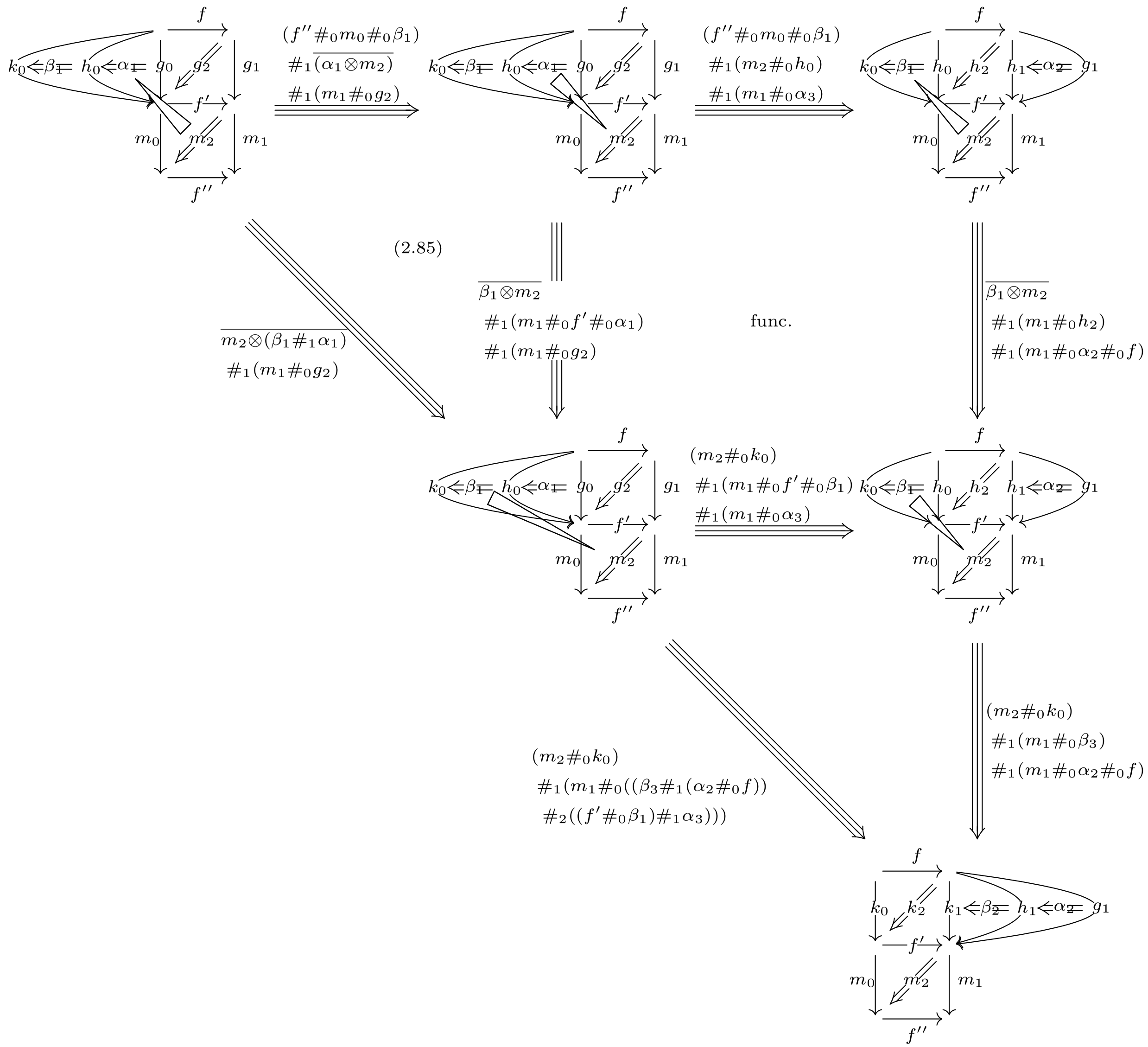
$\beta =$  (4.35)

$\gamma =$  (4.36)

we use (4.17) and the functoriality of the whiskerings in $\mathbb{H}$ to compute:

$$
\begin{aligned}
(\gamma\Box_1\beta)\Box_1\alpha &= \begin{pmatrix} \underbrace{(\gamma_3\#_1(\beta_2\#_0 f))\#_2((f'\#_0\gamma_1)\#_1\beta_3)}_{\omega_3}; \\ \gamma_1\#_1\beta_1, \gamma_2\#_1\beta_2, h_2, m_2; h_0, h_1, m_0, m_1, f, f' \end{pmatrix} \Box_1\alpha \\
&= \begin{pmatrix} (\omega_3\#_1(\alpha_2\#_0 f)) \\ \#_2((f'\#_0(\gamma_1\#_1\beta_1))\#_1\alpha_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
= \begin{pmatrix} (((\gamma_3\#_1(\beta_2\#_0 f))\#_2((f'\#_0\gamma_1)\#_1\beta_3)) \\ \#_1(\alpha_2\#_0 f))\#_2((f'\#_0(\gamma_1\#_1\beta_1))\#_1\alpha_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, \\ g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} &= \begin{pmatrix} (\gamma_3\#_1(\beta_2\#_0 f)\#_1(\alpha_2\#_0 f)) \\ \#_2(((f'\#_0\gamma_1)\#_1\beta_3)\#_1(\alpha_2\#_0 f)) \\ \#_2((f'\#_0(\gamma_1\#_1\beta_1))\#_1\alpha_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; \\ g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \begin{pmatrix} (\gamma_3\#_1((\beta_2\#_1\alpha_2)\#_0 f))\#_2((f'\#_0\gamma_1)\#_1\beta_3\#_1(\alpha_2\#_0 f)) \\ \#_2((f'\#_0\gamma_1)\#_1(f'\#_0\beta_1)\#_1\alpha_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \begin{pmatrix} (\gamma_3\#_1((\beta_2\#_1\alpha_2)\#_0 f))\#_2((f'\#_0\gamma_1) \\ \#_1\underbrace{((\beta_3\#_1(\alpha_2\#_0 f))\#_2((f'\#_0\beta_1)\#_1\alpha_3)))}_{\zeta_3}; \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \begin{pmatrix} (\gamma_3\#_1((\beta_2\#_1\alpha_2)\#_0 f)) \\ \#_2((f'\#_0\gamma_1)\#_1\zeta_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \gamma\Box_1 \begin{pmatrix} \zeta_3; \beta_1\#_1\alpha_1, \beta_2\#_1\alpha_2, \\ g_2, k_2; g_0, g_1, k_0, k_1, f, f' \end{pmatrix} = \gamma\Box_1(\beta\Box_1\alpha)\,. \qquad (4.37)
\end{aligned}
$$

We check that 2-1-whiskering in $\overrightarrow{\mathbb{H}}$ is functorial, that is, $m\Box_0(\beta\Box_1\alpha) = (m\Box_0\beta)\Box_1(m\Box_0\alpha)$. In diagram (4.38) the diagonal is $m\Box_0(\beta\Box_1\alpha)$ and left and down is $(m\Box_0\beta)\Box_1(m\Box_0\alpha)$. 1-2-whiskering in $\overrightarrow{\mathbb{H}}$ is functorial by duality.

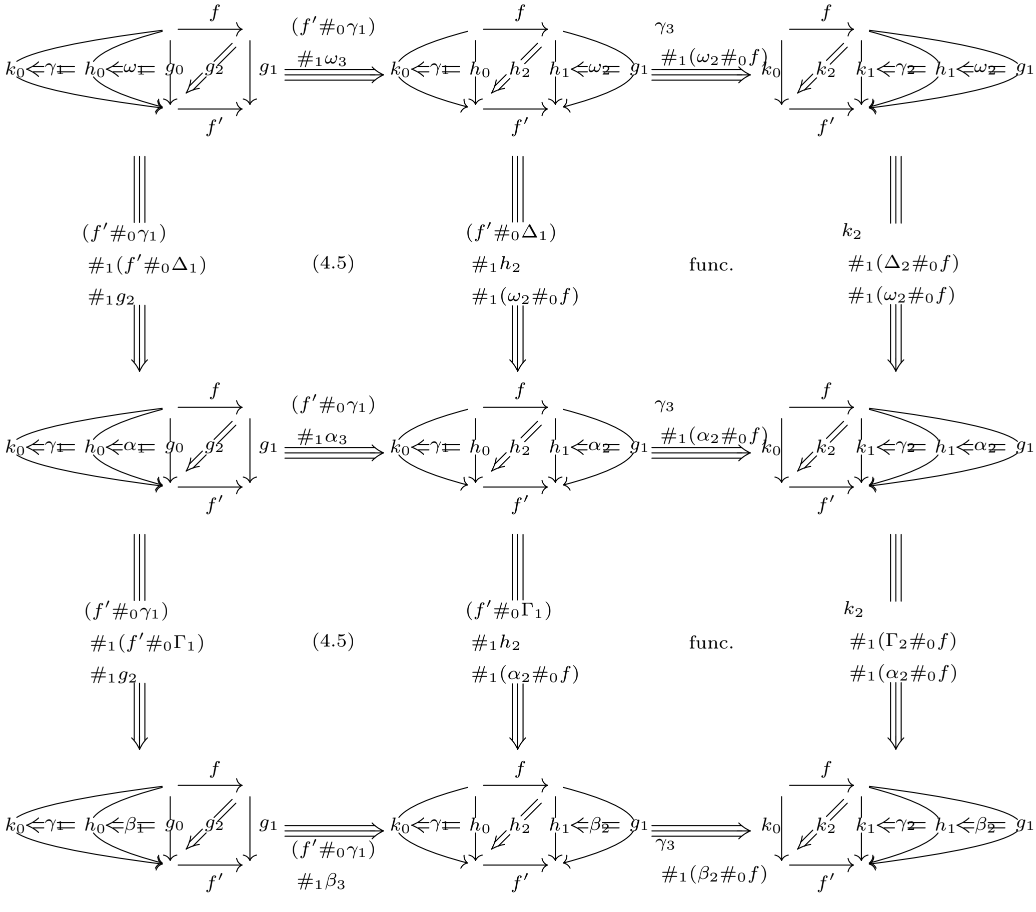

(f″#₀m₀#₀β₁)
#₁(α₁⊗m₂)
#₁(m₁#₀g₂)
(f″#₀m₀#₀β₁)
#₁(m₂#₀h₀)
#₁(m₁#₀α₃)
(2.85)
β₁⊗m₂
#₁(m₁#₀f′#₀α₁)
#₁(m₁#₀g₂)
func.
β₁⊗m₂
#₁(m₁#₀h₂)
#₁(m₁#₀α₂#₀f)
m₂⊗(β₁#₁α₁)
#₁(m₁#₀g₂)
(m₂#₀k₀)
#₁(m₁#₀f′#₀β₁)
#₁(m₁#₀α₃)
(m₂#₀k₀)
#₁(m₁#₀β₃)
#₁(m₁#₀α₂#₀f)
(m₂#₀k₀)
#₁(m₁#₀((β₃#₁(α₂#₀f))
#₂((f′#₀β₁)#₁α₃)))

(4.38)

It is obvious that 3-1-whiskering is 2-functorial, that is,

$$
\begin{aligned}
&(m_0, m_1, m_2)\Box_0((\Delta_1, \Delta_2)\Box_2(\Gamma_1, \Gamma_2)) \\
&\qquad = (m_0, m_1, m_2)\Box_0(\Delta_1\#_2\Gamma_1, \Delta_2\#_2\Gamma_2) \\
&\qquad = (m_0\#_0(\Delta_1\#_2\Gamma_1), m_1\#_0(\Delta_2\#_2\Gamma_2)) \\
&\qquad = (((m_0\#_0\Delta_1)\#_2(m_0\#_0\Gamma_1)), ((m_1\#_0\Delta_2)\#_2(m_1\#_0\Gamma_2))) \\
&\qquad = ((m_0\#_0\Delta_1), (m_1\#_0\Delta_2))\Box_2((m_0\#_0\Gamma_1), (m_1\#_0\Gamma_2)) \\
&\qquad = ((m_0, m_1, m_2)\Box_0(\Delta_1, \Delta_2))\Box_2((m_0, m_1, m_2)\Box_0(\Gamma_1, \Gamma_2)) . \qquad (4.39)
\end{aligned}
$$

By duality, 1-2-whiskering in $\overrightarrow{\mathbb{H}}$ is functorial as well. And the 3-2-whiskering thus defined is functorial with respect to vertical composition of 3-cells, that is, $\gamma\Box_1(\Gamma\Box_2\Delta) = (\gamma\Box_1\Gamma)\Box_2(\gamma\Box_1\Delta)$, as can seen by inspecting the following diagram.



(4.40)

We see that 2-3-whiskering is functorial:

$$
\begin{aligned}
&(\Delta\Box_1\beta)\Box_2(\gamma\Box_1\Gamma) \\
&\qquad = (\Delta_1\#_1\beta_1, \Delta_2\#_1\beta_2)\Box_2(\gamma_1\#_1\Gamma_1, \gamma_2\#_1\Gamma_2) \\
&\qquad = ((\Delta_1\#_1\beta_1)\#_2(\gamma_1\#_1\Gamma_1), ((\Delta_2\#_1\beta_2)\#_2(\gamma_2\#_1\Gamma_2))
\end{aligned}
$$

$$
\begin{aligned}
&= ((\delta_1 \#_1 \Gamma_1)\#_2(\Delta_1\#_1\alpha_1), (\delta_2\#_1\Gamma_2)\#_2(\Delta_2\#_2\alpha_2)) \\
&= (\delta_1\#_1\Gamma_1, \delta_2\#_1\Gamma_2)\Box_2(\Delta_1\#_1\alpha_1, \Delta_2\#_1\alpha_2) \\
&= (\delta\Box_1\Gamma)\Box_2(\Delta\Box_1\alpha)\,. \qquad (4.41)
\end{aligned}
$$

So we can conclude that $\overrightarrow{\mathbb{H}}$ is locally a 2-category.

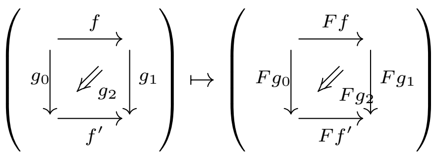
That interchange $\boxtimes$ is natural and functorial in both arguments follows immediately from the respective properties of $\otimes$ in $\mathbb{H}$. Thus we have:

Lemma 32 *The path space $\overrightarrow{\mathbb{H}}$ for a* Gray*-category $\mathbb{H}$ is again a* Gray*-category.* □

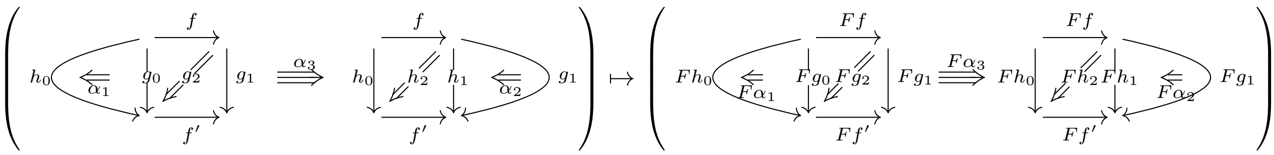
Lemma 33 *Given a* Gray*-functor $F\colon \mathbb{G} \longrightarrow \mathbb{H}$ there is a canonical* Gray*-functor $\overrightarrow{F}\colon \overrightarrow{\mathbb{G}} \longrightarrow \overrightarrow{\mathbb{H}}$.*

Proof The Gray-functor $\overrightarrow{F}$ acts by applying F to all components of the cells of $\overrightarrow{\mathbb{G}}$:

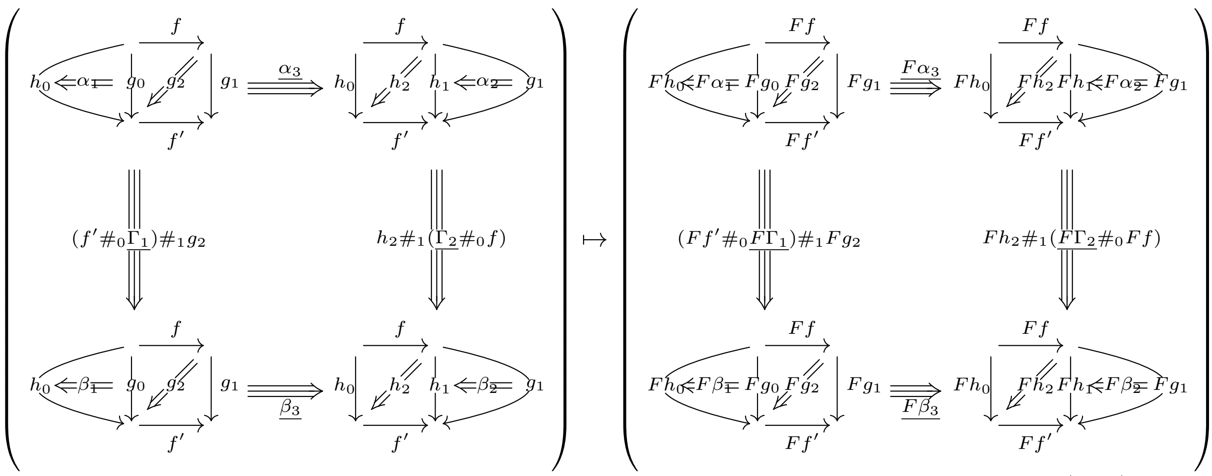
$$\left(x \xrightarrow{f} y\right) \mapsto \left(Fx \xrightarrow{Ff} Fy\right) \qquad (4.42)$$



(4.43)



(4.44)



(4.45)

This preserves the structure of $\overrightarrow{\mathbb{G}}$ since F preserves all commting diagrams on the nose. □

Theorem 34 *Furthermore* $\overrightarrow{(-)}$ *is canonically an endofunctor of* $\mathsf{GrayCat}$.

Proof Obviously $\overrightarrow{GF} = \overrightarrow{G}\,\overrightarrow{F}$. □

We finally note the following:

Lemma 35 *The functor* $\overrightarrow{(-)} \colon \mathsf{GrayCat} \longrightarrow \mathsf{GrayCat}$ *preserves limits.*

Proof This is obviously true for products.

For the equalizer $\mathbb{E}$ of two strict maps F, G we remember that the action of $\overrightarrow{F}$ and $\overrightarrow{G}$ is defined by the component wise action of F and G, that is, a cell of $\overrightarrow{\mathbb{E}}$ is equal under $\overrightarrow{F}$ and $\overrightarrow{G}$ iff its components are so under F and G. □

A straightforward calculation shows how this forms part of an adjunction

$$\mathsf{GrayCat} \underset{\_\otimes\mathbb{I}}{\overset{\overrightarrow{(\_)}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathsf{GrayCat} \tag{4.46}$$

where $\mathbb{I}$ is the free Gray-category on a single 1-cell $(01)\colon 0 \longrightarrow 1$ and $\otimes$ is Crans' tensor of Gray-categories.

Chapter 5

Composition of Paths

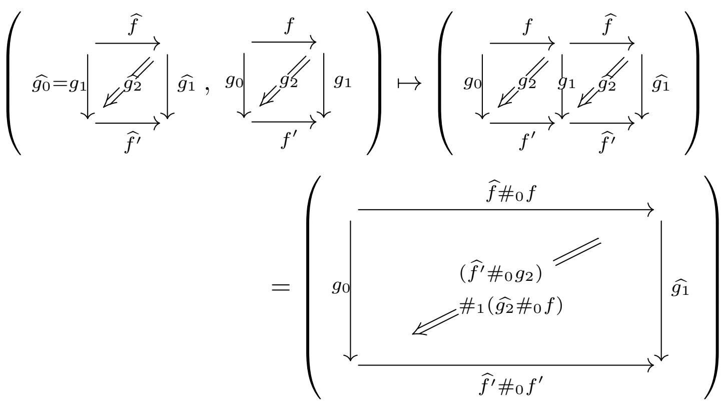
We want to turn the path space that we constructed in the previous section into the arrow part of an internal category, which requires us to define a composition map as follows.

Definition 36 *We define the* **composite of paths** *as a pseudo* Q^1 *graph map* $m\colon \overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}} \nrightarrow \overrightarrow{\mathbb{H}}$ *by horizontal pasting as follows:*

1. *0-cells*

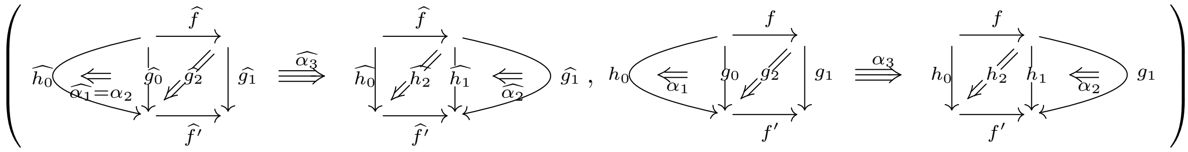
$$\left(y \xrightarrow{\widehat{f}} z \ ,\ x \xrightarrow{f} y \right) \mapsto \left(x \xrightarrow{\widehat{f}\#_0 f} z \right) \tag{5.1}$$

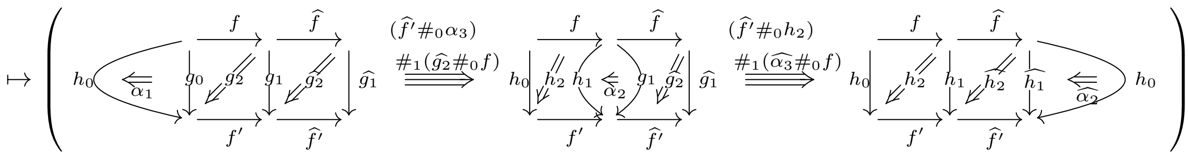
2. *1-cells*



(5.2)

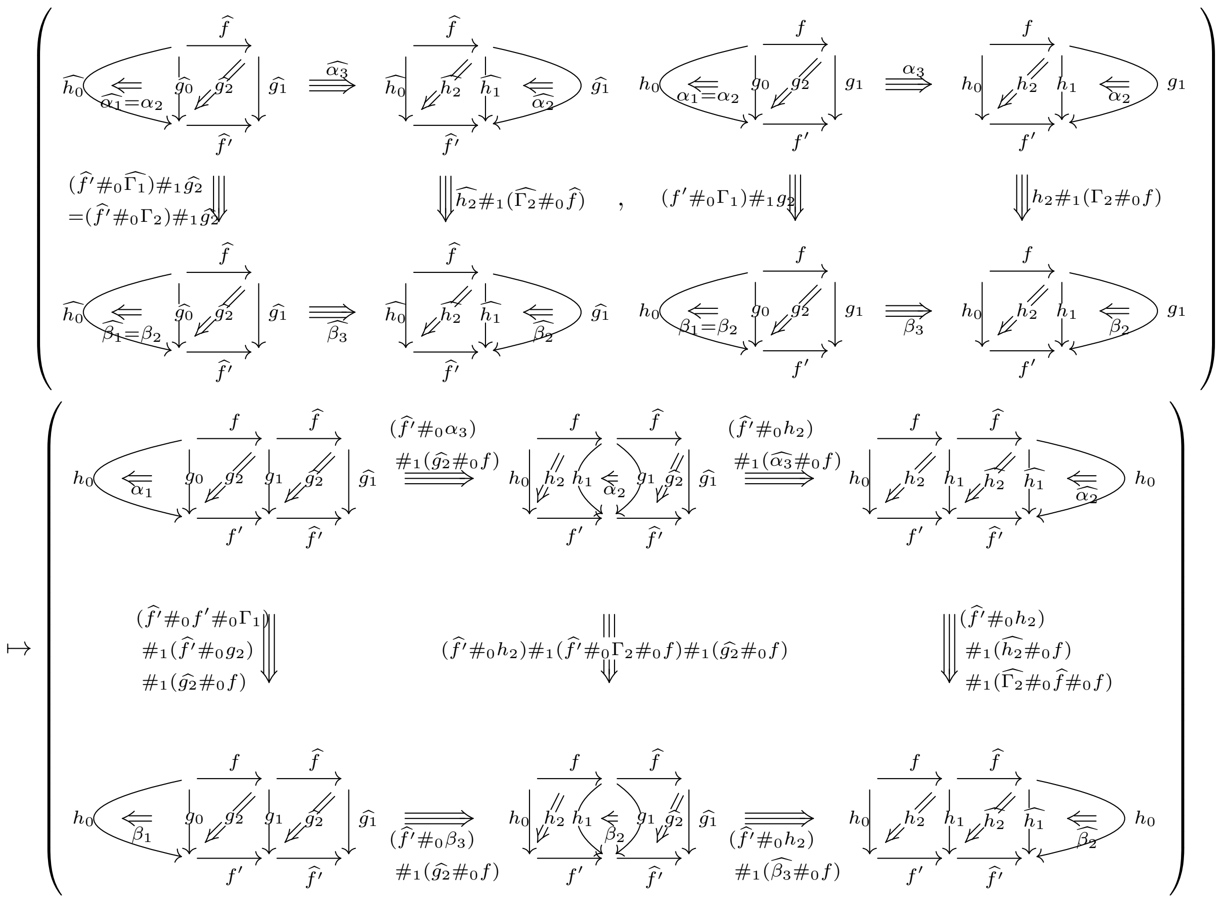
3. *2-cells*





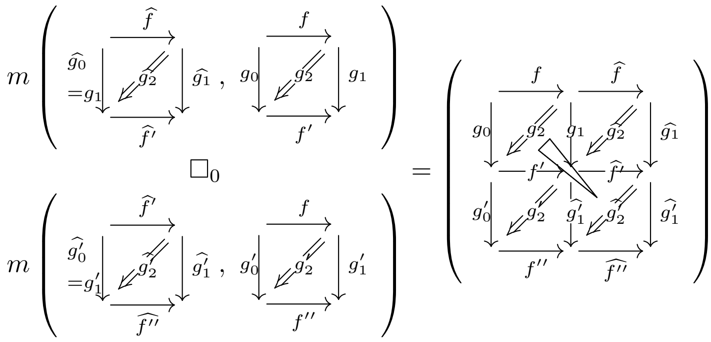
(5.3)

4. 3-cells

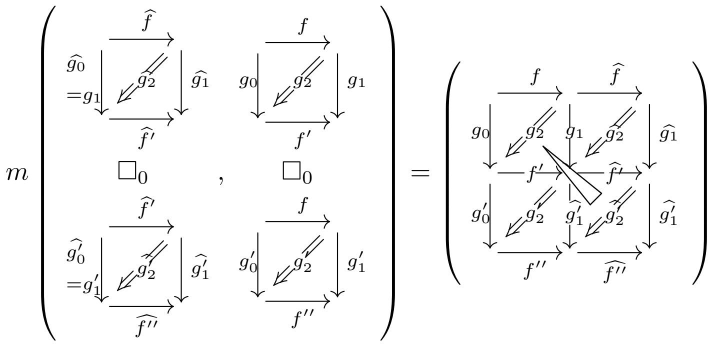


(5.4)

5. the 2-cocycle: for a (vertically) composable pair in $\overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}}$ we have the *composite of the images and the image of the composites under m:*

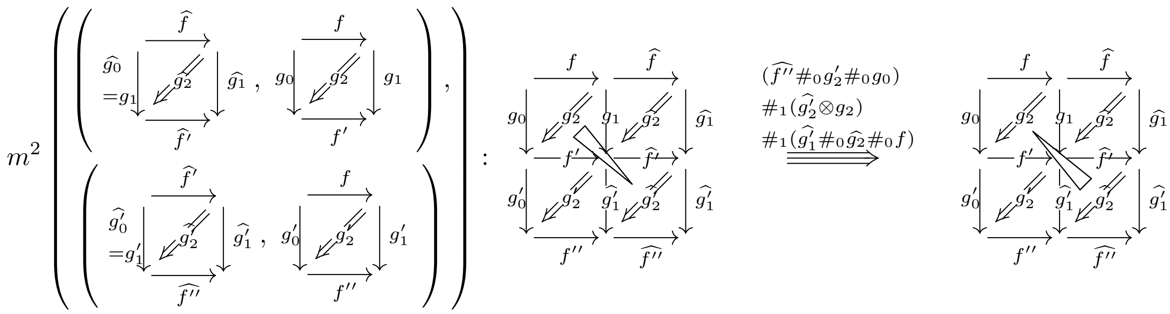


(5.5)



(5.6)

And the 2-cocycle going between them is:



(5.7)

For completeness' sake we give it in the algebraic notation:

$$\begin{pmatrix} (\widehat{f''}\#_0 g_2'\#_0 g_0)\#_1(\widehat{g_2'}\otimes g_2)\#_1(\widehat{g_1'}\#_0\widehat{g_2}\#_0 f); \\ \mathrm{id}_{g_0'\#_0 g_0}, \mathrm{id}_{\widehat{g_1'}\#_0\widehat{g_1}}, \\ (\widehat{f''}\#_0 g_2'\#_0 g_0)\#_1(\widehat{g_2'}\triangleleft g_2)\#_1(\widehat{g_1'}\#_0\widehat{g_2}\#_0 f), \\ (\widehat{f''}\#_0 g_2'\#_0 g_0)\#_1(\widehat{g_2'}\triangleright g_2)\#_1(\widehat{g_1'}\#_0\widehat{g_2}\#_0 f); \\ g_0'\#_0 g_0, \widehat{g_1'}\#_0\widehat{g_1}, g_0'\#_0 g_0, \widehat{g_1'}\#_0\widehat{g_1}, \widehat{f}\#_0 f, \widehat{f''}\#_0 f'' \end{pmatrix} \tag{5.8}$$

Lemma 37 *The map* $m\colon \overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}} \nrightarrow \overrightarrow{\mathbb{H}}$ *is a pseudo* Q^1 *graph map and hence by lemma 23 uniquely defines a pseudo* Gray*-functor.*

Proof As defined above, m is obviously a 3-globular map. We verify that it is locally a sesquifunctor: Let (β^1,β^2) and (α^1,α^2) be two pairs of 2-cells in $\overrightarrow{\mathbb{H}}\times_{\mathbb{H}}\overrightarrow{\mathbb{H}}$ composable along a pair of 1-cells. Then

$$m((\beta^1,\beta^2)\square_1(\alpha^1,\alpha^2)) = m((\beta^1\square_1\alpha^1),(\beta^2\square_1\alpha^2)) = m(\beta^1,\beta^2)\square_1 m(\alpha^1,\alpha^2) \tag{5.9}$$

follows obviously from the fact that in $\mathbb{H}$ 3-cells compose along a 2-cells interchangeably. Let (Δ^1,Δ^2) and (Γ^1,Γ^2) be two pairs of 3-cells in $\overrightarrow{\mathbb{H}}\times_{\mathbb{H}}\overrightarrow{\mathbb{H}}$ composable along a pair of 2-cells. Then

$$\begin{aligned} &m((\Delta^1,\Delta^2)\square_2(\Gamma^1,\Gamma^2)) = m((\Delta^1\square_2\Gamma^1),(\Delta^2\square_2\Gamma^2))\\ &\quad = m((\Delta^1_1\#_2\Gamma^1_1,\Delta^1_2\#_2\Gamma^1_2),(\Delta^2_1\#_2\Gamma^2_1,\Delta^2_2\#_2\Gamma^2_2)) = (\Delta^1_1\#_2\Gamma^1_1,\Delta^2_2\#_2\Gamma^2_2)\\ &\quad = (\Delta^1_1,\Delta^2_2)\square_2(\Gamma^1_1,\Gamma^2_2) = m((\Delta^1_1,\Delta^1_2),(\Delta^2_1,\Delta^2_2))\square_2 m((\Gamma^1_1,\Gamma^1_2),(\Gamma^2_1,\Gamma^2_2))\\ &\qquad\qquad = m(\Delta^1,\Delta^2)\square_2 m(\Gamma^1,\Gamma^2)\,. \end{aligned} \tag{5.10}$$

For the vertical composition of 3-cells see (4.19), their images under m are pastings of commuting diagrams, so preservation is immediate. Preservation of whiskers of 3-cells by 2-cells given for each component of $\overrightarrow{\mathbb{H}}\times_{\mathbb{H}}\overrightarrow{\mathbb{H}}$ in (4.26), again according to 36.4 m pastes two such commuting diagrams horizontally. Preservation of units is trivially satisfied. This concludes verification of 22.1.

We verify that m^2 is a 2-cocycle in (5.11). Note that in the last column (5.11)

$$\begin{pmatrix} (f'''^2\#_0\underline{k^1_2}\#_0h^1_0\#_0g^1_0) \\ \\ \#_1(\overline{k^2_2}\rhd\overline{h^1_2}\#_0g^1_0) \\ \#_1(\overline{k^2_1}\#_0\underline{h^2_2}\rhd\underline{g^1_2}) \\ \\ \#_1(k^2_1\#_0h^2_1\#_0\underline{g^2_2}\#_0f^1) \end{pmatrix} = \begin{pmatrix} (f'''^2\#_0\underline{k^1_2}\#_0h^1_0\#_0g^1_0) \\ \#_1(f'''^2\#_0k^1_1\#_0\underline{h^1_2}\#_0g^1_0) \\ \#_1(\underline{k^2_2}\#_0h^1_1\#_0f'^1\#_0g^1_0) \\ \#_1(\overline{k^2_1}\#_0f''^2\#_0h^1_1\#_0\underline{g^1_2}) \\ \#_1(k^2_1\#_0\underline{h^2_2}\#_0g^1_1\#_0\overline{f^1}) \\ \#_1(k^2_1\#_0\overline{h^2_1}\#_0\underline{g^2_2}\#_0f^1) \end{pmatrix} = \begin{pmatrix} (f'''^2\#_0((\underline{k^1_2}\#_0h^1_0) \\ \#_1(k^1_1\#_0\overline{\underline{h^1_2}}))\#_0g^1_0) \\ \#_1(\underline{k^2_2}\lhd h^1_1\#_0\underline{g^1_2}) \\ \#_1\big(k^2_1\#_0((\underline{h^2_2}\#_0g^1_1) \\ \#_1(h^2_1\#_0\underline{g^2_2}))\#_0f^1) \end{pmatrix}, \tag{5.12}$$

showing how the multiple horizontal composites of squares can be simplified. And the left hand rectangle in (5.11) commutes by local interchange. Also, m^2 is normalized by the unitality of the tensor in $\mathbb{H}$.

We check the coherent preservation of whiskers of 2-cells by 1-cells on the left, that is,

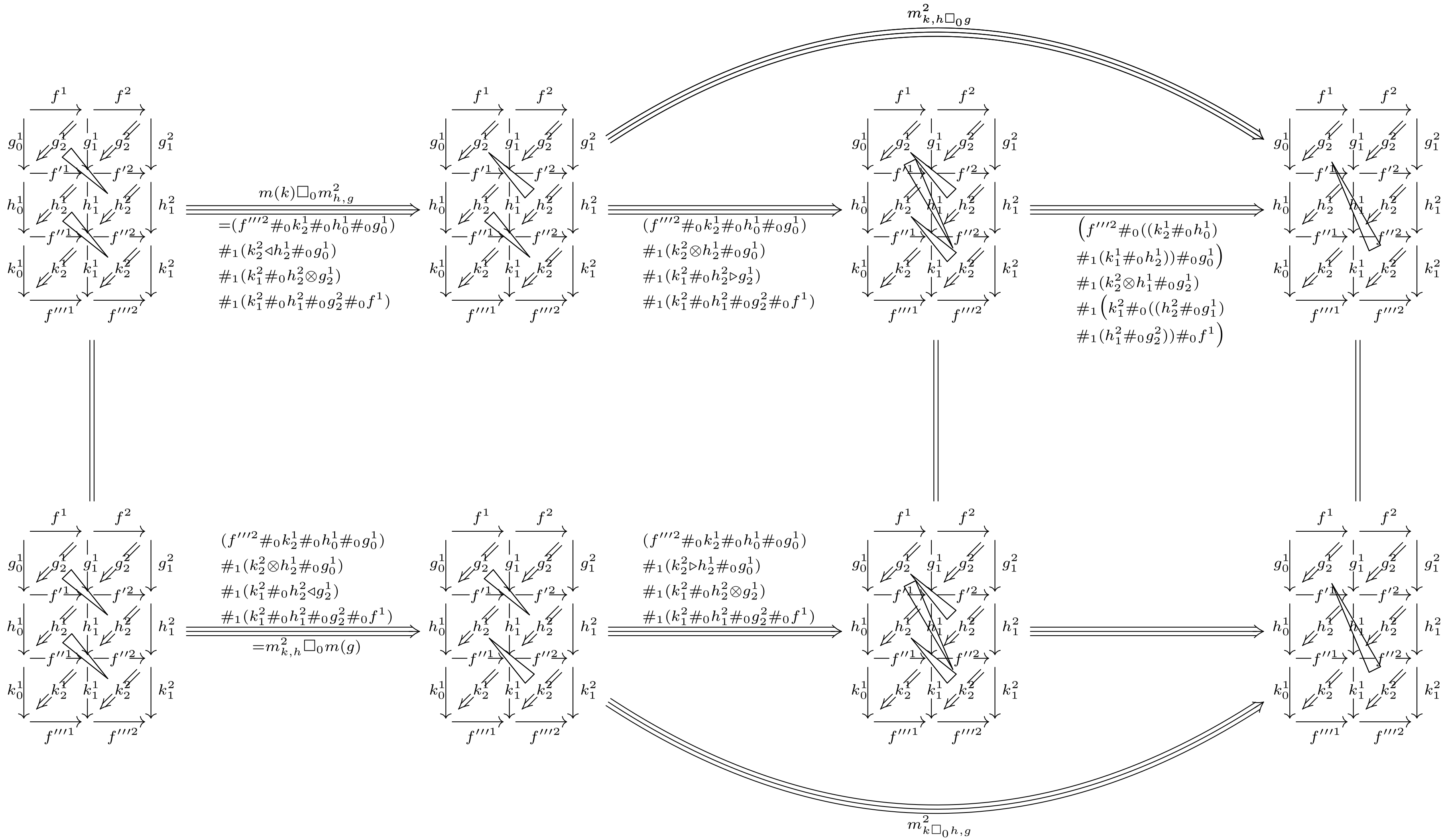
$$m^2_{\tilde{h},g}\square_1(m(\alpha)\square_0 m(g)) = m(\alpha\square_0 g)\square_1 m^2_{h,g} \tag{5.13}$$

in (5.14), where the parts commute by the naturality of the tensor and the local interchange. The corresponding condition for right whiskers is verified similarly. Coherent preservation of whiskers of 3-cells by 1-cells is checked in the same way using in addition the naturality of the horizontal composition of a 3-cell by a 2-cell along a 0-cell. This proves conditions (3.55) and (3.56).

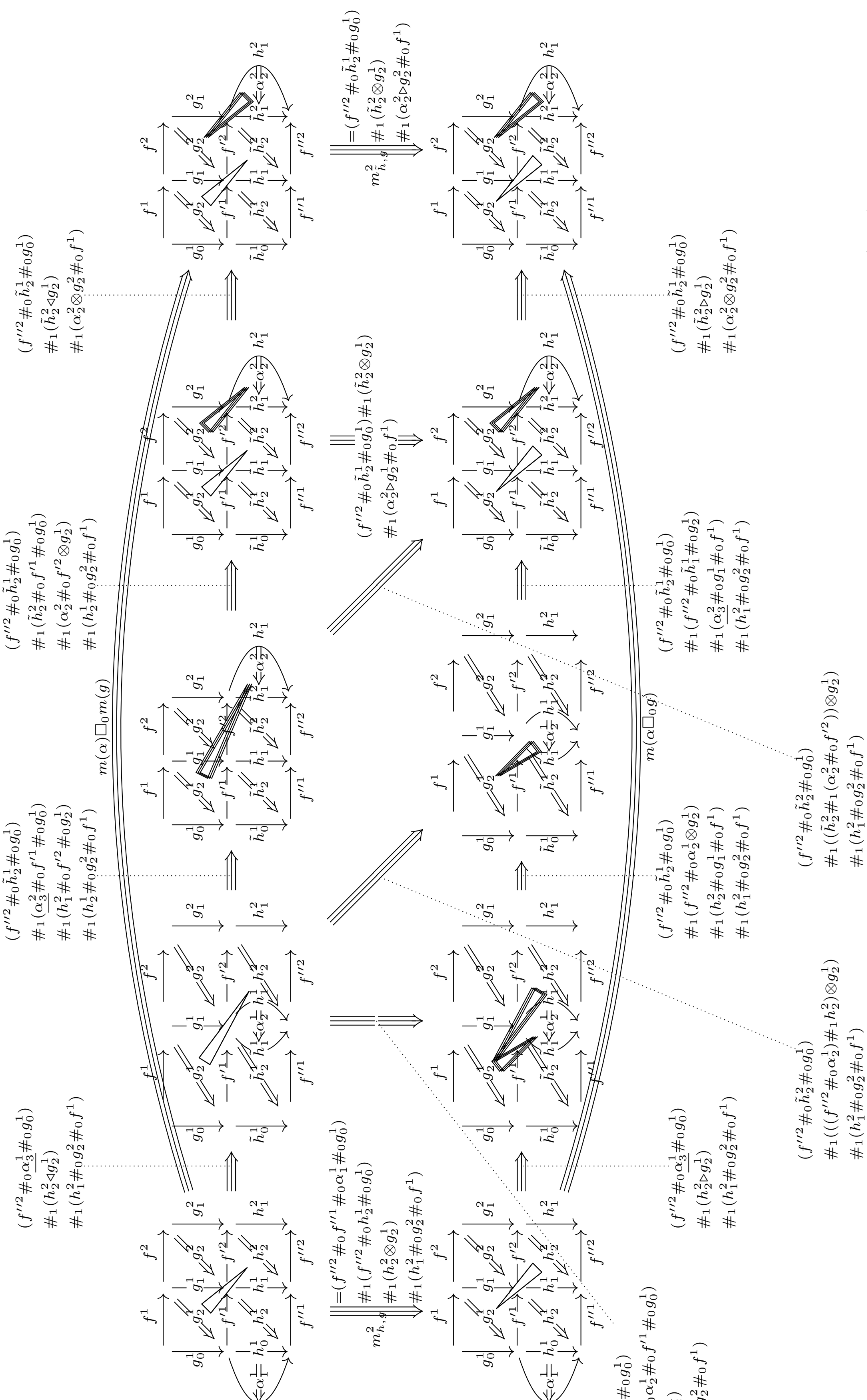
We verify the coherent preservation of tensors, i. e. that

$$m(\beta\boxtimes\alpha)\square_1 m^2_{k,h} = m^2_{\tilde{k},\tilde{h}}\square_1(m(\beta)\boxtimes m(\alpha))\,, \tag{5.16}$$

where $\alpha,\beta,k,h,\tilde{k},\tilde{h}$ are 2- and 1-cells respectively in $\overrightarrow{\mathbb{H}}\times_{\mathbb{H}}\overrightarrow{\mathbb{H}}$. In terms of constituent cells (5.16) can be drawn as (5.17), where the pasting of the center and right squares corresponds to the right hand side of the equation (5.16),

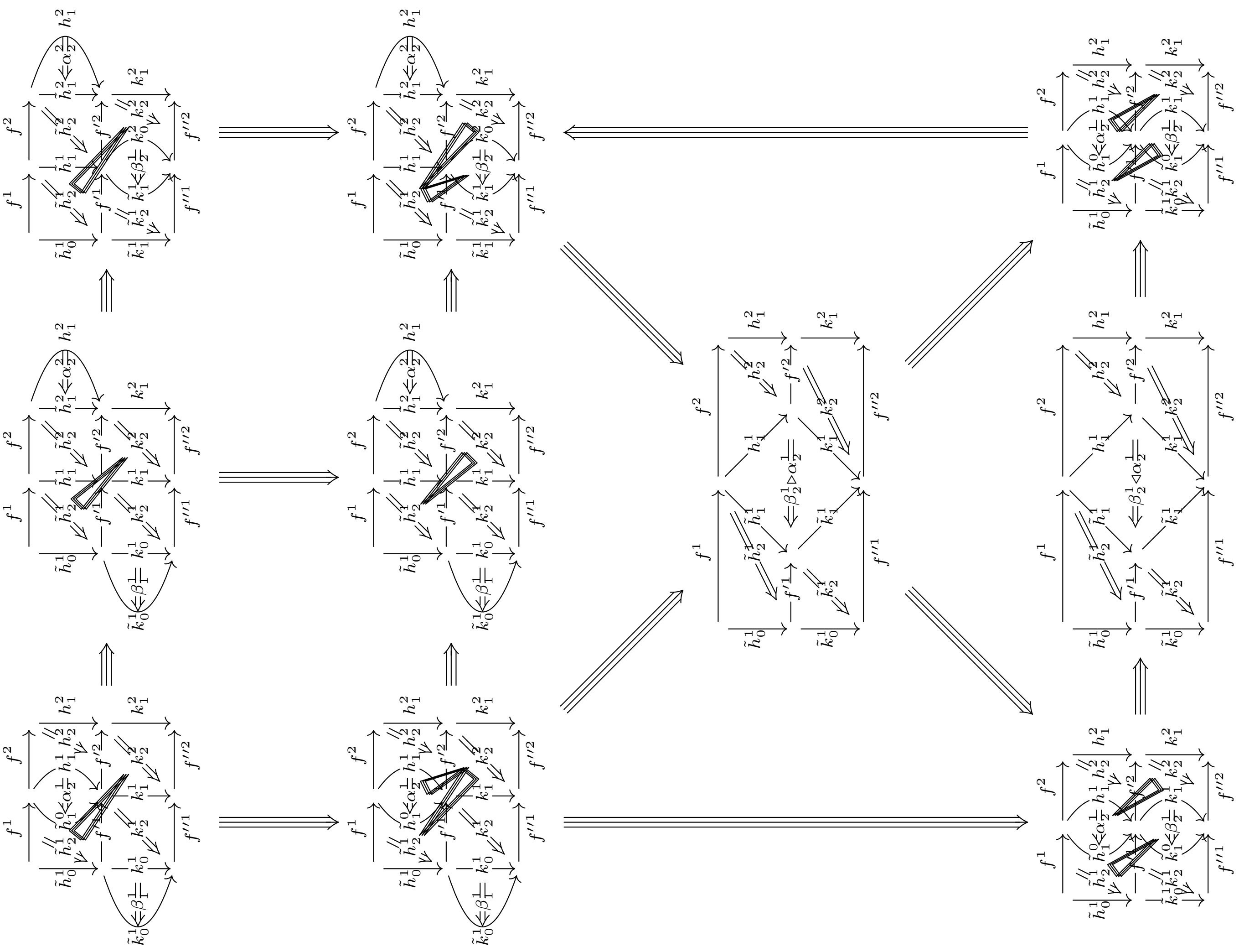

$m^2_{k,h\Box_0 g}$
$m(k)\Box_0 m^2_{h,g}$
$=(f'''^2\#_0 k^1_2\#_0 h^1_0\#_0 g^1_0)$
$\#_1(k^2_2\lhd h^1_2\#_0 g^1_0)$
$\#_1(k^2_1\#_0 h^2_2\otimes g^1_2)$
$\#_1(k^2_1\#_0 h^2_1\#_0 g^2_2\#_0 f^1)$
$(f'''^2\#_0 k^1_2\#_0 h^1_0\#_0 g^1_0)$
$\#_1(k^2_2\otimes h^1_2\#_0 g^1_0)$
$\#_1(k^2_1\#_0 h^2_2\rhd g^1_2)$
$\#_1(k^2_1\#_0 h^2_1\#_0 g^2_2\#_0 f^1)$
$\Big(f'''^2\#_0((k^1_2\#_0 h^1_0)$
$\#_1(k^1_1\#_0 h^1_2))\#_0 g^1_0\Big)$
$\#_1(k^2_2\otimes h^1_1\#_0 g^1_2)$
$\#_1\Big(k^2_1\#_0((h^2_2\#_0 g^1_1)$
$\#_1(h^2_1\#_0 g^2_2))\#_0 f^1\Big)$
$(f'''^2\#_0 k^1_2\#_0 h^1_0\#_0 g^1_0)$
$\#_1(k^2_2\otimes h^1_2\#_0 g^1_0)$
$\#_1(k^2_1\#_0 h^2_2\lhd g^1_2)$
$\#_1(k^2_1\#_0 h^2_1\#_0 g^2_2\#_0 f^1)$
$=m^2_{k,h}\Box_0 m(g)$
$(f'''^2\#_0 k^1_2\#_0 h^1_0\#_0 g^1_0)$
$\#_1(k^2_2\rhd h^1_2\#_0 g^1_0)$
$\#_1(k^2_1\#_0 h^2_2\otimes g^1_2)$
$\#_1(k^2_1\#_0 h^2_1\#_0 g^2_2\#_0 f^1)$
$m^2_{k\Box_0 h,g}$

(5.11)



(5.14)

(5.15)



and the pasting of the left and outer squares corresponds to the left hand side. Equality in (5.16) is equivalent to the top and bottom squares commuting, since the aforementioned ones do so by assumption.

We thus spell out the details of the top and bottom squares in (5.17): The diagram (5.18) shows the details of the top square of (5.17). The central octagon of (5.18) is broken down in (5.15). The parts of these two diagrams commute essentially by the Gray-category axioms and the definitions of 2- and 3-cells in the path space. The bottom square on (5.17) would be analogous.

This proves (3.57).

Furthermore, we check that tensors of cocycle elements are trivial: We calculate according to 4.5:

$$m^2_{f_1,f_2} \boxtimes m^2_{f_3,f_4} = ((m^2_{f_1,f_2})_1 \otimes (m^2_{f_3,f_4})_1, (m^2_{f_1,f_2})_2 \otimes (m^2_{f_3,f_4})_2)\,, \tag{5.19}$$

where according to (5.7) all the arguments on the right are trivial, hence their tensors are trivial, that is, (3.58) holds.

Lastly, images of 2-cells tensor trivially with co-cycle components by the unitality of the tensor in $\mathbb{H}$ and the fact that the 2-cell faces of m^2 are trivial, hence verifying (3.59) and (3.60). □

Theorem 38 *There is a pseudo* Gray*-functor* m *such that*

$$\overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}} \xrightarrow{m} \overrightarrow{\mathbb{H}} \underset{d_0}{\overset{d_1}{\rightrightarrows}} \overset{i}{\leftarrow} \mathbb{H} \tag{5.20}$$

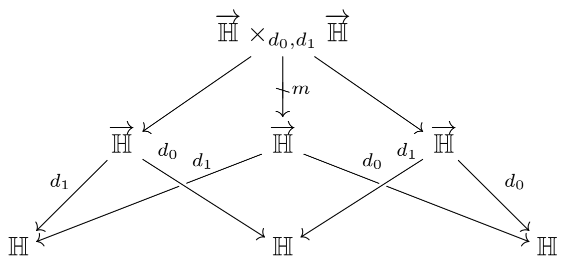
is an internal category object in $\mathsf{GrayCat}_{\mathrm{Q}^1}$*.*

Proof We need to verify that m is an associative and unital operation. We need to check first that

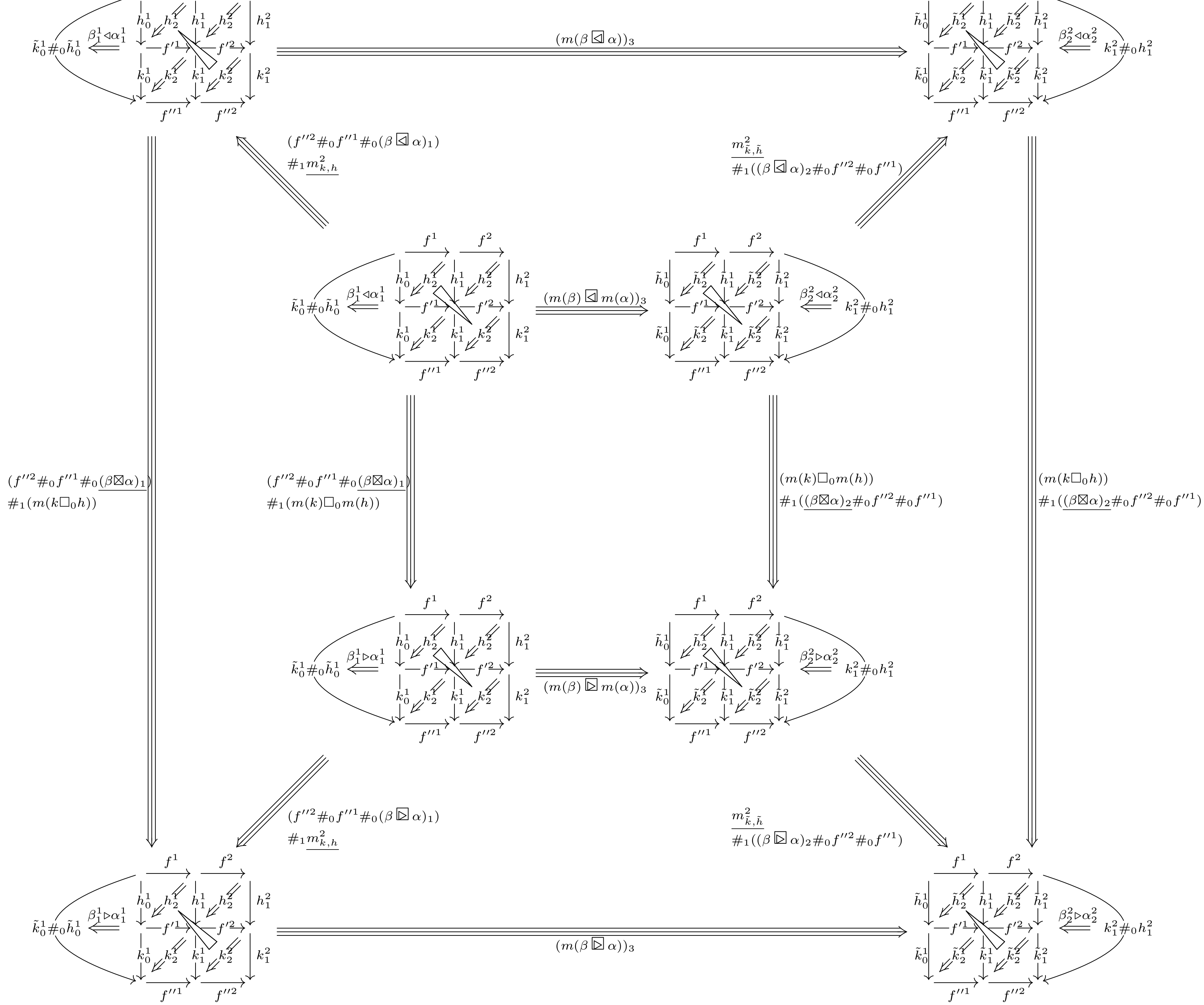
$$\begin{array}{ccc} \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} & \xrightarrow{\overrightarrow{\mathbb{H}}\times m} & \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \\ {\scriptstyle m\times\overrightarrow{\mathbb{H}}}\downarrow & & \downarrow{\scriptstyle m} \\ \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} & \xrightarrow[m]{} & \overrightarrow{\mathbb{H}} \end{array}\quad, \tag{5.21}$$

where $m \times \overrightarrow{\mathbb{H}}$ and $\overrightarrow{\mathbb{H}} \times m$ exist by the observation in 21. On the level of globular maps this is obvious, since it is just pasting according to 36. Proving that the cocylces both ways around are the same, means drawing a diagram that looks like (5.11) with each array transposed.

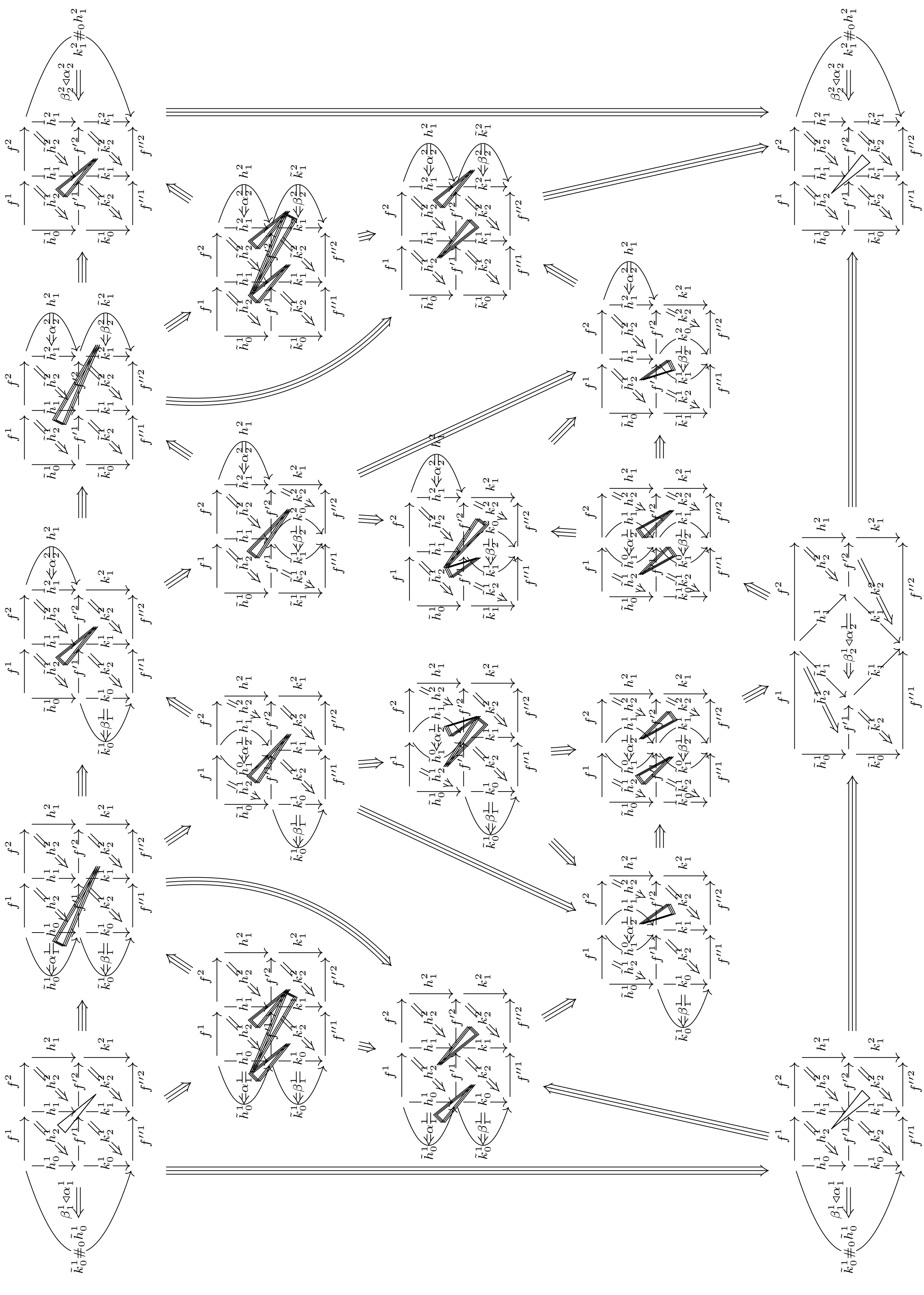
Unitality is obvious, source and target conditions



(5.22)


$(m(\beta \boxed{\lhd} \alpha))_3$
$(f''^2 \#_0 f''^1 \#_0 (\beta \boxed{\lhd} \alpha)_1) \#_1 m^2_{k,h}$
$m^2_{\tilde{k},\tilde{h}} \#_1 ((\beta \boxed{\lhd} \alpha)_2 \#_0 f''^2 \#_0 f''^1)$
$(m(\beta) \boxed{\lhd} m(\alpha))_3$
$(f''^2 \#_0 f''^1 \#_0 (\beta \boxtimes \alpha)_1) \#_1 (m(k \Box_0 h))$
$(f''^2 \#_0 f''^1 \#_0 (\beta \boxtimes \alpha)_1) \#_1 (m(k) \Box_0 m(h))$
$(m(k) \Box_0 m(h)) \#_1 ((\beta \boxtimes \alpha)_2 \#_0 f''^2 \#_0 f''^1)$
$(m(k \Box_0 h)) \#_1 ((\beta \boxtimes \alpha)_2 \#_0 f''^2 \#_0 f''^1)$
$(m(\beta) \boxed{\rhd} m(\alpha))_3$
$(f''^2 \#_0 f''^1 \#_0 (\beta \boxed{\rhd} \alpha)_1) \#_1 m^2_{k,h}$
$m^2_{\tilde{k},\tilde{h}} \#_1 ((\beta \boxed{\rhd} \alpha)_2 \#_0 f''^2 \#_0 f''^1)$
$(m(\beta \boxed{\rhd} \alpha))_3$

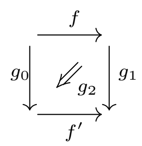
(5.17)



(5.18)

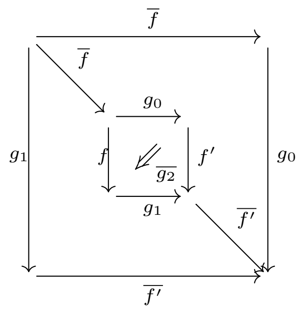
hold by 36. In particular, the 2-cell components of m^2 are trivial, thus $d_0 m$ and $d_1 m$ are strict Gray-functors, even though m is pseudo. □

We can define the 1-cell inverse to



(5.23)

with respect to m as

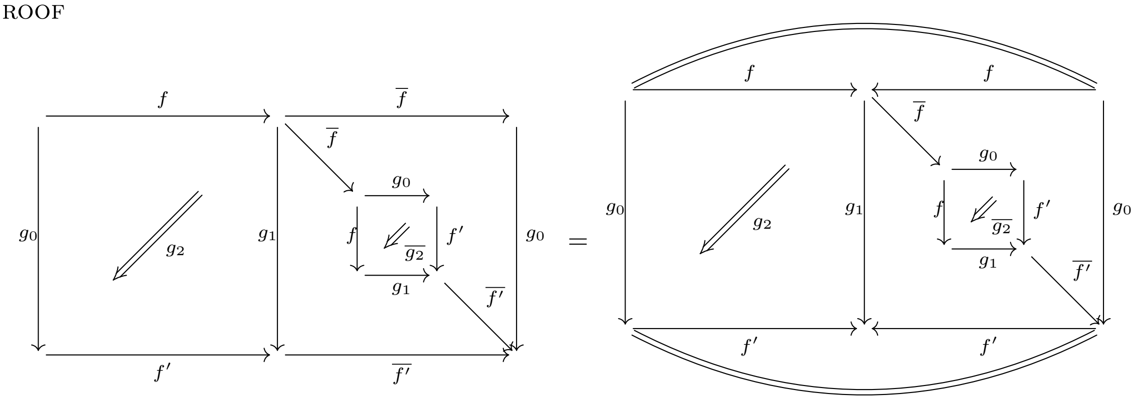


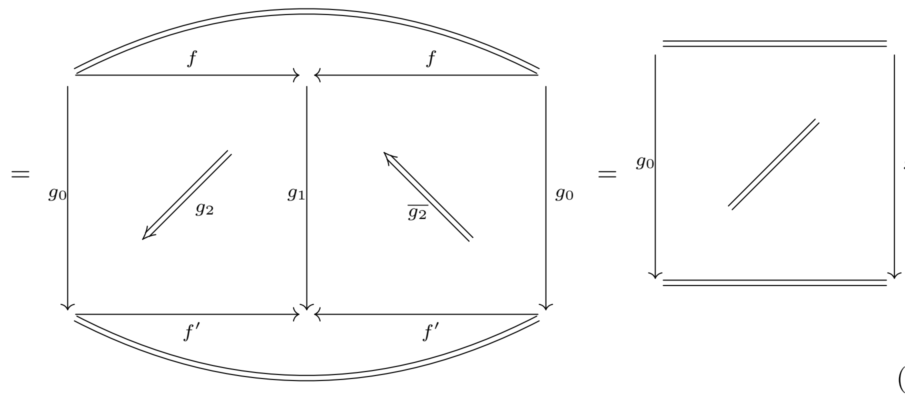
(5.24)

where $\overline{(\_)}$ is the respective vertical inverse in $\mathbb{H}$.

Lemma 39 *The path space 1-cell in* (5.24) *is a left and right inverse to* (5.23) *with respect to* m.

PROOF





(5.25)

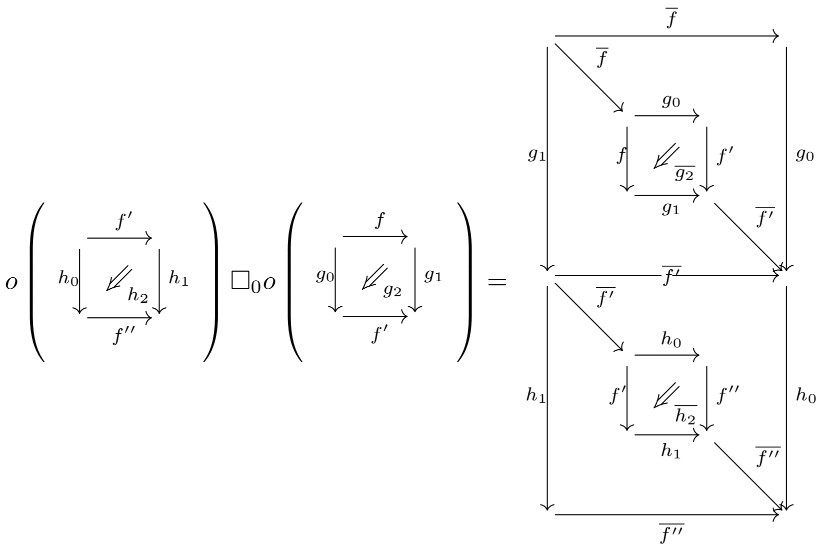
And similarly for the right inverse. □

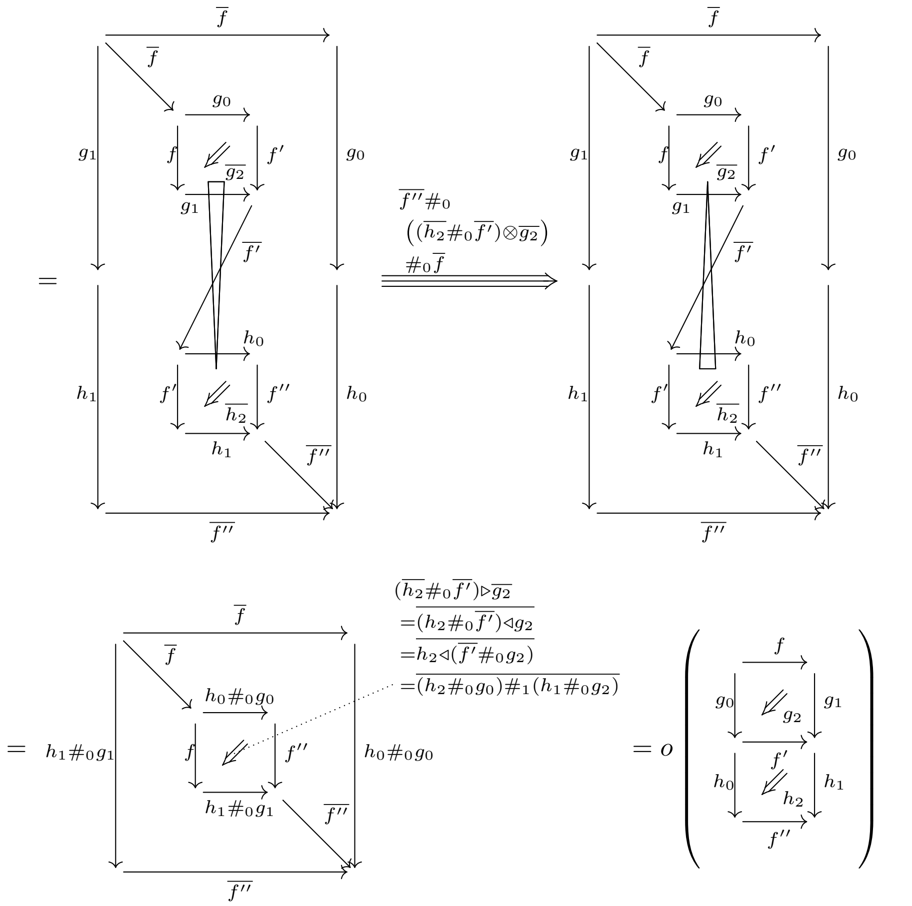
Furthermore these inverses behave well with respect to the internal category structure:

Theorem 40 *Given the situation in* (5.20)*, assume* $\mathbb{H}$ *is a* Gray*-groupoid, then there is a* Q^1*-map* $o\colon \overrightarrow{\mathbb{H}} \twoheadrightarrow \overrightarrow{\mathbb{H}}$ *("opposite") such that* (5.20) *becomes an internal groupoid in* $\mathsf{GrayCat}_{\mathrm{Q}^1}$*.*

Proof The action of o on 0- and 1-cells is already given in (5.24), we need to give its effect on 2- and 3-cells of $\overrightarrow{\mathbb{H}}$:

Furthermore, we need to give a 2-cocycle $o^2_{h,g}\colon o(h)\Box_0 o(g) \longrightarrow o(h\Box_0 g)$ the non-trivial part of which is the following 3-cell:





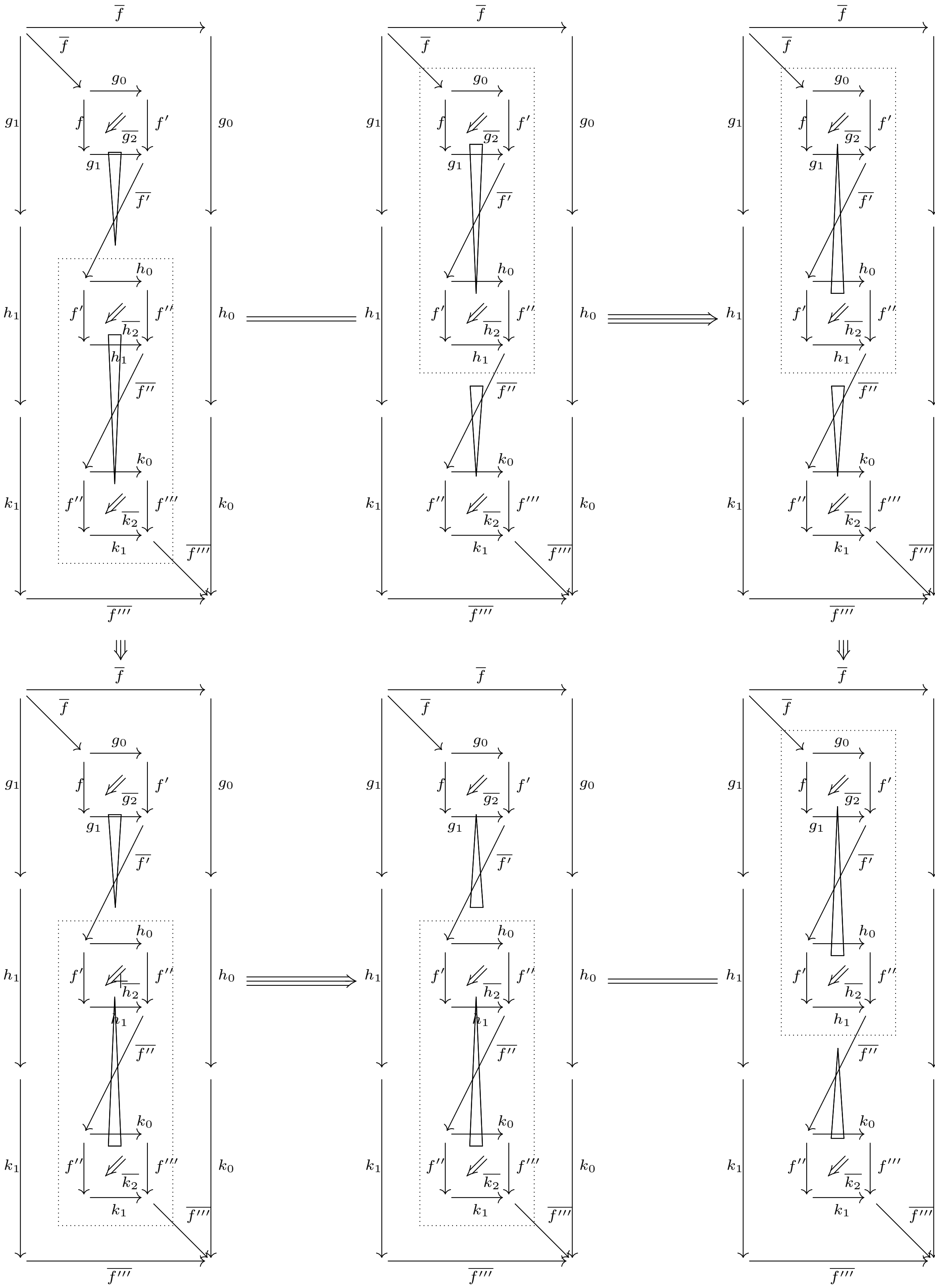
(5.26)

For the relationship between horizontal composition and pasting of squares see remark 31.

We check that o^2 is indeed a 2-cocycle. Given suitably incident 1-cells of $\mathbb{H}$ we need to verify that the analog of (3.53) hold, that is,

$$o^2_{k,h\square_0 g}\square_1(o(k)\square_0 o^2_{h,g}) = o^2_{k\square_0 h,g}\square_1(o^2_{k,h}\square_0 o(g)), \tag{5.27}$$

hence (5.28) commutes. $\square$


$\overline{f}$
g_0
g_1
f
f'
$\overline{g_2}$
$\overline{f'}$
h_0
h_1
f''
$\overline{h_2}$
$\overline{f''}$
k_0
k_1
f'''
$\overline{k_2}$
$\overline{f'''}$

(5.28)

Chapter 6

Higher Cells

In order to describe higher transformations between maps of Gray-categories we construct an internal Gray-category in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ as a substructure of the iterated path space.

6.1 Combining Path Spaces and Resolutions

We begin by describing explicitly the action of $\overrightarrow{e} : \overrightarrow{\mathrm{Q}^1\mathbb{G}} \longrightarrow \overrightarrow{\mathbb{G}}$ as follows:

$$\overrightarrow{e}\left(\xrightarrow{[f_1,\dots,f_{n_f}]} \right) = \left(\xrightarrow{f_1\#_0\cdots\#_0 f_{n_f}} \right) \tag{6.1}$$

$$\overrightarrow{e}\left(\begin{array}{ccc} & \xrightarrow{[f_1,\dots,f_{n_f}]} & \\ {\scriptstyle [g_{0,0},\dots,g_{0,n_{g_0}}]}\Big\downarrow & \Swarrow{\scriptstyle (g_2;[g_{1,0},\dots,g_{1,n_{g_1}},\, f_1,\dots,f_{n_f}],\, [f'_1,\dots,f'_{n_{f'}},\, g_{0,0},\dots,g_{0,n_{g_0}}])} & \Big\downarrow{\scriptstyle [g_{1,0},\dots,g_{1,n_{g_1}}]} \\ & \xrightarrow[{[f'_1,\dots,f'_{n_{f'}}]}]{} & \end{array} \right) = \left(\begin{array}{ccc} & \xrightarrow{f_1\#_0\cdots\#_0 f_{n_f}} & \\ {\scriptstyle g_{0,0}\#_0\cdots\#_0 g_{0,n_{g_0}}}\Big\downarrow & \Swarrow{\scriptstyle g_2} & \Big\downarrow{\scriptstyle g_{1,0}\#_0\cdots\#_0 g_{1,n_{g_1}}} \\ & \xrightarrow[{f'_1\#_0\cdots\#_0 f'_{n_{f'}}}]{} & \end{array} \right) \tag{6.2}$$

$$\overrightarrow{e}\begin{pmatrix} \begin{pmatrix} \alpha_3;[g_{1,1},\dots,g_{1,n_{g_1}},f_{1,1},\dots,f_{1,n_f}], \\ [f'_{1,1},\dots,f'_{1,n_{f'}},h_{0,1},\dots,h_{0,n_{h_0}}] \end{pmatrix}; \\ (\alpha_1;[g_{0,1},\dots,g_{0,n_{g_0}}],[h_{0,1},\dots,h_{0,n_{h_0}}]), \\ (\alpha_2;[g_{1,1},\dots,g_{1,n_{g_1}}],[h_{1,1},\dots,h_{1,n_{h_1}}]), \\ \begin{pmatrix} g_2;[g_{1,1},\dots,g_{1,n_{g_1}},f_{1,1},\dots,f_{1,n_f}], \\ [f'_{1,1},\dots,f'_{1,n_{f'}},g_{0,1},\dots,g_{0,n_{h_0}}] \end{pmatrix}, \\ \begin{pmatrix} h_2;[h_{1,1},\dots,h_{1,n_{h_1}},f_{1,1},\dots,f_{1,n_f}], \\ [f'_{1,1},\dots,f'_{1,n_{f'}},h_{0,1},\dots,h_{0,n_{h_0}}] \end{pmatrix}; \\ [g_{0,1},\dots,g_{0,n_{g_0}}],[g_{1,1},\dots,g_{1,n_{g_1}}], \\ [h_{0,1},\dots,h_{0,n_{h_0}}],[h_{1,1},\dots,h_{1,n_{h_1}}], \\ [f_{1,1},\dots,f_{1,n_f}],[f'_{1,1},\dots,f'_{1,n_{f'}}] \end{pmatrix} = \begin{pmatrix} \alpha_3;\alpha_1,\alpha_2,g_2,h_2; \\ g_{0,1}\#_0\cdots\#_0 g_{0,n_{g_0}},\, g_{1,1}\#_0\cdots\#_0 g_{1,n_{g_1}}, \\ h_{0,1}\#_0\cdots\#_0 h_{0,n_{h_0}},\, h_{1,1}\#_0\cdots\#_0 h_{1,n_{h_1}}, \\ f_{1,1}\#_0\cdots\#_0 f_{1,n_f},\, f'_{1,1}\#_0\cdots\#_0 f'_{1,n_{f'}} \end{pmatrix} \tag{6.3}$$

$$\overrightarrow{e}\begin{pmatrix}(\Gamma_1;\alpha_1,\beta_1,[g_{0,1},\dots,g_{0,n_{g_0}}],[h_{0,1},\dots,h_{0,n_{h_0}}]),\\ (\Gamma_2;\alpha_2,\beta_2,[g_{1,1},\dots,g_{1,n_{g_1}}],[h_{1,1},\dots,h_{1,n_{h_1}}])\end{pmatrix}=(\Gamma_1,\Gamma_2) \tag{6.4}$$

where for the 3-cells we used the abbreviated notation of (4.4).

Lemma 41 *The map* $\overrightarrow{e}\colon\overrightarrow{\mathrm{Q}^1\mathbb{G}}\longrightarrow\overrightarrow{\mathbb{G}}$ *is Cartesian with respect* $(\_)_1$.

Proof $\overrightarrow{e}$ is obviously surjective on 0- and 1-cells and 2-locally an isomorphism. □

Let $F\dashv U\colon \mathsf{Cat}\longrightarrow\mathsf{RGrph}$ be the usual adjunction, then $(\overrightarrow{e})_1\colon\overrightarrow{\mathrm{Q}^1\mathbb{G}_1}\longrightarrow\overrightarrow{\mathbb{G}}_1$ has a splitting $s\colon U(\overrightarrow{\mathbb{G}}_1)\longrightarrow U(\overrightarrow{\mathrm{Q}^1\mathbb{G}_1})$ under U as follows:

$$s\left(\xrightarrow{\;f\;}\right)=\left(\xrightarrow{\;[f]\;}\right) \tag{6.5}$$

$$s\left(\begin{array}{ccc} & \xrightarrow{\;f\;} & \\ g_0\big\downarrow & \swarrow_{g_2} & \big\downarrow g_1 \\ & \xrightarrow[\;f'\;]{} & \end{array}\right)=\left(\begin{array}{ccc} & \xrightarrow{\qquad[f]\qquad} & \\ [g_0]\Bigg\downarrow & (g_2;[g_1,f],[f',g_0]) \swarrow & \Bigg\downarrow[g_1] \\ & \xrightarrow[\qquad{[f']}\qquad]{} & \end{array}\right) \tag{6.6}$$

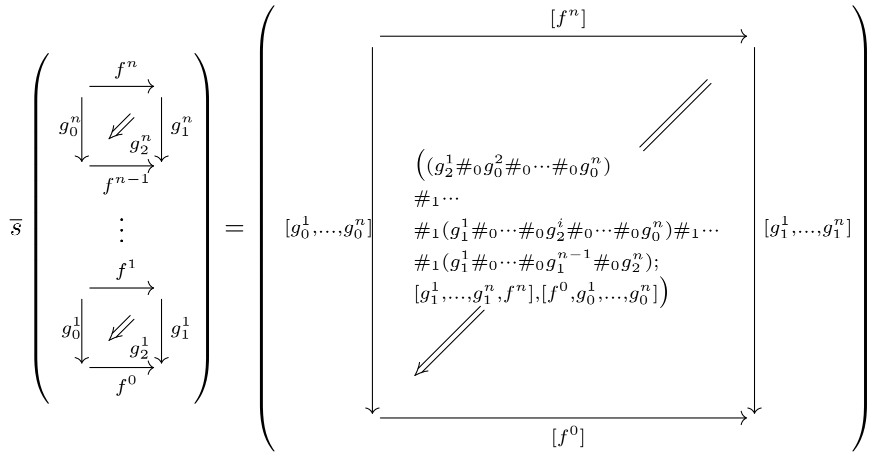
Obviously in RGrph we have $U(\overrightarrow{e}_1)s=\mathrm{id}_{U(\overrightarrow{\mathbb{G}}_1)}$, taking the transpose $\overline{s}$ we get

$$\begin{array}{ccc} FU(\overrightarrow{\mathbb{G}}_1)=\mathrm{Q}^1\overrightarrow{\mathbb{G}}_1 & \xrightarrow{\;\overline{s}\;} & \overrightarrow{\mathrm{Q}^1\mathbb{G}_1} \\ & {\scriptstyle\varepsilon=e_1}\searrow & \big\downarrow{\scriptstyle\overrightarrow{e}_1} \\ & & \overrightarrow{\mathbb{G}}_1 \end{array}\quad, \tag{6.7}$$

since $\overrightarrow{e}$ is Cartesian we can lift $\overline{s}$ through $(\_)_1$ to obtain $\psi\colon \mathrm{Q}^1\overrightarrow{\mathbb{G}}\longrightarrow\overrightarrow{\mathrm{Q}^1\mathbb{G}}$ satisfying

$$\begin{array}{ccc} \mathrm{Q}^1\overrightarrow{\mathbb{G}} & \xrightarrow{\;\psi_{\mathbb{G}}\;} & \overrightarrow{\mathrm{Q}^1\mathbb{G}} \\ & {\scriptstyle e_{\overrightarrow{\mathbb{G}}}}\searrow & \big\downarrow{\scriptstyle \overrightarrow{e_{\mathbb{G}}}} \\ & & \overrightarrow{\mathbb{G}} \end{array}\quad. \tag{6.8}$$

Let us consider the action of $\overline{s}\colon Q^1\overrightarrow{\mathbb{G}}_1 \longrightarrow \overrightarrow{Q^1\mathbb{G}}_1$. On 0-cells it acts just like s, on 1-cells we have the assignment:



(6.9)

Lemma 42 *The family ψ is natural with respect to maps $F\colon \mathbb{G} \longrightarrow \mathbb{H}$.*

Proof Consider the diagram

$$\begin{array}{ccccc} Q^1\overrightarrow{\mathbb{G}} & \xrightarrow{\psi_{\mathbb{G}}} & \overrightarrow{Q^1\mathbb{G}} & \xrightarrow{\overrightarrow{e_{\mathbb{G}}}} & \overrightarrow{\mathbb{G}} \\ \downarrow{\scriptstyle Q^1\overrightarrow{F}} & & \downarrow{\scriptstyle \overrightarrow{Q^1F}} & & \downarrow{\scriptstyle \overrightarrow{F}} \\ Q^1\overrightarrow{\mathbb{H}} & \xrightarrow[\psi_{\mathbb{H}}]{} & \overrightarrow{Q^1\mathbb{H}} & \xrightarrow[\overrightarrow{e_{\mathbb{H}}}]{} & \overrightarrow{\mathbb{H}} \end{array}, \qquad e_{\overrightarrow{\mathbb{G}}}\colon Q^1\overrightarrow{\mathbb{G}}\to\overrightarrow{\mathbb{G}},\quad e_{\overrightarrow{\mathbb{H}}}\colon Q^1\overrightarrow{\mathbb{H}}\to\overrightarrow{\mathbb{H}} \tag{6.10}$$

since the top and bottom triangles as well as the right hand square commute we obtain $\overrightarrow{e_{\mathbb{H}}}\psi_{\mathbb{H}}Q^1\overrightarrow{F} = \overrightarrow{e_{\mathbb{H}}}\overrightarrow{Q^1F}\psi_{\mathbb{G}}$. Since $\psi_1 = \overline{s}$ we need to only verify that $\overline{s}_{\mathbb{H}}(Q^1\overrightarrow{F})_1 = (\overrightarrow{Q^1F})_1\overline{s}_{\mathbb{G}}$, but this is immediate from the action of $\overrightarrow{(\_)}$ and Q^1. Naturality then follows by remark 14. □

It remains to verify that ψ is compatible with the co-multiplication $d\colon Q^1 \longrightarrow Q^1Q^1$, that is,

$$\begin{array}{ccccc} Q^1\overrightarrow{\mathbb{G}} & \xrightarrow{d_{\overrightarrow{\mathbb{G}}}} & Q^1Q^1\overrightarrow{\mathbb{G}} & \xrightarrow{Q^1\psi_{\mathbb{G}}} & Q^1\overrightarrow{Q^1\mathbb{G}} \\ \downarrow{\scriptstyle \psi_{\mathbb{G}}} & & & & \downarrow{\scriptstyle \psi_{Q^1\mathbb{G}}} \\ \overrightarrow{Q^1\mathbb{G}} & & \xrightarrow[\overrightarrow{d_{\mathbb{G}}}]{} & & \overrightarrow{Q^1Q^1\mathbb{G}} \end{array} \tag{6.11}$$

commutes. We will prove this using, again, remark 14 with $\overrightarrow{e}$ and the commutativity of the underlying diagram of categories

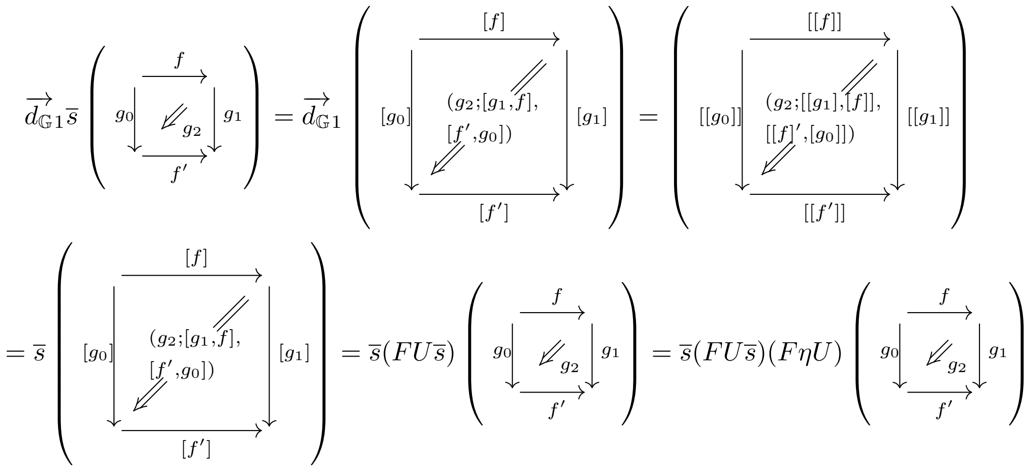
$$\begin{array}{ccccc} FU(\overrightarrow{\mathbb{G}}_1) & \xrightarrow{F\eta U} & FUFU(\overrightarrow{\mathbb{G}}_1) & \xrightarrow{FU\overline{s}} & FU(\overrightarrow{Q^1\mathbb{G}_1}) \\ \downarrow{\scriptstyle \overline{s}} & & & & \downarrow{\scriptstyle \overline{s}} \\ \overrightarrow{Q^1\mathbb{G}_1} & & \xrightarrow[\overrightarrow{d}_{\mathbb{G}1}]{} & & \overrightarrow{Q^1Q^1\mathbb{G}_1} \end{array} \tag{6.12}$$

But because the upper left object is free over the reflexive graph $U(\overrightarrow{\mathbb{G}}_1)$ it is sufficient to check for generating 0- and 1-cells.

For 0-cells we compute:

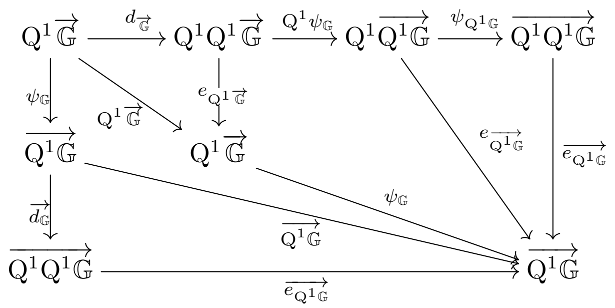
$$\overrightarrow{d}_{\mathbb{G}1}\overline{s}\left(\xrightarrow{f}\right) = \overrightarrow{d}_{\mathbb{G}1}\left(\xrightarrow{[f]}\right) = \left(\xrightarrow{[[f]]}\right)$$
$$= \overline{s}\left(\xrightarrow{[f]}\right) = \overline{s}(FU\overline{s})\left(\xrightarrow{f}\right) = \overline{s}(FU\overline{s})(F\eta U)\left(\xrightarrow{f}\right) \tag{6.13}$$

And likewise for 1-cells:



(6.14)

Furthermore we can check that post-composing (6.11) with $\overrightarrow{e}$ gives a commuting diagram:



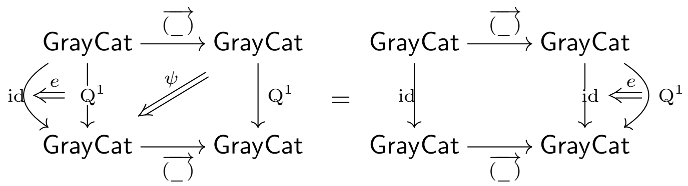
(6.15)

using (6.8), naturality of ψ in lemma 42, and the fact that Q^1 is a comonad. Hence we can cancel $\overrightarrow{e}$ and obtain (6.11).

So, we have proved the following

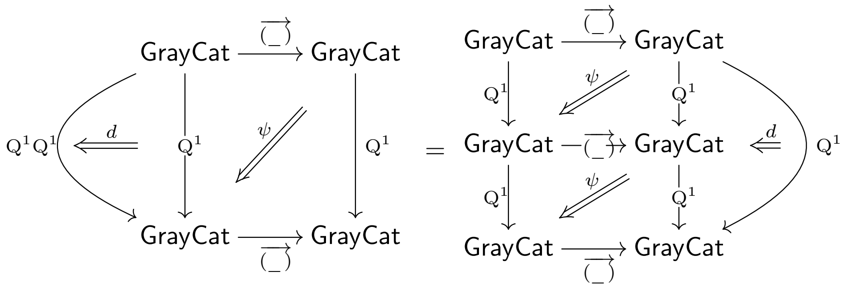
Lemma 43 *There is a natural transformation* $\psi\colon \mathrm{Q}^1\overrightarrow{(\_)} \longrightarrow \overrightarrow{\mathrm{Q}^1(\_)}$ *satisfying properties* (6.8) *and* (6.11). *We call it a* **semi-distributive law**. ☐

Remark 44 *In terms of formal category theory the pair* $(\overrightarrow{(\_)}, \psi)$ *is an endomorphism of the comonad* (Q^1, d, e)*, that is,*



(6.16)

and



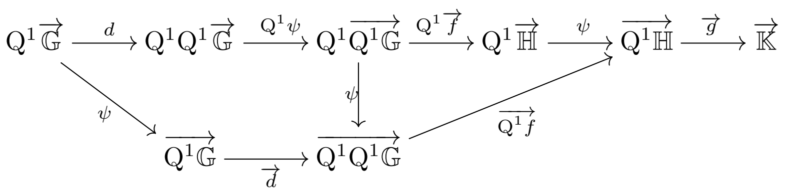
(6.17)

Lemma 45 *The functor* $\overrightarrow{(\_)}$ *extends canonically to an endofunctor* $\mathcal{P}$ *of* $\mathsf{GrayCat}_{\mathrm{Q}^1}$ *by*

$$\mathcal{P}\left(\mathbb{G} \xrightarrow{f} \mathbb{H}\right) = \left(\mathrm{Q}^1\overrightarrow{\mathbb{G}} \xrightarrow{\psi} \overrightarrow{\mathrm{Q}^1\mathbb{G}} \xrightarrow{\overrightarrow{f}} \overrightarrow{\mathbb{H}}\right) = \left(\overrightarrow{\mathbb{G}} \xrightarrow{\mathcal{P}(f)} \overrightarrow{\mathbb{H}}\right). \quad (6.18)$$

Furthermore, it preserves strictness of maps.

Proof We use the properties of ψ to check that this assignment is functorial. Given two maps $f\colon \mathbb{G} \nrightarrow \mathbb{H}$ and $g\colon \mathbb{H} \nrightarrow \mathbb{K}$ we compare $\mathcal{P}(g)\mathcal{P}(f)$ at the top and $\mathcal{P}(gf)$ at the bottom:



(6.19)

The naturality of ψ and (6.11) make sure they are equal. Preservation of units is exactly (6.8).

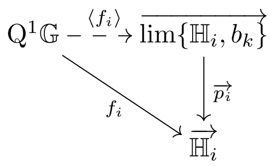
We remember that a strict map in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ is given by $fe_{\mathbb{G}}$ where $f\colon \mathbb{G} \longrightarrow \mathbb{H}$ is from $\mathsf{GrayCat}$ and e is the co-unit of Q^1. Then by (6.8) we get

$$\mathcal{P}(fe_{\mathbb{G}}) = \overrightarrow{f}\,\overrightarrow{e_{\mathbb{G}}}\psi_{\mathbb{G}} = \overrightarrow{f}\,e_{\overrightarrow{\mathbb{G}}}\,, \tag{6.20}$$

Meaning that $\mathcal{P}$ acts on strict maps like $\overrightarrow{(\_)}$, in particular, it takes identities to identities. □

Lemma 46 *The functor* $\mathcal{P}\colon \mathsf{GrayCat}_{\mathrm{Q}^1} \longrightarrow \mathsf{GrayCat}_{\mathrm{Q}^1}$ *preserves limits of diagrams of strict maps.*

Proof Finally, by lemma 35 the restriction $\overrightarrow{(\_)}$ of $\mathcal{P}$ to $\mathsf{GrayCat}$ preserves limits: Let $p_i\colon \lim\{\mathbb{H}_i, b_k\} \longrightarrow \mathbb{H}_i$ be a limit cone in $\mathsf{GrayCat}$, let $f_i\colon \mathbb{G} \nrightarrow \overrightarrow{\mathbb{H}}_i$ be a cone in $\mathsf{GrayCat}_{\mathrm{Q}^1}$.

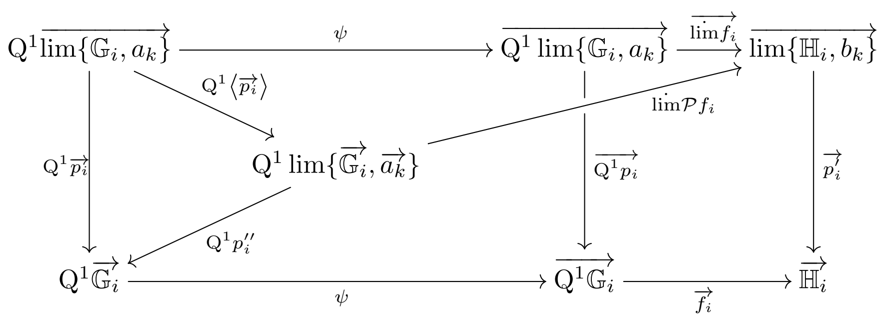


(6.21)

$\overrightarrow{p_i}$ is a limit cone, hence there is the unique weak map $\langle f_i\rangle\colon \mathbb{G} \nrightarrow \overrightarrow{\lim\{\mathbb{H}_i, b_k\}}$. □

Lemma 47 *The functor* $\mathcal{P}\colon \mathsf{GrayCat}_{\mathrm{Q}^1} \longrightarrow \mathsf{GrayCat}_{\mathrm{Q}^1}$ *preserves induced maps of limits of strict diagrams, that is,* $\mathcal{P}(\dot{\lim} f_i) = \dot{\lim}(\mathcal{P}f_i)$.

Proof Consider



(6.22)

using the conventions of 21. Also, note that $\overrightarrow{\dot{\lim} f_i}\psi = \mathcal{P}(\dot{\lim} f_i)$ by definition. $\dot{\lim} f_i$ is the induced arrow for the source $f_i(\mathrm{Q}^1 p_i)$, $\dot{\lim}\mathcal{P}f_i$ is the induced arrow for $\mathcal{P}(f_i)\mathrm{Q}^1(p_i'')$. Since

$$\overrightarrow{p_i}(\dot{\lim}\mathcal{P}f_i)\mathrm{Q}^1\langle\overrightarrow{p_i}\rangle = \overrightarrow{p_i}\,\overrightarrow{\dot{\lim} f_i}\psi \tag{6.23}$$

and $\overrightarrow{p_i}$ is a limit cone we obtain

$$(\dot{\lim}\mathcal{P}f_i)\mathrm{Q}^1\langle\overrightarrow{p_i}\rangle = \overrightarrow{\dot{\lim} f_i}\psi\,. \tag{6.24}$$

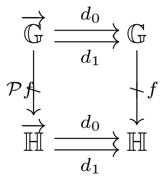
□

If the limit is, for example, a product we may now say that

$$\mathcal{P}(f \dot{\times} g) = \mathcal{P}f \dot{\times} \mathcal{P}g\,. \qquad (6.25)$$

From now on however we shall use $\times$ for the product of arrows in $\mathsf{GrayCat}_{\mathrm{Q}^1}$.

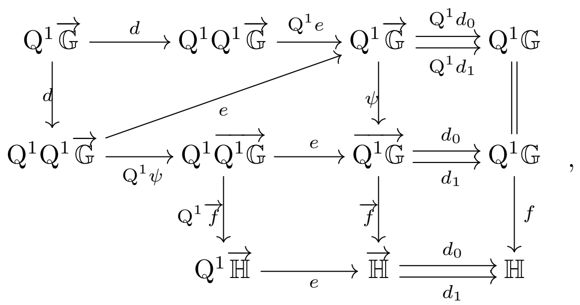
Lemma 48 *The face maps are natural with respect to weak maps, that is*



(6.26)

commutes.

Proof We write (6.26) in terms of its underlying maps:

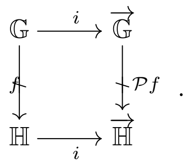


(6.27)

that is, (6.26) commuting is equivalent to the outer frame in (6.27) commuting. All parts are given by naturality and the co-unit laws of Q^1, except the upper right square.

We use remark 14 to conclude $d_0\psi = \mathrm{Q}^1 d_0$ and $d_1\psi = \mathrm{Q}^1 d_1$: By naturality and semi-distributivity we get $ed_0\psi = d_0 \overrightarrow{e}\psi = d_0 e = e\mathrm{Q}^1 d_0$, furthermore $(d_0\psi)_1 = (\mathrm{Q}^1 d_0)_1$ is immediate from the definition of ψ. The map d_1 is obviously treated in the same way. □

Lemma 49 *The degeneracy maps of the path space are natural with respect to weak maps:*



(6.28)

PROOF Consider

$$\begin{array}{ccccccc}
Q^1\mathbb{G} & \xrightarrow{d} & Q^1Q^1\mathbb{G} & \xrightarrow{Q^1 e} & Q^1\mathbb{G} & \xrightarrow{Q^1 i} & Q^1\overrightarrow{\mathbb{G}} \\
\Big\downarrow{\scriptstyle d} & & & & \Big\| & & \Big\downarrow{\scriptstyle \psi} \\
Q^1Q^1\mathbb{G} & & \xrightarrow{\quad e \quad} & & Q^1\mathbb{G} & \xrightarrow{i} & \overrightarrow{Q^1\mathbb{G}} \\
\Big\downarrow{\scriptstyle Q^1 f} & & & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \overrightarrow{f}} \\
Q^1\mathbb{H} & & \xrightarrow[\quad e \quad]{} & & \mathbb{H} & \xrightarrow[i]{} & \overrightarrow{\mathbb{H}}
\end{array} \quad . \tag{6.29}$$

We conclude that then top right square commutes by computing $\overrightarrow{e}i = ie = e\mathrm{Q}^1 i = \overrightarrow{e}\psi \mathrm{Q}^1 i$ and checking that $(\psi \mathrm{Q}^1 i)_1 = i_1$ and again applying remark 14 together with lemma 41. □

The functor $\mathcal{P}$ can also be applied to Q^1-graph maps by setting $\mathcal{P}' = (\mathcal{P}\tilde{G})^\vee$; see lemma 23 for the notation. For the sake of completeness we describe briefly the effect of $\mathcal{P}'$ at the level of 1-cells as well as its 2-co-cycle. Let $G\colon \mathbb{G} \longrightarrow \mathbb{H}$ be a Q^1-graph map. We take a 1-cell $g\colon f \longrightarrow f'$ from $\overrightarrow{\mathbb{G}}$ and calculate:

$$(\mathcal{P}'G)(g) = \left(\overrightarrow{G}\psi\right)^\vee(g) = \overrightarrow{G}\psi \left[\begin{array}{ccc} & \xrightarrow{f} & \\ g_0 \Big\downarrow & \overset{g_2}{\swarrow\!\!\!\swarrow} & \Big\downarrow g_1 \\ & \xrightarrow[f']{} & \end{array} \right]$$

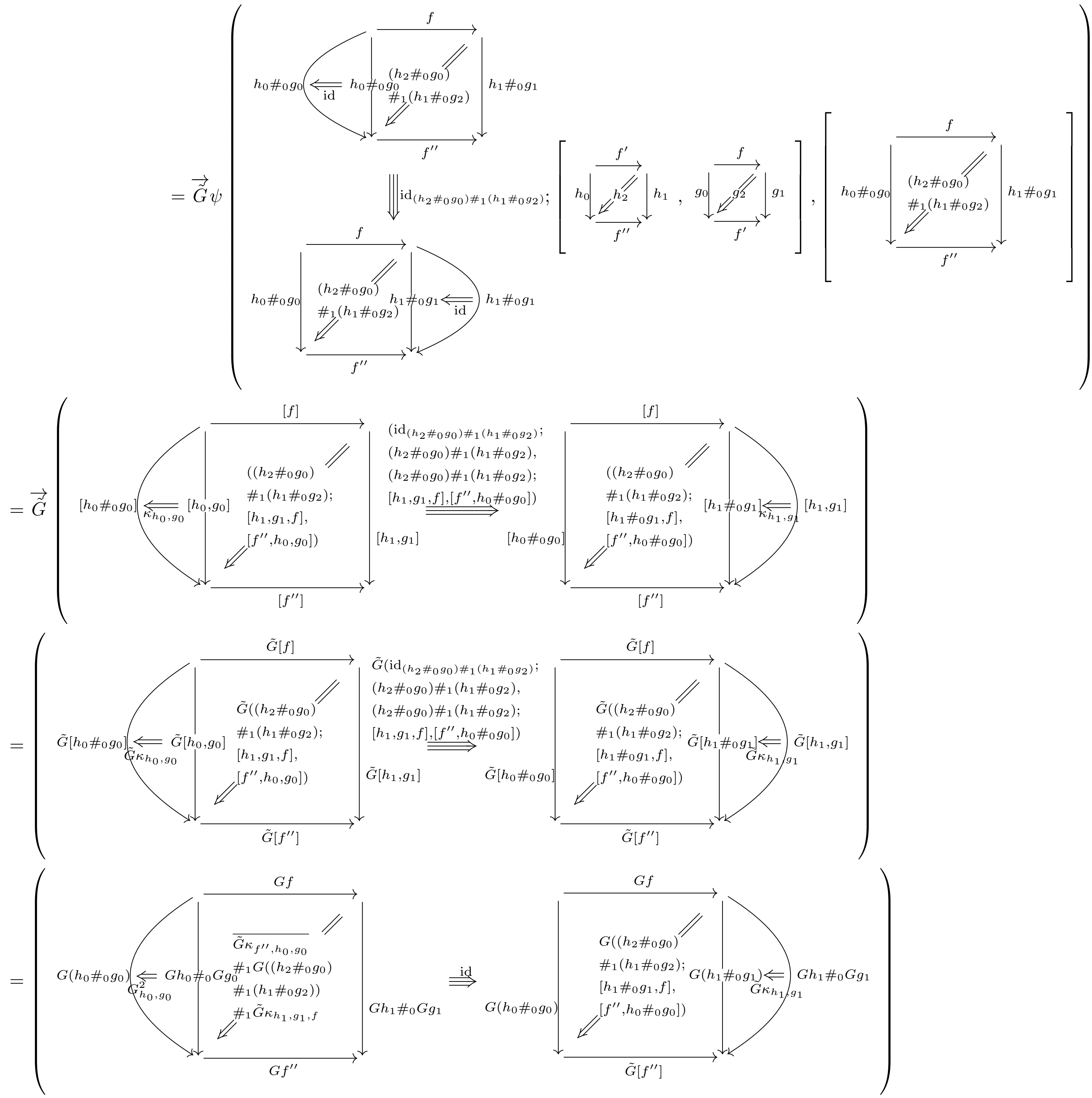
$$= \left(\begin{array}{ccc} & \xrightarrow{\tilde{G}[f]} & \\ \tilde{G}[g_0] \Big\downarrow & \begin{array}{c} \tilde{G}(g_2;[g_1,f], \\ [f,g_0]) \ \swarrow\!\!\!\swarrow \end{array} & \Big\downarrow \tilde{G}[g_1] \\ & \xrightarrow[\tilde{G}[f']]{} & \end{array} \right) = \left(\begin{array}{ccc} & \xrightarrow{Gf} & \\ Gg_0 \Big\downarrow & \begin{array}{c} \overline{G^2_{f',g_0}} \\ \#_1 Gg_2 \\ \swarrow\!\!\!\swarrow \#_1 G^2_{g_1,f} \end{array} & \Big\downarrow Gg_1 \\ & \xrightarrow[Gf']{} & \end{array} \right) \tag{6.30}$$

Taking two composable 1-cells $g\colon f \longrightarrow f'$ and $h\colon f' \longrightarrow f''$ of $\overrightarrow{\mathbb{G}}$ we get a 2-cocycle with components as shown in (6.31), where in the end the $\tilde{G}\kappa_{\dots}$ are iterated 2-cocycles of G.

6.2 Iterating the Path Space Construction

Remark 50 *As a consequence on lemma 48 and lemma 49 The maps i, d_0, d_1 and m for all* Gray*-categories $\mathbb{H}$ constitute natural transformations with respect to strict maps.*

$$((\mathcal{P}'G)^\vee)^2_{h,g} = ((\overrightarrow{\tilde{G}}\psi)^\vee)^2_{h,g} = \overrightarrow{\tilde{G}}\psi(\kappa_{h,g})$$



(6.31)

For reference, this means that for all $f\colon \mathbb{H} \longrightarrow \mathbb{K}$ the following diagram commutes sequentially:

$$\begin{array}{ccccc} \overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}} & \xrightarrow{m} & \overrightarrow{\mathbb{H}} & \overset{d_1}{\underset{d_0}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xleftarrow{i}}}}} & \mathbb{H} \\ \downarrow{\scriptstyle \overrightarrow{f}\times\overrightarrow{f}} & & \downarrow{\scriptstyle \overrightarrow{f}} & & \downarrow{\scriptstyle f} \\ \overrightarrow{\mathbb{K}} \times_{\mathbb{K}} \overrightarrow{\mathbb{K}} & \xrightarrow[m]{} & \overrightarrow{\mathbb{K}} & \overset{d_1}{\underset{d_0}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xleftarrow{i}}}}} & \mathbb{K} \end{array} \tag{6.32}$$

Iterating the arrow construction yields an internal cubical set, so it allows us to talk about higher cells in the internal language of GrayCat. But since we want to construct an internal Gray-category we need to restrict to cubical cells with certain degeneracies. The general recipe beyond the construction in section 4 is to apply $\overrightarrow{(\_)}$ and squash the excess faces given by $\overrightarrow{d_{0,1}}$ so that the only non-trivial faces of each cubical element are the ones given by $d_{0,1}$.

This general procedure will canonically yield an internal reflexive n-graph, we will furthermore have to provide the operations in each degree to actuallty obtain a Gray-category. We carry out this construction for the degrees 2 and 3 in 6.2 and 6.2.

2-Paths

We construct the space of 2-paths $\overline{\overline{\mathbb{H}}}$ over $\overrightarrow{\mathbb{H}}$ and give the vertical composition of 2-paths and their whiskers by 1-paths.

The 0-cells in $\overrightarrow{\overrightarrow{\mathbb{H}}}$ are squares, and we want to filter out those square that are actually bigons, that is, have identity arrows as left and right sides. That is exactly what we get by forming the double pullback on the left:

$$\begin{array}{ccccc} \overline{\overline{\mathbb{H}}} & \xrightarrow{j} & \overrightarrow{\overrightarrow{\mathbb{H}}} & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & \overrightarrow{\mathbb{H}} \\ {\scriptstyle \overline{d_0}}\downdownarrows{\scriptstyle \overline{d_1}} & & {\scriptstyle \overrightarrow{d_0}}\downdownarrows{\scriptstyle \overrightarrow{d_1}} & & {\scriptstyle d_0}\downdownarrows{\scriptstyle d_1} \\ \mathbb{H} & \xrightarrow[i]{} & \overrightarrow{\mathbb{H}} & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & \mathbb{H} \end{array} \tag{6.33}$$

where $\overline{\overline{\mathbb{H}}}$ is the intersection of the pullbacks of d_0 and d_1 along i. Let $d_0^j = d_0 j$ and $d_1^j = d_1 j$.

Lemma 51 *The diagram*

$$\overline{\overline{\mathbb{H}}} \overset{d_1^j}{\underset{d_0^j}{\rightrightarrows}} \overrightarrow{\mathbb{H}} \overset{d_1}{\underset{d_0}{\rightrightarrows}} \mathbb{H} \tag{6.34}$$

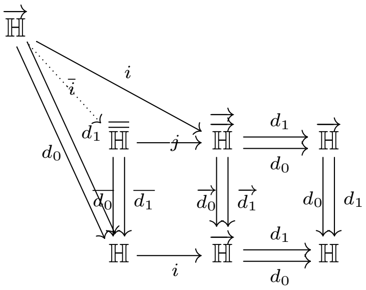
is a globular object, i.e. $d_0 d_0^j = d_0 d_1^j$ *and* $d_1 d_0^j = d_1 d_1^j$.

Proof Using the naturality of d_0 and d_1 we calculate:

$$d_0 d_0^j = d_0 d_0 j = d_0 \overrightarrow{d_0} j = d_0 i \overline{d_0} = d_1 i \overline{d_0} = d_1 \overrightarrow{d_0} j = d_0 d_1 j = d_1 d_0^j\,, \tag{6.35}$$

and similarly for d_1. □

To get a unit for $\overline{\overline{\mathbb{H}}}$, that is, an identity 2-paths for 1-paths, we consider the following diagram:



(6.36)

the upper left span is a compatible source by the naturality of i. The induced arrow $\bar{i}$ is a joint section of d_0^j and d_1^j. Hence we get:

Lemma 52 *The diagram*

$$\overline{\overline{\mathbb{H}}} \underset{d_0^j}{\overset{d_1^j}{\underset{\bar{i}}{\leftrightarrows}}} \overrightarrow{\mathbb{H}} \tag{6.37}$$

is a reflexive graph. □

Lemma 53 *The mapping* $\overline{\overline{(-)}}$ *extends to a sub-functor of* $\overrightarrow{\overrightarrow{(-)}}\colon \mathsf{GrayCat} \longrightarrow \mathsf{GrayCat}$ *with natural embedding* j.

Proof For each $\mathbb{H}$ the map j is a monomorphism by construction and $\overline{\overline{(-)}}$ extends to morphisms by the universal property. □

Lemma 54 *There is a multiplication*

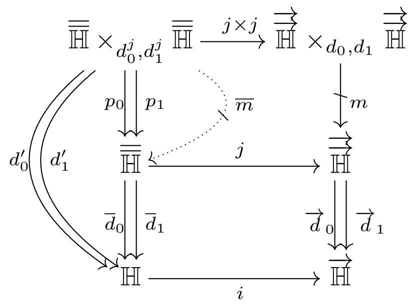
$$\overline{\overline{\mathbb{H}}} \times_{d_0^j, d_1^j} \overline{\overline{\mathbb{H}}} \xrightarrow{\overline{m}} \overline{\overline{\mathbb{H}}} \tag{6.38}$$

with

$$\begin{aligned} d_0^j \overline{m} &= d_0^j p_1 \\ d_1^j \overline{m} &= d_1^j p_0 \end{aligned} \tag{6.39}$$

uniquely induced by $m_{\overrightarrow{\mathbb{H}}}$.

Proof All we need to show is that $m(j \times j)$ factors through j, that is, show that the two outer rectangles commute:



(6.40)

that is, we shall verify that

$$\overrightarrow{d_0} m(j \times j) = id'_0 \tag{6.41}$$

$$\overrightarrow{d_1} m(j \times j) = id'_1 \tag{6.42}$$

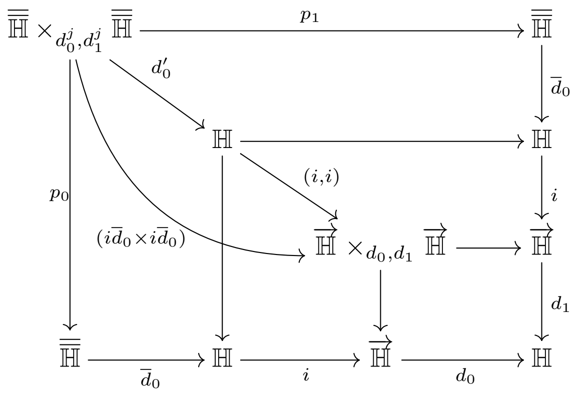
in order to obain $\overline{m}$ as a universally induced arrow.

First we prove that $\overline{d}_0 p_0 = \overline{d}_0 p_1$:

$$\overline{d}_0 p_0 = d_0 i \overline{d}_0 p_0 = d_0 \overrightarrow{d}_0 j p_0 = d_0 d_0 j p_0 = d_0 d_0^j p_0 = d_0 d_1^j p_1 = d_0 d_0^j p_1 = \overline{d}_0 p_1 \tag{6.43}$$

which holds by (6.37), (6.34) and (6.33). Similarly $\overline{d}_1 p_0 = \overline{d}_1 p_1$. Thus we may define $d'_0 = \overline{d}_0 p_0$ and $d'_1 = \overline{d}_1 p_0$. Note that $j \times j$ is universally induced by $d_0 j p_0 = d_1 j p_1$.

Furthermore we need that $(i\overline{d}_0 \times i\overline{d}_0) = (i,i)d'_0$ and $(i\overline{d}_1 \times i\overline{d}_1) = (i,i)d'_1$. Consider



(6.44)

The top and left squares commute by (6.43) and (6.34) makes the pair $(i\overline{d}_0 p_0, i\overline{d}_0 p_1)$ a Compatible source for lower right pullback square. The universality thus proves our equation.

Finally we verify that

$$\overrightarrow{d}_0 m(j \times j) = m(\overrightarrow{d}_0 \times \overrightarrow{d}_0)(j \times j) = m(\overrightarrow{d}_0 j \times \overrightarrow{d}_0 j) = m(i\overline{d}_0 j \times i\overline{d}_0 j) = m(i,i)d'_0 = id'_0 \tag{6.45}$$

By the same token $d_1 m(j \times j) = id'_1$ hence we get the desired $\overline{m}$.

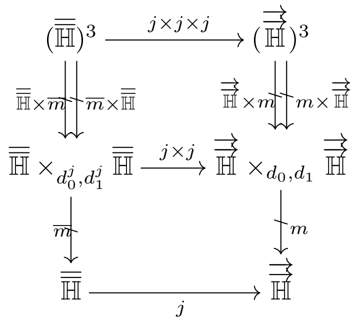
To check (6.39) we calculate:

$$d_0^j \overline{m} = d_0 j \overline{m} = d_0 m(j \times j) = d_0 p_1 (j \times j) = d_0 j p_1 = d_0^j p_1 \,.$$

□

Lemma 55 *The composition $\overline{m}$ is unital and associative, that is, it makes (6.37) a category.*

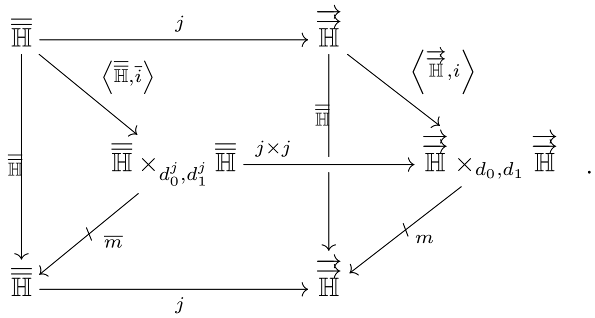
PROOF Obvious since $m_{\overrightarrow{\mathbb{H}}}$ is so: Using the notation of (6.40) we can formulate the associativity condition as the two composites in the left hand column being equal:



(6.46)

whence we conclude that $j\overline{m}(\overline{\overline{\mathbb{H}}} \times \overline{m}) = j\overline{m}(\overline{m} \times \overline{\overline{\mathbb{H}}})$, and by j mono we get the desired $\overline{m}(\overline{\overline{\mathbb{H}}} \times \overline{m}) = \overline{m}(\overline{m} \times \overline{\overline{\mathbb{H}}})$.

For the unit we can argue in the same manner:



(6.47)

□

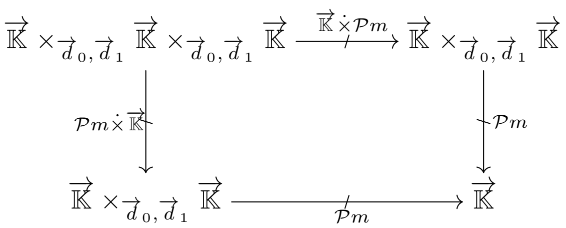
Lemma 56 *Applying $\mathcal{P}$ to an internal category*

$$\mathbb{K} \times_{d_0,d_1} \mathbb{K} \xrightarrow{m} \mathbb{K} \underset{d_0}{\overset{d_1}{\underset{\longleftarrow i}{\rightrightarrows}}} \mathbb{H} \tag{6.48}$$

yields an internal category

$$\overrightarrow{\mathbb{K}} \times_{\overrightarrow{d_0},\overrightarrow{d_1}} \overrightarrow{\mathbb{K}} = \overrightarrow{\mathbb{K} \times_{d_0,d_1} \mathbb{K}} \xrightarrow{\mathcal{P}m} \overrightarrow{\mathbb{K}} \underset{\overrightarrow{d_0}}{\overset{\overrightarrow{d_1}}{\underset{\longleftarrow \overrightarrow{i}}{\rightrightarrows}}} \overrightarrow{\mathbb{H}}\,. \tag{6.49}$$

Proof This is true since $\mathcal{P}$ is an endofunctor of $\mathsf{GrayCat}_{\mathbb{Q}^1}$ that by 35 preserves pullbacks of strict diagrams. In particular



(6.50)

commutes since by (6.25) $\mathcal{P}(\mathbb{K}\dot{\times}m) = \overrightarrow{\mathbb{K}}\dot{\times}\mathcal{P}m$. □

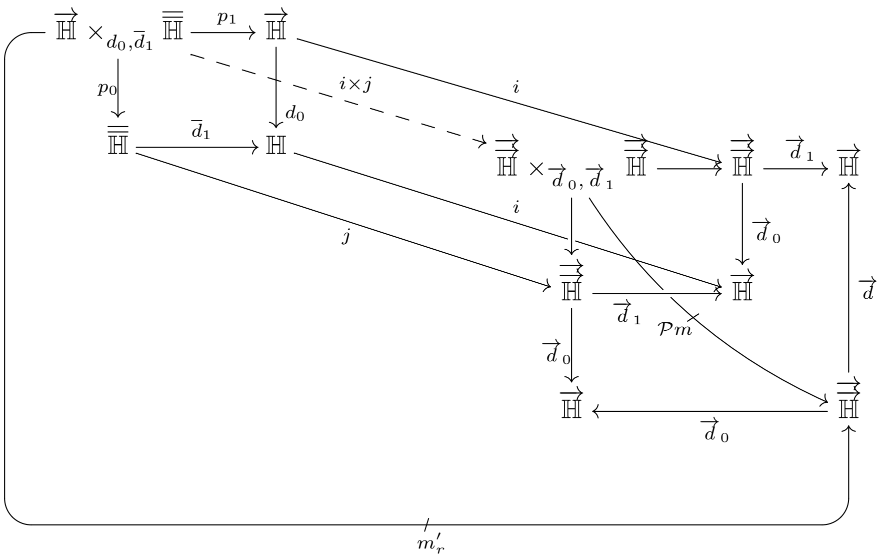
Lemma 57 *There are left and right whiskering maps*

$$\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},d_1} \overrightarrow{\mathbb{H}} \xrightarrow{w_\ell} \overline{\overline{\mathbb{H}}} \tag{6.51}$$

$$\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} \xrightarrow{w_r} \overline{\overline{\mathbb{H}}} \tag{6.52}$$

induced uniquely by $\mathcal{P}(m)$.

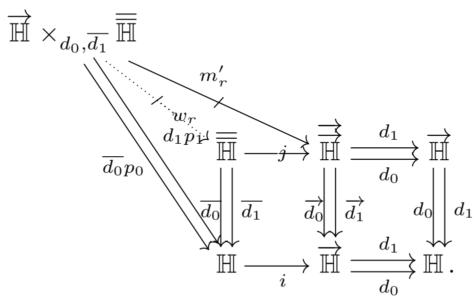
Proof We construct a restricted horizontal composition $m'_r\colon \overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} \nrightarrow \overrightarrow{\overrightarrow{\mathbb{H}}}$ in the following diagram:



(6.53)

where $i\times j$ is universally induced and m'_r is defined as the composite $\mathcal{P}(m)(i\times j)$. We need to show that m'_r factors through $\overline{\overline{\mathbb{H}}}$.

Consider the defining pullback for $\overline{\overline{\mathbb{H}}}$:



(6.54)

We need to show that $\overrightarrow{d}_0 m'_r = i\overline{d}_0 p_0$ and $\overrightarrow{d}_1 m'_r = i d_1 p_1$ to obtain a universal w_r, hence we calculate:

$$\overrightarrow{d}_0 m'_r = \overrightarrow{d}_0\mathcal{P}(m)(i\times j) = \overrightarrow{d}_0 j p_0 = \overline{d}_0 p_0 \tag{6.55}$$

$$\overrightarrow{d}_1 m'_r = \overrightarrow{d}_1 \mathcal{P}(m)(i \times j) = \overrightarrow{d}_1 i p_1 = \overline{d}_1 p_1 \tag{6.56}$$

using the definitions of $i \times j$ and j as well as the naturality of i.

For w_ℓ there is a corresponding argument. □

Lemma 58 *Left and right whiskering are compatible and associative, that is, the diagrams*

$$\begin{array}{ccc}
\overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \times_{d_0,\overline{d}_1} \overline{\overline{\mathbb{H}}} & \xrightarrow{\overrightarrow{\mathbb{H}} \times w_r} & \overrightarrow{\mathbb{H}} \times_{d_0,\overline{d}_1} \overline{\overline{\mathbb{H}}} \\
{\scriptstyle m \times \overrightarrow{\mathbb{H}}} \downarrow & & \downarrow {\scriptstyle w_r} \\
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d}_1} \overline{\overline{\mathbb{H}}} & \xrightarrow[w_r]{} & \overline{\overline{\mathbb{H}}}
\end{array} \tag{6.57}$$

$$\begin{array}{ccc}
\overline{\overline{\mathbb{H}}} \times_{\overline{d}_0,d_1} \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} & \xrightarrow{w_\ell \times \overrightarrow{\mathbb{H}}} & \overline{\overline{\mathbb{H}}} \times_{\overline{d}_0,d_1} \overrightarrow{\mathbb{H}} \\
{\scriptstyle \overrightarrow{\mathbb{H}} \times m} \downarrow & & \downarrow {\scriptstyle w_\ell} \\
\overline{\overline{\mathbb{H}}} \times_{\overline{d}_0,d_1} \overrightarrow{\mathbb{H}} & \xrightarrow[w_\ell]{} & \overline{\overline{\mathbb{H}}}
\end{array} \tag{6.58}$$

$$\begin{array}{ccc}
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d}_1} \overline{\overline{\mathbb{H}}} \times_{\overline{d}_0,d_1} \overrightarrow{\mathbb{H}} & \xrightarrow{w_r \times \overrightarrow{\mathbb{H}}} & \overline{\overline{\mathbb{H}}} \times_{\overline{d}_0,d_1} \overrightarrow{\mathbb{H}} \\
{\scriptstyle \overrightarrow{\mathbb{H}} \times w_\ell} \downarrow & & \downarrow {\scriptstyle w_\ell} \\
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d}_1} \overline{\overline{\mathbb{H}}} & \xrightarrow[w_r]{} & \overline{\overline{\mathbb{H}}}
\end{array} \tag{6.59}$$

commute.

Proof The objects in the above diagram embed into pullbacks of $\overrightarrow{\overrightarrow{\mathbb{H}}}$ by j and these pullbacks being preserved by $\mathcal{P}$ and the monicity of j yield the desired result. □

Lemma 59 *w_ℓ and w_r extend m. That is*

$$\begin{array}{ccc}
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} & \xrightarrow{w_r} & \overline{\overline{\mathbb{H}}} \\
{\scriptstyle \overrightarrow{\mathbb{H}} \times d_0^j} \downarrow \; \uparrow {\scriptstyle \overrightarrow{\mathbb{H}} \times \overline{i}} \; \downarrow {\scriptstyle \overrightarrow{\mathbb{H}} \times d_1^j} & & {\scriptstyle d_0^j} \downarrow \; \uparrow {\scriptstyle \overline{i}} \; \downarrow {\scriptstyle d_1^j} \\
\overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} & \xrightarrow[m]{} & \overrightarrow{\mathbb{H}}
\end{array}
\qquad
\begin{array}{ccc}
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},d_1} \overrightarrow{\mathbb{H}} & \xrightarrow{w_\ell} & \overline{\overline{\mathbb{H}}} \\
{\scriptstyle d_0 \times \overrightarrow{\mathbb{H}}} \downarrow \; \uparrow {\scriptstyle \overrightarrow{\mathbb{H}} \times \overline{i}} \; \downarrow {\scriptstyle d_1 \times \overrightarrow{\mathbb{H}}} & & {\scriptstyle d_0^j} \downarrow \; \uparrow {\scriptstyle \overline{i}} \; \downarrow {\scriptstyle d_1^j} \\
\overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} & \xrightarrow[m]{} & \overrightarrow{\mathbb{H}}
\end{array} \tag{6.60}$$

commute serially, and the outside 0-faces are preserved:

$$\begin{aligned}
\overline{d_0} w_r &= \overline{d_0} p_1 & \qquad \overline{d_0} w_\ell &= d_0 p_1 \\
\overline{d_1} w_r &= d_1 p_0 & \qquad \overline{d_1} w_\ell &= \overline{d_1} p_0
\end{aligned} \tag{6.61}$$

Proof Considering the proof of lemma 57 we calculate:

$$d_0^j w_r = d_0 j w_r = d_0 m_r' = d_0 \mathcal{P} m(i \times j) = m(d_0 \times d_0)(i \times j) = m(\overrightarrow{\mathbb{H}} \times d_0^j)\,. \quad (6.62)$$

Similarly for d_1^j and w_ℓ.

The equations (6.61) hold by the construction as given in (6.54). □

Lemma 59 allows us to define left and right horizontal composites. Call the composite along the middle in the following diagram $h_\ell\colon \overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} \nrightarrow \overline{\overline{\mathbb{H}}}$:

$$\begin{array}{ccccc}
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} & \overset{w_r}{\nrightarrow} & \overline{\overline{\mathbb{H}}} & & \\
\uparrow{\scriptstyle d_0^j \times \overline{\overline{\mathbb{H}}}} & & \uparrow & & \\
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} & \dashrightarrow & \overline{\overline{\mathbb{H}}} \times_{d_0^j,d_1^j} \overline{\overline{\mathbb{H}}} & \overset{\overline{m}}{\nrightarrow} & \overline{\overline{\mathbb{H}}} \\
\downarrow{\scriptstyle \overline{\overline{\mathbb{H}}} \times d_1^j} & & \downarrow & & \\
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},d_1} \overrightarrow{\mathbb{H}} & \underset{w_\ell}{\nrightarrow} & \overline{\overline{\mathbb{H}}} & &
\end{array} \quad , \quad (6.63)$$

and correspondingly $h_r\colon \overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} \nrightarrow \overline{\overline{\mathbb{H}}}$:

$$\begin{array}{ccccc}
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},d_1} \overrightarrow{\mathbb{H}} & \overset{w_\ell}{\nrightarrow} & \overline{\overline{\mathbb{H}}} & & \\
\uparrow{\scriptstyle \overline{\overline{\mathbb{H}}} \times d_1^j} & & \uparrow & & \\
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} & \dashrightarrow & \overline{\overline{\mathbb{H}}} \times_{d_0^j,d_1^j} \overline{\overline{\mathbb{H}}} & \overset{\overline{m}}{\nrightarrow} & \overline{\overline{\mathbb{H}}} \\
\downarrow{\scriptstyle d_0^j \times \overline{\overline{\mathbb{H}}}} & & \downarrow & & \\
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} & \underset{w_r}{\nrightarrow} & \overline{\overline{\mathbb{H}}} & &
\end{array} \quad . \quad (6.64)$$

Lemma 60 *Left and right horizontal composites give a globular object*

$$\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} \underset{h_r}{\overset{h_\ell}{\rightrightarrows}} \overline{\overline{\mathbb{H}}} \underset{d_0^j}{\overset{d_1^j}{\rightrightarrows}} \overrightarrow{\mathbb{H}} \quad . \quad (6.65)$$

Proof We calculate:

$$d_0^j h_\ell \overset{(6.63)}{=} d_0 j \overline{m} \left\langle w_r(d_0^j \times \overline{\overline{\mathbb{H}}}), w_\ell(\overline{\overline{\mathbb{H}}} \times d_1^j) \right\rangle \quad (6.66)$$

$$\overset{(6.40)}{=} d_0 m(j \times j) \left\langle w_r(d_0^j \times \overline{\overline{\mathbb{H}}}), w_\ell(\overline{\overline{\mathbb{H}}} \times d_1^j) \right\rangle \quad (6.67)$$

$$\overset{(6.54)}{=} d_0 p_0 \left\langle m_r'(d_0^j \times \overline{\overline{\mathbb{H}}}), m_\ell'(\overline{\overline{\mathbb{H}}} \times d_1^j) \right\rangle \quad (6.68)$$

$$= d_0 m_r'(d_0^j \times \overline{\overline{\mathbb{H}}}) \quad (6.69)$$

$$= d_0 \mathcal{P} m(i \times j)(d_0^j \times \overline{\overline{\mathbb{H}}}) \quad (6.70)$$

$$\overset{(6.26)}{=} m(d_0 \times d_0)(i \times j)(d_0^j \times \overline{\overline{\mathbb{H}}}) \quad (6.71)$$

$$= m(d_0^j \times d_0^j) \tag{6.72}$$

and by the same token

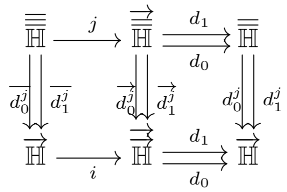
$$d_0^j h_r = m(d_0^j \times d_0^j)\,. \tag{6.73}$$

Analogously for d_1^j. □

3-Paths

We proceed to construct the internal 3-path object and the operations involving 3-cells. Note that the $\overline{(\_)}$ and $\widetilde{(\_)}$ used in this section is not at all a functor.

We apply the construction in (6.33) to (6.37) as follows:



(6.74)

By (6.37) we get a reflexive graph

$$\overline{\overline{\overline{\mathbb{H}}}} \underset{d_0^j}{\overset{d_1^j}{\rightrightarrows}} \overline{\overline{\mathbb{H}}} \quad (\text{with } \bar{i} : \overline{\overline{\mathbb{H}}} \to \overline{\overline{\overline{\mathbb{H}}}}) \tag{6.75}$$

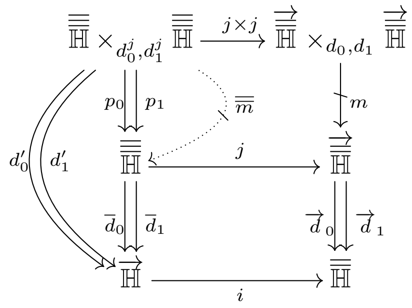
where by (6.34)

$$\overline{\overline{\overline{\mathbb{H}}}} \underset{d_0^j}{\overset{d_1^j}{\rightrightarrows}} \overline{\overline{\mathbb{H}}} \underset{d_0^j}{\overset{d_1^j}{\rightrightarrows}} \overrightarrow{\mathbb{H}} \underset{d_0}{\overset{d_1}{\rightrightarrows}} \mathbb{H} \tag{6.76}$$

is a 3-globular object. Furthermore, by applying the reasoning of lemma 54 we get a vertical multiplication map

$$\overline{\overline{\overline{\mathbb{H}}}} \times_{d_0^j, d_1^j} \overline{\overline{\overline{\mathbb{H}}}} \xrightarrow{\overline{m}} \overline{\overline{\overline{\mathbb{H}}}} \tag{6.77}$$

arising as a restriction of $m_{\overline{\overline{\mathbb{H}}}}$:



(6.78)

where $d_0' = \overline{d_0} p_0$ and $d_1' = \overline{d_1} p_1$.

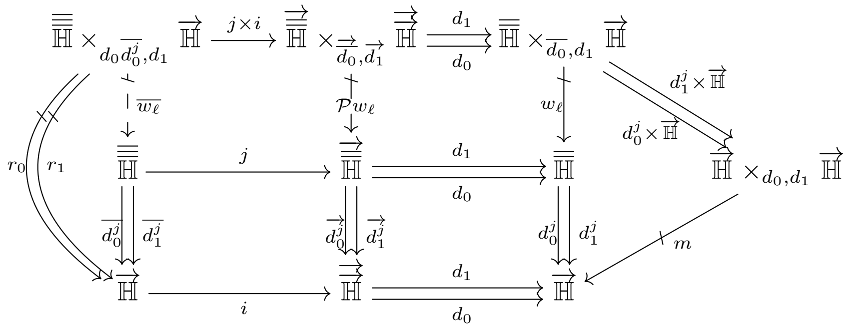
Lemma 61 *There are left and right whiskering maps*

$$\overline{\overline{\overline{\mathbb{H}}}} \times_{d_0\overline{d_0^j},d_1} \overrightarrow{\mathbb{H}} \xrightarrow{\overline{w_\ell}} \overline{\overline{\overline{\mathbb{H}}}} \tag{6.79}$$

$$\overrightarrow{\mathbb{H}} \times_{d_0,d_1\overline{d_1^j}} \overline{\overline{\overline{\mathbb{H}}}} \xrightarrow{\overline{w_r}} \overline{\overline{\overline{\mathbb{H}}}} \tag{6.80}$$

induced uniquely by $\mathcal{P}w_\ell$ and $\mathcal{P}w_r$.

Proof We define $\overline{w_\ell}$ as the universally induced arrow in the following diagram:



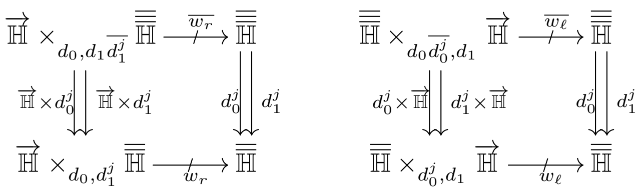
(6.81)

where $r_0 = m(\overline{d_0^j} \times \overrightarrow{\mathbb{H}})$ and $r_1 = m(\overline{d_1^j} \times \overrightarrow{\mathbb{H}})$. We calculate

$$\begin{aligned} & ir_0 \\ & = im(\overline{d_0^j}\times\overrightarrow{\mathbb{H}}) = \mathcal{P}m(i\times i)(\overline{d_0^j}\times\overrightarrow{\mathbb{H}}) = \mathcal{P}m(i\overline{d_0^j}\times i) = \mathcal{P}m(\overrightarrow{d_0^j}j\times i) = \mathcal{P}(d_0^j w_\ell)(j\times i) \\ & \qquad = \overrightarrow{d_0^j}\mathcal{P}w_\ell(j\times i)\,, \end{aligned} \tag{6.82}$$

and likewise for r_1 and $\overrightarrow{d_1^j}$. And hence we obtain $\overline{w_\ell}$, and $\overline{w_r}$ by analogy. □

Lemma 62 *$\overline{w_\ell}$ and $\overline{w_r}$ extend w_ℓ and w_r respectively. That is*



(6.83)

commute serially.

Proof Inspecting (6.81) we can calculate

$$\begin{aligned} & d_0^j\overline{w_\ell} \\ & \qquad = d_0 j\overline{w_\ell} = d_0\mathcal{P}(w_\ell)(j\times i) = w_\ell d_0(j\times i) = w_\ell(d_0\times d_0)(j\times i) \\ & \qquad = w_\ell(d_0^j\times\overrightarrow{\mathbb{H}})\,. \end{aligned} \tag{6.84}$$

And likewise for the other squares in (6.83). □

Lastly, we need the whiskering of a 3-path by a 2-path along a 1-path. We can reapply the basic scheme of 57.

Lemma 63 *There are left and right whiskering maps*

$$\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0^j}, d_1^j} \overline{\overline{\mathbb{H}}} \xrightarrow{\tilde{w}_\ell} \overline{\overline{\overline{\mathbb{H}}}} \tag{6.85}$$

$$\overline{\overline{\mathbb{H}}} \times_{d_0^j, \overline{d_1^j}} \overline{\overline{\overline{\mathbb{H}}}} \xrightarrow{\tilde{w}_r} \overline{\overline{\overline{\mathbb{H}}}} \tag{6.86}$$

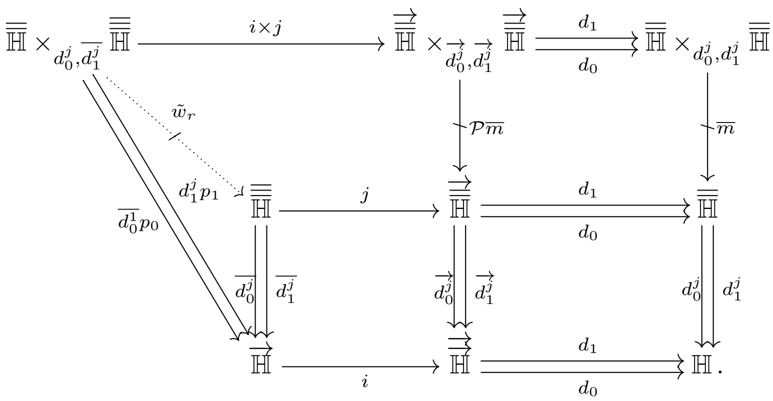
induced uniquely by $\mathcal{P}(\overline{m})$.

And these extend $\overline{m}$, that is

$$d_0^j \tilde{w}_r = \overline{m}(\overline{\overline{\overline{\mathbb{H}}}} \times d_0^j) \qquad d_1^j \tilde{w}_r = \overline{m}(\overline{\overline{\overline{\mathbb{H}}}} \times d_1^j) \tag{6.87}$$

$$d_0^j \tilde{w}_\ell = \overline{m}(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}) \qquad d_1^j \tilde{w}_\ell = \overline{m}(d_1^j \times \overline{\overline{\overline{\mathbb{H}}}}) \tag{6.88}$$

Proof The desired map arises as a universal arrow in the following diagram:



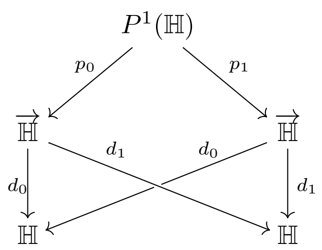
. (6.89)

Now, we can verify $i\overline{d_0^j}p_0 = \overrightarrow{d_0^j}jp_0 = \overrightarrow{d_0^j}p_0(i \times j) = \overrightarrow{d_0^j}\mathcal{P}\overline{m}(i \times j)$ and $id_1^jp_1 = \overrightarrow{d_1^j}jp_1 = \overrightarrow{d_1^j}p_1(i \times j) = \overrightarrow{d_1^j}\mathcal{P}\overline{m}(i \times j)$.

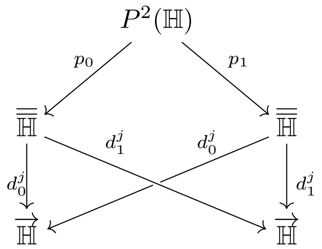
The equations (6.87) are now immediate. □

6.3 The Space of Parallel Cells

For a Gray-category $\mathbb{H}$ we define the space of parallel 1-cells $P^1(\mathbb{H})$ as the following limit:



(6.90)



(6.91)

Lemma 64 *The canonical map* $\left\langle d_0^j, d_1^j \right\rangle \colon \overline{\overline{\overline{\mathbb{H}}}} \longrightarrow P^2(\mathbb{H})$ *is 1-Cartesian.*

PROOF Consider the following cells in $\overline{\overline{\overline{\mathbb{H}}}}$

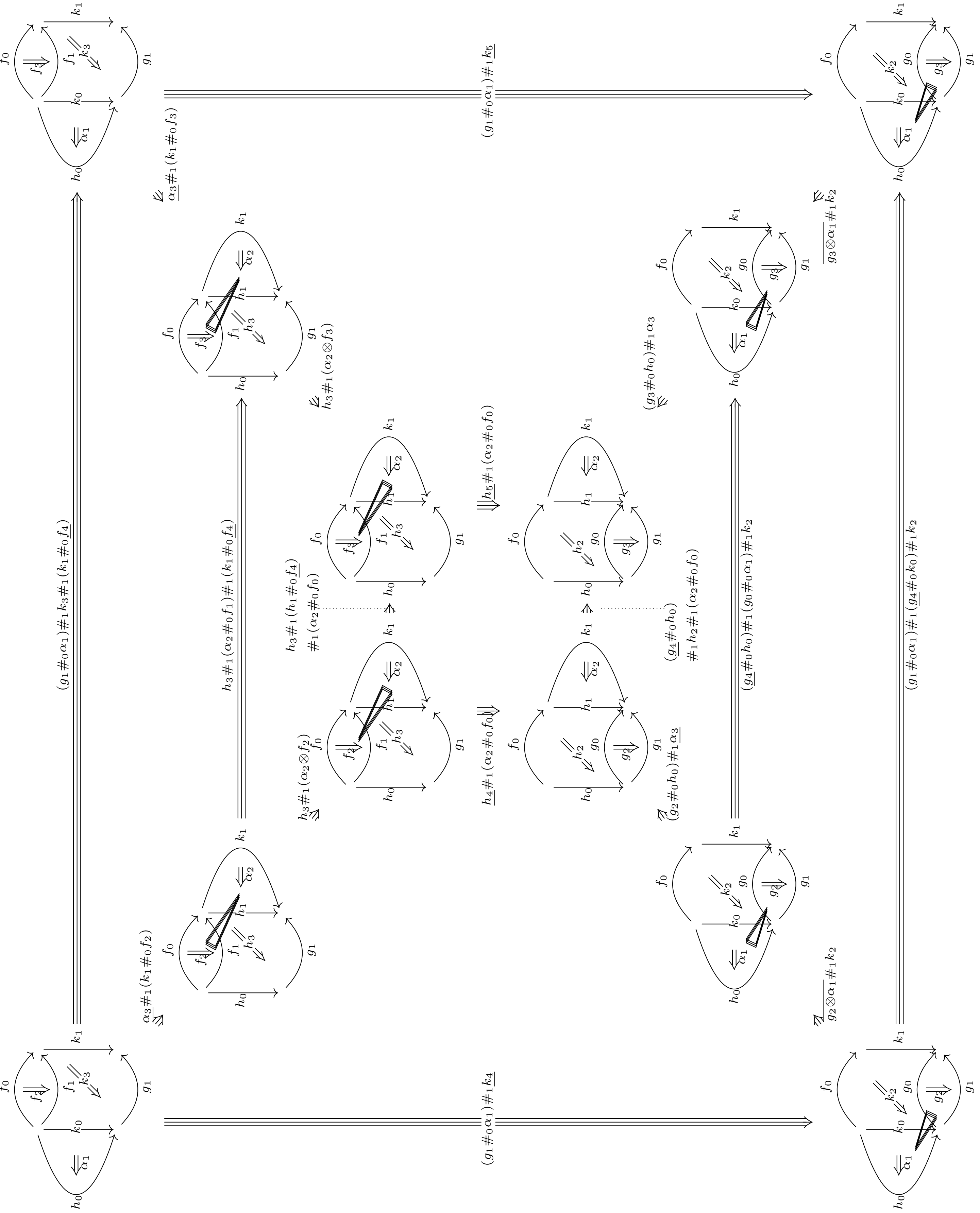
$$f = (f_4; f_2, f_3; f_0, f_1) \tag{6.92}$$
$$g = (g_4; g_2, g_3; g_0, g_1) \tag{6.93}$$
$$h = (h_4, h_5; h_2, h_3; h_0, h_1) \colon f \longrightarrow g \tag{6.94}$$
$$k = (k_4, k_5; k_2, k_3; k_0, k_1) \colon f \longrightarrow g \tag{6.95}$$
$$\alpha = (\alpha_3; \alpha_1, \alpha_2) \colon h \Longrightarrow k \tag{6.96}$$

By construction the map $\left\langle d_0^j, d_1^j \right\rangle$ acts on this data as follows:

$$f \mapsto ((f_2; f_0, f_1), (f_3; f_0, f_1)) \tag{6.97}$$
$$g \mapsto ((g_2; g_0, g_1), (g_3; g_0, g_1)) \tag{6.98}$$
$$h \mapsto ((h_4; h_2, h_3; h_0, h_1), (h_5; h_2, h_3; h_0, h_1)) \tag{6.99}$$
$$k \mapsto ((k_4; k_2, k_3; k_0, k_1), (k_5; k_2, k_3; k_0, k_1)) \tag{6.100}$$
$$\alpha \mapsto ((\alpha_3; \alpha_1, \alpha_2), (\alpha_3; \alpha_1, \alpha_2)) \tag{6.101}$$

where on the right we find parallel pairs of cells from $\overline{\overline{\mathbb{H}}}$, that is, in (6.102) the central square, the outer square, and the left and right trapezoids commute by assumption.

The requisite compatibility conditions for f, g, h, k, α to be cells of $\overline{\overline{\overline{\mathbb{H}}}}$ are displayed in (6.102). We obverse that the remaining trapezoids at the top and the bottom commute by naturality of $\#_1$ and $\otimes$ in $\mathbb{H}$. Hence we conclude that given 1-cells h, k in $\overline{\overline{\overline{\mathbb{H}}}}$ all higher cells, including 3-cells, between them are determined by their image under $\left\langle d_0^j, d_1^j \right\rangle$. □



(6.102)

Lemma 65 *The 3-paths compose horizontally along 2-paths, that is,*

$$\begin{array}{ccc}
\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0^j},\overline{d_1^j}} \overline{\overline{\overline{\mathbb{H}}}} & \xrightarrow{\left\langle \tilde{w_\ell}(\overline{\overline{\overline{\mathbb{H}}}}\times d_1^j),\tilde{w_r}(d_1^j 0\times\overline{\overline{\overline{\mathbb{H}}}}) \right\rangle} & \overline{\overline{\overline{\mathbb{H}}}} \times_{d_0^j,d_1^j} \overline{\overline{\overline{\mathbb{H}}}} \\
{\scriptstyle \left\langle \tilde{w_r}(d_1^j\times\overline{\overline{\overline{\mathbb{H}}}}),\tilde{w_\ell}(\overline{\overline{\overline{\mathbb{H}}}}\times d_0^j) \right\rangle}\Big\downarrow & & \Big\downarrow{\scriptstyle \overline{\overline{m}}} \\
\overline{\overline{\overline{\mathbb{H}}}} \times_{d_0^j,d_1^j} \overline{\overline{\overline{\mathbb{H}}}} & \xrightarrow[\overline{\overline{m}}]{} & \overline{\overline{\overline{\mathbb{H}}}}
\end{array} \tag{6.103}$$

commutes. □

6.4 The Tensor Map

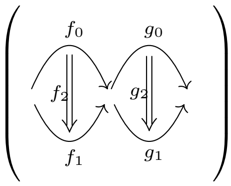
Given that by lemma 64 we have a 1-Cartesian map $\left\langle d_0^j, d_1^j \right\rangle \overline{\overline{\overline{\mathbb{H}}}} \longrightarrow P^2(\mathbb{H})$ we consider the following diagram in $\mathsf{GrayCat}_{\mathrm{Q}^1}$

$$\begin{array}{ccc}
\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}} & & \\
 & {\scriptstyle t}\searrow \quad \searrow{\scriptstyle \langle h_\ell, h_r\rangle} & \\
 & \overline{\overline{\overline{\mathbb{H}}}} \xrightarrow[\left\langle d_0^j, d_1^j \right\rangle]{} & P^2(\mathbb{H})
\end{array} \tag{6.104}$$

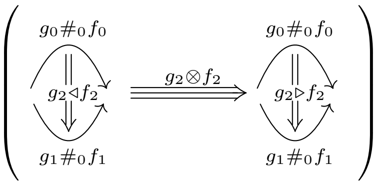
where h_ℓ and h_r are given by (6.63) and (6.64) respectively. By (6.65) we know that (h_ℓ, h_r) is a source for (6.91) hence we obtain $\langle h_\ell, h_r\rangle$.

There is a map $t_1 \colon (\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}})_1 \longrightarrow (\overline{\overline{\overline{\mathbb{H}}}})_1$ in $\mathsf{Cat}_{\mathrm{Q}^1}$ given by:

$$(g, f) = ((g_2; g_0, g_1), (f_2; f_0, f_1)) =$$



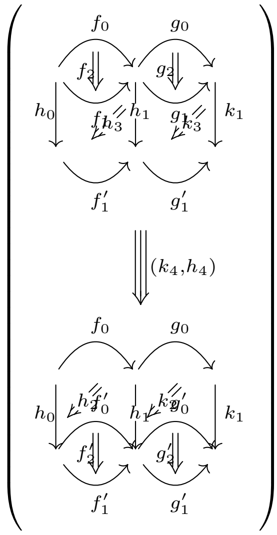
$$\mapsto (g_2 \otimes f_2; g_2 \triangleleft f_2, g_2 \triangleright f_2; g_0 \#_0 f_0, g_1 \#_0 f_1) =$$



(6.105)

and

$$\big((k,h)\colon (g,f)\longrightarrow (g',f')\big) = \begin{pmatrix}(k_4;k_2,k_3;h_1,k_1),\\ (h_4;h_2,h_3;h_0,h_1)\end{pmatrix} =$$



$$\mapsto \begin{pmatrix}\omega_1,\omega_2;(g_0'\#_0h_2)\#_1(k_2\#_0f_0),\\ (g_1'\#_0h_3)\#_1(k_3\#_0f_1);h_0,k_1\end{pmatrix}, \quad (6.106)$$

where ω_1 and ω_2 are defined as the vertical composites in (6.107), by definition these constitute the components of a 1-cell in $\overline{\overline{\overline{\mathbb{H}}}}$.

such that

Lemma 66 $\langle h_\ell, h_r\rangle_1 = \left\langle d_0^j, d_1^j\right\rangle_1 t_1$ *in* RGrph.

Proof One checks that $(h_\ell)_1 = (d_0^j t)_1$ and $(h_r)_1 = (d_1^j t)_1$ as graph maps using definitions (6.63) and (6.64). □

Lemma 67 *The 3-globular set*

$$P^2(\mathbb{H}) \underset{p_0}{\overset{p^1}{\underset{\longleftarrow\Delta\longrightarrow}{\longrightarrow}}} \overline{\overline{\mathbb{H}}} \underset{d_0}{\overset{d^1}{\underset{\longleftarrow i\longrightarrow}{\longrightarrow}}} \overrightarrow{\mathbb{H}} \underset{d_0}{\overset{d^1}{\underset{\longleftarrow i\longrightarrow}{\longrightarrow}}} \mathbb{H} \quad (6.108)$$

is an internal Gray*-category.*

Proof We already know that its three lower stages constitute a sesqui-catgory. The three top parts are trivially a 2-category. The tensor map is given by

$$\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} \xrightarrow{\langle h_\ell,h_r\rangle} P^2(\mathbb{H}) \quad (6.109)$$

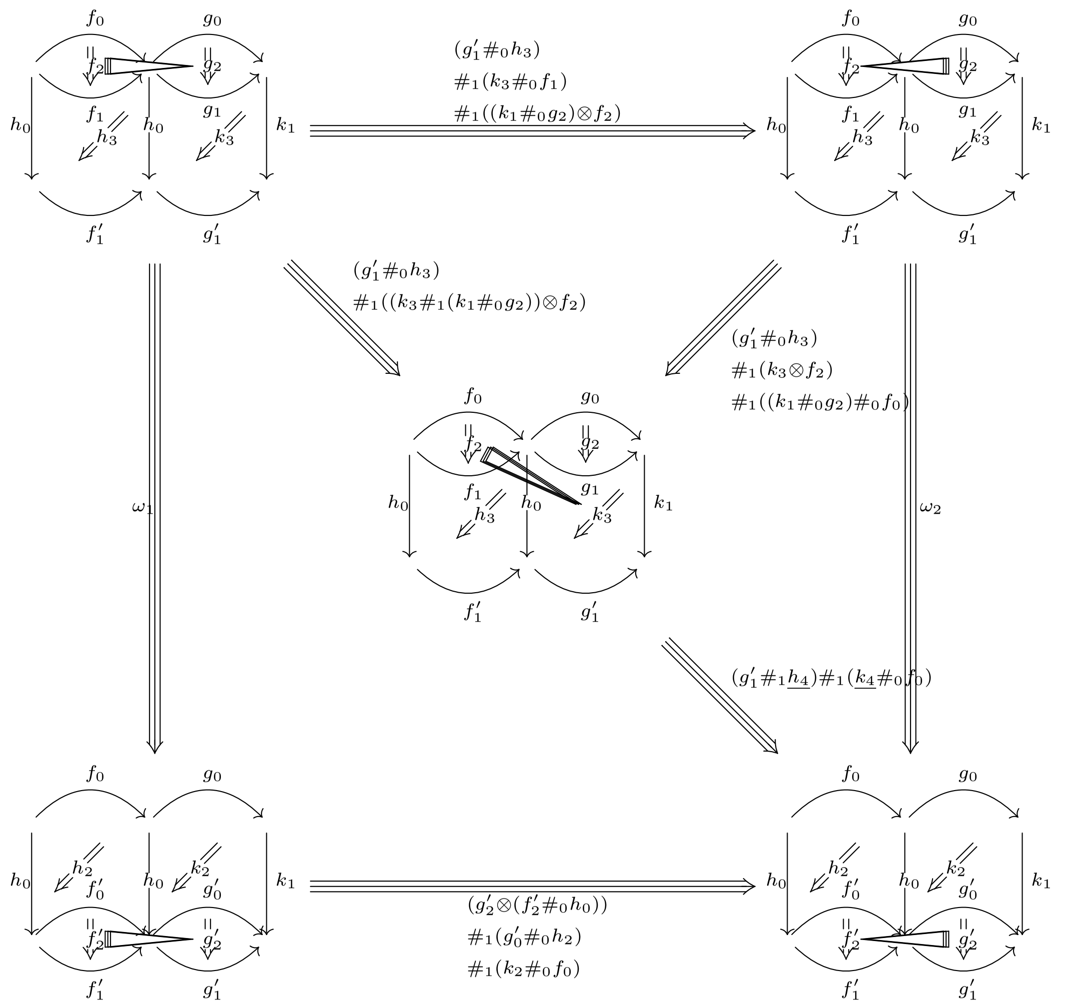
which satisfies the tensor axioms by construction. □

We can finally prove our desired theorem:

Theorem 68 *Given a* Gray*-category* $\mathbb{H}$ *there is an internal* Gray*-category in* $\mathsf{GrayCat}_{\mathrm{Q}^1}$

$$\overline{\overline{\overline{\mathbb{H}}}} \underset{d_0}{\overset{d^1}{\underset{\longleftarrow i\longrightarrow}{\longrightarrow}}} \overline{\overline{\mathbb{H}}} \underset{d_0}{\overset{d^1}{\underset{\longleftarrow i\longrightarrow}{\longrightarrow}}} \overrightarrow{\mathbb{H}} \underset{d_0}{\overset{d^1}{\underset{\longleftarrow i\longrightarrow}{\longrightarrow}}} \mathbb{H} \quad (6.110)$$

with composition operations $m, \overline{m}, \overline{\overline{m}}, w_\ell, w_r, \overline{w_\ell}, \overline{w_r}, \tilde{w}_\ell, \tilde{w}_r$, *and tensor* t.


$(g'_1 \#_0 h_3)$
$\#_1(k_3 \#_0 f_1)$
$\#_1((k_1 \#_0 g_2) \otimes f_2)$
$(g'_1 \#_0 h_3)$
$\#_1((k_3 \#_1 (k_1 \#_0 g_2)) \otimes f_2)$
$(g'_1 \#_0 h_3)$
$\#_1(k_3 \otimes f_2)$
$\#_1((k_1 \#_0 g_2) \#_0 f_0)$
$(g'_1 \#_1 \underline{h_4}) \#_1 (\underline{k_4} \#_0 f_0)$
ω_1
ω_2
$(g'_2 \otimes (f'_2 \#_0 h_0))$
$\#_1(g'_0 \#_0 h_2)$
$\#_1(k_2 \#_0 f_0)$

(6.107)

PROOF We have a globular map

$$\begin{array}{ccccccc}
\overline{\overline{\overline{\mathbb{H}}}} & \overset{d^1}{\underset{d_0}{\overset{\longleftarrow i \longrightarrow}{\longrightarrow}}} & \overline{\overline{\mathbb{H}}} & \overset{d^1}{\underset{d_0}{\overset{\longleftarrow i \longrightarrow}{\longrightarrow}}} & \overrightarrow{\mathbb{H}} & \overset{d^1}{\underset{d_0}{\overset{\longleftarrow i \longrightarrow}{\longrightarrow}}} & \mathbb{H} \\
\left\langle d_0^j, d_1^j \right\rangle \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
P^2(\mathbb{H}) & \overset{p^1}{\underset{p_0}{\overset{\longleftarrow \Delta \longrightarrow}{\longrightarrow}}} & \overline{\overline{\overline{\mathbb{H}}}} & \overset{d^1}{\underset{d_0}{\overset{\longleftarrow i \longrightarrow}{\longrightarrow}}} & \overrightarrow{\mathbb{H}} & \overset{d^1}{\underset{d_0}{\overset{\longleftarrow i \longrightarrow}{\longrightarrow}}} & \mathbb{H}
\end{array} \tag{6.111}$$

This globular map is an internal sesqui-functor in the lower and at the upper degrees, and by (6.104) it preverses the tensor:

$$\begin{array}{ccc}
\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0}, \overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}} & \overset{t}{\longrightarrow\!\!\!\!\!\!\!/\;\;\;} & \overline{\overline{\overline{\mathbb{H}}}} \\
\Big\downarrow & & \Big\downarrow \left\langle d_0^j, d_1^j \right\rangle \\
\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0}, \overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}} & \overset{\langle h_\ell, h_r \rangle}{\longrightarrow\!\!\!\!\!\!\!/\;\;\;} & P^2(\mathbb{H})
\end{array} \tag{6.112}$$

Using the results of 5 and 6 this proves that (6.110) is an internal Gray-category, that is, all the axioms of definition 4 hold. □

Chapter 7

The Internal Hom Functor

We finally define the internal hom of $\mathsf{GrayCat}_{\mathrm{Q}^1}$

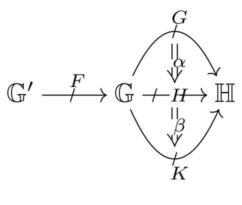
$$[\mathbb{G},\mathbb{H}] = \left(\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},\overline{\overline{\overline{\mathbb{H}}}}) \underset{d_{0*}}{\overset{d_{1*}}{\underset{\longleftarrow i_* \longrightarrow}{\longrightarrow}}} \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},\overline{\overline{\mathbb{H}}}) \underset{d_{0*}}{\overset{d_{1*}}{\underset{\longleftarrow i_* \longrightarrow}{\longrightarrow}}} \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},\overrightarrow{\mathbb{H}}) \underset{d_{0*}}{\overset{d_{1*}}{\underset{\longleftarrow i_* \longrightarrow}{\longrightarrow}}} \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},\mathbb{H}) \right) \tag{7.1}$$

by applying $\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},-)$ to the diagram (6.110), where the lower star means action by post-composition. This includes the various induced composition operations m_*, $\overline{m}_*$, $\overline{\overline{m}}_*$, $w_{\ell *}$, w_{r*} and t_*. Because $\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G},-)$ by definition preserves limits in the second variable, it takes internal Gray-categories in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ to such in Set, that is, to ordinary Gray-categories. In analogy with our earlier notation we write the compositions on $[\mathbb{G},\mathbb{H}]$ as $*_n$ where n is the dimension of the incident cell, we use $*$ for the tensor of transformations incident on a functor.

Theorem 69 *Given a morphism $F\colon \mathbb{G}' \twoheadrightarrow \mathbb{G}$ in $\mathsf{GrayCat}_{\mathrm{Q}^1}$, the map*

$$F^* = [F,\mathbb{H}]\colon [\mathbb{G},\mathbb{H}] \longrightarrow [\mathbb{G}',\mathbb{H}]$$

acting by pre-composition is a Gray-functor, that is, a strict morphism.

Proof Assume a situation  then we have

$$\begin{aligned} F^*(\beta *_0 \alpha) &= (\beta *_0 \alpha)F = m\langle \beta,\alpha\rangle F \\ &= m\langle \beta F,\alpha F\rangle = (\beta F) *_0 (\alpha F) = (F^*\beta) *_0 (F^*\alpha)\,. \end{aligned} \tag{7.2}$$

Also, for identity transformations we have:

$$F^*\mathrm{id}_G = iGF = \mathrm{id}_{GF}\,, \tag{7.3}$$

hence F^* is a functor. By the same reasoning the higher operations including the tensor, are preserved as well. □

Remark 70 *This way* $[-,\mathbb{H}]\colon \mathsf{GrayCat}_{\mathrm{Q}^1} \longrightarrow \mathsf{GrayCat}_{\mathrm{Q}^1}$ *is an endofunctor for each* $\mathbb{H}$*.*

Remark 71 *The* Gray*-category* $[\mathbb{G},\mathbb{H}]$ *is a* Gray*-groupoid if* $\mathbb{H}$ *is one.*

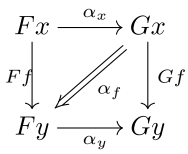
Chapter 8

Putting it all together

Definition 72 *A* **lax transformation** $\alpha\colon F \longrightarrow G$ *between pseudo-functors* $F, G\colon \mathbb{G} \twoheadrightarrow \mathbb{H}$ *of* Gray*-categories is a pseudo-functor* $\alpha\colon \mathbb{G} \twoheadrightarrow \overrightarrow{\mathbb{H}}$ *such that* $d_0\alpha = F$ *and* $d_1\alpha = G$.

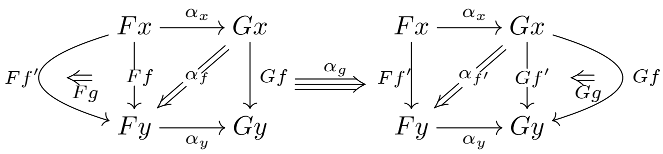
Remark 73 *Using the definition of path spaces in 25 and the characterization of pseudo-maps in 22 we note for reference that a lax transformation* α *is given by the following underlying data:*

1. *for each 0-cell* x *of* $\mathbb{G}$ *a 1-cell* $\alpha_x\colon Fx \longrightarrow Gx$,
2. *for each 1-cell* $f\colon x \longrightarrow y$ *of* $\mathbb{G}$ *a 2-cell*



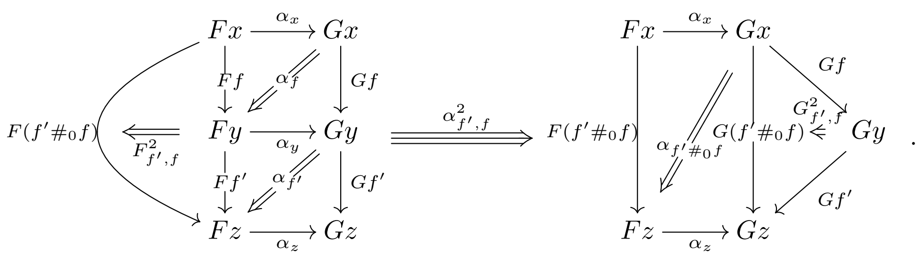
(8.1)

3. *for each 2-cell* $g\colon f \longrightarrow f'$ *of* $\mathbb{G}$ *a 3-cell of* $\mathbb{H}$



(8.2)

4. *for each pair of composable 1-cells* $f\colon x \longrightarrow y$, $f'\colon y \longrightarrow z$ *an invertible 3-cell*



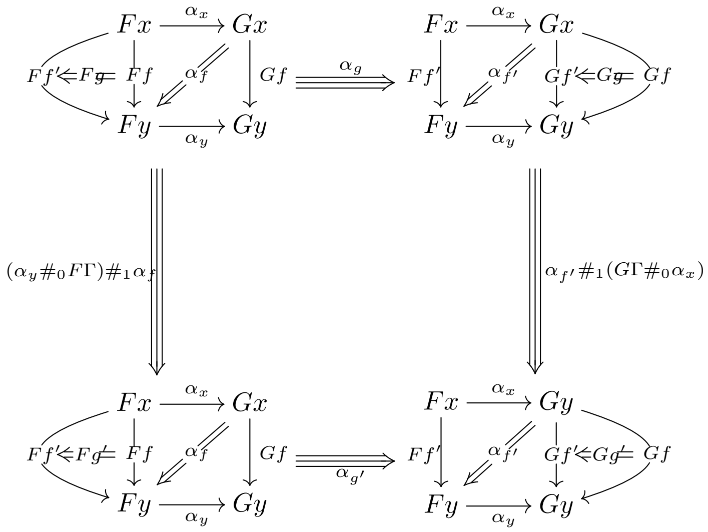
(8.3)

Furthermore, these data have to satisfy the following equations:

1. *On identities of 0-cells:*

$$\alpha_{\mathrm{id}_x} = \mathrm{id}_{\alpha_x} \tag{8.4}$$

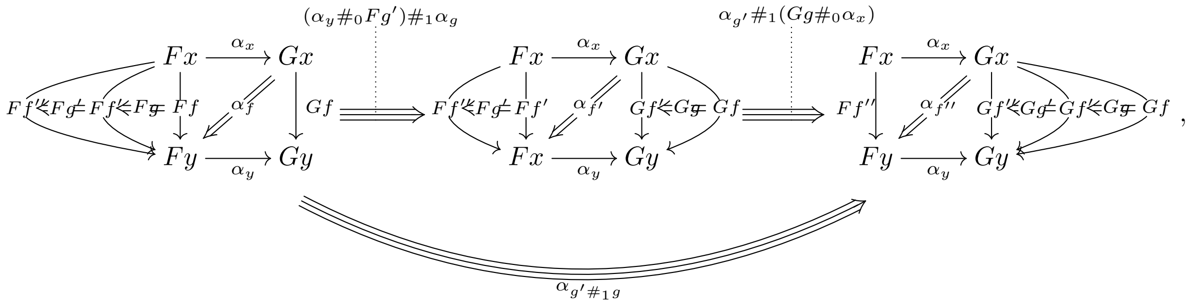
2. *for each 3-cell* $\Gamma\colon g \longrightarrow g'$ *the square of 3-cells in* $\mathbb{H}$



(8.5)

commutes. This condition obviously comes from the definition of 3-cells in the path space.

3. *For every pair* $g\colon f \Longrightarrow f', g'\colon f' \Longrightarrow f''$*:*



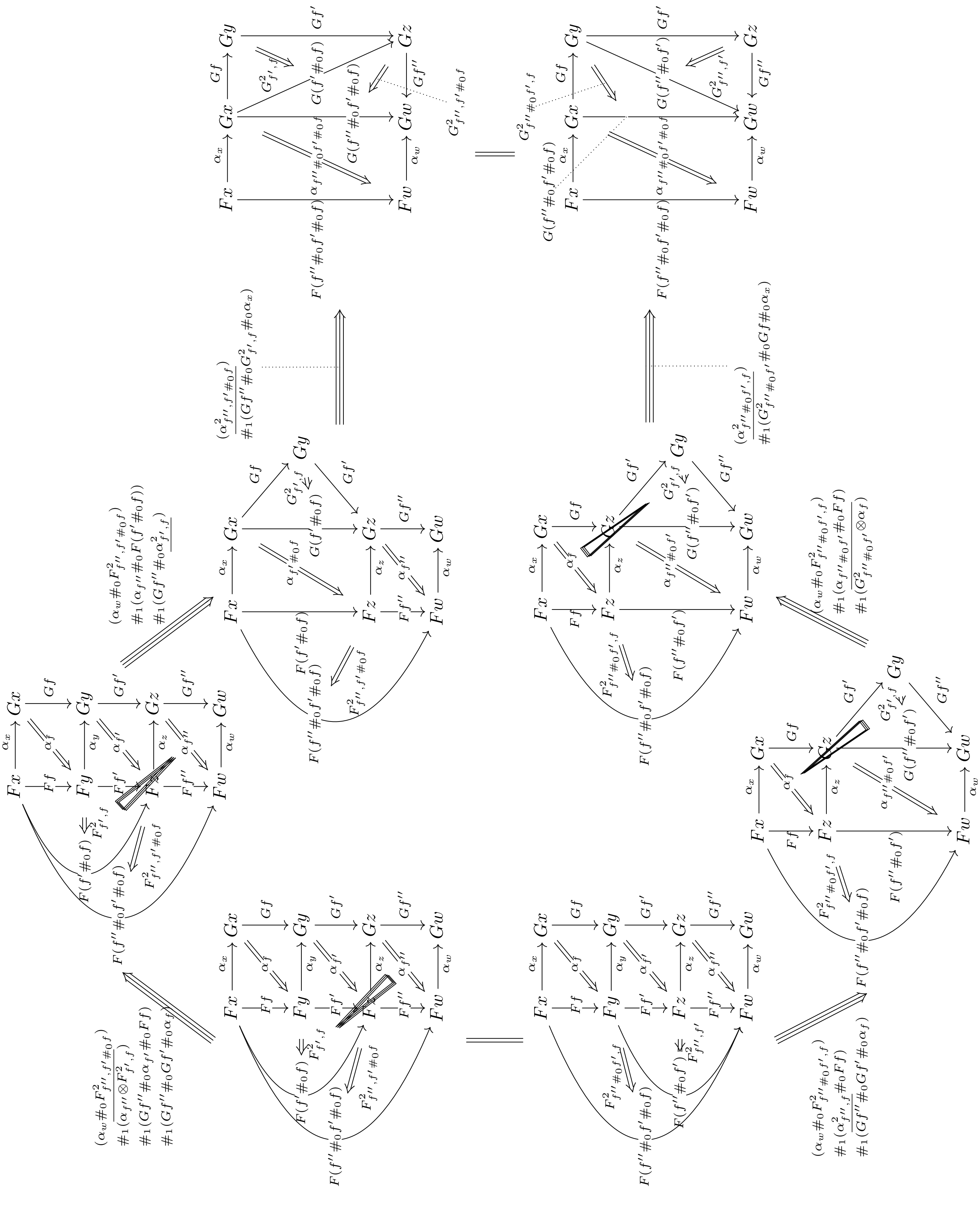
(8.6)

and for identity 2-cells $\mathrm{id}_f\colon f \Longrightarrow f$ *we have an identity 3-cell*

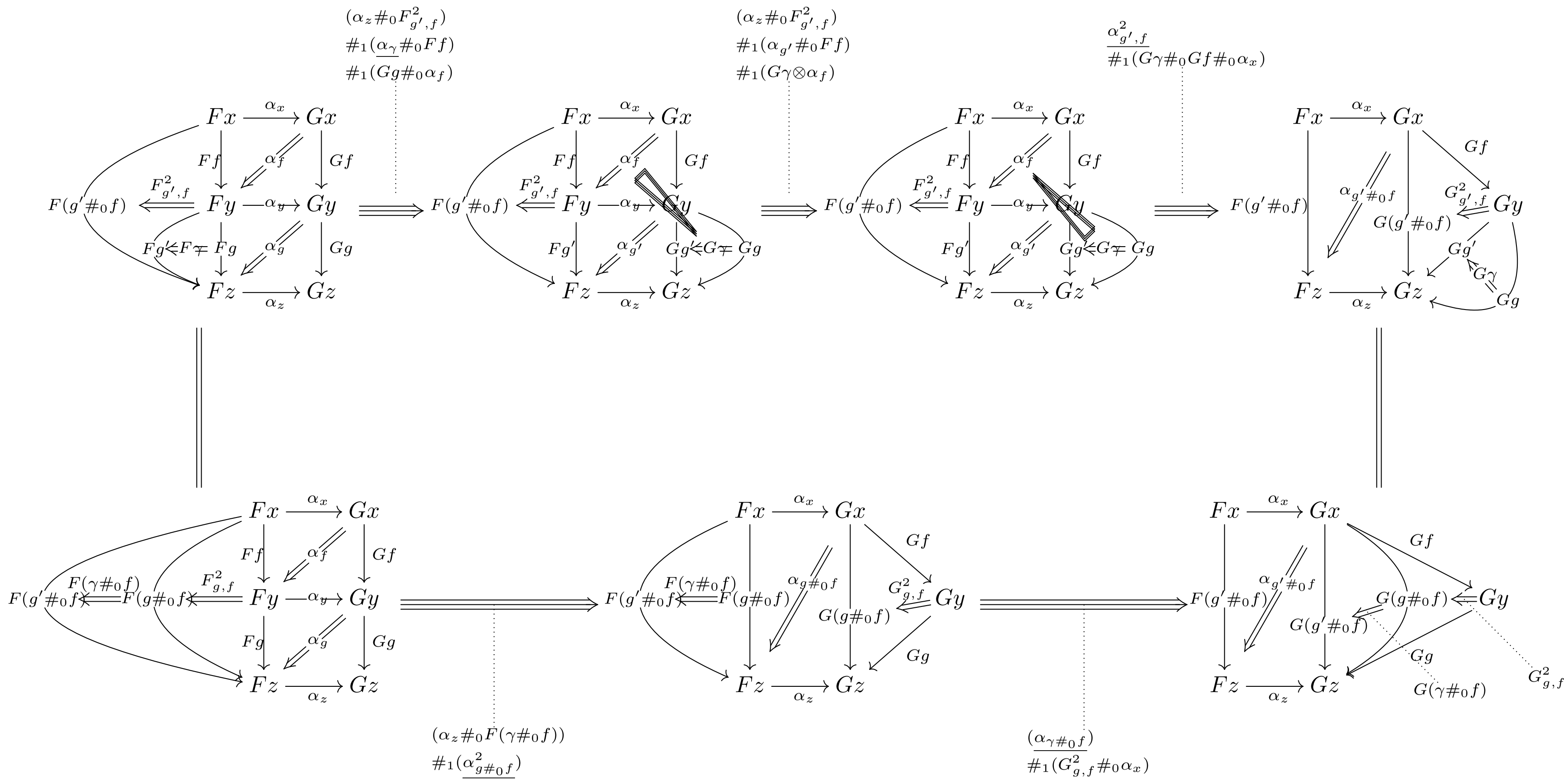
$$\alpha_{\mathrm{id}_f} = \mathrm{id}_{\alpha_f} \,. \tag{8.7}$$

4. *The family of 3-cells has to satisfy a kind of cocycle condition: For a composable triple* f, f', f'' *of 1-cells* α^2 *has to satisfy equation (8.8). furthermore,* α^2 *has to satisfy the normalization condition:*

$$\alpha^2_{f',f} = \begin{cases} \mathrm{id}_{\alpha_{f'}} & \text{if } f' = \mathrm{id}_y \\ \mathrm{id}_{\alpha_f} & \text{if } f = \mathrm{id}_x \end{cases} \tag{8.9}$$

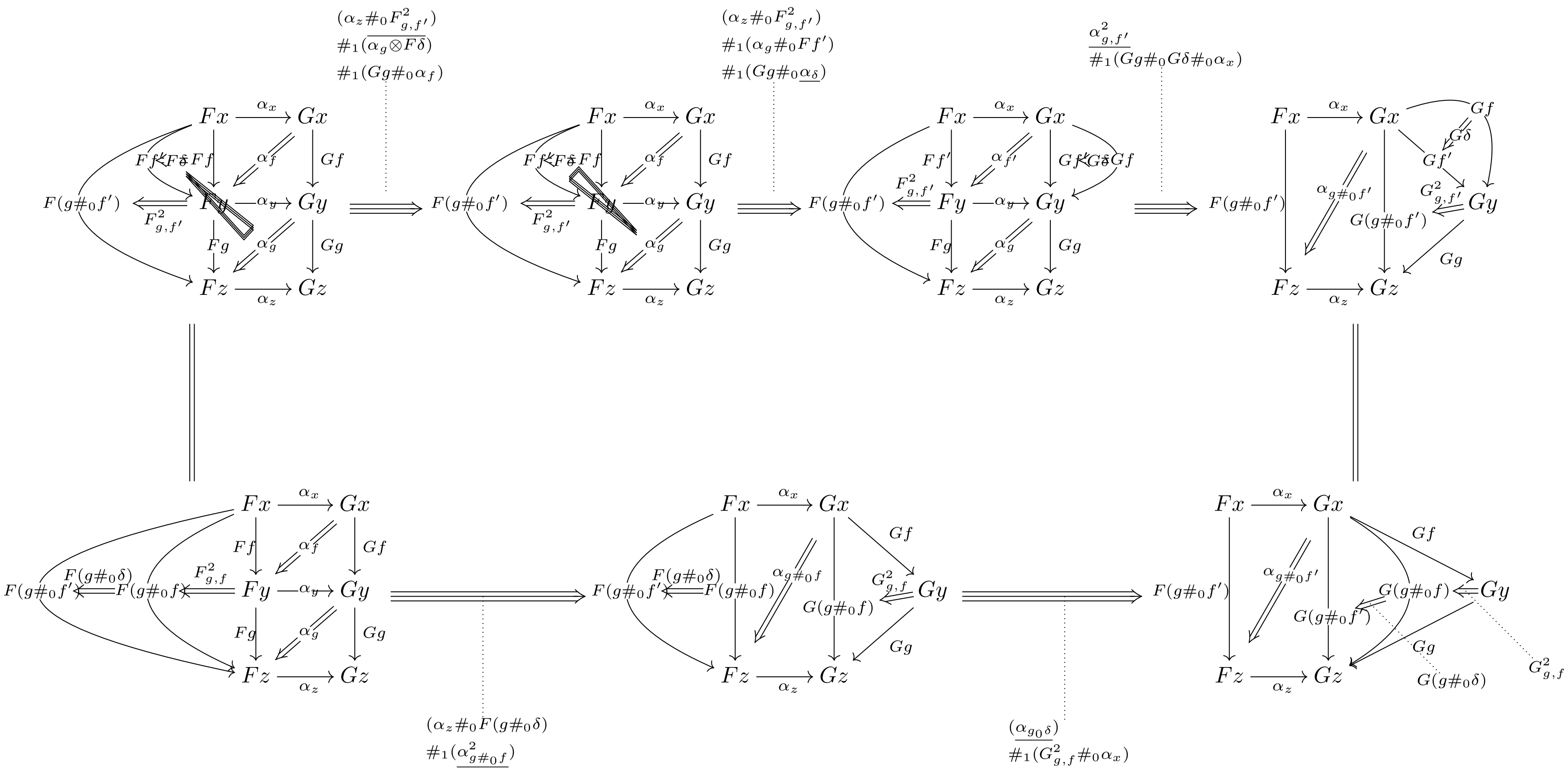


(8.8)



(8.10)

Compatibility of the cocycle α^2 with left whiskers $\gamma \#_0 f$.



(8.11)

Compatibility of the cocycle α^2 with right whiskers $g\#_0\delta$.

5. *The family of 3-cells α^2 has to be compatible with left and right whiskering according to (8.10) and (8.11).*

*These conditions are derived from ones in the definition of pseudo-*Gray*-functors 22. Note how conditions 4, 5, 6 of 22 are trivially satisfied for transformations.*

Definition 74 *A transformation $\alpha\colon F \longrightarrow G$ where the cocycle α^2 has only trivial components we call a* **stiff transformation**.

Lemma 75 *A stiff transformation $\alpha\colon F \longrightarrow G$ with F and G strict* Gray*-functors is a 1-transfor in the sense of [Crans 1999].* □

Remark 76 *Given two lax-transformations $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ their composite $\beta * \alpha$ given by $m\langle\beta,\alpha\rangle$ and has the following components:*

1. *for each 0-cell x of $\mathbb{G}$ the 1-cell*

$$Fx \xrightarrow{(\beta*\alpha)_x} Hx \;=\; Fx \xrightarrow{\alpha_x} Gx \xrightarrow{\beta_x} Hx\,, \tag{8.12}$$

2. *for each 1-cell $f\colon x \longrightarrow y$ of $\mathbb{G}$ the 2-cell*

$$\begin{array}{ccc} Fx & \xrightarrow{(\beta*\alpha)_x} & Hx \\ {\scriptstyle Ff}\downarrow & \swArrow{\scriptstyle (\beta*\alpha)_f} & \downarrow{\scriptstyle Hf} \\ Fy & \xrightarrow[(\beta*\alpha)_y]{} & Hy \end{array} \;=\; \begin{array}{ccccc} Fx & \xrightarrow{\alpha_x} & Gx & \xrightarrow{\beta_x} & Hx \\ {\scriptstyle Ff}\downarrow & \swArrow{\scriptstyle \alpha_f} & \downarrow{\scriptstyle Gf} & \swArrow{\scriptstyle \beta_f} & \downarrow{\scriptstyle Hf} \\ Fy & \xrightarrow[\alpha_y]{} & Gy & \xrightarrow[\beta_y]{} & Hy \end{array} \tag{8.13}$$

3. *for each 2-cell $g\colon f \longrightarrow f'$ of $\mathbb{G}$ the 3-cell of $\mathbb{H}$ shown in* (8.14)

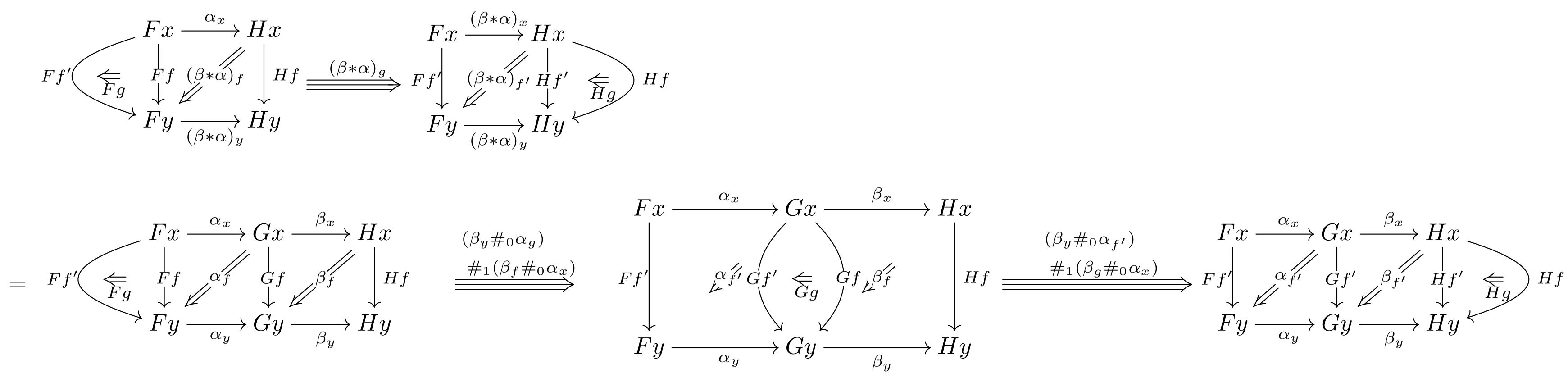
4. *for each pair of composable 1-cells $f\colon x \longrightarrow y$, $f'\colon y \longrightarrow z$ a 3-cell shown in* (8.15)

Definition 77 *Assuming α and β are as in definition 72 and F and G are pseudo-functors $\mathbb{G} \twoheadrightarrow \mathbb{H}$, a* **modification** *$A\colon \alpha \longrightarrow \beta\colon F \longrightarrow G$ is a pseudo-functor $A\colon \mathbb{G} \twoheadrightarrow \overline{\overline{\overline{\mathbb{H}}}}$, such that $d_0A = \alpha$ and $d_1A = \beta$.*

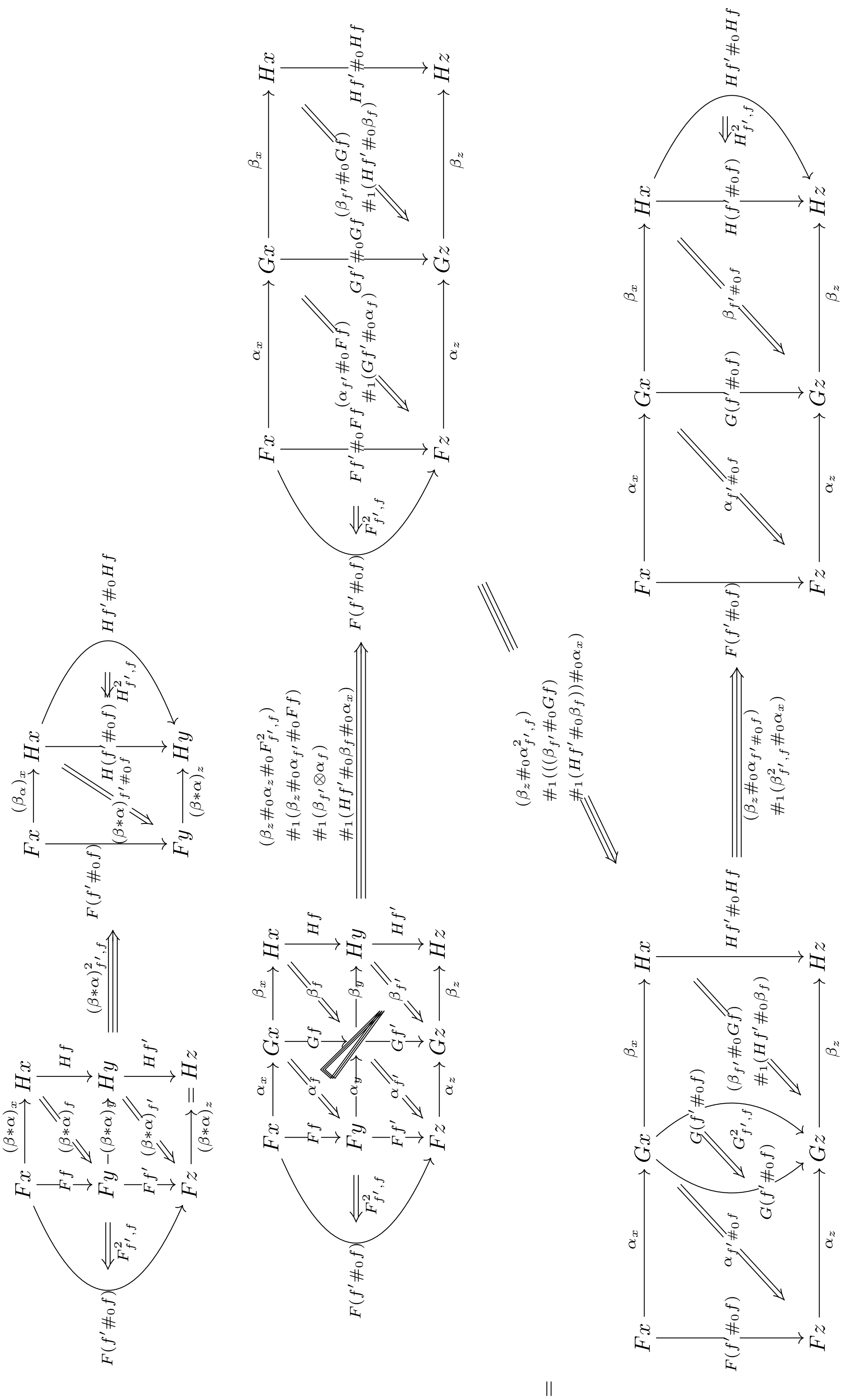
Remark 78 *A modification $A\colon \alpha \longrightarrow \beta$ according to 77 and 22 is given by the following data:*

1. *For every 0-cell x in $\mathbb{G}$ a 2-cell*

$$Fx \underset{\beta_x}{\overset{\alpha_x}{\rightrightarrows}} Gx \qquad A_x\colon \alpha_x \Rightarrow \beta_x \tag{8.16}$$

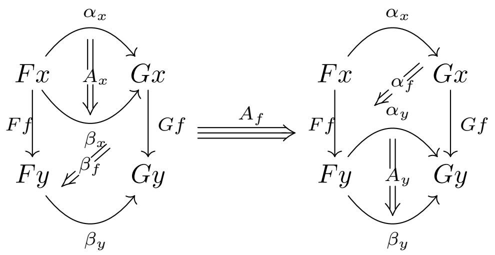

Fx
αx
Hx
Ff′
Fg
Ff
(β∗α)f
Hf
(β∗α)g
Fy
(β∗α)y
Hy
(β∗α)x
(β∗α)f′
Hf′
Hg
=
Gx
βx
αf
Gf
βf
αy
Gy
βy
(βy#0αg)
#1(βf#0αx)
αf′
Gf′
Gg
(βy#0αf′)
#1(βg#0αx)
βf′

(8.14)



(8.15)

2. *For every 1-cell $f\colon x \longrightarrow y$ a 3-cell in $\mathbb{H}$*



(8.17)

This data has to satisfy the following conditions:

1. *Units are preserved:*

$$A_{\mathrm{id}_x} = \mathrm{id}_{A_x} \tag{8.18}$$

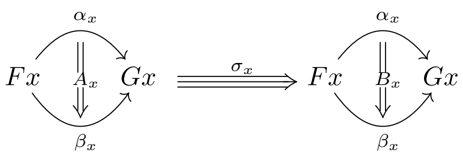
2. *Compatibility with the cocycles of F, G, α, β according to (8.19)*

3. *For 2-cells $g\colon f \Longrightarrow f'$ in $\mathbb{G}$ the images under F and G as well the data of A, α and β are compatible as shown in (8.20)*

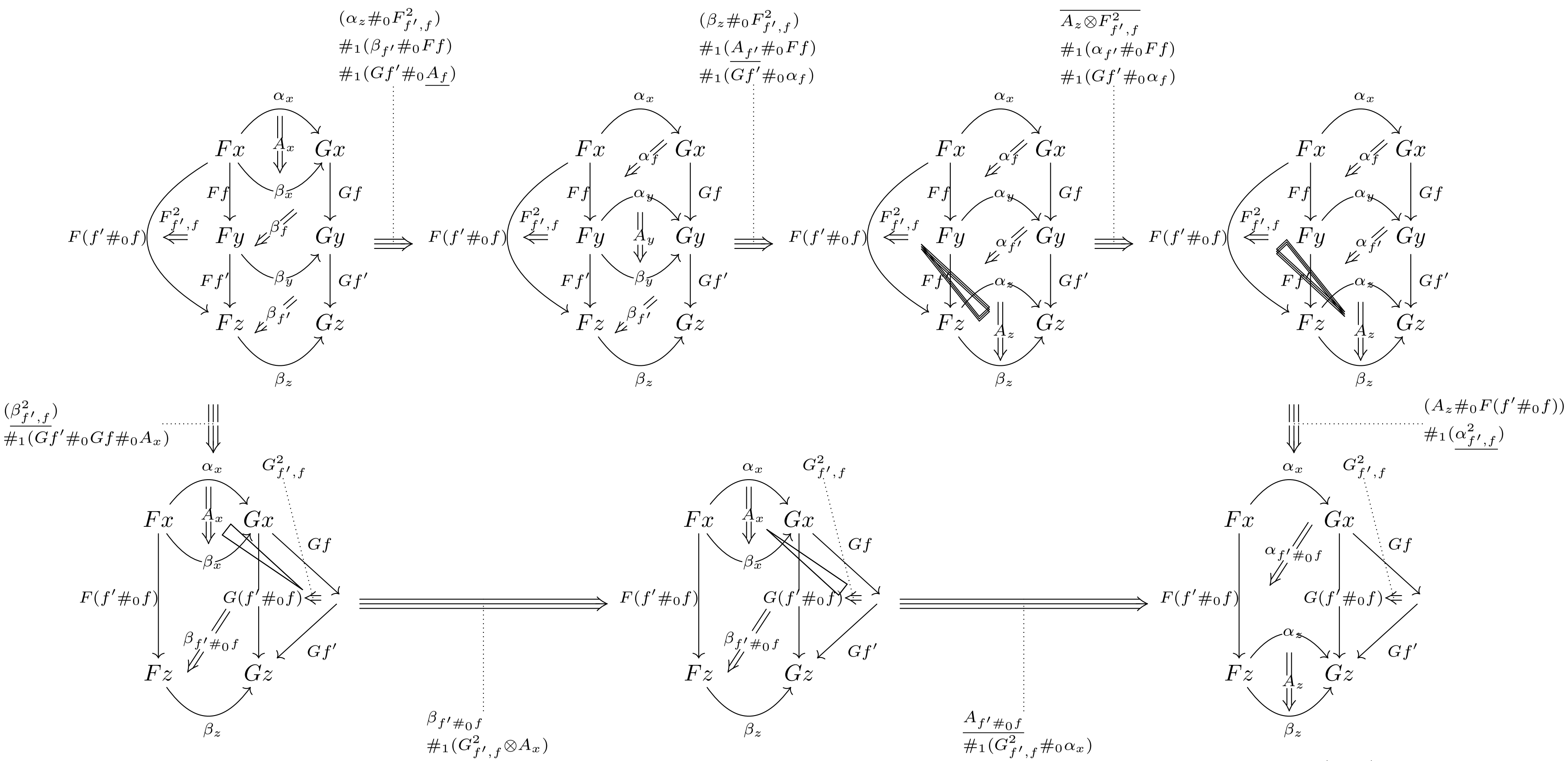
Lemma 79 *A transformation $A\colon \alpha \longrightarrow \beta$ where $\alpha, \beta\colon F \longrightarrow G$ are stiff and F, G are strict is a 2-transfor in the sense of [Crans 1999].* □

Definition 80 *Given modifications $A, B\colon \alpha \longrightarrow \beta$ a* **perturbation** *is a pseudo-*Gray*-functor $\sigma\colon \mathbb{G} \nrightarrow \overline{\overline{\overline{\mathbb{H}}}}$ such that $d_0\sigma = A$ and $d_1\sigma = B$.*

Remark 81 *According to 80 a perturbation is given by a 3-cell in $\mathbb{H}$*

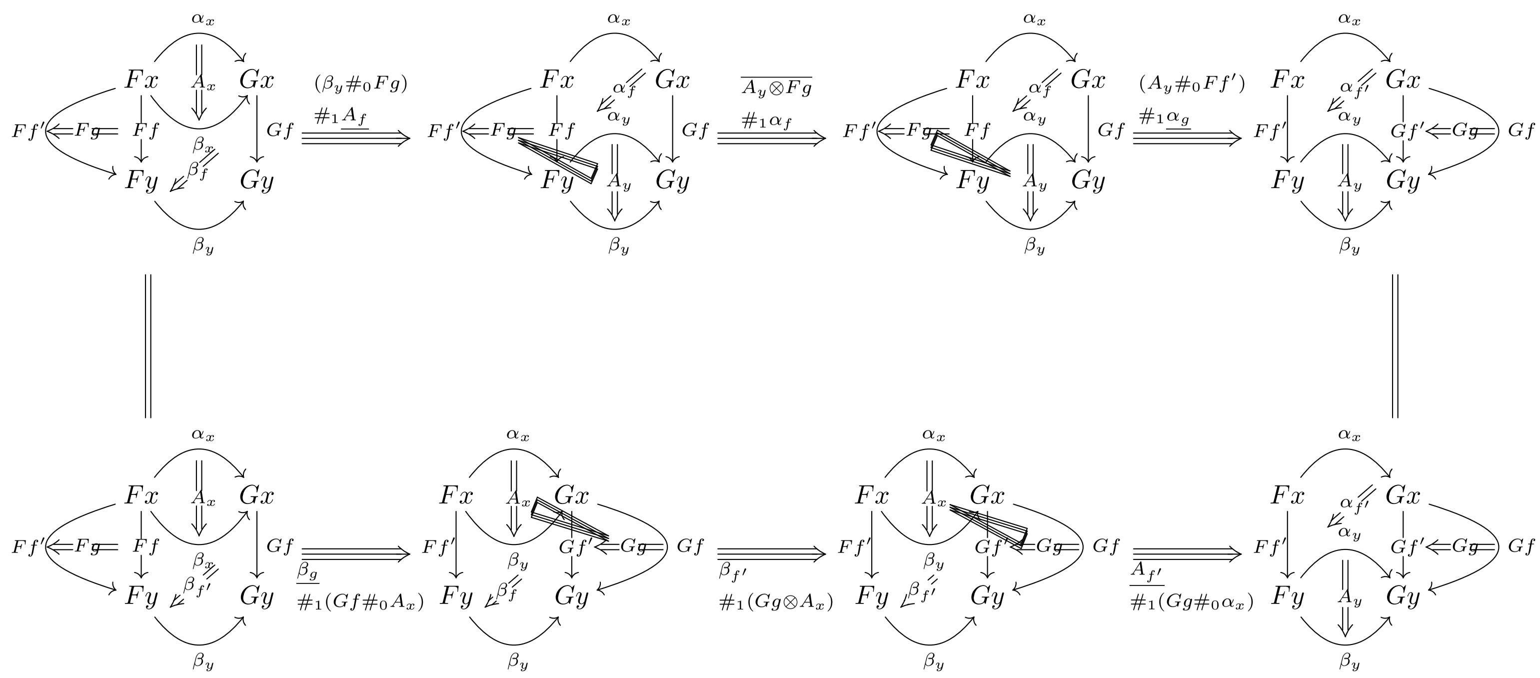


(8.21)



(8.19)

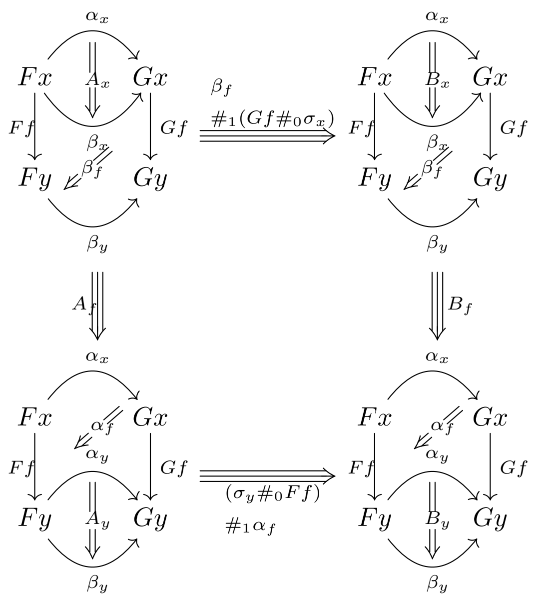
Compatibility of the modification A with the cocycles of F, G, α, β



(8.20)

Compatibility of 2-cells with A, α and β

for each 0-cell x in $\mathbb{G}$ such that



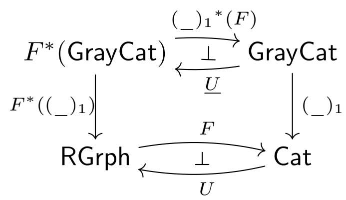
(8.22)

commutes.

Lemma 82 *A perturbation $\sigma\colon A \longrightarrow B$ fulfilling the conditions of 79 is a 3-transfor in the sense of [Crans 1999].* $\square$

Adjunctions

We can embed the ideas developed in section 3 in a more global picture. The functor $\mathrm{Q}^1\colon \mathsf{GrayCat} \longrightarrow \mathsf{GrayCat}$ is part of the following adjunction of fibered categories:



(23)

where F means "free category over a reflexive graph" and U means "underlying reflexive graph of a category", $(\_)_1$ means "underlying category of a Gray-category. According to [Hermida 1999, 4.1] the adjunction $F \dashv U$ lifts canonically to an adjunction $({(\_)_1}^*(F), F) \dashv (\underline{U}, U)$ of fibered categories. Which means in particular that ${(\_)_1}^*(F) \dashv \underline{U}$ is an adjunction and our Q^1 can be defined as ${(\_)_1}^*(F)\underline{U}$.

The objects of $\mathsf{Graph} \times \mathsf{GrayCat}$ might be called 1-free Gray-categories.

We can construct a further resolution which we call Q^2.

Remark 83 *Let $P\colon \mathcal{E} \longrightarrow \mathcal{B}$ be a 2-fibration in the sense of Hermida [1999]. Given $u\colon I \longrightarrow PX$ and $u'\colon I' \longrightarrow PX$ for X an object in $\mathcal{E}$; and an equivalence*

$h\colon I \longrightarrow I'$ such that $u'h = u$. Then the unique filler $\widehat{h}$ over h is an equivalence as well.

In particular, given the comparison functor $K\colon \mathsf{X}_{FU} \longrightarrow \mathsf{A}$ for the comonad induced by $F \dashv U\colon \mathsf{A} \longrightarrow \mathsf{X}$ lifts to a comparison functor $\widehat{K}$.

Lemma 84 *If F is comonadic, then so is $((\_)_1{}^*(F), F)$.*

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