General Relativity in the framework of exact gravito-electromagnetic analogies

Luis Filipe Costa

Abstract

In this work we studied the several gravito-electromagnetic analogies in the literature, and presented a new exact one, based on tidal tensors. We clarified the relation between the different analogies, and further worked out some of them; namely the analogy based on inertial gravitational fields (GEM fields), which was reformulated and extended. The gravito-electromagnetic analogy based on tidal tensors stems from the tidal dynamics of the two theories: the analogy for electric type tidal tensors is manifest in the geodesic deviation, and in the analogous electromagnetic worldline deviation; the analogy for magnetic-type tidal tensors is manifest in the force exerted on spinning particles (magnetic dipoles/gyroscopes). It extends to the field equations: the gravitational field equations may be cast as a set of four algebraic equations for tidal tensors and sources, analogous to Maxwell's equations in this formalism; plus two additional equations with no electromagnetic counterpart. This analogy is ideally suited to compare the tidal dynamics of the two interactions; key differences are made transparent in the symmetries and time projections of the tidal tensors, which are related to the phenomenon of electromagnetic induction, and the absence of analogous effects in gravity. This is extensively explored in the context of the dynamics of spinning multipole particles — a natural application of the formalism. The analogy based on inertial GEM fields stems from the space part of the geodesic equation, which can be cast exactly as consisting of a Lorentz-like part where two spatial vector fields — a gravitoelectric and a gravitomagnetic field — mimic the electromagnetic fields, plus a term involving the shear/expansion of the observer congruence which has no electromagnetic analogue. This analogy also extends to the field equations, where we have on the gravitational side six equations, four of which, again in this formalism, exhibit many similarities with Maxwell's equations; the similarity gets particularly close for rigid frames and stationary fields. This formalism also leads to an exact gravito-electromagnetic analogy for the "precession" of the spin vector of a spinning particle (this analogy, together with the tidal tensor one, means that both equations of motion for a spinning dipole particle — the spin evolution, and the force — can be cast in exact gravito-electromagnetic analogies). At the heart of these analogies is the Mathisson-Pirani (MP) spin condition; however this condition is usually portrayed in the literature as problematic, due to its degeneracy and the famous helical motions it allows, which have been deemed unphysical. We address the problem of the spin condition and the definition of center of mass in General Relativity, and show that these claims are but misconceptions: not only the MP condition is as valid as any other, as it is the most suitable one (through the non-helical solution) for many practical applications. As for the helical motions it equally allows, we show that they are just alternative (but equivalent) descriptions, dynamically consistent and explained through the concept of hidden momentum — analogous to the hidden momentum of electromagnetic systems. We discuss the different forms of hidden momentum, and unveil some of its counter-intuitive features. A number of other issues not well understood in the literature were clarified in the course of this work; namely the physical meaning of the magnetic part of the Riemann tensor, and the problem of the covariant equations of motion for spinning particles in electromagnetic fields, and their interpretation.

Resumo

Neste trabalho estudamos as várias analogias gravito-electromagnéticas existentes na literatura, e apresentamos uma nova, baseada em "tensores de marés". Clarificamos a relação entre as várias analogias, e evoluímos algumas delas, em particular a analogia baseada em campos de forças inerciais (campos GEM), que foi reformulada e estendida. A analogia gravito-electromagnética baseada nos tensores de marés emana da dinâmica dos efeitos "de marés" das duas teorias: a analogia entre tensores de marés do tipo eléctrico manifesta-se na equação de desvio geodésico, e na equação de desvio análoga no electromagnetismo; a analogia dos tensores do tipo magnético manifesta-se na força exercida em partículas em rotação (dipolos magnéticos/giroscópios). A analogia estende-se às equações de campo: as equações relativistas do campo gravitacional podem ser retratadas como um sistema de quatro equações algébricas envolvendo apenas tensores de marés e termos de fonte, análogas às equações de Maxwell quando escritas neste formalismo, mais um par de equações adicionais que não têm análogo electromagnético. Esta analogia é ideal para comparar as duas interacções; diferenças chave são transparentes nas simetrias e projecções temporais dos tensores de marés, que estão relacionadas com os fenómenos de indução electromagnéticos, e a ausência de efeitos análogos na gravidade. Estas diferenças são exploradas com grande detalhe no contexto da dinâmica de partículas com momentos multipolares — uma aplicação natural do formalismo. A analogia baseada nos campos de forças inerciais emana da parte espacial da equação das geodésicas, que pode ser exactamente descrita como consistindo de uma parte semelhante à força de Lorentz, onde dois vectores espaciais — os campos "gravitoeléctrico" e "gravitomagnético" — mimetizam os campos electromagnéticos, mais um termo adicional envolvendo o "shear"/expansão da congruência de observadores, que não tem análogo electromagnético. Esta analogia também se estende às equações de campo, onde obtemos do lado gravitacional seis equações, das quais quatro, neste formalismo também, manifestam várias semelhanças com as equações de Maxwell; a semelhança é particularmente próxima no caso de referenciais rígidos em campos estacionários. Este formalismo leva ainda a uma analogia exacta para a "precessão" do vector de spin de uma partícula em rotação (juntando esta analogia à dos tensores de maré, temos que ambas as equações de movimento para partículas pólo-dipolo — a equação de evolução do spin, e a da força — podem ser retractadas em analogias gravito-electromagnéticas exactas). No coração destas analogias está a condição de spin de Mathisson-Pirani (MP); todavia esta condição é vista na literatura como problemática, devido à sua degenerescência e aos famosos movimentos helicoidais que ela admite, que foram considerados não físicos. Nós abordamos o problema da condição de spin, e da definição de centro de massa em Relatividade Geral, e mostramos que tais afirmações não passam de um mal entendido: não só a condição MP é tão válida como qualquer outra, como é mesmo a mais adequada (através da sua solução não helicoidal) para várias aplicações práticas. Quanto às soluções helicoidais que ela igualmente admite, mostramos que são apenas descrições alternativas (mas equivalentes), dinâmicamente consistentes e explicadas pelo conceito de "hidden momentum" — análogo ao hidden momentum dos sistemas electromagnéticos. Discutimos também as várias de formas hidden momentum, e revelamos alguns dos seus efeitos contra-intuitivos. Várias outras questões que não eram bem compreendidas na literatura foram sendo clarificadas no decurso deste trabalho; nomeadamente o significado físico da parte magnética do tensor de Riemann, e o problema das equações de movimento covariantes para partículas multipolares (em rotação) sob acção de campos electromagnéticos, e a sua interpretação.

Contents

1 Description of this document

This is a "contextualization" document as required for a PhD work in physics in the form of article compilation.

It is organized as follows. In Sec. 2 the articles produced in this thesis are listed; it contains five research papers, three of them published in international journals with peer review, the other two available on-line in the form of preprint, to be submitted soon for publication in refereed journals.

In Sec. 3 I briefly review what *initially* motivated this work, was then the state of the art.

Sec. 4, "Roadmap to the papers", is a quick guide for the papers, explaining their motivation and aim, how they fit in the context of this work, and briefly describing their main outcomes.

In Sec. 5 a more detailed summary and discussion of each paper is given. It is not meant to substitute the introduction and conclusion of the papers though, to which I refer the reader for an even more comprehensive account of the results in each paper, as well as a detailed literature review. In this section the main results of each paper are outlined, but the way they are presented does not always follow rigorously the original text (this is the case with the older papers #1 and #2). Paper #1 was where we first presented the analogy based on tidal tensors; but I have been using and developing this formalism since then, my knowledge advanced accordingly, in the light of the more recent papers #3, #4 and #5; thus I describe the same results from the perspective I have today, and refer the reader to the latest papers where the ideas in Paper #1 are put on firm grounds and further developed. In Paper #2 we studied the conditions under which a similarity between gravity and electromagnetism occurs, in view of astrophysical applications, and at an approximate level; we studied the dependence on the reference frame in particular detail. This issue has been revisited, with an exact approach, in the recent Paper #5, where the results in Paper #2 were generalized and understood at a more fundamental level; and I make use of that in the discussion of this paper.

2 Article compilation — index

Papers published in international journals with peer review

1. L. Filipe Costa, C. Herdeiro, "Gravitoelectromagnetic analogy based on tidal tensors", *Physical Review D* **78**, 024021 (2008)

2. L. Filipe Costa, C. Herdeiro, "Reference frames and the physical gravito-electromagnetic analogy", *Proceedings of the International Astronomical Union (IAU)* vol **5** (Cambridge U. Press) p 31. Preprint [arXiv:0912.2146] (2009)

3. L. Filipe Costa, C. Herdeiro, J. Natário, M. Zilhão, "Mathisson's helical motions for a spinning particle: Are they unphysical?", *Physical Review D* **85**, 024001 (2012)

Papers published in the form of preprint, to be submitted to refereed journals

4. L. Filipe Costa, J. Natário, M. Zilhão, "Spacetime dynamics of spinning particles — exact gravito-electromagnetic analogies", *Preprint* [arXiv:1207.0470]

5. L. Filipe Costa, J. Natário, "Gravito-electromagnetic analogies", *Preprint* [arXiv:1207.0465]

Further publications on the material of this thesis exist in the form of conference proceedings, both in international journals (*in Proceedings*) and in book chapters; they are listed in Sec. 7, but not included in the compilation as, except for one of them, they do not contain significant new material compared with the five research papers above (they are essentially a different way of presenting the same results; they have a few new figures and equations, but I include them in this document in Sec. 5).

Note that Paper #2 above, in spite of being a Proceedings paper, and indeed being associated with a conference (the International Astronomical Union Symposium), is indeed a research paper (original material is encouraged in this publication), and has been subject to scientific refereeing.

3 Introduction and motivation

The material in this thesis can be split in two main topics: gravito-electromagnetic analogies (with their very broad range, and related subjects), and the dynamics of spinning multipole test particles in general relativity (to which we were led in the course of this work).

The pursue for analogue models to describe gravity has a long history, and many different types of models have been proposed, both classical (based on e.g. fluid dynamics, electromagnetism, light propagation in dielectric media) and quantum (based on e.g. Bose-Einstein condensates); for a review and references see [6]. In this work we have studied a special class of these analogies, the ones between the gravitational and electromagnetic interactions. The parallelism is in this case drawn between two relativistic fields theories (the two classical interactions), and (in the case of the the ones dubbed "physical" below) one compares effects "alike". In this sense one might argue that these analogies have a stronger physical component than (most) other analogue models. It is my view that they are interesting not only for providing intuition and a familiar formalism to treat otherwise more complicated gravitational problems (which is a common goal to other analogue models), but also (and especially) for the prospect of yielding a formalism allowing for a direct comparison of the two interactions, from which one might learn fundamental aspects about *both* of them.

When we started this work, the state of the art in the field of the gravito-electromagnetic (GEM) analogies, known as "gravitoelectromagnetism", was that there were several different analogies in the literature, but the relation between them was unclear. The best known ones were the analogy between linearized gravity and electromagnetism in Lorentz frames (based on suitably defined gravitational 3-vector fields, the "GEM fields", that mimic the electromagnetic ones), e.g. [7, 8, 9, 10, 11, 12], and the analogies between the Maxwell and the Weyl tensor (namely their decomposition on electric and magnetic parts, the scalar invariants they form, and the Maxwell-like "higher order field equations"), e.g. [31, 32, 34, 37, 46]; there were also *exact* analogies based on GEM fields, e.g. [18, 19, 20, 22, 23, 25], and an exact mapping, via the Klein Gordon equation, between ultrastationary spacetimes and magnetic fields in curved spacetimes [28]. Not only the relation between the different approaches had not been established, as they seemed to lead to differing views, as for instance spacetimes which were purely magnetic from the point of view of the analogies based on GEM fields, turned out to be purely electric from the point of view of the Weyl tensor (that is the case, for instance, of the Godel and Heisenberg spacetimes). And there were issues within each of them, such as the lack of a consistent physical interpretation for the magnetic parts of the Riemann and Weyl tensors, or the regime of validity of the analogies drawn in linearized theory. I briefly review these analogies in Sec. 3.1 below.

In Sec. 3.2 I review the most relevant literature on the relativistic multipole approaches to the description of the motion of test bodies, and discuss the then open questions when we started this work. They have to do essentially with the problem of the spin supplementary condition (and the center of mass definition in relativity), and with the electromagnetic equations. The latter (perhaps surprisingly) were less well understood than their gravitational counterparts. In this work considerable effort is put in the clarification of these issues. The problem of the spin condition is important because the existence of gravito-electromagnetic analogies in the equations of motion for spinning particles requires a particular spin condition to hold — the Mathisson-Pirani condition, which was poorly understood and portrayed as problematic in most literature. As for the electromagnetic equations and their interpretation, since one of the motivations for studying gravito-electromagnetic analogies is the hope of applying intuition from electromagnetism to the description and understanding of gravitational phenomena, then it is crucial to correctly understand the electromagnetic analogue first.

3.1 The Gravito-electromagnetic analogies in the literature

3.1.1 Analogy based on inertial forces from linearized gravity

This is the oldest and best known gravito-electromagnetic analogy, and it has been presented in different forms and conventions, see e.g. [7, 8, 9, 10, 11, 12, 13, 14]. Herein we will follow the conventions used in papers #1 and #2 of this compilation, which are common in the literature, e.g. [12] (up to the different signature).

One considers perturbations around Minkowski spacetime: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, usually assumed to be such that the line element has the form:

$$ds^2 = -c^2 (1 - 2\Phi) dt^2 - 4\mathcal{A}_j dt dx^j + \delta_{ij} [1 + 2\Phi] dx^i dx^j \ . \tag{3.1}$$

A parallelism is drawn between the perturbations Φ, $\vec{\mathcal{A}}$ (dubbed the "GEM potentials") and the components of the electromagnetic vector potential $A^\alpha = (\phi, \vec{A})$, to define "GEM fields" in analogy with electric and magnetic fields $\vec{E}$ and $\vec{B}$ of electromagnetism. If ones considers stationary perturbations, as is more usual[1] (e.g. [7, 15, 14, 66, 10, 101]), these GEM fields are (up to numerical factors in the different definitions)

$$\vec{E}_{\rm G} = -\nabla\Phi; \qquad \vec{B}_{\rm G} = \nabla \times \vec{\mathcal{A}} \ ,$$

and the motivation for this parallelism and these definitions is that they play in the gravitational equations roles analogous to the fields $\vec{E}$ and $\vec{B}$ of electromagnetostatics. The space part of the equation of geodesics $dU^\alpha/d\tau = -\Gamma^\alpha_{\beta\gamma} U^\beta U^\gamma$ is given, *to first order* in the perturbations and in test particle's velocity $\vec{v}$ (and, again, if the metric is stationary),

[1] If the fields depend on time, different definitions of $\vec{E}_{\rm G}$ exist in the literature, as a fully working analogy, holding simultaneously for geodesics and for the field equations, is not possible. This is discussed in Sec. 5 of Paper #5 of this compilation [5].

by the Lorentz-like expression:

$$\frac{d^2\vec{x}}{dt^2} = -\vec{E}_G - 2\vec{v} \times \vec{B}_G \ . \tag{3.2}$$

The evolution of the spin vector S^α of a torque-free gyroscope, described[2] by the Fermi-Walker transport law

$$\frac{DS^\alpha}{d\tau} = S_\sigma a^\sigma U^\alpha \tag{3.3}$$

U^α is the gyroscope's 4-velocity, $a^\alpha = DU^\alpha/d\tau$), reads in space components, to linear order, and with respect to the basis vectors of the coordinate system in (3.1),

$$\frac{d\vec{S}}{dt} = -\vec{S} \times \vec{B}_G \ , \tag{3.4}$$

similar to the well known text book expression [99] for the precession of a magnetic dipole under a magnetic field, $d\vec{S}/dt = \vec{\mu} \times \vec{B}$. This phenomenon is the so-called "gyroscope precession".

The force exerted on a gyroscope, given by the Mathisson-Papapetrou equation (Eq. (5.4) below), is, to linear order, for a gyroscope at *rest* [10] in the stationary field

$$\vec{F} = -\nabla(\vec{S} \cdot \vec{B}_G) \ , \tag{3.5}$$

similar to the force on a magnetic dipole $\vec{F}_{\rm EM} = \nabla(\vec{\mu} \cdot \vec{B})$, e.g. [99].

Finally, introducing the quantity $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \eta_{\mu\nu} h^\alpha_{\ \alpha}/2$, the Einstein field equations

$$R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T^\gamma_{\ \gamma} \right) \ , \tag{3.6}$$

in the linear regime, and imposing the harmonic gauge condition $\bar{h}_{\alpha\beta}^{\ \ ;\beta} = 0$, reduce to a set of four equations

$$\Box \bar{h}^{\alpha 0} = -16\pi J^\alpha \tag{3.7}$$

closely analogous to the Maxwell equations in the Lorentz gauge, $\Box A^\alpha = -4\pi j^\alpha$. $J^\alpha = -T^{\alpha\beta} U_\beta$ is the mass/energy density current measured by the observers of zero 3-velocity in the coordinate system of (3.1). In terms of the GEM fields, Eqs. (3.7) take the form:

$$\begin{array}{ll} \text{i)}\ \nabla \cdot \vec{E}_G = 4\pi\rho_m & \text{ii)}\ \nabla \times \vec{E}_G = 0 \\ \text{iii)}\ \nabla \cdot \vec{B}_G = 0 & \text{iv)}\ \dfrac{1}{2}\nabla \times \vec{B}_G = \dfrac{4\pi}{c}\vec{J} \end{array} \tag{3.8}$$

very similar to Maxwell's equations for electromagnetostatics *in Lorentz frames.*

In some literature time-dependent versions of the analogy are proposed, e.g. [12, 13, 16, 17], however there was no general agreement[3] on its limit of validity and physical content, with other authors arguing that the physical analogy holds only for stationary phenomena [9, 8, 11].

[2] If the Mathisson-Pirani spin condition holds; see Secs. 5.4 and 5.5 for more details.

[3] The debate was centered about the implications of the gauge condition $\bar{h}_{\alpha\beta}^{\ \ ;\beta} = 0$, and gets somewhat

3.1.2 Exact analogy based on inertial forces

Exact approaches to a gravito-electromagnetic analogy based on GEM inertial fields also existed in the literature, albeit much less known than the approach based on linearized theory of the previous section. Exact GEM fields, for the case of stationary space-times, were introduced by Landau-Lifshitz [18], and further worked out by other authors [20, 19, 21, 23, 24, 22]; this is sometimes called "the quasi-Maxwell" formalism, and can be summarized as follows. One considers a stationary spacetime, whose line element is generically described by:

$$ds^2 = -e^{2\Phi}(dt - A_i dx^i)^2 + \gamma_{ij} dx^i dx^j \tag{3.9}$$

where γ_{ij} is an arbitrary spatial metric (not flat, in general) . Then one defines [19] the *exact* GEM fields as

$$\vec{G} = -\tilde{\nabla}\Phi \qquad \vec{H} = e^{\Phi}\tilde{\nabla} \times \vec{A} \ , \tag{3.10}$$

dubbed, respectively, gravitoelectric and gravitomagnetic fields. $\tilde{\nabla}$ denotes the covariant differentiation operator *with respect to the spatial metric* γ_{ij}. The motivation for these definitions is the following. Let $u^{\alpha} = (u^0, 0)$ be the 4-velocity of the stationary observers whose worldlines are tangent to the time Killing vector $\partial/\partial t$, and consider an orthonormal tetrad field $e_{\hat{\alpha}}$ adapted to these observers, such that $\mathbf{e}_{\hat{0}} = \mathbf{u}$, and the spatial triad $e_{\hat{\imath}}$ is fixed with respect to the coordinate basis of γ_{ij} (i..e. points towards fixed neighboring observers). Let $U^{\hat{\alpha}} = (U^{\hat{0}},\ \vec{U})$ be the 4-velocity of some free test particle; the space part of the geodesic equation $DU^{\hat{\alpha}}/d\tau = 0$ can be exactly written in this frame as:

$$\frac{D\vec{U}}{d\tau} = 0 \Leftrightarrow \frac{\tilde{D}\vec{U}}{d\tau} = U^{\hat{0}}\left(U^{\hat{0}}\vec{G} + \vec{U} \times \vec{H}\right) \tag{3.11}$$

where $\tilde{D}X^{\hat{\imath}}/d\tau = dX^{\hat{\imath}}/d\tau + \Gamma^{\hat{\imath}}_{\hat{\jmath}\hat{k}} U^{\hat{\jmath}} X^{\hat{k}}$ denotes the 3-D covariant derivative *with respect to the spatial manifold* γ_{ij}, of a spatial vector $\vec{X}$, along the projected curve (parametrized by τ) on γ_{ij}. This equation is very similar to the electromagnetic Lorentz force law, which, is this notation, reads:

$$\frac{D\vec{U}}{d\tau} = \frac{q}{m_0}\left(U^{\hat{0}}\vec{E} + \vec{U} \times \vec{B}\right)$$

Also, according to Eq. (3.3), the "precession" of a gyroscope at rest ($U^i = 0$) is given by :

$$\frac{D\vec{S}}{d\tau} = 0 \Leftrightarrow \frac{\tilde{D}\vec{S}}{d\tau} = \frac{1}{2}\vec{S} \times \vec{H} \tag{3.12}$$

muddy; to follow it see [16], [9], [8] p. 163, [17] and [11], by this order. Herein I shall not give more details on it because we argue this is an unnecessary complication; this gauge choice is not in fact necessary to obtain the equations for the GEM fields (only for Eqs. (3.7) above); one just needs to linearize Eqs. (3.6), identify the inertial GEM fields from the geodesic equation, and express the equations in terms of them. In this way one obtains the desired equations in a transparent fashion, in terms of quantities whose physical role is clear in advance, and avoiding the subtleties of the harmonic gauge. See Sec. 5 of Paper #5 for more details.

which is similar to the precession of a magnetic dipole under a magnetic field $D\vec{S}/dt = \vec{\mu} \times \vec{B}$.

The force on a gyroscope *at rest* is given by [19]

$$\vec{F}_G = \frac{1}{2}\left[\tilde{\nabla}(\vec{H}\cdot\vec{S}) - \vec{S}(\tilde{\nabla}\cdot\vec{H}) - 2(\vec{S}\cdot\vec{H})\vec{G}\right] . \tag{3.13}$$

Apart from the last term, this is similar to the expression for the force on magnetic dipole that one obtains by integrating the force density $\vec{F} = \int \vec{j} \times \vec{B} d^3x$ (see [99], p. 189):

$$\vec{F}_{EM} = \nabla(\vec{\mu}\cdot\vec{B}) - \vec{\mu}(\nabla\cdot\vec{B}) \tag{3.14}$$

The term $\vec{\mu}(\nabla\cdot\vec{B})$ is usually not written in the literature since, due to the absence of magnetic monopoles, $\nabla\cdot\vec{B} = 0$ in Lorentz frames.

In terms of the GEM fields $\vec{G}$, $\vec{H}$, the Einstein equations read, in their time-time, time-space, and space-space components, respectively:

$$\tilde{\nabla}\cdot\vec{G} = -4\pi(2\rho + T^{\alpha}_{\ \alpha}) + \vec{G}^2 + \frac{1}{2}\vec{H}^2 \tag{3.15}$$

$$\tilde{\nabla}\times\vec{H} = 2\vec{G}\times\vec{H} - 16\pi\vec{J} \tag{3.16}$$

$$\tilde{\nabla}_i G_j - G_i G_j + \frac{1}{2}\vec{H}^2\gamma_{ij} + \tilde{R}_{ij} = 8\pi\left(\frac{1}{2}\gamma_{ij}T^{\alpha}_{\ \alpha} + T_{ij}\right) \tag{3.17}$$

The first two equations resemble, respectively, the Gauss and Ampere laws of electromagnetostatics, but contain non-linear terms that have no counterpart in the latter (in their usual textbook form, valid for Lorentz frames).

It is worth noting that $\vec{G}$ and $\vec{H}$ are simply, respectively, minus the acceleration and twice vorticity of the stationary observers (of zero 3-velocity $u^i = 0$ in the coordinates system of metric (3.9)):

$$G^{\alpha} = -\frac{Du^{\alpha}}{d\tau} \equiv -a^{\alpha} \tag{3.18}$$

$$H^{\alpha} = 2\epsilon^{\alpha}_{\ \beta\gamma\delta}u^{\gamma;\beta}u^{\delta} \equiv 2\omega^{\alpha} \tag{3.19}$$

This justifies our dubbing of these fields as "inertial fields" (or fields of inertial forces). This observation is also important for a more general formulation of the GEM fields, not relying on the "potentials" $\Phi, \vec{\mathcal{A}}$ of the stationary metric above. Indeed an exact formulation holding for arbitrary fields and observer congruences exists in the literature [25] (albeit little studied). The treatment in that case is much more complicated and for this reason I shall not give it herein, instead I refer the reader to Sec. 3 of Paper #5 [5], where that analogy is reformulated and further worked out.

Finally, this analogy has a very straightforward relation with the approach from the linearized theory (unlike the situation between the other analogies in this section), in spite of the fact that works establishing it are almost non-existing. Linearizing the GEM fields (3.10) herein, one obtains, in terms of the GEM fields of Sec. 3.1.1,

$$\vec{G} \approx -\vec{E}_{\rm G}; \qquad \vec{H} \approx -2\vec{B}_{\rm G} .$$

Thus, up to the different conventions for the GEM fields, linearizing the equations above one obtains the corresponding equations of Sec. 3.1.1.

3.1.3 The exact analogies between the Weyl and the Maxwell tensors

There is a set of analogies, based on exact expressions, relating the Maxwell tensor $F^{\alpha\beta}$ and the Weyl tensor $C_{\alpha\beta\gamma\delta}$. These analogies rest on the fact that: 1) they both irreducibly decompose in an electric and a magnetic type spatial tensors; 2) these tensors obey differential equations — Maxwell's equations and the so called "higher order" gravitational field equations — which are formally analogous to a certain extent [31, 32, 41, 34, 35]; and 3) they form invariants in a similar fashion [31, 32, 36, 47]. I will briefly review these analogies below.

The Maxwell tensor splits, with respect to a unit time-like vector U^α, in its electric and magnetic parts:

$$E^\alpha = F^\alpha_{\ \beta} U^\beta \ , \qquad B^\alpha = \star F^\alpha_{\ \ \beta} U^\beta \ , \tag{3.20}$$

i.e., the electric and magnetic fields as measured by the observers of 4-velocity U^α. These are spatial vectors: $E^\alpha U_\alpha = B^\alpha U_\alpha = 0$, thus possessing 3+3 independent components, which completely encode the 6 independent components of $F_{\mu\nu}$. The explicit decomposition is

$$F_{\alpha\beta} = 2U_{[\alpha}E_{\beta]} + \epsilon_{\alpha\beta\gamma\delta}U^\delta B^\gamma \ . \tag{3.21}$$

In spite of their dependence on U^α, one can use E^α and B^β to define two tensorial quantities which are U^α independent, namely

$$E^\alpha E_\alpha - B^\alpha B_\alpha = -\frac{F_{\alpha\beta}F^{\alpha\beta}}{2} \ , \qquad E^\alpha B_\alpha = -\frac{F_{\alpha\beta} \star F^{\alpha\beta}}{4} \ ; \tag{3.22}$$

these are the only algebraically independent invariants one can define from the Maxwell tensor.

Let $\top^\alpha_{\ \beta}$, $h^\alpha_{\ \beta}$ be the time and space projectors — i.e., parallel and orthogonal to U^α:

$$\top^\alpha_{\ \beta} \equiv (\top^U)^\alpha_{\ \beta} = -U^\alpha U_\beta; \qquad h^\alpha_{\ \beta} \equiv (h^U)^\alpha_{\ \beta} = U^\alpha U_\beta + \delta^\alpha_{\ \beta}. \tag{3.23}$$

Substituting decomposition (3.21) in the Maxwell equations

$$F^{\alpha\beta}_{\ \ \ ;\beta} = 4\pi j^\alpha \quad (a); \qquad \star F^{\alpha\beta}_{\ \ \ ;\beta} = 0 \quad (b), \tag{3.24}$$

and taking time and space projections one expresses them in terms of the electric and magnetic fields measured by the observers of 4-velocity U^α:

$$\hat{\nabla}_\mu E^\mu = 4\pi\rho_c + 2\omega_\mu B^\mu \ ; \tag{3.25}$$

$$\epsilon^{\alpha\gamma\beta}B_{\beta;\gamma} = \frac{D_F E^\alpha}{d\tau} - \sigma^\alpha_{\ \beta}E^\beta + \frac{2}{3}\theta E^\alpha - \epsilon^\alpha_{\ \beta\gamma}\omega^\beta E^\gamma + \epsilon^\alpha_{\ \beta\gamma}B^\beta a^\gamma + 4\pi j^{\langle\alpha\rangle} \ ; \tag{3.26}$$

$$\hat{\nabla}_\mu B^\mu = -2\omega_\mu E^\mu \ ; \tag{3.27}$$

$$\epsilon^{\alpha\gamma\beta}E_{\beta;\gamma} = -\frac{D_F B^\alpha}{d\tau} + \sigma^\alpha_{\ \beta}B^\beta - \frac{2}{3}\theta B^\alpha + \epsilon^\alpha_{\ \beta\gamma}\omega^\beta B^\gamma + \epsilon^{\alpha\mu\sigma}E_\mu a_\sigma \ . \tag{3.28}$$

where $\epsilon_{\mu\nu\rho} \equiv \epsilon_{\mu\nu\rho\tau}U^\tau$, $\hat{\nabla}_\alpha A^{\beta_1...\beta_n} \equiv h^\lambda_{\ \alpha}h^{\beta_1}_{\ \rho_1}...h^{\beta_n}_{\ \rho_n}\nabla_\lambda A^{\rho_1...\rho_n}$ denotes the spatially projected covariant derivative of a tensor $A^{\beta_1...\beta_n}$, and $D_F/d\tau$ is the Fermi-Walker derivative.

The quantities θ, $\sigma_{\mu\nu} \equiv h^{\alpha}_{\ \mu} h^{\beta}_{\ \nu} U_{\alpha;\beta}$, $\omega^{\alpha} \equiv \epsilon^{\alpha}_{\ \beta\gamma} U_{\gamma;\beta}/2$ and a^{α}, are, respectively, the expansion, shear, vorticity and acceleration of the congruence of observers with 4-velocity U^{α}.

The Weyl tensor has a decomposition formally similar to (3.20)-(3.21). With respect to a unit time-like vector U^{α}, it splits irreducibly in its electric $\mathcal{E}_{\alpha\beta}$ and magnetic $\mathcal{H}_{\alpha\beta}$ parts:

$$\mathcal{E}_{\alpha\beta} \equiv C_{\alpha\gamma\beta\sigma} U^{\gamma} U^{\sigma}, \quad \mathcal{H}_{\alpha\beta} \equiv \star C_{\alpha\gamma\beta\sigma} U^{\gamma} U^{\sigma}. \tag{3.29}$$

These two spatial tensors, both of which are symmetric and traceless (hence have 5 independent components each), completely encode the 10 independent components of the Weyl tensor. The explicit decomposition is,

$$C_{\alpha\beta}^{\ \ \gamma\delta} = 4\left\{2U_{[\alpha}U^{[\gamma} + g_{[\alpha}^{[\gamma}\right\}\mathcal{E}_{\beta]}^{\ \delta]} + 2\left\{\epsilon_{\alpha\beta\mu\nu}U^{[\gamma}\mathcal{H}^{\delta]\mu}U^{\nu} + \epsilon^{\gamma\delta\mu\nu}U_{[\alpha}\mathcal{H}_{\beta]\mu}U_{\nu}\right\} \ . \tag{3.30}$$

Again, in spite of their dependence on U^{α}, one can use $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$ to define the two tensorial quantities which are U^{α} independent,

$$\mathcal{E}^{\alpha\beta}\mathcal{E}_{\alpha\beta} - \mathcal{H}^{\alpha\beta}\mathcal{H}_{\alpha\beta} = \frac{C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}}{8} \ , \qquad \mathcal{E}^{\alpha\beta}\mathcal{H}_{\alpha\beta} = \frac{C_{\alpha\beta\mu\nu}\star C^{\alpha\beta\mu\nu}}{16} \ , \tag{3.31}$$

which are *formally* analogous to the electromagnetic scalar invariants (3.22) (it should be noted however that, by contrast with the latter, these are not the only independent scalar invariants one can construct from $C_{\alpha\beta\mu\nu}$; there are also two cubic invariants, see [32, 45, 46]).

As stated above, these tensors obey also differential equations which have some formal similarities with Maxwell's; such equations, dubbed the "higher order field equations" are obtained from the differential Bianchi identities $R_{\sigma\tau[\mu\nu;\alpha]} = 0$. These, together with the field equations (5.7a), lead to:

$$C^{\mu}_{\ \nu\sigma\tau;\mu} = 8\pi\left(T_{\nu[\tau;\sigma]} - \frac{1}{3}g_{\nu[\tau}T_{;\sigma]}\right) \ , \tag{3.32}$$

Expressing $C_{\alpha\beta\delta\gamma}$ in terms of $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$ using (3.30), and taking time and space projections of (3.32) using the projectors (5.105), one obtains, assuming a *perfect fluid*, the set of equations

$$\hat{\nabla}_{\mu}\mathcal{E}_{\nu}^{\ \mu} = \frac{8\pi}{3}\hat{\nabla}_{\nu}\rho + 3\omega^{\mu}\mathcal{H}_{\nu\mu} + \epsilon_{\nu\alpha\beta}\sigma^{\alpha}_{\ \gamma}\mathcal{H}^{\beta\gamma} \tag{3.33}$$

$$\text{curl}\mathcal{H}_{\mu\nu} = \frac{D_F}{d\tau}\mathcal{E}_{\mu\nu} + \mathcal{E}_{\mu\nu}\theta - 3\sigma_{\tau\langle\mu}\mathcal{E}_{\nu\rangle}^{\ \tau} - \omega^{\tau}\epsilon_{\tau\rho(\mu}\mathcal{E}_{\nu)}^{\ \rho} - 2a^{\rho}\epsilon_{\rho\tau(\mu}\mathcal{H}_{\nu)}^{\ \tau} + 4\pi(\rho+p)\sigma_{\mu\nu} \tag{3.34}$$

$$\hat{\nabla}_{\mu}\mathcal{H}_{\nu}^{\ \mu} = 8\pi(\rho+p)\omega_{\nu} - 3\omega^{\mu}\mathcal{E}_{\nu\mu} - \epsilon_{\nu\alpha\beta}\sigma^{\alpha}_{\ \gamma}\mathcal{E}^{\beta\gamma} \tag{3.35}$$

$$\text{curl}\mathcal{E}_{\mu\nu} = \frac{D_F}{d\tau}\mathcal{H}_{\mu\nu} - \mathcal{H}_{\mu\nu}\theta + 3\sigma_{\tau\langle\mu}\mathcal{H}_{\nu\rangle}^{\ \tau} + \omega^{\tau}\epsilon_{\tau\rho(\mu}\mathcal{H}_{\nu)}^{\ \rho} - 2a^{\rho}\epsilon_{\rho\tau(\mu}\mathcal{E}_{\nu)}^{\ \tau} \ , \tag{3.36}$$

where $\text{curl}A_{\alpha\beta} \equiv \epsilon^{\mu\nu}{}_{(\alpha}A_{\beta)\nu;\mu}$, and the index notation $\langle\mu\nu\rangle$ stands for the spatially projected, symmetric and trace free part of a rank two tensor (cf. definitions in [34]):

$$A_{\langle\mu\nu\rangle} \equiv h_{(\mu}{}^{\alpha}h^{\beta}_{\nu)}A_{\alpha\beta} - \frac{1}{3}h_{\mu\nu}h_{\alpha\beta}A^{\alpha\beta} .$$

The analogy with Maxwell's equations is closer if one considers the case of vacuum, and takes the linear regime; in this case Eqs. (3.33)-(3.36) become the equations in the right column of Table 3.1, originally found by Matte [31] (see also [32, 33]), which are formally similar to Maxwell's equations in Lorentz frames, only with the electric and magnetic parts of the Riemann tensor $\mathbb{E}_{\alpha\beta} = R_{\alpha\mu\beta\nu}U^{\mu}U^{\nu}$, $\mathbb{H}_{\alpha\beta} = \star R_{\alpha\mu\beta\nu}U^{\mu}U^{\nu}$ in the place of the electromagnetic fields (note that, in vacuum, $R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}$).

Table 3.1: Analogy between Maxwell's equations in Lorentz frames and Matte's equations.

Electromagnetism		Linearized Gravity	
Maxwell's Equations		Matte's Equations	
$E^{i}{}_{,i} = 0$	(3.1.1a)	$\mathbb{E}^{ij}{}_{,i} = 0$	(3.1.1b)
$B^{i}{}_{,i} = 0$	(3.1.2a)	$\mathbb{H}^{ij}{}_{,i} = 0$	(3.1.2b)
$\epsilon^{ikl}E_{l,k} = -\frac{\partial B^i}{\partial t}$	(3.1.3a)	$\epsilon^{ikl}\mathbb{E}^{j}{}_{l,k} = -\frac{\partial \mathbb{H}^{ij}}{\partial t}$	(3.1.3b)
$\epsilon^{ikl}B_{l,k} = \frac{\partial E^i}{\partial t}$	(3.1.4a)	$\epsilon^{ilk}\mathbb{H}^{j}{}_{l,k} = \frac{\partial \mathbb{E}^{ij}}{\partial t}$	(3.1.4b)

3.1.4 Some then open questions to be addressed

As mentioned above, the limit of applicability of the usual analogies based on the linearized theory was unclear; new approaches allowing for a transparent assessment of the actual physical similarities between linearized gravity and electromagnetism, and under which precise conditions they occur, were for this reason needed.

The physical content of the analogy in Sec. 3.1.3 was an unanswered question in the literature, and that is mainly due to the fact that the magnetic part of the Weyl (and Riemann) tensor was not well understood (the electric part $\mathcal{E}_{\mu\nu}$ was reasonably well understood due its role in the geodesic deviation equation); in the literature concerning this approach its physical significance was either presented as an open question [42, 44, 43], or given inconsistent interpretations. It was suggested in some works to be associated with rotation [37, 38, 34, 39, 40] and gravitational radiation [43, 48, 49, 50, 51, 40]. However, immediately contradictions arise [37, 38, 34]: there are many known examples of rotating spacetimes where the magnetic part of the Weyl tensor vanishes; amongst them is the notorious example of the Gödel Universe. It is also clear that gravitational waves cannot be the sole source for $\mathcal{H}_{\mu\nu}$, since the latter is generically non-vanishing in most stationary spacetimes.

The relationship between the different analogies (namely, the relationship between the electric and magnetic parts of the Weyl tensor, and the GEM fields) was another issue in need for a clarification, as they lead to seemingly different, even opposite, views of the same problems. Consider, for example, the Heisenberg spacetime (the same conclusions would be reached with e.g. the Godel Universe; I choose Heisenberg's because it is possible to analyze it both with exact and linearized theories), whose line element is given by

$$ds^2 = \left(dt - ar^2 d\phi\right)^2 + dr^2 + r^2 d\phi^2 + dz^2 \ .$$

Both according to the approach based on linearized theory of Sec. 3.1.1, and with the exact approach of Sec. 3.1.2, this spacetime has *zero* gravitoelectric field: $\vec{G} = \vec{E}_G = 0$, and a *non-zero* (uniform) gravitomagnetic field; $\vec{B}_{\rm G} = -a\vec{e}_z$ in the definitions of Sec. 3.1.1, or $\vec{H} = 2a\vec{e}_z$ in the definitions of Sec. 3.1.2. However according to the analogy based on the Weyl tensor in Sec. 3.1.3, this is a *purely electric* spacetime! Indeed, for the observers $u^i = 0$ (the same observers measuring the GEM fields above), the magnetic part of the Weyl (and Riemann) tensor *vanishes*: $\mathcal{H}_{\alpha\beta} = \mathbb{H}_{\alpha\beta} = 0$, and it is the electric part of the Weyl tensor that is *non-zero*: $\mathcal{E}_{\alpha\beta} \neq 0$ (the non-zero components are $\mathcal{E}_{rr} = 2a^2/3$, $\mathcal{E}_{\phi\phi} = 2r^2a^2/3$, $\mathcal{E}_{zz} = -4a^2/3$). And when observers exist for which $\mathcal{E}_{\alpha\beta} \neq 0$, $\mathcal{H}_{\alpha\beta} = 0$, the spacetime is in this framework classified as purely electric, see [45, 46].

3.2 Spinning multipole particles in general relativity

The classical equations of motion for spinning charged particles (possessing only charge and intrinsic magnetic dipole moment) under the action of electromagnetic fields were first derived in the framework of Special Relativity by Frenkel [55] (see also [131]), and subsequently[4] by Bhabha-Corben [56, 82, 83] (for particles with both electric and magnetic moments) and Weyssenhoff-Raabe [76]. Later, more rigorous treatments were put forth by Dixon [59] and Gralla *et al* [64]; in [64] particles with electric and magnetic dipole moments are considered, and [59] gives equations valid to *arbitrary order* in the multipole expansion.

In the presence of a gravitational field, the equations of motion for spinning multipole particles were first derived by Mathisson [61], for zero electromagnetic field, and accurate to quadrupole order; these equations were then re-derived by Papapetrou [79], who carried out a derivation exact (in the external field) *at each step,* but for *pole-dipole* particles only. Tulczyjew [62], Taub [117], Dixon [60] and Souriau [118, 119], carried out derivations covariant *at each step,* again for pole-dipole. The latter two, unlike the former, include also the electromagnetic field. Equations with both electromagnetic and gravitational fields valid first to quadrupole order [81], and then to arbitrary order [106], were given by Dixon. Some recent treatments re-derive these equations; Gralla *et al* [66] obtained equations to quadrupole order in Dixon's scheme (based on the "generalized Killing vectors" of [81]);

[4]There were also the famous treatments by Thomas [128] and by Bargmann-Michel-Teledgi [127]; these are not covariant treatments, however, and only take into account the particle's spin in the equation for the spin evolution, and not in the force equation.

and Natário [65], by a totally independent method (based on a Lagrangian approach to the problem of the Euler top in General Relativity) derived, for arbitrary dimension, the equations for pole-dipole particles in a gravitational field.

In-between these works there are treatments on *free* spinning pole-dipole particles in *flat spacetime,* most notably the work by Mathisson [74] where the famous helical motions were discovered, further worked out and re-derived by Weyssenhoff-Raabe [75, 76], and the important treatment by Möller [78], where first light was shed on the issue of the spin supplementary condition[5], the helical motions, and the relation with the problem of defining a center of mass for spinning bodies in relativity.

The treatments are all very different (and lead also to seemingly different results, which was one of the issues we needed to clarify in our work as explained below), some more rigorous than the others, thus I cannot go through the details of each of them; so below I very briefly outline the main ideas of the multipole scheme for test bodies in general relativity, and I take the physically more consistent viewpoint of *extended* test bodies (not point particles), and describe them in terms of a covariant multipole expansion. Thus I follow a scheme that is closer to Dixon's, e.g. [59, 60], yet simplified and already using our formalism and notation of papers #3 [3] and #4 [4].

A test particle is described by the moments of its charge current density 4-vector j^α (the "electromagnetic skeleton"), and the moments of $T^{\alpha\beta}$ (the "gravitational skeleton" [61]). In flat spacetime (and in Lorentz coordinates) they are, respectively,

$$\mathcal{J}^{\alpha_1...\alpha_n\mu}(\tau) \equiv \int_{\Sigma(\tau,U)} r^{\alpha_1}...r^{\alpha_n} j^\mu w^\sigma d\Sigma_\sigma \ ; \tag{3.37}$$

$$t^{\alpha_1...\alpha_n\mu\nu}(\tau) \equiv \int_{\Sigma(\tau,U)} r^{\alpha_1}...r^{\alpha_n} T^{\mu\nu} w^\sigma d\Sigma_\sigma \ . \tag{3.38}$$

These moments are taken with respect to a reference worldline $z^\alpha(\tau)$, of proper time τ and (unit) tangent vector $U^\alpha \equiv dz^\alpha/d\tau$, and a hypersurface of integration $\Sigma(\tau,u)$. $\Sigma(\tau,u) \equiv \Sigma(z(\tau),u)$ is the spacelike hypersurface generated by all geodesics orthogonal to some time-like vector u^α at the point $z^\alpha(\tau)$; $r^\alpha \equiv x^\alpha - z^\alpha(\tau)$, where $\{x^\alpha\}$ is a chart on spacetime; $d\Sigma_\gamma \equiv -u_\gamma d\Sigma$, and $d\Sigma$ is the 3-volume element on $\Sigma(\tau,u)$. w^α is a vector such that displacement of every point by $w^\gamma d\tau$ maps $\Sigma(\tau)$ into $\Sigma(\tau+d\tau)$, see [115, 59, 60] for more details. In a strongly curved spacetime (or if one uses a non-rectangular coordinate system) one needs to refine the expressions above in terms of bitensors (see the formulations in [60, 81, 105, 106, 66]), as not only $r^\alpha = x^\alpha - z^\alpha(\tau)$ is not a vector[6] (only to first order), but also the integrals (3.37)-(3.38) above would make no sense, as they would amount to adding tensor components at different points. If only lower order moments are to be kept, and if the gravitational field is not too strong, one can still to a good approximation

[5] One can say that this important work by Möller laid the foundations for our contribution, Paper #3 [3].

[6] In the general case of a curved spacetime, the distance between two points is the length of the geodesic connecting them; r^α is not even (exactly) a vector, and the closest notion to a separation vector is the bitensor $-\sigma^\kappa(x,z)$, which is the vector tangent to the geodesic at z^α whose length equals that of the geodesic connecting the two points [60, 81]. Also one must distinguish a coordinate system at x^α from the one at z^α, as the basis vectors change from point to point in a curved spacetime.

[81, 140, 116, 4] set up a locally nearly Lorentz frame and compute the moments from the expressions (3.37)-(3.38). In particular, to dipole order (which is the case we are mostly interested in), the bitensors are by definition redundant, as the approximation amounts to considering $T^{\alpha\beta}$ and j^{α} non-vanishing only in a very small region around $z^{\alpha}(\tau)$, so that only terms linear in r are kept; *to first order,* r^{α} is a vector (see e.g. Eq. (7) of [96]), and $\sqrt{r^{\alpha}r_{\alpha}}$ the distance between two points. Also, to first order, spacetime can always be taken as flat, thus these integrals are meaningful mathematical operations and do indeed define tensors, just like in flat spacetime.

3.2.1 Equations of motion for pole-dipole particles

This is the simplest case next to the monopole particle (whose sole equation of motion is the Lorentz force), and it is perhaps surprising that the problem of the equations of motion for it is still not well understood, with different methods and derivations leading to different versions of the equations, and the relation between them not being clear. And that it is the electromagnetic (not the gravitational) field that has been posing more problems.

The many existing approaches are very different, so I will state the general problem in the covariant multipole scheme of the previous section. Truncating the expansion to dipole order amounts to keep only two moments of $T^{\alpha\beta}$: $t^{\alpha\beta}$, $t^{\alpha\beta\gamma}$, and two moments of j^{α}: $\mathcal{J}^{\alpha}$, $\mathcal{J}^{\alpha\beta}$. In the integrals (3.37)-(3.38), to dipole order, $w^{\alpha} \simeq U^{\alpha}$. The equations of motion will follow from the charge conservation $j^{\alpha}_{\ ;\alpha} = 0$, and from the conservation of the total energy-momentum tensor [59, 81],

$$(T_{\rm tot})^{\alpha\beta}_{\ \ \ ;\beta} = 0 \quad \Leftrightarrow \quad T^{\alpha\beta}_{\ \ ;\beta} = F^{\alpha\beta} j_{\beta} \ , \tag{3.39}$$

where $F^{\alpha\beta}$ is the Maxwell tensor of the *external* (background) electromagnetic field and $T^{\alpha\beta}$ the energy-momentum tensor of the particle. A straightforward solution of this apparently simple problem, in term of quantities whose physical meaning is clear at each step, is yet to be given in the literature.

The following form of the equations is popularized in the literature, both concerning general relativistic treatments, e.g. [81, 106, 66]

$$\frac{DP^{\alpha}}{d\tau} = qF^{\alpha\beta}U_{\beta} + \frac{1}{2}F^{\mu\nu;\alpha}Q_{\mu\nu} - \frac{1}{2}R^{\alpha}_{\ \beta\mu\nu}S^{\mu\nu} \ , \tag{3.40}$$

$$\frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha}U^{\beta]} + 2Q^{\theta[\beta}F^{\alpha]}_{\ \ \theta} \ , \tag{3.41}$$

or, for the case $R^{\alpha}_{\ \beta\mu\nu} = 0$, in special relativistic treatments, e.g. [56, 83, 59]. $q \equiv \int_{\Sigma} j^{\alpha} d\Sigma_{\alpha}$ is the charge, $Q_{\alpha\beta}$ is the electromagnetic dipole moment tensor,

$$Q^{\alpha\beta} = 2d^{[\alpha}U^{\beta]} + \epsilon^{\alpha\beta\gamma\delta}\mu_{\gamma}U_{\delta} \ , \tag{3.42}$$

where d^{α} and μ^{α} are the electric and magnetic dipole moments measured by the observer of 4-velocity U^{α} (i.e., comoving with the reference worldline). These can be written in terms of the moments $\mathcal{J}^{\alpha\beta}$ defined in Eq. (3.37), taking $u^\alpha = U^\alpha$:

$$d^\alpha = -\mathcal{J}^{\alpha\beta} U_\beta \ , \tag{3.43}$$

$$\mu^\alpha = \frac{1}{2}\epsilon^\alpha{}_{\beta\gamma\delta} U^\delta \mathcal{J}^{\beta\gamma} \ . \tag{3.44}$$

But other (less well known) version of these equations exists in the literature [60, 77]:

$$\frac{DP^\alpha}{d\tau} = qF^\alpha{}_\beta U^\beta + \frac{1}{2}F^{\mu\nu;\alpha}\mu_{\mu\nu} - \frac{1}{2}R^\alpha{}_{\beta\mu\nu}S^{\mu\nu}U^\beta + F^\alpha{}_{\gamma;\beta}U^\gamma d^\beta + F^\alpha{}_\beta \frac{Dd^\beta}{d\tau} \ , \tag{3.45}$$

$$\frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha}U^{\beta]} + 2\mu^{\theta[\beta}F^{\alpha]}{}_\theta + 2d^{[\alpha}F^{\beta]}{}_\gamma U^\gamma \ , \tag{3.46}$$

where

$$\mu_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta}\mu^\gamma U^\delta \ .$$

P^α and $S^{\alpha\beta}$ are taken to be the momentum and angular momentum of the particle. But clearly one cannot be dealing with the same quantities in Eqs. (3.40)-(3.41) and (3.45)-(3.46), as the equations would not be compatible. This is the case in particular with P^α (in the case of $S^{\alpha\beta}$, to dipole order, it makes no difference, see Appendix A of Paper #4 [4]); contracting (3.41) with U^α, one obtains the expression

$$P^\alpha = mU^\alpha - \frac{DS^{\alpha\beta}}{d\tau}U_\beta - \epsilon^\alpha{}_{\theta\mu\sigma}d^\theta B^\mu U^\sigma + \epsilon^\alpha{}_{\theta\lambda\tau}\mu^\theta E^\lambda U^\tau \tag{3.47}$$

whereas contracting (3.46) with U^α leads to

$$P^\alpha = mU^\alpha - \frac{DS^{\alpha\beta}}{d\tau}U_\beta + \epsilon^\alpha{}_{\theta\lambda\tau}\mu^\theta E^\lambda U^\tau \ , \tag{3.48}$$

so indeed these expressions correspond to different quantities. We argue in Paper #4 [4] that it is expression (3.48) that is the *physical* momentum momentum of the particle, understood as the integral on a spacelike hypersurface of the particle's energy-momentum tensor

$$P^\alpha = \int_{\Sigma(\tau,U)} T^{\alpha\beta} d\Sigma_\beta \ . \tag{3.49}$$

As for the expression (3.47), which hereafter I will denote by $P^\alpha_{\rm Dix}$ ("Dixon's momentum"), we argue, following [110], that it is a part of the *canonical momentum* $P^\alpha_{\rm can} = P^\alpha_{\rm Dix} + qA^\alpha$ associated to the Lagrangian of the system. In many treatments it is simply not clear what P^α or $P^\alpha_{\rm Dix}$ is. In Dixon's treatments [81, 106, 59] it is clear from the beginning that $P^\alpha_{\rm Dix}$ is not (3.49), as electromagnetic terms are explicitly added to its definition, see e.g. Eq. (5.1) of [81]. The problem in this case is in the physical interpretation, as $P^\alpha_{\rm Dix}$ is nevertheless taken therein as the physical momentum, which leads to inconsistencies, as is explained in detail in Appendix A of Paper #4 [4]. Therein the relationship between the two formulations, and how to obtain one from the other, is clarified.

The spin supplementary condition. — Many other issues were unclear in the literature concerning these equations. Firstly, in order for Eqs. (3.45)-(3.46) to be equations of motion, the reference worldline $z^\alpha(\tau)$ must be chosen as some representative point through the body (so that its tangent U^α is the body's 4-velocity). Even in the case $F^{\alpha\beta} = 0$, the system above is undetermined, as it has 3 more unknowns than equations. The system can be closed[7] by imposing a condition of the type $S^{\alpha\beta}u_\beta = 0$, for some time-like unit vector field u^α, which effectively kills off 3 components of the angular momentum. The role of the condition $S^{\alpha\beta}u_\beta = 0$ is to specify the representative point of the body; more precisely, to choose it as being *the center of mass as measured by some observer of 4-velocity* u^α. The choice of the vector field u^α has been subject of a long debate in the literature, sometimes put in terms of which are the "correct" and the "wrong" conditions for each type of particle (since its status as a mere gauge choice is still not generally well understood; see introduction of [87] for a comprehensive review). The three best known ones are the Corinaldesi-Papapetrou condition [89], where $u^\alpha \propto \partial/\partial t$ corresponds to the static observers in Schwarzschild spacetime (but can be easily generalized to other stationary spacetimes), the Frenkel-Mathisson-Pirani condition [55, 61, 80] (hereafter Pirani's condition, as it is best known), where $u^\alpha = U^\alpha$ (i.e., the center of mass is computed in its own rest frame), and the Tulczyjew-Dixon condition [62, 60, 81], where u^α is taken parallel to P^α. It is the point of view in most of the literature that it is preferable to have equations of motion depending not on a center of mass measured by some particular observer, but instead one that is defined only in terms of properties "intrinsic" to the particle. The latter two conditions accomplish that, and are the most widely used (especially the Dixon-Tulczyjew); the differences between the two have been discussed, and again subject of a number of misunderstandings (for a review and clarification, I refer the reader to Appendix C.2 of Paper 4 # [4]).

Especially poorly understood is the Mathisson-Pirani condition, which happens to be the most important one in the context of this work, as the exact gravito-electromagnetic analogies for spinning particles require this condition to be used. This condition is degenerate, and allows for exotic helical solutions, even for a free particle in flat spacetime (in addition to the uniform straightline motion, which is also a solution). In the zero 3-momentum frame ($P^i = 0$), these are circular motions of radius

$$R = \frac{v\gamma^2 S}{m}$$

where $S = \sqrt{S^{\alpha\beta}S_{\alpha\beta}}/2$, $m = -P^\alpha U_\alpha$ is the particle's "proper mass", and $\gamma = 1/\sqrt{1-v^2}$; for more details see Sec. 5.40 below. The motions have been dubbed unphysical (see e.g. [76, 77]), due to the belief that R, for a given particle, can be arbitrarily large, based on the fact that γ can be arbitrarily large. That is, the representative point of a finite free body might move along circular trajectories with any radius, which is obviously contradicted by experiment and seemingly would invalidate the interpretation of this spin condition as a center of mass choice. It was our goal in Paper #3 [3] to clarify the misconception at the origin of these assertions, see Sec. 5.4.4 below, demystify the helical motions, and prove

[7] In the case $F^{\alpha\beta} \neq 0$, one needs also to provide evolution laws for the moments d^α and μ^α.

that there is nothing wrong with this spin condition.

The hidden momentum. — Eq. (3.48) above tells us that the particle's momentum P^α is not parallel to the 4-velocity U^α — the particle is said to possess "hidden momentum". This was another issue that was not well understood, and yet to be discussed in the framework of General Relativity when we started this work. The reason for the denomination hidden momentum is as follows: since P^α is not parallel to U^α, in the center of mass frame (where $\vec{U} = 0$) there will be a non-vanishing spatial momentum $\vec{P} \neq 0$; since in this frame the body is, by definition, at rest, this momentum must be hidden somehow. The third term in (3.48) is what we dub the "electromagnetic" hidden momentum. It is a still not well known feature of relativistic electrodynamics, despite its discovery by Schockley & James [90] dating back from the 60's, and having since been discussed in number of papers, e.g. [91, 90, 92, 93]. It is originated by the action of the electromagnetic field on the particle, and it is for that reason that we dub it "electromagnetic"; but it should be noted that it is *purely mechanical in nature*, see e.g. the simple physical model in Fig. 9 of [100] (see also Appendix D of Paper #4 [4]).

The second term in (3.48) is what we dub "inertial" hidden momentum, a concept that did not exist yet when we started this work. It was introduced only recently by Gralla-Harte-Wald in [66] (where it was dubbed "kinematical" hidden momentum), which was also the first work where the problem of the non-parallelism of U^α and P^α was addressed in a fully relativistic approach. This hidden momentum originates from the spin supplementary condition (i.e., from the field vector u^α with respect to which the center of mass is computed). We further worked out these ideas in papers #3 (where we shown that the hidden momentum explains the dynamical consistency of Mathisson's Helical motions) and #4, and a paper [120] is now in preparation where an exact formulation in terms of the GEM inertial fields of Sec. 3 of Paper #5 [5], yielding the hidden momentum of the particle when its center of mass is computed in an arbitrary frame (that is, applying to arbitrary spin conditions), is presented.

Finally, it should be noted that the hidden momentum (and now in particular the electromagnetic one) is not a feature one only needs to care about in sophisticated relativistic treatments; indeed it affects the textbook expressions for the forces exerted on a particle with electromagnetic moments. Take the case of the force exerted on a magnetic dipole; there has been a long debate in the literature concerning the correct equation for this force, and how it changes according to the two concurrent dipole models (the current loop, and the pair of monopoles; the result will differ because the former, but not the later, possesses hidden momentum); see e.g. [91, 122, 123, 124, 125]. I will not go through the details of each of these works (some of them is fair to say are not very rigorous, and contain even some mistakes). For a purely magnetic dipole ($d^\alpha = 0$, $q = 0$) in flat spacetime, and in the particle's rest frame, the space part of Eq. (3.45) above reads

$$\vec{F}_{\rm EM} = \frac{D\vec{P}}{d\tau} = \nabla(\vec{B} \cdot \vec{\mu}) \ , \tag{3.50}$$

where $\nabla^i(\vec{B} \cdot \vec{\mu}) \equiv B^{j,i}\mu_j$. One is used to see this expression in textbooks, e.g. [99], and it is *correct,* for a magnetic dipole taken as a small *current loop.* But what many authors are not aware of is that this is *not* $m\vec{a}$, because such particle possesses hidden momentum $\vec{P}_{\rm hid} = \vec{\mu} \times \vec{E}$; c.f. Eq. (3.48). And the force Eq. (3.50) above is the derivative of the *total momentum* of the particle (including the hidden one). The acceleration equation reads

$$m\vec{a} = \vec{F}_{\rm EM} - \frac{D\vec{P}_{\rm hid}}{d\tau} = \nabla(\vec{B}\cdot\vec{\mu}) - \frac{D}{d\tau}(\vec{\mu}\times\vec{E}) = (\vec{\mu}\cdot\nabla)\vec{B} - \frac{D\vec{\mu}}{d\tau}\times\vec{E} \tag{3.51}$$

which is in agreement with [91, 124, 125]. Here $(\vec{\mu}\cdot\nabla^i)\vec{B} \equiv B^{i,j}\mu_j$, and in the last equality I used Maxwell vacuum equation $\nabla\times\vec{B} = \partial\vec{E}/\partial t$. In some textbooks, e.g. [126], we find the expression $\vec{F} = (\vec{\mu}\cdot\nabla)\vec{B}$ for the force on a magnetic dipole; it is thus not true in general. Such expression can only yield either $D\vec{P}/d\tau$ for vacuum electrostatics (so that $\nabla\times\vec{B} = 0 \Rightarrow (\vec{\mu}\cdot\nabla)\vec{B} = \nabla(\vec{B}\cdot\vec{\mu})$), or $m\vec{a}$ for the case of a fixed dipole $D\vec{\mu}/d\tau = 0$.

3.2.2 Equations of motion to quadrupole order

The relativistic equations of motion for spinning particles, to quadrupole order, were given first by Mathisson [61] (see also [97]) for purely gravitational fields; then Dixon derived the equations in the presence of an electromagnetic field, first in flat spacetime [59], and later for curved spacetime [81, 106]. Dixon's treatments in [59, 106] are actually valid to arbitrary multipole order, and based on *exact* multipole moments, which in the case of curved spacetime require the use of bitensors. The theory of bitensors is given in [129]; see also the brief reviews in [81, 105]. As explained above, to quadrupole order, and to a good approximation, the bitensors may be dropped and one may define the moments in a way similar to the case of flat spacetime. The equations are [59, 106, 66],

$$\begin{aligned}\frac{DP^{\alpha}_{\rm Dix}}{d\tau} &= qF^{\alpha\beta}U_\beta + \frac{1}{2}F^{\mu\nu;\alpha}Q_{\mu\nu} - \frac{1}{2}R^{\alpha}{}_{\beta\mu\nu}S^{\mu\nu} \\ &\quad + \frac{1}{3}Q_{\beta\gamma\delta}F^{\gamma\delta;\beta\alpha} - \frac{1}{6}J_{\beta\gamma\delta\sigma}R^{\beta\gamma\delta\sigma;\alpha} \ ,\end{aligned} \tag{3.52}$$

$$\begin{aligned}\frac{DS^{\alpha\beta}_{\rm can}}{d\tau} &= 2(P_{\rm Dix})^{[\alpha}U^{\beta]} + 2Q^{\theta[\beta}F^{\alpha]}{}_{\theta} \\ &\quad + 2m^{[\alpha}{}_{\rho\mu}F^{\beta]\mu;\rho} + \frac{4}{3}J^{\mu\nu\rho[\alpha}R^{\beta]}{}_{\rho\mu\nu} \ ,\end{aligned} \tag{3.53}$$

where $m^{\alpha\beta\gamma} \equiv 4Q^{(\alpha\beta)\gamma}/3$, $Q^{\alpha\beta\gamma}$ is an electromagnetic quadrupole moment, and $J^{\alpha\beta\gamma\delta}$ a quadrupole moment of $T^{\alpha\beta}$, see Sec. VI of Paper #4 [4] for their definitions. The question mark is on the physical interpretation of these equations. In the literature these are portrayed as giving the "force" and the "torque" [81, 66] on the spinning particle, up to quadrupole order. We argue that they do not yield the actual force and torque on the body, because these are not equations for the *physical* momentum P^α, defined by Eq. (3.49), and angular momentum $S^{\alpha\beta}$, defined by

$$S^{\alpha\beta} \equiv 2\int_{\Sigma(\tau,U)} r^{[\alpha}T^{\beta]\gamma}d\Sigma_\gamma \ .$$

For this reason, following the discussion of the previous section, I denote the "momentum" in Eqs. (3.52)-(3.53) by $P^{\alpha}_{\rm Dix}$, and the angular momentum by $S^{\alpha\beta}_{\rm can}$, because it is argued in to be the "canonical" angular momentum [110] (for more details see Sec. 5.5.4.1). P^{α} is not the same as $P^{\alpha}_{\rm Dix}$ when an electromagnetic field is present, which is already manifest to dipole order as discussed above; and to quadrupole order, $S^{\alpha\beta}_{\rm can}$ cannot also be taken as $S^{\alpha\beta}$, as shown by Eq. (5.93) below. Moreover, unlike the issue with the definition of momentum, which does not affect the main results in Paper #4 [4] as we deal mostly therein with magnetic dipoles (for which $P^{\alpha}_{\rm Dix} = P^{\alpha}$), this one affects any spinning charged body, and also the interpretation of the work of the force, and of the proper mass variation, that we obtain to *dipole order* (as explained in Sec. 5.5.4.1 below). On top of that, unlike the equations to dipole order, for which the alternative version (3.45)-(3.46), in terms of what we call the physical momenta, is known, in the quadrupole case, by contrast, Eqs. (3.52)-(3.53) above (given in e.g. [59, 81, 106, 66]), were the only ones available in the literature that take into account the electromagnetic field (in the framework of a covariant multipole approach).

The inadequacy of Eq. (3.53), taken as a physical torque, becomes clear if one considers a spherical, uniformly charged body in flat spacetime. In that case (in vacuum) the term $m^{[\alpha}{}_{\rho\mu}F^{\beta]\mu;\rho}$ vanishes (see Sec. VIA of Paper #4 for details); all that remains is the dipole term $2\mu^{\theta[\beta}F^{\alpha]}{}_{\theta}$, present in Eqs. (3.46) or (3.41), which does not change the magnitude of $S^{\alpha\beta}_{\rm can}$ (if one also assumes $\mu^{\alpha\beta} = \sigma S^{\alpha\beta}_{\rm can}$, as done in [81, 66]), and moreover vanishes if the magnetic field B^{α} is aligned with $S^{\alpha}_{\rm can}$. That contradicts what we know from elementary arguments: if an electric field with a curl is present, it must torque the charged body; that is the case when the magnetic field is time-dependent, by Faraday's law of induction $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$. That torque has actually been computed in some non-relativistic treatments [107, 108, 109], where the following expression is presented for the torque exerted on a spinning charged ball in an electromagnetic field, e.g. Eq. (1) of [107]:

$$\vec{\tau} = \vec{\mu} \times \vec{B} - \sigma I \frac{d\vec{B}}{dt} \ .$$

Here $\sigma = q/2m$ is the gyromagnetic ratio, and I the moment of inertia of the sphere about an axis passing through its center. The first term is the usual torque on the magnetic dipole, the second is the torque due to the induced electric field, which we dub $\vec{\tau}_{\rm ind}$. If the sphere is uniform, $\sigma I = q^{\alpha}{}_{\alpha}/3$, where $q_{\alpha\beta}$ is the charge quadrupole, given by definition (5.94) below. Thus $\vec{\tau}_{\rm ind}$ is manifest to quadrupole order.

Due to its importance in the context of this work — since a fundamental difference revealed in the tidal tensor formalism put forth in Paper #1 [1] is the absence of gravitational effects analogous to electromagnetic induction — this was a problem in need to be addressed: obtain the equations for the *physical* quadrupole torque, single out the torque due to electromagnetic induction and clarify how it fits in Dixon's multipole scheme. That is done in Paper #4 [4], where, as discussed in Sec. 5.5.4.1 below, it is shown that indeed it is the part of the torque ignored in Eq. (3.53) (i.e., the space part of $-DS'^{\alpha\beta}/d\tau$, see notation therein) that encodes the torque due to Faraday's induction on an arbitrary charged body (not only spherical).

3.2.3 The gravito-electromagnetic analogies for spinning particles in the literature

Analogies between the equations of motion for gyroscopes in a gravitational field and magnetic dipoles in an electromagnetic field have been unveiled in different forms throughout the years. This is the case for both the force equation (center of mass motion) and the spin evolution equation of these test particles in external fields. Below I shall describe what was the state of the art prior to our paper devoted to the subject, Paper #4 [4].

The analogy for the force was first found by Wald [10] in the framework of linearized theory, who showed that the gravitational force exerted on a spinning pole-dipole test particle (hereafter a gyroscope), whose center of mass is *at rest* in a *stationary field*, takes the form (3.5), analogous to the force on a magnetic dipole. The analogy was later cast in an *exact* form, Eq. (3.13), by Natário [19], using the *exact* gravitoelectromagnetic (GEM) inertial fields from the so-called 1+3 "quasi-Maxwell" formalism that I briefly review in Sec. 3.1.2. The force was seen therein to consist of an electromagnetic-like part in the form $\tilde{\nabla}(\vec{H}\cdot\vec{S})-\vec{S}(\tilde{\nabla}\cdot\vec{H})$, plus a term $(\vec{S}\cdot\vec{H})\vec{G}$ interpreted as the weight of the energy of the gravitomagnetic dipole; the limit of validity of the analogy was thereby extended to arbitrarily strong *stationary* fields *and* when the gyroscope's worldline *is tangent to any time-like Killing vector field* (it comprehends e.g. circular trajectories with arbitrary speed in axisymmetric spacetimes). And in Paper #1 [1] we put forth the exact, *covariant* and *fully general* analogy relating the two forces, made explicit in the tidal tensor formalism.

The analogy between the so-called "precession" of a gyroscope in a gravitational field and the precession of a magnetic dipole under the action of a magnetic field was noticed long ago, in the framework of linearized theory, see Eq. (3.4), by a number of authors [140, 147, 7, 141]. The analogy was later cast in an *exact* form, Eq. (3.12), in the framework of the GEM inertial fields, e.g. [140, 25, 19]; it is not covariant, holding only in a specific frame comoving with the particle, but, in the formulation in [25] (see Paper #4 [4] for details), the test particle can be moving with arbitrary velocity in an *arbitrary field.*

Finally, it had recently been found by Gralla-Harte-Wald [66] an analogy, at an approximate level, between what we call the "inertial" hidden momentum $P^{\alpha}_{\rm hidI}$ (dubbed "kinematical" in [66]), under Dixon-Tulczyjew spin condition $S^{\alpha\beta}P_{\beta}=0$, and the electromagnetic hidden momentum $P^{\alpha}_{\rm hidEM}$. The approximate expressions are $\vec{P}_{\rm hidI}\approx-(1/M)\vec{S}\times\vec{F}$ (with $\vec{F}=D\vec{P}/d\tau$), and $\vec{P}_{\rm hidEM}\approx\vec{\mu}\times\vec{E}$ (we show in Paper #4 [4] that the analogy can be cast in an *exact* form, using the Mathisson-Pirani condition instead).

4 Roadmap to the papers

4.1 The exact GEM analogy based on tidal tensors

In an approach started in [1], and completed in [5], an exact analogy between gravity and electromagnetism, based on what we called the "tidal tensors", was found. We were seeking a formalism that would allow for a transparent comparison between the two interactions. Clearly, relativistic gravity and electromagnetism are very different theories; the reasoning was that, in order to separate the differences in the description from the actual differences in the physics, the comparison should be based on *physical* forces that could be invariantly defined in both theories. There is no physical, covariant, gravitational force analogous to the Lorentz force of electromagnetism, due to the equivalence principle (which can be stated as gravity being "pure geometry"); that is, the gravitational field does not exert forces on monopole particles, which move along geodesics. Tidal forces, by their turn, are covariantly present in both theories, and for this reason they are the basis of this approach. They manifest themselves in two basic effects:

- the relative acceleration of two nearby (monopole) test particles; this is described by the geodesic deviation equation in gravity, and by the analogous worldline deviation of electromagnetism (for particles with the same ratio q/m);
- the net force exerted on test particles with dipole moments. There is no gravitational analogue to the intrinsic electric dipole, as there are no negative masses; but there is an analogue to the magnetic dipole moment. The intrinsic magnetic dipole moment is the dipole moment of the spatial (with respect to the particle's proper frame) charge current density about the particle's center of mass. Its gravitational analogue is the particle's angular momentum about its center of mass, i.e. the dipole moment of the spatial mass/energy current (usually dubbed "spin" tensor/vector). A spinning particle whose only gravitational moments are the momentum and the spin, is what we dub a (ideal) gyroscope.

When one compares the exact equations describing these effects, an exact dynamical analogy emerges. Both in gravity and electromagnetism, the relative acceleration of neighboring monopole test particles (momentarily with the same velocity, and the same q/m in the electromagnetic case) are given by a contraction of an electric-type tidal tensor with the separation vector. And, *if the Mathisson-Pirani spin condition is employed*, both the force on a gyroscope and the force on a magnetic dipole, are given by a contraction of a magnetic type tidal tensor with the magnetic moment / spin vector of the particle.

This analogy extends to the field equations; the Maxwell equations (the source Eqs. (5.5a), plus the Bianchi identity (5.5b))

$$F^{\alpha\beta}{}_{;\beta} = 4\pi j^{\alpha} \quad (a); \qquad \star F^{\alpha\beta}{}_{;\beta} = 0 \quad (b). \tag{4.1}$$

can be cast, decomposing them in their time and space projections with respect to a unit time-like vector U^{α}, as a set of four algebraic equations involving only tidal tensors and sources. The gravitational field equations (Einstein Eqs. with sources, plus the algebraic Bianchi identity)

$$R^{\gamma}{}_{\alpha\gamma\beta} \equiv R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T^{\gamma}{}_{\gamma}\right) \quad (a); \qquad \star R^{\gamma\alpha}{}_{\gamma\beta} = 0 \quad (b), \tag{4.2}$$

by their turn, taking time and space projections, decompose into a set of six algebraic equations, four of which analogous to the Maxwell equations (and likewise being algebraic equations involving only tidal tensors and sources) plus two additional equations which have no electromagnetic counterpart, and involve a third rank 2 tensor that encodes the purely spatial curvature. Conversely, in the four equations with an electromagnetic counterpart, there are missing terms comparing with electromagnetism; these correspond to the antisymmetric parts / time projections (with respect to the observer congruence) of the electromagnetic tidal tensors, and encode the laws of electromagnetic induction, which have no gravitational counterpart. This is a fundamental difference between the two interactions that is herein made transparent, and is extensively explored in this work [4, 5].

4.2 The exact GEM analogy based on fields of inertial forces

This approach has a different philosophy from the tidal tensor analogy, as it does not compare physical forces from both theories; instead it explores a parallelism between the electromagnetic fields and fields of inertial forces arising from the so-called "1+3 splitting" of spacetime. That is, physical forces on the electromagnetic side, and reference frame artifacts on the gravitational side. This is more in the spirit of the well known analogy between linearized gravity and electromagnetism, where one looks for some "vector" fields playing in gravity a role analogous to the electromagnetic ones — actually those fields are but a special limit of the exact ones herein.

This analogy emerges as follows. We take an arbitrary orthonormal reference frame, which can be thought as a continuous field of orthonormal tetrads, or, alternatively, as consisting of a congruence of observers, each of them carrying an orthonormal tetrad whose time axis is the observer's 4-velocity, and the spatial triads, spanning the local rest space of the observers, are for now left arbitrary (namely their rotation with respect to Fermi-Walker transport). The mixed time-space part of the connections associated to this frame encode four spatial kinematical fields: the acceleration a^{α}, vorticity ω^{α} and shear $K_{(\alpha\beta)}$ of the observer congruence, plus the rotation Ω^{α} of the spatial triads with respect to Fermi-Walker (FW) transport. Writing the geodesic equation in such frame, one obtains for the spatial part an equation resembling the Lorentz force, with $-\vec{a} \equiv \vec{G}$

in the role of an electric field, $\vec{\Omega} + \vec{\omega} \equiv \vec{H}$ in the role of a magnetic field, plus a third term with no electromagnetic analogue, involving $K_{(\alpha\beta)}$. And, *if the Mathisson-Pirani spin condition holds*, gyroscopes are seen to precess relative to this frame at frequency $-\vec{\Omega}$ (like a magnetic dipole precesses at a rate $-\vec{B}$ under a magnetic field).

In this approach we finally understood the true nature of the so-called "gravitomagnetic field" ($\vec{H}$) — it is a reference frame artifact that comes from two separate parts of distinct origin: the vorticity of the observer congruence, plus the rotation (relative to FW transport) of the tetrads they carry. Only the latter is implied in the gyroscope "precession".

Doing a time plus space splitting of the Maxwell equations (5.5), and expressing them in terms of the electric and magnetic fields measured with respect to the observer congruence, we obtain a set of four equations which are a generalization of the usual textbook expressions (valid in Lorentz frames) to frames with arbitrary acceleration, rotation, and shear. And doing a similar splitting of the gravitational field equations (5.7), and expressing them in this frame — that is, in in terms of the "gravitoelectromagnetic" (GEM) fields $\vec{G}$ and $\vec{H}$, the shear $K_{(\alpha\beta)}$, and a suitably defined "spatial curvature" — we again obtain a set of six equations, four of which analogous to Maxwell's, plus two additional ones with no electromagnetic counterpart. The four equations which are analogous to Maxwell equations present many formal similarities with the latter; and the similarity gets closer when one considers stationary fields and rigid frames ($K_{(\alpha\beta)} = 0$), where we obtain the so-called "quasi-Maxwell" regime. *In this regime,* an exact analogy emerges also in this formalism between the electromagnetic force on a magnetic dipole and the gravitational force on a gyroscope (it is not, however, as general as the one from the tidal tensor formalism).

The well known analogies between linearized gravity and electromagnetism, where the notion of GEM fields first came about, are recovered as a limiting case of this exact formalism (for the weak field, slow motion regime).

This formalism is very powerful (cf. discussion in the conclusion of [5]); and it is even more so when used together with the tidal tensor formalism — for this reason the connection between the two, established in [5], is of primary importance.

4.3 Gravity contrasted with electromagnetism — where can they be similar

It is important to realize that the existence of the *exact* analogies above does *not* mean that the interactions are similar. Both the analogy based on tidal tensors and the analogy based on inertial GEM fields are functional analogies; the gravitational tidal tensors and inertial fields, despite playing *roles dynamically analogous* to the electromagnetic tidal tensors and fields, are themselves in general very different from the latter. (Even for seemingly analogous setups).

What these formalisms provide is a set of tools to compare the two interactions, and tell us where a similarity is to be expected. In what concerns the concrete effects, the precise conditions (namely regarding the time dependence of the fields) for occurrence of a gravito-electromagnetic similarity are specific to the type of effect [2, 5]. More generally one can say that the similarity may occur in weak *stationary* fields, and only for special frames therein. In [2] we discussed this issue in the framework of linearized theory and Post-Newtonian approximation, in view of astrophysical applications of present interest, namely the rotational and translational gravitomagnetic effects, subject of experimental scrutiny in past, recent and ongoing space experiments [54, 69]. A distinction, from the point of view of the analogy with electrodynamics, between effects related to (stationary) rotational mass currents, and those arising from translational mass currents, was found to exist: while the former are clearly analogous to magnetism, in the case of the latter the analogy is not so close. In Sec. 7 of [5] we generalize this analysis in terms of exact equations.

4.4 The problem of the center of mass in general relativity; Mathisson-Pirani spin condition

Both the magnetic parts of the analogy based on tidal tensors (i.e., the force on the spinning particle), and of the one based on inertial fields (both the equation for the force and for the particle's spin precession), require the Mathisson-Pirani condition to hold [4]. However this spin condition is usually portrayed in the literature as problematic; that led us to address and clarify this issue in [3].

The problem of the spin condition can be stated as follows: the equations of motion for spinning pole-dipole test particles in general relativity, which follow from the conservation of the energy-momentum tensor, are undetermined until one specifies the reference worldline (relative to which the moments are taken). A spin condition of the type $S^{\alpha\beta}u_\beta = 0$ for some unit time-like vector u^α has the role of requiring the reference worldline to be the center of mass as measured by some observer of 4-velocity u^α. This is because (unlike in Newtonian mechanics) in Relativity the center of mass is an observer dependent point, see Fig. 1 of [3].

The Mathisson-Pirani condition states that the reference worldline is the center of mass as measured in its own rest frame. It is however poorly understood in the literature — in particular its degeneracy, and the exotic helical motions it allows even for a free particle in flat spacetime (in addition to the expected straightline motions). These motions have been deemed unphysical, due to the belief that the radius of the helices was arbitrarily large (see e.g. Refs [4,5,15,28,29] of [3]), and for this reason a lot of skepticism has been drawn into this spin condition.

In [3] we show that these claims are a misconception, arising from a subtle (but crucial) mistake in some derivations in the literature. The radius of the helices is finite and always contained within the "disk of centroids" (the disk formed by all the possible positions of the centers of mass measured by the different observers), and the helical solutions are just equivalent, albeit more complicated (comparing with the non-helical one, that the same condition equally allows) descriptions of the motion of a spinning body. Interestingly, the dynamics of the helical motions (as well as other exotic motions allowed by the infinite possible spin conditions) are seen to be explained through the same concept of hidden momentum that was recently proposed in Ref. [20] of [3] as an explanation for the bobbings observed in numerical simulations of binary systems. And another exact analogy, in the framework of the GEM inertial fields, is seen to emerge [4], between this type of hidden momentum and the one present in electromagnetic systems. Such analogy proves insightful for the understanding of the helical motions, see Fig. 1 of [26].

However it is not only because of the analogies that this spin condition is interesting; indeed as we argue in [3], and exemplify in [4], in spite of its degeneracy, it is this spin condition, through its *non-helical* solution, that in many applications yields the simplest (especially in comparison with the more popular Dixon-Tulczyjew condition), clearest, and physically more sound description.

4.5 Spinning test particles in general relativity

The dynamics of spinning particles with multipole electromagnetic and inertial/gravitational moments are one of the most natural applications of the tidal tensor formalism, and an ideally suited one to illustrate the similarities and differences between the two interactions, that are encoded in the symmetries and time projections of the tidal tensors. This is studied in detail in [4]. The gravitational tidal tensors are symmetric in vacuum; the electromagnetic ones however possess an antisymmetric part, which encodes the laws of electromagnetic induction. A number of physical consequences are explored. It is seen that whenever a magnetic dipole moves in a non-homogeneous electromagnetic field it always measures a non-vanishing magnetic tidal tensor $B_{\alpha\beta}$, and therefore a force is exerted on it. In gravity by contrast, that is not the case: the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}$ can be zero for observers moving in non-homogeneous fields; geodesic motions for spinning particles were even found to exist in Schwarzschild (radial geodesics) and in Kerr-dS (circular equatorial geodesics) spacetimes. Also, due to the fact that the electric tidal tensor $E_{\alpha\beta}$ possesses an antisymmetric part (encoding the curl of the induced electric field), a time-varying electromagnetic field torques a spherical charged body; the gravitational field, by contrast, never torques a "spherical" body (i.e., a body whose multipole moments in an orthonormal frame match the ones of a sphere in flat spacetime).

The time-projections of the tidal tensors in a given frame are related with the rate of work done on the test particle by the external fields; in order to obtain that relationship, we start by writing the general equation yielding the variation of energy of a particle with multipole structure with respect to an arbitrary congruence of observers. This generalizes, for the case of non-spatial forces, and test particles possessing hidden momentum and varying mass, the previous results in the literature (which apply only to spatial forces on monopole particles). An interesting reciprocity is found to exist: in a frame comoving with the particle, the electromagnetic field, unlike the gravitational field, does work on the particle, causing a variation of its proper mass; conversely, for "static observers", a stationary gravitomagnetic field (but not a magnetic field) does work on mass currents — there is actually a spin-curvature potential energy, which quantitatively accounts for the Hawking-Wald spin-spin interaction energy.

4.6 Other issues clarified in the course of this work

One of our most relevant contributions to the field was the clarification of the relationship between the several gravito-electromagnetic analogies that have been unveiled throughout the years (including the new one based on tidal tensors that we proposed), which is done in [5].

A related open question in the literature was the physical significance of the magnetic part of the Riemann tensor (i.e., the gravitomagnetic tidal tensor); it is now clear from the analysis in [1, 4, 5], and the interpretation therein solves the inconsistencies previously found in the literature (see Refs. in Sec. 4.3 of [27]).

Another issue to which we gave a contribution is on the very problem of the covariant equations describing the motion of spinning particles subject to gravitational and electromagnetic fields, which even today is not generally well understood, with different methods and derivations leading to different versions of the equations, the relation between them not being clear. Perhaps surprisingly, it is the electromagnetic (not the gravitational) field that has been posing more problems. In [4] these differences are clarified, and the various terms in Dixon's equations are physically interpreted. The dynamical differences between the different dipole models (the current loop, and the charge models), the physical justification of the variation of the particle's proper mass, the work done on it by the external fields, and the case of the (apparently) missing electromagnetic induction torque in the quadrupole order equation of motion for the spin vector are also clarified.

A central issue in this context, which is not well understood, and leads to countless misconceptions in the literature, is the fact that the momentum of a spinning particle with multipole moments is not parallel to its 4-velocity — that is, it possesses hidden momentum. In this work we discussed several types of hidden momentum: the pure gauge "inertial" hidden momentum [3, 4], and the gauge invariant hidden momentum generated by the electromagnetic and gravitational fields (the latter being of quadrupole order) [4]; some further counter-intuitive consequences of it were unveiled — such as the fact that bodies can accelerate in opposite direction to the force in very simple systems.

4.7 Outcome and future directions

In this work we found a new exact analogy between gravity and electromagnetism based tidal tensors and we also studied and further developed other analogies in the literature; in particular, the not well known, but extremely powerful, analogy based on exact GEM fields. The main outcome of each approach, as well as the exact results one can obtain from application of each analogy, are summarized in the conclusion of [5]. We gave a contribution to the understanding of the equations of motion of multipole particles in gravitational and electromagnetic fields, and revealed some fundamental, yet not well known, aspects of both interactions. Moreover, we addressed the old problem of Mathisson helical motions, the issue of the various types of hidden momentum, and the physical meaning of the Dixon's equations.

The main points of this work on the gravito-electromagnetic analogies is that 1) there is a lot to be learned from a comparative study of the two interactions; and 2) the analogies are useful from a practical point of view, as they provide intuition and a familiar formalism to treat otherwise more complicated gravitational problems. Indeed, the formalisms developed, by being exact and general, provide a powerful set of tools that allows to study gravitomagnetic effects in arbitrarily strong fields, and also add new phenomena to the "gravitoelectromagnetism" category — a major addition being the motion of spinning pole-dipole particles in General Relativity, which can be *exactly* described in the framework of gravito-electromagnetic analogies, as we have shown in [4]. This is just the first major application of the formalism, on which we plan to build on (a first glimpse of the application of the tidal tensor formalism to the study of gravitational radiation, and the physical insight it brings, is given in [5]; and a paper on the use of the curvature scalar invariants will soon be published [30]), with many further applications being planned for the coming years.

5 The papers summarized and discussed

5.1 Notation and conventions

1. *Signature and signs.* We use the signature $-+++$; $\epsilon_{\alpha\beta\sigma\gamma} \equiv \sqrt{-g}[\alpha\beta\gamma\delta]$ denotes the Levi-Civita tensor, and we follow the orientation $[1230] = 1$ (i.e., in flat spacetime, $\epsilon_{1230} = 1$). $\epsilon_{ijk} \equiv \epsilon_{ijk0}$ is the 3-D alternating tensor. We use the convention for the Riemann tensor: $R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + ...$

2. Sometimes we use the abbreviation $\epsilon_{\alpha\beta\gamma} \equiv \epsilon_{\alpha\beta\gamma\delta}U^{\delta}$, where U^{α} is the 4-velocity of the test particle's CM.

3. Greek letters α, β, γ, ... denote 4-D spacetime indices, Roman letters $i, j, k, ...$ denote 3-D spatial indices. Following the usual practice, sometimes we use component notation $T^{\alpha\beta}$ to refer to a tensor $\mathbf{T}$. We use arrows for 3-vectors $\vec{v}$ except in Paper #2 where bold fonts $\mathbf{v}$ are used instead. In papers #3-#5 we use bold fonts to denote tensors $\mathbf{T}$ (including 4-vectors $\mathbf{U}$) .

4. *Time and space projectors.* $(\top^u)^{\alpha}{}_{\beta} \equiv -u^{\alpha}u_{\beta}$, $(h^u)^{\alpha}{}_{\beta} \equiv u^{\alpha}u_{\beta} + g^{\alpha}{}_{\beta}$ are, respectively, the projectors parallel and orthogonal to a unit time-like vector u^{α}; may be interpreted as the time and space projectors in the local rest frame of an observer of 4-velocity u^{α}. $\langle\alpha\rangle$ denotes the index of a spatially projected tensor: $A^{\langle\alpha\rangle\beta...} \equiv (h^u)^{\alpha}{}_{\beta}A^{\mu\beta...}$.

5. *Tensors resulting from a measurement process.* $(A^u)^{\alpha_1..\alpha_n}$ denotes the tensor $\mathbf{A}$ as measured by an observer $\mathcal{O}(u)$ of 4-velocity u^{α}. For example, $(E^u)^{\alpha} \equiv F^{\alpha}{}_{\beta}u^{\beta}$, $(E^u)_{\alpha\beta} \equiv F_{\alpha\gamma;\beta}u^{\gamma}$ and $(\mathbb{E}^u)_{\alpha\beta} \equiv R_{\alpha\nu\beta\nu}u^{\nu}u^{\mu}$ denote, respectively, the electric field, electric tidal tensor, and gravito-electric tidal tensor as measured by $\mathcal{O}(u)$. Analogous forms apply to their magnetic/gravitomagnetic counterparts.
   For 3-vectors we use notation $\vec{A}(u)$; for example, $\vec{E}(u)$ denotes the electric 3-vector field as measured by $\mathcal{O}(u)$ (i.e., the space part of $(E^u)^{\alpha}$, *written in a frame* where $u^i = 0$). When $u^{\alpha} = U^{\alpha}$ (i.e., the particle's CM 4-velocity) we drop the superscript (e.g. $(E^U)^{\alpha} \equiv E^{\alpha}$), or the argument of the 3-vector: $\vec{E}(U) \equiv \vec{E}$.

6. *Electromagnetic field.* The Maxwell tensor $F^{\alpha\beta}$ and its dual $\star F^{\alpha\beta}$ decompose in terms of the electric $(E^u)^{\alpha} \equiv F^{\alpha}{}_{\beta}u^{\beta}$ and magnetic $(B^u)^{\alpha} \equiv \star F^{\alpha}{}_{\beta}u^{\beta}$ fields measured by an observer of 4-velocity u^{α} as

$$F_{\alpha\beta} = 2u_{[\alpha}(E^u)_{\beta]} + \epsilon_{\alpha\beta\gamma\delta}u^{\delta}(B^u)^{\gamma} \; ; \tag{5.1}$$

$$\star F_{\alpha\beta} = 2u_{[\alpha}(B^u)_{\beta]} - \epsilon_{\alpha\beta\gamma\sigma}u^{\sigma}(E^u)^{\gamma} \; . \tag{5.2}$$

7. *Static observers.* In stationary, asymptotically flat spacetimes, we dub "static observers" the *rigid* congruence of observers whose worldlines are tangent to the temporal Killing vector field $\xi = \partial/\partial t$; may be interpreted as the set of points rigidly fixed to the "distant stars" (the asymptotic inertial rest frame of the source). For the case of Kerr spacetime, these correspond to the observers of zero 3-velocity in Boyer-Lindquist coordinates. This agrees with the convention in e.g. [73]. Note however that the denomination "static observers" is employed with a different meaning in some literature, e.g. [94], where it designates rigid, *vorticity-free* congruences (existing only in *static spacetimes*). In the case of the electromagnetic systems in flat spacetimes, by static observers we mean the globally inertial rest frame of the sources.

8. *GEM and GEM fields.* GEM is the acronym for "gravitoelectromagnetism". By "inertial GEM fields", we mean the fields of inertial forces that arise from the 1+3 splitting of spacetime: the gravitoelectric field, which plays in this framework a role analogous to the electric field of electromagnetism, and the gravitomagnetic field, analogous to the magnetic field. Different notations and conventions are used in the literature for these fields; and it is also the case for the papers in this compilation. In papers #1 and #2, the gravitoelectric and gravitomagnetic fields were denoted, respectively, by $\vec{E}_{\rm G}$ and $\vec{B}_{\rm G}$, and defined (following e.g. [12]) such that the geodesic equation in linearized *stationary* fields reads, in its space components,

$$\frac{d^2x}{dt^2} = -\vec{E}_{\rm G} - 2\vec{v} \times \vec{B}_{\rm G} \ ,$$

cf. Eq. (3.2) above. In papers #3-#5, we denoted the exact GEM fields by $\vec{G}$ and $\vec{H}$, and used a convention (following e.g. [19]) such that the *exact* geodesic equation, for stationary fields, has space components

$$\frac{\tilde{D}\vec{U}}{d\tau} = U^{\hat{0}} \left(U^{\hat{0}} \vec{G} + \vec{U} \times \vec{H} \right) \ ;$$

cf. Eq. (3.11). In linear regime, the correspondence between the two definitions is

$$\vec{G} \approx -\vec{E}_{\rm G}; \qquad \vec{H} \approx -2\vec{B}_{\rm G} \ .$$

Accordingly, different conventions for the GEM "potentials" were used. Φ, $\vec{\mathcal{A}}$, Θ_{ij} of papers #1 and #2 correspond to, respectively, $-\Phi$, $-\vec{\mathcal{A}}/2$, $-\xi_{ij}$ of paper #5. In what follows I will discuss each paper using its *original* notation.

5.2 Paper #1 — "Gravitoelectromagnetic analogy based on tidal tensors"

In this paper we first presented an exact analogy between gravity and electromagnetism that stems from the tidal dynamics of both theories, based on mathematical objects that we dubbed "tidal tensors".

We were interested in comparing the two interactions in a way as transparent as possible; the rationale behind our approach was that such comparison should based on *physical,* covariant forces that are present in *both* theories. The electromagnetic Lorentz force has no physical counterpart in gravity, as monopole point test particles in a gravitational field move along geodesics, without any real force being exerted on them. Also the spin vector of an ideal gyroscope in a gravitational field undergoes Fermi-Walker transport (i.e., follows the compass of inertia) without any real torque being exerted on it. Thus the analogies between the Lorentz force and the geodesic equation, and between the precession of a magnetic dipole and the "precession" of a gyroscope, which are well known from the linearized theory approaches reviewed in Sec. 3.1.1, and exist also in exact versions (reviewed in Sec. 3.1.2), are not suited for our purpose. They draw a parallelism between the electromagnetic fields E^α, B^α and fields of inertial forces G^α, H^α (fictitious forces, that vanish in locally inertial frames), i.e., they compare physical forces from one theory, with reference frame artifacts from the other.

Tidal forces, by their turn, are covariantly present in both theories, and their mathematical description in terms of "tidal tensors" is the basis of this approach. Tidal forces manifest themselves in two basic effects: the relative acceleration of two nearby monopole test particles, and in the net force exerted on dipoles. These notions of multipole moments are the ones given in Sec. 3.2, i.e., the multipole moments of the current density vector $j^\alpha = (\rho_c, \vec{j})$ in electromagnetism, and the moments of the energy momentum tensor $T_{\alpha\beta}$ in gravity. From the latter, only the moments of the projection $J^\alpha = -T^{\alpha\beta}U_\beta$ (i.e., the moments of the mass/energy 4-current density) have an electromagnetic counterpart. Monopole particles in the context of electromagnetism are those whose only non-vanishing moment is the total charge; dipole particles are particles with nonvanishing electric and magnetic dipole moments (i.e., respectively, the dipole moments of ρ_c and $\vec{j}$). Monopole particles in gravity are particles whose only non-vanishing moment of $T^{\alpha\beta}$ is $t_{\alpha\beta}$ (see definition (3.38) above), of which only the momentum $P^\alpha = -t^\alpha{}_\beta U^\beta$ contributes to the equations of motion; they correspond to the usual notion of *point test particle*s, which move along geodesics. There is no gravitational analogue of the intrinsic electric dipole, as there are no negative masses; but there is an analogue of the magnetic dipole moment, which is the "intrinsic" angular momentum (i.e. the angular momentum about the particle's center of mass), usually dubbed spin vector/tensor. A particle possessing only pole-dipole gravitational moments corresponds to the notion of an ideal gyroscope. We thus have two physically analogous effects suited to compare gravitational and electromagnetic tidal forces: worldline deviation of nearby monopole test particles, and the force exerted on magnetic dipoles/gyroscopes. An exact gravito-electromagnetic analogy, summarized in Table 5.1 , emerges from this comparison.

Eqs. (5.1.1) are the worldline deviations for nearby test particles with the same[1] tangent

[1] We want to emphasize this point, which, even today, is not clear in the literature. Eqs. (5.1.1) apply only to the instant where the two particles have the same (or infinitesimally close, in the gravitational case) tangent vector. For both electromagnetism and gravity, in the more general case that where velocity of the two particles is not infinitesimally close, the deviation equations include more terms (which depend on both particles' 4-velocity, thus in their relative velocity); the relative acceleration is not, in either case, given by a simple contraction of a tidal tensor with a separation vector, see [1, 27, 96]. A more

Table 5.1: The gravito-electromagnetic analogy based on tidal tensors.

Electromagnetism		Gravity	
Worldline deviation:		Geodesic deviation:	
$\frac{D^2\delta x^\alpha}{d\tau^2} = \frac{q}{m}E^\alpha{}_\beta \delta x^\beta, \;\; E^\alpha{}_\beta \equiv F^\alpha{}_{\mu;\beta}U^\mu$	(5.1.1a)	$\frac{D^2\delta x^\alpha}{d\tau^2} = -\mathbb{E}^\alpha{}_\beta \delta x^\beta, \;\; \mathbb{E}^\alpha{}_\beta \equiv R^\alpha{}_{\mu\beta\nu}U^\mu U^\nu$	(5.1.1b)
Force on magnetic dipole:		Force on gyroscope:	
$F^\beta_{EM} = B_\alpha{}^\beta \mu^\alpha, \;\; B^\alpha{}_\beta \equiv \star F^\alpha{}_{\mu;\beta}U^\mu$	(5.1.2a)	$F^\beta_G = -\mathbb{H}_\alpha{}^\beta S^\alpha, \;\; \mathbb{H}^\alpha{}_\beta \equiv \star R^\alpha{}_{\mu\beta\nu}U^\mu U^\nu$	(5.1.2b)
Maxwell Equations:		Eqs. Grav. Tidal Tensors:	
$E^\alpha{}_\alpha = 4\pi\rho_c$	(5.1.3a)	$\mathbb{E}^\alpha{}_\alpha = 4\pi\left(2\rho_m + T^\alpha_{\ \alpha}\right)$	(5.1.3b)
$E_{[\alpha\beta]} = \frac{1}{2}F_{\alpha\beta;\gamma}U^\gamma$	(5.1.4a)	$\mathbb{E}_{[\alpha\beta]} = 0$	(5.1.4b)
$B^\alpha{}_\alpha = 0$	(5.1.5a)	$\mathbb{H}^\alpha{}_\alpha = 0$	(5.1.5b)
$B_{[\alpha\beta]} = \frac{1}{2}\star F_{\alpha\beta;\gamma}U^\gamma - 2\pi\epsilon_{\alpha\beta\sigma\gamma}j^\sigma U^\gamma$	(5.1.6a)	$\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma}J^\sigma U^\gamma$	(5.1.6b)

$\rho_c = -j^\alpha U_\alpha$ and j^α are, respectively, the charge density and current 4-vector; $\rho_m = T_{\alpha\beta}U^\alpha U^\beta$ and $J^\alpha = -T^\alpha_\beta U^\beta$ are the mass/energy density and current (quantities measured by the observer of 4-velocity U^α); $T_{\alpha\beta} \equiv$ energy-momentum tensor; S^α, μ^α are the spin and magnetic moment 4-vectors; $\star \equiv$ Hodge dual. We use $\epsilon_{1230} = \sqrt{-g}$.

vector (and the same ratio charge/mass in the electromagnetic case), separated by the infinitesimal vector δx^α. They tell us that the so-called electric part of the Riemann tensor $\mathbb{E}^\alpha{}_\beta \equiv R^\alpha{}_{\mu\beta\nu}U^\mu U^\nu$ plays in the geodesic deviation equation (5.1.1b) the same physical role as the tensor $E_{\alpha\beta} \equiv F_{\alpha\gamma;\beta}U^\gamma$ in the electromagnetic worldline deviation (5.1.1a). $E_{\alpha\beta}$ describes the tidal effects produced by the electric field $E^\alpha = F^\alpha_{\ \gamma}U^\gamma$ *as measured by the test particle* of 4-velocity U^α. We can define it as a covariant derivative of the electric field as measured in the inertial frame momentarily comoving with the particle: $E_{\alpha\beta} = E_{\alpha;\beta}|_{U=const}$. Hence we dub it the "electric tidal tensor", and its gravitational counterpart the "gravitoelectric tidal tensor".

There is a magnetic counterpart to this analogy. Consider a purely magnetic dipole, that is, a particle whose only non-vanishing electromagnetic moment, as measured in the particle's CM frame, is the magnetic moment $\mu^\alpha = (0, \vec{\mu})$, defined by Eq. (3.44) above. The force exerted on it is [59, 60, 63, 66]

$$F^\alpha_{\rm EM} = \frac{DP^\alpha}{d\tau} = F^{\lambda\nu;\alpha}\mu_{\lambda\nu} = B^{\beta\alpha}\mu_\beta \,, \tag{5.3}$$

where $\mu_{\alpha\beta} \equiv \epsilon_{\alpha\beta\gamma\delta}\mu^\gamma U^\delta$ and $B_{\alpha\beta} \equiv \star F_{\alpha\gamma;\beta}U^\gamma$ is the "magnetic tidal tensor", describing the tidal effects produced by the magnetic field $B^\alpha = \star F^\alpha_{\ \gamma}U^\gamma$ *as measured by the particle* of 4-velocity U^α.

detailed discussion of this important issue will be presented elsewhere.

The gravitational force exerted on a gyroscope (i.e., a spinning pole-dipole particle) is given by the Mathisson-Papapetrou equation:

$$F_{\rm G}^{\alpha} \equiv \frac{DP^{\alpha}}{d\tau} = -\frac{1}{2} R^{\alpha}{}_{\beta\mu\nu} U^{\beta} S^{\mu\nu}. \tag{5.4}$$

If the Mathisson-Pirani spin condition $S^{\alpha\beta}U_{\beta} = 0$ holds, we have $S^{\mu\nu} = \epsilon^{\mu\nu\tau\lambda} S_{\tau} U_{\lambda}$, where S^{α} is the spin 4-vector, defined as being the 4-vector with components $(0, \vec{S})$ in the CM frame; substituting in Eq. (5.4) above, we obtain Eq. (5.1.2b) of Table 5.1, revealing the physical analogy $B_{\alpha\beta} \leftrightarrow \mathbb{H}_{\alpha\beta}$. That is, the magnetic part of the Riemann tensor $\mathbb{H}^{\alpha}{}_{\beta} \equiv \star R^{\alpha}{}_{\mu\beta\nu} U^{\mu} U^{\nu}$ plays in the gravitational force (5.1.2b) the same physical role as $B_{\alpha\beta}$ in the electromagnetic force (5.1.2a); for this reason we dub it "gravitomagnetic tidal tensor". Note the relative minus sign between Eqs. (5.1.2a) and (5.1.2b); it reflects the fact that masses/charges of the same sign attract/repel, implying that parallel charge/mass currents attract/repel.

In Paper #1 [1] this magnetic analogy was the final addition to the results of Table 5.1, and was presented as an application of the formalism (i.e., an analogy based "derivation" of Mathisson-Papapetrou-Pirani equation). There were however relevant issues left unaddressed, and for this reason the treatment therein is not fully satisfactory. The gravitational part of the analogy was based on the well established Eq. (5.4); however this equation needs to be supplemented by a spin condition as explained in Sec. 3.2.1; and this is an old, but important problem in this context because, in order to be *exact,* the analogy requires the Mathisson-Pirani spin condition to hold, as explained above. It turns out that this condition was poorly understood and portrayed as problematic in the literature, due to its degeneracy and the exotic helical solutions it allows. In the later work [3] (Paper #3 [3]) we clarify these issues, explain the meaning of the spin condition, and demystify the helical motions. The electromagnetic part of the analogy, Eq. (5.1.2b) of Table 5.1, was obtained therein from a covariantization of the textbook, non-relativistic, 3-D expression $\vec{F}_{\rm EM} = \nabla(\vec{\mu} \cdot \vec{B})$; and then the result checked with expression (11.26) of [63]. However in the latter result (in spite of being relativistic) only space components of the force were given; furthermore, in the derivation in [63], the physical meaning of the moments of the current defined therein is not clear (as customary, unfortunately, in most literature concerning multipole equations), and given by guessing. Especially because the physical content explored in Sec. V of [1] concerns mostly the time components of the forces, a more solid foundation for these results was in order. This is done in the recent work [4] (Paper #4), where the same exact analogy is shown to stem from the rigorous equations of motion for pole-dipole particles in gravitational and electromagnetic fields, and a number of issues and subtleties involving them (some of them not well understood in the literature, even today) are clarified. Namely the physical meaning of the forces above (which are not trivial, $DP^{\alpha}/d\tau \neq ma^{\alpha}$ in general!), as well as the momentum (which is not mU^{α}, due to the "hidden momentum") and the mass of the test particle (which is not a constant).

5.2.1 The gravitational analogue of Maxwell's Equations

If we take the traces and anti-symmetric parts of the electromagnetic tidal tensors, we obtain Eqs. (5.1.3a)-(5.1.6a) of Table 5.1, which are Maxwell's equations

$$F^{\alpha\beta}{}_{;\beta} = 4\pi j^{\alpha} \quad (a); \qquad \star F^{\alpha\beta}{}_{;\beta} = 0 \quad (b), \tag{5.5}$$

written in tidal tensor form. That is, Eqs. (5.1.3a) and (5.1.6a) are, respectively, the time and space projections (with respect to U^{α}) of the Maxwell equations with sources (5.5a); and (5.1.4a) and (5.1.5a) are, respectively, the space and time projections of the source-free equations (5.5a) (i.e., the electromagnetic Bianchi identity). This is explicitly shown using the projectors in [5] (Paper #5).

Eqs. (5.1.3a)-(5.1.6a) may be cast as equations involving only tidal tensors and sources, which can be seen decomposing:

$$F_{\alpha\beta;\gamma} = 2U_{[\alpha}E_{\beta]\gamma} + \epsilon_{\alpha\beta\mu\sigma}B^{\mu}{}_{\gamma}U^{\sigma}$$

or by noting that the pair of Eqs. (5.1.4a), (5.1.6a), may be condensed in the equivalent pair

$$\epsilon^{\beta\gamma}{}_{\alpha\delta}U^{\delta}E_{[\gamma\beta]} = -B_{\alpha\beta}U^{\beta}; \quad \text{(a)} \qquad \epsilon^{\beta\gamma}{}_{\alpha\delta}U^{\delta}B_{[\gamma\beta]} = E_{\alpha\beta}U^{\beta} + 4\pi j_{\alpha} \quad \text{(b)} \tag{5.6}$$

In a Lorentz frame in flat spacetime, since $U^{\alpha}{}_{;\beta} = U^{\alpha}{}_{,\beta} = 0$, we have $E_{\gamma\beta} = E_{\gamma;\beta}$, $B_{\gamma\beta} = B_{\gamma;\beta}$; and (using $U^{\alpha} = \delta^{\alpha}_{0}$) Eqs. (5.6) can be written in the familiar textbook forms $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$ and $\nabla \times \vec{B} = \partial\vec{E}/\partial t + 4\pi\vec{j}$, respectively. Likewise, Eqs. (5.1.3a) and (5.1.5a) reduce in this frame to the familiar forms $\nabla\cdot\vec{E} = 4\pi\rho_c$ and $\nabla\cdot\vec{B} = 0$, respectively.

By performing, on the gravitational tidal tensors, the same operations that lead to Eqs. (5.1.3a)-(5.1.6a) (i.e., taking the traces and anti-symmetric parts) we obtain the analogous set of Eqs. (5.1.3b)-(5.1.6b). Eqs. (5.1.3b) and (5.1.6b) turn out to be *exactly* the time-time and and time-space projections of Einstein equations with sources (5.7a):

$$R^{\gamma}{}_{\alpha\gamma\beta} \equiv R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T^{\gamma}{}_{\gamma}\right) \quad (a); \qquad \star R^{\gamma\alpha}{}_{\gamma\beta} = 0 \quad (b). \tag{5.7}$$

And Eqs. (5.1.4b) and (5.1.5b) are, respectively, the time-time and space-time projections of the algebraic Bianchi identity $R_{[\alpha\beta\gamma]\delta} = 0 \Leftrightarrow \star R^{\gamma\alpha}{}_{\gamma\beta} = 0$. Again, this is explicitly shown using the projectors in [5].

5.2.2 Gravity vs Electromagnetism

Eqs. (5.13a)-(5.16a) are strikingly similar to Eqs. (5.13b)-(5.16b) when the fields do not vary along the observer's worldline. Otherwise, they tell us that the two interactions must differ significantly, since the tidal tensors do not have the same symmetries.

Sources — Eqs. (5.1.3) tell us that the source of $E_{\alpha\beta}$ is ρ_c, and its gravitational analogue, the source of $\mathbb{E}_{\alpha\beta}$, is $2\rho + T^{\alpha}_{\ \alpha}$ ($\rho + 3p$ for a perfect fluid), manifesting the well known fact that in gravity, by contrast with electromagnetism, pressure and stresses act as sources of the field. The magnetic/gravitomagnetic tidal tensors are analogously sourced by the charge/mass-energy spatial currents $j^{\langle\mu\rangle}/J^{\langle\mu\rangle}$, as shown by Eqs. (5.1.6). Note that, when the fields do not vary along the observer's worldline, $\star F_{\alpha\beta;\gamma}U^{\gamma}$ vanishes and equations (5.1.6a) and (5.1.6b) match up to a factor of 2, identifying $j^{\mu} \leftrightarrow J^{\mu}$.

Symmetries and time projections of tidal tensors — The gravitational and electromagnetic tidal tensors do not generically exhibit the same symmetries; moreover, the former are spatial, whereas the latter have a time projection (with respect to the observer measuring them), signaling fundamental differences between the two interactions. In the general case of fields that are time dependent in the observer's rest frame (that is the case of an intrinsically non-stationary field, or an observer moving in a stationary field), $E_{\alpha\beta}$ possesses an antisymmetric part, which is the covariant derivative of the Maxwell tensor along the observer's worldline; $\mathbb{E}_{\alpha\beta}$, by contrast, is always symmetric. As discussed above, $E_{\alpha\beta}$ is a covariant derivative of the electric field as measured in the the momentarily comoving reference frame (MCRF); and Eq. (5.6a) is a covariant way of writing the Maxwell-Faraday equation $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$. Therefore, the statement encoded in the equation $\mathbb{E}_{[\alpha\beta]} = 0$ is that there is no *physical*, gravitational analogue to Faraday's law of induction (in the language of GEM vector fields of Paper #5 [5], we can say the the curl of the gravitoelectric field $\vec{G}$ does not manifest itself in the tidal forces, unlike its electromagnetic counterpart). To see a physical consequence, let δx^{α} in Eq. (5.1.1a) — the separation vector between a pair of particles with the same q/m and the same 4-velocity U^{α} — be spatial with respect to U^{α} ($\delta x^{\alpha}U_{\alpha} = 0$); and note that the spatially projected antisymmetric part of $E_{\mu\nu}$ can be written in terms of the dual spatial vector α^{μ}: $E_{[\langle\mu\rangle\langle\nu\rangle]} = \epsilon_{\mu\nu\gamma\delta}\alpha^{\gamma}U^{\delta}$. Then the spatial components (5.1.1a) can be written as (using $E_{\langle\mu\rangle\langle\nu\rangle} = E_{(\langle\mu\rangle\langle\nu\rangle)} + E_{[\langle\mu\rangle\langle\nu\rangle]}$):

$$\frac{D^2\delta x_{\langle\mu\rangle}}{d\tau^2} = \frac{q}{m}\left[E_{(\langle\mu\rangle\langle\nu\rangle)}\delta x^{\nu} + \epsilon_{\mu\nu\gamma\delta}\alpha^{\gamma}U^{\delta}\delta x^{\nu}\right] \quad\Leftrightarrow\quad \frac{D^2\delta\vec{x}}{d\tau^2} = \frac{q}{m}\left[\overleftrightarrow{E}\cdot\delta\vec{x} + \delta\vec{x}\times\vec{\alpha}\right], \tag{5.8}$$

the second equation holding in the frame $U^i = 0$, where we used the dyadic notation $\overleftrightarrow{E}$ of e.g. [67]. From the form of the second equation we see that $q\vec{\alpha}/m$ is *minus* an angular acceleration. Using relation (5.6), we see that $\alpha^{\mu} = -B^{\mu}{}_{\beta}U^{\beta}$; and in an inertial frame $\vec{\alpha} = \partial\vec{B}/\partial t = -\nabla\times\vec{E}$. In the gravitational case, since $\mathbb{E}_{\mu\nu} = \mathbb{E}_{(\mu\nu)} = \mathbb{E}_{\langle\mu\rangle\langle\nu\rangle}$, we have

$$\frac{D^2\delta x_{\langle\mu\rangle}}{d\tau^2} = \frac{D^2\delta x_{\mu}}{d\tau^2} = -\mathbb{E}_{(\mu\nu)}\delta x^{\nu} \quad\Leftrightarrow\quad \frac{D^2\delta\vec{x}}{d\tau^2} = -\overleftrightarrow{\mathbb{E}}\cdot\delta\vec{x}\ . \tag{5.9}$$

That is, given a set of neighboring charged test particles, the electromagnetic field "shears" the set via $E_{(\mu\nu)}$, and induces an accelerated rotation[2] via the laws of electromagnetic induction encoded in $E_{[\mu\nu]}$. The gravitational field, by contrast, only shears[3] the set, since

[2] By rotation we mean here absolute rotation, i.e, measured with respect to a comoving Fermi-Walker transported frame. See Paper #5 [5].

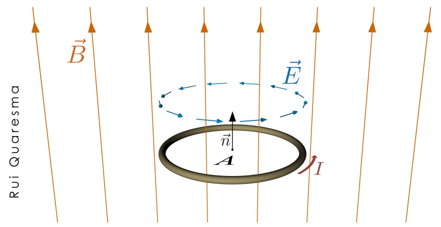
[3] If the two particles were connected by a "rigid" rod then the symmetric part of the electric tidal tensor would also, in general, torque the rod; hence in such system we would have a rotation even in the gravitational case, see [140] pp. 154-155. The same is true for a quasi-rigid extended body; however, even in this case the effects due to the symmetric part are very different from the ones arising from electromagnetic induction: first, the former do not require the fields to vary along the particle's worldline, they exist even if the body is at rest in a stationary field; second, they vanish if the body is spherical, which does not happen with the torque generated by the induced electric field, see [4].

$\mathbb{E}_{[\mu\nu]} = 0$.

Further physical evidence for the absence of a *physical* gravitational analogue for Faraday's law of induction is given in Sec. VI. of paper #4 [4]: consider a spinning spherical charged body in an electromagnetic field; and choose the MCRF (in order to keep things simple; see [4] for the general covariant treatment); if the magnetic field is not constant in this frame, by virtue of equation $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$, a torque will in general be exerted on the body by the induced electric field, changing its angular momentum and kinetic energy of rotation. By contrast, *no gravitational torque* is exerted in a spinning "spherical" body (i.e., a particle whose multipole moments in a local orthonormal frame match the ones of a spherical body in flat spacetime) placed in an *arbitrary* gravitational field; its angular momentum and kinetic energy of rotation are *constant*.

There is also an antisymmetric contribution $\star F_{\alpha\beta;\gamma}U^{\gamma}$ to $B_{\alpha\beta}$; in vacuum, Eq. (5.1.6a) is a covariant form of $\nabla \times \vec{B} = \partial \vec{E}/\partial t$; hence the fact that, in vacuum, $\mathbb{H}_{[\alpha\beta]} = 0$, means that there is no gravitational analogue to the antisymmetric part $B_{[\alpha\beta]}$ (i.e., the curl of $\vec{B}$) induced by the time varying field $\vec{E}$. Some physical consequences of this fact are as follows. Eq. (5.1.6a) implies, via (5.1.2a), that whenever a magnetic dipole moves in a non-homogeneous field, it measures a non vanishing $B_{[\alpha\beta]}$ (thus also $B_{\alpha\beta} \neq 0$), and therefore (except for very special orientations of the dipole moment μ^{α}) a force will be exerted on it; in the gravitational case, by contrast, the gravitational force on a gyroscope is not constrained to be non-vanishing when it moves in a non-homogeneous field; it is found that it may actually move along geodesics, as is the case of radial motion in Schwarzschild spacetime, or circular geodesics in Kerr-dS spacetime. For more details see Paper #4 [4].

The spatial character of the gravitational tidal tensors, contrasting with their electromagnetic counterparts, is another difference in the tensorial structure related to the difference in the symmetries (and thus to the laws of electromagnetic induction), as can be seen from Eqs. (5.6) and (5.1.6a). Physically, this is manifest (for instance) in the fact that the electromagnetic force on a magnetic dipole has a non-vanishing projection along the particle's 4-velocity U^{α}, which is the rate of work done on it by the induced electric field (and is reflected in a variation of the particle's proper mass, as shown in Paper #4 [4]). This is more easily seen be seen if we imagine the dipole as a small current loop, as depicted below:



$$F^{\alpha}_{EM}U_{\alpha} = \frac{\partial \vec{B}}{\partial t}.\vec{\mu} = \frac{\partial \Phi}{\partial t} I = -I \oint_{loop} \vec{E}.\vec{ds}$$

$\vec{\mu} = IA\vec{n}$ (magnetic moment), $\Phi \equiv \vec{B}.\vec{n}A$ (magnetic flux),
$\vec{E} \equiv$ induced electric field, $A \equiv$ area of the loop.

where Φ is the magnetic flux through the loop and $\vec{E}$ is the induced electric field. Thus $F^{\alpha}_{EM}U_{\alpha}$ is *minus* the power transferred to the dipole by Faraday's induction, due to a time varying magnetic field $\vec{B}$ (as measured in the particle's proper frame). An equivalent, but manifestly covariant derivation of this result is given in Paper #4 [4].

This effect has no counterpart in gravity. Since $\mathbb{H}_{\alpha\beta}$ is a spatial tensor, we *always have*

$$F_G^{\alpha} U_{\alpha} = 0$$

which means that no work is done on the gyroscope, *as measured in its proper frame* (and its proper mass is constant). Hence, the spatial character of the gravitational tidal tensors *precludes* induction effects analogous to the electromagnetic ones.

5.2.3 Matching between tidal tensors

It is important to realize that the existence of this *exact* gravito-electromagnetic analogy does *not* mean that the interactions are similar. This is a *functional analogy*: despite playing analogous roles in the dynamics of both theories, gravitational and electromagnetic tidal tensors are generically very different. In their symmetries and time projections, as we have seen above, and also in the more obvious fact that the electromagnetic tidal tensors are linear in terms of the electromagnetic potential (and fields), whereas the gravitational tidal tensors are non-linear in the metric tensor (and in the "gravitoelectromagnetic" fields of Paper #5 [5]). Nevertheless, we found that a matching occurs in certain special cases: linearized gravity *under certain conditions*, and an exact matching in the so-called ultra-stationary spacetimes.

5.2.3.1 Linearized Gravity

Take arbitrary perturbations around Minkowski spacetime in the form,

$$ds^2 = -(1-2\Phi)\,dt^2 - 4\mathcal{A}_j dt dx^j + [\hat{g}_{ij} + 2\Theta_{ij}]\,dx^i dx^j \tag{5.10}$$

($\hat{g}_{ij} \equiv$ euclidean metric in an arbitrary coordinate system), and an arbitrary electromagnetic field $A^{\alpha} = (\phi, \vec{A})$ in Minkowski spacetime $ds^2 = -dt^2 + \hat{g}_{ij}(x^k) dx^i dx^j$. The explicit expressions for gravitational and electromagnetic tidal tensors from these setups, as measured by an arbitrary observer of 4-velocity $u^{\alpha} = (u^0, u^i)$, are given in Eqs. (11)-(12) and (14)-(20) of Paper #1 [1]. Comparing these expressions we see that they will in general be very different, even to linear order. But if one takes time independent electromagnetic potentials/gravitational perturbations, and a "static observer", $U^{\mu} = \delta^{\mu}_0$ (i.e., an observer with zero 3-velocity in the coordinates of (5.10)) then the *linearized* gravitational tidal tensors match their electromagnetic counterparts:

$$\mathbb{E}_{ij} \simeq -\Phi_{;ij} \overset{\Phi \leftrightarrow \phi}{=} E_{ij}, \quad \mathbb{H}_{ij} \simeq \hat{\epsilon}_i^{\ lk} \mathcal{A}_{k;lj} \overset{\mathcal{A} \leftrightarrow A}{=} B_{ij}\ . \tag{5.11}$$

This can be illustrated by an elementary example of analogous physical systems — the gravitational field of a spinning mass and the electromagnetic field of a spinning charge: in the far field limit (where the non-linearities of the gravitational field are negligible), for an observer at rest with respect to the central body, the gravitational tidal tensors asymptotically match the electromagnetic ones, cf. Eqs. (8)-(9) of [1]. But if the observer moves, the electromagnetic tidal tensors it measures will be very different from the gravitational ones (for explicit expressions, see [27], sec. 2.2.1; or, for the case of non-spinning bodies, Eqs. (2.11)-(2.16) of Paper #2 [2]).

5.2.3.2 Ultrastationary Spacetimes

Ultra-stationary spacetimes are stationary spacetimes admitting *rigid* geodesic time-like congruences. In the coordinate system adapted to such congruence, the metric is generically given by:

$$ds^2 = -\left(dt + \mathcal{A}_i(x^k)dx^i\right)^2 + \gamma_{ij}(x^k)dx^i dx^j \ . \qquad (5.12)$$

The Klein-Gordon equation in these spacetimes reduces to a time-independent Schrödinger equation

$$H\psi = \epsilon\psi; \quad H = \frac{(\vec{P} + E\vec{\mathcal{A}})^2}{2m} \ , \quad \epsilon = \frac{E^2 - m^2}{2m} \ ,$$

for a particle with "charge" $-E$ and mass m, living in a *curved* 3-space with metric γ_{ij}, under the action of a "gravitomagnetic field"

$$\vec{H} = \tilde{\nabla} \times \vec{\mathcal{A}} \qquad (5.13)$$

where $(\tilde{\nabla} \times \vec{\mathcal{A}})^i = \hat{\epsilon}^{il}{}_k \tilde{\nabla}_l \mathcal{A}^k$. $\tilde{\nabla}$ denotes 3-D covariant derivatives with respect to γ_{ij}. The covariant derivative of $\vec{H}$ turns out to be, up to a factor of 2, the *exact* gravitomagnetic tidal tensor of (5.12) as measured by a static observer ($u^\alpha = \delta^\alpha_0$):

$$\tilde{\nabla}_j H_i = \hat{\epsilon}_{lki} \tilde{\nabla}_j \tilde{\nabla}^l \mathcal{A}^k = 2\mathbb{H}_{ij} \ . \qquad (5.14)$$

That is, we have an *exact* matching between tidal tensors from a linear theory (Electromagnetism) in a curved spacetime, and a non-linear one (General Relativity). This provides valuable insight for the understanding of the magnetic part of the Riemann/Weyl tensors ($\mathbb{H}_{\alpha\beta}/\mathcal{H}_{\alpha\beta}$) of these spacetimes, whose interpretation had been posing difficulties in the literature. It has been suggested (see e.g. [37, 38, 34, 39]) that "rotation" (here one should in rigor read vorticity $\omega^\alpha = \epsilon^{\alpha\beta\gamma\delta} u_{\gamma;\beta} u^\delta/2$ of the congruence of $u^i = 0$ observers, due to the frame dragging effect) sources the magnetic part of the Weyl tensor; but immediately contradictions arise. Whereas a number of examples, like the Kerr metric or the Van-Stockum interior solution [37], are known to support the idea that rotation sources $\mathcal{H}_{\mu\nu}$, there are also well known counterexamples, like the Gödel universe [37, 38, 34]. Indeed the vorticity does not directly relate to $\mathbb{H}_{\alpha\beta}$ or $\mathcal{H}_{\alpha\beta}$, but to the gravitomagnetic field[4] $\vec{H}$ of the corresponding frame. This makes a crucial difference. Take first the case of the interior Van Stockum metric (e.g. [37]), describing the interior of an infinitely long rotating cylinder of dust[5]; the line element can be put in the form (5.12), with

$$A_i dx^i = -ar^2 d\phi \ , \quad \gamma_{ij} dx^i dx^j = e^{-a^2 r^2} dr^2 + r^2 d\phi^2 + e^{-a^2 r^2} dz^2 \ , \qquad (5.15)$$

where a is a constant. The gravitomagnetic field for this setup is, cf. Eq. (5.13), $\vec{H} = -2ae^{r^2 b^2} \vec{e}_z$. The gravitomagnetic tidal tensor has non-vanishing components (for

[4] More precisely, with the definition (5.13) for $\vec{H}$, it is minus one half of the vorticity: $\vec{H} = -2\vec{\omega}$, as discussed in Sec. 3 of Paper #5 [5], or in e.g. [19])

[5] Note: this example was not kept in the final published version, but was part of an earlier preprint [27].

an observer with 4-velocity $u^\mu = \delta^\mu_0$): $\mathbb{H}_{rz} = \mathbb{H}_{zr} = -a^3 r$. The general relation between $\mathbb{H}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$ is (cf. Eq. (10) of [27])

$$\mathbb{H}_{\alpha\beta} = \mathcal{H}_{\alpha\beta} - 4\pi\epsilon_{\alpha\beta\sigma\gamma}J^\sigma U^\gamma \ ; \tag{5.16}$$

so in this case $\mathbb{H}_{\alpha\beta} = \mathcal{H}_{\alpha\beta}$, and from Eq. (5.14) we can write $\mathcal{H}_{ij} = \tilde{\nabla}_j H_i/2 = \tilde{\nabla}_j \omega_i/4$. Thus in this spacetime, there is a simple relation between the magnetic part of the Weyl tensor, and the *gradient* of the vorticity.

Now take the case of the Gödel Universe, which is portrayed in the literature as a homogeneous rotating universe (a difficult notion, means rotation around *any* point!). The metric can be written in the form (5.12), as described below:

$$\mathcal{A}_i dx^i = e^{\sqrt{2}\omega x} dy,$$
$$\gamma_{ij} dx^i dx^j = dx^2 + \frac{1}{2} e^{2\sqrt{2}\omega x} dy^2 + dz^2;$$
$$\vec{H} \equiv \tilde{\nabla} \times \vec{\mathcal{A}} = 2\omega \vec{e}_z$$

The gravitomagnetic field $\vec{H}$ is uniform; thus its tidal tensor vanishes $\mathbb{H}_{ij} = \tilde{\nabla}_j H_i/2 = 0$. This is the physical interpretation for the vanishing of the magnetic part of the Weyl tensor in this metric (since, again, $\mathbb{H}_{\alpha\beta} = \mathcal{H}_{\alpha\beta}$). That is, in other words: frame-dragging is present, the $u^i = 0$ observers have vorticity, but it is uniform, and for this reason $\mathcal{H}_{\alpha\beta} = 0$.

This allows us also to easily visualize the concept of homogeneous rotation, in analogy with a gas of charged particles subject to an uniform magnetic field (see picture above): there are Larmor orbits around any point.

In Paper #5 [5] we revisit these spacetimes in the framework of another exact GEM analogy (based on exact inertial GEM fields, not tidal tensors), with which we get more insight into the mapping presented above via Klein-Gordon equation, and where we also address the non-vanishing gravitoelectric tidal tensor in these spacetimes, which has no counterpart in the electromagnetic analogue, and was a question left unanswered in Paper #1.

5.2.4 Conclusion. Where does it stand in the context of the literature.

In this paper we introduced the analogy based on tidal tensors, which was then a new addition to the list of gravito-electromagnetic analogies that have been unveiled throughout the years. One may split these in two classes: physical and purely formal. In the second category falls the analogy between the electric and magnetic parts of the Weyl and Maxwell tensors, reviewed in Sec. 3.1.3. We classify this one as purely formal because it is not based on objects that play analogous physical roles in the two theories; it compares gravitational tidal tensors with electromagnetic fields (not tidal tensors; see discussion in Sec. 4.3 of [27]). Amongst the physical analogies, the best known are the ones from the Post-Newtonian and linearized theory approaches, briefly reviewed in Sec. 3.1.1, where a parallelism is drawn between the electromagnetic fields and fields of inertial forces — the so-called "gravitoelectromagnetic" (GEM) fields, which are reference frame artifacts. The latter play in the equations of motion (3.2), (3.4), and in the field equations (3.7), a role analogous to their electromagnetic counterparts. There are exact versions of this analogy, discussed in Sec. 3.1.2 (and that we further work out in Paper #5 [5]). Our tidal tensor analogy is also a physical one (i.e., it is based on objects playing analogous dynamical roles), and it is exact; its distinctive feature is that it is based on *physical*, covariant forces of *both* theories. The connection between the various analogies is established and discussed in detail in [5].

The tidal tensor formalism is ideally suited to compare the two interactions (the primary motivation of this approach). We have shown that Maxwell equations can be cast as equations for the traces and antisymmetric parts of the electromagnetic tidal tensors (and involving only tidal tensors and sources); taking the traces and antisymmetric parts of the gravitational tidal tensors, we obtain a set of exact equations which we argue to be their *physical* gravitational analogues. By comparing them we found striking similarities, and key differences between the interactions; namely in the phenomenon of electromagnetic induction, the way it manifests itself in the tidal forces, and the absence of analogous phenomena in gravity.

In this paper we just started exploring this comparison. Based on Eqs. (5.1.6) of Table 5.1, we concluded that, due to the symmetries of the tidal tensors, a similarity between the gravitational force on a gyroscope Eq. (5.1.2b) and the electromagnetic force on a magnetic dipole (5.1.2a) may occur only when the fields do not vary along the particle's worldline. The detailed comparison of the two forces in setups of interest, and the physical consequences of the symmetries of the tidal tensors, are then given in Paper #4 [4]. We explored the physical content of the (temporal) projection of these forces along the particle's worldline, which was seen to be the power transferred to the magnetic dipole by Faraday's law of induction, and is zero in the gravitational case (i.e., the force, just like the gravitomagnetic tidal tensor governing it, are spatial with respect to the particle's 4-velocity, which precludes an analogous gravitational induction effect).

We compared the tidal tensors in some special types of fields. We considered weak gravitational fields (i.e., linear regime), where we have found close similarities in the tidal tensors, as expected from the similarities that have been long known in the literature (and that I review in Sec. 3.1.1). Generically one can say that, in this regime, the electromagnetic and gravitational tidal forces are similar if there is a frame where the fields are stationary, and for observers at rest therein. Otherwise very important differences arise, that are not negligible in any consistent linear approximation (even in the slow motion regime), and that are usually overlooked in the literature. One may see this also as a statement about the physical content of the above mentioned gravito-electromagnetic analogies from linearized theory. In those approaches, as explained in Sec. 3.1.1, a parallelism is drawn between the metric perturbations $\{\Phi, \vec{\mathcal{A}}\}$ and the electromagnetic 4-potential $A^{\alpha} = (\phi, \vec{A})$, and GEM fields $\vec{E}_{\rm G}$ and $\vec{B}_{\rm G}$ are defined from the former in analogy with the electric $\vec{E}$ and magnetic $\vec{B}$ fields. $\vec{E}_{\rm G}$ and $\vec{B}_{\rm G}$ are fields of inertial forces (i.e., fictitious forces, as mentioned above); however they have been used to describe also (through their derivatives) tidal effects, like the force applied on a gyroscope Eq. (3.5). These are covariant effects, implying *physical* gravitational forces. In electromagnetism (in inertial frames) the tidal effects are always given by the derivative of the fields $E_{ij} = E_{i;j}$; $B_{ij} = B_{i;j}$; but the derivatives of $\vec{E}_G$ and $\vec{B}_G$ will only yield the correct tidal forces $\mathbb{E}_{ij}$, $\mathbb{H}_{ij}$, in the conditions mentioned above (see also in this respect Paper #5 [5]). In particular, Eq. (3.5) is true only when the particle is *at rest in a stationary field.*

We considered also ultrastationary metrics, whose gravitomagnetic tidal tensor was found to match exactly the one of a magnetic field in a curved 3-space. This allowed us to clarify some conceptual difficulties in the literature concerning these spacetimes, in particular the vanishing of the magnetic part of the Weyl tensor ($\mathcal{H}_{\alpha\beta}$) in some of them [37, 38, 34]: it is a magnetic type tidal tensor, thus it does not measure rotation itself, but *differential* rotation, which vanishes if the spacetime is homogeneous. $\mathcal{H}_{\alpha\beta}$ is actually linear in this spacetimes, and has a simple relation with the *gradient* of the vorticity. The connection for general spacetimes and arbitrary frames is made in Paper #5 [5]. But quite generally we can say that the analogy between the magnetic part of the Riemann tensor and the magnetic tidal tensor of electromagnetism gives a physical interpretation to the former (and to the magnetic part of the Weyl tensor as well), which was then an open question in the literature [42, 44, 43, 40].

5.2.5 Erratum for Paper #1

1. Missing terms in expression (20) of [1] for $\mathbb{H}_{ij}$; should read instead

$$\begin{aligned}\mathbb{H}_{ij} \;=\;& \hat{\epsilon}_i^{\;lk}\left(\mathcal{A}_{k;lj} + \dot{\Theta}_{jk;l}\right)(U^0)^2 + 2\hat{\epsilon}_i^{\;lm}\Theta_{l[k;j]m}U^0U^k \\ &+ \hat{\epsilon}_{ik}^{\;\;l}\left(\Phi_{;jl} + 2\dot{\mathcal{A}}_{(j;l)} + \ddot{\Theta}_{lj}\right)U^0U^k - 2\hat{\epsilon}_{ik}^{\;\;m}\left(\mathcal{A}_{[l;j]m} + \dot{\Theta}_{m[l;j]}\right)U^kU^l\end{aligned}$$

2. Type mistakes in unnumbered expressions in Sec. IV; they should read instead:

$$\mathbb{L} \simeq \frac{6m^2}{r^6} \overset{m\leftrightarrow q}{\simeq} L, \quad \mathbb{M} \simeq \frac{18Jm}{r^7}\cos\theta \overset{\{m,J\}\leftrightarrow\{q,\mu\}}{=} M\,.$$

(down to some mistype in the editing process, as the expressions were correct in the preprint).

5.3 Paper #2 — Reference frames and the physical gravito-electromagnetic analogy

Most general relativistic effects accessible to experimental (or observational) testing are within the realm of the linear or Post Newtonian approximations. Some of them are described in the framework of gravito-electromagnetic analogies that (as explained in the previous section) are drawn in these approaches. In this paper, building up on what we learned in Paper #1 [1], we addressed the following questions: under which specific conditions is there a similarity between weak field gravity and electromagnetism; what is the realm of applicability of the analogies from the linearized and Post-Newtonian approaches; in view of the astrophysical applications of present interest, how does the answer depend on the (Post-Newtonian) reference frame, and is there a difference, from the point of view of the analogies, between “rotational” and “translational” gravitomagnetism?

5.3.1 Linearized theory

We start by studying gravity in the linear regime, and comparing with electromagnetism. As in Sec. 5.2.3.1 above, we take arbitrary perturbations around Minkowski spacetime,

$$ds^2 = -c^2\left(1-2\frac{\Phi}{c^2}\right)dt^2 - \frac{4}{c}\mathcal{A}_j dt dx^j + \left[\delta_{ij} + 2\frac{\Theta_{ij}}{c^2}\right]dx^i dx^j \ , \qquad (5.17)$$

but now, in order to facilitate the comparison with the relevant literature, we keep track of the c factors (i.e., we are not putting $c = 1$). Again we consider an arbitrary electromagnetic field $A^\alpha = (\phi, \mathbf{A})$ in flat spacetime. Next we compare tidal forces of the two theories (governed by the tidal tensors); and also other types of analogous effects, which in electromagnetism consist of physical forces and torques (the Lorentz force, and the precession of a magnetic dipole, governed by the EM fields $\vec{E}$, $\vec{B}$), but whose gravitational counterpart are reference frame artifacts (the inertial “forces” in the geodesic equation, and the gyroscope “precession”, governed by the GEM fields $\mathbf{E_G}$, $\mathbf{B_G}$).

Tidal effects. — As we have seen in Sec. 5.2.3.1 above, the electromagnetic and gravitational tidal tensors from these setups will be in general very different, cf. Eqs. (11)-(12), (14)-(20) of Paper #1 [1]. But that if one considers time independent fields, and a static observer of 4-velocity $U^\mu \simeq c\delta^\mu_0$, then the *linearized* gravitational tidal tensors match their electromagnetic counterparts identifying $(\phi, A^i) \leftrightarrow (\Phi, \mathcal{A}^i)$, cf. Eq. (5.11); that suggests a physical analogy between these potentials, and defining the “gravitoelectromagnetic fields” $\mathbf{E_G} = -\nabla\Phi$ and $\mathbf{B_G} = \nabla \times \boldsymbol{\mathcal{A}}$, in analogy with the electromagnetic fields $\mathbf{E} = -\nabla\phi$, $\mathbf{B} = \nabla \times \mathbf{A}$. In terms of these fields we have $\mathbb{E}_{ij} \simeq (E_G)_{i,j}$ and $\mathbb{H}_{ij} \simeq (B_G)_{i,j}$, similar to the electromagnetic tidal tensors in Lorentz frames, $E_{ij} = E_{i,j}$ and $B_{ij} = B_{i,j}$.

The matching (5.11) means that a gyroscope at rest (relative to static observers) will feel a force F^α_G similar to the electromagnetic force F^α_{EM} on a magnetic dipole, which in this case take the very simple forms (time components are zero):

$$\mathbf{F_{EM}} = \frac{q}{2mc}\nabla(\mathbf{B}.\mathbf{S}); \qquad F^j_G = -\frac{1}{c}\mathbb{H}^{ij}S_i \approx -\frac{1}{c}(B_G)^{i,j}S_i \ \Leftrightarrow\ \mathbf{F_G} = -\frac{1}{c}\nabla(\mathbf{B_G}.\mathbf{S}) \ . \quad (5.18)$$

Had we considered gyroscopes/dipoles with different 4-velocities, not only the expressions for the forces would be more complicated, but also the gravitational force would significantly differ from the electromagnetic one, as one may check comparing Eqs. (12) with (17)-(20) of [1]. This will be exemplified in section 5.3.2 below.

The matching (5.11) also means, by similar arguments, that the relative acceleration between two neighboring masses $D^2\delta x^i/d\tau^2 = -\mathbb{E}^{ij}\delta x_j$ is similar to the relative acceleration between two charges (with the same q/m): $D^2\delta x^i/d\tau^2 = E^{ij}\delta x_j(q/m)$, *at the instant* when the test particles have 4-velocity $U^\alpha \simeq c\delta^\alpha_0$ (i.e., are *at rest* relative to the static observer $\mathcal{O}$).

Geodesics. — The space part of the equation of geodesics $U^{\alpha}_{\ ,\beta}U^{\beta} = -\Gamma^{\alpha}_{\beta\gamma}U^{\beta}U^{\gamma}$ is given, to first order in the perturbations and in test particle's velocity, by ($a^i \equiv d^2x^i/dt^2$):

$$\mathbf{a} = \nabla\Phi + \frac{2}{c}\frac{\partial \boldsymbol{\mathcal{A}}}{\partial t} - \frac{2}{c}\mathbf{v}\times(\nabla\times\boldsymbol{\mathcal{A}}) - \frac{1}{c^2}\left[\frac{\partial\Phi}{\partial t}\mathbf{v} + 2\frac{\partial {\Theta^i}_j}{\partial t}v^j\mathbf{e_i}\right] \tag{5.19}$$

$$\equiv -\mathbf{E_G} - \frac{2}{c}\mathbf{v}\times\mathbf{B_G} - \frac{1}{c^2}\left[\frac{\partial\Phi}{\partial t}\mathbf{v} + 2\frac{\partial {\Theta^i}_j}{\partial t}v^j\mathbf{e_i}\right] \tag{5.20}$$

where we defined the time-dependent gravitoelectric field[6] $\mathbf{E_G} \equiv -\nabla\Phi - (2/c)\partial\boldsymbol{\mathcal{A}}/\partial t$. Comparing with the electromagnetic Lorentz force:

$$\mathbf{a} = \frac{q}{m}\left[-\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} + \frac{\mathbf{v}}{c}\times(\nabla\times\mathbf{A})\right] = \frac{q}{m}\left[\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right], \tag{5.21}$$

these equations do not manifest, in general, a close analogy, due to the extra terms in (5.20), which have no electromagnetic counterpart. Note that, for the problem at hand in the next section, they are of the same order of magnitude as the gravitomagnetic term $\mathbf{v}\times\mathbf{B_G}/c$. But when one considers stationary fields, then (5.20) takes the form $\mathbf{a} = -\mathbf{E_G} - 2\mathbf{v}\times\mathbf{B_G}/c$ analogous to (5.21).

Note the difference between this analogy and the one from the tidal effects considered above: in the case of the latter, since are the tidal tensors *as measured by the test particle* that determine the forces, the similarity occurs only when the electromagnetic/gravitational potentials *and* their gradients, are time-independent in the frame comoving with the *test particle* (by comoving frame herein I mean the comoving Post-Newtonian frame [72, 68], or, for electromagnetism in flat spacetime, the momentarily comoving inertial frame); this is clear from Eqs. (11)-(12), (14)-(20) of Paper #1 [1]. In the case of the analogy for the geodesics, it is when the gravitational *potentials* (Φ, Θ_{ij}) are time-independent in *the observers'* (not the test particle's!) frame.

As mentioned above, the gravitational "acceleration" (5.20) is a reference frame artifact; it is worth noting that, as explained in detail in Paper #5 [5], $E^i_{\rm G} = Du^i/d\tau$ is the acceleration and $B^i_{\rm G} = (\nabla\times\mathbf{u})^i/2$ the vorticity of the observer congruence (the observers of zero 3-velocity $u^i = 0$ in the coordinate system of (5.17)). The last term of (5.20) arises from the fact that this congruence is a shearing one, since, to linear order, we have $-\partial\Theta_{ij}/\partial t = u_{(i;j)} = \sigma_{ij} + \theta\delta_{ij}/3$, where σ_{ij} is the traceless shear tensor and θ the expansion scalar.

[6] In some works, e.g. [12], the gravitoelectric field is given a different definition: $\mathbf{E_G}' = -\nabla\Phi - (1/2c)\partial\vec{\mathcal{A}}/\partial t$. The motivation of such definition is to obtain linearized field equations more similar to the electromagnetic ones. In particular, taking $\Theta_{ij} = \Phi\delta_{ij}$ (the so-called GEM regime [5]), and choosing the harmonic gauge condition, which implies $c\nabla\cdot\vec{\mathcal{A}} = -2\partial\Phi/\partial t$, one obtains with such definition $\nabla\cdot\mathbf{E_G}' = 4\pi\rho_m$, cf. Eq. (16) of [12]. But but, on the other hand, an extra "non-Lorentz-like" term appears in the equation for geodesics (5.20). With our definition above, the geodesic equation is more similar to the Lorentz force, but on the other hand $\nabla\cdot\mathbf{E_G} = 4\pi\rho_m - 3\partial^2\Phi/\partial t^2$. It is not possible to find a fully working analogy, based on GEM fields, when they depend on time, see Sec. 5 of Paper #5 [5].

Gyroscope precession. — The evolution of the spin vector of the gyroscope is given by the Fermi-Walker transport law, which, for a gyroscope at rest reads $DS^i/d\tau = 0$; hence, we have, in the coordinate basis,

$$\frac{dS^i}{dt} = -c\Gamma^i_{0j}S^j = -\frac{1}{c}\left[(\mathbf{S}\times\mathbf{B_G})^i + \frac{1}{c}\frac{\partial\Theta^{ij}}{\partial t}S_j\right] \ . \tag{5.22}$$

Comparing with the equation for the precession of a magnetic dipole under the action of a magnetic field

$$\frac{d\mathbf{S}}{dt} = \frac{1}{c}\mu\times\mathbf{B} \tag{5.23}$$

we see that there is an extra term in the gravitational equation, arising from the shear of the reference frame (that generically exists when the gravitational field is time-dependent), as discussed above. But this term can be made to vanish by choosing a suitable frame. Consider the *orthonormal* tetrad $\mathbf{e}_{\hat{\alpha}}$, and let $e^{\alpha}_{\ \hat{\alpha}}$ denote the transformation matrix relating it to the coordinate basis $\mathbf{e}_\alpha \equiv \partial_\alpha$: $\mathbf{e}_{\hat{\alpha}} = e^{\alpha}_{\ \hat{\alpha}}\mathbf{e}_\alpha$. Take $\mathbf{e}_{\hat{\alpha}}$ such that $\mathbf{e}_{\hat{0}}$ is the observers 4-velocity u^α, and the spatial triads $\mathbf{e}_{\hat{i}}$ follow as much as possible the coordinate basis vectors $\mathbf{e}_i$. That is, the $\mathbf{e}_{\hat{i}}$ co-rotate with the observer congruence (i.e., they rotate with respect to Fermi-Walker transport with an angular velocity that equals the vorticity of the congruence, see Secs. 3 and 5 of Paper #5 [5] for detailed explanation), but without suffering the shear and expansion effects of the later. To linear order, $e^{\alpha}_{\ \hat{\alpha}}$, and its inverse $e^{\hat{\alpha}}_{\ \alpha}$, are given by

$$\begin{aligned}
&\mathbf{e}_{\hat{0}} = (1-\phi)\mathbf{e}_0 \ ; \qquad \mathbf{e}_{\hat{i}} = \mathbf{e}_i + \frac{\Theta_i^{\ \hat{j}}}{c^2}\mathbf{e}_j + 2\frac{\mathcal{A}_i}{c}\mathbf{e}_0 \ ; \\
&\mathbf{e}_0 = (1+\phi)\mathbf{e}_{\hat{0}} \ ; \qquad \mathbf{e}_i = \mathbf{e}_{\hat{i}} - \frac{\Theta_i^{\ \hat{j}}}{c^2}\mathbf{e}_{\hat{j}} - 2\frac{\mathcal{A}_i}{c}\mathbf{e}_{\hat{0}} \ .
\end{aligned} \tag{5.24}$$

Thus $e^{i}_{\ \hat{i}} = \delta^{i}_{\ \hat{i}} - \Theta^{i}_{\ \hat{i}}/c^2$; expressing $\mathbf{S}$ in the tetrad, $S^i = S^{\hat{i}}e^{i}_{\ \hat{i}}$, we obtain an expression similar to the electromagnetic one (5.23):

$$\frac{dS^{\hat{i}}}{dt} = -\frac{1}{c}(\mathbf{S}\times\mathbf{B_G})^{\hat{i}} \ . \tag{5.25}$$

Thus, in the special case of gyroscope precession, the linear gravito-electromagnetic analogy holds even if the fields vary with time. This might be somewhat surprising, because if we think about the magnetic dipole as a spinning charged body in a time dependent magnetic field, an induced electric field would arise that should torque the body, changing it angular momentum S. Which, in the light of the conclusions of Paper #1, does not have a gravitational counterpart. As discussed in detail in Sec. VI of Paper #4 [4], the apparent paradox is an artifact of the dipole approximation, which neglects the torque exerted by the induced electric field. The latter depends on the particle's second moment of the charge, that is of quadrupole order. To dipole order the total torque reduces to Eq. (5.23), which preserves the magnitude of S, only causing the dipole to precess (regardless of the time-dependence of the field). That is: the analogy holds because electromagnetic induction effects are neglected in this degree of approximation.

5.3.2 Translational vs. Rotational Mass Currents

The existence of a similarity between gravity and electromagnetism, both in terms of tidal and inertial effects, requires time independent fields, as shown above. To what pertains gravitomagnetic effects, this is a statement about the time dependence of the mass currents in the chosen reference frame: if the they are (nearly) stationary, for example from a rotating celestial body, the gravitational field generated is analogous to a magnetic field; such is the field detected on LAGEOS Satellites data [52], in the Gravity Probe B mission [53], and presently under experimental scrutiny by the LARES mission [54]. But when they vary with time — e.g. the ones resulting from translation of the celestial body, considered in [69] — then the dynamics differ significantly.

Rotational Currents. — We start by the basic example of analogous systems already considered in Sec. III of Paper #1 [1]: the electromagnetic field of a spinning charge (charge Q, magnetic moment μ_s) and the gravitational field (in the far region $r \to \infty$) of a rotating celestial body (mass m, angular momentum J), see Fig. 5.1.

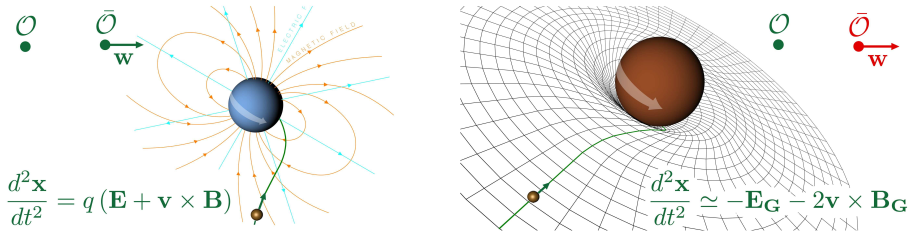


Figure 5.1: Spinning charge vs. spinning mass

The electromagnetic field of the spinning charge is described by the 4-potential $A^\alpha = (\phi, \mathbf{A})$, given by (5.26a). The spacetime around the spinning mass is asymptotically described by the linearized Kerr solution, obtained by putting in (5.17) the perturbations (5.26b) :

$$\phi = \frac{Q}{r}\ , \quad \mathbf{A} = \frac{1}{c}\frac{\boldsymbol{\mu}_\mathbf{s} \times \mathbf{r}}{r^3} \quad (a); \qquad \Phi = \frac{M}{r}\ , \quad \boldsymbol{\mathcal{A}} = \frac{1}{c}\frac{\mathbf{J} \times \mathbf{r}}{r^3}\ , \quad \Theta_{ij} = \Phi\delta_{ij} \quad (b). \qquad (5.26)$$

Let $\mathcal{O}$ be a static observer (i.e., at rest with respect to the central bodies); the gravitational tidal tensors it measures asymptotically match the electromagnetic ones, identifying the appropriate parameters:

$$\mathbb{E}_{ij} \simeq \frac{M}{r^3}\delta_{ij} - \frac{3Mr_ir_j}{r^5} \overset{M \leftrightarrow Q}{=} E_{ij}; \qquad \mathbb{H}_{ij} \simeq \frac{3}{c}\left[\frac{(\mathbf{r}.\mathbf{J})}{r^5}\delta_{ij} + 2\frac{r_{(i}J_{j)}}{r^5} - 5\frac{(\mathbf{r}.\mathbf{J})r_ir_j}{r^7}\right] \overset{J \leftrightarrow \mu_s}{=} B_{ij}$$

(all the time components are zero for this observer). This means that $\mathcal{O}$ will find a similarity between *physical* (i.e., tidal) gravitational and electromagnetic forces: the gravitational force $F^i_G = -\mathbb{H}^{ji}S_j/c$ exerted on a gyroscope carried by $\mathcal{O}$ is similar to the force $F^i_{EM} = B^{ji}\mu_j/c$ on a magnetic dipole; and the worldline deviation $D^2\delta x^i/d\tau^2 = -\mathbb{E}^{ij}\delta x_i$ of two masses dropped from rest is similar to the deviation between two charged particles with the same q/m.

Moreover, in the frame of the static observers $\mathcal{O}$, test particles will be seen moving on geodesics described by equations analogous to the electromagnetic Lorentz force, see Fig. 5.1.

Translational Currents. — For observers $\bar{\mathcal{O}}$ moving relative to the mass/charge of Fig. 5.1, however, the electromagnetic and gravitational interactions will look significantly different. Consider the frame obtained from the frame of the static observers $\mathcal{O}$ by applying a boost of constant *coordinate* velocity $\mathbf{w}$; and let us denote[7] the boosted frame by $\bar{\mathcal{O}}$. For simplicity we specialize here to the case where $\mathbf{J} = \boldsymbol{\mu} = 0$, so that the mass/charge currents in the frame $\bar{\mathcal{O}}$ arise solely from translation. To obtain the electromagnetic 4-potential $A^{\bar{\alpha}}$ in $\bar{\mathcal{O}}$, we apply the boost $A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\ \alpha} A^{\alpha} = (\bar{\phi}, \bar{\mathbf{A}})$, where $\Lambda^{\bar{\alpha}}_{\ \alpha} \equiv \partial \bar{x}^{\bar{\alpha}} / \partial x^{\alpha}$, using the expansion of Lorentz transformation (as done in e.g. [72]):

$$t = \bar{t}\left(1 + \frac{w^2}{2c^2} + \frac{3w^4}{8c^4}\right) + \left(1 + \frac{w^2}{2c^2}\right)\frac{\bar{\mathbf{x}}.\mathbf{w}}{c^2}; \quad \mathbf{x} = \bar{\mathbf{x}} + \frac{1}{2c^2}(\bar{\mathbf{x}}.\mathbf{w})\mathbf{w} + \left(1 + \frac{w^2}{2c^2}\right)\mathbf{w}\bar{t}\ , \tag{5.27}$$

yielding, to order c^{-2}, $A^{\bar{\alpha}} = (\bar{\phi}, \bar{\mathbf{A}})$, with $\bar{\phi} = Q(1 + w^2/2c^2)/r$ and $\bar{\mathbf{A}} = -Q\mathbf{w}/rc$. To obtain $A^{\bar{\alpha}}$ in the coordinates $(\bar{x}^i, \bar{t})$ of $\bar{\mathcal{O}}$, we must also express r (which denotes the distance between the source and the point of observation, in the frame $\mathcal{O}$) in terms of $R \equiv |\bar{\mathbf{r}} + \mathbf{w}\bar{t}|$, i.e., the distance between the source and the point of observation in the frame $\bar{\mathcal{O}}$. Using transformation (5.27), we obtain: $r^{-1} = R^{-1}[1 - (\mathbf{w}.\mathbf{R})^2/(2R^2c^2)]$, and finally the electromagnetic potentials measured in $\bar{\mathcal{O}}$:

$$\bar{\phi} = \frac{Q}{R}\left(1 + \frac{w^2}{2c^2} - \frac{(\mathbf{w}.\mathbf{R})^2}{4R^2c^2}\right); \qquad \bar{\mathbf{A}} = -\frac{1}{c}\frac{Q}{R}\mathbf{w}\ . \tag{5.28}$$

The metric of the spacetime around a point mass, in the coordinates of $\bar{\mathcal{O}}$, is also obtained using transformation (5.27), which is accurate to Post Newtonian order, by an analogous procedure. First we apply the boost $g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}_{\ \bar{\alpha}} \Lambda^{\beta}_{\ \bar{\beta}} g_{\alpha\beta}$ to the metric (5.26) (with $\mathcal{A} = 0$); then, expressing r in terms of R, we finally obtain (note that, although we are not putting the bars therein, indices $\alpha = 0, i$ in the following expressions refer to the coordinates of $\bar{\mathcal{O}}$):

$$\begin{aligned} g_{00} &= -1 + 2\frac{M}{Rc^2} + \frac{4Mw^2}{Rc^4} - \frac{M(\mathbf{w}.\mathbf{R})^2}{c^4R^3} \equiv -1 + \frac{2\bar{\Phi}}{c^2}; \\ g_{0i} &= \frac{4Mw_i}{Rc^3} \equiv -\frac{2\bar{\mathcal{A}}_i}{c^2}; \qquad g_{ij} = \left[1 + 2\frac{M}{Rc^2}\right]\delta_{ij} \equiv \left[1 + 2\frac{\bar{\Theta}}{c^2}\right]\delta_{ij}\ , \end{aligned} \tag{5.29}$$

where we retained terms up to c^{-4} in g_{00}, up to c^{-3} in g_{i0}, and c^{-2} in g_{ij}, as usual in Post-Newtonian approximation. Note that in the fields (5.28)-(5.29) we kept terms to second order in w (and the transformation (5.27) originating them was non-linear). This is because, in order to consistently take into account the gravitomagnetic force on a test

[7] This is a slight notation abuse, as one should in general distinguish observer from frame, see Sec. 3.1 of Paper #5 [5] where this issue is discussed in detail. Herein the situation is simple because we are dealing with *Post-Newtonian frames [68],* differing between each other only by boosts, thus both the observer congruence and the corresponding spatial frame are always well defined.

particle, $-2\mathbf{v}\times\mathbf{B_G}/c$, even to first order in the velocity *of the test particle* v, one needs to keep terms up to second order[8] in the translational velocity *of the source* $(-\mathbf{w})$; see Eq. (5.36) below. We neglected non-linear terms in M, as done in [69]; the metric (5.29) is equivalent to expressions (11) of [69] (where an additional gauge choice, Eq. (19) of [72], was made), for the case of a single source. It also matches Eqs. (5) of [70] to linear order in M (and again for the case of one single source). Note that the metric (5.29), like the electromagnetic potential (5.28), is now time dependent, since $\mathbf{R}(\bar{t})=\bar{\mathbf{r}}+\mathbf{w}\bar{t}$.

The gravitational tidal tensors measured by the observers $\bar{\mathcal{O}}$ are ($\mathbb{E}_{\alpha 0}=\mathbb{E}_{0\alpha}=\mathbb{H}_{\alpha 0}=\mathbb{H}_{0\alpha}=0$):

$$\begin{aligned}
\mathbb{E}_{ij} &= -\bar{\Phi}_{,ij}-\frac{2}{c}\frac{\partial}{\partial\bar{t}}\bar{\mathcal{A}}_{(i,j)}-\frac{1}{c^2}\frac{\partial^2}{\partial\bar{t}^2}\bar{\Theta}\delta_{ij}\\
&= \frac{M\delta_{ij}}{R^3}\left[1+\frac{3w^2}{c^2}-\frac{9}{2}\frac{(\mathbf{R}.\mathbf{w})^2}{c^2R^2}\right]-\frac{3MR_iR_j}{R^5}\left[1+\frac{2w^2}{c^2}-\frac{5(\mathbf{R}.\mathbf{w})^2}{2c^2R^2}\right]\\
&\quad -\frac{3Mw_iw_j}{c^2R^3}+\frac{6Mw_{(i}R_{j)}(\mathbf{R}.\mathbf{w})}{c^2R^5}; \qquad (5.30)\\
\mathbb{H}_{ij} &= \epsilon_i^{\ lk}\bar{\mathcal{A}}_{k,lj}-\frac{1}{c}\epsilon_{ij}^{\ \ l}\frac{\partial\bar{\Theta}_{,l}}{\partial\bar{t}} = \frac{M}{cR^3}\left[3\epsilon_{ij}^{\ \ k}w_k-\frac{3}{R^2}(\mathbf{R}.\mathbf{w})\epsilon_{ij}^{\ \ k}R_k-\frac{6}{R^2}(\mathbf{R}\times\mathbf{w})_iR_j\right] \qquad (5.31)
\end{aligned}$$

which significantly differ from the electromagnetic ones ($E_{0\alpha}=B_{0\alpha}=0$):

$$\begin{aligned}
E_{ij} &= -\bar{\phi}_{,ij}-\frac{1}{c}\frac{\partial}{\partial\bar{t}}\bar{A}_{i;j} = E_{i,j}\\
&= \frac{Q\delta_{ij}}{R^3}\left[1+\frac{w^2}{2c^2}-\frac{3}{4}\frac{(\mathbf{R}.\mathbf{w})^2}{c^2R^2}\right]-\frac{3QR_iR_j}{R^5}\left[1+\frac{w^2}{2c^2}-\frac{5(\mathbf{R}.\mathbf{w})^2}{4c^2R^2}\right]\\
&\quad -\frac{Qw_iw_j}{2c^2R^3}+\frac{3Qw_{[i}R_{j]}(\mathbf{R}.\mathbf{w})}{c^2R^5}; \qquad (5.32)\\
E_{i0} &= -\frac{1}{c}\frac{\partial}{\partial\bar{t}}\bar{\phi}_{;i}-\frac{1}{c^2}\frac{\partial^2\bar{A}_i}{\partial\bar{t}^2} \equiv \frac{1}{c}\frac{\partial E_i}{\partial\bar{t}} = \frac{Q}{cR^3}\left[w_i-\frac{3(\mathbf{R}.\mathbf{w})R_i}{R^2}\right]; \qquad (5.33)\\
B_{ij} &= \epsilon_i^{\ lm}\bar{A}_{m;lj} \equiv B_{i,j}=\frac{Q}{cR^3}\left[\epsilon_{ij}^{\ \ k}w_k-\frac{3}{R^2}(\mathbf{R}\times\mathbf{w})_iR_j\right]; \qquad (5.34)\\
B_{i0} &= \frac{1}{c}\frac{\partial B_i}{\partial\bar{t}} = -\frac{3Q}{c^2R^5}(\mathbf{R}.\mathbf{w})(\mathbf{R}\times\mathbf{w})_i\ . \qquad (5.35)
\end{aligned}$$

In particular, unlike their gravitational counterparts, $E_{\alpha\beta}$ and $B_{\alpha\beta}$ are not symmetric, and have non-zero time components. Note that the differing terms causing this are of the same order of magnitude as the others, thus cannot be neglected in any consistent approximation.

The space part of the geodesic equation for a test particle of velocity $\mathbf{v}$ is:

$$\begin{aligned}
\mathbf{a} &= \nabla\bar{\Phi}+\frac{2}{c}\frac{\partial\bar{\boldsymbol{\mathcal{A}}}}{\partial\bar{t}}-2\mathbf{v}\times(\nabla\times\bar{\boldsymbol{\mathcal{A}}})-\frac{3}{c^2}\frac{\partial}{\partial\bar{t}}\left(\frac{M}{R}\right)\mathbf{v} \qquad (5.36)\\
&= -\frac{M}{R^3}\left[1+\frac{2w^2}{c^2}-\frac{3(\mathbf{R}.\mathbf{w})^2}{2c^2R^2}\right]\mathbf{R}+\frac{3M(\mathbf{R}.\mathbf{w})}{c^2R^3}\mathbf{w}-\frac{4M}{c^2R^3}\mathbf{v}\times(\mathbf{R}\times\mathbf{w})+\frac{3}{c^2}\frac{M}{R^3}(\mathbf{R}.\mathbf{w})\mathbf{v}\ ,
\end{aligned}$$

[8] Otherwise, if we assumed $w^2\approx 0$ together with $v^2\approx 0$, then $vw\approx 0$, and Eq. (5.36) would reduce to the first term (the Newtonian acceleration)

which matches equation (10) of [70], or (7) of [71], again, in the special case of only one source, and keeping therein only linear terms in the perturbations and test particle's velocity $\mathbf{v}$.

Comparing with its electromagnetic counterpart

$$\left(\frac{m}{q}\right)\mathbf{a}=\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}=\frac{Q}{R^3}\left[1+\frac{w^2}{2c^2}-\frac{3(\mathbf{R}.\mathbf{w})^2}{4c^2R^2}\right]\mathbf{R}-\frac{1}{2}\frac{Q(\mathbf{R}.\mathbf{w})}{c^2R^3}\mathbf{w}+\frac{Q}{c^2R^3}\mathbf{v}\times(\mathbf{R}\times\mathbf{w})$$

we find them similar to a certain degree (up to some factors), except for the last term of (5.36). That term signals a difference between the two interactions, because it means that there is a velocity dependent "acceleration" which is parallel to the velocity; that is in contrast with the situation in electromagnetism, where the velocity dependent accelerations arise from magnetic forces, and are thus always perpendicular to $\mathbf{v}$.

As expected from Eqs. (5.25) (and by contrast with the other effects), the precession of a gyroscope carried by $\bar{\mathcal{O}}$, Eq. (5.37b) takes a form analogous to the precession of a magnetic dipole,

$$\frac{d\mathbf{S}}{d\bar{t}}=\frac{q}{2m}\frac{Q}{c^2R^3}\left[\mathbf{S}\times(\mathbf{R}\times\mathbf{w})\right] ,$$

if we express $\mathbf{S}$ in the local orthonormal triad $\mathbf{e}^{\hat{i}}$ as defined in Sec. 5.3.1, such that $S^i=(1-M/R)S^{\hat{i}}$:

$$\frac{dS^{\hat{i}}}{d\bar{t}}=\frac{2M}{c^2R^3}\left[(\mathbf{R}\times\mathbf{w})\times\mathbf{S}\right]^{\hat{i}} . \tag{5.37}$$

The triad of axes $\mathbf{e}^{\hat{i}}$ is in this case fixed relative to the "distant stars" (i.e., non rotating relative to inertial frames at infinity); thus Eq. (5.37) yields the precession of the gyroscope relative to the distant stars, which is the situation of interest for astrophysical applications, namely the measurements performed by the Gravity Probe B (GPB) [53]. Indeed, the precession angular velocity above,

$$\mathbf{\Omega}=\frac{2M}{c^2R^3}\mathbf{R}\times\mathbf{w} ,$$

or equivalently, $\mathbf{\Omega}=\mathbf{B_G}/c$, cf. Eq. (5.25), can be regarded as the sum of two terms: the *geodetic* (or de Sitter) precession, measured by the GPB (in addition to the Lense-Thirring one), which amounts to $3\mathbf{\Omega}/4$, plus the *Thomas precession*, which amounts to $\mathbf{\Omega}/4$. See e.g. Eqs. (40.33) of [73], or Eqs. (3.4.38) of [7]; see also Sec. IVB of Paper #4 [4] where a related issue is discussed.

Finally, a problem that was not addressed in this paper, and is usually overlooked in the literature concerning both the linearized and Post-Newtonian approaches, is the following: we said above that Eq. (5.37) yields the precession of a gyroscope with respect to a system of axes $\mathbf{e}^{\hat{i}}$ that is non-rotating with respect to an inertial frame at infinity; but how can one ensure that, as it amounts to comparing systems of vectors at different points in a curved spacetime? This is an highly non-trivial problem, which is studied in a exact approach in Secs. 3.1 and 3.3 of Paper #5. The conclusion is that one may determine the relative rotation of two tetrads at different points, exactly, only in spacetimes admitting shearfree observer congruences. In the problem at hand, the boosted frame indeed shears, but not to this degree of accuracy, as explained above; the traceless shear is neglected, only the expansion (which preserves angles) is manifest in the boosted metric (5.29).

5.3.3 Conclusion

In this work, and in view of the analogies from linearized theory reviewed in Sec. 3.1.1, whose limit of applicability is not always clear, we dissected under which specific conditions gravitational dynamics (for weak fields) becomes similar to electromagnetism, with a special emphasis to setups of recent and present experimental interest.

We have concluded that the actual physical similarities between gravity and electromagnetism (on which the physical content of such approaches relies) occur only on very special conditions. In the framework of linearized theory and Post-Newtonian approximations (and with the type of frames commonly used therein), this is a requirement of time-independence of the fields; the frame in which such independence is required depends on the type of effect. For tidal effects, like the forces on gyroscopes/dipoles, the similarity manifest in Eqs. (5.18) (and the analogy based on GEM fields therein) holds only when both the potentials (gravitational/electromagnetic) *and* their gradients are time-independent in the *test particle's* frame. In the example of analogous systems considered in Sec. 5.3.2, this means that the center of mass of the gyroscope/magnetic dipole must not move relative to the central body. In the case of the analogy between the equation for the geodesics and the Lorentz force law (see Fig. 5.1), as manifest in equation (5.19), it is in the *the observers'* (not the test particle!) frame, that the time independence of the *potentials*, is required. In the case of the tidal effects, as mentioned above, the restriction is not only on the potentials, but also in its gradients; here I would like to remark that this makes a difference. Consider this basic example, a test particle in circular motion around the Coulomb field of a point charge. In the inertial frame momentarily comoving (MCRF) with the particle, the potential ϕ is constant; but *not* the electric field $\mathbf{E}$ (it is constant in magnitude, but time-varying in direction), and due to that the electromagnetic tidal tensors can no longer be similar to the gravitational ones (for instance of the analogous situation, a particle in circular motion around a Schwarzschild black hole). In this framework this can be understood as follows: take the magnetic tidal tensor $B_{\alpha\beta}$; as discussed in Sec. 5.2, it is a covariant derivative of the magnetic field as measured in the MCRF: $B_{\alpha\beta} = B_{\alpha;\beta}|_{U=const} = (B_{\text{MCRF}})_{\alpha;\beta}$. Since in this frame the electric field is time-varying, by virtue of Maxwell equation $\nabla \times \vec{B} = \partial \vec{E}/\partial t$, this means that $\vec{B}$ has a curl, thus $B_{[\alpha\beta]} \neq 0$. By contrast with the gravitational analogue, where $\mathbb{H}_{[\alpha\beta]} = 0$. This can be stated in this way: the symmetries of the electromagnetic tidal tensors differ from the gravitational ones when $F_{\alpha\beta;\gamma}U^{\gamma} \neq 0$, i.e., when the electromagnetic field is not covariantly constant along the observer's (in this case the particle's) worldline; this is what Eqs. (5.1.6a), (5.1.4a) of Table 5.1 tell us. And indeed for circular motions around a coulomb charge, $F_{\alpha\beta;\gamma}U^{\gamma} \neq 0$, despite $d\phi/d\tau = \phi_{;\alpha}U^{\alpha} = 0$.

In Sec. 7 of Paper #5 [5] the analysis herein is refined and generalized for the exact case.

Finally, as a consequence of this analysis, a distinction, from the point of view of the analogy with electrodynamics, between effects related to (stationary) rotational mass currents, and those arising from translational mass currents, becomes clear: albeit in the literature both are dubbed “gravitomagnetism”, one must note that, while the former are clearly analogous to magnetism, in the case of the latter the analogy is not so close.

5.4 Paper #3 — Mathisson’s helical motions for a spinning particle: Are they unphysical?

Both the analogy between the electromagnetic force on a magnetic dipole and the gravitational force on a gyroscope (a spinning pole-dipole particle), introduced in Paper #1 (and put on firm ground in Paper #4), and the analogy between the precession of a magnetic dipole in a electromagnetic field and the “precession” of a gyroscope, known from the approaches based on exact GEM inertial fields of Sec. 3.1.2 (see also papers #4 and #5), require the so-called Mathisson-Pirani spin condition to hold. The very notions of rotation and compass of inertia in relativity, and the physical meaning of the Fermi-Walker transport law, rely also on it. It turns out that the problem of the spin supplementary condition is an old one, and not well understood even today. This is even more so in the case of Mathisson-Pirani condition, due to its degeneracy and the exotic solutions it allows. In particular the famous helical motions for a free particle in flat spacetime (where the particle’s center of mass accelerates without the action of any force), which were regarded with a lot of skepticism, and deemed unphysical, due to the belief that the radius of the helices could be arbitrarily large [76, 75, 60, 77].

In this work, which initially started out as an Appendix of Paper #4 [4], we clarify these issues, explain the helical motions, and show that there is nothing wrong or unphysical with the Mathisson-Pirani condition.

5.4.1 Equations of motion for free spinning particles in flat spacetime. Mathisson’s helical motions.

As explained in Sec. 3.2, in a multipole expansion, a body is represented by a set of moments of $T^{\alpha\beta}$, called “inertial” or “gravitational” moments (forming the so called [61] “gravitational skeleton”), and the moments of j^{α} (the electromagnetic skeleton). In this work we are interested in free particles, so only the former contribute to the equations of motion. The moments are taken about a reference worldline $z^{\alpha}(\tau)$, which could in principle be arbitrary, but will be chosen below as a suitably defined center of mass. The case of pole-dipole particles corresponds to truncating the expansion at dipole order. In this case the equations of motion involve only two moments of $T^{\alpha\beta}$, the momentum P^{α}, and the angular momentum $S^{\alpha\beta}$ defined as (see e.g. [97, 60, 59]):

$$P^{\alpha} \equiv \int_{\Sigma(\tau,u)} T^{\alpha\beta} d\Sigma_{\beta} \ , \tag{5.38}$$

$$S^{\alpha\beta} \equiv 2\int_{\Sigma(\tau,u)} r^{[\alpha}T^{\beta]\gamma} d\Sigma_{\gamma} \ . \tag{5.39}$$

Here $P^\alpha(\tau)$ is the 4-momentum of the body; $S^{\alpha\beta}(\tau)$ is the angular momentum about a point $z^\alpha(\tau)$ of the reference worldline; $\Sigma(\tau,u) \equiv \Sigma(z(\tau),u)$ is the spacelike hypersurface generated by all geodesics orthogonal to some time-like vector u^α at the point $z^\alpha(\tau)$; $r^\alpha \equiv x^\alpha - z^\alpha(\tau)$, where $\{x^\alpha\}$ is a chart on spacetime; $d\Sigma_\gamma \equiv -u_\gamma d\Sigma$, and $d\Sigma$ is the 3-volume element on $\Sigma(\tau,u)$.

In the case of *free* particles in flat spacetime (i.e., without any further fields), the equations of motion that follow from the conservation law $T^{\alpha\beta}_{\ \ ;\beta} = 0$ are [76, 79, 60, 59, 81, 97]:

$$\frac{DP^\alpha}{d\tau} = 0 \quad (a), \qquad \frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha}U^{\beta]} \quad (b). \tag{5.40}$$

Contracting (5.40b) with U^α we obtain an expression for the momentum:

$$P^\alpha = mU^\alpha - \frac{DS^{\alpha\beta}}{d\tau}U_\beta \ , \tag{5.41}$$

where $m \equiv -P^\alpha U_\alpha$. Eqs. (5.40) form an indeterminate system. Indeed, there are 13 unknowns (P^α, 3 independent components of U^α, and 6 independent components of $S^{\alpha\beta}$) for only 10 equations. This is where the spin condition comes into play. A supplementary spin condition of the type $S^{\alpha\beta}u_\beta = 0$, for some unit timelike vector $u^\alpha(\tau)$, effectively kills off 3 components of the angular momentum, thereby closing the system. Such condition has the role of specifying the representative point of the body (i.e., the worldline of reference relative to which $S^{\alpha\beta}$ is taken); as I shall show below, it demands it to be the center of mass as measured in the rest frame of the observer of velocity u^α.

In this way U^α is the center of mass 4-velocity and m denotes the *proper mass*, i.e., the energy of the body as measured in the center of mass frame. Note from Eq. (5.41) that the momentum P^α is not, in general, parallel to U^α; the spinning particle is said to possess "hidden momentum" [66, 4], which plays a key role in this discussion, as explained in Sec. 5.4.5 below.

Mathisson's helical solutions [74] arise when one uses the condition $S^{\alpha\beta}U_\alpha = 0$. In this case $S^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}S_\mu U_\nu$; (5.40c) becomes $P^\alpha = mU^\alpha + S^{\alpha\beta}a_\beta$, where $a^\alpha = DU^\alpha/d\tau$; and $DS^\alpha/d\tau = 0$. The solution of (5.40) under this condition turns out to be degenerate; it describes the famous helical motions, which, *in the* $P^i = 0$ *frame*, correspond to *clockwise* (i.e. *opposite* to the spin direction) circular motions with radius

$$R = \frac{v\gamma^2 S}{m} \tag{5.42}$$

and speed v on the xy plane. Taking their center as the spatial origin of the frame, they read:

$$z^\alpha(\tau) = \left(\gamma\tau, -R\cos\left(\frac{v\gamma}{R}\tau\right), R\sin\left(\frac{v\gamma}{R}\tau\right), 0\right) \tag{5.43}$$

These motions were interpreted in [74] (for the case of an electron) as the classical counterpart of the Dirac equation 'zitterbewegung'. However, the fact that γ can be arbitrarily large has led some authors (see e.g. [76, 77]) to believe that, according to (5.43), a given free body might move along circular trajectories with any radius; for this reason these

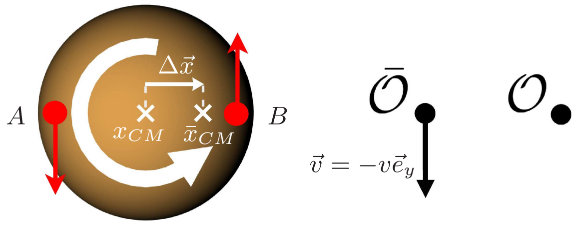


Figure 5.2: Center of mass of a spinning particle ($\vec{S} = S\vec{e}_z$, orthogonal to the page) as evaluated by two different observers. Observer $\mathcal{O}$ of 4-velocity $u^\alpha = P^\alpha/M$ is at rest with respect to center of mass $x^i_{CM} \equiv x^i_{CM}(u)$ it measures (i.e., x^i_{CM} is a proper center of mass). Observer $\bar{\mathcal{O}}$, moving with velocity $\vec{v} = -v\vec{e}_y$ relative to $\mathcal{O}$, sees the points on the right hemisphere (e.g. point B) moving faster than the points in the left hemisphere (e.g. point A), and, therefore, for $\bar{\mathcal{O}}$, the right hemisphere will be more massive than the left one. This means that the center of mass $\bar{x}^i_{CM} \equiv x^i_{CM}(\bar{u})$ as evaluated in the moving frame of $\bar{\mathcal{O}}$ is shifted to the right (relative to x^i_{CM}). The shift is exactly $\Delta\vec{x} = \vec{S}_\star \times \vec{v}/M$.

solutions have been deemed *unphysical*. The same arguments were used to imply that the the frequency of these motions, given by

$$\omega = \frac{m}{\gamma^2 S} \ , \tag{5.44}$$

only coincides, for the case of an electron, with Dirac's zitterbewegung frequency $\omega = 2M_e/\hbar$, in the limit $\gamma \to 1$. Both these assessments are misconceptions as shown below.

5.4.2 Center of mass. Significance of the spin condition.

In order for (5.40) to be equations of motion for the body, $z^\alpha(\tau)$ must be taken as its representative point. The natural choice for such point would be the body's center of mass (CM); however, in relativity, the CM of a spinning body is an observer dependent point. This is illustrated in Fig. 5.2. As mentioned above, a spin condition of the type $S^{\alpha\beta}u_\beta = 0$ (for some unit time-like vector u^α) amounts to choosing $z^\alpha(\tau)$ as the center of mass $x^\alpha_{\rm CM}(u)$ measured by the observer $\mathcal{O}(u)$ of 4-velocity u^α. This is easily seen in the rest frame of $\mathcal{O}(u)$ (the $u^i = 0$ frame). In such frame $S^{\alpha\beta}u_\beta = S^{\alpha 0}u_0$; thus, from Eq. (5.39):

$$S^{i0} = 2\int_{\Sigma(\tau,u)} r^{[i}T^{0]\gamma}d\Sigma_\gamma = \int x^i T^{00} d^3x - m(u) z^i \ , \tag{5.45}$$

where $m(u) \equiv -P^\alpha u_\alpha$ denotes the mass as measured in the frame $\mathcal{O}$. The first term of (5.45) is by definition $m(u)x^i_{CM}(u)$, where $x^i_{CM}(u)$ are the coordinates of the center of mass as measured by $\mathcal{O}$, and so

$$x^i_{CM}(u) - z^i = \frac{S^{i0}}{m(u)} \Leftrightarrow x^\alpha_{CM}(u) - z^\alpha = -\frac{S^{\alpha\beta}u_\beta}{m(u)} \ . \tag{5.46}$$

Figure 5.3: Kinematical explanation of the helical motions allowed by $S^{\alpha\beta}U_\beta = 0$: every point within a disk of radius $S_\star/M$, centered at $x^\alpha_{\rm CM}(P)$, is a *centroid* corresponding to some observer; and it is also a *proper* center of mass *if it rotates with angular velocity* $\omega = M/S_\star$ *in the opposite sense of the spinning body* (solid red lines).

Thus the condition $S^{\alpha\beta}u_\beta = 0$ is precisely the condition $x^\alpha_{CM}(u) = z^\alpha$, i.e., that the reference worldline is the center of mass as measured in this frame. In order to see how the center of mass changes with the observer, take $u^\alpha = P^\alpha/M$, i.e., the reference worldline is the center of mass as measured in the zero 3-momentum frame, that we denote by $x^\alpha_{\rm CM}(P)$. And let $S^{\alpha\beta}_\star$ denote the angular momentum with respect to $x^\alpha_{CM}(P)$. Consider now another observer $\bar{\mathcal{O}}$ moving relative to $\mathcal{O}$ with 4-velocity $\bar{u}^\alpha = \bar{u}^0(1,\vec{v})$; for this observer the center of mass will be at a different position, as depicted in Fig. 5.2. It is displaced by a vector $\Delta x^\alpha = -S^{\alpha\beta}_\star\bar{u}_\beta/m(\bar{u})$ relative to the reference worldline z^α, where $m(\bar{u}) \equiv -P^\gamma\bar{u}_\gamma$ denotes the mass of the particle as measured by $\bar{\mathcal{O}}$. That is,

$$\Delta x^i = \frac{(\vec{S}_\star \times \vec{v})^i}{M} , \tag{5.47}$$

with $M \equiv \sqrt{-P^\alpha P_\alpha}$. Hence the set of all possible CM's measured by all observers $\mathcal{O}(\bar{u})$ fills a disk of radius $R_{max} = S_*/M$ centered at $x^\alpha_{\rm CM}(P)$. This is the minimum size a particle can have without violating the dominant energy condition (i.e., without possessing matter/energy flowing faster than light). The latter implies $\rho \geq |\vec{J}|$, where $\rho \equiv T^{00}$ and $J^i \equiv T^{0i}$; let b be the largest dimension of the body. Using the definition of $S^{\alpha\beta}_\star$ in [3], we may write, in the $P^i = 0$ frame,

$$S_\star = \left|\int \vec{r}\times\vec{J}d^3x\right| \leq \int r|\vec{J}|d^3x \leq \int \rho r d^3x \leq Mb \;\Leftrightarrow\; b \geq \frac{S_\star}{M} = R_{max} \, . \tag{5.48}$$

Thus the disk of CM's, within which all the helical motions are contained, is always smaller than the body.

5.4.3 Kinematical interpretation of the helical motions

The Mathisson-Pirani condition $S^{\alpha\beta}U_\alpha = 0$ amounts to choosing for z^α the center of mass $x^\alpha_{\rm CM}(U)$ as measured in its own rest frame, i.e., the frame $U^i = 0$. Such CM is dubbed a "proper center of mass". It turns out that, contrary to what one might expect, such point is not unique, as we shall now show. The CM measured in the $P^i = 0$ frame ($x_{\rm CM}$ in Fig. 5.2) is one of the proper CM's, as it is clearly at rest in this inertial frame, and thus has uniform straightline motion in any other inertial frame. It corresponds to the solution $R = 0$ in Eq. (5.43). Let again $\bar{\mathcal{O}}$ be an observer moving relative to the $P^i = 0$ frame with 3-velocity $\vec{v}$. The 4-velocity of $\bar{\mathcal{O}}$ is $\bar{u}^\alpha = \gamma(u^\alpha + v^\alpha)$, where $\gamma \equiv -u_\alpha \bar{u}^\alpha$ and v^α is the relative velocity vector which is spatial with respect to $u^\alpha = P^\alpha/M$. As we have seen above, the center of mass measured by $\bar{\mathcal{O}}$ is at a new point $x^i_{\rm CM}(\bar{u}) \equiv \bar{x}^i_{\rm CM}$, shifted by a vector Δx^i relative to $x^i_{\rm CM}(P)$, cf. Fig. 5.2 and Eq. (5.47). But this new point in principle is *not* at rest relative to the observer $\bar{\mathcal{O}}$ measuring it, i.e., it is *not* a proper CM. In order to see that, we take as reference worldline $z^\alpha(\tau_P) = x^\alpha_{\rm CM}(P)$, and compute the evolution of the shift vector along it:

$$\frac{D\Delta x^\alpha}{d\tau_P} = -\frac{S_\star^{\alpha\beta}}{M}\frac{Dv_\beta}{d\tau_P} \ \ (a) \quad \Leftrightarrow \quad \frac{d\vec{\Delta x}}{dt} = \frac{1}{M}\vec{S}_\star \times \frac{d\vec{v}}{dt} \ \ (b). \tag{5.49}$$

The second equation holds in the rest frame of $\mathcal{O}$ (the frame $u^i = 0 = P^i$), where the time coordinate is $t = \tau_P$. If $\bar{\mathcal{O}}$ is inertial, $Dv^\alpha/d\tau = 0$, and $x^\alpha_{\rm CM}(\bar{u})$ is a point at rest relative to $x^\alpha_{\rm CM}(P)$, thus *not* at rest relative to $\bar{\mathcal{O}}$ (it moves relative to it at speed $-\vec{v}$). The set of CM's measured by all the possible *inertial* observers forms a disk of points *all at rest* with respect to each other and (again, for a *free particle in flat spacetime*) with respect to $x^\alpha_{CM}(P)$, around which the disk is centered.

But if $\vec{v}$ is not constant, then the shift Δx^α varies accordingly, and the centroid $x^\alpha_{\rm CM}(\bar{u})$ acquires a non-trivial velocity $\vec{v}_{\rm CM} = d\vec{\Delta x}/dt$ (as measured in the $P^i = 0$ frame). If $\bar{\mathcal{O}}$ itself moves with $\vec{v} = \vec{v}_{\rm CM}$, i.e., if the observer velocity $\vec{v}$ is a solution of the equation

$$\vec{v} = \frac{1}{M}\vec{S}_\star \times \frac{d\vec{v}}{dt}\ , \tag{5.50}$$

then $x^\alpha_{\rm CM}(\bar{u})$ it is at rest relative to $\bar{\mathcal{O}}$ (it is a proper center of mass). The solutions (in the $P^i = 0$ frame) are circular motions in the plane orthogonal to $\vec{S}_\star$, with radius

$$R = \Delta x = \frac{|\vec{v} \times \vec{S}_\star|}{M} = \frac{vS_\star}{M} \tag{5.51}$$

and constant (independent of R) angular velocity

$$\omega = -\frac{M}{S_\star} \tag{5.52}$$

in the *opposite sense* to the rotation of the body, as illustrated in Fig. 5.3. This is origin of the helical motions, as argued in [78]; our analysis in this section is equivalent to the one therein (only stated and derived in a different form). Indeed the solutions of (5.50) are precisely (5.43), and the expressions for the radius and angular velocity (5.51)-(5.52) are equivalent to (5.42), (5.44), as shown below. Hence the radius of the helical motions is not arbitrarily large; they are contained within the disk of CM's, of radius $R_{max} = S_\star/M$, which is always smaller than the body as shown by inequality (5.48) above.

5.4.4 The misconception in the literature

So what is then the origin of the apparent paradox? It all comes down to a misunderstanding of what it means to consider different solutions corresponding to the same particle; in particular which are the parameters that must be fixed.

Different representations of the same extended body must yield the same moments (P^α and $S^{\alpha\beta}$) with respect to the *same observer* and the *same reference worldline*. When one changes from one helical representation to the other, one is changing the point about which the angular momentum is taken; and also changing the CM velocity U^α, which means that the proper mass $m = -P^\alpha U_\alpha$ will also be different. Let U^α and $\bar{U}^\alpha$ denote the 4-velocity vectors of two different helical representations. Clearly m must be different from $\bar{m} \equiv -P^\alpha U_\alpha$. Also the tensor $S^{\alpha\beta}$, obeying $S^{\alpha\beta}U_\beta = 0$, must be, in general, different from the tensor $\bar{S}^{\alpha\beta}$, obeying $\bar{S}^{\alpha\beta}\bar{U}_\beta = 0$, *if* $S^{\alpha\beta}$ *and* $\bar{S}^{\alpha\beta}$ *are to represent the same body*, since the former is the angular momentum about the point $x^\alpha_{CM}(U)$, and the latter about the point $x^\alpha_{CM}(\bar{U})$. As we show in detail in Sec. IV of Paper #3 [3], the magnitude S of the spin vector of any helical solution obeys

$$S = \frac{S_\star}{\gamma} \ .$$

Thus, it is the quantities $S_\star = \gamma S$ and $M = m/\gamma$, not m and S, that we must fix in order to ensure that we are dealing with the same particle. Therefore, $R = v\gamma^2 S/m = vS_\star/M \leq R_{max}$, for all the helical representations corresponding to a given particle. Moreover, the frequency $\omega = m/\gamma^2 S = M/S_\star$ is the same for all helices corresponding to the same particle, and coincides exactly (even in the relativistic limit) with Dirac's zitterbewegung frequency, identifying $S_\star = \hbar/2$ and $M = M_e$.

5.4.5 Dynamical Interpretation of the Helical Motions

We see from Eq. (5.49) that the CM $x^\alpha_{\rm CM}(u)$ is not at rest in the $\vec{P} = 0$ frame when the 4-velocity u^α of the observer measuring it changes; conversely, $\vec{P}$ will not be zero in the CM frame (where, by definition, the particle is at rest); thus P^α is not parallel to U^α, and the particle is said to possess hidden momentum [66]. This is a key concept for the understanding of the dynamics of the helical solutions; namely how the CM of a spinning particle can accelerate in the absence of any force without violating the conservation laws. Consider a generic spin condition $S^{\alpha\beta}u_\beta = 0$; contracting (5.40b) with u_β, leads to

$$S^{\alpha\beta}\frac{Du_\beta}{d\tau} = \gamma(u,U)P^\alpha - m(u)U^\alpha; \tag{5.53}$$

where $\gamma(u,U) \equiv -U^\beta u_\beta$ and $m(u) \equiv -P^\beta u_\beta$. We then split the momentum P^α in two parts:

$$P^\alpha = P_{\rm kin} + P^\alpha_{\rm hid}; \qquad P^\alpha_{\rm kin} = mU^\alpha; \qquad P^\alpha_{\rm hid} = \frac{1}{\gamma(u,U)}(h^U)^\alpha{}_\sigma S^{\sigma\beta}\frac{Du_\beta}{d\tau} \tag{5.54}$$

The "kinetic momentum" is the projection of P^α along U^α; and the projection orthogonal to U^α, $P^\alpha_{\rm hid} \equiv (h^U)^\alpha{}_\beta P^\beta$, is the hidden momentum. Hence, if $Du_\beta/d\tau = 0$, that is, if we take as $z^\alpha(\tau)$ the CM measured by an observer $\mathcal{O}(u)$ *such that* u^α *is parallel transported along it* (e.g., an inertial observer in flat spacetime), then $P^\alpha \parallel U^\alpha$, and $P^\alpha_{\rm hid} = 0$. Otherwise, $P^\alpha_{\rm hid} \neq 0$ in general. This is the reciprocal of Eq. (5.49b); one can obtain one effect from the other, see [3] for details.

Notice the important message encoded herein: in relativity, the motion of a spinning particle is not determined by the force laws given the initial position and velocity; one needs also to determine the field of vectors u^α relative to which the CM is computed; the variation of u^α along $z^\alpha(\tau)$ is enough to possibly cause the CM to accelerate, even in the absence of any force. In this case the variation of $P^\alpha_{\rm kin}$ is compensated by an opposite variation of $P^\alpha_{\rm hid}$, keeping P^α constant. If u^α varies in a way such that the signal in Eq. (5.49b) oscillates, we may have a bobbing; or if it is such that $\mathcal{O}(u)$ sees its CM to be at rest ($u^\alpha = U^\alpha$, i.e, its 3-velocity $\vec{v}$, in the frame $P^i = 0$, is a solution of $\vec{v} = \vec{v}_{\rm CM}$, Eq. (5.49c)), so that the condition $S^{\alpha\beta}u_\beta = S^{\alpha\beta}U_\beta = 0$ is obeyed, then we have a helical solution. In this special case the hidden momentum takes the form

$$P^\alpha_{hid} = S^{\alpha\beta}a_\beta = \epsilon^\alpha{}_{\beta\gamma\delta}a^\beta S^\gamma U^\delta \ ,$$

which in vector notation reads $\vec{P}_{hid} = -\vec{S} \times_U \vec{a} = \vec{S} \times_U \vec{G}$, where $\vec{G}$ is the "gravitoelectric field" as measured in the CM frame (see Paper #4 [4]). This is formally analogous to the hidden momentum $P^\alpha_{\rm hid} = \epsilon^\alpha{}_{\beta\gamma\delta}E^\gamma\mu^\beta U^\delta$ of electromagnetic systems; in vector notation $\vec{P}_{\rm hid} = \vec{\mu} \times_U \vec{E}$, see [4]. The dynamics of the helical representations may actually be cast as analogous to the bobbing [66] of a magnetic dipole orbiting a cylindrical charge, as explained in Fig. 5.4.

5.4.6 Conclusion

In this work we studied the problem of the spin supplementary condition and the center of mass definition in relativity, and in special detail the Mathisson-Pirani condition, and its famous helical solutions. We concluded that there is nothing wrong with this spin condition, it is as valid as the Tulczyjew-Dixon [62, 60, 81], the Papapetrou-Corinaldesi [89], or any other physically reasonable condition (there are infinite possibilities), the choice between them being just a matter of convenience. Actually, in some applications, the Mathisson-Pirani condition is the most suitable one, many applications in Paper #4 [4] are examples of that; this is discussed in detail in Appendix C therein. It is degenerate, and the helical solutions it allows for a free particle (in addition to the expected uniform straightline motion), are just *alternative* and *physically consistent* descriptions of the motion (only more complicated). The claims in the literature that these solutions are unphysical because the radius of the helices is arbitrary large were shown to arise from a subtle (but crucial) mistake in earlier derivations. Indeed the radius is finite and always contained within the disk of centroids (i.e., the disk formed by all possible positions of the centers of mass as measured by the different observers), which is typically much smaller than the body, and even in the case that the body rotates with relativistic velocity, it is always contained within its convex hull.

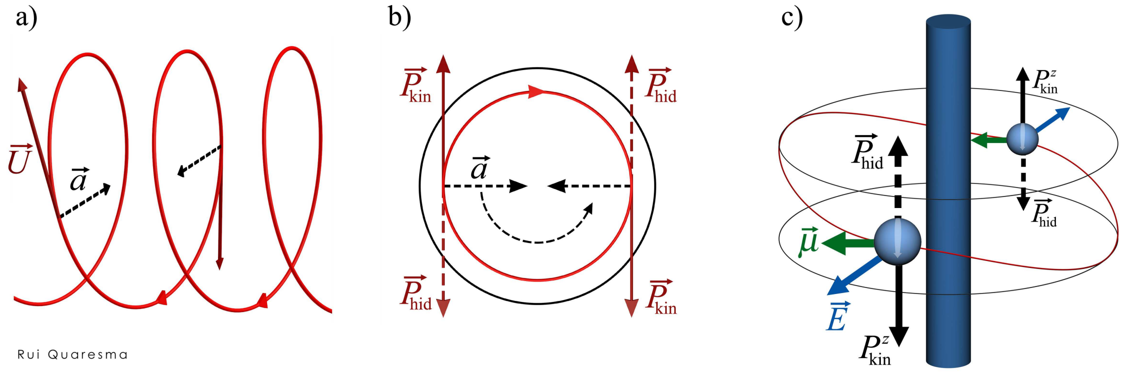


Figure 5.4: Hidden momentum provides dynamical interpretation for the helical motions: the acceleration results from an interchange between kinetic $P^\alpha_{kin} = mU^\alpha$ and hidden "inertial" momentum $P^\alpha_{hid} = S^{\alpha\beta}a_\beta$, which occurs in a way that their variations cancel out at every instant, keeping P^α constant. This is made manifest in b) panel, representing the $\vec{P} = 0$ frame, wherein $\vec{P}_{hid} = \vec{a} \times_U \vec{S} = -m\vec{U} = -\vec{P}_{kin}$. Panel c) represents an electromagnetic analogue [66]: a (negatively) charged test particle possessing magnetic dipole moment $\vec{\mu} = (\mu^x, \mu^y, 0)$, orbiting a cylindrical (positively) charged body. The cylinder is along the z axis, and $\vec{E}$ is the electric field it produces (measured in the particle's CM frame). The z component of the force vanishes for this setup; hence $P^z = 0 = constant$. But the particle possesses a hidden momentum [66, 4] $\vec{P}_{hid} = \vec{\mu} \times_U \vec{E}$; as it orbits the line charge, $\vec{P}_{hid}$ oscillates between positive and negative values along the z-axis, implying the particle to bob up and down in order to keep the total momentum along z constant: $P^z = P^z_{kin} + P^z_{hid} = 0$. (Note however the important distinction: $\vec{a} \times_U \vec{S}$, but *not* $\vec{\mu} \times_U \vec{E}$, is pure gauge).

Interestingly, the dynamical consistency of the helical motions (as well as other exotic motions which are allowed by the infinite possible spin conditions, and where the center of mass accelerates without any force) can be explained through the same concept of "hidden momentum" that was recently discussed in [66], and argued therein to play a key role in the bobbings observed in numerical simulations of binary systems. This hidden momentum may be cast as analogous to the hidden momentum of electromagnetic systems (another not well understood feature of relativistic electrodynamics, despite its discovery dating back from the 60's [90]), that we discuss in detail in Paper #4 (see also in this respect [91, 66]).

Regarding the suggestions that Mathisson's helical motions (for an electron) are the classical limit of the zitterbewegung of Dirac's equation, originally put forth by Mathisson [74] (and later supported by other authors [82, 83, 79, 84, 85, 98]), they have been rejected by other authors [75, 76, 60, 77], based on two arguments: one is again the supposedly arbitrarily large radius of the helical motions (which would be contradicted by experimental evidence), the other is that the frequency of the helical motions only coincides with the zitterbewegung frequency of Dirac's equation in the non-relativistic limit. As a byproduct of our analysis, both turn out to be misconceptions, the second argument originating precisely from the same mistake that led to the former. Indeed, identifying the appropriate parameters, all the classical helical representations for the free electron have the frequency $\omega = 2M_e/\hbar$, which is precisely Dirac's frequency. With the no-go arguments against the correspondence now put aside, the way is now open for a demonstration of the existence (or not) of a deep connection between the two phenomena.

Finally, the outcome of this paper, and the validation of the Mathisson-Pirani condition, is of special importance in the context of this work, since, as mentioned at the start of this section, part of the gravito-electromagnetic analogies, in order to be exact, rely on this condition. That is the case of the analogies for both the force and the spin evolution of a spinning particle, studied in the other four papers in this compilation, and also of a third one — the *exact* analogy relating the "electromagnetic" hidden momentum and the "inertial" hidden momentum of a spinning particle — which is unveiled in Paper #4 [4].

5.5 Paper #4 — Spacetime dynamics of spinning particles – exact gravito-electromagnetic analogies

In this paper we studied several aspects of the motion of spinning multipole test particles in electromagnetic and gravitational fields, in the framework of exact gravito-electromagnetic analogies. This is the first major application of the tidal tensor formalism developed in Paper #1 [1], as the forces on the test particles, and the dynamical implications of the symmetries and time-projections of the tidal tensors (key distinctive features between electromagnetism and gravity found in [1]) are discussed in depth. The main point of this work is that there is a lot to be learned (about *both* of them) from a comparative study of the gravitational and electromagnetic interactions.

Below I will (very) briefly review the main results in this paper; I refer the reader for the introduction and conclusion of the paper for a comprehensive account of the results and issues addressed; and I also would like to draw the reader's attention to the boxes at the end of Secs. III, IV, V and IV therein, summarizing the main results and conclusions therein.

5.5.1 Equations of motion and the exact analogies

In most of this work we deal with the dynamics of the so-called pole-dipole spinning test particles. We consider systems composed of a test body plus background gravitational and electromagnetic fields. Let $(T_{\rm tot})^{\alpha\beta} = \Theta^{\alpha\beta} + (T_{matter})^{\alpha\beta}$ denote the total energy momentum tensor, which splits into the electromagnetic stress-energy tensor $\Theta^{\alpha\beta}$ and the energy-momentum of the matter $(T_{matter})^{\alpha\beta}$. Moreover, let $T^{\alpha\beta}$ and j^α denote, respectively, the energy momentum tensor and the current density 4-vector *of the test body.* We also consider that the only matter and currents present are the ones arising from the test body: $(T_{matter})^{\alpha\beta} = T^{\alpha\beta}$, $(j_{\rm tot})^\alpha = j^\alpha$. In this case the conservation of total energy-momentum tensor yields:

$$(T_{\rm tot})^{\alpha\beta}{}_{;\beta} = 0 \;\Rightarrow\; T^{\alpha\beta}{}_{;\beta} = -\Theta^{\alpha\beta}{}_{;\beta} \;\Leftrightarrow\; T^{\alpha\beta}{}_{;\beta} = F^{\alpha\beta} j_\beta \ , \tag{5.55}$$

where $F^{\alpha\beta}$ is the Maxwell tensor of the *external* (background) electromagnetic field.

As discussed in Sec. 3.2, in a multipole expansion the body is represented by the moments of j^α, and a set of moments of $T^{\alpha\beta}$. Truncating the expansion at dipole order, the equations of motion for such a particle involve only two moments of $T^{\alpha\beta}$, which are P^α and $S^{\alpha\beta}$, Eqs. (5.38)-(5.39) above, and the electromagnetic moments:

$$q \equiv \int_\Sigma j^\alpha d\Sigma_\alpha \ , \tag{5.56}$$

$$d^\alpha \equiv \int_{\Sigma(\tau,U)} r^\alpha j^\sigma d\Sigma_\sigma \ , \tag{5.57}$$

$$\mu^\alpha \equiv \frac{1}{2}\epsilon^\alpha{}_{\beta\gamma\delta}U^\delta \int_{\Sigma(\tau,U)} r^\beta j^\gamma U^\sigma d\Sigma_\sigma \ . \tag{5.58}$$

As already discussed in Secs. 3.2 and 5.4, the moments are taken with respect to a reference worldline $z^\alpha(\tau)$, of proper time τ and (unit) tangent vector $U^\alpha \equiv dz^\alpha/d\tau$, and a hypersurface of integration $\Sigma(\tau, u)$. Following [59] we take $u^\alpha = U^\alpha$; $d\Sigma_\gamma \equiv -U_\gamma d\Sigma$, where $d\Sigma$ is the 3-volume element on $\Sigma(\tau, U)$; q denotes the total charge, which is an invariant (making Σ in this case arbitrary), $d^\alpha(\tau)$ and $\mu^\alpha(\tau)$ are, respectively, the *intrinsic* electric and magnetic dipole moments about the point $z^\alpha(\tau)$ of the reference worldline. It is useful to introduce also the magnetic dipole 2-form $\mu_{\alpha\beta}$ by

$$\mu_{\alpha\beta} \equiv \epsilon_{\alpha\beta\gamma\delta}\mu^\gamma U^\delta; \qquad \mu^\alpha = \frac{1}{2}\epsilon^\alpha{}_{\beta\gamma\delta}U^\beta \mu^{\gamma\delta} \ . \tag{5.59}$$

The motion of the test particle is described by the reference worldline $z^\alpha(\tau)$, which is prescribed by the spin supplementary condition (as explained in Sec. 5.4.2 above). In most of Paper #4 we choose the Mathisson-Pirani spin condition $S^{\alpha\beta}U_\beta = 0$, under which the *exact* gravito-electromagnetic analogies studied in this work arise. The rigorous equations of motion, that follow from the conservation equation (5.55) and the charge conservation $j^{\alpha}_{;\alpha} = 0$, have been derived in a number of independent treatments [60, 59, 81, 106, 64, 66]; the relationship between them, as well as the physical interpretation of the terms involved is an important clarification made in this work; it is discussed detail in Appendixes A (see also B) of Paper #4 [4]. Writing them in terms of the physical momentum and angular momentum, given by definitions above, they read, *in tidal tensor form*:

$$\frac{DP^{\alpha}}{d\tau} = qE^{\alpha} + B^{\beta\alpha}\mu_{\beta} - \mathbb{H}^{\beta\alpha}S_{\beta} + E^{\alpha\beta}d_{\beta} + F^{\alpha}_{\ \beta}\frac{Dd^{\beta}}{d\tau} \ ; \tag{5.60}$$

$$\frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha}U^{\beta]} + 2\mu^{\theta[\beta}F^{\alpha]}_{\ \ \theta} + 2d^{[\alpha}F^{\beta]}_{\ \ \gamma}U^{\gamma} \ , \tag{5.61}$$

where $F^{\alpha\beta}$ is the background Maxwell tensor. The first term in (5.60) is the Lorentz force; the second and third terms terms are the forces discussed in Paper #1 [1]: the force $B^{\beta\alpha}\mu_{\beta} \equiv F^{\alpha}_{\rm EM}$ due to the tidal coupling of the electromagnetic field to the magnetic dipole moment, and the third, $-\mathbb{H}^{\beta\alpha}S_{\beta} \equiv F^{\alpha}_{G}$, is the Mathisson-Papapetrou spin-curvature force. The last two terms are the force exerted on the electric dipole, consisting of a tidal term $E^{\alpha\beta}d_{\beta}$ governed by the electric tidal tensor, and of a *non tidal* term $F^{\alpha}_{\ \beta}Dd^{\beta}/d\tau$.

Analogy based on tidal tensors.— The force equation (5.60) manifests the physical analogy

$$B_{\alpha\beta} \longleftrightarrow \mathbb{H}_{\alpha\beta}$$

we found in Paper #1 (now being extracted from the rigorous, fully covariant equations of motion): both the electromagnetic force on a magnetic dipole and the gravitational force on a gyroscope are determined by a contraction of the spin/magnetic dipole 4-vector with a magnetic type tidal tensor. $B_{\alpha\beta}$ is a covariant derivative, keeping U^{α} fixed (covariantly constant), of the magnetic field $B^{\alpha} = \star F^{\alpha}_{\ \beta}U^{\beta}$ measured by the test particle: $B_{\alpha\beta} = B_{\alpha;\beta}|_{U=const.}$; i.e., it is a derivative of B^{α} as measured in the *inertial* frame *momentarily* comoving with the particle.

Precession analogy based on GEM fields.— Another exact analogy arises from the spin evolution equation (5.61). Take now, for simplicity, *purely magnetic dipoles* (i.e., $d^{\alpha} = 0$); if the Mathisson-Pirani condition $S^{\alpha\beta}U_{\beta} = 0$ holds, Eq. (5.61) can be written as

$$\frac{D_F S_{\mu}}{d\tau} = \epsilon_{\mu\alpha\beta\nu}U^{\nu}\mu^{\alpha}B^{\beta}, \tag{5.62}$$

where B^{α} is the magnetic field *as measured by the test particle,* $B^{\alpha} = \star F^{\alpha\beta}U_{\beta}$, and $D_F/d\tau$ denotes the Fermi-Walker covariant derivative (see e.g. [73, 7, 63] for an explanation of this derivative). This is the relativistic generalization of the the familiar textbook torque $\tau = \vec{\mu} \times \vec{B}$. Consider now an orthonormal frame $e_{\hat{\alpha}}$ comoving with the test particle, i.e.

$\mathbf{U} = \mathbf{e}_{\hat{0}}$. In such frame, $S^{\hat{0}} = 0$ and $U^{\hat{\alpha}} = \delta^{\hat{\alpha}}_{\hat{0}}$, and equation (5.62) reduces to:

$$\frac{DS^{\hat{i}}}{d\tau} = (\vec{\mu} \times \vec{B})^{\hat{i}} \;\Leftrightarrow\; \frac{dS^{\hat{i}}}{d\tau} = \left(\vec{S} \times \vec{\Omega} + \vec{\mu} \times \vec{B}\right)^{\hat{i}} \tag{5.63}$$

where $\vec{\Omega}$ is angular velocity of rotation of the spatial axes $\mathbf{e}_{\hat{i}}$ relative to the tetrad Fermi-Walker transported along the particle's CM worldline (may be interpreted as their rotation relative to a system of local comoving guiding gyroscopes, defining the so-called *compass of inertia*). This equation manifests the analogy $\vec{\Omega} \leftrightarrow \vec{B}$. If instead of a local tetrad one considers an extended frame, such that the time axis of the tetrads is tangent to a congruence of observers, and the spatial triads $\mathbf{e}_{\hat{i}}$ co-rotate with the observers (this is set up by demanding $\vec{\Omega} = \vec{\omega}$, where $\vec{\omega}$ is the vorticity of the congruence), which is physically the most natural and relevant frame[9], then $\vec{\Omega}$ becomes one half of the gravitomagnetic field $\vec{H}$ of the corresponding frame. The details on this are given in Sec. 3 of Paper #5 [5]. In this way we obtain a generalized version (now valid for arbitrary fields) of the analogy in Sec. (3.1.2)

$$\frac{dS^{\hat{i}}}{d\tau} = \left(\frac{1}{2}\vec{S} \times \vec{H} + \vec{\mu} \times \vec{B}\right)^{\hat{i}} \tag{5.64}$$

Momentum of the particle: analogy based on GEM fields.— The momentum of the particle is not parallel to its 4-velocity, it is said to possess "hidden momentum", $P^{\alpha}_{\rm hid} = (h^U)^{\alpha}{}_{\beta} P^{\beta}$. Contracting (5.61) with U^{α}, and using the spin condition $S^{\alpha\beta}U_{\beta} = 0$, one obtains an expression for P^{α}:

$$P^{\alpha} = P^{\alpha}_{\rm kin} + P^{\alpha}_{\rm hidI} + P^{\alpha}_{\rm hidEM};$$

$$P^{\alpha}_{\rm kin} = mU^{\alpha}; \quad P^{\alpha}_{\rm hidEM} \equiv \epsilon^{\alpha}{}_{\beta\gamma\delta}\mu^{\beta}E^{\gamma}U^{\delta}; \quad P^{\alpha}_{\rm hidI} \equiv -\epsilon^{\alpha}{}_{\beta\gamma\delta}S^{\beta}a^{\gamma}U^{\delta} \; . \tag{5.65}$$

Thus the hidden momentum consists of two parts: the "inertial" one $P^{\alpha}_{\rm hidI}$ discussed in Sec. 5.4.5 above, and which is pure gauge, and another part $P^{\alpha}_{\rm hidEM}$ that arises in when an electromagnetic field is present (albeit being purely mechanical in nature, see e.g. the model in Fig. 9 of [100]). E^{α} is the electric field as measured in the particle's CM frame; and $-a^{\alpha} = G^{\alpha}$ is the gravitoelectric field in this frame, see Eq. (3.18) above. Thus we have another exact analogy based on GEM fields: the inertial hidden momentum is analogous to the "electromagnetic" hidden momentum, with $\vec{S}$ playing the role of $\vec{\mu}$, and the gravitoelectric field playing the role of the electric field. To make it more explicit, we write the momentum in the particle's CM frame (where $U^i = 0$), and in vector notation ($P^0_{\rm hid} = 0$):

$$\vec{P}_{\rm hid} = \vec{P} = -\vec{S} \times \vec{a} + \vec{\mu} \times \vec{E} = \vec{S} \times \vec{G} + \vec{\mu} \times \vec{E} \; . \tag{5.66}$$

[9] It is this type of frame that is useful in the experimental detection of gravitomagnetism, such as in the Gravity Probe B mission [53], where one needs to define a frame whose axes are everywhere fixed to the "distant stars". For shear-free congruences, this amounts to say that the triads $\mathbf{e}_{\hat{i}}$ point to fixed neighboring observers, as is the case of the frames defined in Sec. 3.1.2 for the case of stationary spacetimes.

Table 5.2: Analogy between the electromagnetic force on a magnetic dipole and the gravitational force on a gyroscope

Electromagnetic Force on a Magnetic Dipole		Gravitational Force on a Spinning Particle	
$F^{\beta}_{EM} = B_{\alpha}{}^{\beta}\mu^{\alpha}$;	(5.2.1a)	$F^{\beta}_{G} = -\mathbb{H}_{\alpha}{}^{\beta}S^{\alpha}$;	(5.2.1b)
$B^{\alpha}{}_{\beta} \equiv \star F^{\alpha}{}_{\mu;\beta}U^{\mu}$		$\mathbb{H}^{\alpha}{}_{\beta} \equiv \star R^{\alpha}{}_{\mu\beta\nu}U^{\mu}U^{\nu}$	
Eqs. Magnetic TT		Eqs. Gravitomagnetic TT	
$B^{\alpha}{}_{\alpha} = 0$	(5.2.2a)	$\mathbb{H}^{\alpha}{}_{\alpha} = 0$	(5.2.2b)
$B_{[\alpha\beta]} = \frac{1}{2}\star F_{\alpha\beta;\gamma}U^{\gamma} - 2\pi\epsilon_{\alpha\beta\sigma\gamma}j^{\sigma}U^{\gamma}$	(5.2.3a)	$\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma}J^{\sigma}U^{\gamma}$	(5.2.3b)
$B_{\alpha\beta}U^{\alpha} = 0;\quad B_{\alpha\beta}U^{\beta} = \epsilon^{\beta\gamma}{}_{\alpha\delta}E_{[\beta\gamma]}U^{\delta}$	(5.2.4a)	$\mathbb{H}_{\alpha\beta}U^{\alpha} = \mathbb{H}_{\alpha\beta}U^{\beta} = 0$	(5.2.4b)

Note that this analogy holds for the Mathisson-Pirani condition, with other spin conditions $P^{\alpha}_{\rm hidI}$ has a different form, cf. Eq. (5.54).

Mass of the particle. — The "proper mass" of the particle is defined as the scalar $m = -P^{\alpha}U_{\alpha}$, and represents the energy of the particle as measured in the frame where its center of mass is at rest. It is conserved in a purely gravitational field, if the Mathisson-Pirani condition holds; but it is not conserved in general in the presence of an electromagnetic field. Using (5.60), we obtain (see Paper #4 for details)

$$\frac{dm}{d\tau} = -\mu_{\gamma}\frac{DB^{\gamma}}{d\tau} + E_{\gamma}\frac{Dd^{\gamma}}{d\tau}\ . \tag{5.67}$$

The first term is *essentially*[10] the rate of work done on the magnetic dipole by Faraday's law of induction, already discussed in Sec. 5.2.2 above (see figure therein). We shall see below that if the test particle is a rigid spinning particle, this corresponds to a variation of kinetic energy of rotation. The second term corresponds to the work done on the magnetic dipole by the electric field when the dipole vector varies, e.g., when the dipole rotates. Note the important difference between the two terms: the first is non-zero only when *the magnetic field* varies along the particle's worldline; the second has nothing to do with induction effects, it is non-zero only when *the dipole* varies (regardless of the variation of the electric field).

5.5.2 Dynamical implications of the symmetries of the tidal tensors

According to Table 5.2, both in the case of the electromagnetic force on a magnetic dipole, and in the case of the gravitational force on a gyroscope, it is the magnetic tidal tensor,

[10] If $P^{\alpha}_{\rm hidEM} = 0$, then $\mu_{\mu}DB^{\mu}/d\tau = B^{\gamma\alpha}U_{\alpha}\mu_{\gamma} = F^{\alpha}_{\rm EM}U_{\alpha}$; otherwise there is an extra term (quadratic in μ) originating from the electromagnetic hidden momentum.

as seen by the test particle of 4-velocity U^α, that determines the force exerted upon it. The explicit analogy in Table 5.2 is thus ideally suited to compare the two forces, because in this framework it amounts to comparing $B_{\alpha\beta}$ to $\mathbb{H}_{\alpha\beta}$. The most important differences between them are: i) $B_{\alpha\beta}$ is linear in the electromagnetic potentials and vector fields, whereas $\mathbb{H}_{\alpha\beta}$ is not linear in the metric tensor, nor in the GEM "vector" fields (for a detailed discussion of this aspect, see Sec. 3 of [5]); ii) *in vacuum*, $\mathbb{H}_{[\alpha\beta]} = 0$ (symmetric tensor), whereas $B_{\alpha\beta}$ is generically not symmetric, even in vacuum; iii) time components: $\mathbb{H}_{\alpha\beta}$ is a spatial tensor (with respect to the observer measuring it), whereas $B_{\alpha\beta}$ is not. These last two differences, which are clear from equations (5.2.3)-(5.2.4), are the ones in which we are mostly interested in the present work. In this section we start by the physical consequences of the symmetries, and in the next section we discuss the time projections.

Eq. (5.2.3a), in vacuum ($j^\alpha = 0$) reduces to

$$B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^\gamma \ . \tag{5.68}$$

There is an important statement encoded in this equation that can be stated as follows: since it is the tensor $B_{\alpha\beta}$ measured by the particle that yields the force $F^\alpha_{\rm EM}$, whenever the particle sees a varying field ($DF^{\alpha\beta}/d\tau \neq 0$), $B_{\alpha\beta}$ is non-vanishing ($B_{[\alpha\beta]} \neq 0 \Rightarrow B_{\alpha\beta} \neq 0$) and therefore a force will be exerted on it (except possibly for very special orientations of $\vec{\mu}$). In particular, whenever a magnetic dipole moves in a non-homogeneous field, a force will be exerted on it (again, except for very special $\vec{\mu}$'s).

In the gravitational case, since $\mathbb{H}_{[\alpha\beta]} = 0$, analogous effects to not occur, and therefore, even in non-homogeneous fields, there are velocity fields for which $\mathbb{H}_{\alpha\beta} = 0$, i.e., for which *gyroscopes feel no force.* There are even geodesic motions for spinning particles. This is exemplified below.

5.5.2.1 Radial motion in Schwarzschild spacetime

Consider a magnetic dipole in the field of a static point charge Q, and with a purely radial initial velocity $U^\alpha = U^0(1, \vec{v})$. The particle sees a varying field ($DF^{\alpha\beta}/d\tau \neq 0$); thus, by virtue of Eq. (5.68) it measures a non-vanishing tensor $B_{\alpha\beta}$, and therefore (except for the special case $\vec{v} \parallel \vec{\mu}$) a net force will be exerted on the dipole; explicitly:

$$F^0_{\rm EM} = 0; \quad F^i_{\rm EM} = B^{[\alpha i]}\mu_\alpha = \frac{\gamma Q}{r^3}(\vec{v} \times \vec{\mu})^i \ . \tag{5.69}$$

This is unlike what one might naively expect, as the radially moving dipole sees a vanishing magnetic field B^α; taking the perspective of the frame comoving with the particle, this is explained through the laws of electromagnetic induction: the moving dipole "sees" a time-varying a electric field; by virtue of Eq. (5.68) (which is a covariant form for $\nabla \times \vec{B} = \partial \vec{E}/\partial t$), that will induce a curl in $\vec{B}$, i.e., an antisymmetric part in the magnetic tidal tensor $B_{\alpha\beta}$. For this configuration, actually $B_{\alpha\beta} = B_{[\alpha\beta]}$, i.e., the force $F^\alpha_{\rm EM}$ comes *entirely* from the antisymmetric part of $B_{\alpha\beta}$.

Therefore, since $\mathbb{H}_{\alpha\beta}$ is symmetric (in vacuum), $\mathbb{H}_{[\alpha\beta]} = 0$, we expect, in the spirit of the analogy, the force to vanish in the analogous gravitational setup. This is exactly the

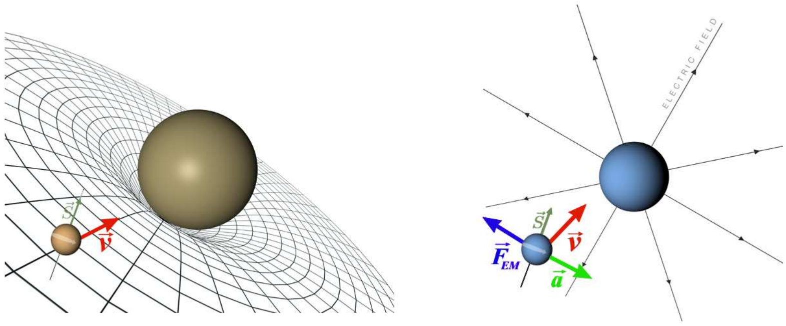


Figure 5.5: An illustration of the physical consequences of the different symmetries of the tidal tensors. A gyroscope dropped from rest in Schwarzschild spacetime will move radially along a geodesic towards the source, with no force exerted on it. A magnetic dipole in (initially) radial motion in a Coulomb field, by contrast, feels a force. Due to the hidden momentum, the force is approximately *opposite* to the acceleration!

case. If one considers a gyroscope *in radial motion* in Schwarzschild spacetime, the force is exactly zero:

$$F_G^\alpha = -\mathbb{H}^{\beta\alpha} S_\beta = 0$$

This means that a gyroscope in radial motion moves along a geodesic (for instance, a gyroscope dropped from rest will fall into the singularity moving in a straight line).

Finally (this is not the topic of this section, but is nevertheless interesting), we note, in the electromagnetic system, this counterintuitive consequence of the hidden momentum P^α_{hidEM}: if one assumes $\vec{\mu} = \sigma \vec{S}$, the acceleration is

$$m_0 a^\alpha \;\; = \;\; B^{[\alpha\beta]}\mu_\beta + \epsilon^\alpha{}_{\beta\gamma\delta}\frac{D}{d\tau}(S^\beta a^\gamma)U^\delta \approx B^{[\alpha\beta]}\mu_\beta = -F^\alpha_{\text{EM}} \; ,$$

approximately *opposite* to the force!

5.5.2.2 Equatorial motion in Kerr and Kerr-de-Sitter spacetimes

We found another manifestation of the absence of a gravitational counterpart to the antisymmetric part of the magnetic tidal tensor $B_{[\alpha\beta]}$ (i.e., of induction effects analogous to the electromagnetic ones) comparing the forces on gyroscopes in equatorial motions in Kerr and Kerr-de-Sitter spacetimes, to the ones of magnetic dipoles in equatorial motions in the field of a spinning charge.

In the equatorial plane of the Kerr, and Kerr-dS spacetimes, there are observers for which $\mathbb{H}_{\alpha\beta} = 0$; it is so when the observer's angular velocity is

$$v^\phi = \frac{U^\phi}{U^t} = \frac{a}{a^2 + r^2} \equiv v^\phi_{(\mathbb{H}=0)} \; . \tag{5.70}$$

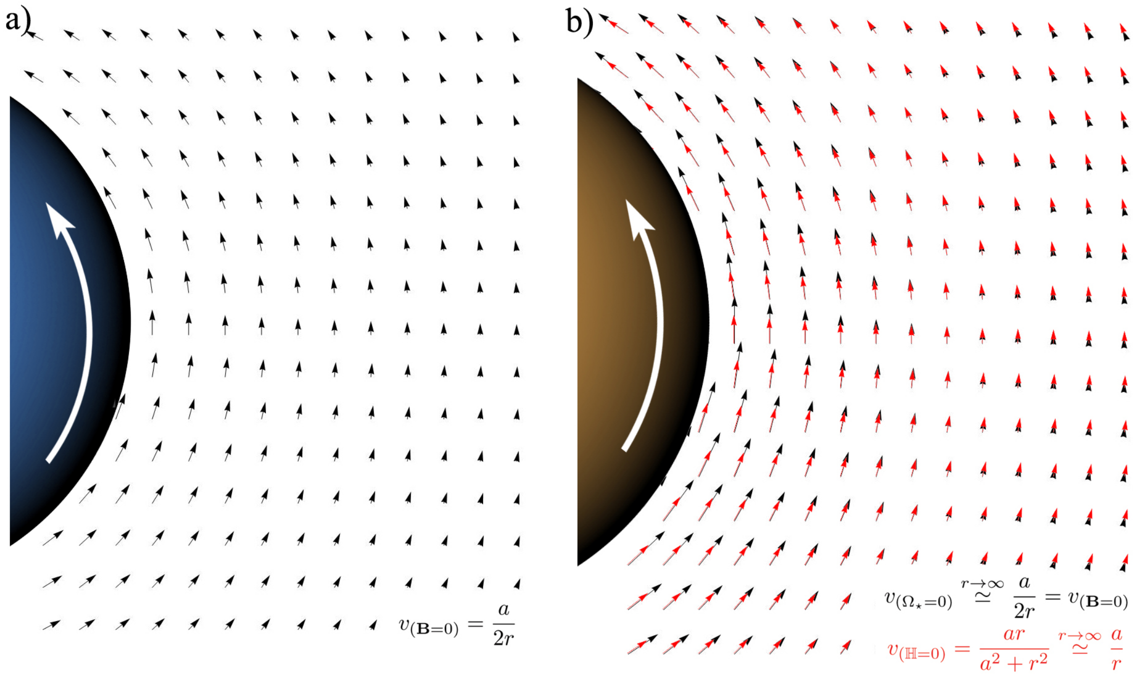


Figure 5.6: a) Equatorial plane of a spinning charge. Black arrows: velocity field $\vec{v}_{(\mathbf{B}=0)}$, which makes the *magnetic field* B^α vanish; magnetic dipoles in straightline motion, *momentarily* with such velocities, do not precess relative to the distant stars. b) Equatorial plane of Kerr spacetime. Black arrows: velocity field $\vec{v}_{(\Omega_\star=0)}$ for which a gyroscope in straightline motion momentarily does not "precess" (with respect to the distant stars); asymptotically it matches its electromagnetic counterpart. Red arrows: velocity field $\vec{v}_{(\mathbb{H}=0)}$ which makes the *gravito-magnetic tidal tensor* $\mathbb{H}_{\alpha\beta}$ vanish; this means that gyroscopes moving along trajectories tangent to $\vec{v}_{(\mathbb{H}=0)}$ feel no force, $F^\alpha_{\rm G} = 0$. If $\Lambda > 0$ (Kerr-dS spacetime) circular geodesics for gyroscopes do even exist. $\vec{v}_{(\mathbb{H}=0)}$ has no electromagnetic analogue: for a moving dipole *always* $B_{\alpha\beta} \neq 0$, by virtue of $B_{[\alpha\beta]} = \star DF_{\alpha\beta}/d\tau$, generically implying $F^\alpha_{\rm EM} \neq 0$.

This means that gyroscopes carried by such observers feel no force (regardless of the orientation of the spin vector $\vec{S}$). The angular velocity $v^{\phi}_{(\mathbb{H}=0)}$ does not correspond to any circular geodesic for material particles in the Kerr spacetime; circular geodesics "are too fast", their angular velocity dies off as $r^{-3/2}$, whereas $v^{\phi}_{(\mathbb{H}=0)}$ goes as r^{-2}. But in Kerr-dS, the repulsive Λ "slows down" the circular geodesics, and makes possible the existence of circular geodesics for which $\mathbb{H}_{\alpha\beta} = 0$. The angular velocity that makes $\mathbb{H}_{\alpha\beta}$ vanish in the equatorial plane of Kerr-dS is the same Eq. (5.70) above; and it is a rotation in the same sense of the black hole; thus the geodesics obeying this condition are found equating (5.70) to the equation for *prograde* circular geodesics in Kerr-dS,

$$(v^{\phi}_{\rm geo})_{+} = \frac{-Ma + \frac{\Lambda}{3}ar^3 \pm \sqrt{Mr^3 - \frac{\Lambda}{3}r^6}}{r^3 - a^2M + \frac{\Lambda}{3}a^2r^3}\,.$$

We show the solutions to exist numerically; that is, in Kerr-dS, it is possible for gyroscopes to move along geodesic orbits (for arbitrary $\vec{S}$).

This type of motions, and the velocity field (5.70) have no electromagnetic counterpart; due to the laws of electromagnetic induction, cf. Eq. (5.68), $B_{\alpha\beta}$, can never vanish for a moving particle (nor does it vanish for a particle at rest in this field), hence a force will always be exerted on the dipole (except for very special, fixed, orientations of $\vec{\mu}$).

The formal analogy between the scalar invariants of **F** *and* **R** *as a guide.* — In our study of the equatorial motions, and in particular to find the velocity field (5.70) above we made use of the formal analogy between the electromagnetic invariants (3.22) and the quadratic invariants of the Weyl tensor, expressions (3.31) above. The electromagnetic invariants have the following physical interpretation [63, 18, 30]:

1. if $E^{\alpha}B_{\alpha} \neq 0$, then the electric E^{α} and magnetic B^{α} fields are both non-vanishing for all observers;

2. if $E^{\alpha}E_{\alpha} - B^{\alpha}B_{\alpha} > 0$ (< 0) and $E^{\alpha}B_{\alpha} = 0$, there are observers u^{α} for which the magnetic field B^{α} (the electric field (E^{α}) is zero.

In the case of the Weyl tensor (or the Riemann tensor, in vacuum) the quadratic invariants (3.31) are not sufficient for an analysis like the one we did for the electromagnetic case. Whereas the invariants (3.22) are the only two independent scalar invariants of $F^{\alpha\beta}$, in the case of $R_{\alpha\beta\gamma\delta}$ there are 14 independent invariants in general, which in vacuum reduce to four: the invariants (3.31), plus two *cubic* invariants, given by

$$A \equiv \frac{1}{16}R^{\alpha\beta}{}_{\lambda\mu}R^{\lambda\mu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}, \qquad B \equiv \frac{1}{16}R^{\alpha\beta}{}_{\lambda\mu}R^{\lambda\mu}{}_{\rho\sigma}\star R^{\rho\sigma}{}_{\alpha\beta}$$

and usually combined in the complex quantity $J \equiv A - iB$. Define also the complex quantity $I \equiv (\mathbf{R}\cdot\mathbf{R} + i\star\mathbf{R}\cdot\mathbf{R})/8$. It turns out (cf. [45, 46, 30]) that one obtains formally equivalent statements to 1-2 above, replacing **F** by **R**, *provided that the condition* $\mathbb{M} \equiv I^3/J^2 - 6 \geq 0$ (real or infinite) *is added to* 2); that is:

1. $\star\mathbf{R}\cdot\mathbf{R}\neq 0 \Rightarrow \mathbb{E}_{\alpha\gamma}$ and $\mathbb{H}_{\alpha\gamma}$ are both non-vanishing for all observers;

2. $\star\mathbf{R}\cdot\mathbf{R}=0$, $\mathbf{R}\cdot\mathbf{R}>0$, with $\mathbb{M}\geq 0 \Rightarrow$ there are observers for which $\mathbb{H}_{\alpha\gamma}$ vanishes ("Purely Electric" spacetime)[11].

Further details and comments on this classification shall be given in [30]. The analysis above is for vacuum, where $\mathbf{C}=\mathbf{R}$; in order to obtain similar statements valid generically, one only has to replace $\mathbf{R}$ by $\mathbf{C}$.

The invariant structure of the electromagnetic of a spinning charge (charge Q, magnetic moment μ_s) is:

$$\begin{cases} \vec{E}^2-\vec{B}^2=\dfrac{Q^2}{r^4}-\dfrac{\mu_s^2(5+3\cos 2\theta)}{2r^6}>0\ , \\ \\ \vec{E}\cdot\vec{B}=\dfrac{2\mu_s Q\cos\theta}{r^5}\ (=0 \text{ in the equatorial plane}) \end{cases} \tag{5.71}$$

which tells us that in the equatorial plane $\theta=\pi/2$ there are observers measuring the magnetic field B^α (*not* the tidal tensor!) to be *locally* zero, since $\vec{E}\cdot\vec{B}=0$ and $\vec{E}^2-\vec{B}^2>0$ therein. The angular velocity of such observers is $U^\phi/U^t=\mu_s/(Qr^2)$. If we additionally assume that the charge and mass are identically distributed in the body, its gyromagnetic ratio is $\mu_s/J=Q/2M$, and we obtain the angular velocity

$$v^\phi=\frac{U^\phi}{U^t}=\frac{J}{2Mr^2}\equiv v^\phi_{(\mathbf{B}=0)}\ , \tag{5.72}$$

which asymptotically matches, up to a factor of 2, the velocity (5.70) above.

This analogy proves illuminating for the gravitational problem. In the case of Kerr spacetime, which is of Petrov type D, the condition $\mathbb{M}\geq 0$ (real) is satisfied, since $I^3=6J^2$ (see e.g. [46]). Thus one only has to worry about the invariants (3.31), which have the structure:

$$\begin{cases} \mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma}-\mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma}\approx\dfrac{6M^2}{r^6}>0 \\ \\ \mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma}\approx\dfrac{18JM\cos\theta}{r^7}\ \ (=0\ \text{ in the equatorial plane}) \end{cases} \tag{5.73}$$

formally analogous to its electromagnetic counterpart (5.71). Note in particular that the result $\mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma}=0$ for the equatorial plane ($\theta=\pi/2$) is *exact*. It was in this way that we concluded that *in the equatorial plane*, *there are observers for which the magnetic tidal tensor vanishes,* whose angular velocity we found to be given by Eq. (5.70)above (see Paper #4 [4] for more details).

At this point it is important to stress the following: in spite of the striking similarities in the invariant structures (5.71) and (5.73), and in the velocity fields (5.72) and (5.70),

[11] The case $\star\mathbf{R}\cdot\mathbf{R}=0$, $\star\mathbf{R}\cdot\mathbf{R}<0$ *would* mean that there would be observers for which $\mathbb{E}_{\alpha\gamma}$ vanishes (but no "Purely Magnetic" vacuum solutions are known, and it has been conjectured that they do not exist, see e.g. [111, 112]).

this is a purely formal analogy; in one case we are talking about velocities for which the *magnetic field* B^α vanishes, in the other about the vanishing of the *gravito-magnetic tidal tensor* $\mathbb{H}_{\alpha\beta}$. The physical effects are actually opposite: in the first case magnetic dipoles with such velocities do not undergo Larmor precession, $D\vec{S}/d\tau = 0$, cf. Eq. (5.62), but they feel a force $F^\alpha_{EM} \neq 0$ ($B_{\alpha\beta}$ *never* vanishes, as discussed above). In the gravitational case, the gyroscope feels no force: $F^\alpha_G = 0$, but it precesses (relative to the local comoving tetrad *non-rotating relative to the distant stars*): $\tilde{D}\vec{S}/d\tau = \vec{S} \times \vec{H}$, cf. Eq. (5.63).

For completeness, we also investigated the *physical* gravitational counterpart of the velocity for which magnetic dipoles do not precess; it involves some subtleties (it comes down to finding the velocity that a gyroscope, momentarily in "straightline" motion, must have in order to not precess relative to the distant stars) but asymptotically it matches (5.72).

5.5.3 Time projections of the forces and work done on a test particle

A fundamental difference between the gravitational and electromagnetic interactions concerns the time projections of the forces $F^\alpha_{\rm G}$ and $F^\alpha_{\rm EM}$ in the different frames. These encode the work done by the force in the given frame.

In order to understand it, and its relation with the particle's energy, consider a congruence of observers $\mathcal{O}(u)$ with 4-velocity u^α, and let U^α denote the 4-velocity of a test particle. The following relation generically holds [25]:

$$U^\alpha = \gamma(u^\alpha + v^\alpha); \quad \gamma \equiv -u^\alpha U_\alpha = \frac{1}{\sqrt{1 - v^\alpha v_\alpha}} \,, \tag{5.74}$$

where $v^\alpha = U^\alpha/\gamma - u^\alpha$ is the velocity of the test particle relative to the observers $\mathcal{O}(u)$; in the frame $u^i = 0$, v^i is the ordinary 3-velocity. The energy of the test particle relative to $\mathcal{O}(u)$ is $E \equiv -P^\alpha u_\alpha$, and its rate of change (the "power equation")

$$\frac{dE}{d\tau} = -F^\alpha u_\alpha - P^\alpha u_{\alpha;\beta} U^\beta \,, \tag{5.75}$$

where $F^\alpha \equiv DP^\alpha/d\tau$ denotes the 4-force. Thus we see that the variation energy of the particle relative to $\mathcal{O}(u)$ consists of two terms: the time projection of F^α along u^α, plus a term depending on the variation of u^α along the test particle worldline. The first term is interpreted as the rate of work, *as measured by* $\mathcal{O}(u)$, done by the force on the test particle. Using $u^\alpha = U^\alpha/\gamma - v^\alpha$, we can write it as

$$-F^\alpha u_\alpha = -\frac{F^\alpha U_\alpha}{\gamma} + F^\alpha v_\alpha \,. \tag{5.76}$$

In the simplest case of a point particle with no internal structure (a monopole particle) the first term is zero, since the momentum is parallel to the 4-velocity ($P^\alpha = mU^\alpha$), and its mass is a constant, $m = m_0$; hence the force is parallel to the acceleration and orthogonal to U^α. Such force is said to be *spatial* with respect to U^α. Thus $-F^\alpha u_\alpha = F^\alpha v_\alpha$, telling us that the time-projection of the 4-D force F^α is the familiar power (i.e., the rate of work per unit of proper time τ) transferred to the particle by the 3-D force $(h^U)^\alpha_\mu F^\mu$ (see e.g. [61, 114]). If the frame is inertial, so that the second term of (5.75) vanishes, then $-F^\alpha u_\alpha = dE/d\tau = m_0 d\gamma/d\tau$, i.e., $F^\alpha v_\alpha = m_0 d\gamma/d\tau$ is the rate of variation of kinetic energy of translation of the particle's center of mass. This is the type of force we are more familiar with; an example is the Lorentz force on a charged particle, $DP^\alpha/d\tau = qF^{\alpha\beta}U_\beta$, whose projection along u^α reads $-u^\alpha DP_\alpha/d\tau = \gamma q v^\alpha E_\alpha$, yielding the rate of work (per unit proper time) done by the electric field on the particle moving with velocity v^α relative to $\mathcal{O}(u)$.

However, if the particle has internal structure, as in the problem at hand (spinning multipole particles), its internal degrees of freedom may store energy, which in general will be exchanged with the energy of the external fields and the kinetic energy of the center of mass. Therefore, the proper mass of the particle $m = -P^\alpha U_\alpha$ no longer has to be a constant, cf. Eq. (5.67). Also the momentum is not be parallel to U^α, as the particle in general possesses hidden momentum, cf. Eq. (5.65). These, together, endow F^α with a nonvanishing time projection: $F^\alpha U_\alpha \neq 0$.

Let us turn our attention now to the second term of Eq. (5.75). Decomposing (e.g. [25, 5, 34])

$$u_{\alpha;\beta} = -a(u)_\alpha u_\beta + \omega_{\alpha\beta} + \theta_{\alpha\beta} \ , \qquad (5.77)$$

where $a(u)^\alpha = u^\alpha_{\ ;\beta}u^\beta$ is the acceleration of $\mathcal{O}(u)$ (*not* the particle's!), $\omega_{\alpha\beta} \equiv (h^u)^\lambda_\alpha (h^u)^\nu_\beta u_{[\lambda;\nu]}$ the vorticity, and $\theta_{\alpha\beta} \equiv (h^u)^\lambda_\alpha (h^u)^\nu_\beta u_{(\lambda;\nu)}$ the *total* shear of the congruence ($\theta_{\alpha\beta} \equiv \sigma_{\alpha\beta} + \theta(h^u)_{\alpha\beta}/3$, with $\sigma_{\alpha\beta}$ as usual the traceless shear and θ the expansion scalar). $G(u)^\alpha = -a(u)^\alpha$ is thus the gravitoelectric field measured in the frame $u^i = 0$. Decomposing $P^\alpha = mU^\alpha + P^\alpha_{\rm hid}$, and decomposing U^α using (5.74), the second term of Eq. (5.75) becomes:

$$\begin{aligned} -P^\alpha u_{\alpha;\beta}U^\beta = {} & m\gamma^2[G(u)_\alpha - \theta_{\alpha\beta}v^\beta]v^\alpha \\ & + \gamma P^\alpha_{\rm hid}\left[G(u)_\alpha - (\omega_{\alpha\beta} + \theta_{\alpha\beta})\, v^\beta\right] . \end{aligned} \qquad (5.78)$$

This part of $dE/d\tau$ depends only on the kinematical quantities of the congruence. That is, unlike the term (5.76), which arises from the 4-force F^α, the term (5.78) does not depend on any physical quantity one can *locally* measure; it is locally an artifact of the reference frame, which can always be made to vanish by choosing a locally inertial one. Its importance (in a non-local sense) should not however be overlooked. To understand it, consider a simple example, a monopole particle in Kerr spacetime, from the point of view of the congruence of *static observers* (i.e., u^α parallel to the time-like Killing vector field $\boldsymbol{\xi} \equiv \partial/\partial t$, in Boyer Lindquist coordinates). Since the congruence is rigid, $\theta_{\alpha\beta} = 0$; also, for a monopole particle, $P^\alpha_{\rm hid} = 0$, and, in a gravitational field, $F^\alpha = 0$ (the particle moves along a geodesic). Therefore, the energy variation reduces to $dE/d\tau = -P^\alpha u_{\alpha;\beta}U^\beta = m\gamma^2 G(u)_\alpha v^\alpha$; which is the rate of "work" (per unit proper time τ) done by the gravitoelectric "force" [19, 5, 25] $m\gamma^2 G(u)^\alpha$. (In the Newtonian limit, reduces to the work of the Newtonian force $m\vec{G}$.) Hence we see that (5.78) is the part of (5.75) that encodes the change in translational kinetic energy of a particle (relative to static observers) which occurs due to the gravitational field, without the action of any (physical, covariant) force, i.e., for particles in geodesic motion.

The system of Eqs. (5.75), (5.76) and (5.78) are a generalization, for the case of test particles with varying m and hidden momentum, of the "power" equation (6.12) of [25] (the latter applying to monopole particles only).

5.5.3.1 Time components in test particle's frame

One fundamental difference between the tensorial structure of $\mathbb{H}_{\alpha\beta}$ and $B_{\alpha\beta}$ is that whereas the former a spatial tensor, in *both* indices, with respect to the observer u^{α} measuring it: $(\mathbb{H}^u)_{\alpha\beta}u^{\beta} = (\mathbb{H}^u)_{\alpha\beta}u^{\alpha} = 0$ (this follows from the symmetries of the Riemann tensor), the latter is not: $(B^u)_{\alpha\beta}u^{\alpha} = 0$ but $(B^u)_{\alpha\beta}u^{\beta} = \star F_{\alpha\gamma;\beta}u^{\gamma}u^{\beta} \neq 0$ in general. That means that whereas F_G^{α} is a spatial force, $F_{\rm EM}^{\alpha}$ has a non vanishing time projection in the particle's proper frame,

$$F_{\rm EM}^{\alpha}U_{\alpha} = B^{\beta\alpha}U_{\alpha}\mu_{\beta} = \epsilon_{\beta\delta\mu\nu}U^{\delta}E^{[\mu\nu]}\mu^{\beta} \ . \tag{5.79}$$

We already discussed the meaning of $-F_{\rm EM}^{\alpha}U_{\alpha} \equiv \mathcal{P}_{\rm ind}$ in Sec. 5.2.2 (see figure therein; in Paper #4 we present an equivalent derivation of the same result, only covariant at each step); it is the rate of work transferred to the dipole by Faraday's law of induction. It consists in the variation of the proper mass m, *minus* the projection along U^{α} of the derivative of the hidden momentum (to which only the "electromagnetic" hidden momentum contributes):

$$\mathcal{P}_{\rm ind} = \frac{dm}{d\tau} - \frac{DP_{\rm hid}^{\alpha}}{d\tau}U_{\alpha} = \frac{dm}{d\tau} - \frac{DP_{\rm hidEM}^{\alpha}}{d\tau}U_{\alpha} \, . \tag{5.80}$$

Also, we can say that $\mathcal{P}_{\rm ind}$ is the variation of the dipole's energy $E = -P_{\alpha}U^{\alpha}$ *as measured in a momentarily comoving inertial frame.* It is worth mentioning that the failure to notice that $F_{\rm EM}^{\alpha}$ has a time projection, and that m varies, has led to the difficulties in [102, 103] to covariantly describe the force on a magnetic dipole (namely to the claim that no covariant description of such forces is this scheme is possible).

As discussed already in Sec 5.2.2 above, the induction phenomenon in (5.79) has no counterpart in gravity. Since $\mathbb{H}_{\alpha\beta}$ is a spatial tensor, we *always have*

$$F_G^{\alpha}U_{\alpha} = 0 \tag{5.81}$$

which means that the *energy* of the gyroscope, *as measured in a momentarily comoving inertial frame,* is constant. The proper mass m is also constant, since, in the gravitational case, $P^{\alpha}a_{\alpha} = -U_{\alpha}DP_{\rm hid}^{\alpha}/d\tau$ is always zero.

5.5.3.2 Time components as measured by static observers

Electromagnetism. — With respect to an arbitrary congruence of observers u^{α}, the time projection of the force exerted on a magnetic dipole is, cf. Eq. (5.76):

$$-F_{\rm EM}^{\alpha}u_{\alpha} = -\frac{F_{\rm EM}^{\beta}U_{\beta}}{\gamma} + F_{\rm EM}^{\alpha}v_{\alpha} = \frac{\mathcal{P}_{\rm ind}}{\gamma} + F_{\rm EM}^{\alpha}v_{\alpha} \tag{5.82}$$

where, in accordance with discussion above, we identify $\mathcal{P}_{\rm ind} = -F^{\beta}_{\rm EM}U_{\beta}$ as the power transferred to the dipole by Faraday's induction, and $F^{\alpha}_{\rm EM}v_{\alpha}$ is the power transferred by the 3-force $(h^U)^{\alpha}_{\mu}F^{\mu}_{\rm EM}$ exerted on it. Consider now a congruence of observers u^{α} for which the fields are covariantly constant, $F^{\alpha\beta}_{\ \ ;\gamma}u^{\gamma} = 0$, which we dub in this context "*static* observers"[12]. The time projection of the force in this frame vanishes:

$$-F^{\alpha}_{\rm EM}u_{\alpha} \equiv -\frac{DP_{\alpha}}{d\tau}u^{\alpha} = \star F_{\alpha\beta;\gamma}U^{\beta}\mu^{\alpha}u^{\gamma} = 0. \tag{5.83}$$

That tells us that the total work done on the dipole, *as measured in this frame*, is zero. The scalar $E = -P_{\alpha}u^{\alpha}$ (the energy of the particle in the $u^i = 0$ frame) is thus a conserved quantity, and using $P^{\alpha} = mU^{\alpha} + P^{\alpha}_{\rm hid}$, we can write it in the form

$$E = m + T + E_{\rm hid} = \text{constant}, \tag{5.84}$$

where we dub $E_{\rm hid} \equiv -P^{\alpha}_{\rm hid}u_{\alpha}$ as the "hidden energy" (i.e., the time component of the hidden momentum in the $u^i = 0$ frame), and $T \equiv (\gamma - 1)m$ is the kinetic energy of translation of the center of mass (cf. e.g. [114] p. 70), as measured in this frame. The reason for the later denomination is seen taking the Newtonian regime, where $T \approx mv^2/2$. Thus the constancy of the energy of the particle comes from an exchange of energy between three parts: T, proper mass m (which, for a rigid particle, is kinetic energy of rotation about the CM as I shall show below), and $E_{\rm hid}$. The latter is the most unfamiliar; a suggestive setup where it plays a role are the bobbings of a particle with magnetic dipole moment orbiting a cylindrical charge in Fig. 5.4 above.

In this work we are especially interested in the case: $P^{\alpha}_{\rm hid} = 0$ ($\Rightarrow E_{\rm hid} = 0$); in this case the energy exchange occurs only between proper mass and translational kinetic energy, $m + T = \text{constant}$. Also, we have $\mathcal{P}_{\rm ind} = dm/d\tau$, and for static observers in flat spacetime $u_{\alpha;\beta} = 0$, so that the second term of (5.75) vanishes. We can thus write:

$$\frac{dE}{d\tau} = -F^{\alpha}_{\rm EM}u_{\alpha} = \mathcal{P}_{\rm ind} + \mathcal{P}_{\rm trans} = 0\ , \tag{5.85}$$

where $\mathcal{P}_{\rm trans} \equiv dT/d\tau$. An example is the problem depicted in Fig. 5.7a): a magnetic dipole falling along the symmetry axis of the field generated by a strong magnet ($P^{\alpha}_{\rm hid} = 0$ for this configuration). From the point of view of the static observers, $\vec{E} = 0$ and only magnetic field $\vec{B}$ is present; since the latter does no work in any charge/current distribution, naturally $F^{\alpha}_{\rm EM}u_{\alpha} = 0$. According to Eq. (5.85), this arises from an *exact cancellation* between $\mathcal{P}_{\rm trans}$ and $\mathcal{P}_{\rm ind}$: on the one hand there is an attractive spatial force $F^{i}_{\rm EM}$ causing the dipole to gain translational kinetic energy; on the other hand there is a variation of its internal energy (proper mass m) by induction, which allows for the total work to vanish.

[12] The reason for this denomination is the fact that the condition $F^{\alpha\beta}_{\ \ ;\gamma}u^{\gamma} = 0$ corresponds to the observers *at rest* with respect to the sources in the electromagnetic applications herein. Note that for e.g. observers u'^{α} in circular orbits around a Coulomb charge we have $F^{\alpha\beta}_{\ \ ;\gamma}u'^{\gamma} \neq 0$, even though u'^{α} is in that case a symmetry of $F_{\alpha\beta}$: $\mathcal{L}_{u'}F_{\alpha\beta} = 0$, and the latter is explicitly time-independent in a rotating frame.

Gravity. — In gravity, where the induction effects are absent ($F^{\alpha}_{\rm G}U_{\alpha}=0$), we have for arbitrary u^{α}

$$-F^{\alpha}_{G}u_{\alpha}=F^{\alpha}_{G}v_{\alpha}\ . \tag{5.86}$$

This implies in particular that a cancellation similar to the one above does not occur (even for some "static observers"); unless $\vec{F}_{\rm G}\cdot\vec{v}=0$, F^{α}_{G} does work on the test particle, *by contrast with its electromagnetic counterpart.* That is, a stationary gravitomagnetic tidal field does work on mass currents; and there is actually a potential energy associated with it, as we shall now show. A conserved quantity for a spinning particle in a stationary spacetime is (e.g. [81, 66, 87])

$$E_{\rm tot}=-P^{\alpha}\xi_{\alpha}+\frac{1}{2}\xi_{\alpha;\beta}S^{\alpha\beta}={\rm constant} \tag{5.87}$$

where $\boldsymbol{\xi}\equiv\partial/\partial t$ is the time-like Killing vector field. Consider the congruence of the static observers[13] $\mathcal{O}(u)$, defined as the unit-time like vectors u^{α} tangent to ξ^{α}; we may write $\xi^{\alpha}=\xi u^{\alpha}$, with $\xi\equiv\sqrt{-\xi^{\alpha}\xi_{\alpha}}$. The first term of (5.87), $-P^{\alpha}\xi_{\alpha}=E\xi$ is the "Killing energy", a conserved quantity for the case of a non-spinning particle, which yields its energy E *with respect to the static observer at infinity.* It can be interpreted as its "total energy" (rest mass + kinetic + "Newtonian potential energy") in a gravitational field (e.g. [15]). The energy $E_{\rm tot}$ can likewise be interpreted as the energy at infinity for the case of a spinning particle. To see the interpretation of the second term in (5.87), $V\equiv\xi_{[\alpha;\beta]}S^{\alpha\beta}/2$, consider the case that $P^{\alpha}_{\rm hid}=0$. We have

$$0=\frac{dE_{\rm tot}}{d\tau}\ =\ -F^{\alpha}_{G}\xi_{\alpha}-mU^{\alpha}U^{\beta}\xi_{\alpha;\beta}+\frac{dV}{d\tau} \tag{5.88}$$

$$\Leftrightarrow\xi F^{\alpha}_{G}u_{\alpha}\ =\ \frac{dV}{d\tau}\ ; \tag{5.89}$$

$-\xi F^{\alpha}_{G}u_{\alpha}=\xi F^{\alpha}_{G}v_{\alpha}$ is the rate work of F^{α}_{G}, *as measured by the static observers at infinity,* and thus V is the spin-curvature *potential energy* associated with that work. In order to compare with the electromagnetic Eq. (5.84), note that $d\xi/d\tau=-\gamma G(u)_{\alpha}v^{\alpha}$, and that for $P^{\alpha}_{\rm hid}=0$, $E=\gamma m=m+T$, thus we can write $dE_{\rm tot}/d\tau=d(\xi E+V)d\tau$ in the form

$$\xi\frac{dT}{d\tau}-\xi m\gamma^{2}G_{\alpha}v^{\alpha}+\frac{dV}{d\tau}=0. \tag{5.90}$$

The second term accounts for the "power" of the gravitoelectric "force" $m\gamma^{2}\vec{G}(u)$ (which, as explained above, is *not* a physical force, it arises from the acceleration of the observer congruence, being non-zero even for geodesic motion; in the weak field limit it reduces to the variation of Newtonian potential energy). Eq. (5.90) tells us that the variation of translational kinetic energy T comes from the potential energy V, and the power of $\vec{G}(u)$ (m being constant); this contrasts with the case of the magnetic dipole discussed above,

[13] See point 7 of Sec. 5.1. In stationary asymptotically flat spacetimes, such as the Kerr metric studied below, these are observers rigidly fixed to the asymptotic inertial rest frame of the source. They are thus the closest analogue of the flat spacetime notion of observers at rest relative to the source in the electromagnetic systems above.

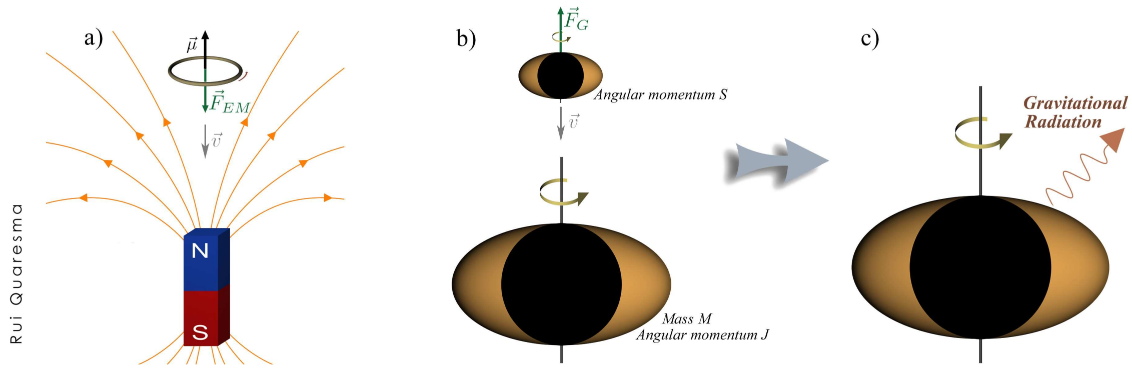


Figure 5.7: a) A magnetic dipole in the inhomogeneous magnetic field of a strong magnet, from the point of view of the static observers u^α; the work is zero ($-F^\alpha_{\rm EM}u_\alpha = \mathcal{P}_{\rm ind} + \mathcal{P}_{\rm trans} = 0$), manifesting that a stationary magnetic field does no work. Arises from an exact cancellation between $\mathcal{P}_{\rm ind}$ and $\mathcal{P}_{\rm trans}$. b) Gyroscope (small Kerr black hole) in the field of a large Kerr black hole; b) Black hole merger. Evidence that, unlike its electromagnetic counterpart, gravitomagnetic tidal field *does* work: spin-dependent part of the energy released is the work (as measured by the static observers at infinity) of F^α_G.

where (also for $P^\alpha_{\rm hid} = 0$) the kinetic energy comes from the variation of proper mass m, no potential energy being involved. In terms of the work done on the particle, F^α_G is thus more similar to the electromagnetic force exerted on a monopole charge, where the proper mass is constant and the energy exchange is between T and potential energy.

There is a known consequence of the fact that F^α_G does work (and of the interaction energy V): the spin dependence of the upper bounds for the energy released by gravitational radiation (Fig. 5.7c) when two black holes collide, obtained by Hawking [104] from the area law.

In order to see that, consider the apparatus in Fig. 5.7b: two Kerr black holes with spins aligned, a large one (mass M, spin $J = aM$) which is our source, and small one (4-velocity U^α, spin $S \equiv \sqrt{S^\alpha S_\alpha}$) which we take as test particle, falling into the former along the symmetry axis. For axial fall, and given that $\vec{S}$ is also along the axis, $P^\alpha_{\rm hid}(= P^\alpha_{\rm hidI}) = 0$. (Consider again the frame of the static observers $\mathcal{O}(u)$, $u^\alpha = \xi^\alpha/\xi$, i.e. the observers with zero 3-velocity in Boyer-Lindquist coordinates). For this configuration, V is a pure spin-spin potential energy; it reads:

$$V(r) = \pm\frac{2aMSr}{(a^2+r^2)^2} = \int_\infty^{\tau(r)} \xi F^\alpha_G u_\alpha d\tau \ ,$$

the $+/-$ sign applying to the case that $\vec{S}$ and $\vec{J}$ are parallel/antiparallel. The second equality follows directly from Eq. (5.89), and can also be easily checked noting that, in Boyer-Lindquist coordinates, $\xi F^\alpha_G u_\alpha = (F_G)_0$, and computing explicitly the time component $(F_G)_0$ for axial fall, Eq. (37) of [10]. Thus we see that $V(r)$ is *minus* the work[14]

[14]Note that computing the work of F^α_G does not amount to integrate the power measured by the local static

done by F_G^α as the particle goes from infinity to r. Let us now analyze the problem of the black hole merger. The increase of translational kinetic energy of the small black hole during the fall is given by Eq. (5.90) above (or by the power Eq. (5.75); note in this case that $\theta_{\alpha\beta} = 0$, since $\mathcal{O}(u)$ is a rigid congruence). The second term of (5.90) is the gain in kinetic energy due to the "Newtonian" attraction, and exists regardless of S^α; V by its turn is a spin-spin energy; thus the kinetic energy of the particle (and therefore energy available to be released by gravitational radiation in the black hole merger) depends on S. Upper bounds for this energy which are, accordingly, spin dependent, were obtained in [104] by a totally independent method. From these limits, and for the case of the setup in Fig. 5.7, Wald [10] obtained an expression (Eq. (35) therein) for the amount of energy ΔE_s by which the upper bound is increased/reduced when $\vec{S}$ is parallel/antiparallel to $\vec{J}$, comparing with the case $S = 0$ (fall along a geodesic). This energy is precisely *minus* the value of $V(r)$ at the horizon r_+: $\Delta E_s = -V(r_+)$; that is, it is *the work done by* F_G^α on the small black hole as it comes from infinity to the horizon: $\Delta E_s = \int_\infty^{\tau(r_+)}(-\xi F_G^\alpha u_\alpha)d\tau$.

5.5.4 Beyond pole-dipole: the torque on the spinning particle

It follows, from Eqs. (5.61) or (5.62), that for purely magnetic dipoles ($d^\alpha = 0$), if we assume $\vec{\mu} = \sigma\vec{S}$, S^2 is a constant of the motion. This might be somewhat surprising. If one imagines the magnetic dipole as a spinning charged body, one would expect that, in a time-varying magnetic field, the induced electric field will in general exert a net torque on it, which will accelerate[15]the rotation of the body. But in Eq. (5.63) we only find the term $\vec{\mu} \times \vec{B}$, coupling the field to the dipole moment (which is there in any case, even if the field is constant), and no term coupling to the *derivatives* of the electromagnetic fields; i.e., no trace of induction phenomena. Moreover, we have seen that the induced electric field does work on the spinning particle, cf. Eq. (5.79), contributing to a variation $dm/d\tau = -\vec{\mu}\cdot D\vec{B}/d\tau$ of its proper mass m. That work has been shown, in non-relativistic treatments [107, 109] (for the case of a rigid spherical body), to equal the variation of the particle's rotational kinetic energy. Thus we expect it to be associated as well to a variation of the spinning angular velocity, and hence of S^2.

As shown below, this apparent inconsistency is an artifact inherent to the pole-dipole approximation, where terms $\mathcal{O}(R^2)$ (being R the size of the particle), which are of quadrupole type, are neglected; indeed, whereas the contribution of induction to the body's energy is of the type $\vec{\mu} \cdot \vec{B}$, i.e., of dipole order, the associated torque involves *the trace of* the quadrupole moment of the charge distribution. And there is no analogous torque in the gravitational case, confirming the absence of an analogous induction effect.

For clarity, I will treat the two interactions (electromagnetic and gravitational) separately.

observers, $-F_G^\alpha u_\alpha = F_G^\alpha v_\alpha$ (i.e., to sum up the work elements $dW \equiv F_G^\alpha v_\alpha d\tau$), as that would mean summing up energies measured by different observers; but to integrate instead the quantity $\xi F_G^\alpha v_\alpha$, which can be thought as summing work elements measured by static the observer *at infinity.*

[15] Unlike the torque due to the magnetic field, the torque due the induced electric field will not be orthogonal to $\vec{S}$, and hence will in general change its magnitude. For instance, in the application in Fig. 5.8, $\vec{E}_{\rm ind}$ has circular lines around $\vec{S}$, so that $\vec{\tau}_{\rm ind} \parallel \vec{S}$.

5.5.4.1 Electromagnetic torque

The relativistic equations of motion for spinning multipole particles in an electromagnetic field, accurate to quadrupole order, were obtained by Dixon [59, 105, 106]; the equation for the spin evolution (sometimes dubbed the "torque" [66] on the particle) reads, in our notation

$$\frac{D(S_{\rm can})^{\alpha\beta}}{d\tau} \quad = \quad 2(P_{\rm Dix})^{[\alpha}U^{\beta]} + 2Q^{\theta[\beta}F^{\alpha]}{}_{\theta} + 2m^{[\alpha}{}_{\rho\mu}F^{\beta]\mu;\rho} \ . \tag{5.91}$$

$Q^{\alpha\beta}$ is the electromagnetic dipole moment tensor, encoding the electric and magnetic dipole moments:

$$Q^{\alpha\beta} = 2d^{[\alpha}U^{\beta]} + \epsilon^{\alpha\beta\gamma\delta}\mu_{\gamma}U_{\delta} \ , \tag{5.92}$$

and $m^{\alpha\beta\gamma}$ is a quadrupole type moment whose definition is given in Sec. VIA of Paper #4 [4]. These equations follow from rigorous derivations in [59, 106, 66], and are thus well established. The question mark in the literature is the physical interpretation of the quantities $(P_{\rm Dix})^{\alpha}$ and $(S_{\rm can})^{\alpha\beta}$; this is the first issue we had to address, as it is crucial in this context. $(S_{\rm can})^{\alpha\beta}$ is not the angular momentum $S^{\alpha\beta}$ of def. (5.39) (which we argue to be the *physical* angular momentum of the particle), but instead a quantity obtained from it by adding certain electromagnetic functions, cf. Eqs. (7.1)-(7.2) and (7.6)-(7.7) of [59]. To quadrupole order

$$(S_{\rm can})^{\alpha\beta} = S^{\alpha\beta} + S'^{\alpha\beta} \ , \qquad S'^{\alpha\beta} = F^{[\alpha}{}_{\sigma}q^{\beta]\sigma} \ , \tag{5.93}$$

where

$$q^{\alpha\beta} \equiv \int_{\Sigma(\tau,U)} r^{\alpha}r^{\beta}j^{\gamma}d\Sigma_{\gamma} \tag{5.94}$$

is the second moment of the charge (the charge "quadrupole").

In order to see how important it is to distinguish between $(S_{\rm can})^{\alpha\beta}$ and $S^{\alpha\beta}$, take for example a spinning spherical, uniformly charged body. In this case, as shown in Paper #4 , $2m^{[\alpha}{}_{\rho\mu}F^{\beta]\mu;\rho} = 0$ (if external currents are present), and the space part of (5.91) reduces to $\epsilon_{\alpha\beta}{}^{\sigma\delta}U_{\delta}D(S_{\rm can})^{\alpha\beta}/d\tau = \epsilon^{\sigma}{}_{\alpha\beta\nu}U^{\nu}\mu^{\alpha}B^{\beta}$, i.e., it is the familiar *dipole order* torque on the magnetic dipole ($\vec{\mu}\times\vec{B}$ in vector notation). If one takes $\vec{\mu}\parallel\vec{B}$, it is zero: $\epsilon_{\alpha\beta}{}^{\sigma\delta}U_{\delta}D(S_{\rm can})^{\alpha\beta}/d\tau = 0$. Thus $(S_{\rm can})^{\alpha\beta}$ is certainly not the physical momentum, nor its variation the physical torque, as we know that if the magnetic field varies with time, it induces an electric field with a curl that torques the body. And it is also known, from non-relativistic treatments of this problem [107, 108, 109], that such torque depends on the trace of the charge quadrupole $q^{\alpha\beta}$.

$(S_{\rm can})^{\alpha\beta}$ is shown in [110] to be instead the canonical angular momentum, and $(P_{\rm Dix})^{\alpha} + qA^{\alpha} \equiv (P_{\rm can})^{\alpha}$ the canonical momentum associated to the Lagrangian of the system[16] (which is the quantity conserved in collisions, and whose time part is the energy scalar conserved in time-constant fields). We note also that the vector $(S_{\rm can})^{\gamma} \equiv \epsilon^{\gamma}{}_{\mu\alpha\beta}(S_{\rm can})^{\alpha\beta}U^{\mu}/2$,

[16] I thank A. I. Harte for discussions on this point, and drawing our attention to Ref. [110].

in the frame $U^i = 0$, also coincides with the canonical angular momentum obtained by differentiating the non-relativistic Lagrangian, $\partial\mathcal{L}/\partial\vec{\omega}$ (cf. Eq. (31) of [109]).

Having this issue clarified, one computes the physical torque — that is, the Fermi-Walker derivative of the physical angular momentum vector S^α,

$$\tau^\alpha \equiv \frac{D_F S^\alpha}{d\tau} \Rightarrow \tau^\sigma = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\sigma\delta}U_\delta \frac{DS^{\alpha\beta}}{d\tau} \ , \tag{5.95}$$

by subtracting the contribution of $S'^{\alpha\beta}$ from Eq. (5.91) above. It reads:

$$\tau^\sigma = \epsilon^\sigma{}_{\alpha\beta\nu}U^\nu \mu^\alpha B^\beta + \epsilon_{\alpha\beta}{}^{\sigma} m^{[\alpha}{}_{\rho\mu} F^{\beta]\mu;\rho} - \frac{1}{2}\epsilon^\sigma{}_{\alpha\beta}F^\alpha{}_\gamma \frac{Dq^{\beta\gamma}}{d\tau} + \tau^\sigma_{\text{ind}} \ , \tag{5.96}$$

where the first term is the dipole torque, cf. Eq. (5.62), and the next terms are quadrupole contributions; the second term is the one that vanishes for a spherical body (thus not the one we are interested in), and the fourth term is the result we were looking for,

$$\tau^\alpha_{\text{ind}} = \frac{1}{2}\epsilon^\sigma{}_{\mu\nu}E^{[\mu\nu]}\left[q^\alpha{}_\sigma - \delta^\alpha{}_\sigma q^\gamma{}_\gamma\right] \ , \tag{5.97}$$

the torque due to the laws of electromagnetic induction, governed by the antisymmetric part of the electric tidal tensor. Indeed in the Lorentz frame momentarily comoving with the particle, we can write

$$\tau^i_{\text{ind}} = -\frac{1}{2}(\nabla\times\vec{E}_{\text{CM}})^j\left[q^i{}_j - \delta^i{}_j q^\gamma{}_\gamma\right]$$

where $(\nabla\times\vec{E}_{\text{CM}})^j$ is the curl of the electric field at the particle's center of mass. This is precisely what one obtains computing explicitly the torque, to quadrupole order, from the integral $\vec{\tau}_{\text{ind}} = \int \rho_c \vec{r}\times\vec{E}_{\text{ind}}d^3x$, see Paper #4 [4] for details. Using Eqs. (5.6), we write (5.97) in the equivalent form

$$\tau^\alpha_{\text{ind}} = \frac{1}{2}B^\sigma{}_\beta U^\beta\left[q^\alpha{}_\sigma - \delta^\alpha{}_\sigma q^\gamma{}_\gamma\right] \ . \tag{5.98}$$

Now, if the spinning body is "quasi-rigid", we have

$$\mu^\alpha = \frac{\Omega^\beta}{2}\left(\delta^\alpha_\beta q^\gamma{}_\gamma - q^\alpha{}_\beta\right) \tag{5.99}$$

where Ω^α is the body's angular velocity relative to a system of comoving Fermi-Walker transported axes. Therefore, the rate of work done on this body by the induction torque τ^α_{ind}, $\mathcal{P} = \tau^\alpha_{\text{ind}}\Omega_\alpha$, is:

$$\tau^\alpha_{\text{ind}}\Omega_\alpha = -B^\alpha{}_\beta U^\beta \mu_\alpha = -F^\alpha_{\text{EM}}U_\alpha \tag{5.100}$$

i.e., it equals the time projection, in the particle's proper frame, of the dipole force F^α_{EM} (in other words, the work done by the dipole force, as measured in the particle's frame). This confirms that the work transferred to the particle by Faraday's induction, that we

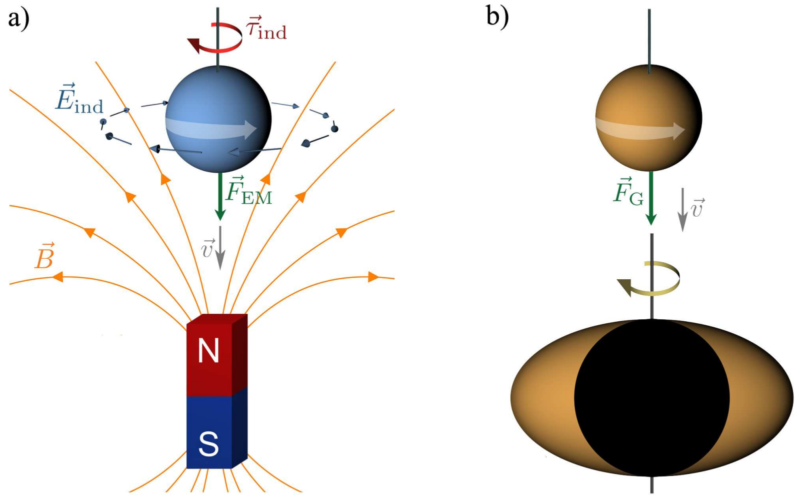


Figure 5.8: a) A spinning, positively charged spherical body being pulled by a strong magnet; $\vec{E}_{\rm ind} \equiv$ electric field induced in the body's CM frame. b) A spinning spherical body falling into a Kerr Black hole. As the spinning charge moves in the inhomogeneous magnetic field $\vec{B}$, a torque $\tau^{\alpha}_{\rm ind}$, Eq. (5.97), is exerted on it due to $\vec{E}_{\rm ind}$, i.e. to the antisymmetric part $E_{[\alpha\beta]}$. That causes $S \equiv \sqrt{S^{\alpha}S_{\alpha}}$, and the body's angular velocity $\Omega = S/I$ to vary. $\tau^{\alpha}_{\rm ind}$ does work at a rate $\tau^{\alpha}_{\rm ind}\Omega_{\alpha} = \mathcal{P}_{\rm ind}$, which exactly matches time projection of the dipole force $F^{\alpha}_{\rm EM}$. That causes the kinetic energy of rotation about the CM to decrease, manifest in a variation of proper mass $dm/d\tau$, and cancels out the gain in translational kinetic energy, so that the total work transfer is zero (cf. Sec. 5.5.3.2). In the gravitational case no analogous induction effects occur (as expected since $\mathbb{E}_{[\alpha\beta]} = \mathbb{H}_{\alpha\beta}U^{\beta} = 0$): no torque is exerted on the spinning particle, its angular momentum S, angular velocity Ω, and proper mass m, are constant; and there is a net work done on it by F^{α}_{G} at a rate $\mathcal{P}_{\rm tot} = -F^{\alpha}_{G}u_{\alpha}$, corresponding to an increase of translational kinetic energy.

discussed in Sec. 5.5.3.1, is indeed associated to a torque, which causes S^2 to vary as expected (since $\tau^{\alpha}_{\rm ind}$ is not orthogonal to S^{α} in general).

It is also associated with a variation of kinetic energy of rotation. In order to more easily see that, consider now a spherical body, so that the second term in (5.96) vanishes; and a configuration where the electromagnetic hidden momentum $\vec{P}_{\rm hidEM} = \vec{\mu} \times \vec{E}$ vanishes, for example the setup in Fig. 5.8a), which causes (see Paper #4 for details) the third term of (5.96) to vanish also; in this case the total torque reduces to

$$\tau^{\sigma} = \epsilon^{\sigma}{}_{\alpha\beta\nu} U^{\nu} \mu^{\alpha} B^{\beta} + \tau^{\sigma}_{\rm ind} \ . \tag{5.101}$$

Assuming $\mu^{\alpha} = \sigma S^{\alpha}$, and since $S^{\alpha} = I\Omega^{\alpha}$ ($I \equiv$moment of inertia of the sphere with respect to an axis passing through its center), it follows that the power of the total torque is $\tau^{\sigma}\Omega_{\sigma} = \tau^{\sigma}_{\rm ind}\Omega_{\sigma}$. Since also, from Eq. (5.95), $\tau^{\sigma} = ID_F\Omega^{\sigma}/d\tau$, we have that the power of $\tau^{\alpha}_{\rm ind}$ equals the rate of variation of the particle's rotational kinetic energy, $I\Omega^2/2$:

$$\tau^{\alpha}_{\rm ind}\Omega_{\alpha} = \tau^{\sigma}\Omega_{\sigma} = \frac{1}{2}\frac{d\Omega^2}{d\tau} I \ .$$

Thus we can write

$$\frac{1}{2} I \frac{d\Omega^2}{d\tau} = -F^{\alpha}_{\rm EM} U_{\alpha} = -\frac{DB^{\alpha}}{d\tau}\mu_{\alpha} \ ,$$

where in the last equality I used again $P^{\alpha}_{\rm hidEM} = 0$. That is, for this setup, the variation of the body's kinetic energy of rotation is the projection, along its worldline, of the dipole force F^{α}_{EM}. And finally, this result confirms also that (for a purely magnetic dipole, $d^{\alpha} = 0$) the variable part of the particle's mass that we obtained in the dipole approximation, Eq. (5.67), is *kinetic energy of rotation* (*not* potential energy, as claimed in some literature, e.g. [81, 76, 77]) — this confirms, in a relativistic covariant formulation, and in the context of Dixon's multipole approach, the claims in [107, 108, 109, 113].

5.5.4.2 Gravitational torque

The equation for the spin evolution of an extended spinning body in a gravitational field is, up to quadrupole order [106, 66],

$$\frac{DS^{\kappa\lambda}}{d\tau} = 2P^{[\kappa}U^{\lambda]} + \frac{4}{3} J^{\mu\nu\rho[\kappa} R^{\lambda]}{}_{\rho\mu\nu} \tag{5.102}$$

leading to

$$\tau^{\alpha} \equiv \frac{D_F S^{\sigma}}{d\tau} = \frac{4}{6} J^{\mu\nu\rho[\kappa} R^{\lambda]}{}_{\rho\mu\nu} \epsilon_{\kappa\lambda}{}^{\sigma\delta} U_{\delta} \ , \tag{5.103}$$

where $J^{\alpha\beta\gamma\delta}$ is a quadrupole moment of the energy-momentum tensor $T^{\alpha\beta}$, see Paper #4 for more details. Our goal herein is to consider the gravitational analogue of the problem in Sec. 5.5.4.1. Therein we considered a spherical charged body in flat spacetime; we prescribe the analogous test body for the gravitational problem by demanding it to have *an analogous multipole structure* (i.e., its "gravitational skeleton" [61]) to its electromagnetic counterpart (rather than demanding its shape to be "spherical", which in a general curved spacetime is not a well defined notion. More precisely: in a local orthonormal tetrad $\hat{e}_\alpha$, such that $\hat{e}_0 = U^\alpha$ (i.e., the triad $\hat{e}_i$ spans the rest space of the center of mass), the moments of J^α are the same as for a sphere in flat spacetime (thus have the same structure as the moments of j^α in the electromagnetic problem above). The moments of the space part $T^{\langle\alpha\rangle\langle\beta\rangle}$ are negligible to a good approximation. For this type of body we have (see Sec. VIB of Paper #4 [4] for details)

$$\tau^\sigma = 0 \ ,$$

the equality holding for vacuum ($R^{\mu\nu} = 0$), which (as in the electromagnetic case) is the problem at hand. Thus, no gravitational torque is exerted, up to quadrupole order, in a spinning spherical body. Therefore, as expected from the discussion in the previous sections, there is no gravitational counterpart to the torque $\tau^\alpha_{\rm ind}$ that comes from the antisymmetric part of the electric tidal tensor $E_{\alpha\beta}$ (or, equivalently, from the time projection $B_{\alpha\beta}U^\beta$), and which is due to Faraday's law of induction. The comparison with the electromagnetic analogue makes this result natural, since the gravito-electric tidal tensor $\mathbb{E}_{\alpha\beta}$ is symmetric, and the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}$ is spatial, meaning that the dynamical effects which, in electromagnetism, are caused by the curl of the electric field $\vec{E}$, have no counterpart in the *physical* gravitational forces and torques.

5.5.4.3 Summarizing with a simple realization

The main ideas in Secs. 5.5.3 and 5.5.4 can be summarized in the example of analogous systems in Fig. 5.8: a spinning spherical charge moving in the field of a strong magnet (or another spinning charged body), and a spinning "spherical" mass moving in Kerr spacetime. Starting by the electromagnetic system, a force $F^\alpha_{\rm EM}$, Eq. (5.2.1a) of Table 5.2, will be exerted on the particle, causing it to move and gain translational kinetic energy. And as it moves in an inhomogeneous magnetic field, a torque $\tau^\alpha_{\rm ind}$ is exerted upon it, due, from the the viewpoint of the observer comoving with the particle, to the electric field induced by the time-varying magnetic field. That torque will cause a variation of angular momentum S^α, and therefore of the angular velocity $\Omega^\alpha = S^\alpha/I$ of the particle (measured with respect to the comoving Fermi-Walker transported tetrad). Clearly, as we see from Eq. (5.101), S^2 *is not conserved* (as would be the case in a pole-dipole approximation, see Eq. (5.62)). The variation of the magnitude Ω of the angular velocity also implies a variation of rotational kinetic energy of the particle; that variation is the projection of $F^\alpha_{\rm EM}$ along the particle's worldline, and is reflected in a variation of proper mass $dm/d\tau$. With respect to the "static observers" u^α, the variation of rotational kinetic energy is exactly canceled out by the variation of translational kinetic energy, ensuring that a static magnetic field does not do work, so that $F^\alpha_{\rm EM}u_\alpha = 0$, and the total energy of the particle, $E = -P_\alpha u^\alpha$, is conserved.

In the gravitational case, there is also a net force $F^\alpha_{\rm G}$ on the body, cf. Eq. (5.2.1b) of Table 5.2, causing it to gain kinetic energy at a rate $\mathcal{P}_{\rm trans} = F^i_{\rm G}v_i$. But no torque is exerted on it; up to quadrupole order we have:

$$\frac{D_F S^\alpha}{d\tau} = 0; \quad S^2 = \text{constant}$$

(i.e., the spin vector of the spinning spherical mass is Fermi-Walker transported), implying also $\Omega =$ constant. This is consistent with the constancy of the proper mass (and the fact that $F_{\rm G}^{\alpha}$ is spatial, meaning that no work is done by induction), because, since there is no torque, the kinetic energy of rotation is constant. Thus in this case, from the point of view of the static observers, the gain in translational kinetic energy is not canceled out by any variation of rotational kinetic energy, and therefore the stationary gravitomagnetic (tidal) field does a net work on the particle.

5.5.5 Conclusion

In this work we explored the exact gravito-electromagnetic analogies in the equations of motion for spinning particles in gravitational and electromagnetic fields, that were seen to arise when the Mathisson-Pirani spin condition is employed. In special detail we explored the analogy based on tidal tensors for the force equations, that was put forth in Paper #1 [1]. We also studied the analogy for the spin precession, based on GEM fields, and, also based on it, found a new one, for the particle's hidden momentum.

A point that it is never too much to emphasize is that the existence of these *exact* analogies does *not* mean that the interactions are similar. These are *functional* analogies: $B_{\alpha\gamma}$ plays in the equation (5.2.1a) for the force exerted on a magnetic dipole the same role as $\mathbb{H}_{\alpha\gamma}$ in Eq. (5.2.1b) for the gravitational force exerted on a gyroscope; also, in the appropriate frame, the gravitomagnetic field $\vec{H}$ plays in the "precession" of the gyroscope a role analogous to $\vec{B}$ in the precession of a magnetic dipole, cf. Eq. (5.64). But the analogies *do not imply* that these objects themselves are similar, in fact they are in general very different even in seemingly analogous setups (we give in this work many examples of that). The exact analogies are suited instead for a comparison between the interactions, as it amounts to comparing mathematical objects that play analogous dynamical roles in both theories. Such comparison unveils suggestive similarities, useful in terms of the intuition they provide. But, and especially in the case of the tidal tensor analogy, it was the differences it makes transparent that proved particularly illuminating.

We had found in Paper #1 [1] that the key differences, in terms of tidal forces, between gravity and electromagnetism, are the fact that $\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ are symmetric (the latter in vacuum) whereas their electromagnetic counterparts $E_{\alpha\beta}$ and $B_{\alpha\beta}$ are not. These differences were seen to be related with the phenomenon of electromagnetic induction, and the way *it manifests itself in the electromagnetic tidal forces*, which has no analogue in gravity. In this work we explored the physical consequences for the dynamics of test particles. The results in Sec. 5.5.3, concerning the time components of the force, and in Sec. 5.5.4, concerning the torque exerted on the spinning particle, are manifestations of the absence of a gravitational counterpart to the antisymmetric part of $E_{\alpha\beta}$ (or, equivalently, to the projection of $B_{\alpha\beta}$ along U^{α}); $E_{[\alpha\beta]}$ encodes the Maxwell-Faraday law $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$; the gravitoelectric tidal tensor, by contrast, is symmetric: $\mathbb{E}_{[\alpha\beta]} = 0$, translating in an absence of analogous induction effects in the *physical* gravitational forces and torques. And the results in Sec. 5.5.2, showing that in a non-homogeneous gravitational field there are moving observers for which $\mathbb{H}_{\alpha\beta} = 0$, so that gyroscopes can actually move along radial or circular geodesics (in Schwarzschild and Kerr-dS spacetimes, respectively), manifest that there is no gravitational analogue the antisymmetric part $B_{[\alpha\beta]}$, encoding Maxwell Eq. $\nabla \times \vec{B} = \partial \vec{E}/\partial t$. In electromagnetic systems, due to this law (more precisely, in covariant form $2B_{[\alpha\beta]} = \star F_{\alpha\beta;\gamma}U^{\gamma}$), $B_{\alpha\beta}$ is non-vanishing whenever the dipole "sees" a varying field (as is the case when the particle moves in a non-homogeneous field), and therefore (except for some special orientations of $\vec{\mu}$) an electromagnetic force $F^{\alpha}_{\rm EM} \neq 0$ is exerted on it.

We have studied in detail the work done by the fields on the particle from the point of view of different frames, which is an important physical content encoded in the time projections of the forces; and its relation with the particle's energy (in the given frame) and proper mass. For that we needed to generalize the power law existing in the literature, extending it to the case of particles with multipole structure (which possess hidden momentum and varying mass). An interesting reciprocity was found to exist: in a frame comoving with the particle, the electromagnetic (but not the gravitational) field does work on it, causing a variation of its proper mass; conversely, for "static observers", a stationary gravitomagnetic (but not a magnetic) field does work on the particle. We shown that there is actually a potential energy associated with this work which embodies the Hawking-Wald spin-spin interaction energy [10] (that had been found to exist in the special case of an axial fall in a Kerr black hole).

In the course of this work a number of issues had to be clarified, the first of them being the equations of motion themselves, both to dipole and quadrupole order, and the physical meaning of the quantities involved. In particular the misconceptions in the literature regarding the problem of the spin supplementary condition, and the difficulties in the electromagnetic part of the equations. Some of this problems are briefly reviewed in Sec. 3.2 above (for a more comprehensive summary I refer the reader to the conclusion of Paper #4).

5.6 Paper #5 — Gravito-electromagnetic analogies

This work has two main goals: 1) establish the connection between the several gravito-electromagnetic analogies existing in the literature, and in particular between the tidal tensors and the exact GEM fields; 2) further develop these two approaches.

As for the approach based on tidal tensors, we complete the tidal tensor formulation of the gravitational field equations started in Paper #1 [1]. Using the time and space projectors, we do a full splitting of the gravitational field equations (the Einstein equations with sources, plus the algebraic Bianchi identities), obtaining six equations (5 independent), four of which are the ones first derived in [1], which are analogous to Maxwell's equations in this formalism, and two additional ones with no electromagnetic counterpart that are not given in [1]. And we add to the list of analogies in this formalism the one we found to exist in the "differential precession" of gyroscopes/magnetic dipoles.

As for the analogy based on exact GEM fields, we take its most general form in the literature [25], valid for arbitrary fields, reformulate and further generalize it for arbitrary frames. We discuss in detail the inertial forces that arise in the different frames, and the origin of the GEM fields; and we derive a general expression for them which generalizes the previous results in terms of an arbitrary transport law for the spatial frame. The gravitomagnetic field (i.e., the field that yields the Coriolis-like "acceleration" in the geodesic equation) is seen to consist of two contributions of *independent* origin: the vorticity of the observer congruence, and the rotation of the spatial triads relative to Fermi-Walker transport. This definition encompasses the many gravitomagnetic fields that have been defined in the literature. As for the field equations, we do, again, a full splitting of the gravitational and electromagnetic equations, and express them in this formalism. It turns out that from the set of six gravitational equations, four are seen to exhibit many similarities with the electromagnetic equations (that is, it is not only in the tidal tensor formalism; the similarity occurs in this formalism as well). Restricting the approach to stationary fields, we obtain on the gravitational side the "quasi-Maxwell" field equations of Sec. 3.1.2 above; and in the electromagnetic side, the equations for the electric and magnetic fields in the analogous situation: for arbitrarily accelerated and rotating frames (not in Lorentz frames, as is the usual comparison in the literature), which unveils a much closer analogy. We also build up on the work in [19] — where an analogy was found between the gravitational force on a gyroscope written in terms of GEM fields, Eq. (3.13), and the electromagnetic force on a magnetic dipole at rest in a Lorentz frame, Eq. (3.14) — by adding the corresponding electromagnetic force in the analogous conditions, i.e., in terms of the fields measured in the arbitrarily accelerating and rotating frame where the particle is at rest. Again the analogy is seen to get strikingly closer.

5.6.1 Analogy based on tidal tensors

In Paper #5 [5] we revisited the analogy based on tidal tensors introduced in Paper #1 [1], and completed the tidal tensor formulation of the gravitational field equations. In [1] it was shown that, by taking the traces and antisymmetric parts of the electromagnetic tidal tensors, one obtains the Maxwell equations, and performing the same operations on the gravitational tidal tensors leads to a strikingly similar set of equations, which turn out to be some projections of the gravitational field equations.

In [5], using a more robust approach, we extend this formalism to the full gravitational field equations — Einstein equations with sources plus the algebraic Bianchi identities,

$$R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T^{\gamma}_{\ \gamma}\right) \quad (a); \qquad \star R^{\gamma\alpha}_{\ \ \ \gamma\beta} = 0 \quad (b)\ , \tag{5.104}$$

which I summarize in what follows. Using the time and space projections with respect to a unit time-like vector U^{α},

$$\top^{\alpha}_{\ \beta} \equiv (\top^{U})^{\alpha}_{\ \beta} = -U^{\alpha}U_{\beta}; \qquad h^{\alpha}_{\ \beta} \equiv (h^{U})^{\alpha}_{\ \beta} = U^{\alpha}U_{\beta} + \delta^{\alpha}_{\ \beta}. \tag{5.105}$$

we first do a full decomposition of the Riemann tensor

$$R^{\alpha\beta\gamma\delta} = \left(\top^{\alpha}_{\ \rho} + h^{\alpha}_{\ \rho}\right)...\left(\top^{\delta}_{\ \sigma} + h^{\delta}_{\ \sigma}\right)R^{\rho..\sigma}\ ,$$

obtaining[17]

$$\begin{aligned} R^{\alpha\beta}{}_{\gamma\delta} &= 4\mathbb{E}^{[\alpha}{}_{[\gamma}U_{\delta]}U^{\beta]} + 2\left\{\epsilon^{\mu\chi}{}_{\gamma\delta}U_{\chi}\mathbb{H}_{\mu}{}^{[\beta}U^{\alpha]} + \epsilon^{\mu\alpha\beta\chi}U_{\chi}\mathbb{H}_{\mu[\delta}U_{\gamma]}\right\} \\ &\quad + \epsilon^{\alpha\beta\phi\psi}U_{\psi}\epsilon^{\mu\nu}{}_{\gamma\delta}U_{\nu}\mathbb{F}_{\phi\mu}. \end{aligned} \quad (5.106)$$

This equation tells us that the Riemann tensor decomposes, with respect to U^{α}, in three *spatial* tensors: the gravitoelectric tidal tensor $\mathbb{E}_{\alpha\beta}$, the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}$, plus a third tensor

$$\mathbb{F}_{\alpha\beta} \equiv \star R \star_{\alpha\gamma\beta\delta} U^{\gamma}U^{\delta} = \epsilon^{\mu\nu}{}_{\alpha\gamma}\epsilon^{\lambda\tau}{}_{\beta\delta}R_{\mu\nu\lambda\tau}U^{\gamma}U^{\delta} \ ,$$

introduced by Bel [138], which encodes the purely spatial curvature with respect to U^{α}, and has no electromagnetic analogue. $\mathbb{E}_{\alpha\beta}$ and $\mathbb{F}_{\alpha\beta}$ are symmetric (and spatial), and therefore have 6 independent components each; $\mathbb{H}_{\alpha\beta}$, is traceless (and spatial), and so has 8 independent components. Therefore these three tensors together encode the 20 independent components of the Riemann tensor.

Substituting decomposition (5.106) and its Hodge dual in Eqs. (5.104a), and decomposing in time and space projections, we obtain, respectively, the time-time, time-space, and space-space projections:

$$\mathbb{E}^{\alpha}{}_{\alpha} = 4\pi\left(2\rho_m + T^{\alpha}{}_{\alpha}\right) \ ; \quad (5.107)$$

$$\mathbb{H}_{[\sigma\tau]} = -4\pi\epsilon_{\lambda\sigma\tau\gamma}J^{\lambda}U^{\gamma} \, ; \quad (5.108)$$

$$\mathbb{F}^{\alpha}{}_{\beta} + \mathbb{E}^{\alpha}{}_{\beta} - \mathbb{F}^{\sigma}{}_{\sigma}h^{\alpha}{}_{\beta} = 8\pi\left[\frac{1}{2}T^{\gamma}{}_{\gamma}h^{\alpha}{}_{\beta} - T^{\langle\alpha\rangle}{}_{\langle\beta\rangle}\right] \quad (5.109)$$

(since Eq. (5.104a) is symmetric, these are the only non-trivial projections). Here $J^{\alpha} \equiv -T^{\alpha\beta}U_{\beta}$, $\rho_m \equiv T^{\alpha\beta}U_{\beta}U_{\alpha}$ are, respectively, the mass/energy current and density, as measured by an observer of 4-velocity U^{α}; and $T_{\langle\theta\rangle}^{\langle\lambda\rangle} \equiv h^{\lambda}{}_{\delta}h^{\beta}{}_{\theta}T_{\beta}{}^{\delta}$.

Repeating the procedure in Eqs. (5.104b), we obtain the time-time (which is the same as the space-space), time-space, and space-time projections, respectively:

$$\mathbb{H}^{\alpha}{}_{\alpha} = 0; \quad \text{(a)} \qquad \mathbb{F}_{[\alpha\beta]} = 0; \quad \text{(b)} \qquad \mathbb{E}_{[\alpha\beta]} = 0 \quad \text{(c)}. \quad (5.110)$$

Turning now to the electromagnetic field equations (source equations, plus Bianchi identity),

$$F^{\alpha\beta}{}_{;\beta} = 4\pi j^{\alpha} \quad (a); \qquad \star F^{\alpha\beta}{}_{;\beta} = 0 \quad (b), \quad (5.111)$$

we decompose $F_{\alpha\beta;\gamma}$ and its dual in terms of the electromagnetic tidal tensors,

$$F_{\alpha\beta;\gamma} = 2U_{[\alpha}E_{\beta]\gamma} + \epsilon_{\alpha\beta\mu\sigma}U^{\sigma}B^{\mu}{}_{\gamma} \ ; \quad (5.112)$$

$$\star F_{\alpha\beta;\gamma} = 2U_{[\alpha}B_{\beta]\gamma} - \epsilon_{\alpha\beta\mu\sigma}U^{\sigma}E^{\mu}{}_{\gamma} \ . \quad (5.113)$$

[17] The characterization of the Riemann tensor by these three spatial rank 2 tensors is known as the "Bel decomposition", even though the explicit decomposition (5.106) is not presented in any of Bel's papers (e.g. [138]). To the author's knowledge, an equivalent expression (Eq. (4.6) therein) can only be found in [144].

Table 5.3: Tidal tensor formulation of the electromagnetic and gravitational field equations.

Electromagnetism		Gravity	
Maxwell Source Equations $F^{\alpha\beta}{}_{;\beta} = 4\pi J^{\beta}$		Einstein Equations $R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\alpha}{}_{\alpha}\right)$	
• Time Projection: $E^{\alpha}{}_{\alpha} = 4\pi\rho_c$	(5.3.3a)	• Time-Time Projection: $\mathbb{E}^{\alpha}{}_{\alpha} = 4\pi\left(2\rho + T^{\alpha}{}_{\alpha}\right)$	(5.3.3b)
• Space Projection: $B_{[\alpha\beta]} = \frac{1}{2}\star F_{\alpha\beta;\gamma}U^{\gamma} - 2\pi\epsilon_{\alpha\beta\sigma\gamma}j^{\sigma}U^{\gamma}$	(5.3.6a)	• Time-Space Projection: $\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma}J^{\sigma}U^{\gamma}$	(5.3.6b)
No electromagnetic analogue		• Space-Space Projection: $\mathbb{F}^{\alpha}{}_{\beta} + \mathbb{E}^{\alpha}{}_{\beta} - \mathbb{F}^{\sigma}{}_{\sigma}h^{\alpha}{}_{\beta} = 8\pi\left[\frac{1}{2}T^{\gamma}{}_{\gamma}h^{\alpha}{}_{\beta} - T^{\langle\alpha\rangle}{}_{\langle\beta\rangle}\right]$	(5.3.7)
Bianchi Identity $\star F^{\alpha\beta}{}_{;\beta} = 0 \quad (\Leftrightarrow F_{[\alpha\beta;\gamma]} = 0\ )$		Algebraic Bianchi Identity $\star R^{\gamma\alpha}{}_{\gamma\beta} = 0 \quad (\Leftrightarrow R_{[\alpha\beta\gamma]\delta} = 0)$	
• Time Projection: $B^{\alpha}{}_{\alpha} = 0$	(5.3.5a)	• Time-Time (or Space-Space) Proj: $\mathbb{H}^{\alpha}{}_{\alpha} = 0$	(5.3.5b)
• Space Projection: $E_{[\alpha\beta]} = \frac{1}{2}F_{\alpha\beta;\gamma}U^{\gamma}$	(5.3.4a)	• Space-Time Projection: $\mathbb{E}_{[\alpha\beta]} = 0$	(5.3.4b)
No electromagnetic analogue		• Time-Space Projection: $\mathbb{F}_{[\alpha\beta]} = 0$	

Then, substituting in Eqs. (5.111), and splitting in the time and space projections, we obtain the set of four electromagnetic equations already presented in Paper #1 [1], Eqs. (5.1.3a)-(5.1.6a) of Table 5.1, and that we summarize again in Table 5.3. That is, the time and space projections of (5.111a) are, respectively Eqs. (5.3.3a) and (5.3.6a) of Table 5.3. The same procedure applied to Eq. (5.111b) yields Eqs. (5.3.4a) and (5.3.5a) as the time and space projections, respectively. We re-write them in the form

$$E^{\alpha}{}_{\alpha} = 4\pi\rho_c\ ; \tag{5.114}$$

$$E_{[\alpha\beta]} = U_{[\alpha}E_{\beta]\gamma}U^{\gamma} + \frac{1}{2}\epsilon_{\alpha\beta\mu\sigma}U^{\sigma}B^{\mu\gamma}U_{\gamma}\ ; \tag{5.115}$$

$$B^{\alpha}{}_{\alpha} = 0\ ; \tag{5.116}$$

$$B_{[\alpha\beta]} = U_{[\alpha}B_{\beta]\gamma}U^{\gamma} - \frac{1}{2}\epsilon_{\alpha\beta\mu\sigma}U^{\sigma}E^{\mu\gamma}U_{\gamma} - 2\pi\epsilon_{\alpha\beta\sigma\gamma}j^{\sigma}U^{\gamma}\ , \tag{5.117}$$

to note that indeed Maxwell's equations may be cast as algebraic equations involving only the two tidal tensors and the sources.

Thus, as illustrated in Table 5.3, the gravitational field equations consist of four equations with an electromagnetic analogue, plus two equations — the space-space projection of (5.104a), Eq. (5.109), and the time-space projection of (5.104b), Eq. (5.110b), which have no electromagnetic analogue. However this is not a set of six independent equations, as Eqs. (5.110b), (5.110c) and (5.109) are not independent; using the latter, together with (5.110b)/(5.110c), one can obtain the remaining one, (5.110c)/(5.110b).

Eq. (5.109) involves, as a source, the space-space part of the energy momentum tensor, $T^{\langle\alpha\rangle\langle\beta\rangle}$, which, unlike the energy current 4-vector $J^\alpha = -T^{\alpha\beta}U_\beta$ (analogous to the charge current 4-vector j^α) has no electromagnetic counterpart. It has a fundamental difference[18] with respect to the other gravitational field equations in Table 5.3 (and their electromagnetic analogues): the latter are algebraic equations involving only the traces and antisymmetric parts of the tidal tensors (or of $\mathbb{F}_{\alpha\beta}$), plus the source terms; they impose no condition on the symmetric parts. But Eq. (5.109), by contrast, is an equation for the symmetric parts of the tensors $\mathbb{E}_{\alpha\beta}$ and $\mathbb{F}_{\alpha\beta}$. It can be split in two parts. Taking the trace, and using (5.107), one obtains the source equation for $\mathbb{F}_{\alpha\beta}$:

$$\mathbb{F}^\sigma_{\ \sigma} = 8\pi\rho \ ; \tag{5.118}$$

substituting back in (5.109) we get:

$$\mathbb{F}^\alpha_{\ \beta} + \mathbb{E}^\alpha_{\ \beta} = 8\pi\left[h^\alpha_{\ \beta}\left(\frac{1}{2}T^\gamma_{\ \gamma} + \rho\right) - T^{\langle\alpha\rangle}_{\ \langle\beta\rangle}\right]. \tag{5.119}$$

This equation tells us that the tensor $\mathbb{F}^\alpha_{\ \beta}$ is not an extra (comparing with electrodynamics) *independent* object; given the sources and the gravitoelectric tidal tensor $\mathbb{E}_{\alpha\beta}$, $\mathbb{F}_{\alpha\beta}$ is completely determined by (5.119).

In vacuum ($T^{\alpha\beta} = 0$, $j^\alpha = 0$), the Riemann tensor becomes the Weyl tensor: $R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}$; due to the self duality property of the latter: $C_{\alpha\beta\gamma\delta} = -\star C\star_{\alpha\beta\gamma\delta}$, it follows that $\mathbb{F}_{\alpha\beta} = -\mathbb{E}_{\alpha\beta}$.

The gravitational field equations are summarized and contrasted with their electromagnetic counterparts in Table 5.3. Eqs. (5.3.3b)-(5.3.6b) are very similar in form to Maxwell Eqs. (5.3.3b)-(5.3.6b); they are their *physical* gravitational analogues, since both are the traces and antisymmetric parts of tensors $\{E_{\alpha\beta},\ B_{\alpha\beta}\} \leftrightarrow \{\mathbb{E}_{\alpha\beta},\ \mathbb{H}_{\alpha\beta}\}$, which we know, from equations (5.1.1) and (5.1.2) of Table 5.1, to play analogous physical roles in the two theories. Note this interesting aspect of the analogy: if one replaces, in Eqs. (5.114)-(5.117), the electromagnetic tidal tensors ($E_{\alpha\beta}$ and $B_{\alpha\beta}$) by the gravitational ones ($\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$), and the charges by masses (i.e., charge density ρ_c and current j^α, by mass/energy density ρ and current J^α), one *almost* obtains Eqs. (5.3.3b)-(5.3.6b), apart from a factor of 2 in the source term in (5.3.6b) and the difference in the source of Eq. (5.3.3b), signaling that in gravity pressure and stresses contribute as sources. This happens because, since $\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ are spatial tensors, all the contractions with U^α present in Eqs. (5.115) and (5.117) vanish.

[18]We thank João Penedones for drawing our attention to this point.

5.6.1.1 The analogy for differential precession

In Paper #1 [1] we gave a physical interpretation for the tensors $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ as the tensors which, when contracted with a magnetic/gravitomagnetic dipole vector, yield the force exerted on magnetic dipoles/gyroscopes. In Paper #5 we unveil a new exact analogy (again relying on the Mathisson-Pirani spin condition) relating these two tensors, which are cast as tensors of "relative", or "differential", precession for spinning particles; i.e., tensors that, when contracted with a separation vector δx^{β}, yield the angular velocity of precession of a spinning particle at given point $\mathcal{P}_2$ relative to a system of axes anchored to spinning particles, with *the same 4-velocity* (and gyromagnetic ratio σ, if an electromagnetic field is present), at the *infinitesimally close* point $\mathcal{P}_1$. (This is somewhat analogous to the electric tidal tensors $E_{\alpha\beta}$ and $\mathbb{E}_{\alpha\beta}$, which, when contracted with δx^{β}, yield the relative acceleration of two infinitesimally close test particles with the same 4-velocity).

On the gravitational side, the equation yielding the precession of a gyroscope at $\mathcal{P}_2$, moving with 4-velocity U^{α}, with respect to a system of axes locked to comoving (i.e., moving with the same U^{α}) guiding gyroscopes at $\mathcal{P}_1$, is

$$\frac{D\vec{S}}{d\tau} = \delta\vec{\Omega}_{\rm G} \times \vec{S} \ ; \qquad \delta\Omega^{i}_{\rm G} \equiv \mathbb{H}^{i}{}_{l}\delta x^{l} \ , \tag{5.120}$$

where $\mathbb{H}_{\alpha\beta} = R_{\alpha\mu\beta\nu}U^{\mu}U^{\nu}$. See Paper #5 [5] for details. This is not an original result: it was was obtained first in a recent work [142], but through a derivation that, from our point of view, is not satisfactory, and makes an assumption — that the gyroscopes at $\mathcal{P}_1$ and $\mathcal{P}_2$ have the same acceleration — that turns out to be unnecessary; they only need to be momentarily comoving (i.e., have the same 4-velocity) in order for (5.120) to hold. In Sec. 2.3 of Paper #5 we proposed a new, more straightforward derivation of the same result. In order to find the precession of a gyroscope at $\mathcal{P}_2$ relative to the comoving system of gyroscopes at $\mathcal{P}_1$, one just has to setup a system of Fermi coordinates (see e.g. [129, 145]) with origin along the worldline of the system of guiding gyroscopes passing trough the location $\mathcal{P}_1$, and compute the angular velocity of rotation (let me denote it by $-\delta\vec{\Omega}$) of *its* basis vectors $\mathbf{e}_{\alpha}$ *at* $\mathcal{P}_2$, *relative to the comoving gyroscopes therein* (i.e., relative to the Fermi-Walker transported tetrad of the momentarily comoving worldline passing through $\mathcal{P}_2$). This angular velocity is given by the connection coefficient Γ^{i}_{0j} of this frame at ($\mathcal{P}_2$), which is well known to be $\Gamma^{i}_{0j}(\mathcal{P}_2) = R^{i}{}_{jk0}\delta x^{k}$, e.g. [145]. Then the angular velocity of precession of a gyroscope at $\mathcal{P}_2$ is simply *minus* the former, i.e., $\delta\vec{\Omega}$.

It is interesting that, considering the electromagnetic analogue, the exact physical analogy $\mathbb{H}_{\alpha\beta} \leftrightarrow B_{\alpha\beta}$ emerges again. The precession of a magnetic dipole at $\mathcal{P}_2$, with 4-velocity U^{α}, with respect to a system of axis fixed to magnetic dipoles (with the same gyromagnetic ratio σ) at $\mathcal{P}_1$ and comoving with the same U^{α}, is given by

$$\frac{D\vec{S}}{d\tau} = \delta\vec{\Omega}_{\rm EM} \times \vec{S} \ ; \qquad \delta\Omega^{i}_{\rm EM} = -\sigma B^{i}{}_{\gamma}\delta x^{\gamma} \ . \tag{5.121}$$

5.6.2 The exact analogy based on GEM fields

In our approach one starts with an arbitrary orthonormal reference frame, which can be thought as consisting of a congruence of observers, each of them carrying an orthonormal tetrad whose time axis is the observer's 4-velocity (i.e. the tangent to the congruence) and the spatial triads spanning the local rest spaces of the observers. We choose the frame orthonormal because the connection coefficients associated to it are very simply related with the inertial fields. Along the observer's worldline, the variation of the basis vectors $\mathbf{e}_{\hat{\alpha}}$ is given by the rotation matrix $\Omega_{\alpha\beta}$:

$$\nabla_{\mathbf{u}}\mathbf{e}_{\hat{\beta}} = \Omega^{\hat{\alpha}}_{\ \hat{\beta}}\mathbf{e}_{\hat{\alpha}}; \quad \Omega^{\alpha\beta} = 2u^{[\alpha}a^{\beta]} + \epsilon^{\alpha\beta}_{\ \ \mu\nu}\Omega^{\mu}u^{\nu} \ , \tag{5.122}$$

which encodes the observer's acceleration a^{α}, and the angular velocity of rotation Ω^{α} of the spatial triad $\mathbf{e}_{\hat{i}}$ relative to Fermi-Walker transport. Since

$$\nabla_{\mathbf{e}_{\hat{\beta}}}\mathbf{e}_{\hat{\gamma}} = \Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}}\mathbf{e}_{\hat{\alpha}}$$

and $\mathbf{u} = \mathbf{e}_{\hat{0}}$, this defines the connection coefficients

$$\Gamma^{\hat{i}}_{\hat{0}\hat{0}} = a^{\hat{i}} = \Gamma^{\hat{0}}_{\hat{0}\hat{i}} \ ; \qquad \Gamma^{\hat{i}}_{\hat{0}\hat{j}} \ = \ \epsilon_{\hat{i}\hat{k}\hat{j}}\Omega^{\hat{k}} \ . \tag{5.123}$$

The coefficients $\Gamma^{\hat{0}}_{\hat{\alpha}\hat{0}}$ are trivially zero:

$$\Gamma^{\hat{0}}_{\hat{\alpha}\hat{0}} = -\mathbf{e}_{\hat{0}} \cdot \nabla_{\mathbf{e}_{\hat{\alpha}}}\mathbf{e}_{\hat{0}} = -\frac{1}{2}\nabla_{\mathbf{e}_{\hat{\alpha}}}(\mathbf{e}_{\hat{0}} \cdot \mathbf{e}_{\hat{0}}) = 0 \ ,$$

and the remaining mixed time-space connection coefficients encode kinematical quantities of the observer congruence:

$$\Gamma^{\hat{0}}_{\hat{j}\hat{i}} = \nabla_{\hat{j}}u_{\hat{i}} \equiv K_{\hat{i}\hat{j}} = \Gamma^{\hat{i}}_{\hat{j}\hat{0}}. \tag{5.124}$$

where the tensor $K^{\alpha\beta} \equiv (h^u)^{\alpha}_{\ \lambda}(h^u)^{\beta}_{\ \tau}u^{\lambda;\tau}$ decomposes in the the vorticity $\omega_{\alpha\beta} = K_{[\alpha\beta]}$, and the shear/expansion tensor $K_{(\alpha\beta)}$.

The kinematics of the congruence may be written as

$$u_{\alpha;\beta} = -a(u)_{\alpha}u_{\beta} - \epsilon_{\alpha\beta\gamma\delta}\omega^{\gamma}u^{\delta} + K_{(\alpha\beta)}$$

where ω^{α} is the vorticity vector

$$\omega^{\alpha} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}u_{\gamma;\beta}u^{\delta} = -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\omega_{\alpha\beta}u^{\delta}. \tag{5.125}$$

It is useful to split $K_{(\alpha\beta)}$ in the traceless shear $\sigma_{\alpha\beta}$ and expansion $\theta \equiv u^{\alpha}_{\ ;\alpha}$:

$$K_{(\alpha\beta)} = \sigma_{\alpha\beta} + \frac{1}{3}(h^u)_{\alpha\beta}\theta \ .$$

The transport law for the spatial triads $\mathbf{e}_{\hat{i}}$ along the congruence is up until now still arbitrary; it is generically independent of the congruence and its kinematics. But some preferred choices exist. Let $Y^\alpha = (h^u)^\alpha_{\ \beta} X^\beta$ be the projection orthogonal to the congruence of a vector X^α connecting two neighboring observers. The condition that X^α is a connecting vector is the vanishing of its Lie derivative along u^α: $\mathcal{L}_{\mathbf{u}} X^\alpha = 0$. The evolution of Y^α, in tetrad components, is given by (see Paper #5 [5] for details)

$$\dot{Y}_{\hat{i}} = \left(\sigma_{\hat{i}\hat{j}} + \frac{1}{3}\theta\delta_{\hat{i}\hat{j}} + \omega_{\hat{i}\hat{j}} - \Omega_{\hat{i}\hat{j}} \right) Y^{\hat{j}}, \tag{5.126}$$

where dot denotes the ordinary derivative along $\mathbf{u}$: $\dot{A}_{\hat{\alpha}} \equiv A_{\hat{\alpha},\hat{\beta}} u^{\hat{\beta}}$. Eq. (5.126) tells us that for a shear-free congruence ($\sigma_{\hat{i}\hat{j}} = 0$), if we lock the rotation $\vec{\Omega}$ of the tetrad to the vorticity $\vec{\omega}$ of the congruence, $\Omega_{\hat{i}\hat{j}} = \omega_{\hat{i}\hat{j}}$, the connecting vector's direction is fixed on the tetrad (and if in addition $\theta = 0$, i.e., a rigid congruence, the connecting vectors have constant components on the tetrad). This is a very important result in the context of this work, as it tells us how to setup (in non-shearing spacetimes) frames whose local spatial axes $\mathbf{e}_{\hat{i}}$ are all locked one to another. A familiar example is the rigidly rotating frame in flat spacetime; by choosing[19] $\vec{\Omega} = \vec{\omega}$, one is demanding that the spatial triads $\mathbf{e}_{\hat{i}}$ carried by the observers co-rotate with the congruence; hence it is clear that the axes $\mathbf{e}_{\hat{i}}$ always point to the same neighboring observers. Another example, of central importance in the context of gravitomagnetism, is the so-called frame of the distant stars, relative to which the gyroscope "precession" is measured; this is discussed in Sec. 5.6.2.2 below.

The choice $\vec{\Omega} = \vec{\omega}$ is argued in [136, 134] to be the most natural generalization of the non-relativistic concept of reference frame; we dub the frames verifying this transport law "*congruence adapted frames*". But other choices are possible. An also natural one would be $\vec{\Omega} = 0$, i.e., the triad $\mathbf{e}_{\hat{i}}$ does not rotate relative to Fermi-Walker transport along the congruence (which means that it is fixed relative to local guiding gyroscopes, as discussed in Sec. 5.6.2.2 below). Another usual choice corresponds to the frames sometimes employed in the in the context of black hole physics and astrophysics [140, 132, 133]: the tetrads carried by hypersurface orthogonal observers, whose spatial axis are taken to be fixed to the background symmetries; for instance, in the Kerr spacetime, the congruence are the zero angular momentum observers (ZAMOS, see e.g. [73, 140]), and the spatial triads are fixed to the Boyer-Lindquist spatial coordinate basis vectors. This tetrad field has been dubbed in some literature "locally non-rotating frames" [133, 132] (somewhat erroneously, as such tetrads *do rotate* with respect to the local compass of inertia, since they are not Fermi-Walker transported in general) or "proper reference frames of the fiducial observers" [140]. It is regarded as important for black hole physics because it is a reference frame that is defined everywhere (unlike for instance the star fixed static observers $u^\alpha \propto \partial/\partial t$ of Kerr spacetime, that do not exist past the ergosphere).

[19] Note that for relativistic rotation, the vorticity $\vec{\omega}$ is not constant, and not equal to the (constant) angular velocity of the rigidly rotating observers; but it is the condition $\vec{\Omega} = \vec{\omega}$ (not $\vec{\Omega}$ equal to the angular velocity) that that ensures that the tetrads are rigidly anchored to the observer congruence.

5.6.2.1 Inertial forces — "gravitoelectromagnetic (GEM) fields"

The spatial part of the geodesic equation for a test particle of 4-velocity U^α, $\nabla_{\mathbf{U}} U^\alpha \equiv DU^\alpha/d\tau = 0$, reads, in the frame $e_{\hat{\alpha}}$:

$$\frac{dU^{\hat{i}}}{d\tau} + \Gamma^{\hat{i}}_{\hat{0}\hat{0}}(U^{\hat{0}})^2 + \left(\Gamma^{\hat{i}}_{\hat{0}\hat{j}} + \Gamma^{\hat{i}}_{\hat{j}\hat{0}}\right) U^{\hat{0}} U^{\hat{j}} + \Gamma^{\hat{i}}_{\hat{j}\hat{k}} U^{\hat{k}} U^{\hat{j}} = 0 \; .$$

Substituting (5.123) and (5.124), we have

$$\frac{\tilde{D}\vec{U}}{d\tau} = U^{\hat{0}} \left[U^{\hat{0}} \vec{G} + \vec{U} \times \vec{H} - \sigma^{\hat{i}}{}_{\hat{j}} U^{\hat{j}} \mathbf{e}_{\hat{i}} - \frac{1}{3}\theta \vec{U} \right] \tag{5.127}$$

where

$$\frac{\tilde{D}U^{\hat{i}}}{d\tau} = \frac{dU^{\hat{i}}}{d\tau} + \Gamma^{\hat{i}}_{\hat{j}\hat{k}} U^{\hat{k}} U^{\hat{j}} \;\equiv F^{\hat{i}}_{\rm GEM} \; . \tag{5.128}$$

Eq. (5.127) has formal similarities with the electromagnetic Lorentz force, which, in this frame, reads

$$\frac{D\vec{U}}{d\tau} = \frac{q}{m}\left(U^{\hat{0}} \vec{E} + \vec{U} \times \vec{B} \right) \; . \tag{5.129}$$

The derivative operator $\tilde{D}/d\tau$

As shown in detail in Paper #5 [5], $\tilde{D}/d\tau$ is a spatial covariant derivative operator along a curve (parametrized by τ), *preserving the spatial metric* $(h^u)_{\alpha\beta}$, for spatial vectors. It can be understood as follows. The variation the components of a vector A^α (or an arbitrary tensor) along a curve of tangent $\mathbf{U}$ must be measured with respect to a system of axes; the ordinary covariant derivative $DA^\alpha/d\tau$ yields the variation with respect to a system of axes parallel transported along the curve; but this is not the only covariant derivative one can define along the curve. Another covariant derivative, the Fermi-Walker derivative $D_F A^\alpha/d\tau$, yields the variation of A^α with respect to a system of Fermi-Walker transported axes. The derivative we are interested in is a different one, it is one that measures the variation of a spatial vector X^α along the curve, but with respect to a system of axes undergoing the arbitrarily prescribed transport law (5.122) *along the congruence* (i.e., a connection for which the spatial triad vectors $\mathbf{e}_{\hat{i}}$ are constant along $\mathbf{u}$). It is set up as follows. First we note that the spatial projection (i.e., orthogonal to $\mathbf{u}$, for this reason we denote it by $\nabla^\perp$) of the ordinary covariant derivative of a spatial vector X^α

$$\nabla^\perp_\alpha X^\beta \equiv (h^u)^\beta{}_\gamma \nabla_\alpha X^\gamma,$$

when taken along the congruence: $\nabla^\perp_{\mathbf{u}} X^\beta$, yields the Fermi-Walker derivative along the congruence

$$\nabla^\perp_{\mathbf{u}} X^\alpha = (h^u)^\alpha_\beta \frac{DX^\beta}{d\tau_u} = \frac{D_F X^\alpha}{d\tau_u} \; . \tag{5.130}$$

Thus we dub $\nabla^\perp$ the *Fermi-Walker connection.* The connection $\tilde{\nabla}$ we are looking for is such that it coincides with the ordinary covariant derivative in the directions orthogonal to the congruence (thus $\tilde{\nabla}_{\mathbf{X}} = \nabla^{\perp}_{\mathbf{X}}$ if $X^\alpha u_\alpha = 0$), but such that $\tilde{\nabla}_{\mathbf{u}}\mathbf{e}_{\hat{i}} = 0$. In tetrad components such connection is written as

$$\tilde{\nabla}_{\hat{\alpha}} X^{\hat{i}} = \nabla^{\perp}_{\hat{\alpha}} X^{\hat{i}} - \delta^{\hat{0}}_{\hat{\alpha}} \Omega^{\hat{i}}_{\ \hat{j}} X^{\hat{j}} \ . \tag{5.131}$$

The derivative $\tilde{D}/d\tau$ is the one determined by this connection along a curve of tangent $\mathbf{U}$, $\tilde{D}/d\tau \equiv \tilde{\nabla}_{\mathbf{U}}$. Its action on a spatial vector X^α is thus, in tetrad components

$$\frac{\tilde{D}X^{\hat{i}}}{d\tau} = \nabla^{\perp}_{\mathbf{U}} X^{\hat{i}} - \Omega^{\hat{i}}_{\ \hat{j}} U^{\hat{0}} X^{\hat{j}} = \frac{dX^{\hat{i}}}{d\tau} + \Gamma^{\hat{i}}_{\hat{j}\hat{k}} U^{\hat{j}} X^{\hat{k}},$$

or, in manifestly covariant notation:

$$\frac{\tilde{D}X^\alpha}{d\tau} = (h^u)^\alpha_{\ \beta} \frac{DX^\alpha}{d\tau} + \gamma \epsilon^\alpha_{\ \beta\gamma\delta} u^\delta X^\beta \Omega^\gamma \ .$$

Acting on the spatial velocity $U^{\langle\alpha\rangle} \equiv (h^u)^\alpha_{\ \beta} U^\beta$ of the test particle, $\tilde{D}U^{\langle\alpha\rangle}/d\tau$, yields the result we want: the "acceleration" with respect to the chosen frame, whose triad vectors $\mathbf{e}_{\hat{i}}$ rotate along the congruence (relative the Fermi-Walker transport) with an angular velocity $\vec{\Omega}$ that one may arbitrarily specify.

$\tilde{D}U^{\hat{i}}/d\tau$ has a particularly simple interpretation when the congruence is rigid, and one chooses the congruence adapted frame $\vec{\Omega} = \vec{\omega}$ (see previous section) which was already given in Sec. 3.1.2 above: in this case there is a well defined space manifold in the quotient space, of (time-independent) metric $(h^u)_{ij} = \gamma_{ij}$ (c.f. notation of Sec. 3.1.2); and $\tilde{D}U^{\hat{i}}/d\tau$ is just 3-D covariant acceleration of the projected curve (parametrized by τ) on γ_{ij}.

Exact GEM fields

Eq. (5.127), which we can write in the manifestly covariant form

$$F^\alpha_{\rm GEM} \equiv \frac{\tilde{D}U^{\langle\alpha\rangle}}{d\tau} = \gamma \left[\gamma G^\alpha + \epsilon^\alpha_{\ \beta\gamma\delta} u^\delta U^\beta H^\gamma - K^{(\alpha\beta)} U_\beta \right] \ , \tag{5.132}$$

is a very important result in the context of this work, as it yields the inertial forces, in the GEM language, of an arbitrary frame. G^α and $H^\alpha$are, respectively, the *"gravitoelectric"* and *"gravitomagnetic"* fields, defined by

$$G^\alpha = -\nabla_{\mathbf{u}} u^\alpha \equiv -a^\alpha(u); \qquad H^\alpha \equiv \omega^\alpha + \Omega^\alpha \ . \tag{5.133}$$

These designations are due to the analogy with the roles that the electric $(E^u)^\alpha = F^{\alpha\beta} u_\beta$ and magnetic $(B^u)^\alpha = \star F^{\alpha\beta} u_\beta$ fields play in the electromagnetic Lorentz force

$$\frac{DU^\alpha}{d\tau} = \frac{q}{m} \left[\gamma (E^u)^\alpha + \epsilon^\alpha_{\ \beta\gamma\delta} u^\delta U^\beta (B^u)^\gamma \right] \ ,$$

or, in the tetrad, Eq. (5.129). The gravitomagnetic field H^α consists of two parts of different origins: the angular velocity Ω^α of rotation of the tetrads relative to Fermi-Walker transport (i.e., the local guiding gyroscopes), which is *independent* of the congruence, plus the vorticity ω^α of the congruence of observers u^α. In a congruence adapted frame, where the rotation of the tetrad is locked to the vorticity of the congruence, $\Omega^\alpha = \omega^\alpha$, the gravitomagnetic field becomes simply twice the vorticity: $H^\alpha = 2\omega^\alpha$. This is the case of the gravitomagnetic field of the quasi-Maxwell formalism reviewed in Sec. 3.1.2 above.

The last two terms of (5.127) have no electromagnetic counterpart; they consist of the shear/expansion tensor $K_{(\alpha\beta)}$. It corresponds to the time derivative of the spatial metric $(h^u)_{\alpha\beta}$ that locally measures the spatial distances between neighboring observers. This can be seen noting that $K_{(\alpha\beta)} = \mathcal{L}_{\mathbf{u}}(h^u)_{\alpha\beta} = (h^u)_{\alpha\beta,0}u^0$, the last equality holding in the $u^i = 0$ frame. $K_{(\alpha\beta)}$ is sometimes called the *second fundamental form* of the distribution of hyperplanes orthogonal to $\mathbf{u}$. If this distribution is integrable (that is, if $\omega^\alpha = 0$) then $K_{(\alpha\beta)}$ is just the extrinsic curvature of the time slices orthogonal to $\mathbf{u}$.

Simple examples in flat spacetime

An important result in this paper is the above clarification of the origin of the so-called gravitomagnetic field, as arising from the two independent parts Ω^α and ω^α, and its general formulation applying to arbitrary frames, given by Eq. (5.127). In order to see how these things play out, and the relationship with the inertial forces we are familiar with from the textbooks on classical mechanics, e.g. [137], we consider, in flat spacetime, the straightline geodesic motion of a free test particle, from the point of view of three distinct frames: a) a frame whose time axis is the velocity of a congruence of observers at rest, but whose spatial triads rotate uniformly with angular velocity $\vec{\Omega}$; b) a frame composed of a congruence of rigidly rotating observers (vorticity $\vec{\omega}$), but carrying Fermi-Walker transported spatial triads ($\vec{\Omega} = 0$); c) a rigidly rotating frame, that is, a frame composed of a congruence of rigidly rotating observers, carrying spatial triads co-rotating with the congruence $\vec{\Omega} = \vec{\omega}$ (i.e., "adapted" to the congruence). This is depicted in Fig. 5.9.

In the first case there we have a vanishing gravitoelectric field $\vec{G} = 0$, and a gravitomagnetic field $\vec{H} = \vec{\Omega}$ arising solely from the rotation (with respect to Fermi-Walker transport) of the spatial triads; thus the only inertial force is the gravitomagnetic force $\vec{F}_{\text{GEM}} = \gamma\vec{U} \times \vec{\Omega}$. In the frame b), there is a gravitoelectric field $\vec{G} = \vec{\omega} \times (\vec{r} \times \vec{\omega})$ due the observers acceleration, and a also gravitomagnetic field $\vec{H} = \vec{\omega}$, which originates solely from the vorticity of the observer congruence. That is, there is gravitomagnetic force $\gamma\vec{U} \times \vec{\omega}$ which reflects the fact that the relative velocity $v^\alpha = U^\alpha/\gamma - u^\alpha$ (or $\vec{v} = \vec{U}/\gamma$ in the frame $\vec{u} = 0$) between the test particle and the observer it is passing by changes in time. The total inertial forces are in this frame

$$\vec{F}_{\text{GEM}} = \gamma\left[\gamma\vec{\omega} \times (\vec{r} \times \vec{\omega}) + \vec{U} \times \vec{\omega}\right].$$

In the frame c), which is the relativistic version of the classical rigid rotating frame, one has the effects of 1) and 2) combined: a gravitoelectric field $\vec{G} = \vec{\omega} \times (\vec{r} \times \vec{\omega})$, plus a gravitomagnetic field $\vec{H} = \vec{\omega} + \vec{\Omega} = 2\vec{\omega}$, the latter leading to the gravitomagnetic force $2\vec{U} \times \vec{\omega}$, which is the relativistic version of the well known Coriolis acceleration, e.g. [137].

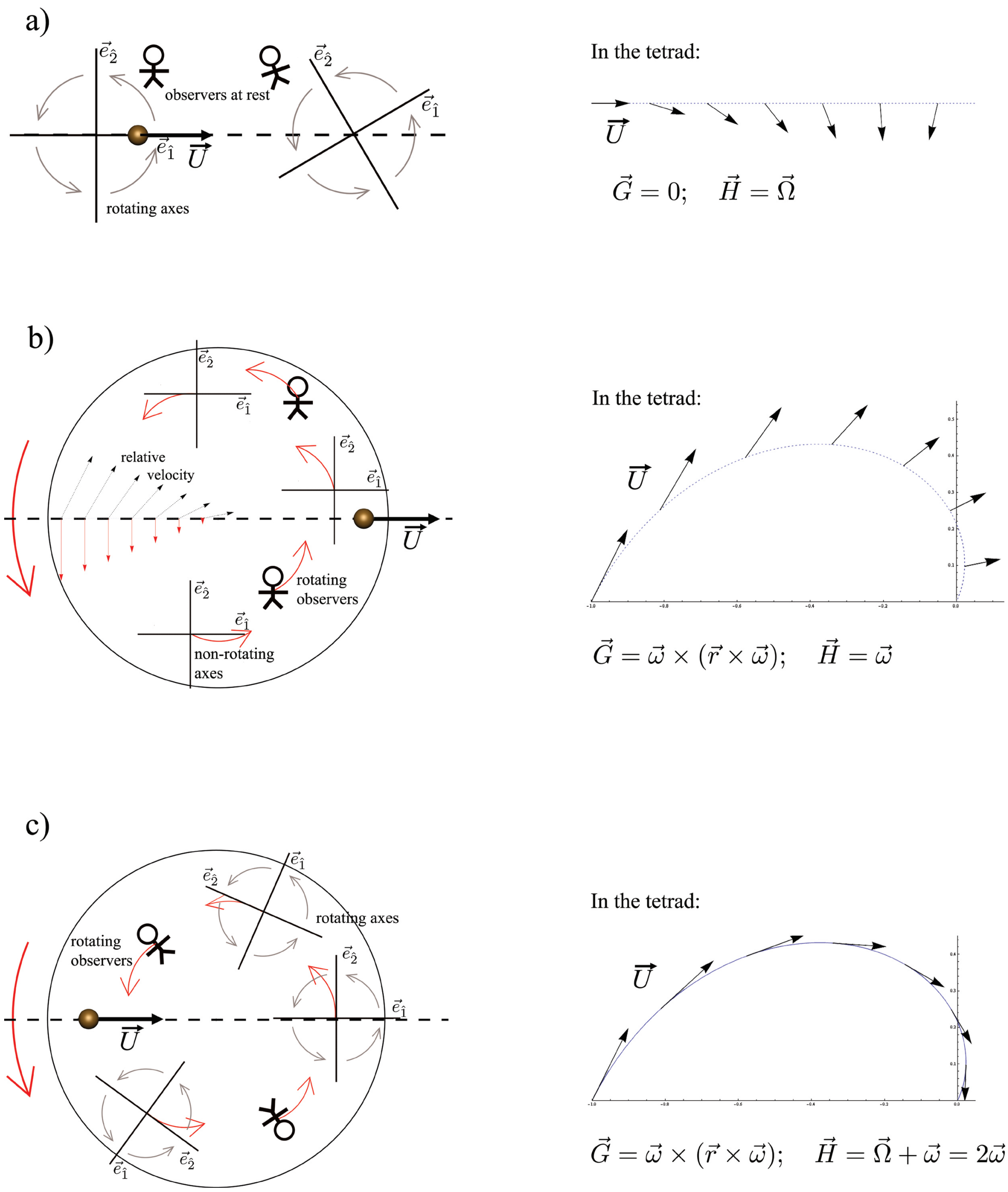


Figure 5.9: A test particle in uniform motion in flat spacetime from the point of view of three different frames: a) a frame composed of observers at rest, but carrying spatial triads that rotate with uniform angular velocity $\vec{\Omega}$; b) a frame consisting of a congruence of rigidly rotating observers (vorticity $\vec{\omega}$), but each of them carrying a non-rotating spatial triad (i.e., that undergoes Fermi-Walker transport); c) a rigidly rotating frame (a frame adapted to a congruence of rigidly rotating observers); the spatial triads co-rotate with the congruence, $\vec{\Omega}=\vec{\omega}$. *Note:* by observer's rotation we mean their circular motion around the center; and by axes rotation we mean their rotation (relative to FW transport) about the frame's origin.

The total inertial force is in this frame

$$\vec{F}_{\rm GEM} = \gamma\left[\gamma\vec{\omega}\times(\vec{r}\times\vec{\omega}) + 2\vec{U}\times\vec{\omega}\right]$$

which is the relativistic generalization of the inertial force in e.g. Eq. (4-107) of [137]. Moreover, in this case the spatial connection coefficients $\Gamma^{\hat{i}}_{\hat{j}\hat{k}}$ equal the ones of the 3-D spatial metric (i.e. of the spatial manifold associated to the quotient of the spacetime by the congruence), there is a well defined 3-D curve obtained by projecting the particle's worldline on the space manifold, $\vec{U}$ is the vector tangent to it (see Fig. 5.9c) and $\vec{F}_{\rm GEM} = \tilde{D}\vec{U}/d\tau$, cf. Eq. (5.128), is simply the acceleration of the curve.

Finally, let me make these remarks on the usefulness of Eq. (5.127), and of our general definition of $\vec{H}$. Although the congruence adapted frame, $\vec{\Omega} = \vec{\omega} = \vec{H}/2$, might seem the most natural frame associated to a given family of observers, other frames are useful and are used in the literature, and the gravitomagnetic effects of such frames discussed therein. Eq. (5.127) yields the inertial forces of any of such frames, in particular our general definition of $\vec{H}$ encompasses all the gravitomagnetic fields defined in the different approaches. That includes the case of the "locally non-rotating frames" [133, 132], or "proper frames of the fiducial observers" [140] in Kerr spacetime discussed above, for which $\vec{\omega} = 0$, and $\vec{H} = \vec{\Omega} = N^{-1}\tilde{\nabla}\times\vec{\beta}$; that is, all the gravitomagnetic accelerations come from Ω^α (N, $\vec{\beta}$ denote, respectively, the lapse function and the shift vector [140]). Frames corresponding to a congruence with vorticity, but where the spatial triads are chosen to be Fermi-Walker transported, $\vec{\Omega} = 0$, have also been considered; in such frames $\vec{H} = \vec{\omega}$ (dubbed the "Fermi-Walker gravitomagnetic field" [25]).

5.6.2.2 Gyroscope precession

Another main result of this GEM formalism is the exact analogy between the so-called gyroscope "precession" and the precession of a magnetic dipole, that I already presented in Sec. 5.5.1 above. Herein I will further elaborate on this subject, and show how the formalism in Paper #5 [5] helps clarifying the precise meaning of the gyroscope precession, what is it in the exact theory, and how can one setup a frame that allows us to determine the rotation of a vector relative to an inertial frame at infinity.

As discussed in Sec. 5.5, if the Mathisson-Pirani condition holds, the spin vector of an ideal gyroscope (that is, a spinning pole-dipole particle) in a gravitational field is Fermi-Walker transported:

$$\frac{DS^\alpha}{d\tau} = S_\nu a^\nu\, U^\alpha\ , \qquad (5.134)$$

where U^α is its center of mass 4-velocity. This is the natural result: a gyroscope, which is understood as an object that opposes to changes in direction of its spin axis $\vec{S}$, has it fixed with respect to the mathematical definition of a comoving non-rotating frame. (This emphasizes the importance of acknowledging the physical validity of the Mathisson-Pirani condition, that we addressed in Paper #3 [3]). In a comoving orthonormal tetrad $e_{\hat{\alpha}}$

(where $U^{\hat{i}} = 0$, and $S^{\hat{0}} = 0$) we can write:

$$\frac{D\vec{S}}{d\tau} = 0 \Leftrightarrow \frac{dS^{\hat{i}}}{d\tau} = -\Gamma^{\hat{i}}_{\hat{0}\hat{k}} S^{\hat{k}} = \left(\vec{S} \times \vec{\Omega}\right)^{\hat{i}} .$$

So ideal gyroscopes in a gravitational field are torque-free, and do *not precess* (relative to non-rotating frames). What one means in the literature by gyroscope "precession", e.g. [7, 8, 73], and which has been measured by the Gravity Probe B mission [53], is a precession with respect to the distant stars, that is, with respect to the axes of an inertial frame at infinity. And this has a physical meaning, as it may detect the presence of frame dragging (and also the Thomas precession). That is, *locally* $\vec{\Omega}$ has no meaning; if $\vec{\Omega} \neq 0$, that tells us only that we are using a rotating frame to describe the motion of the gyroscope. But if the frame we are choosing has its axes fixed to an inertial frame at infinity, and still $\vec{\Omega} \neq 0$, then non-rotating frames at different points rotate one relative to another which indicates frame dragging (if the Thomas precession can be ruled out, e.g. if the gyroscopes are in geodesic motion).

This notion of "frame of the distant stars" obviously applies only to asymptotically flat spacetimes. The question now is how can one compare systems of axes at different points in a curved spacetime, in order to determine if one rotates or not one relative to another. The answer is given by Eq. (5.126) above. As discussed above, if a rigid congruence of observers exists (as is the case in stationary spacetime), setting $\mathbf{e}_{\hat{0}}$ as the tangent to the congruence u^{α}, and locking the rotation of the spatial triads to the vorticity, $\vec{\Omega} = \vec{\omega}$ (i.e., choosing the congruence adapted frame), the connecting vectors between neighboring observers obey

$$\dot{Y}^{\hat{i}} = 0 \ ;$$

that is, the tetrad vectors point to fixed neighboring observers. Thus we have a frame in which the local spatial triads carried by the observers are all locked one to another. Therefore measuring the angular velocity rotation of a vector relative to the local system of axes at point, effectively amounts to measure it with respect to any tetrad at another point. Now consider the spacetime to be asymptotically flat (besides stationary). In this case there are the so-called "static observers" (cf. point 7 of Sec. 5.1), the *rigid* congruence of observers whose worldlines are tangent to the time-like Killing vector field, and that at infinity coincides with the asymptotic inertial rest frame of the source — the axes of the latter define the directions fixed relative to the distant stars. Setting up the frame adapted to this congruence as explained above, yields a frame with axes everywhere locked to the distant stars, thus by measuring the precession of a gyroscope relative to any local tetrad of this frame one is in fact measuring it relative to the former.

The analysis above, based on rigid congruences, applies to stationary spacetimes, such as the Kerr metric or the (approximate) gravitational field of spinning bodies, which is was the problem at hand in the Gravity Probe B mission. But what about the gravitational field generated by a system of translating bodies, which have been studied in the Post-Newtonian approximation (and whose gravitomagnetic field has also been subject of experimental test, e.g. [69, 71, 70])? These are not stationary spacetimes. However a similar analysis for gyroscope precession can be done in these spacetimes, because, as can be seen from the line element (5.29) above (and discussed in detail in Paper #5 [5]), to post Newtonian order (as well as in the "gravitomagnetic limit" of linearized theory) the shear of the PN frame is negligible, only the expansion remains. In this case, for a congruence adapted frame, Eq. (5.126) reads

$$\dot{Y}^{\hat{i}} \simeq \frac{1}{3}\theta Y^{\hat{i}} \ ,$$

which again means that tetrad vectors point to fixed neighboring observers, and by the same construction above one can show that the so-called PN frames (the frames adapted to the $u^i = 0$ observers in the PN metrics) are frames fixed to the distant stars.

Finally, it should be noted that in most literature dealing with GEM analogies, the precession of the gyroscope is cast as being governed by the same gravitomagnetic field that yields the Coriolis acceleration $\vec{U} \times \vec{H}$ in the geodesic equation (5.127), see e.g. Eqs. (3.4), (3.2) and (3.12), (3.11), only with a relative factor of 2 between the two. It is clear in the general formulation herein that the fields involved in these effects are not the same; the field $\vec{H}$ leading to the Coriolis acceleration arises not only from the rotation $\vec{\Omega}$ of the frame relative to a local Fermi-Walker transported tetrad (that yields the gyroscope "precession"), but also from the vorticity $\vec{\omega}$ of the congruence. In this sense, one can say that the Lense-Thirring effect detected in the LAGEOS satellite data [52] (and currently under scrutiny by LARES mission [54]), measuring $\vec{H}$ from test particle's deflection, is of a different mathematical origin from the one which was under scrutiny by the Gravity Probe B mission [53], measuring $\vec{\Omega}$ from gyroscope precession, the two being made to match by measuring both effects relative to the "frame of the distant stars" (verifying $\vec{\Omega} = \vec{\omega}$, and thus in this case the fields differ only by a factor of 2). It is important to bear this in mind, as in the literature GEM fields of frames which are not congruence adapted are discussed; for instance the "Fermi-Walker gravitomagnetic field" defined in [25], which is the $\vec{H}$ of a frame corresponding to a congruence with vorticity, but where the spatial triads are chosen to be Fermi-Walker transported: $\vec{\Omega} = 0$. Thus there is a non-vanishing $\vec{H} = \vec{\omega}$ in this frame, whereas at the same time gyroscopes do not precess relative to it.

5.6.2.3 Field equations

In a parallelism to what is done in Sec. 5.6.1, we split the Einstein and the Maxwell equations in their time and space projections with respect to the observer congruence, but now expressing them not in tidal tensors, but instead in terms of the EM/GEM fields as measured in such frame.

I start by the electromagnetic equations. Using decomposition (5.1), we write Maxwell's Eqs. (5.111) in terms of the electric and magnetic fields $(E^u)^\alpha = F^\alpha_{\ \beta} u^\beta$ and $(B^u)^\alpha = \star F^\alpha_{\ \beta} u^\beta$ measured by the congruence of observers of 4-velocity u^α. All the fields below are measured with respect to this congruence, so we may drop the superscripts: $(E^u)^\alpha \equiv E^\alpha$, $(B^u)^\alpha \equiv B^\alpha$. For simplicity, below I choose the congruence adapted frame ($\vec{\omega} = \vec{\Omega} = \vec{H}/2$); and I refer the reader to Paper #5 for the general expressions. The time and space projections with respect to u^α of Eq. (5.5a) read, in tetrad components, respectively,

$$\tilde{\nabla}\cdot\vec{E} = 4\pi\rho_c + \vec{H}\cdot\vec{B}\,, \quad (5.135)$$

$$\tilde{\nabla}\times\vec{B} = \dot{\vec{E}} + \vec{G}\times\vec{B} + 4\pi\vec{j} - K^{(\hat{i}\hat{j})}E_{\hat{j}}\vec{e}_{\hat{i}} + \theta\vec{E}\,. \quad (5.136)$$

The time and space projections of (5.5b) are, in the tetrad,

$$\tilde{\nabla}\cdot\vec{B} = -\vec{H}\cdot\vec{E}\,, \quad (5.137)$$

$$\tilde{\nabla}\times\vec{E} = -\dot{\vec{B}} + \vec{G}\times\vec{E} + K^{(\hat{i}\hat{j})}B_{\hat{j}}\vec{e}_{\hat{i}} - \theta\vec{B}\,. \quad (5.138)$$

$\tilde{\nabla}$ is the connection defined in (5.131); since herein we are dealing with derivatives along the spatial directions, and for spatial vectors, it could be taken also as the spatial projection of the ordinary covariant derivative, since, for spatial X^α, $\tilde{\nabla}_\alpha X^\beta = (h^u)_\gamma^{\ \beta}\nabla_\alpha X^\gamma$ (or, in the tetrad, $\tilde{\nabla}_{\hat{i}}X^{\hat{j}} = \nabla_{\hat{i}}X^{\hat{j}}$).

Eqs. (5.135)-(5.138) are equivalent to Eqs. (3.25)-(3.28), only written in a different form. In the special case of a rigid frame ($K^{(\hat{i}\hat{j})} = \theta = 0$) and time-independent fields ($\dot{\vec{E}} = \dot{\vec{B}} = 0$), these equations yield Eqs. (5.4.4a)-(5.4.8a) of Table 5.4.

Turning now to the gravitational equations, using $T^{\hat{0}\hat{0}} = \rho$ and $T^{\hat{0}\hat{i}} = J^{\hat{i}}$, and the expressions for the Riemann and Ricci tensors in terms of GEM fields given in Paper #5 [5], the time-time, time-space, and space-space components of the Einstein field equations with sources, Eq. (5.104a), read, respectively:

$$\tilde{\nabla}\cdot\vec{G} = -4\pi(2\rho + T^\alpha_{\ \alpha}) + \vec{G}^2 + \frac{1}{2}\vec{H}^2 - \dot{\theta} - K^{(\hat{i}\hat{j})}K_{(\hat{i}\hat{j})}\,; \quad (5.139)$$

$$\tilde{\nabla}\times\vec{H} = -16\pi\vec{J} + 2\vec{G}\times\vec{H} + 2\tilde{\nabla}\theta - 2\tilde{\nabla}_{\hat{j}}K^{(\hat{j}\hat{i})}\vec{e}_{\hat{i}}\,; \quad (5.140)$$

$$\begin{aligned} 8\pi\left(T_{\hat{i}\hat{j}} - \frac{1}{2}\delta_{\hat{i}\hat{j}}T^\alpha_{\ \alpha}\right) &= \tilde{R}_{\hat{i}\hat{j}} + \tilde{\nabla}_{\hat{i}}G_{\hat{j}} - G_{\hat{i}}G_{\hat{j}} + \dot{K}_{(\hat{i}\hat{j})} + K_{(\hat{i}\hat{j})}\theta \\ &+ \frac{1}{2}\left[\dot{H}_{\hat{i}\hat{j}} + H_{\hat{i}\hat{j}}\theta + \vec{H}^2\delta_{\hat{i}\hat{j}} - H_{\hat{i}}H_{\hat{j}} + K_{(\hat{i}\hat{l})}H^{\hat{l}}_{\ \hat{j}} - H_{\hat{i}}^{\ \hat{l}}K_{(\hat{l}\hat{j})}\right] \end{aligned} \quad (5.141)$$

where $H_{ij} = \epsilon_{ijk}H^k$ is the dual of $\vec{H}$. Eqs. (5.139)-(5.140) are the gravitational analogues of the electromagnetic equations (5.135) and (5.136), respectively; Eq. (5.141) has no electromagnetic counterpart.

As for the the algebraic Bianchi identities (5.7b), the time-time (equal to space-space, as discussed in Sec. 5.7), space-time and time-space components become, respectively:

$$\tilde{\nabla}\cdot\vec{H} = -\vec{G}\cdot\vec{H}\,; \quad (5.142)$$

$$\tilde{\nabla}\times\vec{G} = -\dot{\vec{H}} - \vec{H}\theta + H_{\hat{j}}K^{(\hat{i}\hat{j})}\vec{e}_{\hat{i}}\,; \quad (5.143)$$

$$K_{(ij)}H^j = -\star\tilde{R}^j_{\ ji}\,. \quad (5.144)$$

Eqs. (5.142)-(5.143) are the gravitational analogues of the time and space projections of the electromagnetic Bianchi identities, Eqs. (5.136)-(5.138), respectively[20]; Eq. (5.144) has no electromagnetic analogue.

The 3-D curvature tensor $\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}$ in the equations above is the restriction to the spatial directions of the curvature of the connection $\tilde{\nabla}$, given in the tetrad by

$$\tilde{R}_{\hat{i}\hat{j}\ \ \hat{k}}^{\ \ \hat{l}} \equiv \Gamma^{\hat{l}}_{\hat{j}\hat{k},\hat{i}} - \Gamma^{\hat{l}}_{\hat{i}\hat{k},\hat{j}} + \Gamma^{\hat{l}}_{\hat{i}\hat{m}}\Gamma^{\hat{m}}_{\hat{j}\hat{k}} - \Gamma^{\hat{l}}_{\hat{j}\hat{m}}\Gamma^{\hat{m}}_{\hat{i}\hat{k}} - C^{\hat{m}}_{\hat{i}\hat{j}}\Gamma^{\hat{l}}_{\hat{m}\hat{k}} \ , \tag{5.145}$$

and $\tilde{R}_{\hat{i}\hat{j}} \equiv \tilde{R}^{\hat{l}}_{\ \hat{i}\hat{l}\hat{j}}$ is the Ricci tensor associated to it; this tensor is *not* symmetric in the general case of a congruence possessing both vorticity and shear. Eq. (5.144) states that if the observer congruence has both vorticity and shear/expansion, then $\tilde{R}_{ijkl}$ does not obey the algebraic Bianchi identities for a 3D curvature tensor. In some special regimes the interpretation of $\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}$ is simple. In the quasi-Maxwell limit of Sec. 3.1.2 — that is, rigid ($K_{(\alpha\beta)} = 0$), congruence adapted ($\vec{\Omega} = \vec{\omega}$) frames — it is the curvature tensor of the spatial metric γ_{ij} (which yields the constant infinitesimal distances between neighboring observers of the congruence). In the case that the vorticity is zero ($\vec{\omega} = 0$), the congruence is hypersurface orthogonal, and $\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}$ gives the curvature of these hypersurfaces.

This remarkable aspect should be noted: all the terms in the Maxwell equations (5.135), (5.136) and (5.138) have a gravitational counterpart in (5.139), (5.142) and (5.143), respectively, substituting $\{\vec{E}, \vec{B}\} \to \{\vec{G}, \vec{H}\}$ (up to some numerical factors). As for (5.136), there are clear gravitational analogues in (5.140) to the terms $\vec{G} \times \vec{B}$ and the current $4\pi\vec{j}$, but not to the remaining terms. It should nevertheless be noted that, as shown in Paper #5, in the Post-Newtonian regime (or in the "GEM limit" of linearized theory), the term $2\tilde{\nabla}\theta$ of (5.140) embodies a contribution analogous to the displacement current term $\dot{\vec{E}}$ of (5.136). The gravitational equations contain, as one might expect, terms with no parallel in electromagnetism, most of them involving the shear/expansion tensor $K_{(\alpha\beta)}$.

Special cases. "Quasi-Maxwell" regime (1+3 formalism)

Since most of the differing terms involve $K_{(\alpha\beta)}$, the similarity gets closer if we take the "quasi-Maxwell" regime, i.e., stationary fields, and a frame adapted to a rigid congruence of stationary observers $K_{(\alpha\beta)} = \theta = 0$. The field equations in this regime are given in Table 5.4 . Therein we drop the hats in the indices, for the following reason: as discussed in Sec. 5.6.2.1, in this regime there is a natural 3-D Riemannian manifold on the quotient space (measuring the *fixed* distance between neighboring observers). This manifold has metric γ_{ij}, in the notation of Sec. 3.1.2. We thus interpret the spatial fields $\vec{G}$ and $\vec{H}$ as vector fields on this 3-D Riemannian manifold. The operator $\tilde{\nabla}$ becomes the covariant derivative of γ_{ij} (as $\Gamma^i_{jk} =^{(3)} \Gamma^i_{jk}$, i.e., the 4-D spatial connection coefficients equal the connection coefficients for γ_{ij}), and $\tilde{R}_{ij}$ its Ricci tensor, which is symmetric (contrary to the general case). The equations in this "quasi-Maxwell" regime exhibit a striking

[20] Eqs. (5.142)-(5.143) are equivalent to Eqs. (7.3) of [25]; therein they are obtained through a different procedure, not by projecting the identity $\star R^{\gamma\alpha}_{\ \ \ \gamma\beta} = 0 \Leftrightarrow R_{[\alpha\beta\gamma]\delta} = 0$, but instead from the splitting of the identity $d^2\mathbf{u} = 0 \Leftrightarrow u_{[\alpha;\beta\gamma]} = 0$. Noting that $u_{[\alpha;\beta\gamma]} = -R_{[\alpha\beta\gamma]\lambda}u^\lambda$, we see that the latter is indeed encoded in the time-time and space-time parts (with respect to u^α) of the former.

Table 5.4: GEM formulation of the electromagnetic and gravitational field equations, for *stationary fields.*

Stationary fields, rigid, congruence adapted frame: $\vec{\Omega} = \vec{\omega} = \vec{H}/2$ (quasi-Maxwell formalism)			
Electromagnetism		Gravity	
Maxwell Source Equations		Einstein Equations	
$F^{\alpha\beta}_{\ \ ;\beta} = 4\pi J^{\beta}$		$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\alpha}_{\ \alpha}\right)$	
• Time Component:		• Time-Time Component:	
$\tilde{\nabla}\cdot\vec{E} = 4\pi\rho_c + \vec{H}\cdot\vec{B}$	(5.4.4a)	$\tilde{\nabla}\cdot\vec{G} = -4\pi(2\rho + T^{\alpha}_{\ \alpha}) + \vec{G}^2 + \frac{1}{2}\vec{H}^2$	(5.4.4b)
• Space Components:		• Time-Space Components:	
$\tilde{\nabla}\times\vec{B} = \vec{G}\times\vec{B} + 4\pi\vec{j}$	(5.4.5a)	$\tilde{\nabla}\times\vec{H} = 2\vec{G}\times\vec{H} - 16\pi\vec{J}$	(5.4.5b)
		• Space-Space Component:	
No electromagnetic analogue		$\tilde{\nabla}_i G_j - G_i G_j + \frac{1}{2}\vec{H}^2\gamma_{ij} + \tilde{R}_{ij} = 8\pi\left(\frac{1}{2}\gamma_{ij}T^{\alpha}_{\ \alpha} + T_{ij}\right)$	(5.4.6)
Bianchi Identity		Algebraic Bianchi Identity	
$\star F^{\alpha\beta}_{\ \ ;\beta} = 0 \quad (\Leftrightarrow F_{[\alpha\beta;\gamma]} = 0\ )$		$\star R^{\gamma\alpha}_{\ \ \ \gamma\beta} = 0 \quad (\Leftrightarrow R_{[\alpha\beta\gamma]\delta} = 0)$	
• Time Component:		• Time-Time (or Space-Space) Component:	
$\tilde{\nabla}\cdot\vec{B} = -\vec{H}\cdot\vec{E}$	(5.4.7a)	$\tilde{\nabla}\cdot\vec{H} = -\vec{H}\cdot\vec{G}$	(5.4.7b)
• Space Components:		• Space-Time Components:	
$\tilde{\nabla}\times\vec{E} = \vec{G}\times\vec{E}$	(5.4.8a)	$\tilde{\nabla}\times\vec{G} = 0$	(5.4.8b)

similarity with their electromagnetic counterparts, Eqs. (5.4.4a)-(5.4.8a) of Table 5.4, in spite of some natural differences that remain — numerical factors, the source and terms in (5.4.4b) with no electromagnetic counterpart. We note in particular that, by simply replacing $\{\vec{E},\vec{B}\} \to \{\vec{G},\vec{H}\}$ in (5.4.5a)-(5.4.8a), one obtains, up to some numerical factors, Eqs. (5.4.5b), (5.4.7b)-(5.4.8b). Of course, the electromagnetic terms involving products of GEM fields with EM fields, are mimicked in gravity by second order terms in the gravitational field. This is intrinsic to the non-linear nature of the gravitational field, and may be thought of as manifesting the fact that the gravitational field sources itself.

The results in Table 5.4 complete the usual approach in the literature dealing with this regime, e.g. [19, 22, 23], where the gravitational Eqs. (5.4.4b)-(5.4.8b) are presented, but not the electromagnetic equations (5.4.4a)-(5.4.8a); the former are usually compared with the Maxwell equations in Lorentz frames. In Table 5.4, by contrast, *analogous situations are compared*: gravitational and Maxwell's equations in terms of fields both measured in accelerating and rotating frames.

Finally, it should be mentioned that there is another notable limit of Eqs. (5.139)-(5.144), which the case that the frame is adapted to an hypersurface orthogonal (i.e., vorticity free) congruence, leading to the well known ADM "3+1 formalism" (see e.g. [143, 25]), obtained by setting $\vec{H} = 0$ in the equations above. Namely Eq. (5.139) becomes the so-called "Hamiltonian constrain", Eq. (5.140) the "momentum constrain", and Eq. (5.141) the equation for the evolution of the extrinsic curvature $K_{(\alpha\beta)}$ of the hypersurfaces orthogonal to the congruence; see Paper #5 [5] for details.

5.6.2.4 Relation with tidal tensor formalism

One of the motivations of this work was to establish the connection between the inertial GEM fields herein and the tidal tensors of Secs. 5.2 and 5.6.1. The two analogies are intrinsically different; the latter stems from tensor equations, whereas the former from fields of inertial forces, i.e., artifacts of the reference frame. A relationship between the two formalisms exists nevertheless, and we are finding it of great interest, due the importance of using the two formalisms together in some applications, to be presented elsewhere (e.g. [30]).

In an arbitrary frame one can express the gravitational tidal tensors in terms of the GEM fields, using the expressions for the tetrad components of Riemann tensor given in Sec. 3.4.2 of Paper #5 [5]. The expressions obtained are to be compared with the analogous electromagnetic situation, i.e., the electromagnetic tidal tensors computed from the fields as measured in an arbitrarily accelerating, rotating, and shearing frame (in flat or curved spacetime). General expressions for an arbitrary choice of the spatial frame are given in [5], herein I will assume the congruence adapted frame ($\vec{\omega} = \vec{\Omega} = \vec{H}/2$).

I will start by the electromagnetic tidal tensors; since in this section all the fields and tensors will be measured with respect to the congruence of observers u^α, I use the abbreviated notation $E_{\alpha\beta} \equiv (E^u)_{\alpha\beta} = F_{\alpha\mu;\beta}u^\mu$, and $B_{\alpha\beta} \equiv (E^u)_{\alpha\beta} = \star F_{\alpha\mu;\beta}u^\mu$. It follows that

$$E_{\alpha\gamma} = E_{\alpha;\gamma} - F_{\alpha\beta}u^\beta_{\ ;\gamma}\ ; \qquad B_{\alpha\gamma} = B_{\alpha;\gamma} - \star F_{\alpha\beta}u^\beta_{\ ;\gamma}\ .$$

Using decompositions (5.1), we obtain the tetrad components ($E_{\hat{0}\hat{i}} = B_{\hat{0}\hat{i}} = 0$):

$$E_{\hat{i}\hat{j}} = \tilde{\nabla}_{\hat{j}}E_{\hat{i}} - \frac{1}{2}\left[\vec{B}\cdot\vec{H}\delta_{\hat{i}\hat{j}} - B_{\hat{j}}H_{\hat{i}}\right] - \epsilon_{\hat{i}}^{\ \hat{l}\hat{m}}B_{\hat{m}}K_{(\hat{l}\hat{j})}\ ; \tag{5.146}$$

$$B_{\hat{i}\hat{j}} = \tilde{\nabla}_{\hat{j}}B_{\hat{i}} + \frac{1}{2}\left[\vec{E}\cdot\vec{H}\delta_{\hat{i}\hat{j}} - E_{\hat{j}}H_{\hat{i}}\right] + \epsilon_{\hat{i}}^{\ \hat{l}\hat{m}}E_{\hat{m}}K_{(\hat{l}\hat{j})}\ ; \tag{5.147}$$

$$E_{\hat{i}\hat{0}} = \frac{dE_{\hat{i}}}{d\tau} + \frac{1}{2}(\vec{H}\times\vec{E})_{\hat{i}} + (\vec{G}\times\vec{B})_{\hat{i}}\ ; \tag{5.148}$$

$$B_{\hat{i}\hat{0}} = \frac{dB_{\hat{i}}}{d\tau} + \frac{1}{2}(\vec{H}\times\vec{B})_{\hat{i}} - (\vec{G}\times\vec{E})_{\hat{i}}\ . \tag{5.149}$$

Turning now to gravitational tidal tensors, again we use the abbreviated notation $\mathbb{E}_{\alpha\beta} \equiv (\mathbb{E}^u)_{\alpha\beta} = R_{\alpha\mu\beta\nu}u^\mu u^\nu$, $\mathbb{H}_{\alpha\beta} \equiv (\mathbb{H}^u)_{\alpha\beta} = \star R_{\alpha\mu\beta\nu}u^\mu u^\nu$. Using the tetrad components of the

Riemann tensor given in Paper #5 [5], we obtain ($\mathbb{E}_{\hat{0}\hat{\alpha}} = \mathbb{E}_{\hat{\alpha}\hat{0}} = \mathbb{H}_{\hat{0}\hat{\alpha}} = \mathbb{H}_{\hat{\alpha}\hat{0}} = 0$):

$$\begin{aligned} \mathbb{E}_{\hat{i}\hat{j}} &= -\tilde{\nabla}_{\hat{j}} G_{\hat{i}} + G_{\hat{i}} G_{\hat{j}} + \frac{1}{4}\left(\vec{H}^2 \gamma_{ij} - H_j H_i\right) + \frac{1}{2}\epsilon_{\hat{i}\hat{j}\hat{k}} \frac{dH^{\hat{k}}}{d\tau} + \epsilon^{\hat{l}}{}_{\hat{j}\hat{m}} H^{\hat{m}} K_{(\hat{i}\hat{l})} \\ &\quad - \frac{d}{d\tau} K_{(\hat{i}\hat{j})} - \delta^{\hat{l}\hat{m}} K_{(\hat{i}\hat{l})} K_{(\hat{m}\hat{j})} ; \end{aligned} \tag{5.150}$$

$$\mathbb{H}_{\hat{i}\hat{j}} = -\frac{1}{2}\left[\tilde{\nabla}_j H_i + (\vec{G}\cdot\vec{H})\gamma_{ij} - 2G_j H_i\right] + \epsilon_{\hat{i}}{}^{\hat{l}\hat{m}} \tilde{\nabla}_{\hat{l}} K_{(\hat{j}\hat{m})} . \tag{5.151}$$

Note the formal similarities with the electromagnetic analogues (5.146)-(5.147). All the terms present in E_{ij} and B_{ij}, except for the last term of the latter, have a correspondence in their gravitational counterparts $\mathbb{E}_{ij}$, $\mathbb{H}_{ij}$, substituting $\{\vec{E},\vec{B}\} \to -\{\vec{G},\vec{H}\}$ and correcting some factors of 2. However, the gravitational tidal tensors contain additional terms, which (together with the differing numerical factors) encode the crucial differences in the tidal dynamics of the two interactions. The fourth and fifth terms in (5.150) have the role of canceling out the antisymmetric part of $\tilde{\nabla}_{\hat{j}} G_{\hat{i}}$, that is, canceling out the contribution of the curl of $\vec{G}$ to the gravitoelectric tidal tensor, as can be seen from Eq. (5.143). Note in particular the term $-\dot{H}^i$, which has no counterpart in the electric tidal tensor (5.146); in Eq. (5.143), that term shows up "inducing" the curl of $\vec{G}$, in a role analogous to $\dot{B}$ in the equation (5.138) for $\tilde{\nabla}\times\vec{E}$, which might lead one to think about gravitational induction effects in analogy with Faraday's law of electromagnetism. The fact that it is being subtracted in (5.150), means, however, that the curl of $\vec{G}$ does not translate into *physical, covariant* forces. For instance, it does not induce rotation in a set of free neighboring particles (see Eq. (5.9) above and discussion therein), nor does it torque an extended rigid body, as discussed in Sec. 5.5.4 (see sec. VI of Paper #4 [4] for more details).

There are some interesting special regimes where the relation between the tidal tensors and the inertial fields becomes simpler. One is the "quasi-Maxwell" regime of of Secs. 3.1.2 and 5.6.2.3; i.e., *stationary spacetimes*, and a *frame adapted to a rigid congruence of stationary observers.* The gravitational tidal tensors *as measured in such frame* can be expressed entirely in terms of the gravito-electric $\vec{G}$ and gravitomagnetic $\vec{H}$ fields; the non-vanishing components are:

$$\mathbb{E}_{ij} = -\tilde{\nabla}_j G_i + G_i G_j + \frac{1}{4}\left(\vec{H}^2\gamma_{ij} - H_j H_i\right) ; \tag{5.152}$$

$$\mathbb{H}_{ij} = -\frac{1}{2}\left[\tilde{\nabla}_j H_i + (\vec{G}\cdot\vec{H})\gamma_{ij} - 2G_j H_i\right] \tag{5.153}$$

(in accordance with the discussion in Sec. 5.6.2.3, the hats in the indices are dropped since these tensors may be expressed in an arbitrary, coordinate or not, basis on the spatial manifold γ_{ij}).

The non-vanishing components of the electromagnetic tidal tensors are, under the same conditions,

$$E_{ij} = \tilde{\nabla}_j E_i - \frac{1}{2}\left[\vec{B}\cdot\vec{H}\gamma_{ij} - B_j H_i\right] \quad \text{(a)} \qquad E_{i0} = \frac{1}{2}(\vec{H}\times\vec{E})_i + (\vec{G}\times\vec{B})_i \quad \text{(b)} \tag{5.154}$$

$$B_{ij} = \tilde{\nabla}_j B_i + \frac{1}{2}\left[\vec{E}\cdot\vec{H}\gamma_{ij} - E_j H_i\right] \quad \text{(a)} \qquad B_{i0} = \frac{1}{2}(\vec{H}\times\vec{B})_i - (\vec{G}\times\vec{E})_i \quad \text{(b)} \quad (5.155)$$

Thus again, even in the stationary regime, the electromagnetic tidal tensors have non-vanishing time components, unlike their gravitational counterparts. The spatial parts, however, are very similar in form; note that replacing $\{\vec{E},\vec{B}\} \to -\{\vec{G},\vec{H}/2\}$ in (5.155), the time components vanish, and one *almost* obtains the space part (5.153), apart from the factor of 2 in the third term; and that a similar substitution in (5.154) *almost* leads to (5.152), apart from the term $G_i G_j$, which has no electromagnetic counterpart. The gravitational and electromagnetic tidal tensors are nevertheless very different, even in this regime; namely in their symmetries. E_{ij} is not symmetric, whereas $\mathbb{E}_{ij}$ is (the second and third terms in (5.152) are obviously symmetric; and that the first one also is can be seen from Eq. (5.4.8b) of Table 5.4). As for the magnetic tidal tensors, note that, by virtue of Eq. (5.4.5b), the last term of (5.153) ensures that, in vacuum, the antisymmetric part $H_{[i;j]}$ (i.e., the curl of $\vec{H}$) is subtracted from $H_{i;j}$ in (5.31), thus keeping $\mathbb{H}_{ij}$ symmetric, by contrast with B_{ij}. This can be seen explicitly by noting that *in vacuum* (5.153) can be put in the equivalent form:

$$\mathbb{H}_{ij} = -\frac{1}{2}\left[H_{i;j} - H_{[i;j]} + (\vec{G}\cdot\vec{H})\gamma_{ij} - 2G_{(j}H_{i)}\right] \ ,$$

where we used $H_{[i;j]} = 2G_{[j}H_{i]}$, as follows from Eq. (5.4.5b).

Another interesting regime to consider is the weak field limit, where the non-linearities of the gravitational field are negligible, and compare with electromagnetism in inertial frames. From Eqs. (5.146)-(5.149), the non-vanishing components of the electromagnetic tidal tensors measured by observers at rest in an inertial frame are:

$$E_{ij} = E_{i,j} \ ; \qquad E_{i0} = \frac{dE_i}{d\tau} \ ; \qquad B_{ij} = B_{i,j} \ ; \qquad B_{i0} = \frac{dB_i}{d\tau} \ ,$$

i.e., they reduce to ordinary derivatives of the electric and magnetic fields. The linearized gravitational tidal tensors are, from Eqs. (5.150)-(5.151):

$$\mathbb{E}_{ij} \approx -G_{i,j} + \frac{1}{2}\epsilon_{ijk}\frac{dH^k}{d\tau} - \frac{d}{d\tau}K_{(ij)} \ ; \quad \text{(a)} \qquad \mathbb{H}_{ij} \approx -\frac{1}{2}H_{i,j} + \epsilon_i^{\ lm}K_{(jm),l} \ . \quad \text{(b)} \quad (5.156)$$

Thus, even in the linear regime, the gravitational tidal tensors cannot, in general, be regarded as derivatives of the gravitoelectromagnetic fields $\vec{G}$ and $\vec{H}$. As discussed in Sec. 5.6.2.3, $K_{(ij)}$ is the time derivative of the spatial metric; thus we see that *only if the fields are time independent* in the chosen frame do we have $\mathbb{E}_{ij} \approx -G_{i,j}$, $\mathbb{H}_{ij} \approx -\frac{1}{2}H_{i,j}$.

5.6.2.5 Force on a gyroscope

In the framework of the inertial GEM fields, there is also an analogy, based on exact equations, relating the gravitational force on a gyroscope and the electromagnetic force on a magnetic dipole. It is different from the analogy based on tidal tensors, Eqs. (5.1.2) of Table 5.1, and not as general. The gravitational force on a gyroscope was first written in terms of GEM fields in [19], where Eq. (3.13) was obtained, valid for a gyroscope at rest with respect to some rigid congruence of observers in a stationary spacetime. It was therein compared to the textbook expression (3.14) for the force on a magnetic dipole, written in terms of the fields measured in the *inertial* frame *momentarily* comoving with it. Herein we compare the gravitational force to its electromagnetic counterpart *on analogous conditions*— in terms of the fields measured in an arbitrarily accelerating and rotating frame where the particle is at rest. The material below may be seen as a continuation of the work in [19], and unveils an analogy even stronger than initially found.

We start with equations (5.1.2) of Table 5.1, which tell us that the forces are determined by the magnetic/gravitomagnetic tidal tensors as *seen by the particle*. For the spatial part of the forces, only the space components of the tidal tensors, as measured in the particle's proper frame, contribute. Comparing Eqs. (5.147) and (5.151), which yield the tidal tensors in terms of the electromagnetic/gravitoelectromagnetic fields, we see that a close formal analogy is possible only when $K_{(\alpha\beta)} = 0$ in the chosen frame. Thus, a close analogy between the forces in this formalism can hold only when the particle is at rest with respect to a congruence for which $K_{(\alpha\beta)} = 0$; that is, a rigid congruence. The rigidity requirement can be satisfied only in special spacetimes [146]; it is ensured in the "quasi-Maxwell" regime — that is, stationary spacetimes, and congruences tangent to time-like Killing vector fields therein.

I start by the electromagnetic problem — a magnetic dipole at rest in a rigid, but arbitrarily accelerating and rotating frame. Since the dipole is at rest in that frame, $\mu^\alpha = (0, \mu^i)$; hence the spatial part of the force is $F^i_{EM} = B^{ji}\mu_j$. Substituting (5.155a) in this expression yields the force exerted on the dipole, in terms of the electric and magnetic fields *as measured in its proper frame*:

$$\vec{F}_{EM} = \tilde{\nabla}(\vec{B}\cdot\vec{\mu}) + \frac{1}{2}\left[\vec{\mu}(\vec{E}\cdot\vec{H}) - (\vec{\mu}\cdot\vec{H})\vec{E}\right] . \tag{5.157}$$

Using $\vec{H}\cdot\vec{E} = -\tilde{\nabla}\cdot\vec{B}$, cf. Eq. (5.4.7a) of Table 5.4, we can re-write this expression as

$$\vec{F}_{EM} = \tilde{\nabla}(\vec{B}\cdot\vec{\mu}) - \frac{1}{2}\left[\vec{\mu}(\tilde{\nabla}\cdot\vec{B}) + (\vec{\mu}\cdot\vec{H})\vec{E}\right] . \tag{5.158}$$

Consider now the analogous gravitational situation: a gyroscope at rest (i.e., with zero 3-velocity, $U^i = 0$) with respect to stationary observers (arbitrarily accelerated and rotating) in a stationary gravitational field; from Eqs. (5.1.2b) and (5.153), the force exerted on it is given by:

$$\vec{F}_G = \frac{1}{2}\left[\tilde{\nabla}(\vec{H}\cdot\vec{S}) + \vec{S}(\vec{G}\cdot\vec{H}) - 2(\vec{S}\cdot\vec{H})\vec{G}\right] . \tag{5.159}$$

From Eq. (5.4.7b) we have $\vec{G}\cdot\vec{H} = -\tilde{\nabla}\cdot\vec{H}$; substituting yields [19]:

$$\vec{F}_G = \frac{1}{2}\left[\tilde{\nabla}(\vec{H}\cdot\vec{S}) - \vec{S}(\tilde{\nabla}\cdot\vec{H}) - 2(\vec{S}\cdot\vec{H})\vec{G}\right] . \tag{5.160}$$

Note that replacing $\{\vec{\mu}, \vec{E}, \vec{B}\} \to \{\vec{S}, \vec{G}, \vec{H}/2\}$ in Eq. (5.157) one *almost* obtains (5.159), except for a factor of 2 in the last term. The last term of (5.159)-(5.160), in this framework, can be interpreted as the "weight" of the dipole's energy [19]. It plays, together with

Eq. (5.4.5b), a crucial role in the dynamics, as it cancels out the contribution of the curl of $\vec{H}$ to the force, ensuring that, in the tidal tensor form (5.1.2b), it is given by a contraction of S^{α} with a *symmetric* tensor $\mathbb{H}_{\alpha\beta}$ (see the detailed discussion in Sec. 5.6.2.4). This contrasts with the electromagnetic case, where the curl of $\vec{B}$ is manifest in $B_{\alpha\beta}$ (which has an antisymmetric part) and in the force.

5.6.3 Conclusion

In this section I summarized most of the main results in Paper #5 [5]. Herein I will briefly summarize some other results in the paper. As described in Sec. 5.6.2.4, in this work we established the connection between the GEM inertial fields and the tidal tensor formalism. But other important connections were made. In section 5 of [5], the popular linearized theory analogies are obtained as a limiting case of the results from the theory based on exact GEM fields of Sec. 5.6.2 above. In the case of the tidal effects, they were also obtained as a special case of the tidal tensor analogy. That material is not being included in this document since its presentation in [5] is already succinct enough; but it is nevertheless of relevance. Obtaining the linear regime from the rigorous, exact approaches, clarifies the limit of validity of the equations usually presented which was not clear in the literature, as discussed in Sec. 3.1. It also allows one to work with quantities whose physical meaning is clear, which is not the case with the GEM fields of the usual linearized theory. In the way they are usually presented (as described in Sec. 3.1.1 above) they are somewhat naively derived from the temporal components of the metric tensor (drawing a parallelism with the electromagnetic potentials), their status as artifacts of the reference frame not being transparent, and in particular their relation with the kinematical quantities associated to the observer's congruence. The exact approach yields a more accurate account of some subtleties involved, which are overlooked in the linear approach, for instance the effects concerning gyroscope "precession" relative to the "distant stars" (such as the Lense-Thirring and the geodetic precessions) — the question arising of how can one talk about the "precession" of a local gyroscope relative to the distant stars, as it amounts to comparing systems of vectors at different points in a curved spacetime? The answer is given in Sec. 5.6.2.2. A correct understanding of the linear gravitomagnetic effects is of primary importance in the context of experimental astrophysics nowadays, as it pertains all gravitomagnetic effects detected to date [52, 14, 53, 69, 149, 70], and the ones we hope to detect in the near future [54].

In section 7 of [5] we discussed under which conditions gravity can be similar to electromagnetism (as already emphasized in Sec. 5.5.5, one must bear in mind that the existence of exact analogies does *not* mean that the interactions are similar). The exact inertial GEM formalism, together with the tidal tensor formalism, provide a suitable "set of tools" to make such comparison. The precise conditions for occurrence of a close gravito-electromagnetic similarity are seen to be specific to the type of effect; the results can be seen as a generalization of the ones obtained in Paper #2 [2] in the framework of linearized theory and Post-Newtonian regimes.

Table 5.5: What can be computed by direct application of the GEM analogies

Result	**Approach**
• Geodesic deviation equation (5.1.1b) of Table 5.1: -Replacing $\{q, E_{\alpha\beta}\} \to \{m, -\mathbb{E}_{\alpha\beta}\}$ in (5.1.1a). • Force on a gyroscope (5.1.1b) of Table 5.1: -Replacing $\{\mu^\alpha, B_{\alpha\beta}\} \to \{S^\alpha, -\mathbb{H}_{\alpha\beta}\}$ in (5.1.1a). • Differential precession of gyroscopes (5.120): -Replacing $\{\sigma, B_{\alpha\beta}\} \to \{1, -\mathbb{H}_{\alpha\beta}\}$ in (5.121). • Gravitational field equations (5.1.3b)-(5.1.6b) of Table 5.1: -Replacing $\{E_{\alpha\beta}, B_{\alpha\beta}\} \to \{\mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta}\}$ in Eqs. (5.114)-(5.117), *and* $\rho_c \to 2\rho + T^{\alpha}_{\ \alpha}$ in (5.114), $j^\alpha \to 2J^\alpha$ in (5.117).	**Tidal tensor analogy** (Exact, general results)
• Geodesic Equation (3.11) (*stationary* fields) -Replacing $\{\vec{E}, \vec{B}\} \to \{\vec{G}, \vec{H}\}$ in (5.129), multiplying by γ. • Gyroscope "precession" Eq. (5.64) (*arbitrary* fields): -Replacing $\{\vec{\mu}, \vec{B}\} \to \{\vec{S}, \vec{H}/2\}$ in (5.64). • Force on gyroscope Eq. (5.159) (*stationary* fields, particle's worldline tangent to time-like Killing vector): -Replacing $\{\vec{\mu}, \vec{E}, \vec{B}\} \to \{\vec{S}, \vec{G}, \vec{H}/2\}$ in (5.157), factor of 2 in the last term.	**Inertial "GEM fields" analogy** (Exact results, require special frames)
• Higher order field equations (3.1.1b)-(3.1.4b) of Table 3.1: -Replacing $\{\vec{E}, \vec{B}\} \to \{\mathbb{E}_{ij}, \mathbb{H}_{ij}\}$ in Eqs. (3.1.1a)-(3.1.4a). • Equations of gravitational waves (5.161): -Replacing $\{\vec{E}, \vec{B}\} \to \{\mathbb{E}_{ij}, \mathbb{H}_{ij}\}$ in Eqs. (5.162).	**Analogy** $F_{\mu\nu} \leftrightarrow C_{\alpha\beta\gamma\delta}$ (Linearized theory)

We also revisited (Sec. 6 of [5]) the formal analogies between the Weyl and the Maxwell tensors, that I briefly review in Sec. 3.1.3 above. The tidal tensor approach in Papers #1 and #4 (and also the discussion in Sec. 5.6.1.1 above) gives a physical interpretation to the magnetic part of the Riemann tensor, which was not well understood as explained above. In [5] we put this together with the Matte equations of Table 3.1 to propose a suggestive interpretation of gravitational radiation, and of its interaction with matter. Taking curls of Eqs. (3.1.3b)-(3.1.4b) of Table 3.1, we obtain the wave equations

$$\left(\frac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) \mathbb{E}_{ij} = 0; \quad (a) \qquad \left(\frac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) \mathbb{H}_{ij} = 0 \quad (b) \ , \tag{5.161}$$

formally analogous to the electromagnetic waves

$$\left(\frac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) E^i = 0; \quad (a) \qquad \left(\frac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) B^i = 0 \quad (b) \ , \tag{5.162}$$

only with the gravitational tidal tensors $\{\mathbb{E}_{\alpha\beta},\ \mathbb{H}_{\alpha\beta}\}$ in the place of $\{\vec{E},\ \vec{B}\}$. We arrive at an interpretation of gravitational radiation as a pair of traveling orthogonal (since $\mathbb{E}_{\alpha\beta}\mathbb{H}^{\alpha\beta} = 0$ for gravitational radiation) tidal tensors, propagating by mutually inducing each other, just like $\vec{E}$ and $\vec{B}$ in the electromagnetic waves. And we know how these tensors act on test bodies: the tensor $\mathbb{E}_{\alpha\beta}$ causes a *relative* acceleration between two (monopole) test masses; and $\mathbb{H}_{\alpha\beta}$ causes a force on a spinning particle; but none of them can couple to a monopole particle. It is well known (e.g. [7]) that a gravitational wave (by contrast with an electromagnetic wave) does not displace the center of mass of an approximately monopole particle; this could be interpreted as being the reason. The coupling of the gravitational wave to test bodies, through its multipole moments, is readily established using Eqs (3.52)-(3.53) of Sec. 3.2.1, and thereby also compared with the action of the electromagnetic wave. The interaction of the electromagnetic field with the multipole particle starts at monopole order (Lorentz force), then there is the force and torque on dipoles, and so on to higher orders; the gravitational interaction starts only at dipole order (the spin-curvature force), and continues to quadrupole order (force and torque on the body), and so on. See [5] for more details.

Given the interpretation above of gravitational radiation as a pair of propagating tidal tensors, just like the pair $\{\vec{E},\ \vec{B}\}$ in the electromagnetic waves, it is natural to suppose that gravitational waves might carry some quantity formally similar to the energy and momentum densities $\rho_{EM} \propto E^2 + B^2$, $p_{EM}^{\langle\alpha\rangle} \propto \epsilon^{\alpha}_{\ \mu\nu\sigma} u^\sigma E^\mu B^\nu$; but with $\{\mathbb{E}_{\alpha\beta},\ \mathbb{H}_{\alpha\beta}\}$ in the place of $\{\vec{E},\ \vec{B}\}$. Such quantities turn out to be the well known "super-energy" and "super-momentum" densities defined from the Bel-Robinson tensor, e.g. [32, 34, 148, 144], which arise naturally in this framework. For this reason, that tensor and the concept of super-energy are also briefly discussed.

Paper #5 [5] concludes with a brief discussion of the main outcomes of the gravito-electromagnetic analogies existing in the literature; I reproduce here the Table 5.5, and refer the reader to the discussion therein.

Bibliography

[1] L. Filipe O. Costa, Carlos A. R. Herdeiro, *Phys. Rev. D.* **78** (2008) 024021

[2] L. F. Costa, C. Herdeiro *Proc. IAU* vol **5** (Cambridge U. Press) p 31 (2009) [arXiv:0912.2146]

[3] L. F. Costa, C. Herdeiro, J. Natário, M. Zilhão, *Phys. Rev. D* **85**, 024001 (2012)

[4] L. F. Costa, J. Natário, M. Zilhão, "Spacetime dynamics of spinning particles — exact gravito-electromagnetic analogies", *preprint* [arXiv:1207.0470]

[5] L. F. Costa, José Natário, "Gravito-electromagnetic analogies", *preprint* [arXiv:1207.0465]

[6] C. Barceló, S. Liberati, M. Visser, "Analogue Gravity", Living Rev. Relativity 8, (2005), 12.
http://www.livingreviews.org/lrr-2005-12

[7] I. Ciufolini, J. A. Wheeler, "Gravitation and Inertia," Princeton Series in Physics (1995)

[8] Hans C. Ohanian, Remo Ruffini, "Gravitation and Spacetime," W.W. Norton & Company, Second Edition (1994)

[9] Edward G. Harris, Am. J. Phys. **59** (5) (1991) 421

[10] Robert M. Wald, Phys. Rev. D **6** (1972) 406

[11] S. J. Clark, R. W. Tucker, Class. Quant. Grav. **17** (2000) 4125 [arXiv:gr-qc/0003115]

[12] M. L. Ruggiero and A. Tartaglia, Il Nuovo Cimento B, **117** (2002) 743 [arXiv:gr-qc/0207065].

[13] B. Mashhoon, in "The Measurement of Gravitomagnetism: A Challenging Enterprise", ed. L. Iorio (Nova Science, New York, 2007), pp. 29-39 [arXiv:gr-qc/0311030].

[14] I. Ciufolini, *Nature* **449** (2007) 41

[15] Robert M. Wald, "General Relativity," The University of Chicago Press (1984)

[16] V. B. Braginsky, C. M. Caves, Kip S. Thorne, Phys. Rev. D **15** (1977) 2047

[17] J. -F. Pascual Sánchez, Il Nuovo Cimento B, **115**, p. 725 (2000) [arXiv:gr-qc/0010075].

[18] L. Landau, E. Lifshitz, "The classical theory of fields", 4^{th} Ed., BH-Elsevier (1975)

[19] José Natário, Gen. Rel. Grav. **39** (2007) 1477 [arXiv:gr-qc/0701067]

[20] J. Costa, J. Natário, J. Math. Phys. 46, 082501 (2005)

[21] F. C. Mena, J. Natário, J. Geom. Phys. **59** (2009) 448

[22] W. Oliva, "Geometric mechanics", Springer (2002); Lect. Not. Math. **1798**, 223 (2004)

[23] D. Lynden-Bell, M. Nouri-Zonoz, Rev. Mod. Physics **70** (1998)

[24] M Nouri-Zonoz, Phys. Rev. D **60**, 024013 (1999)

[25] R. T. Jantzen, P. Carini, D. Bini, Ann. Phys. **215** (1992) 1 [arXiv:gr-qc/0106043]

[26] L. F. Costa, J. Natário, M. Zilhão, *Proc. Spanish. Rel. Meeting 2011*, AIP Conf. Proc. 1458 (2011) 367-370. *Preprint* [arXiv:1206.7093]

[27] L. F. Costa, C. A. R. Herdeiro, [arXiv:gr-qc/0612140]

[28] N. Drukker, B. Fiol, J. Simon, JCAP **0410** (2004) 012; C. Herdeiro, Ph.D. thesis, University of Cambridge, 2001

[29] I. Ciufolini, E. Pavlis, *Nature* **431** (2004) 958; F. Everitt *et al*, Phys. Rev. Lett. **106**, 221101 (2011); I. Ciufolini et al, *Space Sci. Rev.* **148**, 71 (2009); LARES mission website: http://www.lares-mission.com/

[30] L. F. O. Costa, J. Natário, L. Wylleman, *work in progress* (for a brief summary of the main results, see *J. Phys.: Conf. Ser.* **314** 012072)

[31] A. Matte, Canadian J. Math. **5** (1953) 1.

[32] L. Bel, Cahiers de Physique, **16** (1962) 59; eng. trans. Gen. Rel. Grav., **32** (2000) 2047.

[33] W. Campbell, T. Morgan, Am. J. Phys. **44** (1976) 356.

[34] R. Maartens and B. A. Bassett, Class. Quant. Grav. **15** (1998) 705 [arXiv:gr-qc/9704059].

[35] G. F. R. Ellis, P. A. Hogan, Gen. Rel. Grav. **29** (1997) 235.

[36] W. B. Bonnor, Class. Quant. Grav. **12** (1995) 499.

[37] W. B. Bonnor, Class. Quant. Grav. **12** (1995) 1483.

[38] C. Lozanovski and C. B. G. Mcintosh, Gen. Rel. Grav. **31** (1999) 1355.

[39] G. Fodor, M. Marklund and Z. Perjés, Class. Quant. Grav. **16** (1999) 453 [arXiv:gr-qc/9805017].

[40] S. Matarrese, O. Pantano and D. Saez, Phys. Rev. Lett. **72** (1994) 320 [arXiv:astro-ph/9310036].

[41] G. F. R. Ellis, "General relativity and Cosmology," Proceedings of the International School of Physics 'Enrico Fermi, Course XLVII, Edited by B. K. Sachs (1971).

[42] C. Lozanovski and M. Aarons, Class. Quantum Grav. **16** (1999) 4075.

[43] N. Van den Bergh, Class. Quant. Grav. **20** (2003) L1 [arXiv:gr-qc/0211025].

[44] P. K. S. Dunsby, B. A. C. Bassett and G. F. R. Ellis, Class. Quant. Grav. **14** (1997) 1215 [arXiv:gr-qc/9811092].

[45] C. McIntosh, R. Arianhod, S. T. Wade, C. Hoenselaers, Class. Quant. Grav. **11** (1994) 1555

[46] L. Wylleman, N. Van den Bergh, *Phys. Rev. D.* **74** (2006) 084001

[47] C. Cherubini, D. Bini, S. Capozziello, R. Ruffini, Int. J. Mod. Phys. D **11** (6) (2002) 827 [arXiv:gr-qc/0302095].

[48] S. Matarrese, O. Pantano and D. Saez, Phys. Rev. D **47** (1993) 1311.

[49] W. M. Lesame, G. F. R. Ellis and P. K. S. Dunsby, Phys. Rev. D **53** (1996) 738 [arXiv:gr-qc/9508049].

[50] R. Maartens, W. M. Lesame and G. F. R. Ellis, Phys. Rev. D **55** (1997) 5219 [arXiv:gr-qc/9703080].

[51] R. Maartens, W. M. Lesame and G. F. R. Ellis, Class. Quant. Grav. **15** (1998) 1005 [arXiv:gr-qc/9802014].

[52] I. Ciufolini, E. Pavlis, *Nature* **431** (2004) 958; Ciufolini I., Pavlis E., Peron R., *New Astron.* 11, 527-550 (2006).

[53] http://einstein.stanford.edu/
"Review of Gravity Probe B", National Academy Press (1995); F. Everitt *et al*, Phys. Rev. Lett. **106**, 221101 (2011)

[54] http://www.lares-mission.com/
I. Ciufolini et al, Space Sci. Rev. **148**, 71 (2009)

[55] J. Frenkel, Z. Phys. **37**, 243 (1926); *idem* Nature **117,** 514 (1926)

[56] H. J. Bhabha, H. C. Corben, Proc. R. Soc. London A **178,** 273 (1941)

[57] H. C. Corben, Physical Review **121,** 1833 (1961)

[58] H. C. Corben, "Classical and Quantum Theories of Spinning Particles", Holden Day, Inc., San Francisco (1968)

[59] W. G. Dixon, J. Math. Phys. **8**, 1591 (1967)

[60] W. G. Dixon, Il Nuovo Cimento, **34**, 317 (1964)

[61] M. Mathisson, Acta Phys. Pol. **6** (1937) 163;
Eng. Trans.:Gen. Rel. Grav. **42** (2010) 1011

[62] W. Tulczyjew, Acta Phys. Pol. **18** (1959) 393

[63] H. Stephani, "Relativity," Cambridge UP, third ed. (2004)

[64] S. Gralla, A. I. Harte, R. M. Wald, Phys. Rev. D **80** (2009) 024031

[65] José Natário, Commun. Math. Phys. **281** (2008) 387

[66] S. E. Gralla, A. I. Harte, R. M. Wald, Phys. Rev. D **81,** 104012 (2010)

[67] W. Campbell, J. Macek, T. Morgan, Phys. Rev. D **15**, 2156 (1977)

[68] D. Bini, P. Carnini, R. T. Jantzen, D. Wilkins, Phys. Rev. D **49** (1994) 2820

[69] K. Nordtvedt, *Int. J. Theoretical Physics* **27** (1988) 2347

[70] M. Soffel, S. Klioner, J. Muller, L. Biskupek, *Phys. Rev. D* **78** (2008) 024033

[71] K. Nordtvedt, *Phys. Rev. D* **7** (1973) 2347

[72] C. Will, K. Nordvedt, APJ **177** (1972) 757

[73] Charles W. Misner, Kip. S. Thorne, John A. Wheeler "Gravitation," W. H Freeman and Company, San Francisco (1973)

[74] M. Mathisson, Acta Phys. Pol. **6,** 218 (1937)

[75] J. Weyssenhoff, Nature **157,** 766 (1946)

[76] J. Weyssenhoff, A. Raabe, Acta Phys. Pol. **9,** 7 (1947)

[77] W. G. Dixon, Il Nuovo Cimento, **38,** 1616 (1965)

[78] C. Moller, Ann. Inst. Henri Poincaré **11**, 251 (1949)

[79] A. Papapetrou, Proc. R. Soc. London A **209,** 248 (1951)

[80] F.A.E. Pirani, Acta Phys. Pol. **15,** 389 (1956)

[81] W. G. Dixon, Proc. Roy. Soc. Lond. A. **314,** 499 (1970)

[82] H. C. Corben, Physical Review **121,** 1833 (1961)

[83] H. C. Corben, “Classical and Quantum Theories of Spinning Particles”, Holden Day, Inc., San Francisco (1968)

[84] H. C. Corben, Phys. Rev. D **30** (1984) 2683

[85] P. Horvathy, Acta Phys. Pol. B **34**, 2611 (2003)

[86] K. Kyrian, O. Semerák, Mon. Not. R. Soc. **382,** 1922 (2007)

[87] O. Semerák, Mon. Not. R. Soc. **308,** 863 (1999)

[88] Lev Vaidman, Am. J. Phys. **58,** 978 (1990)

[89] E. Corinaldesi, A. Papapetrou, Proc. R. Soc. London A **209,** 259 (1951).

[90] W. Shockley, R. P. James, Phys. Rev Lett **18** (1967) 876

[91] Lev Vaidman, Am. J. Phys. **58** (1990) 978

[92] V. Hnizdo, Am. J. Phys **65** (1997) 92

[93] S. Coleman, J. H. Van Vleck, Phys. Review **171** (1968) 1370

[94] L. Wylleman, D. Beke, Phys. Rev. D **81,** 104038 (2010)

[95] Kip S. Thorne, R. H. Price, D. A. Macdonald, “Black Holes, the Membrane Paradigm”, Yale Univ. Press, 1986

[96] I. Ciufolini, Phys. Rev. D **34** (1986) 1014.

[97] W. G. Dixon, Acta Phys. Pol. B, Proc. Supplement, 27 (2008)

[98] R. F. O’ Connell, in “General Relativity and John Archibald Wheeler”, Eds. I. Ciufolini, R. Matner, Springer (2010), p. 325. Preprint [arXiv:1009.4401]

[99] J. D. Jackson, “Classical Electrodynamics”, 3rd Edition, John Wiley & Sons (1999)

[100] D. Babson, S. P. Reynolds, R. Bjorquist, D. J. Griffiths, Am. J. Phys. **77,** 826 (2009)

[101] L. Iorio, H. Lichtenegger, M. L. Ruggiero, C. Corda, Astrophys. Space Sci **331**, 351 (2010)

[102] A. A. Pomeranskii, I. B. Khriplovich, JETP **86,** 839 (1998);
A. A. Pomeranskii, R. A. Sen’Kov, I. B. Khriplovich, Phys. Uspekhi 43 1055 (2000)

[103] I. B. Khriplovich, Acta Phys. Pol. B, proceedings supplement, **1,** 197 (2008) [arXiv:0801.1881]

[104] S. W. Hawking, Phys Rev. Lett. **26** (1971)

[105] W. G. Dixon, Proc. Roy. Soc. Lond. A. **319** (1970) 509

[106] W. G. Dixon, Phil. Trans. R. Soc. Lond. A **277** (1974) 59

[107] R. H. Young, Am. J. Phys **44** (1976) 583

[108] R. H. Young, Am. J. Phys **66** (1998) 1043

[109] R. J. Deissler, Phys. Rev. E **77** (2008) 036609

[110] I. Bailey, W. Israel, Annals of Physics **130** (1980) 188

[111] A. Barnes, Proc. XXVIIth Spanish Relativity Meeting, Eds. J. Miralles, J. Font, J. Pons, Universidad de Alicante (2004) [gr-qc/0401068]

[112] N. Van den Bergh, Class. Quant. Grav. **20** (2003) L165

[113] C. A. Coombes, Am. J. Phys. **47** (1979) 915

[114] C. Moller, "The theory of relativity", Oxford at the Clarendon Press 1960

[115] M. Mathisson, Proc. Cambrige Phil. Soc. **36**, 331 (1940)

[116] K. S. Thorne, J. B. Hartle, Phys. Rev. D **31** (1985) 1815

[117] A. H. Taub, J. Math. Phys. **5,** 112 (1964)

[118] J-M Souriau, Annales de l' I. H. P., Sec. A, **20,** 315 (1974)

[119] J-M Souriau, C. R. Acad. Sc. **271,** 751 (1970)

[120] L. F. Costa, to appear in the *Proceedings of the Spanish Relativity Meeting* 2012, Springer Proceedings in Mathematics

[121] D. Bini, G. Gemelli, R. Ruffini, Phys. Rev. D **61**, 064013 (2000)

[122] Timothy H. Boyer, Am. J. Phys. **56** (1988) 688

[123] B. D. H. Tellegen, Am. J. Phys. **30** (1962)

[124] H. Haus, P. Penfield, Phys. Lett. A **26** (1968) 412

[125] H. Haus, P. Penfield, "Electrodynamics of moving media", MIT Research Monographs (1967)

[126] E. M. Purcell, "Electricity and Magnetism" (Berkeley Physics Course, Vol. 2)

[127] V. Bargman, L. Michel, V. L. Teledgi, *Phys. Rev. Lett.* **2**, 435 (1959)

[128] L. H. Thomas, *Nature* **117**, 514 (1926)

[129] J. L. Synge, "Relativity: the general theory", North-Holland Publ. Co., Amsterdan (1960)

[130] A. Harte, Class.Quant.Grav. **26**, 155015 (2009)

[131] I. M. Ternov, V. A. Bordovitsyn, Sov. Phys. Usp. **23**, 679 (1980)

[132] J. M. Bardeen, APJ **162** (1970) 71

[133] Bardeen *et al.*, APJ **178** (1972) 347

[134] E. Massa, C. Zordan, Meccanica **10** (1975) 27

[135] E. Massa., Gen. Rel. Grav. **5** (1974) 555

[136] E. Massa., Gen. Rel. Grav. **5** (1974) 573

[137] H. Goldstein, "Classical Mechanics", Addison-Wesley (1959)

[138] L. Bel, C. R. Acad. Sci. Paris **246** (1958) 3015.

[139] L. Bel, Annales de l' I. H. P. **17** (1961) 37

[140] Kip S. Thorne, R. H. Price, D. A. Macdonald, "Black Holes, the Membrane Paradigm", Yale Univ. Press, 1986

[141] R. F. O'Connell, "Spin, Rotation and C, P, and T Effects in the Gravitational Interaction and Related Experiments," in Experimental Gravitation: Proc. of Course 56 of the Int. School "Enrico Fermi" (Academic Press, 1974), p.496..

[142] Nichols et al., Phys. Rev. D 84, 124014 (2011)

[143] Eric Gourgoulhon, *"3+1 Formalism and Bases of Numerical Relativity"*, [gr-qc/0703035]

[144] A. García-Parrado Gómez-Lobo, Class. Quant. Grav. **25** (2008) 015006

[145] A. I. Nesterov, Class. Quant. Grav. **16,** 465 (1999)

[146] D. Mason, C. Pooe, J. Math. Phys. **28**, 2705 (1987)

[147] Kip S. Thorne in "Near Zero: New Frontiers of Physics", Eds. J. D. Fairbank, B. S. Deaver Jr., C. W. F. Everitt, P. F. Michelson, W. H. Freeman and Company, NY (1988)

[148] J. M. M. Senovilla Class. Quantum Grav. **17**, 2799 (2000)

[149] T. W. Murphy Jr., K. Nordtvedt, S. G. Turyshev, *Phys. Rev. Lett.* **98** (2007) 071102

6 Communications on the material of this thesis

6.1 Invited Department talks

- "A gravito-electromagnetic analogy based on tidal tensors"
  Luís Filipe P. O. Costa (Speaker), Carlos A. R. Herdeiro;
  - Invited Talk at **Departamento de Matemática Aplicada, Facultad de Ciencias Universidad de Valladolid**, Valladolid – **Spain**, 11 December 2008.
  - Invited Seminar at **Centro Multidisciplinar de Astrofísica (CENTRA), Instituto Superior Técnico**, Lisboa – **Portugal**, 04 December 2008.
    URL: http://centra.ist.utl.pt/index.php?option=com_jcalpro&Itemid=72&extmode=view&extid=28
- "A gravito-electromagnetic analogy based on tidal tensors — the physical meaning of the second order scalar invariants of the Riemann tensor"
  L. Filipe O. Costa
  Invited Seminar at **Dipartimento di Ingegneria Aerospaziale e Astronautica, Sapienza Universita' di Roma**, Roma – **Italy**, 1 July 2009.
- "Spacetime dynamics of spinning particles — exact gravito-electromagnetic analogies"
  L. Filipe Costa (Speaker), J. Natário, M. Zilhão
  Invited Seminar at **Department of Mathematical Analysis, Ghent University,** Ghent-**Belgium**, 21 February 2012

6.2 Communications in International Conferences

- "A gravito-electromagnetic analogy based on tidal tensors"
  L. Filipe P. O. Costa (Speaker), Carlos A. R. Herdeiro
  - Oral Communication at **Seventh Alexander Friedmann International Seminar Gravitation and Cosmology**, João Pessoa – **Brazil**, 30 June to 04 July 2008.
    URL: http://www.fisica.ufpb.br/eventos/friedmann2008/friedmann2008new2.htm

  - Poster presented at **Spanish Relativity Meeting 2008 (ERE2009)**, Salamanca – **Spain**, 14-20 September 2008.
    URL: www.usal.es/ere2008
  - Lecture presented at **VIII SIGRAV Graduate School in Contemporary Relativity and Gravitational Physics**, Villa Olmo (Como) – **Italy**, 11-15 May 2009
    URL: http://www.centrovolta.it/sigrav2009/
  - Oral Communication at **12th Marcel Grossman Meeting** (Parallel Session MAGT3 B — Theoretical Issues in General Relativity), Paris – **France**, 12-18 July 2009
    URL: http://www.icra.it/MG/mg12/en/

- "Reference frames and the physical gravito-electromagnetic analogy"
  L. Filipe P. O. Costa, Carlos A. R. Herdeiro
  Poster presented at **IAU** (International Astronomical Union) **Symposium 261** — *"Relativity in Fundamental Astronomy: Dynamics, Reference Frames, and Data Analysis"*, Virginia Beach – **USA**, 27 April - 1 May 2009
  Link: http://www.aas.org/divisions/meetings/iau/programme.php

- "Tidal tensor approach to gravitomagnetism"
  L. Filipe P. O. Costa (Speaker), Carlos A. R. Herdeiro
  Lecture presented at **1st LARES** (Laser Relativity Satellite) **Workshop**, Roma – **Italy**, 3-4 July 2009
  Talk available online at: http://www.lares-mission.com/ILSW2009.html

- "The gravitational force on a gyroscope and the electromagnetic force on a magnetic dipole as analogous tidal effects"
  L. Filipe P. O. Costa (Speaker), Carlos A. R. Herdeiro
  Oral Communication at **Spanish Relativity Meeting 2009 (ERE2009)**, Bilbao – **Spain**, 7-11 September 2009.
  URL: http://www.ehu.es/ere2009/website/modules/pageworks/index.php?page=8&ve=95

- "Aspects of the motion of gyroscopes around Schwarzschild and Kerr black holes — exact gravito-electromagnetic analogies"
  L. Filipe P. O. Costa (Speaker), Carlos A. R. Herdeiro
  Oral communication at **II Workshop on Black Holes**, Instituto Superior Técnico, Lisboa – **Portugal**, 21-22 December 2009.
  Talk available on-line at http://centra.ist.utl.pt/~bhw/

- "Spinning test particles in general relativity — exact gravito-electromagnetic analogies"
  L. Filipe Costa (Speaker) and C. A. R. Herdeiro
  Oral communication at **19th International Conference on General Relativity and Gravitation** (GR19), **Mexico City** – **Mexico**, July 2010

- "Gravitomagnetism and the significance of the curvature scalar invariants"
  Luis Filipe Costa (Speaker), C. A. R. Herdeiro, and Lode Wylleman (2010)
  Oral Communication at **Spanish Relativity Meeting 2010 (ERE2010)**, **Granada – Spain**, 6-10 September 2010.
  URL: http://www.iaa.es/ere2010/website/modules/pageworks/index.php?page=8&ve=269

- L. Filipe Costa (Speaker), C. Herdeiro, J. Natário, M. Zilhão
  "Mathisson's helical motions — are they unphysical?"
  Oral Communication at **Spanish Relativity Meeting 2011 (ERE2011)**, **Madrid – Spain**, 29th August - 2nd September 2011
  Talk available online at http://teorica.fis.ucm.es/ERE2011/Program.html

- L. Filipe Costa
  "Hidden momentum in general relativity"
  Oral Communication at **Spanish Relativity Meeting 2012 (ERE2012)**, **Guimarães – Portugal**, 3-7 September 2012
  URL: http://w3.math.uminho.pt/~ERE2012/website/modules/tinyd0/

6.3 Other oral communications

- "Spacetime dynamics of spinning particles — exact gravito-electromagnetic analogies"
  L. Filipe O. Costa (Speaker), J. Natário, M. Zilhão
  CFP Journal Club - **Centro de Física do Porto**, Porto - **Portugal**, 9 May 2012
  Link: http://faraday.fc.up.pt/cfp/events/journal-clubs/spacetime-dynamics
  -of-spinning-particles-exact-gravito-electromagnetic-analogies/

- "Aspects of the motion of gyroscopes around Schwarzschild and Kerr black holes — exact gravito-electromagnetic analogies"
  L. Filipe O. Costa (Speaker), Carlos A. R. Herdeiro
  Talk presented at **MAP-FIS PhD Research Conference 2009/2010**, **University of Aveiro**, **Aveiro – Portugal**, 15 January 2010
  Link: http://www.map.edu.pt/fis/Workshop

- "A gravito-electromagnetic analogy based on tidal tensors"
  L. Filipe O. Costa (Speaker), Carlos A. R. Herdeiro
  Talk presented at **MAP-FIS PhD Research Conference 2008**, **University of Minho**, **Braga – Portugal**, 16-17 January 2009
  Link: http://www.map.edu.pt/fis/Workshop

- "Uma analogia gravito-electromagnética baseada nas forças de maré"
  L. Filipe O. Costa (Speaker), Carlos A. R. Herdeiro
  Series of seminars presented at Journal Club - **Centro de Física do Porto**, Porto – **Portugal**

- Part I: 22 February 2007
  Link: http://faraday.fc.up.pt/cfp/events/journal-clubs/2007/uma-analogia-gravito-electromagnetica-baseada-nas-forcas-de-mare-i/
- Part II: 03 March 2007
  Link: http://faraday.fc.up.pt/cfp/events/journal-clubs/2007/uma-analogia-gravito-electromagnetica-baseada-nas-forcas-de-mare-ii/
- Part III: "O movimento não geodésico do giroscópio", 24 July 2008
  Link: http://faraday.fc.up.pt/cfp/events/journal-clubs/2008/uma-analogia-gravito-electromagnetica-baseada-nas-forcas-de-mares

7 Further publications on the material of this thesis (conference proceedings)

- L. Filipe O. Costa, Carlos A. R. Herdeiro,
  "Tidal tensor approach to gravitoelectromagnetism",
  International Journal of Modern Physics A **24** 1695 (2009)
  DOI: 10.1142/S0217751X0904525X

- L. Filipe O. Costa, Carlos A. R. Herdeiro
  "The gravitational force on a gyroscope and the electromagnetic force on a magnetic dipole as analogous tidal effects"
  Proceedings of the Spanish Relativity Meeting (ERE2009), 7–11 September 2009, Bilbao, Spain
  Journal of Physics Conf.. Ser. **229**, 012031 (2010)
  DOI: 10.1088/1742-6596/229/1/012031

- L. Filipe O. Costa, Carlos A. R. Herdeiro
  "Analogy between general relativity and electromagnetism based on tidal tensors"
  Proceedings of the 12th Marcel Grossmann Meeting, Paris-France, 12-18 July 2009, Edited by T. Damour, R.T. Jantzen, R. Ruffini, World Scientific (2012)

- L. Filipe O. Costa, Carlos A. R. Herdeiro, Lode Wylleman
  "Electromagnetic and Gravitational Invariants"
  Proceedings of the Spanish Relativity Meeting (ERE2010), 6–10 September 2010, Granada, Spain
  Journal of Physics: Conf. Ser. 314, 012072 (2011)
  DOI: 10.1088/1742-6596/314/1/012072

- L. Filipe Costa, J. Natário, M. Zilhão
  "Mathisson's helical motions demystified"
  Proceedings of the Spanish Relativity Meeting 2011 (ERE2011), Madrid – Spain, 29th August - 2nd September 2011; AIP Conf. Proc. 1458 (2011) 367-370. DOI: 10.1063/1.4734436. *Preprint* [arXiv:1206.7093]

Acknowledgments

I am deeply grateful to all those who, in one way or another, helped me in this journey. Those, both within the academia and amongst my family and friends, not being mentioned below, but whom I do not forget, will, I am sure, forgive me for individualizing the following acknowledgments.

I thank João Penedones and José Natário, for their valuable scientific advice. I thank João his continued support and genuine interest in my work, the enlightening discussions and helpful suggestions in my papers. I thank Zé for having been my "super-collaborator", for taking me into another level, and for the long hours of passionate, ever thrilling, discussions. I thank my Professor, former supervisor (of my undergraduate thesis, and my first research Grant) and ever inspiration, João Lopes dos Santos. For the long exciting discussions on Relativity when I started as an undergraduate, which was where this dream started coming to life; for the encouragement and comfort in the critical moments, and for the continued and valuable input even when working in different fields. I thank my Co-Supervisor José Sande Lemos for his support. I thank my collaborators Lode Wylleman and Miguel Zilhão; I thank Norbert Van den Bergh, Ignazio Ciufolini, and José Pascual-Sanchéz for their interest in my work, their kind invitations and hospitality when I visited Ghent University, Sapienza University of Rome, and Universidad de Valladolid. I thank my friend Rui Quaresma for the great illustrations he furnished me during these years, which helped me so much in my papers and presentations. I would like to say a big thanks to all those colleagues (whose names I will not try to list as I would be bound to forget someone) I met at the conferences and seminars abroad with whom I interacted and helped me improve my work. Finally, I thank my father, and a very special thanks has to go to my mother and my girlfriend Marta, for their endless support and devotion. There is no way I could have gone through all this without them.