SEMIGROUP ACTIONS OF EXPANDING MAPS

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Abstract. We consider semigroups of Ruelle-expanding maps, parameterized by random walks on the free semigroup, with the aim of examining their complexity and exploring the relation between intrinsic properties of the semigroup action and the thermodynamic formalism of the associated skew-product. In particular, we clarify the connection between the topological entropy of the semigroup action and the growth rate of the periodic points, establish the main properties of the dynamical zeta function of the semigroup action and relate these notions to recent research on annealed and quenched thermodynamic formalism. Meanwhile, we examine how the choice of the random walk in the semigroup unsettles the ergodic properties of the action.

1. Introduction

In the mid seventies the thermodynamic formalism was brought from statistical mechanics to dynamical systems by the pioneering work of Sinai, Ruelle and Bowen [20, 4, 5, 24]. The correspondence between one-dimensional lattices and uniformly hyperbolic dynamics conveyed several notions from one setting to the other, introducing, via Markov partitions, Gibbs measures and equilibrium states into the realm of dynamical systems. Within non-invertible dynamics, a complete description of the thermodynamic formalism has been established for Ruelle-expanding maps [21] and for expansive maps with a specification property [23, 15]. In particular, it is known that, for every potential under some regularity condition, there exists a unique equilibrium state, which is a Gibbs measure and has exponential decay of correlations. The classical strategy to prove these properties ultimately relies on the analysis of the spectral properties of the Perron-Frobenius transfer operator, and it is known how to extend this method to finitely generated group actions. Yet, the attempts to generalize the previous results have so far been riddled with difficulties, and a global theory is still out of reach. Some success has been registered for continuous actions of finitely generated abelian groups. More precisely, the statistical mechanics of expansive $\mathbb{Z}^d$-actions satisfying a specification property has been studied by Ruelle in [19], after introducing a suitable notion of pressure and discussing its link with measure theoretical entropy and free energy. The key ingredient in this context has been the fact that continuous $\mathbb{Z}^d$-actions on compact spaces admit probability measures invariant under every continuous map involved in the group action. With it, Ruelle proved a variational principle for the topological pressure and built equilibrium states as the class of pressure maximizing invariant probability measures. This duality between topological and measure theoretical complexity of the dynamical system has been later used by Eizenberg, Kifer and Weiss [13] to establish large deviations principles for $\mathbb{Z}^d$-actions satisfying a specification property.

A unified approach to the thermodynamic formalism for continuous group actions in the absence of probability measures invariant under all elements of the group is still unknown. Although these actions are not dynamical systems, a few definitions of topological pressure have been proposed, although most of them unrelated and assuming either abelianity, amenability or some growth rate of the corresponding group. Inspired by the notion of complexity presented by Bufetov in [7] in the context of skew-products, where no commutativity or conditions on the semigroup growth rate are required, the second and third named authors introduced in [18] a notion of topological pressure for semigroup actions, and showed that it reflects the complex behavior of the action. In the

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present paper we push this analysis further, considering finitely generated semigroups of Ruelle-expanding maps. We relate the notion of topological entropy of a semigroup action introduced in [18] with the growth of the set of periodic orbits, the concepts of fibred and relative entropies and the radius of convergence of a dynamical zeta function for the semigroup action. Moreover, we generalize the classical Ruelle-Perron-Frobenius transfer operator and construct equilibrium states for semigroup actions of C^2 expanding maps. In the meantime, we will verify the impact of changing the random walk inside the semigroup on the dynamical and ergodic attributes of the semigroup action.

2. Setting

Let M be a compact metric space and $C^0(M)$ denote the space of all continuous observable functions $\psi : M \to \mathbb{R}$. Given a finite set of continuous maps $g_i : M \to M$, $i \in \{1, 2, \dots, p\}$, $p \geq 1$, and the finitely generated semigroup $(G, \circ)$ with the finite set of generators $G_1 = \{id, g_1, g_2, \dots, g_p\}$, write $G = \bigcup_{n\in\mathbb{N}_0} G_n$, with $G_0 = \{id\}$, and $\underline{g} \in G_n$ if and only if $\underline{g} = g_{i_n} \dots g_{i_2} g_{i_1}$, with $g_{i_j} \in G_1$ (for notational simplicity's sake we will use $g_j\, g_i$ instead of the composition $g_j \circ g_i$). Set $G_1^* = G_1 \setminus \{id\}$. As a semigroup can have multiple generating sets, we will assume that the generator set G_1 is minimal, meaning that no function g_j, for $j = 1, \dots, p$, can be expressed as a composition from the remaining generators.

Free semigroups. In G, one considers the semigroup operation of concatenation defined as usual: if $\underline{g} = g_{i_n} \dots g_{i_2} g_{i_1}$ and $\underline{h} = h_{i_m} \dots h_{i_2} h_{i_1}$, where $n = |\underline{g}|$ and $m = |\underline{h}|$, then $\underline{g}\,\underline{h} = g_{i_n} \dots g_{i_2} g_{i_1} h_{i_m} \dots h_{i_2} h_{i_1} \in G_{m+n}$. Each element $\underline{g}$ of G_n may be seen as a word that originates from the concatenation of n elements in G_1. Clearly, different concatenations may generate the same element in G. Nevertheless, in most of the computations to be done, we shall consider different concatenations instead of the elements in G they create. One way to interpret this statement is to consider the itinerary map

$$\begin{array}{cccc} \iota : & F_p & \to & G \\ & \underline{i} = i_n \dots i_1 & \mapsto & \underline{g}_{\underline{i}} := g_{i_n} \dots g_{i_1} \end{array}$$

where F_p is the free semigroup with p generators, and to regard concatenations on G as images by ι of paths on F_p. Set $G_1^* = G_1 \setminus \{id\}$ and, for every $n \geq 1$, let G_n^* denote the space of concatenations of n elements in G_1^*. To summon each element $\underline{g}$ of G_n^*, we will write $|\underline{g}| = n$ instead of $\underline{g} \in G_n^*$.

Semigroup actions. We say that the finitely generated semigroup G induces a *semigroup action* $S : G \times M \to M$ in M defined by $S(\underline{g}, x) = \underline{g}(x)$ if, for any $\underline{g}$, $\underline{h} \in G$ and all $x \in M$, we have $S(\underline{g}\,\underline{h}, x) = S(\underline{g}, S(\underline{h}, x))$. The action S is said to be continuous if, for any $\underline{g} \in G$, the map $\underline{g} : M \to M$ given by $\underline{g}(x) = S(\underline{g}, x)$ is continuous. A point $x \in M$ is a *fixed point* of $\underline{g} \in G$ if $\underline{g}(x) = x$; the set of these fixed points will be denoted by $\mathrm{Fix}(\underline{g})$. A point $x \in M$ is a *periodic point with period* n of the action S if there exists $\underline{g} \in G_n^*$ such that $\underline{g}(x) = x$. We let $\mathrm{Per}(G_n) = \bigcup_{|\underline{g}|=n} \mathrm{Fix}(\underline{g})$ denote the set of all these periodic points with period n. Accordingly, $\mathrm{Per}(G) = \bigcup_{n\geq 1} \mathrm{Per}(G_n)$ will stand for the set of periodic points of the whole semigroup action. We observe that, when $G_1^* = \{f\}$, this definition coincides with the usual one for the single dynamical system f.

Ruelle-expanding maps. Let (X, d) be a compact metric space and $T : X \to X$ a continuous map. T is said to be *expansive* if there exists $\varepsilon > 0$ such that, for any $x \neq y \in X$, $\sup_{n\in\mathbb{N}}\ d(T^n(x), T^n(y)) > \varepsilon$. The map T is *locally expanding* if there exist $\lambda > 1$ and $\delta > 0$ such that $d(T(x), T(y)) > \lambda\, d(x, y)$ for all $x, y \in X$ with $d(x, y) < \delta$ These two notions are bonded: every locally expanding system is expansive and an expansive map is locally expanding with respect to an adapted metric (cf. [10]).

Definition 2.1. The system (X, T) is *Ruelle-expanding* if T is locally expanding and open, i.e.:
(i) there exists $c > 0$ such that, for all $x, y \in X$ with $x \neq y$, $T(x) = T(y)$ implies $d(x, y) > c$.
(ii) there are $r > 0$ and $0 < \rho < 1$ such that, for each $x \in X$ and all $a \in T^{-1}(\{x\})$ there is a

map $\varphi : B_r(x) \to X$, defined on the open ball centered at x with radius r, such that $\varphi(x) = a$, $T \circ \varphi(z) = z$ and, for all $z, w \in B_r(x)$, we have $d(\varphi(z), \varphi(w)) \leq \rho\, d(z, w)$.

Examples of Ruelle-expanding maps include one-sided Markov subshifts of finite type, determined by aperiodic square matrices with entries in $\{0, 1\}$, and C^1-expanding maps on compact manifolds. We remark that, due to the spectral decomposition, if the domain of a Ruelle-expanding map is connected, then it is topologically mixing. For more details we refer the reader to [5, 21, 9].

3. Main results

We start with a topological description of finitely generated semigroups of uniformly expanding maps. Later, we will assume that G_1 is either a finite subset of Ruelle-expanding maps acting on a compact connected metric space M or a finite subset of the space $End^2(M)$ of non-singular C^2 endomorphisms in a compact connected manifold M. In this setting, we will show that the set of periodic points with period n for such a semigroup dynamics has a definite exponential growth rate with the period n, which is given by the topological entropy of the semigroup $h_{\text{top}}(S)$ (see Definition 6.1). Moreover, we will prove that this entropy is equal to the logarithm of the spectral radius of a suitable transfer operator $\mathbf{L}_0$ (cf. Subsections 5 and 6).

Theorem A. *Let G be the semigroup generated by $G_1 = \{Id, g_1, \dots, g_p\}$, where G_1^* is a set of Ruelle-expanding maps on a compact connected metric space M, and let $S : G \times M \to M$ be its continuous semigroup action. Then*

$$0 < h_{top}(S) = \lim_{n\to\infty} \frac{1}{n} \log\Big(\frac{1}{p^n} \sum_{|\underline{g}|=n} \sharp\, Fix(\underline{g})\Big) = \log sp\,(\mathbf{L}_0).$$

The second part of this work concerns the asymptotic growth of the periodic points of a semigroup action. Inspired by the Artin-Mazur zeta function (cf. [1]), we define the *zeta function* associated to the continuous semigroup action $S : G \times M \to M$ by the formal power series

$$z \in \mathbb{C} \;\mapsto\; \zeta_S(z) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n(G)}{n}\, z^n\right), \tag{1}$$

where $N_n(G) = \frac{1}{p^n}\sum_{|\underline{g}|=n} \sharp\,\mathrm{Fix}(\underline{g})$. (We refer the reader to [8] for an account on random zeta functions.) Our next result relates the regularity of ζ_S to the exponential growth rate of periodic points, given by

$$\wp(S) = \limsup_{n\to+\infty} \frac{1}{n} \log\left(\max\{N_n(G), 1\}\right).$$

Theorem B. *Let G be the semigroup generated by $G_1 = \{Id, g_1, \dots, g_p\}$, where G_1^* is a set of Ruelle-expanding maps on a compact connected metric space M, and $S : G \times M \to M$ be the corresponding continuous semigroup action on M. Then ζ_S is rational and its radius of convergence ρ_S is equal to $e^{-\wp(S)} = e^{-h_{top}(S)}$.*

Afterwards, we will study the ergodic properties of semigroup actions of maps in $End^2(M)$. Although one does not expect to find an absolutely continuous common invariant probability measure, we may hope to discover some probability measure which reflects an averaged distribution of the Lebesgue measure under the action of the semigroup. We say that $R_{\underline{\theta}}$ is a *random walk on* G if $R_{\underline{\theta}} = \iota_*\theta^{\mathbb{N}}$, where θ is a probability measure on $\iota^{-1}(G_1^*) = \{1, \dots, p\}$, the set of generators of F_p. For instance, if $\theta(i) = \frac{1}{p}$ for any $i \in \{1, \dots, p\}$, then $\eta_{\underline{p}} = \theta^{\mathbb{N}}$ is the equally distributed Bernoulli probability measure on the Borel sets of the unilateral shift $\Sigma_p^+ = \{1, \dots, p\}^{\mathbb{N}}$. If, instead, $\theta(i) = a_i > 0$ for each $i \in \{1, \dots, p\}$, then $\underline{a} = (a_1, a_2, \dots, a_p)$ is a non-trivial probability vector and $\eta_{\underline{a}} = \theta^{\mathbb{N}}$ will stand for the Bernoulli probability measure $\theta^{\mathbb{N}}$ on Σ_p^+, while $R_{\underline{a}} = \iota_*(\eta_{\underline{a}})$ will denote the corresponding random walk $R_{\underline{\theta}}$ on G. More generally, we may take a σ-invariant probability measure η on Σ_p^+ and consider the associated random walk $R_\eta = \iota_*(\eta)$. Given a σ-invariant probability measure η, a probability measure ν on M is said to be R_η-*stationary* if

$$\nu = \int \underline{g}_*\nu \; dR_\eta(\underline{g}). \tag{2}$$

As probability measures invariant under all elements of the semigroup are unlikely to exist, the concept of stationary measure is the most natural to be addressed while studying ergodic properties of semigroup actions. However, there is evidence that this is not the suitable notion for describing maximal entropy measures for semigroup actions (cf. Section 9). Finally, we will discuss a thermodynamic formalism and find an adequate definition of measure of maximal entropy for a semigroup action with respect to a fixed random walk.

Theorem C. *Let G be the semigroup generated by $G_1 = \{id, g_1, \dots, g_p\}$, where G_1^* is a set of C^2 expanding maps on a compact connected manifold M. Consider the corresponding continuous semigroup action $S : G \times M \to M$ and the random walk $R_{\underline{p}} = \iota_*(\eta_{\underline{p}})$. Then the semigroup action has a probability measure of maximal entropy which can be computed as the weak*-limit of an averaged distribution of either pre-images or periodic points.*

4. Overview

To complement the classical approach of random dynamical systems, we are mainly interested in determining intrinsic objects for the dynamics of the semigroup action. Ruelle-Perron-Frobenius transfer operators suitable for the analysis of semigroup actions will be defined in Section 5. These operators are in a strong relation with the classical transfer operator for a locally constant skew-product dynamics. For that reason, we also discuss the dependence of the ergodic properties of the skew-product dynamics on the chosen probability measure in the underlying shift, which describes the random walk on the free semigroup. In Sections 6 and 7 we will justify the choice of the notion of topological entropy for semigroup actions: we shall prove not only that the entropy can be computed by the growth rate of the mean number of periodic orbits but also that it arises naturally as the radius of convergence of the zeta function of the semigroup action (Theorems A and B). The rationality of the zeta function for semigroup actions of expanding maps will follow as a consequence of Lemma 7.1. In Section 8 we focus on building a bridge between the several concepts of the thermodynamic formalism for skew-products and the intrinsic objects for semigroup actions. In particular, we compare the notion of classical topological entropy with the fibred and relative entropies, and relate the quenched and annealed equilibrium states for symmetric and non-symmetric random walks. For instance, we prove that the topological entropy of the semigroup coincides with Ledrappier-Walters' entropy, introduced in [17], if and only if all maps have the same degree (Proposition 8.2); and that this fibred entropy coincides with a quenched pressure in random dynamics (Proposition 8.8). Additionally, the topological entropy of the semigroup is showed to coincide with an annealed pressure in random dynamical systems (Corollary 8.7), and to be equal to the classical topological pressure of a suitable potential for the skew-product dynamics (cf. (18)). In Section 9 we will introduce a procedure to select measures for the semigroup action using some variational insight. More precisely, from connections between the semigroup action, the classical thermodynamic formalism and the annealed pressure function, we will provide an intrinsic construction of probability measures (not necessarily stationary) on the ambient space which arise as marginals of equilibrium states. The two different approaches will allow us to conclude that maximal entropy measures can be computed using either an averaged equidistribution of periodic points or of pre-images (Theorem C).

5. Ruelle-Perron-Frobenius operators

In this section we shall introduce the Ruelle-Perron-Frobenius transfer operator to be assigned to a semigroup action which is a natural extension of the concept of transfer operator for an individual dynamical system. The operator will depend on the chosen set G_1 of generators of G and on the selected random walk R_η on G (we will come back to this subject in Subsection 8.2). Let G be a semigroup generated by a finite subset G_1 of Ruelle-expanding maps acting on a compact connected metric space M and $S : G \times M \to M$ the corresponding continuous semigroup action. Given a continuous observable $\varphi : M \to \mathbb{R}$, let $\mathfrak{L}_{\underline{g},\varphi} : C^0(M) \to C^0(M)$ denote the usual

Ruelle-Perron-Frobenius operator associated to the dynamical system $\underline{g}$ and the observable φ:

$$\mathfrak{L}_{\underline{g},\varphi}(\psi)(x) = \sum_{\underline{g}(y)=x} e^{\varphi(y)}\,\psi(y). \tag{3}$$

It is not hard to check that $\mathcal{L}_{\underline{g},\varphi} = \mathfrak{L}_{g_{i_n},\varphi} \circ \mathfrak{L}_{g_{i_{n-1}},\varphi} \cdots \circ \mathfrak{L}_{g_{i_1},\varphi}$ for all $\underline{g} = g_{i_n}...g_{i_1} \in G_n$ and $n \geq 1$. Consider now the non-stationary dynamical system whose complexity is indexed by the *time* n, corresponding to the "ball of radius n" in the semigroup. Such viewpoint has turned to be very fruitful in the description of the topological entropy and the complexity of semigroup actions [18], and motivates the definition of the following weighted mean sequence of transfer operators.

Definition 5.1. Given a continuous potential $\varphi : M \to \mathbb{R}$ and a shift-invariant probability measure η on Σ_p^+, the *Ruelle-Perron-Frobenius sequence* of bounded linear operators $(\mathbf{L}_{n,\eta,\varphi})_{n\geq 1}$ acting on $C^0(M)$ is defined, for every $n \geq 1$, by

$$\mathbf{L}_{n,\eta,\varphi} = \int_{\Sigma_p^+} \mathcal{L}_{g_{i_n},\varphi} \dots \mathcal{L}_{g_{i_2},\varphi}\mathcal{L}_{g_{i_1},\varphi}\,(\mathbf{1})\, d\eta([i_1,\dots,i_n]).$$

Given $\underline{\varphi} = (\varphi_1,\dots,\varphi_p) \in C^0(M)^p$ and a non-trivial probability vector $\underline{a}$, we define the *integrated transfer operator* $\widetilde{\mathbf{L}}_{\underline{a},\underline{\varphi}} : C^0(M) \to C^0(M)$ by

$$\widetilde{\mathbf{L}}_{\underline{a},\underline{\varphi}}\phi(x) = \sum_{i=1}^{p} a_i \sum_{g_i(y)=x} e^{\varphi_i(y)}\phi(y).$$

For instance, if η is the symmetric random walk $\eta_{\underline{p}}$, then, for all $n \geq 1$,

$$\mathbf{L}_{n,\eta_{\underline{p}},\varphi} = \frac{1}{p^n}\sum_{|\underline{g}|=n} \mathcal{L}_{\underline{g},\varphi} = \widetilde{\mathbf{L}}^n_{\underline{p},\underline{\varphi}}.$$

Taking into account that the semigroup action is naturally associated to the skew-product dynamics

$$\begin{array}{cccc} \mathcal{F}_G: & \Sigma_p^+ \times M & \to & \Sigma_p^+ \times M \\ & (\omega,x) & \mapsto & (\sigma(\omega), g_{\omega_1}(x)) \end{array} \tag{4}$$

where $\omega = (\omega_1,\omega_2,\dots)$, given a Bernoulli probability measure $\eta_{\underline{a}}$ on Σ_p^+ we assign to any $\underline{\varphi} = (\varphi_1,\dots,\varphi_p) \in C^0(\Sigma_p^+ \times M)^p$ the integrated transfer operators $\widehat{\mathbf{L}}_{\underline{a},\underline{\varphi}} : C^0(\Sigma_p^+ \times M) \to C^0(\Sigma_p^+ \times M)$ for $\mathcal{F}_G$, defined by

$$\widehat{\mathbf{L}}_{\underline{a},\underline{\varphi}}\psi(\omega,x) = \sum_{i=1}^{p} a_i\,\mathcal{L}_{g_i,\varphi_i}\psi(i\omega,x) = \sum_{i=1}^{p} a_i \sum_{g_i(y)=x} e^{\varphi_i(i\omega,y)}\psi(i\omega,y) \tag{5}$$

where $\psi \in C^0(\Sigma_p^+ \times M)$ and $i\omega$ stands for the sequence $(i,\omega_1,\omega_2,\dots)$. For instance, if $\underline{\varphi} = (0,\dots,0)$ and $\psi = 1$, one gets $\widehat{\mathbf{L}}_{\underline{a},\underline{\varphi}}1 = \int \deg(g_i)\, d\underline{a}(i)$, where *deg* stands for the degree of the map. Now, for each $\varphi \in C^0(M)$ we may consider the map $\underline{\varphi}(\omega,x) = (\varphi(x),\dots,\varphi(x))$ in $C^0(\Sigma_p^+ \times M)^p$ and, dually, if $\psi \in C^0(\Sigma_p^+ \times M)$ does not depend on ω, we may take $\phi : M \to \mathbb{R}$ defined by $\phi(x) = \psi(\overline{1},x)$. Then, $\widehat{\mathbf{L}}_{\underline{a},\underline{\varphi}}\,\psi = \widetilde{\mathbf{L}}_{\underline{a},\underline{\varphi}}\,\phi$. As $\widehat{\mathbf{L}}_{\underline{a},\underline{\varphi}}$ and $\widetilde{\mathbf{L}}_{\underline{a},\underline{\varphi}}$ are positive operators, the logarithm of each spectral radius is equal to the exponential growth rate of $\|\widehat{\mathbf{L}}^n_{\underline{a},\underline{\varphi}} 1_{\Sigma_p^+\times M}\|_0$ and $\|\widetilde{\mathbf{L}}^n_{\underline{a},\underline{\varphi}} 1_M\|_0$, respectively. Thus, $sp\,(\widehat{\mathbf{L}}_{\underline{a},\underline{\varphi}}) = sp\,(\widetilde{\mathbf{L}}_{\underline{a},\underline{\varphi}})$. A similar link between transfer operators on the phase space and on the skew-product dynamics has been considered previously in [26]. However, in this reference the operators are built averaging *normalized* transfer operators for individual dynamics.

6. Topological entropy of the semigroup action

This section is devoted to the proof of Theorem A. First, let us recall the concept of separated points and topological entropy of a semigroup action adopted in [7, 18]. Given $\varepsilon > 0$ and $\underline{g} := g_{i_n} \dots g_{i_2}\, g_{i_1} \in G_n$, the *dynamical ball* $B(x,\underline{g},\varepsilon)$ is the set $B(x,\underline{g},\varepsilon) := \{y \in X : d(\underline{g}_j(y), \underline{g}_j(x)) \leq \varepsilon, \;\; \text{for every } 0 \leq j \leq n\}$ where, as before, for every $1 \leq j \leq n-1$ we denote by $\underline{g}_j$ the concatenation

$g_{i_j} \dots g_{i_2} g_{i_1} \in G_j$, and $\underline{g}_0 = id$. We also assign a dynamical metric $d_{\underline{g}}$ to M by setting $d_{\underline{g}}(x,y) := \max_{0 \le j \le n} d(\underline{g}_j(x), \underline{g}_j(y))$. Notice that both the dynamical ball and the metric depend on the underlying concatenation of generators $g_{i_n} \dots g_{i_1}$ and not on the group element $\underline{g}$, since the latter may have distinct representations. Given $\underline{g} = g_{i_n} \dots g_{i_1} \in G_n$, we say that a set $K \subset M$ is $(\underline{g}, n, \varepsilon)$-*separated* if $d_{\underline{g}}(x,y) > \varepsilon$ for any distinct $x, y \in K$. The maximal cardinality of a $(\underline{g}, \varepsilon, n)$-separated set on M will be denoted by $s(\underline{g}, n, \varepsilon)$. The topological entropy of a semigroup action estimates the growth rate in n of the number of orbits of length n up to some small error ε.

Definition 6.1. The *topological entropy* of the semigroup action $S : G \times M \to M$ is

$$h_{\text{top}}(S) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \Big(\frac{1}{p^n} \sum_{|\underline{g}|=n} s(\underline{g}, n, \varepsilon) \Big). \tag{6}$$

Ghys, Langevin and Walczak proposed in [14] another definition of topological entropy of a semigroup action using the asymptotic exponential growth rate of points that are separated by some group element. This corresponds to the largest exponential growth rate, while the definition we have adopted observes the growth rates of separated points averaged along semigroup elements.

In the context of a single Ruelle-expanding dynamics, the topological entropy is related to the largest exponential growth rate of periodic points (cf. [9]). This motivates considering the following asymptotic speed.

Definition 6.2. The *periodic entropy* of a semigroup action $S : G \times M \to M$ is

$$\wp(S) = \limsup_{n \to +\infty} \frac{1}{n} \log \left(\max\{N_n(G), 1\} \right)$$

where $N_n(G)$ has been defined in (1).

Observe that in order for $\wp(S)$ to be a finite value, the set $\text{Fix}(\underline{g})$ must be finite for each $\underline{g} \in G \setminus \{id\}$, as happens when $\underline{g}$ is expansive (that is, when there exists $\varepsilon_{\underline{g}} > 0$ such that, whenever $x \neq y \in M$, we have $\max \{d(\underline{g}^\ell(x), \underline{g}^\ell(y)) \colon \ell \in \mathbb{N}_0\} \ge \varepsilon_{\underline{g}}$.) After [18], we know that an action of a finitely generated semigroup of C^1-expanding maps is even strongly δ-expansive for some $\delta > 0$, a notion that we now recall and which will ease the task of computing $h_{\text{top}}(S)$.

Definition 6.3. Given $\delta > 0$, we say that a continuous semigroup action $S : G \times M \to M$ is δ-*expansive* if, whenever $x \neq y \in M$, there exist $\kappa \in \mathbb{N}$ and $\underline{g} \in G_\kappa$ such that $d(\underline{g}(x), \underline{g}(y)) > \delta$. The action S is said to be *strongly δ-expansive* if, for any $\gamma > 0$, there exists $\kappa_\gamma \ge 1$ such that, for every $x \neq y \in M$ with $d(x,y) \ge \gamma$, for all $\kappa \ge \kappa_\gamma$ and any $\underline{g} \in G^*_\kappa$, we have $d_{\underline{g}}(x,y) = \max_{0 \le j \le n} d(\underline{g}_j(x), \underline{g}_j(y)) > \delta$.

Lemma 6.4. [18, Theorem 25] *Let G be the semigroup generated by a set $G_1 = \{Id, g_1, \dots, g_p\}$, where G_1^* is a finite set of Ruelle-expanding maps on a compact metric space M, and $S : G \times M \to M$ its continuous semigroup action. Take* $0 < \varepsilon < \delta$. *Then*

$$h_{top}(S) = \limsup_{n \to \infty} \frac{1}{n} \log \Big(\frac{1}{p^n} \sum_{|\underline{g}|=n} s(\underline{g}, n, \varepsilon) \Big).$$

Additionally, it was proved in [18, Theorem 28] that $h_{\text{top}}(S) \le \wp(S)$. We are left to show the opposite inequality. For that purpose, we will have to specify in advance what is the path, or concatenation of elements, one is interested in tracing.

Definition 6.5. We say that a continuous semigroup action $S : G \times M \to M$, associated to a finitely generated semigroup G, satisfies the *(strong) orbital specification property* if, for any $\delta > 0$, there exists $T(\delta) > 0$ such that, given $k \in \mathbb{N}$, for any $\underline{h}_{p_j} \in G^*_{p_j}$ with $p_j \ge T(\delta)$ for every $1 \le j \le k$, for each choice of k points $x_1, \dots, x_k$ in M, for any natural numbers $n_1, \dots, n_k$ and any semigroup element $\underline{g}_{n_j, j} = g_{i_{n_j}, j} \dots g_{i_2, j}\, g_{i_1, j} \in G_{n_j}$, where $j \in \{1, \dots, k\}$, there exists $x \in M$ such that $d(\underline{g}_{\ell,1}(x),\ \underline{g}_{\ell,1}(x_1)) < \delta$ for all $1 \le \ell \le n_1$ and $d(\underline{g}_{\ell,j}\, \underline{h}_{p_{j-1}} \cdots \underline{g}_{n_2,2}\, \underline{h}_{p_1}\, \underline{g}_{n_1,1}(x),\ \underline{g}_{\ell,j}(x_j)) < \delta$ for all $2 \le j \le k$, $1 \le \ell \le n_j$, where $\underline{g}_{\ell,j} = g_{i_\ell, j} \dots g_{i_1, j}$. The semigroup action satisfies the *periodic orbital specification property* if the point x can be chosen periodic.

6.1. **Proof of Theorem A.** We start proving that the class of Ruelle-expanding maps is closed under concatenation, and so forms a semigroup.

Lemma 6.6. *If each map in the finite set G_1^* is Ruelle-expanding, then $\underline{g}$ is Ruelle-expanding for any $\underline{g} \in G - \{Id\}$. Moreover, there exists $\delta > 0$ such that the semigroup action $S : G \times M \to M$ is strongly δ-expansive.*

Proof. If g_1 and g_2 are Ruelle-expanding maps on a compact metric space, then it is not hard to use uniform continuity to prove that the composition $g_2\, g_1$ is a Ruelle-expanding map: if $\rho_i \in (0,1)$ denotes the backward contraction rate for $g_i \in G_1^*$ and $r_i > 0$ is so that all inverse branches for g_i are defined in balls of radius r_i, then every map $g_{i_2} g_{i_1}$ is Ruelle-expanding and its inverse branches are defined in balls of radius r with backward contraction rates ρ, where

$$r = \min\{r_i : 1 \leq i \leq p\} \quad \text{and} \quad \rho = \min\{\rho_i : 1 \leq i \leq p\}. \tag{7}$$

We now proceed by induction on n. If, for a fixed positive integer n, the concatenation of n Ruelle-expanding maps is Ruelle-expanding, then, considering $n+1$ such maps, say $g_{i_{n+1}}\, g_{i_n} \, \cdots \, g_{i_1}$, we may split their composition into the concatenation of two Ruelle-expanding maps $g_{i_{n+1}}\, (g_{i_n} \, \cdots \, g_{i_1})$ and apply what we have just proved.

Concerning the strong δ-expansiveness of the action S for some $\delta > 0$, take $\delta = \frac{r}{2}$ and, given $\gamma > 0$, let $\kappa_\gamma \geq 1$ be such that $\rho^{\kappa_\gamma} \delta < \gamma$, where ρ is defined by (7). Now, for any points $x \neq y \in X$ with $d(x,y) \geq \gamma$ and any $\underline{g} \in G_\kappa^*$ with $\kappa \geq \kappa_\gamma$, clearly $d_{\underline{g}}(x,y) > \delta$, otherwise we would get $d(x,y) \leq \rho^{\kappa_\gamma} d(\underline{g}(x), \underline{g}(y)) \leq \rho^{\kappa_\gamma} d_{\underline{g}}(x,y) < \gamma$ which leads to a contradiction. $\square$

Recall now, from Lemma 6.4 and the fact that the semigroup action is strongly expansive, that the computation of the topological entropy of the semigroup action may be done with a well chosen, but fixed, ε. More precisely, if $\delta > 0$ is given by the proof of Lemma 6.6 and we fix $0 < \varepsilon < \delta$, then $h_{\text{top}}(S) = \limsup_{n\to\infty} \frac{1}{n} \log \left(\frac{1}{p^n} \sum_{|\underline{g}|=n} s(\underline{g}, n, \varepsilon) \right)$. We claim that for every $\underline{g} \in G$ such that $|\underline{g}| = n$, the set $\text{Fix}(\underline{g})$ is $(\underline{g}, n, \varepsilon)$-separated. Otherwise, there would exist $P \neq Q \in \text{Fix}(\underline{g})$ which were not $(\underline{g}, n, \varepsilon)$-separated, that is, such that $d(\underline{g}_m(P), \underline{g}_m(Q)) < \varepsilon$ for every $0 \leq m \leq n$. But then, if $\gamma = d(P,Q)/2$ and $k_\gamma \geq 1$ is given by Lemma 6.6, we have $|\underline{g}^{\kappa_\gamma}| = n\kappa_\gamma \geq \kappa_\gamma$ and $d_{\underline{g}^{\kappa_\gamma}}(P,Q) = d_{\underline{g}}(P,Q) < \varepsilon$, a contradiction. Therefore, $s(\underline{g}, n, \varepsilon) \geq \sharp\, \text{Fix}(\underline{g})$, for every $\underline{g} \in G$ with $|\underline{g}| = n$, and, consequently,

$$h_{\text{top}}(S) = \limsup_{n\to\infty} \frac{1}{n} \log \left(\frac{1}{p^n} \sum_{|\underline{g}|=n} s(\underline{g}, n, \varepsilon) \right) \geq \limsup_{n\to\infty} \frac{1}{n} \log \left(\frac{1}{p^n} \sum_{|\underline{g}|=n} \text{Fix}(\underline{g}) \right) = \wp(S).$$

To complete the proof we are left to show that the lim sup in the definition of $\wp(S)$ is indeed a limit. First notice that, as G is finitely generated by Ruelle-expanding maps, by [18, Theorem 16] it satisfies the periodic orbital specification property. Fix $\varepsilon \in (0, \delta)$, let $T(\varepsilon/2) \in \mathbb{N}$ be given by this property and take $\underline{g} \in G^*_{m+n+T(\varepsilon/2)}$, to which there exist $\underline{a} \in G_n$, $\underline{b} \in G^*_{T(\varepsilon/2)}$ and $\underline{c} \in G_m$ such that $\underline{g} = \underline{a}\ \underline{b}\ \underline{c}$. Let $\text{Fix}(\underline{c}) = \{P_1, \ldots, P_r\}$ and $\text{Fix}(\underline{a}) = \{Q_1, \ldots, Q_s\}$ be the sets of fixed points of $\underline{c}$ and $\underline{a}$, respectively. By the periodic specification property, for the semigroup elements $\underline{c}, \underline{a}$ and the points $P_i \in \text{Fix}(\underline{c})$ and $Q_j \in \text{Fix}(\underline{a})$ there exists $x_{ij} \in \text{Fix}(\underline{a}\ \underline{b}\ \underline{c})$ such that $d(\underline{c}_\ell(x_{ij}), \underline{c}_\ell(P_i)) < \frac{\varepsilon}{2}$ and $d(\underline{a}_u\ \underline{b}\ \underline{c}\,(x_{ij}), \underline{a}_u(Q_j)) < \frac{\varepsilon}{2}$ for every $\ell = 0, \ldots, m$ and every $u = 0, \ldots, n$. As the set $\text{Fix}(\underline{c})$ is $(\underline{c}, m, \varepsilon)$-separated, we have $x_{i_1 j_1} \neq x_{i_2 j_2}$ for $(i_1, j_1) \neq (i_2, j_2)$. This implies that $\sharp\, \text{Fix}(\underline{g}) \geq \sharp\, \text{Fix}(\underline{a})\, \sharp\, \text{Fix}(\underline{c})$ and so,

$$\sum_{|\underline{g}|=m+n+T(\varepsilon/2)} \sharp\, \text{Fix}(\underline{g}) \geq \sum_{|\underline{c}|=m, |\underline{a}|=n} \sharp\, \text{Fix}(\underline{c})\, \sharp\, \text{Fix}(\underline{a}) = \left(\sum_{|\underline{c}|=m} \sharp\, \text{Fix}(\underline{c}) \right) \left(\sum_{|\underline{a}|=m} \sharp\, \text{Fix}(\underline{a}) \right).$$

This yields

$$\frac{1}{p^{m+n+T(\varepsilon/2)}} \sum_{|\underline{g}|=m+n+T(\varepsilon/2)} \sharp\, \text{Fix}(\underline{g}) \geq \frac{1}{p^{T(\varepsilon/2)}} \left(\frac{1}{p^m} \sum_{|\underline{c}|=m} \sharp\, \text{Fix}(\underline{c}) \right) \left(\frac{1}{p^n} \sum_{|\underline{a}|=n} \sharp\, \text{Fix}(\underline{a}) \right). \tag{8}$$

Write $\alpha_n = \log\left(\frac{1}{p^n}\sum_{|\underline{a}|=n} \sharp\,\mathrm{Fix}(\underline{a})\right)$. The inequality (8) implies that $\alpha_{m+n+T(\varepsilon/2)} \geq \alpha_n + \alpha_m$ for all $m, n \geq 1$. As $T(\varepsilon/2)$ is a fixed constant, by a simple adaptation of the proof of Fekete's Lemma ([28, Theorem 4.9]), it follows that the sequence $(\frac{\alpha_n}{n})_{n\in\mathbb{N}}$ converges to its supremum. Therefore, the lim sup in the definition of $\wp(S)$ may be replaced by a limit.

Finally, by Lemma 6.6, each $\underline{g} \in G$ is a Ruelle-expanding map and $\sharp\,\mathrm{Fix}(\underline{g}) = \deg(\underline{g})$. So, taking the observable $\varphi \equiv 0$, we get $\wp(S) = \log sp\,(\widehat{\mathbf{L}}_{\underline{p},0}) = \log\,(\frac{\sum_{i=1}^{p}\deg(g_i)}{p})$.

7. The zeta function of the semigroup action

Recall that the dynamical Artin-Mazur's zeta function of a dynamical system f computes $\zeta_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\sharp\,\mathrm{Fix}(f^n)}{n}\, z^n\right)$ (cf. [1]). The function ζ_S we associate to the action of a semigroup G, generated by a finite set G_1^* of Ruelle-expanding maps, is linked to the notion of the annealed zeta function introduced in [2, 22] within the context of random families of C^2 expanding maps. The aim of this section is to show that, when we consider Ruelle-expanding maps and the random walk $R_{\underline{p}}$, this annealed zeta function is rational and its radius of convergence is $\exp(-h_{\mathrm{top}}(S))$.

7.1. **Proof of Theorem B.** We will start estimating the radius ρ_S of convergence of ζ_S and relating it with $h_{\mathrm{top}}(S)$. Notice that, as $\lim_{n\to\infty} \sqrt[n]{n} = 1$, then

$$\frac{1}{\rho_S} = \limsup_{n\to\infty} \sqrt[n]{\frac{N_n(G)}{n}} = \limsup_{n\to\infty} \sqrt[n]{\max\{N_n(G), 1\}} = \exp(\wp(S)). \tag{9}$$

Thus, whenever $\wp(S) > 0$, the zeta function ζ_S has a positive radius of convergence and is well defined in $\{z \in \mathbb{C} \colon |z| < \exp(-\wp(S))\}$. Under the assumptions of Theorem A, one also has $\rho_S = \exp(-h_{\mathrm{top}}(S))$. We are left to show that the zeta function of a semigroup of Ruelle-expanding maps is rational.

Lemma 7.1. *The skew-product $\mathcal{F}_G$ is Ruelle-expanding and topologically mixing.*

Proof. Denote by d_M and $d_{\sum}$ the metrics in M and Σ_p^+, respectively. We are considering in $\Sigma_p^+ \times M$ the product topology, which is metrizable; its topology is given, for instance, by the metric $D((\omega^0, x_0), (\omega^1, x_1)) = \max\{d_M(x_0, x_1), d_{\sum}(\omega^0, \omega^1)\}$. As σ and each $g_i \in G_1^*$ are Ruelle-expanding, there exist positive constants c_σ and c_i, for $i \in \{1, \ldots, p\}$, such that: (a) if $x, y \in M$, $x \neq y$, $g_i(x) = g_i(y)$ then $d_M(x, y) > c_i$; and (b) $\omega^0, \omega^1 \in \Sigma_p^+$, $\omega^0 \neq \omega^1$, $\sigma(\omega^0) = \sigma(\omega^1)$ implies $d_{\sum}(\omega^0, \omega^1) > c_\sigma$. Let $(\omega^0, x_0) \neq (\omega^1, x_1)$ be such that $\mathcal{F}_G((\omega^0, x_0)) = \mathcal{F}_G((\omega^1, x_1))$, that is, $\sigma(\omega^0) = \sigma(\omega^1)$ and $g_{\omega_1^0}(x_0) = g_{\omega_1^1}(x_1)$. Then, either $\omega^0 \neq \omega^1$, in which case we have $D((\omega^0, x_0), (\omega^1, x_1)) > c_\sigma$; or else $\omega^0 = \omega^1$, and then $D((\omega^0, x_0), (\omega^1, x_1)) > c_{\omega_1^0}$. Therefore, if $c = \min\{c_\sigma, c_1, c_2, \ldots, c_d\}$ and $(\omega^0, x_0) \neq (\omega^1, x_1)$ are so that $\mathcal{F}_G((\omega^0, x_0)) = \mathcal{F}_G((\omega^1, x_1))$ then $D((\omega^0, x_0), (\omega^1, x_1) > c$. The second property that characterizes Ruelle-expanding maps (cf. Definition 2.1) is proved similarly by considering the minimal expansion rates and minimal radius where convergence holds.

We now proceed showing that $\mathcal{F}_G$ is topologically mixing. Consider a non-empty open subset $\mathcal{W}$ of $\Sigma_p^+ \times M$, and take a cylinder $U = C(1; a_1\, a_2\, \ldots, a_k)$ and an open set V of M such that $U \times V \subset \mathcal{W}$. As the maps σ and $g_{a_k} \ldots g_{a_1}$ are topologically mixing and Ruelle-expanding, there exist positive integers m_U and m_V such that $\sigma^\ell(U) = \Sigma_p^+$ and $(g_{a_k} \ldots g_{a_1})^\ell = M$, for all $\ell \geq m = \max\{m_U, m_V\}$. Hence, for all $\ell \geq m$ we have $\mathcal{F}_G^\ell(U \times V) = \left(\sigma^\ell(U), \bigcup_{\omega\in U} f_\omega^\ell(V)\right) = \Sigma_p^+ \times M$ since V contains all the sequences of $\sum_d^+$ whose k first entries are $a_1\, a_2\, \ldots, a_k$, in particular those which start with this block repeated ℓ times for every $\ell \in \mathbb{N}$. □

After Lemma 7.1, we conclude that the Artin-Mazur zeta function of $\mathcal{F}_G$, say $\zeta_{\mathcal{F}_G}$, is rational (cf. [21, 9]). Moreover, given $n \in \mathbb{N}$, if $\sharp\,\mathrm{Per}_n(\mathcal{F}_G)$ denotes the number of periodic points with period n of $\mathcal{F}_G$, then it is straightforward to check that

$$\sharp\,\mathrm{Per}_n(\mathcal{F}_G) = \sum_{\sigma^n(\omega)=\omega} \sharp\mathrm{Fix}(f_\omega^n) = \sum_{|\underline{g}|=n} \sharp\mathrm{Fix}(\underline{g}) = p^n \times N_n(G).$$

This implies that, for any $z \in \mathbb{C}$,

$$\zeta_S(z) = \exp\Big(\sum_{n=1}^{+\infty} \frac{\sharp\,\mathrm{Per}_n(\mathcal{F}_G)}{n \times p^n}\, z^n\Big) = \exp\Big(\sum_{n=1}^{+\infty} \frac{\sharp\,\mathrm{Per}_n(\mathcal{F}_G)}{n}\, \left(\frac{z}{p}\right)^n\Big) = \zeta_{\mathcal{F}_G}\left(\frac{z}{p}\right)$$

and so ζ_S is a rational function.

8. Intrinsic objects *vs* skew-product dynamics

In this section we will establish a bridge between topological Markov chains and semigroup actions regarding the notions of fibred and relative topological entropies and the concepts of annealed and quenched topological pressures.

8.1. **Thermodynamic formalism for the skew-product.** There have been several approaches to study the thermodynamic formalism of skew-product dynamics. For instance: (i) Ruelle expanding skew-product maps. (ii) Fibred entropy for factor maps. (iii) Quenched and annealed equilibrium states for random dynamical systems. (iv) Relative measures for skew-products. For future use, we briefly collect some of them, referring the reader to [2, 11, 12, 16, 25].

Topological entropy of the skew-product. Since the skew product $\mathcal{F}_G$ is a Ruelle-expanding map (cf. Lemma 7.1), for any Hölder continuous potential φ on $\Sigma_p^+ \times M$ the map $\mathcal{F}_G$ admits a unique equilibrium state. In particular, if $\varphi \equiv 0$, there is a unique measure $\mu_{\underline{m}}$ of maximal entropy of $\mathcal{F}_G$, which is equally distributed along the $\sum_{i=1}^p \deg(g_i)$ elements of the natural Markov partition $\mathcal{Q}$ on $\Sigma_p^+ \times M$ and may be computed by the limit process using distribution by pre-images (cf. [21]). From [26], the projection of $\mu_{\underline{m}}$ in Σ_p^+ is $\eta_{\underline{m}}$, where

$$\underline{m} = \Big(\frac{\deg(g_1)}{\sum_{k=1}^p \deg(g_k)}, \frac{\deg(g_2)}{\sum_{k=1}^p \deg(g_k)}, \dots, \frac{\deg(g_p)}{\sum_{k=1}^p \deg(g_k)}\Big). \tag{10}$$

Moreover, the topological entropy of the skew-product $\mathcal{F}_G$ is given by

$$h_{\mathrm{top}}(\mathcal{F}_G) = h_{\mathrm{top}}(S) + \log p \tag{11}$$

(cf. [7]), so $h_{\mathrm{top}}(\mathcal{F}_G) = \log(\sum_{i=1}^p \deg(g_i))$ and $h_{\mathrm{top}}(S) = \log(\frac{\sum_{i=1}^p \deg(g_i)}{p})$. Notice that the last equality for $h_{\mathrm{top}}(S)$ depends only on the ingredients that set up the semigroup action. More generally (see [21]), if φ is piecewise constant along the $\sum_{i=1}^p \deg(g_i)$ elements of Markov partition $\mathcal{Q}$, then there exists a unique equilibrium state μ_φ for $\mathcal{F}_G$ with respect to φ, it is a Bernoulli measure and satisfies

$$P_{\mathrm{top}}(\mathcal{F}_G, \varphi) = \log\Big(\sum_{Q \in \mathcal{Q}} e^{\varphi(Q)}\Big) \quad \text{and} \quad \mu_\varphi(Q) = e^{-\varphi(Q)} \sum_{\tilde{Q} \in \mathcal{Q}} e^{\varphi(\tilde{Q})}, \;\; \forall\, Q \in \mathcal{Q}.$$

Fibred entropy of the skew-product. Following [17], consider the skew-product $\mathcal{F}_G$ and the projection on the first coordinate, say $\pi : \Sigma_p^+ \times M \to \Sigma_p^+$, $\pi(\omega, x) = \omega$. We say that a subset E of $\pi^{-1}(\omega)$ is (n,ε)-separated if there exists $i \in \{0, \dots, n\}$ such that $d(g_{\omega_i} \dots g_{\omega_1}(x), g_{\omega_i} \dots g_{\omega_1}(y)) \geq \varepsilon$ where $\underline{g} = g_{\omega_n} \dots g_{\omega_1} \in G$ and $|\underline{g}| = n$. Therefore, if $s(n, \varepsilon, \pi^{-1}(\omega))$ is the maximal cardinality of a (n,ε)-separated subset of $\pi^{-1}(\omega)$, then $s(n, \varepsilon, \pi^{-1}(\omega)) = s(\underline{g}, n, \varepsilon)$. Given a σ-invariant probability measure η on Σ_p^+, the map $\omega \mapsto h_{\mathrm{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) := \lim_{\varepsilon \to 0} \limsup_{n\to\infty} \frac{1}{n} \log s(n, \varepsilon, \pi^{-1}(\omega))$ is measurable and (cf. [17])

$$\sup_{\{\mu:\; \mathcal{F}_{G*}(\mu) = \mu,\; \pi_*(\mu) = \eta\}} h_\mu(\mathcal{F}_G) = h_\eta(\sigma) + \int_{\Sigma_p^+} h_{\mathrm{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta(\omega). \tag{12}$$

We will refer to $h_{\mathrm{top}}(\mathcal{F}_G, \pi^{-1}(\omega))$ as the *relative entropy* on the fiber $\pi^{-1}(\omega)$. The *fibred entropy* of the semigroup action S with respect to η is $\int_{\Sigma_p^+} h_{\mathrm{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta(\omega)$.

Quenched and annealed equilibrium states for the skew-product. Following [2], given a continuous potential $\varphi : \Sigma_p^+ \times M \to \mathbb{R}$ and a probability measure $\underline{a}$ on $\{1, \ldots, p\}$, the *annealed topological pressure* of $\mathcal{F}_G$ with respect to φ and $\underline{a}$ is defined by

$$P^{(a)}_{\text{top}}(\mathcal{F}_G, \varphi, \underline{a}) = \sup_{\{\mu\colon \mathcal{F}_{G*}\mu=\mu\}} \Big\{ h_\mu(\mathcal{F}_G) - h_{\pi_*(\mu)}(\sigma) + h^{\underline{a}}(\pi_*(\mu)) + \int \varphi(\omega, x)\, d\mu(\omega, x) \Big\}$$

where $\omega = (\omega_1, \omega_2, \ldots)$, $\pi_*(\mu)$ is the marginal of μ in Σ_p^+ and the *entropy per site* $h^{\underline{a}}(\pi_*(\mu))$ with respect to $\eta_{\underline{a}}$ is given by

$$h^{\eta_{\underline{a}}}(\pi_*(\mu)) = -\int_{\Sigma_p^+} \log \psi_{\pi_*(\mu)}(\omega)\ d\pi_*(\mu)(\omega) = \int_{\Sigma_p^+} -\psi_{\pi_*(\mu)}(\omega) \log \psi_{\pi_*(\mu)}(\omega)\ d\underline{a}(\omega_1)\ d\pi_*(\mu)(\sigma(\omega))$$

if $d\pi_*(\mu)(\omega_1, \omega_2, \ldots) \ll d\underline{a}(\omega_1)\ d\pi_*(\mu)(\omega_2, \omega_3, \ldots)$ and $\psi_{\pi_*(\mu)} := \frac{d\pi_*(\mu)}{d\underline{a}\ d\pi_*(\mu)\circ\sigma}$ denotes the Radon-Nykodin derivative of $\pi_*(\mu)$ with respect to $\underline{a} \times \pi_*(\mu) \circ \sigma$; and $h^{\underline{a}}(\pi_*(\mu)) = -\infty$ otherwise. Recall from [2, page 676] that $h^{\underline{a}}(\pi_*(\mu)) = 0$ if and only if $\pi_*(\mu) = \eta_{\underline{a}}$. According to [2, Equation (2.28)], the annealed pressure can also be evaluated by

$$P^{(a)}_{\text{top}}(\mathcal{F}_G, \varphi, \underline{a}) = \sup_{\{\mu\colon \mathcal{F}_{G*}\mu=\mu\}} \Big\{ h_\mu(\mathcal{F}_G) + \int_{\Sigma_p^+ \times M} \log\big(\underline{a}(\omega_1) e^{\varphi(\omega,x)}\big)\, d\mu(\omega, x) \Big\}. \tag{13}$$

The *quenched topological pressure* of $\mathcal{F}_G$ with respect to φ and $\underline{a}$ is defined by

$$P^{(q)}_{\text{top}}(\mathcal{F}_G, \varphi, \underline{a}) = \sup_{\{\mu\colon \mathcal{F}_{G*}\mu=\mu,\ \pi_*(\mu)=\eta_{\underline{a}}\}} \Big\{ h_\mu(\mathcal{F}_G) - h_{\eta_{\underline{a}}}(\sigma) + \int \varphi(\omega, x)\, d\mu(\omega, x) \Big\}. \tag{14}$$

It follows from the definitions that we always have $P^{(a)}_{\text{top}}(\mathcal{F}_G, \varphi, \underline{a}) \ge P^{(q)}_{\text{top}}(\mathcal{F}_G, \varphi, \underline{a})$.

An $\mathcal{F}_G$-invariant probability measure is said to be an *annealed (resp. quenched) equilibrium state for $\mathcal{F}_G$ with respect to φ and $\underline{a}$* if it attains the supremum in equation (13) (resp. (14)). In the case of finitely generated semigroups of C^2 expanding maps, there exists a unique quenched and a unique annealed equilibrium state for every Hölder continuous observable φ and every $\underline{a}$, and they exhibit an exponential decay of correlations (cf. [2]). For instance, for a semigroup G with generators $G_1 = \{id, g_1, \ldots, g_p\}$ where each g_i is a C^2 expanding map, we may consider the Hölder potential $\varphi(\omega, x) = -\log|\det Dg_{\omega_1}(x)|$; its annealed equilibrium state $\mu^{(a)}_{\varphi,\underline{a}}$ was described in [2, Proposition 2] and is also the quenched equilibrium state for this potential. In less demanding setting of a semigroup action of Ruelle expanding maps, relative (or quenched) equilibrium states have been constructed by Denker, Gordin and Heinemann in [11, 12]. Beyond uniform hyperbolicity, Benedicks and Young [3] constructed absolutely continuous stationary measures using appropriate skew-product dynamics to model the random perturbations of quadratic maps.

Given a continuous potential $\varphi : \Sigma_p^+ \times M \to \mathbb{R}$ and a σ-invariant probability measure η on Σ_p^+, the notion of *relative pressure* of φ on the fiber $\pi^{-1}(\omega)$, denoted by $P_{\text{top}}(\mathcal{F}_G, \varphi, \pi^{-1}(\omega))$, was also studied in [17, Section 2]. In this reference, the authors showed that the relative pressure satisfies the following relative variational principle:

$$\int_{\Sigma_p^+} P_{\text{top}}(\mathcal{F}_G, \varphi, \pi^{-1}(\omega))\, d\eta(\omega) = \sup_{\{\mu\colon \mathcal{F}_{G*}(\mu)=\mu,\ \pi_*(\mu)=\eta\}} \Big\{ h_\mu(\mathcal{F}_G) - h_\eta(\sigma) + \int \varphi\, d\mu \Big\}. \tag{15}$$

In particular, when $\varphi \equiv 0$ and $\eta = \eta_{\underline{a}}$, the equations (14) and (12) imply that the fibred entropy coincides with the quenched pressure, that is,

$$\int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{a}}(\omega) = P^{(q)}_{\text{top}}(\mathcal{F}_G, 0, \underline{a}).$$

Example 8.1. Let G be a semigroup with generators $G_1 = \{id, g_1, \ldots, g_p\}$, where each g_i is a C^2 expanding map. For the potential $\varphi \equiv 0$ we will analyze how the annealed and quenched pressures vary with $\underline{a}$. When $\underline{a} = \underline{p} = (\frac{1}{p}, \ldots, \frac{1}{p})$, we obtain

$$P^{(a)}_{\text{top}}(\mathcal{F}_G, 0, \underline{p}) = \sup_{\{\mu\colon \mathcal{F}_{G*}\mu=\mu\}} \Big\{ h_\mu(\mathcal{F}_G) + \int \log\big(\underline{p}(\omega_1)\big)\, d\mu(\omega, x) \Big\} = h_{\text{top}}(\mathcal{F}_G) - \log p = h_{\text{top}}(S). \tag{16}$$

The corresponding annealed equilibrium state $\mu_{\underline{p}}^{(a)}$ is the measure of maximal entropy $\mu_{\underline{m}}$ of $\mathcal{F}_G$. Moreover, by [2, Equation (2.28)], for any non-trivial probability vector $\underline{a}$ and the corresponding annealed equilibrium state $\mu_{\underline{a}}^{(a)}$ we have

$$h^{\underline{a}}(\pi_*(\mu_{\underline{a}}^{(a)})) = h_{\pi_*(\mu_{\underline{a}}^{(a)})}(\sigma) + \int_{\Sigma_p^+ \times M} \log\big(\underline{a}(\omega_1)\big)\, d\mu_{\underline{a}}^{(a)}(\omega, x)$$

so,

$$h^{\underline{p}}(\pi_*(\mu_{\underline{p}}^{(a)})) = h_{\pi_*(\mu_{\underline{p}}^{(a)})}(\sigma) + \int_{\Sigma_p^+ \times M} \log\big(\underline{p}(\omega_1)\big)\, d\mu_{\underline{p}}^{(a)}(\omega, x) = h_{\eta_{\underline{m}}}(\sigma) - \log p = h_{\eta_{\underline{m}}}(\sigma) - h_{\eta_{\underline{p}}}(\sigma). \tag{17}$$

Concerning the quenched operator, we get

$$P_{\text{top}}^{(q)}(\mathcal{F}_G, 0, \underline{p}) = \sup_{\{\mu\colon \mathcal{F}_{G*}\mu=\mu,\, \pi_*(\mu)=\eta_{\underline{p}}\}} \Big\{h_\mu(\mathcal{F}_G)\Big\} - \log p.$$

Yet, as $\pi_*(\mu_{\underline{m}}) = \eta_{\underline{m}}$, if $\underline{m} \neq \underline{p}$ (which happens if the degrees of g_i are not equal; cf. (10)), then the quenched equilibrium state differs from $\mu_{\underline{p}}^{(a)}$. In general, for a non-trivial probability vector $\underline{a}$, we obtain

$$P_{\text{top}}^{(a)}(\mathcal{F}_G, 0, \underline{a}) = \sup_{\{\mu\colon \mathcal{F}_{G*}\mu=\mu\}} \Big\{h_\mu(\mathcal{F}_G) + \int_{\Sigma_p^+ \times M} \log\big(\underline{a}(\omega_1)\big)\, d\mu(\omega, x)\Big\} = P_{\text{top}}(\mathcal{F}_G, \varphi_{\underline{a}}) \tag{18}$$

where $\varphi_{\underline{a}} : \Sigma_p^+ \times M \to \mathbb{R}$ is the locally constant potential given by $\varphi_{\underline{a}}(\omega, x) = \log \underline{a}(\omega_0)$. Therefore, a quenched equilibrium state $\mu_{\underline{a}}^{(q)}$ for $\varphi \equiv 0$ and $\underline{a}$ satisfies $\pi_*(\mu_{\underline{a}}^{(q)}) = \eta_{\underline{a}}$ and

$$h_{\mu_{\underline{a}}^{(q)}}(\mathcal{F}_G) = \sup_{\{\mu\colon \mathcal{F}_{G*}\mu=\mu,\, \pi_*(\mu)=\eta_{\underline{a}}\}} h_\mu(\mathcal{F}_G). \tag{19}$$

In particular, when $\underline{a} = \underline{m}$, we conclude that

$$\mu_{\underline{m}}^{(q)} = \mu_{\underline{m}} = \mu_{\underline{p}}^{(a)} \qquad \text{and} \qquad P_{\text{top}}^{(q)}(\mathcal{F}_G, 0, \underline{m}) = h_{\text{top}}(\mathcal{F}_G) - h_{\eta_{\underline{m}}}(\sigma). \tag{20}$$

Relative measures for the skew-product. In [26, Theorem 1.3] it was shown that, having fixed a Bernoulli probability measure $\eta_{\underline{a}}$ on Σ_p^+, one may find a self-similar probability measure $\mu_{\underline{a}}$ which is invariant under the skew-product $\mathcal{F}_G$, whose projection to the base space is $\pi_*(\mu_{\underline{a}}) = \eta_{\underline{a}}$ and which satisfies

$$h_{\mu_{\underline{a}}}(\mathcal{F}_G) = \sup_{\{\mu\colon \mathcal{F}_{G*}(\mu)=\mu,\, \pi_*(\mu) = \eta_{\underline{a}}\}} h_\mu(\mathcal{F}_G) = h_{\eta_{\underline{a}}}(\sigma) + \sum_{k=1}^{p} a_k \log \deg(g_k). \tag{21}$$

When $\underline{a} = \underline{m}$, the measure $\mu_{\underline{m}}$ is the unique probability of maximal entropy of the skew-product $\mathcal{F}_G$, and a simple computation indicates that $h_{\text{top}}(\mathcal{F}_G) = \log\big(\sum_{i=1}^{p} \deg(g_i)\big)$. A generalization for other potentials was obtained in [27].

8.2. **Topological entropy of the semigroup action with respect to a random walk.** In what follows we will compare the notions of entropy for the semigroup action with the previous notions of fibred, quenched, annealed and relative pressures.

Fibred entropy for the symmetric random walk. We first study the case of the symmetric random walk, that is, when each semigroup generator receives the same weight $\frac{1}{p}$ and $\eta = \eta_{\underline{p}}$. Under this assumption, the formula (12) becomes

$$\sup_{\{\mu\colon \mathcal{F}_{G*}(\mu)=\mu,\, \pi_*(\mu) = \eta_{\underline{p}}\}} h_\mu(\mathcal{F}_G) = \log p + \int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{p}}(\omega). \tag{22}$$

Thus, taking into account that $h_{\text{top}}(\mathcal{F}_G) = \sup_\mu h_\mu(\mathcal{F}_G)$, we conclude that

$$\log p + \int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{p}} \leq h_{\text{top}}(\mathcal{F}_G) \leq \log p + \sup_{\omega \in \Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)). \tag{23}$$

Besides, (11) together with (23) imply that

$$\int_{\Sigma_p^+} h_{\rm top}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{p}} \le h_{\rm top}(S) \le \sup_{\omega \in \Sigma_p^+} h_{\rm top}(\mathcal{F}_G, \pi^{-1}(\omega)).$$

So, it makes sense to ask under what conditions $h_{\rm top}(S) = \int_{\Sigma_p^+} h_{\rm top}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{p}}$.

Proposition 8.2. *Let $G_1 = \{id, g_1, \cdots, g_p\}$, $p \ge 2$, be a finite set of expanding maps in $End^2(M)$, G be the semigroup generated by G_1 and $\mathcal{F}_G : \Sigma_p^+ \times M \to \Sigma_p^+ \times M$ be the corresponding skew-product. Then the following conditions are equivalent:*

1. $h_{top}(S) = \int_{\Sigma_p^+} h_{top}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{p}}$.
2. $\eta_{\underline{p}} = \pi_*(\mu_{\underline{m}})$*, where* $\mu_{\underline{m}}$ *is the unique maximal entropy measure for* $\mathcal{F}_G$.
3. *The degrees of the maps* g_i *are the same for all* $1 \le i \le p$.

Moreover, if any of these conditions holds, then $h_{top}(S) = \int_{\Sigma_p^+} \log \deg(g_\omega)\, d\eta_{\underline{p}}(\omega)$.

Proof. Recall that $\mathcal{F}_G$ is a topologically mixing Ruelle-expanding map (cf. Lemma 7.1). Hence, taking $\varphi \equiv 0$, we deduce from [21] that $\mathcal{F}_G$ has a unique maximal entropy measure $\mu_{\underline{m}}$. Furthermore, it follows from the variational principle for $\mathcal{F}_G$ and the relations (22) and (11) that

$$h_{\rm top}(S) = h_{\rm top}(\mathcal{F}_G) - \log p \ge \sup_{\substack{\mathcal{F}_{G*}(\mu)=\mu \\ \pi_*(\mu)=\eta_{\underline{p}}}} [h_\mu(\mathcal{F}_G) - \log p] = \int_{\Sigma_p^+} h_{\rm top}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{p}}(\omega).$$

Clearly the previous inequality becomes an equality if an only if $\pi_*(\mu_{\underline{m}}) = \eta_{\underline{p}}$, which proves that (1) is equivalent to (2). The remaining of the proof relies on the property of $\mu_{\underline{m}}$ stated in (10). It implies that $h_{\rm top}(S) = \int_{\Sigma_p^+} h_{\rm top}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{p}}$ if and only the degrees of the maps g_i are the same for all $1 \le i \le p$. This proves that (2) is equivalent to (3). Finally, as $\eta_{\underline{p}}$ is a Bernoulli measure, $h_{\eta_{\underline{p}}}(\sigma) = \log p$ and $h_{\mu_{\underline{m}}}(\mathcal{F}_G) = h_{\rm top}(\mathcal{F}_G)$, then $h_{\rm top}(S) = \int_{\Sigma_p^+} \log \deg(g_{\omega_1})\, d\eta_{\underline{p}}(\omega)$. □

Relative entropy for non-symmetric random walks. If, instead of $\eta_{\underline{p}}$, we take another σ-invariant Borel probability measure η on Σ_p^+, then η portrays an asymmetric random walk on G and it suggests a generalization of the concept of topological entropy of S.

Definition 8.3. The *relative topological entropy of the semigroup action S with respect to η* is given by $h_{\rm top}(S, \eta) = \lim_{\epsilon \to 0} \limsup_{n\to\infty} \frac{1}{n} \log \int_{\Sigma_p^+} s(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon)\, d\eta(\omega)$, where $s(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon)$ is the maximum cardinality of a $(\underline{g}, n, \varepsilon)$-separated set (cf. Section 6).

The previous notion is well defined since the map $\omega \to s(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon)$ is constant on n-cylinders (hence measurable), bounded by $e^{n \max_{i \in \{1,\dots,p\}} \{h_{\rm top}(g_i)\}}$ and $s(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon)$ is monotonic in the variable ϵ. For instance, $h_{\rm top}(S, \eta_{\underline{p}}) = h_{\rm top}(S)$. In view of Definition 8.3, $h_{\rm top}(S, \eta)$ is also given by $h_{\rm top}(S, \eta) = \limsup_{n\to\infty} \frac{1}{n} \log \big(\sum_{|\underline{g}|=n} \iota_*(\eta)(\underline{g})\, \mathcal{L}_{\underline{g},0}(\mathbf{1}) \big)$. (Notice that this formula makes sense since $\underline{g} \mapsto \mathcal{L}_{\underline{g},0}(\mathbf{1})$ is bounded, its values are away from 0 and ∞ and $\iota_*(\eta)$ is a probability measure.) Following an argument analogous to the one used to prove [18, Theorem 25], we obtain:

Corollary 8.4. *Assume that the continuous action of G on the compact metric space M is strongly δ^*-expansive. Then, if $0 < \varepsilon < \delta^*$, we have* $h_{top}(S, \eta) = \lim_{n\to\infty} \frac{1}{n} \log \int_{\Sigma_p^+} s(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon)\, d\eta(\omega)$.

A variational principle for the relative entropy. Denote by $\mathcal{M}_B(\Sigma_p^+)$ the space of Bernoulli measures on Σ_p^+, that is, the probability measures $\eta = \eta_{\underline{a}}$ for some probability vector $\underline{a} = (a_1, \dots, a_p)$, where some of the entries may be zero. This space of σ-invariant measures, which encodes all random walks on the semigroup G we have considered so far, is homeomorphic to the finite dimensional simplex $\{\underline{a} = (a_1, \dots, a_p) \in \mathbb{R}^p : a_i \ge 0 \text{ and } \sum_{i=1}^p a_i = 1\}$, and therefore it is a closed subset of $\mathcal{M}_\sigma(\Sigma_p^+)$. Let $\mathcal{H}$ be the entropy map with respect to the random walks in $\mathcal{M}_B(\Sigma_p^+)$, given by

$$\begin{array}{rccc} \mathcal{H}: & \mathcal{M}_B(\Sigma_p^+) & \to & [0, +\infty] \\ & \eta & \mapsto & h_{\rm top}(S, \eta). \end{array}$$

Lemma 8.5. *The map $\mathcal{H}$ is continuous.*

Proof. Firstly, recall that the action of G on the compact metric space M is strongly δ^*-expansive. To prove that $\mathcal{H}$ is lower semicontinuous, we go back to the proof of Theorem A where it was established that, having fixed $\varepsilon \in (0, \delta_*)$, the periodic specification property produces $T(\varepsilon/2) \in \mathbb{N}$ such that very $\underline{g} \in G^*_{m+n+T(\varepsilon/2)}$ can be written as $\underline{g} = \underline{a}\ \underline{b}\ \underline{c}$ (for $\underline{a} \in G^*_n$ and $\underline{c} \in G^*_m$) and $\sharp\,\mathrm{Fix}(\underline{g}) \geq \sharp\,\mathrm{Fix}(\underline{a})\,\sharp\,\mathrm{Fix}(\underline{c})$. This proves that $\int \sharp\,\mathrm{Fix}(\underline{g})\ d\eta \geq \big(\int \sharp\,\mathrm{Fix}(\underline{c})\ d\eta\ \big)\ \big(\int \sharp\,\mathrm{Fix}(\underline{a})\ d\eta\big)$ and so, as we are restricting to $\mathcal{M}_B(\Sigma_p^+)$, the map

$$\mathcal{H}(\eta) = \sup_{n \geq 1} \frac{1}{n} \log \int \sharp\,\mathrm{Fix}\,(g_{\omega_n} \ldots g_{\omega_1})\ d\eta(\omega) \tag{24}$$

is the supremum of the continuous functions $\eta \mapsto \frac{1}{n} \log \int \sharp\,\mathrm{Fix}\,(g_{\omega_n} \ldots g_{\omega_1})\ d\eta$, hence lower semicontinuous. We proceed by showing that $\mathcal{H}$ is upper semicontinuous as well. This is due to the characterization of of the topological entropy via generating sets. Indeed, the same steps of the proof of Theorem 25 in [18] (where integration is considered with respect to the equidistributed Bernouli measure η_p) imply that, as S is a strongly δ^*-expansive continuous semigroup action, then $h_{\mathrm{top}}(S, \eta) = \limsup_{n\to\infty} \frac{1}{n} \log \int_{\Sigma_p^+} c(g_{\omega_n} \ldots g_{\omega_1}, n, \epsilon) d\eta(\omega)$ for every $0 < \varepsilon < \delta^*$, where $c(g_{\omega_n} \ldots g_{\omega_1}, n, \epsilon) = \inf\{\sharp\mathcal{U} : \mathcal{U} \text{ is a } (g_{\omega_n} \ldots g_{\omega_1}, \varepsilon) - \text{cover}\}$. Take $\varepsilon > 0$, $n \in \mathbb{N}$ and $\underline{g} = g_{\omega_{n+m}} \ldots g_{\omega_1} \in G$, and consider $\underline{\ell} = g_{\omega_{n+m}} \ldots g_{\omega_{n+1}}$ and $\underline{k} = g_{\omega_n} \ldots g_{\omega_1}$. Given an $(\underline{\ell}, \varepsilon)$-cover $\mathcal{U}$ and a $(\underline{k}, \varepsilon)$-cover $\mathcal{V}$, the set $\mathcal{W} = \underline{k}^{-1}(\mathcal{U}) \vee \mathcal{V}$ is a $(\underline{g}, \varepsilon)$-cover with $\sharp\mathcal{W} \leq \sharp\mathcal{U}\ \sharp\mathcal{V}$. This implies that $c(g_{\omega_{n+m}} \ldots g_{\omega_1}, n+m, \epsilon) \leq c(g_{\omega_{n+m}} \ldots g_{\omega_{m+1}}, m, \epsilon)\ c(g_{\omega_n} \ldots g_{\omega_1}, n, \epsilon)$. Since all measures in $\mathcal{M}_B(\Sigma_p^+)$ are Bernoulli, the previous inequality implies that the sequence $(\beta_n)_{n\in\mathbb{N}} = \big(\log \int_{\Sigma_p^+} c(g_{\omega_n} \ldots g_{\omega_1}, n, \epsilon)\ d\eta(\omega)\big)_{n\in\mathbb{N}}$ is subadditive and $\mathcal{H}$ is upper semicontinuous as it can be expressed as the infimum of continuous functions: $\mathcal{H}(\eta) = \inf_{n\geq 1} \frac{1}{n} \log \int c(g_{\omega_n} \ldots g_{\omega_1})\ d\eta(\omega)$. $\square$

Proposition 8.6. *There exists $\eta_0 \in \mathcal{M}_B(\Sigma_p^+)$ such that $\sup_{\eta\in\mathcal{M}_B(\Sigma_p^+)} h_{top}(S, \eta) = h_{top}(S, \eta_0)$. Moreover,*

$$\sup_{\eta\in\mathcal{M}_B(\Sigma_p^+)} h_{top}(S, \eta) = \log\Big(\max_{1\leq i\leq p} \deg\,(g_i)\Big) \tag{25}$$

and $\eta_{\underline{p}}$ attains the supremum if and only if the degrees of the maps g_i are the same for all $1 \leq i \leq p$.

Proof. The first assertion is a direct consequence of the compactness of $\mathcal{M}_B(\Sigma_p^+)$ together with the continuity of the function $\mathcal{H}$. We are left to prove (25). Let $1 \leq j \leq p$ be such that $\deg\,(g_j) = \max_{1\leq i\leq p} \deg\,(g_i)$. Take $\underline{a} = (a_i)_{1\leq i\leq p}$, where $a_i = \delta_{ij}$ is the Kronecker delta function, and $\eta_{\underline{a}} = \delta_{jjj\ldots}$. Then $\sup_{\eta\in\mathcal{M}_B(\Sigma_p^+)} h_{\mathrm{top}}(S, \eta) \geq h_{\mathrm{top}}(S, \eta_{\underline{a}}) = h_{\mathrm{top}}(g_j) = \log\deg\,(g_j) = \log\Big(\max_{1\leq i\leq p} \deg\,(g_i)\Big)$. Conversely, assume that $\sup_{\eta\in\mathcal{M}_B(\Sigma_p^+)} h_{\mathrm{top}}(S, \eta) > \log\Big(\max_{1\leq i\leq p} \deg\,(g_i)\Big)$. Using (24), we may find $\delta > 0$ and $\eta \in \mathcal{M}_B(\Sigma_p^+)$ satisfying

$$\lim_{n\to\infty} \frac{1}{n} \log \int \sharp\mathrm{Fix}(g_{\omega_n} \ldots g_{\omega_1})\ d\eta(\omega) > \log\Big(\max_{1\leq i\leq p} \deg\,(g_i)\Big) + 2\delta.$$

Then, for every large n, there exists $\omega \in \Sigma_p^+$ such that $\sharp\,\mathrm{Fix}\,(g_{\omega_n} \ldots g_{\omega_1}) > e^{\delta n}\,\big(\max_{1\leq i\leq p} \deg\,(g_i)\big)^n$ which contradicts the growth rate of the fixed point sets of the expanding maps $g_{\omega_n} \ldots g_{\omega_1}$. Finally, recall that $h_{\mathrm{top}}(S, \eta_p) = \log\Big(\frac{\sum_{i=1}^p \deg\,(g_i)}{p}\Big)$ so η_p is a maximizing measure for $\mathcal{H}$ if and only if the maps g_i, for $1 \leq i \leq p$, have equal degrees.

From equality (25) we also conclude that any probability measure in $\mathcal{M}_B(\Sigma_p^+)$ that attains the maximum of $\mathcal{H}$ is of the form $\eta_{\underline{a}} \in \mathcal{M}_B(\Sigma_p^+)$ for some $\underline{a}$ in the simplex

$$\Delta = \Big\{\underline{a} = (a_1 \ldots, a_p) \in \mathbb{R}_0^+ : \sum_{i=1}^p a_i = 1 \text{ and } a_k = 0 \text{ whenever } \deg\,(g_k) \neq \max_{1\leq i\leq p} \deg\,(g_i)\Big\}.$$

In particular, we have uniqueness of such maximizing measures if and only if there exists a unique expanding map with largest degree. $\square$

Relative entropy *vs.* annealed and quenched pressure. In order to compare the relative entropies of the semigroup action with the several notions of pressure for skew-product dynamics we shall use the transfer operators defined in Section 5. As previously remarked, when $\eta = \eta_{\underline{a}}$, we have $h_{\text{top}}(S, \eta_{\underline{a}}) = \limsup_{n\to\infty} \frac{1}{n} \log \|\widetilde{\mathbf{L}}^n_{\underline{a},\underline{\varphi}}(\mathbf{1})\|_0 = \limsup_{n\to\infty} \frac{1}{n} \log \|\widehat{\mathbf{L}}^n_{\underline{a},\underline{\varphi}}(\mathbf{1})\|_0$. It follows from [2, Proposition 3.2] that the spectral radius of $\widehat{\mathbf{L}}_{\underline{a},\underline{\varphi}}$ coincides with $\exp(-P^{(a)}_{\text{top}}(\mathcal{F}_G, \underline{\varphi}, \underline{a}))$ and, for that reason,

$$h_{\text{top}}(S, \eta_{\underline{a}}) = P^{(a)}_{\text{top}}(\mathcal{F}_G, 0, \underline{a}). \tag{26}$$

This suggests the following generalization of Bufetov's formula (11).

Corollary 8.7. *Let $G_1 = \{id, g_1, \cdots, g_p\}$, $p \geq 2$, where G_1^* is a finite set of expanding maps in $End^2(M)$. If $h^{\underline{a}}(\pi_*(\mu^{(a)}_{\underline{a}})) = h_{\pi_*(\mu^{(a)}_{\underline{a}})}(\sigma) - h_{\eta_{\underline{a}}}(\sigma)$, then $h_{\mu^{(a)}_{\underline{a}}}(\mathcal{F}_G) = h_{top}(S, \eta_{\underline{a}}) + h_{\eta_{\underline{a}}}(\sigma)$.*

Proof. It is a direct consequence of the equality (26) since, under the assumption on $h^{\underline{a}}(\pi_*(\mu_{\underline{a}}))$, we have

$$P^{(a)}_{\text{top}}(\mathcal{F}_G, 0, \underline{a}) = \sup_{\{\mu\colon \mathcal{F}_{G*}\mu=\mu\}} \Big\{ h_\mu(\mathcal{F}_G) - h_{\pi_*(\mu)}(\sigma) + h^{\underline{a}}(\pi_*(\mu)) \Big\} = h_{\mu^{(a)}_{\underline{a}}}(\mathcal{F}_G) - h_{\eta_{\underline{a}}}(\sigma).$$

Notice that the assumption on $h^{\underline{a}}(\pi_*(\mu^{(a)}_{\underline{a}}))$ is fulfilled when $\underline{a} = \underline{p}$ (cf. (17)). □

Given a non-trivial probability vector $\underline{a}$, for some potentials φ the transfer operator $\widehat{\mathbf{L}}_{\underline{a},\underline{\varphi}}$ coincides with the averaged normalized transfer operator used in [26]. Therefore, we may match the values of the corresponding pressures and their equilibrium states, and deduce the following thermodynamic criterium for the self-similar probability measures constructed in [26].

Proposition 8.8. *Let $G_1 = \{id, g_1, \cdots, g_p\}$, $p \geq 2$, where G_1^* is a finite set of expanding maps in $End^2(M)$. Given a Bernoulli probability measure $\eta_{\underline{a}}$ on Σ_p^+ and the corresponding self-similar probability measure $\mu_{\underline{a}}$, then the following assertions are equivalent:*

1. $h_{\mu_{\underline{a}}}(\mathcal{F}_G) = \sup_{\{\mu\colon \mathcal{F}_{G*}(\mu)=\mu,\, \pi_*(\mu)=\eta_{\underline{a}}\}} h_\mu(\mathcal{F}_G)$.
2. $P^{(q)}_{top}(\mathcal{F}_G, 0, \underline{a}) = \int \log \deg(g_i)\, d\underline{a}(i)$.

Moreover, $\int_{\Sigma_p^+} h_{top}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_a(\omega) = \sum_{k=1}^p a_k \log \deg(g_k) = \int_{\Sigma_p^+} \log \deg(g_{\omega_1})\, d\eta_a(\omega)$.

We observe that the condition (1) is equivalent to say that $\mu_{\underline{a}} = \mu^{(q)}_{\underline{a}}$, where $\mu^{(q)}_{\underline{a}}$ is the unique quenched equilibrium state of $\mathcal{F}_G$ with respect to $\varphi \equiv 0$ and $\underline{a}$; notice also that (1) and (2) happen when $\underline{a} = \underline{m}$.

Proof. Fix $\underline{a} = (a_1, a_2, \ldots, a_p)$ and the potential $\underline{\varphi} = (-\log\deg(g_1), \ldots, -\log\deg(g_p))$. The transfer operator $\widehat{\mathbf{L}}_{\underline{a},\underline{\varphi}}$ is precisely the averaged normalized transfer operator introduced in [26]. Therefore, $\widehat{\mathbf{L}}_{\underline{a},\underline{\varphi}}1 = 1$, and consequently $P^{(a)}_{\text{top}}(\mathcal{F}_G, \underline{\varphi}, \underline{a}) = 0$. Moreover, $\mu_{\underline{a}} = \mu^{(a)}_{\varphi,\underline{a}}$, which is the unique annealed equilibrium state for $\mathcal{F}_G$ with respect to $\underline{\varphi}$ and $\underline{a}$. So, equation (13) yields

$$h_{\mu_{\underline{a}}}(\mathcal{F}_G) = -\int \log(\underline{a}(\omega_1) e^{\varphi(\omega,x)})\, d\mu_{\underline{a}}.$$

On the other hand, the quenched variational principle indicates that

$$\sup_{\{\mu\colon \mathcal{F}_{G*}\mu=\mu,\, \pi_*(\mu)=\eta_{\underline{a}}\}} \Big\{ h_\mu(\mathcal{F}_G) \Big\} - h_{\eta_{\underline{a}}}(\sigma) = P^{(q)}_{\text{top}}(\mathcal{F}_G, 0, \underline{a}).$$

Thus the first condition in the statement of the proposition is equivalent to the equality

$$\begin{aligned} P^{(q)}_{\text{top}}(\mathcal{F}_G, 0, \underline{a}) &= -\int \log(\underline{a}(\omega_1) e^{\varphi(\omega,x)})\, d\mu_{\underline{a}} - h_{\eta_{\underline{a}}}(\sigma) \\ &= \sum_{i=1}^p -a_i \log a_i + \sum_{i=1}^p a_i \log\deg(g_i) + \sum_{i=1}^p a_i \log a_i = \int \log\deg(g_i)\, d\underline{a}(i). \end{aligned}$$

Finally, equations (21) and (12) imply $\int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_a = \sum_{k=1}^p a_k \log\deg(g_k)$. The second equality is immediate from the fact that $\eta_{\underline{a}}$ is a Bernoulli probability measure. □

8.3. **Examples.** Let us analyze two (non-abelian) examples that illustrate the range of applications of our results on semigroup actions, transfer operators and the dynamical zeta function.

Example 8.9. Consider the semigroup G generated by $G_1 = \{id, g_1, g_2\}$ where g_1, g_2 are circle expanding maps given by $g_1(z) = z^2$ and $g_2(z) = z^3$. A simple computation shows that, for every $n \in \mathbb{N}$, we have $N_n(G) = \frac{1}{2^n}\sum_{k=0}^{n}(2^k\,3^{n-k} - 1) = \mathcal{O}\left(\frac{5}{2}\right)^n$; consequently, $h_{\text{top}}(S) = \wp(S) = \log 5 - \log 2$. If $\eta_{\underline{m}}$ is the Bernoulli probability measure on Σ_2^+ determined by the weights $\underline{m} = (\frac{2}{5}, \frac{3}{5}) \equiv (0.4, 0.6)$, equation (21) becomes $h_{\mu_{\underline{m}}}(\mathcal{F}_G) = h_{\text{top}}(\mathcal{F}_G) = \log 5$ and equation (12) informs that

$$\int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{m}}(\omega) = \frac{2}{5}\,\log 2 + \frac{3}{5}\,\log 3.$$

So, $h_{\text{top}}(S) < \int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{m}}(\omega)$. Concerning $\underline{a} = (\frac{1}{2}, \frac{1}{2}) \equiv (0.5, 0.5)$ and $\eta_{\underline{a}} = \eta_{\underline{2}}$, we deduce from Proposition 8.2 that $h_{\text{top}}(S) > \int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{2}}(\omega)$. If $\mu_{\underline{2}}$ is the self-similar measure assigned to $\eta_{\underline{a}}$ in [26], then, again by (21) and (12), we have $h_{\mu_{\underline{2}}}(\mathcal{F}_G) = \log 2 + \frac{\log 2 + \log 3}{2}$ and

$$\sup_{\mu:\ \mathcal{F}_{G*}(\mu)=\mu,\ \pi_*(\mu)=\eta_{\underline{2}}} h_\mu(\mathcal{F}_G) = \log 2 + \int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{2}}(\omega).$$

Consequently,

$$\frac{\log 2 + \log 3}{2} \le \int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{2}}(\omega) < h_{\text{top}}(S) = \log 5 - \log 2.$$

Moreover, from Proposition 8.8, we conclude that $\int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{2}}(\omega) = \frac{\log 2 + \log 3}{2}$ and, more generally, that, for any choice of $\underline{a} = (a_1, a_2)$ with $a_i > 0$ and $a_1 + a_2 = 1$, we have $\int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{a}}(\omega) = a_1 \log 2 + a_2 \log 3$. Therefore, there is a (unique) vector $\underline{a}$ whose corresponding probability measure $\eta_{\underline{a}}$ on Σ_2^+ satisfies $h_{\text{top}}(S) = \int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))\, d\eta_{\underline{a}}(\omega)$, namely $\underline{a} = \left(\frac{\log \frac{6}{5}}{\log \frac{3}{2}}, \frac{\log \frac{5}{4}}{\log \frac{3}{2}}\right) \approx (0.45, 0.55)$.

Example 8.10. Given $q \in \mathbb{N}$, let $A_i \in GL(q, \mathbb{Z})$ induce linear expanding endomorphisms $g_i := g_{A_i}$ on $\mathbb{T}^q$, for $i = 1, \ldots, p$. Consider the locally constant potential $\varphi : \Sigma_p^+ \times \mathbb{T}^q \to \mathbb{R}$ given by $\varphi(\omega, x) = -\log|\det Dg_{\omega_1}(x)| = -\log \deg(g_{\omega_1})$. Then, by [2, Proposition 2], the quenched and the annealed equilibrium states of φ coincide and are SRB measures. In this setting, for any non-trivial probability vector $\underline{a}$, the self-similar $\mathcal{F}_G$-invariant probability measure $\mu_{\underline{a}}$ constructed in [26] coincides with the annealed (hence quenched) SRB measure for $\mathcal{F}_G$ with respect to $\mathcal{F}$, $\underline{\varphi}$ and $\underline{a}$. In particular, we have $h_{\mu_{\underline{a}}}(\mathcal{F}_G) = P^{(q)}_{\text{top}}(\mathcal{F}_G, \underline{\varphi}, \underline{a}) + h_{\eta_{\underline{a}}}(\sigma) - \int \varphi\, d\mu_{\underline{a}}$ so, comparing this equality with (21), we obtain

$$P^{(q)}_{\text{top}}(\mathcal{F}_G, \underline{\varphi}, \underline{a}) = \sum_{i=1}^{p} a_i\, \log \deg(g_i) + \int \varphi\, d\mu_{\underline{a}} = \int \log \deg(g_i)\, d\underline{a}(i) + \int \varphi\, d\mu_{\underline{a}}.$$

9. Selection of measures for semigroup actions

The action of a semigroup generated by more than one dynamics is not a dynamical system, thus it is not straightforward how to define equilibrium states and establish a variational principle that might relate topological and measure theoretical aspects of the semigroup action. Yet, a semigroup action can be embodied into a dynamical system whose topological and measure theoretical properties we may convey to the semigroup action. From the formulas (18) and (26) in Section 8, recall that $h_{\text{top}}(S, \eta_{\underline{a}}) = P^{(a)}_{\text{top}}(\mathcal{F}_G, 0, \underline{a}) = P_{\text{top}}(\mathcal{F}_G, \varphi_{\underline{a}})$. These two different flavored equalities motivate the construction of maximal entropy measures for semigroup actions from the skew-product dynamics, in a way that discloses periodic data and highlights the equidistribution of pre-images.

Projection of the maximal entropy measure of the skew product. Given an Hölder potential $\psi : M \to \mathbb{R}$, consider the fiberwise constant potential in $\Sigma_p^+ \times M$ defined by $\varphi = \varphi_\psi : \Sigma_p^+ \times M \to \mathbb{R}$ such that $\varphi_\psi(\omega, x) = \psi(x)$. Then we say that a probability measure ν on the Borel sets of M is an *equilibrium state for the semigroup action with respect to* ψ if $\nu = (\pi_M)_*(\mu_\varphi)$, where μ_φ is the unique equilibrium state for the (topologically mixing Ruelle-expanding) map $\mathcal{F}_G$ with respect to φ_ψ. Notice, however, that such a ν may not be stationary with respect to a previously fixed random walk η. It is true that, if $\mu_\varphi = (\mu_\omega)_{\omega \in \Sigma_p^+}$ is a Rohlin disintegration of μ_φ along the measurable partition $(\pi^{-1}(\omega))_{\omega \in \Sigma_p^+}$, then $\nu = \int_{\Sigma_p^+} \mu_\omega \, d\eta(\omega)$. But, although this equality resembles the η-stationarity condition (cf. (2)), these two notions do not necessarily coincide. For instance, if the semigroup G is generated by a finite set of C^2-expanding maps and $\psi \equiv 0$, then μ_φ is the measure of maximal entropy of $\mathcal{F}_G$ and it is also its annealed equilibrium state with respect to ψ and $\eta_{\underline{a}}$, where $\underline{a}$ was given in (10). Yet, $(\pi_M)_*(\mu_\varphi)$ is also the measure constructed in [6], and from this reference one infers that $(\pi_M)_*(\mu_\varphi)$ is not, in general, stationary.

Marginals of quenched/annealed equilibrium states of the skew product. The discussion in Subsection 8.1 suggests another approach. Given an Hölder continuous potential $\psi : M \to \mathbb{R}$ and its corresponding $\varphi = \varphi_\psi$, we may associate to a random walk on G, determined by a probability measure $\eta_{\underline{a}}$ on Σ_p^+, two probability measures on M which are the marginals of the unique annealed and quenched equilibrium states $\mu^{(a)}_{\underline{a},\varphi}$ and $\mu^{(q)}_{\underline{a},\varphi}$ of the potential φ_ψ on the skew product. Notice that, even when the quenched and annealed states are different (which may happen; see [2, Section 2.4]), it is not plain that they have different marginals on M. Since the physically observable measures are these M-marginals, it is worth exploring under what conditions on the semigroup action they coincide, in which case it would be natural to say that a probability measure ν on the Borel sets of M is an *equilibrium state for the semigroup action with respect to* ψ *and* $\underline{a}$ if $\nu = (\pi_M)_*(\mu^{(\star)}_{\underline{a},\varphi})$, where $\mu^{(\star)}_{\underline{a},\varphi}$ is either the unique annealed or quenched equilibrium state for the skew-product $\mathcal{F}_G$ with respect to $\varphi = \varphi_\psi$ and $\star \in \{a, q\}$. To find a candidate to be a measure of maximal entropy, the appropriate potential would be $\varphi \equiv 0$ and, indeed, the equalities (16) confirm that the corresponding annealed equilibrium state for $\mathcal{F}_G$ with respect to the symmetric random walk $\eta_{\underline{p}}$ has an annealed pressure equal to the topological entropy of the semigroup. However, as we have verified in Example 8.1, in this case the annealed and quenched equilibrium states may differ. We remark that it may happen that the annealed equilibrium state $\mu^{(a)}_{\underline{a},\varphi}$ for $\mathcal{F}_G$ with respect to a potential φ and a non-trivial probability vector $\underline{a}$ satisfies $\pi_*(\mu^{(a)}_{\underline{a},\varphi}) = \eta_{\underline{a}}$, in which case $h^{\underline{a}}(\pi_*(\mu^{(a)}_{\underline{a},\varphi})) = 0$ and so $\mu^{(a)}_{\underline{a},\varphi}$ is also the quenched equilibrium state of φ with respect to $\underline{a}$ (cf. [2, Proposition 2]). This occurs, for example, when G is generated by a finite set $G_1 = \{id, g_1, \ldots, g_p\}$ of C^2 expanding maps on a Riemannian manifold and the potential φ is defined by $\varphi(\omega, x) = -\log|\det Dg_{\omega_1}|(x)$. In this particular case, the marginal on M of this common equilibrium state has a disintegration which is almost everywhere absolutely continuous with respect to the Lebesgue measure on M (cf. [2, Remark 3.4]). Moreover, for the special vector $\underline{p}$, the transfer operator $\mathbf{L}_{\underline{p},\underline{\varphi}}$ coincides with the averaged Ruelle-Perron-Frobenius of [2]. However, this potential is not a φ_ψ for any observable map ψ on M.

Proof of Theorem C. In the setting of a single topologically mixing Ruelle-expanding map T, the maximal entropy measure is the equilibrium state of the potential $\varphi \equiv 0$ and may be computed as a weak* limit of the sequence of averages of Dirac measures supported on pre-images of any point (cf. [21]), namely $\nu = \nu_n(x) = \frac{1}{\deg(T)^n} \sum_{T^n(y)=x} \delta_y$, where δ_y is the Dirac measure supported on $\{y\}$ and $\deg(T) = \sharp T^{-1}(\{a\})$, a number independent of $a \in M$ and such that $h_\nu(T) = \log(\deg(T))$. This suggests a third approach to find the concept of maximal entropy measure for a semigroup action. Consider a finitely generated semigroup G of of C^2 expanding maps, with a generating set $G_1 = \{id, g_1, \cdots, g_p\}$ where $p \geq 2$, and take the equidistributed Bernoulli measure $\eta_{\underline{p}}$. It follows from [6] that the sequence of measures

$$\frac{1}{\lambda^n} \sum_{|\underline{g}|=n} \sum_{\underline{g}(y)=x} \delta_y$$

where $\lambda = \deg(g_1) + \ldots + \deg(g_p)$, is weak* convergent to a probability measure $\nu_{0,p}$ on M (independently of x). This motivates the choice of $\nu_{0,p}$ as a *measure of maximal entropy for the semigroup action with respect to* $\eta_{\underline{p}}$, with the major advantage of being defined intrinsically, using only the fixed generators of the semigroup.

More generally, for $\eta = \eta_{\underline{a}}$ and an Hölder potential $\underline{\varphi}$ on $\Sigma_p^+ \times M$, there is a natural stationary transfer operator $\widetilde{\mathbf{L}}_{\underline{a},\underline{\varphi}}$ acting on $C^0(M)$ (cf. Section 5) satisfying $sp\,(\widetilde{\mathbf{L}}_{\underline{a},\underline{\varphi}}) = sp\,(\widehat{\mathbf{L}}_{\underline{a},\underline{\varphi}}) = \exp\,(P^{(a)}_{\text{top}}(\mathcal{F}_G, \varphi, \underline{a}))$. Moreover, by [2, Proposition 3.1], the spectral properties of the transfer operators $\widetilde{\mathbf{L}}_{\underline{a},\underline{\varphi}}$ and $\widehat{\mathbf{L}}_{\underline{a},\underline{\varphi}}$ are strongly related. For instance, $\lambda_{\underline{a},\varphi} := \exp\,(P^{(a)}_{\text{top}}(\mathcal{F}_G, \varphi, \underline{a}))$ is the leading eigenvalue for the transfer operator $\widetilde{\mathbf{L}}_{\underline{a},\underline{\varphi}}$ acting on $C^r(M)$ ($r \geq 1$) with a one-dimensional eigenspace generated by some $\rho_{\varphi,\underline{a}} \in C^r(M)$. The dual operator $\widetilde{\mathbf{L}}^*_{\underline{a},\underline{\varphi}}$ defined, for every continuous $\psi : M \to \mathbb{R}$, by $\int \psi \, d\widetilde{\mathbf{L}}^*_{\underline{a},\underline{\varphi}}\eta = \int \widetilde{\mathbf{L}}_{\underline{a},\underline{\varphi}}\psi \, d\eta$ has also a one-dimensional eigenspace associated to the leading eigenvalue $\lambda_{\underline{a},\varphi}$, generated by some probability measure $\gamma_{\varphi,\underline{a}}$ on M. Additionally, $(\pi_M)_* \circ \widehat{\mathbf{L}}^*_{\underline{a},\underline{\varphi}} = \widetilde{\mathbf{L}}^*_{\underline{a},\underline{\varphi}} \circ (\pi_M)_*$ and, for each $x \in M$, the measures obtained by averaging the pre-images of x according to the random walk $\eta_{\underline{a}}$

$$\lambda^{-n}_{\underline{a},\varphi}\,(\widetilde{\mathbf{L}}^n_{\underline{a},\underline{\varphi}})_*\,\delta_x = \frac{1}{\lambda^n_{\underline{a},\varphi}} \int \sum_{g_{\omega_n}\cdots g_{\omega_1}(y)=x} e^{S_n\varphi_{\underline{g}}(y)}\,\delta_y \, d\eta_{\underline{a}}(\omega_1,\ldots,\omega_n) \tag{27}$$

are convergent to the measure $\nu_{\varphi,\underline{a}} = \rho_{\varphi,\underline{a}}(x) \cdot \gamma_{\varphi,\underline{a}}$ on M. Indeed, observe that there exists a Borel probability measure $\hat{\gamma}_{\varphi,\underline{a}}$ on $\Sigma_p^+ \times M$ such that $\widetilde{\mathbf{L}}^*_{\underline{a},\underline{\varphi}}\hat{\gamma}_{\varphi,\underline{a}} = \lambda_{\underline{a},\varphi}\,\hat{\gamma}_{\varphi,\underline{a}}$, $\mu^{(a)}_{\varphi,\underline{a}} = \rho\hat{\gamma}_{\varphi,\underline{a}}$ and $(\pi_M)_*\hat{\gamma}_{\varphi,\underline{a}} = \gamma_{\varphi,\underline{a}}$ (cf. [2, Proposition 3.1(2) and Proposition 3.2]). Thus, the marginal measure of $\mu^{(a)}_{\varphi,\underline{a}}$ on M satisfies $(\pi_M)_*(\mu^{(a)}_{\varphi,\underline{a}}) = \nu_{\varphi,\underline{a}}$. In particular, when $\varphi \equiv 0$, we obtain $\nu_{0,\underline{a}} = \rho_{0,\underline{a}}(x) \cdot \gamma_{0,\underline{a}} = (\pi_M)_*(\mu^{(a)}_{0,\underline{a}})$, for every $x \in M$.

We now show that, having fixed the symmetric random walk $R_{\underline{p}} = \iota_*(\eta_{\underline{p}})$, the probability measure $\nu_{0,\underline{p}}$ also describes the distribution of the periodic points of the semigroup action, that is,

$$\nu_{0,\underline{p}} = \lim_{n\to\infty} e^{-n\,h_{top}(S)} \sum_{\sigma^n(\omega)=\omega} \;\sum_{g_{\omega_n}\cdots g_{\omega_1}(x)=x} \delta_x.$$

By Lemma 7.1, the skew product $\mathcal{F}_G$ is a Ruelle expanding map, so its measure of maximal entropy $\mu_{\underline{m}}$ is the weak* limit of the sequence

$$\nu_n(\omega, x) = \frac{1}{\#\text{Fix}\,(\mathcal{F}^n_G)} \sum_{\mathcal{F}^n_G(\tilde{\omega},y)=(\omega,x)} \delta_{(\tilde{\omega},y)}.$$

Moreover, we have $h_{\text{top}}(S) = h_{\text{top}}(S, \eta_{\underline{p}}) = P_{\text{top}}(\mathcal{F}_G, \varphi_{\underline{p}})$ and $\mu_{\underline{m}} = \mu^{(a)}_{\underline{p}}$. Consequently, by the continuity of the push-forward map $(\pi_M)_*$ and the known asymptotic growth rate of periodic orbits (cf. Theorem A), we conclude that

$$\begin{aligned}
\nu_{0,\underline{p}} = (\pi_M)_*(\mu^{(a)}_{\underline{p}}) = (\pi_M)_*(\mu_{\underline{m}}) &= (\pi_M)_*\Big(\lim_{n\to\infty} \frac{\sum_{(\omega,x)\in\text{Fix}\,(\mathcal{F}^n_G)} \delta_{(\omega,x)}}{\#\text{Fix}\,(\mathcal{F}^n_G)}\Big) \\
&= \lim_{n\to\infty} (\pi_M)_*\Big(\frac{\sum_{\sigma^n(\omega)=\omega}\sum_{g_{\omega_n}\cdots g_{\omega_1}(x)=x} \delta_{(\omega,x)}}{\sum_{\sigma^n(\omega)=\omega} \#\text{Fix}(g_{\omega_n}\cdots g_{\omega_1})}\Big) \\
&= \lim_{n\to\infty} e^{-n\,h_{top}(S)} \sum_{\sigma^n(\omega)=\omega} \;\sum_{g_{\omega_n}\cdots g_{\omega_1}(x)=x} \delta_x.
\end{aligned}$$

In particular, $\nu_{0,\underline{p}}(A) = \lim_{n\to\infty} e^{-n\,h_{top}(S)} \sum_{\sigma^n(\omega)=\omega} \#\big\{\text{Fix}(g_{\omega_n}\cdots g_{\omega_1}) \cap A\big\}$ for any measurable set $A \subset M$ whose boundary has $\nu_{0,\underline{p}}$ measure zero.

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