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Dynamics in Hotelling based Models

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Thesis submitted to Faculty of Sciences of University of Porto,
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Acknowledgments

To my parents, for all.

To Professor Alberto Adrego Pinto, for the possibility to work with him, for his insights and support.

To all those who, one way or the other, helped me to become better.

Abstract

In 1929, Hotelling presented a model for spacial competition between two firms in which they compete in a two-stage game. First, both firms choose their location and after they choose their prices. In this work, we assume that the establishment of prices occurs in a dynamic way instead of a static one. We will use two different dynamics, namely, the Myopic Dynamics and the Best-Response Dynamics. Our main goal, will be to study the stability of the Nash Price Equilibrium. We also study the conditions that make the set of prices, for which every firm has a non-empty market share, a forward invariant set. For last, we will address the extension of the Hotelling model to a network, proposed by A. A. Pinto and T. Parreira. Again, we assume a dynamic establishment of prices and our goal is to study the stability of the Nash Price Equilibrium.

Resumo

Em 1929, Hotelling apresentou um modelo de competição espacial entre duas firmas. Neste modelo, as firmas competem num jogo com duas etapas. Em primeiro lugar, as firmas escolhem a sua localização e, em seguida, escolhem o seu preço. Neste trabalho, nós iremos assumir que a escolha de preços ocorre de forma dinâmica. Iremos usar duas dinâmicas distintas, nomeadamente, Myopic Dynamics e Best-Response Dynamics. O nosso objectivo, é estudar a estabilidade do equilíbrio de Nash em preços. Iremos também estudar as condições que tornam o conjunto de preços, para os quais as firmas partilham o mercado, num conjunto invariante. Por último, iremos estudar a extensão do modelo de Hotelling para uma network, proposto por A. A. Pinto e T. Parreira. De novo, iremos assumir a existência de uma escolha dinâmica de preços. O objectivo, é estudar a estabilidade do equilíbrio de Nash em preços.

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Chapter 1

Introduction

The model of spacial competition, introduced by Hotelling [6], has become an attractive framework to the study of oligopoly markets (see [5], [2], [13], [8] [1]). Hotelling, in his work, presented a city where the consumers buy a commodity from one of the two firms available. This city is represented by a segment line where the consumers are uniformly distributed. The firms, also located in the segment line, compete in a two-stage game. In first place, both firms choose their location and, afterwards, each one establishes the price for the commodity. In his first approach, Hotelling concluded that the firms would have an incentive to establish themselves in the center of the segment line. However, D`Aspremont et al. [5] showed that Hotelling`s conclusion is invalid. In this work, we will focus in the second stage of the Hotelling model by introducing a dynamical component. We consider that the firms continuously adjust the price of the commodity, instead of fixing the prices. Our goal is to study the stability of the Nash Price Equilibrium and the forward invariance of the Duopoly Zone. In addition, we intend to gain an insight in how the price of the commodity and market share of each firm varies with time. We will study the Hotelling model with different transportation and productions costs for both firms. We will utilize two different dynamics for the second sub game, namely, the Myopic Dynamics and the Best-Response Dynamics [9].
A. A. Pinto and T. Parreira [2], presented a model of an Hotelling town that generalizes the Hotelling model to a network, where the consumers and the firms are located. Like in the original Hotelling model, the consumers buy a single commodity from one of the firms available. In addition, the consumers are uniformly distributed along the edges of the network, also called roads. The firms compete, again, in a two stage game, choosing the location in the first sub game and the price

of the commodity in the second sub game. A. A. Pinto and T. Parreira introduced a condition, linking transportation costs, production costs and the topology of the network, that guarantees the existence of a Nash equilibrium in the price sub game.

In the last chapter of this work, we study the stability of the Nash Price Equilibrium in the Hotelling Network, assuming, that every firm adjusts is price accordingly to the principles of the Myopic Dynamics.

Chapter 2

Basics of Game Theory and Dynamics

In this chapter, we will introduce some concepts of Game Theory and Differential Equations, that will be necessary in the rest of the text. Since the results are well known, some of them are presented without proof, but, in such cases, a reference is provided.
We will start by doing a brief introduction to differential equations and qualitative analyses.

2.1 Qualitative Analyses of Differential Equations

A system of differential equations in $\mathbb{R}^n$ is given by a group of linked differential equations

$$\begin{cases} \frac{dx_1}{dt} = f_1(t, x_1, \ldots, x_n) \\ \frac{dx_2}{dt} = f_2(t, x_1, \ldots, x_n) \\ \qquad \ldots \\ \frac{dx_n}{dt} = f_n(t, x_1, \ldots, x_n) \end{cases} \tag{2.1}$$

where each of the functions f_i is defined in a open subset U of $\mathbb{R}^n$, f_i is real valued and at least C^0. The system 2.1 can be written in the abbreviated form

$$X^{`} = F(t, X)$$

where $X = (x_1, \ldots, x_n)$ and $F(t, X) = (f_1(t, X), \ldots, f_n(t, X))$, commonly referred to as a **vector field**. If the system does not depend explicitly of the variable t, then we refer to it as an **au-**

tonomous system. We say that $X(t;X_I)$ is a solution for system 2.1, with initial condition X_I, if

$$X^{`}(t) = F(t, X)$$

and

$$X(0;X_I) = X_I \ .$$

It is not guarantee, from the start, that such a solution exists or even if it is unique. An answer for this problem is given by the **Theorem of Existence and Uniqueness of Solution** (see [10]).

2.1.1 Stability

Definition 1. A point $X_0 \in \mathbb{R}^n$ is an **equilibrium point** for a system of differential equations if

$$F(t, X_0) = 0 \ .$$

Now, we introduce a key notion in qualitative theory, that is the **stability** of an equilibrium point.

Definition 2. Let $X_0 \in \mathbb{R}^n$ be an equilibrium point for a system of differential equations:

- X_0 is **stable** if
  - for all $\varepsilon > 0$ there exists $\delta > 0$ such that, if $X_I \in B(X_0;\delta)$ then $X(t;X_I) \in B(X_0;\varepsilon)$ for all $t \in \mathbb{R}^+$.
- otherwise, X_0 is **unstable**.
- X_0 is **asymptotically stable** if X_0 his stable and
  - there exists $\delta^* > 0$ such that, if $X_I \in B(X_0;\delta^*)$ then $X(t;X_i) \to X_0$ as $t \to +\infty$.

Informally, an equilibrium point X_0 is stable, if a solution for the system that starts close of X_0, remains close to X_0, for all positive t. The point X_0 is asymptotically stable if it is stable and if there exists a neighborhood of X_0, where the solution of the system converges to X_0. A general method for investigating the stability of a fixed point was introduced by Lyapunov.

Definition 3. Let X_0 be an equilibrium point for a system of differential equations $X^{`} = F(t, X)$ where $F : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^n$. A function $V : D \subseteq \mathbb{R}^{n+1} \to \mathbb{R}$ is a **Lyapunov function** for the above system if:

- D is a open subset of $\mathbb{R}^{n+1}$ and $X_0 \in D \subseteq U$.
- $V(X_0) = 0$ and $V(X) > 0$ for all $X \in D - \{X_0\}$.
- $(V(X))^{`} \leq 0$ for all $X \in D - \{X_0\}$.

If the last condition is strict then V is a **strict** Lyapunov function .

Theorem 1. *Let X_0 be an equilibrium point for a system of differential equations, $X^{`} = F(t, X)$ where $F : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^n$.*

- *If there exists a Lyapunov function for the system then the equilibrium point is stable.*
- *If the Lyapunov function is strict then the equilibrium point is asymptotically stable.*

A proof of this result can be seen in [10]. In the particular case where the system is autonomous, and all of the functions f_i are affine, there exists a more direct way for proving the stability of an equilibrium point.

Theorem 2. *Let $X^{`} = F(X)$ be a system of differential equations where $F : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^n$ is an affine vector field and X_0 is an equilibrium point. Consider the matrix A that allows the system to be written as $Y = AY + R$ and let $\{\lambda_i : 1 \leq i \leq n\}$ be set of eigenvalues of A. If $|\lambda_i| < 0$, for all $1 \leq i \leq n$, then X_0 is asymptotically stable.*

Another notion that will play a key role in this work, is the notion of a **forward invariant** set.

Definition 4. Let Γ be a subset of $\mathbb{R}^n$ and $X(t; X_I)$ a solution of a system of differential equations. The set Γ is forward invariant if for all $t \geq 0$ and for all $X_I \in \Gamma$

$$X(t; X_I) \subseteq \Gamma \ .$$

2.1.2 Lotka-Voterra Differential Equations

Now, we make an introduction a well-known class of differential equations called **Lotka-Volterra**. The most common example of a Lotka-Volterra type equation is the predator-prey equation proposed by Vito Volterra. This equation is given by

$$\begin{cases} \frac{dx}{dt} & = x(a - by) \\ \frac{dy}{dt} & = y(-c + dx) \ . \end{cases} \tag{2.2}$$

This equation models the dynamics of interacting populations of predators y and preys x . The parameters a and c represent the intrinsic growth of the respective species, while the parameters b and d represent the effect that one population has on the other population. We are interested in the generalization of this equation for an arbitrary number of species.

Definition 5. The general Lotka-Volterra type equation, for n populations, is of the form

$$\frac{dx_i}{dt} = x_i(r_i + \sum_{j=1}^{n} a_{ij}x_j) \ . \tag{2.3}$$

The coefficients r_i represent the intrinsic growth of the population i and the parameter a_{ij} describe the impact of the j-th population upon the i-th population. The matrix $A = (a_{ij})$ is often called the **interaction matrix**.

This equation has a non-trivial equilibrium point, $x^* = (x_1^*, \ldots, x_n^*)$, determined by the linear system

$$\sum_{j=1}^{n} a_{ij}x_j = -r_i \ . \tag{2.4}$$

If for all $i \in \{1, \ldots, n\}$ the components x_i^* satisfy $x_i^* > 0$ we call this equilibrium point **feasible**. Furthermore if A is non-singular then x^* exists and is unique.

Definition 6. Let A be a symmetric matrix of dimension $n \times n$ and real entries. A is called **negative definite** if all of its eigenvalues are negative. Equivalently, symmetric matrix A is negative definite if

$$x^T A x < 0$$

for all non-zero x in $\mathbb{R}^n$.

There are a number of results regarding the dynamic behavior of this type of differential equation (see for example [7]). In this work, we concentrate our attention in a result proved in 1977 by Goh [4] regarding the stability of a feasible fixed point of this type of differential equations.

Theorem 3. *Assume that the general Lotka-Volterra equation has feasible fixed point $x^* = (x_1^*, \ldots, x_n^*)$. In addition, assume that the interaction matrix A is non-singular. If there exists a constant positive diagonal matrix C, such that, $CA + A^TC$ is negative definite then x^* is asymptotically stable with a Lyapunov function given by*

$$V = \sum_{n}^{n} c_i[x_i - x_i^* - x_i^* ln(x_i/x_i^*)] \ .$$

Proof. Assume that exists a positive diagonal matrix $C = (c_1, \ldots, c_n)$, such that, $CA + A^TC$ is negative definite. The function

$$V = \sum_{i=1}^{n} c_i[x_i - x_i^* - x_i^* ln(x_i/x_i^*)]$$

has a global minimum at $(x_1^*, \ldots, x_n^*)$.

Using 2.4, the derivative of this function computed along the solutions of equation 2.3 is given by

$$\begin{aligned} \frac{dV}{dt}(X) &= \sum_{i=1}^{n} c_i(x_1 - x_i^*)(\frac{dx_i/dt}{x_i}) \\ &= \sum_{i=1}^{n} c_i(x_i - x_i^*)(r_i + \sum_{j=1}^{n} a_{ij}x_j) \text{ by 2.4} \\ &= \sum_{i=1}^{n} c_i(x_i - x_i^*) \sum_{i=1}^{n} a_{ij}(x_j - x_j^*) \\ &= \frac{1}{2}(x - x^*)^T(CA + A^TC)(x - x^*) \ . \end{aligned}$$

Since $CA + A^TC$ is negative definite then $\frac{dV}{dt}(X)$ is negative, for all $x \neq x^*$, and V is indeed a Lyapunov function for 2.3 which concludes the proof. □

This insights regarding Lotka-Volterra equations will be particularly useful in Chapter 4 and 6. The last result of this section, is a theorem proved by Gershgorin (see [15]) concerning the eigenvalues of an arbitrary matrix.

Theorem 4. *Let A be a matrix with dimension $n \times n$. Then all of its eigenvalues lie into, at least, one of the discs*

$$D(a_{ii}, R_i) = |\lambda - a_{ii}| \leq R_i \tag{2.5}$$

where $R_i = \sum_{j \neq i}^{n} |a_{ij}|$.

In the next section, we introduce some useful concepts of Game Theory.

2.2 Basics in Game Theory

The field of Game Theory started from very sporadic considerations, made by authors like E. Zermelo [16], E. Borel [3] or Steinhaus [14], to a widely considered approach to economic theory. Most of this path was walked by John Von Neuwman and Oskar Morgenstern, a collaboration which ultimately led to the publication of the book entitled *Theory of Games and Economic Behavior*. Later on, John Nash gives his contribution to the theory of non cooperative games (see [11]). He was awarded with the Nobel Prize in Economics and since then many more have won this prize for their contributions in this area.

Informally a game can be described as a set of players, who are matched, to take some actions, chosen from a given set. An utility function is attached to every player, that determines his utility, given a combination of actions, taken by all the players. The objective of each player is to maximize his utility function.

Definition 7. A game is described by the quadruple

$$G =< I, M, (S_i)_{i \in I}, (\pi_i)_{i \in I} >$$

where:

1. $I = \{1, 2, \ldots, n\}$ is the set of types of players;
2. $M = \{m_1, m_2, \ldots, m_n\}$ specifies the number of players for type i;
3. for every $i \in I$, S_i is the closed set of strategies (or actions) for each player type;

Defining $m_j(i) = m_j$ if $j \neq i$ and $m_i(i) = m_i - 1$ the function

$$\pi_i : \times_{j \in I} S_j^{m_j(i)} \times S_i \to \mathbb{R}$$

$$\pi_i(s_{-i}; s_i) = \pi_i((s_1^1, \ldots, s_1^{m_1}, \ldots, s_1^{m_n}, \ldots, s_n^{m_n}); s_i)$$

represents the utility obtained by the player i when he plays s_i and the other players play the combination of strategies s_{-i}.

Keep in mind that a player objective, in any game, is to maximize his utility function, thus the next logical step is to know the strategy or strategies that maximizes the utility function, given the other players strategies.

Definition 8. Given a game G and a vector of strategies s_i,

$$BR_i(s_{-i}) = argmax_{s_i \in S} \pi_i(s_{-i}; s_i)$$

is the set of **best response strategies for player** i given the combination of actions s_{-i} taken by the other players.

Let $BR(S) = (BR_1(s_{-1}), \ldots, BR_i(s_{-i}), \ldots)$. We note that the set $BR()$ might not define a best response function, because $BR()$ might be only a correspondence between strategies. Now, we state one of the most important concepts in Game Theory, the **Nash Equilibrium**.

Definition 9. A strategy vector $x^* \in \times_{j \in I} S_j^{m_j(i)} \times S_i$ is a **Nash Equilibrium** if

$$x^* \in BR(x^*) \ .$$

Example 1. A classical example of a game is the so called Prisoner\`s Dilema.
Suppose that two members of a criminal gang are caught by the police and that the prosecutors do not have enough evidences to convict the pair on the charges. In order to solve this problem they put the criminals in different rooms, and simultaneously they offer them the following deal:

- Betray the other criminal, by testifying that they have committed the crime. (B)
- Stay silent. (S)

Formally, we define this game by having two players $\{\alpha, \beta\}$ of the same type with a discrete set of strategies, namely $\{B, S\}$. The payoffs are summarized in table 2.1.

α, β	B	S
B	(1,1)	(0,3)
S	(0,3)	(2,2)

Table 2.1: Utilities for the Prisoner`s Dilemma.

Each pair (i, j), of table 2.1, means that the players α,β will get i and j years in jail, respectively, when they play the corresponding strategies. The Best-Response correspondence is

$$\begin{aligned} BR_\alpha(B) &= S \\ BR_\alpha(S) &= S \\ BR_\beta(B) &= S \\ BR_\beta(S) &= S \end{aligned}$$

Hence, it follows that the Nash Equilibrium is the pair of strategies (S, S) were each of the criminals gets two years in jail. Interestingly, if the players choose to play (B, B) each of them would only get one year in jail, which is better than what they gain by playing (S, S). This is a canonical example, were pure "rational" behaviors, in the sense that players only try to maximize their utilities, might not lead to the best choices.

Definition 10. Given a game G, we say that the players are choosing their strategies accordingly to the **Best-Response Dynamics** if

$$\frac{ds_i}{dt} \in BR(s_{-i}) - s_i \tag{2.6}$$

where $BR(s_{-i})$ denotes the **Best-Response Correspondence** of player i given the strategies of all the other players.

This Best-Response Dynamics was introduced by Matsui [9]. We note that equation 2.6 does not necessarily define a differential equation since $BR()$ may only be a correspondence and not a function. A big number of results concerning the Best Response Dynamics have been proven for general games. A simple result linking the Best-Response Dynamics to the Nash Equilibrium is the following.

Definition 11. Given a game $G =< I, M, (S_i)_{i\in I}, (\pi_i)_{i\in I} >$ then S^* is a Nash Equilibrium if, and only if, S^* is an equilibrium point for the Best-Response Dynamics.

In the case of the Hotelling model that we analyze in this thesis, the Best-Response Dynamics thus in fact define a differential equation. Our objective, is to use the tools exposed in the last section to investigate the stability of the Nash Equilibrium. We will also investigate the asymptotically properties of the solutions in order to obtain some economical insight. This field of Game Theory is normally referred to as Evolutionary Game Theory.

Chapter 3

Hotelling Model

In 1929, H. Hotelling [6] introduced a spacial model of a competition between two firms. In this model firms are located along a segment line of given length l and sell the same commodity at prices p_A and p_B respectively. The consumers are uniformly distributed along the line and buy one unity of product at each unit of time. The price P payed by the consumers is given by the price at which they buy the commodity, p_A or p_B, plus a transportation cost per unit of distance. We will consider that this transportation cost can be different, t_A or t_B, depending of the firm where the consumer buys the product. We consider that each firm has a unitary production cost c_A and c_B corresponding to the price the firms pay per one unity of the commodity. From the point of view of game theory there are two sub games to be considered in this model, first the firms choose their location and afterwards the firms choose their prices. In this work, we will mainly concentrate on the price sub game assuming that the firms are located in the extremes of the line.

3.1 Indifferent Consumer and Profit

In a game the goal of the players is to maximize their utility, with this fact in mind is logic to consider that the firms will try to sell their product at the highest price possible. On the other hand, each consumer will buy at the firm that guaranties him the lowest price P. Now, we introduce the **indifferent consumer** that is important to determine the profits of both firms. If both firms have a non-empty market share the **indifferent consumer** x is the consumer who pays the same price for the commodity whether he buys at firm A or firm B, that is

$$p_A + t_A x = p_B + t_B(l - x) \; . \tag{3.1}$$

If the firm A supplies the whole market then the indifferent consumer is taken to be $x = l$. On other hand if the firm B supplies the whole market then the indifferent consumer is taken to be $x = 0$.

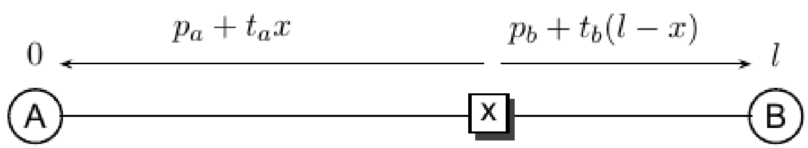


Figure 3.1: Hotelling Model Diagram

Lema 1. *Given a price configuration* (p_A, p_B) *the indifferent consumer is given by*

$$x(p_A, p_B) = \begin{cases} l & \textit{if } p_B - p_A \geq t_A l \\ \frac{p_B - p_A + t_B l}{t_A + t_B} & \textit{if } -t_B l < p_B - p_A < t_A l \\ 0 & \textit{if } p_B - p_A \leq -t_B l \; . \end{cases} \tag{3.2}$$

Proof. In the computation of x there are three cases to consider:

1. The firm A supplies the whole market;
2. Both firms have a non-empty market share, that is $0 < x < l$;
3. The firm B supplies the whole market;

Suppose that firm A supplies the whole market then the price payed by the consumer located in the end of the line satisfies

$$p_A + t_A l \leq p_B \; .$$

If firm B supplies the whole market then the consumer located in the beginning of the line satisfies

$$p_B + t_B l \leq p_A \; .$$

If both firm are sharing the market then the indifferent consumer satisfies

$$p_A + t_A x = p_B + t_B(l - x)\ . \tag{3.3}$$

Solving for x, we obtain

$$x = \frac{p_B - p_A + t_B l}{t_A + t_B}\ . \tag{3.4}$$

In this case x satisfies $0 < x < l$ and so

$$-t_B l < p_B - p_A < t_A l\ .$$

□

Since we are interested that both firms have a non empty market share, that is $0 < x < l$, we introduce the following definition of the **Duopoly Property**.

Definition 12. A price configuration (p_A, p_B) satisfies the **Duopoly Property** if, and only if,

$$-t_B l < p_B - p_A < t_A l\ . \tag{3.5}$$

We observe that $p_A \geq c_A$ and $p_B \geq c_B$ to guarantee have a non-negative profit. The definition of indifferent consumer allows us to define the profit functions for the Hotelling game. Observe that the indifferent consumer $x(p_A, p_B)$ represents the numbers of consumers that buy on firm A. Therefore the profit of a firm A is given by its profit margin $p_A - c_A$ times the number of consumers $x(p_A, p_B)$ that buy at firm A. Hence,

$$\pi_A = \pi_A(p_A, p_B) = \begin{cases} (p_A - c_A)l & \text{if } p_A \leq p_B - t_A l \\ \frac{(p_A - c_A)(p_B - p_A + t_B l)}{t_A + t_B} & \text{if } -t_B l < p_B - p_A < t_A l \\ 0 & \text{if } p_A \geq p_B + t_B l\ . \end{cases} \tag{3.6}$$

The reasoning behind the profit of firm B is analogous, however, we need to have in consideration that the number of clients, in this case, is given by $l - x(p_A, p_B)$. Hence,

$$\pi_B = \pi_B(p_A, p_B) = \begin{cases} (p_B - c_B)l & \text{if } p_B \leq p_A - t_B l \\ \frac{(p_B - c_B)(p_B - p_A + t_B l)}{t_A + t_B} & \text{if } -t_B l < p_B - p_A < t_A l \\ 0 & \text{if } p_B \geq p_A + t_A l\ . \end{cases} \tag{3.7}$$

3.2 The Duopoly Zone

The Duopoly Property in conjunction with $p_A \geq c_A$ and $p_B \geq c_B$, allows us to give a geometric interpretation of the set of price configurations, for which, the firms are sharing the market. From now on, we refer to this set of prices as **Duopoly Zone**. Hence, figure 3.2 represents one of the

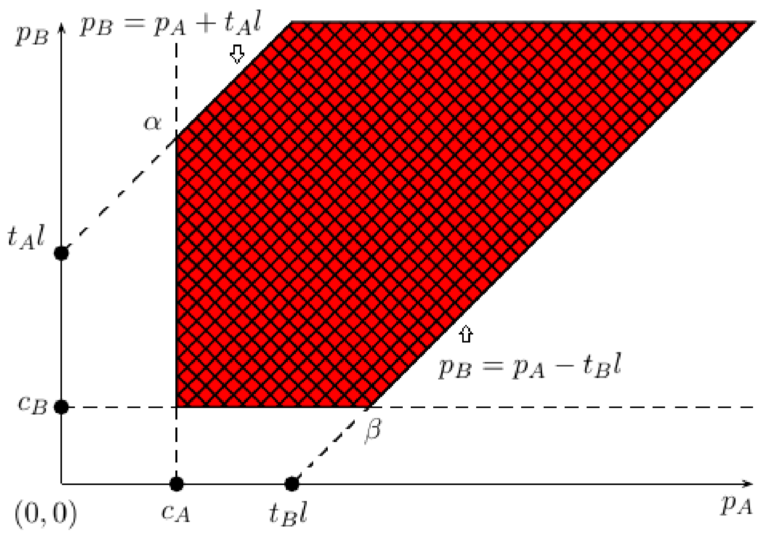


Figure 3.2: Duopoly Zone

possible shapes of the Duopoly Zone. The horizontal and vertical boundaries of the Duopoly Zone are given, respectively, by the sets

$$B_h = \{(p_A, c_B) : p_A \geq c_A \text{ and } (p_A, c_B) \text{ has Duopoly Property } \} \tag{3.8}$$

$$B_v = \{(c_A, p_B) : p_B \geq c_B \text{ and } (c_A, p_B) \text{ has Duopoly Property } \} \tag{3.9}$$

On other hand, the superior and inferior diagonal boundaries of the Duopoly Zone are represented, respectively by

$$B_s = \{(p_A, p_B) : p_B = p_A + t_A l \text{ and } p_A > c_A\} \tag{3.10}$$

$$B_i = \{(p_A, p_B) : p_B = p_A - t_B l \text{ and } p_B > c_B\} \tag{3.11}$$

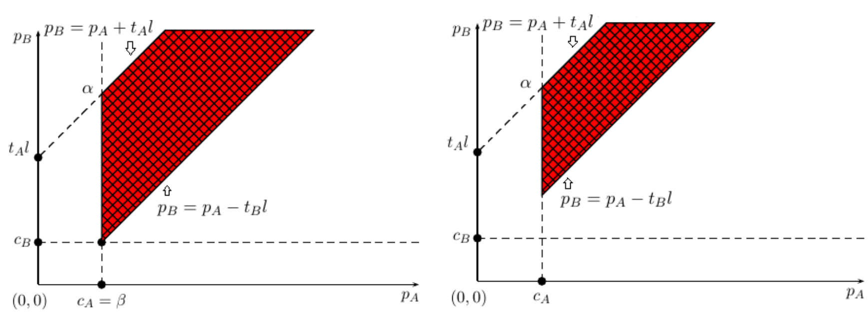
We note that above the boundary B_s, the firm A supplies the whole market and the indifferent consumer is $x = l$. Below the boundary B_i, the firm B supplies the entire market and the indifferent consumer is $x = 0$.

Let $c_B - c_A$ be denoted by Δc.

There are three possibilities for the shape of the Duopoly Zone depending upon the following inequalities:

1. $-t_B l < \Delta c < t_A l$;
2. $\Delta c \leq -t_B l$;
3. $\Delta c \geq t_A l$;

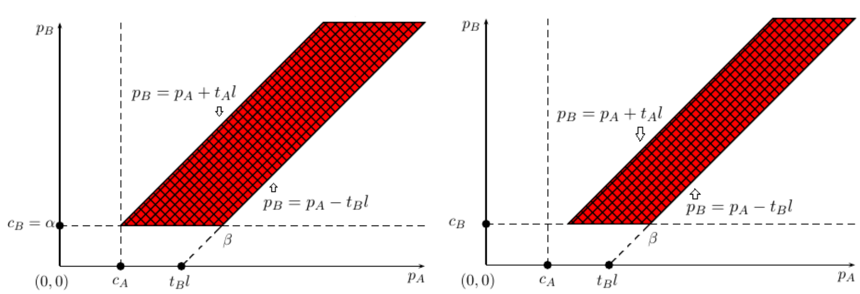
In the first case, point (c_A, c_B) has the Duopoly Property. Hence, the Duopoly Zone has the shape represented by figure 3.2. In the second case, point (c_A, c_B) belongs to the boundary B_i or is below the boundary the B_i. Figure 3.3 represents the shape of the Duopoly Zone in case 2. The last



(a) Case $\Delta c = -t_B l$ (b) Case $\Delta c < -t_B l$

Figure 3.3: Shape of the Duopoly Zone in Case 2

case is symmetric to case 2 and point (c_A, c_B) belongs to the boundary B_s or is above boundary B_s. Figure 3.4 represents the shape of the Duopoly Zone in the last case.



(a) Case $\Delta c = t_A l$ (b) Case $\Delta c > t_A l$

Figure 3.4: Shape of the Duopoly Zone in Case 3

We observe that, in cases 2 and 3 one of the firms has an advantage in the competition. In case 2, the firm B can throw firm A out of the market without having to lower its price to values near its production cost. In case 3, the advantage belongs to firm A, who can throw firm B out of the market without having to lower its price near its production cost.

As a final note, we make a remark regarding the lines with slope one located inside the Duopoly Zone. Along these lines, the market share of each firm remains constant. If the firms start, at some point, to collaborate they can raise the prices by the same amount changing their market-shares. From the consumer point of view this is not good, because they can end paying a lot more for the product. So this discussion justifies the introduction of mechanisms to protect the consumer from this possible firm cooperation, specially in markets where the commodity being sold is essential.

Chapter 4

Best-Response Dynamics

In this chapter, our objective is to find the Best-Response Dynamics for the Hotelling Model. First of all, we need to focus our attention in finding the Best-Response Correspondence of each of the firms. Since the profit functions are similar for both firms, we concentrate in finding the Best-Response Correspondence for firm A.

4.1 Dynamics

We recall that the profit π_A is given by

$$\pi_A = \pi_A(p_A, p_B) = \begin{cases} (p_A - c_A)l & \text{if } p_A \leq p_B - t_A l \\ \frac{(p_A - c_A)(p_B - p_A + t_B l)}{t_A + t_B} & \text{if } -t_B l < p_B - p_A < t_A l \\ 0 & \text{if } p_A \geq p_B + t_B l \ . \end{cases} \tag{4.1}$$

The second branch of the profit function is a concave quadratic function in p_A, with zeros given by $p_A = c_A$ and $p_A = p_B + t_B l$. Consequently, admits a maximum point for $p_A = \frac{p_B + t_B l + c_A}{2}$. However, this point may not be the global optimum for π_A. We are going to show that the form and maximum of π_A will depend on the relative position in the line of the four possible maximum candidates,

$$c_A,\ p_B + t_B l,\ \frac{p_B + t_B l + c_A}{2},\ p_B - t_A l\ . \tag{4.2}$$

Let us suppose, that the smaller value is $p_B + t_B l$ then, we obtain that $p_B + t_B l \leq p_B - t_A l$. This is an absurd, since the transportation costs are positive and not necessarily equally. On the other hand, consider that the smaller value is $\frac{p_B+t_Bl+c_A}{2}$ then, we obtain that $\frac{p_B+t_Bl+c_A}{2} \leq c_A$ and $\frac{p_B+t_Bl+c_A}{2} \leq p_B + t_Bl$. This is an absurd since $\frac{p_B+t_Bl+c_A}{2}$ is the middle point between c_A and $p_B + t_Bl$. Therefore, there are only two cases:

- Case A, where $p_B - t_Bl$ is the smallest maximum candidate in 4.2.
- Case B, where c_A is the smallest maximum candidate in 4.2.

Consider Case A:

There are two hypothesis for the relative position of the maximum candidates in 4.2.

- Case A1

$$p_B - t_Al \leq c_A \leq \frac{p_B + t_Bl + c_A}{2} \leq p_B + t_Bl\ . \tag{4.3}$$

- Case A2

$$p_B - t_Al \leq p_B + t_Bl \leq \frac{p_B + t_Bl + c_A}{2} \leq c_A\ . \tag{4.4}$$

In figure 4.1, we show, in Case A1, the graphic representation of π_A. From the graphic represen-

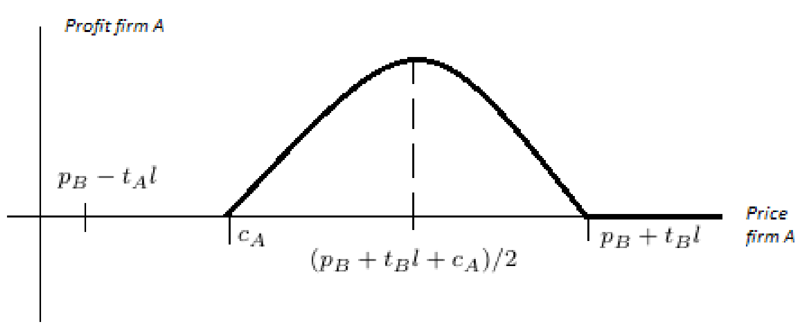


Figure 4.1: Shape of π_A in Case A1

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tation of π_A in Case A1, is possibly to verify that, given a price p_B, the profit of firm A is maximum when $p_A = \frac{p_B+t_Bl+c_A}{2}$, that is

$$arg\ max\ \pi_A(p_A; p_B) = \frac{p_B + t_Bl + c_A}{2}\ . \tag{4.5}$$

Consider Case A2:

Consider figure 4.2. In this picture it is represented the profit of firm A in this case. The profit will be zero independently of the price set by firm A, so the maximum profit is achieved at $p_A = c_A$ and

$$arg\ max\ \pi_A(p_A; p_B) = c_A\ . \tag{4.6}$$

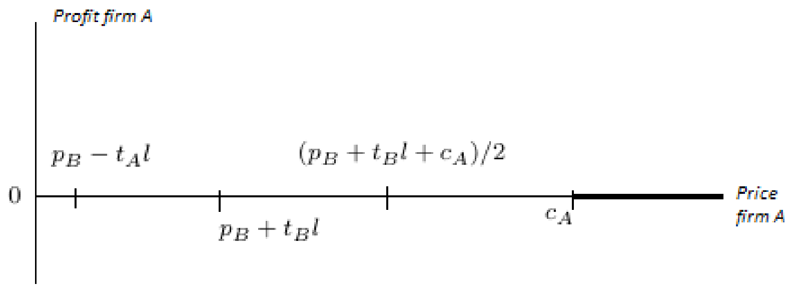


Figure 4.2: Shape of π_A in Case A2

Consider Case B:

Again, we end up with two possibilities.

- Case B1

$$c_A \leq \frac{p_B + t_B l + c_A}{2} \leq p_B - t_A l \leq p_B + t_B l\ . \tag{4.7}$$

- Case B2

$$c_A \leq p_B - t_A l \leq \frac{p_B + t_B l + c_A}{2} \leq p_B + t_B l\ . \tag{4.8}$$

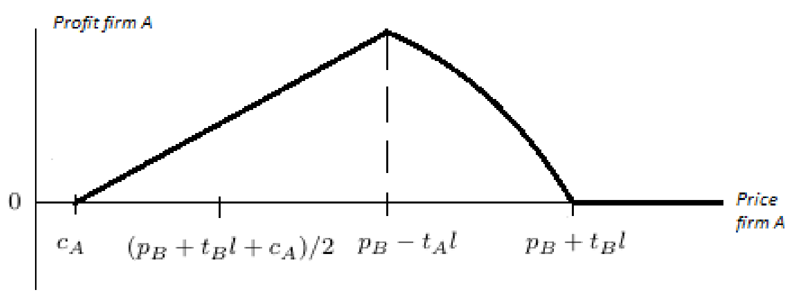


Figure 4.3: Shape of π_A in Case B1

Consider Case B1:

The figure 4.3 shows the graphic representation of π_A in this case. In this picture, we can verify that, given p_B, the profit of firm A is maximum at $p_A = p_B - t_A$, that is

$$arg\ max\ \pi_A(p_A; p_B) = p_B - t_A l\ . \tag{4.9}$$

Consider Case B2:

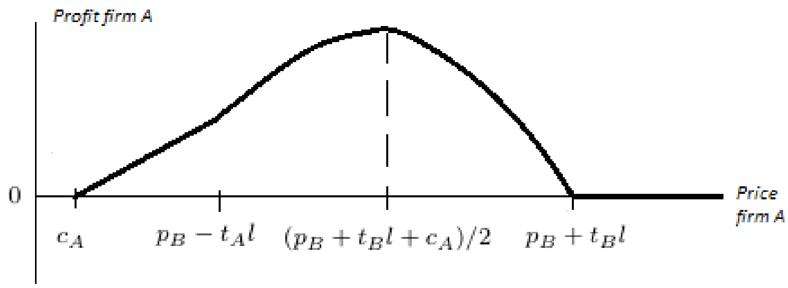


Figure 4.4: Shape of π_A in Case B2

The picture 4.4 shows, that if p_B satisfies Case B2, then firm A achieves is maximum profit by setting is price at $\frac{p_B+t_Bl+c_A}{2}$ and

$$arg\ max\ \pi_A(p_A; p_B) = \frac{p_B + t_B l + c_A}{2}\ . \tag{4.10}$$

The above discussion leaves us in position to compute the Best-Response Correspondence for firm A.

Theorem 5. *The Best-Response Correspondence for firm A, $Br_A(p_B)$, is:*

$$Br_A(p_B) = \begin{cases} p_B - t_A l & \text{if } p_B \geq c_A + t_B l + 2t_A l \\ \frac{p_B + c_A + t_B l}{2} & \text{if } c_A - t_B l < p_B < c_A + t_B l + 2t_A l \\ c_A & \text{if } p_B \leq c_A - t_B l\ . \end{cases}$$

Proof. Given a price p_B, it was already shown that there are four different cases for the shape of firm`s A profit function. The four cases can be joined, accordingly to the point where the global maximum is achieved. We end up with three groups:

- Case B1 - π_A as a global maximum at $p_A = p_B - t_A l$.
- Case B2 and Case A1 - π_A as a global maximum at $p_A = (p_B + t_B l + c_A)/2$.
- Case A2 - π_B as a global maximum at $p_A = c_A$.

Case B1 is characterized by

$$c_A \leq \frac{p_B + t_B l + c_A}{2} \leq p_B - t_A l \leq p_B + t_B l \ .$$

Since $p_B - t_A l \leq p_B + t_B l$ and that $(p_B + t_B l + c_A)/2$ is the middle point between c_A and $p_B + t_B l$, we obtain that Case B1 is also defined by the system of inequalities given by

$$\begin{cases} c_A & \leq p_B + t_B l \\ p_B & \geq c_A + t_B l + 2 t_A l \ . \end{cases}$$

Since $p_B \geq c_A + t_B l + 2 t_A l$ implies $c_A \leq p_B + t_B l$ then Case B1 is fully characterized only by the first inequality.

Therefore, if $p_B \geq c_A + t_B l + 2 t_A l$ then

$$arg\ max\ \pi_A(p_A; p_B) = p_B - t_A l \ .$$

Case A2 is characterized by

$$p_B - t_A l \leq p_B + t_B l \leq \frac{p_B + t_B l + c_A}{2} \leq c_A \ .$$

Again, considering that $p_B - t_A l \leq p_B + t_B l$ and since, $(p_B + t_B l + c_A)/2$ is the middle point between c_A and $p_B + t_B l$, we obtain that Case A2 is also fully characterized by

$$p_B + t_B l \leq \frac{p_B + t_B l + c_A}{2} \ .$$

With some computations, we verify that Case A2 is characterized only by

$$p_B \leq c_A - t_B l \ .$$

Therefore, if $p_B \leq c_A - t_B l$ then

$$arg\ max\ \pi_A(p_A; p_B) = c_A \ .$$

Consider now Case B2 and Case A1 characterized by

$$c_A \leq p_B - t_A l \leq \frac{p_B + t_B l + c_A}{2} \leq p_B + t_B l$$

or

$$p_B - t_A l \leq c_A \leq \frac{p_B + t_B l + c_A}{2} \leq p_B + t_B l \ .$$

For the same reasons pointed in the previous cases, we verify that Case B2 and Case A1 are characterized by

$$c_A - t_B l \leq p_B \leq c_A + t_B l + 2t_A l \; .$$

Hence,

$$arg\ max\ \pi_A(p_A; p_B) = \frac{p_B + t_B l + c_A}{2} \; .$$

□

By symmetry, we set the Best-Response Correspondence for firm B.

Theorem 6. *The Best-Response Correspondence for firm B, $Br_B(p_A)$, is given by*

$$Br_B(p_A) = \begin{cases} p_A - t_B l & \text{se } p_A \geq c_B + 2t_B l + t_A l \\ \frac{p_A + c_B + t_A l}{2} & c_B - t_A l < p_A < c_B + 2t_B l + t_A l \\ c_B & \text{se } p_A \leq c_B - t_A l \; . \end{cases}$$

Recall the definition of the Best-Response Dynamics given in chapter 1. As a corollary of the last two theorems we set the main result of this section.

Corollary 1. *In the Hotelling model, the Best-Response Dynamics is given by*

$$\frac{dp_A}{dt} = \begin{cases} p_B - t_A l - p_A & \text{if } p_B \geq c_A + t_B l + 2t_A l \\ \frac{p_B - 2p_A + c_A + t_B l}{2} & \text{if } c_A - t_B l < p_B < c_A + t_B l + 2t_A l \\ c_A - p_A & \text{if } p_B \leq c_A - t_B l \end{cases}$$

$$\frac{dp_B}{dt} = \begin{cases} p_A - t_B l - p_B & \text{if } p_A \geq c_B + 2t_B l + t_A l \\ \frac{p_A - 2p_B + c_B + t_A l}{2} & \text{if } c_B - t_A l < p_A < c_B + 2t_B l + t_A l \\ c_B - p_B & \text{if } p_A \leq c_B - t_A l \; . \end{cases}$$

The Best-Response Dynamics separates the set of price configurations into nine different regions.

- Region A_{11} given by $[0, c_B - t_A l] \times [0, c_A - t_B l]$.
- Region A_{12} given by $[c_B - t_A l, c_B + t_A l + 2t_B l] \times [0, c_A - t_B l]$.
- Region A_{13} given by $[c_B + t_A l + 2t_B l, +\infty[\times [0, c_A - t_B l]$.

- Region A_{21} given by $[0, c_B - t_A l] \times [c_A - t_B l, c_A + 2t_A l + t_B l]$.
- Region A_{22} given by $[c_B - t_A l, c_B + t_A l + 2t_B l] \times [c_A - t_B l, c_A + 2t_A l + t_B l]$.
- Region A_{23} given by $[c_B + t_A l + 2t_B l, +\infty[\times [c_A - t_B l, c_A + 2t_A l + t_B l]$.
- Region A_{31} given by $[0, c_B - t_A l] \times [c_A + t_B l + 2t_A l, +\infty[$.
- Region A_{32} given by $[c_B - t_A l, c_B + t_A l + 2t_B l] \times [c_A + t_B l + 2t_A l, +\infty[$.
- Region A_{33} given by $[c_B + t_A l + 2t_B l, +\infty[\times [c_A + t_B l + 2t_A l, +\infty[$.

Figure 4.5 shows the prices partition in these regions. Recall that the dynamical equilibrium points

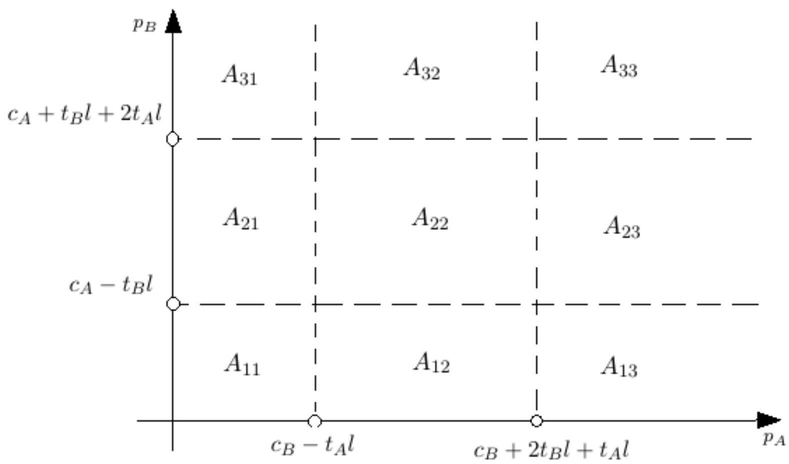


Figure 4.5: Partition of the Price Configurations Set

of the Best-Response Dynamics are the Nash equilibrium points. We can find the Nash Price Equilibrium using the above prices partition.

Theorem 7. *In the Hotelling model, there exists a Nash Price Equilibrium with Duopoly Property if, and only if,*

$$-2t_B l - t_A l < \Delta c < 2t_A l + t_B l \; .$$

In addition, it is unique and is located in Region A_{22}.

Proof. In Region A_{11} the equilibrium point is given by

$$\begin{cases} c_A - p_A & = 0 \\ c_B - p_B & = 0 \ . \end{cases}$$

Consequently, (c_A, c_B) is an equilibrium point for the system. However, this point is not necessarily located inside Region A_{11}. In fact, (c_A, c_B) is only located in this zone if, and only if,

$$c_A \leq c_B - t_A l \text{ and } c_B \leq c_A - t_B l,$$

which is absurd, hence there are no equilibrium points in this zone. Utilizing similar arguments, we conclude that in Regions A_{12},A_{13},A_{21},A_{23},A_{31},A_{32} and A_{33} there are no equilibrium points for the Best-Response Dynamics.

Consider Region A_{22}. The equilibrium points are given by

$$\begin{cases} \frac{p_B - 2p_A + c_A + t_B l}{2} & = 0 \\ \frac{p_A - 2p_B + c_B + t_A l}{2} & = 0 \ . \end{cases}$$

Therefore, the equilibrium point is

$$P^* = \left(\frac{2c_A + c_B + 2t_B l + t_A l}{3}, \frac{2c_B + c_A + 2t_A l + t_B l}{3} \right) \ .$$

This point falls inside Region A_{22} if, and only if,

$$\begin{aligned} c_B - t_A l \quad &< \tfrac{2c_A + c_B + 2t_B l + t_A l}{3} < t_A l + 2t_B l + c_B \\ c_A - t_B l \quad &< \tfrac{2c_B + c_A + 2t_A l + t_B l}{3} < 2t_A l + t_B l + c_A \ . \end{aligned}$$

Hence, there exits a Nash Price Equilibrium if, and only if,

$$-2t_B l - t_A l < \Delta c < 2t_A l + t_B l \ . \tag{4.11}$$

We only need to verify, that under the above condition, this Nash Price Equilibrium has Duopoly property. In fact, the indifferent consumer at P^* satisfies

$$0 < x(P^*) < l \ .$$

□

4.2 Stability of the Nash Price Equilibrium

In this section, we proceed under the assumption that the Nash Price Equilibrium exists. The first question we want to answer, regarding the price evolution under the Best-Response Dynamics is:

- Consider that the initial prices are sufficiently close to the Nash Price Equilibrium, which are the trajectories followed by the prices?

This question is mathematically corresponds to study of the stability of an equilibrium point, presented in chapter 2.

Theorem 8. *The only Nash Price Equilibrium point*

$$P^* = \left(\frac{2c_A + c_B + 2t_B l + t_A l}{3}, \frac{2c_B + c_A + 2t_A l + t_B l}{3}\right)$$

is asymptotically stable.

Proof. Inside Region A_{22} the Best-Response Dynamics can be written as

$$\begin{pmatrix} \frac{dp_A}{dt} \\ \frac{dp_B}{dt} \end{pmatrix} = \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} p_A \\ p_B \end{pmatrix} + \begin{pmatrix} \frac{c_A + t_B l}{2} \\ \frac{c_B + t_B l}{2} \end{pmatrix}$$

Hence, the Best-Response Dynamics in Region A_{22} is affine. The system matrix has two negative eigenvalues, $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = -\frac{3}{2}$. This implies that the Nash Price Equilibrium is asymptotically stable. □

Theorem 8 does not imply that, for every initial condition inside the Duopoly Zone or even Region A_{22}, the price trajectory will converge into the equilibrium point. Since the Best-Response Dynamics is defined taking into account different zones of the plane, at this point, we can only guarantee that exists an open set of price configurations around the Nash Price Equilibrium, for which the price trajectory will converge to the equilibrium point.

4.3 Invariance of the Duopoly Zone

The question we want to answer in this section is:

- Given a price a configuration in the Duopoly Zone, does any of the firms ends with an empty market share?

Mathematically, we address this question as follows:

- Is the Duopoly Zone a forward invariant set for the Best-Response Dynamics?

First we state a result about the invariance of the set of prices given by $[c_A, +\infty[\times[c_B, +\infty[$.

Lema 2. *The set of prices, defined by $[c_A, +\infty[\times[c_B, +\infty[$, is a forward invariant set for the Best-Response Dynamics.*

Proof. The proof of this result follows directly from the fact that the Best-Response Correspondence never drops bellow c_A for firm A and bellow c_B for firm B. The direction of the vector field along $p_A = c_A$ is fully characterized by the sign of dp_A/dt. This derivative is given by

$$\frac{dp_A}{dt} = \begin{cases} p_B - t_A l - c_A & \text{if } p_B \geq c_A + t_B l + 2t_A l \\ \frac{p_B - c_A + t_B l}{2} & \text{if } c_A - t_B l < p_B < c_A + t_B l + 2t_A l \\ 0 & \text{if } p_B \leq c_A - t_B l \ . \end{cases}$$

Therefore, $dp_A/dt \geq 0$ along the line $p_A = c_A$. This implies that a solution of the Best-Response Dynamics never crosses the line $p_A = c_A$. A similar argument allows to prove that a solution of the Best-Response Dynamics never crosses the line $p_B = c_B$. The combination of these two results proves the claim of the lema. □

Lema 2 has a very acceptable and expected economic interpretation. Given any initial price configuration, non of the firms has an advantage in dropping the price bellow their respective production cost.

We also observe, that theorem 2 implies that a solution of the Best-Response Dynamics, with initial condition inside the Duopoly Zone, never leaves this zone by the boundaries B_h and B_v. The next result determines sufficient conditions for the invariance of the Duopoly Zone under the Best-Response Dynamics.

Theorem 9. *If a Nash Price Equilibrium with Duopoly Property exists, then the Duopoly Zone is a forward invariant set for the Best-Response Dynamics.*

Proof. To prove this theorem, we only need to prove, that if a Nash Price Equilibrium with Duopoly Property exists then the Best-Response Dynamics vector field points inwards, along the boundaries B_s and B_i of the Duopoly Zone. Now, by Corollary 1 the components of the vector field along B_s are given by

$$\frac{dp_A}{dt} = \begin{cases} 0 & \text{if } p_A \geq c_A + t_B l + t_A l \\ \frac{-p_A + c_A + t_B l + t_A l}{2} & \text{if } c_A - t_B l - t_A l < p_A < c_A + t_B l + t_A l \\ c_A - p_A & \text{if } p_A \leq c_A - t_B l - t_A l \end{cases}$$

and

$$\frac{dp_B}{dt} = \begin{cases} -t_B l - t_A l & \text{if } p_A \geq c_B + 2t_B l + t_A l \\ \frac{-p_A + c_B - t_A l}{2} & \text{if } c_B - t_A l < p_A < c_B + 2t_B l + t_A l \\ c_B - p_A - t_A l & \text{if } p_A \leq c_B - t_A l \ . \end{cases}$$

We will study, separately, three different cases:

1. $-t_B l < \Delta c < t_A l$.
2. $-2t_B l - t_A l < \Delta c \leq -t_B l$.
3. $t_A l \leq \Delta c < 2t_A l + t_B l$.

Consider that $-t_B l < \Delta c < t_A l$. The following relation of order is satisfied

$$c_A - t_B l - t_A l \leq c_B - t_A l \leq c_A \leq t_B l + t_A l + c_A \leq t_A l + 2t_B l + c_B \ .$$

The last inequality divides the line $p_B = p_A + t_A l$ in six segments, where different branches of the Best-Response Dynamics are valid. The vector field along those segments is given by:

1. if $p_A \in]-\infty, c_A - t_B l - t_A l]$ then
$$\begin{cases} \frac{dp_A}{dt} & = c_A - p_A \\ \frac{dp_B}{dt} & = c_B - p_A - t_A l \ . \end{cases}$$
2. if $p_A \in]c_A - t_B l - t_A l, c_B - t_A l]$ then
$$\begin{cases} \frac{dp_A}{dt} & = \frac{-p_A + c_A + t_A l + t_B l}{2} \\ \frac{dp_B}{dt} & = c_B - p_A - t_A l \ . \end{cases}$$

3. if $p_A \in]c_B - t_A l, c_A]$ then

$$\begin{cases} \frac{dp_A}{dt} &= \frac{-p_A+c_A+t_Al+t_Bl}{2} \\ \frac{dp_B}{dt} &= \frac{-p_A+c_B-t_Al}{2} \end{cases} .$$

4. if $p_A \in]c_A, c_A + t_B l + t_A l]$ then

$$\begin{cases} \frac{dp_A}{dt} &= \frac{-p_A+c_A+t_Al+t_Bl}{2} \\ \frac{dp_B}{dt} &= \frac{-p_A+c_B-t_Al}{2} \end{cases} .$$

5. if $p_A \in]c_A + t_B l + t_A l, t_A l + 2t_B l + c_B]$ then

$$\begin{cases} \frac{dp_A}{dt} &= 0 \\ \frac{dp_B}{dt} &= \frac{-p_A+c_B-t_Al}{2} \end{cases} .$$

6. if $p_A \in]t_A l + 2t_B l + c_B, +\infty[$ then

$$\begin{cases} \frac{dp_A}{dt} &= 0 \\ \frac{dp_B}{dt} &= \frac{-p_A+c_B-t_Al}{2} \end{cases} .$$

Considering that B_s is limited by $p_A = c_A$, we only need to consider the last three cases. If $p_A \in]t_A l + 2t_B l + c_B, +\infty[$ or $p_A \in]c_A + t_B l + t_A l, t_A l + 2t_B l + c_B]$, the first component of the vector field is zero, therefore the vector field point inwards if, and only if, $dp_B/dt \leq 0$. If $p_A \in]t_A l + 2t_B l + c_B, +\infty[$ we obtain

$$\frac{dp_B}{dt} = -(t_A + t_B)l < 0 \ ,$$

and the vector field points inwards. If $p_A \in]c_A + t_B l + t_A l, t_A l + 2t_B l + c_B]$ and since $p_A \geq c_A$, we obtain

$$\frac{dp_B}{dt} = \frac{-p_A + c_B - t_A l}{2}$$
$$\Rightarrow \frac{dp_B}{dt} \leq \frac{\Delta c - t_A l}{2} \ .$$

Therefore, under the condition $-t_B l < \Delta c < t_A l$, we obtain that $dp_B/dt < 0$ and the vector field points inwards.

Finally, consider that $p_A \in]c_A, c_A + t_B l + t_A l]$. Under the condition $-t_B l < \Delta c < t_A l$ we obtain

$$\frac{dp_B}{dt} = \frac{-p_A + c_B - t_A l}{2}$$
$$\Rightarrow \frac{dp_B}{dt} \leq \frac{\Delta c - t_A l}{2} \ .$$

If p_A is between c_A and $c_A + t_A l + t_B l$ we obtain

$$\frac{dp_A}{dt} = \frac{-p_A + c_A + t_A l + t_B l}{2} \Rightarrow \frac{dp_A}{dt} \geq 0 \ .$$

This implies that Best-Response Dynamics vector field points inwards along the boundary B_s. A similar argument, allows to prove that the Best-Response Dynamics vector field, also points inwards in the case where $-2t_B l - t_A l < \Delta c \leq -t_B l$ and $t_A l \leq \Delta c < 2t_A l + t_B l$. By symmetry, the Best-Response Dynamics vector field, also points towards the interior of the Duopoly Zone along the boundary B_i. Since, a solution of the Best-Response Dynamics never crosses the lines $p_A = c_A$ and $p_B = c_B$, the prove follows. □

The fact that the Duopoly Zone is a forward invariant set for the Best-Response Dynamics brings very interesting consequences from the economic point view. First of all, if a Nash Price Equilibrium with Duopoly Property exists, non of the firms has an incentive to supply the whole market. This behavior makes the market very resistant to fluctuations resulting from outside factors. Facing an unexpected fluctuation in the market share a firm never drops out the market as long it keeps a tiny number of customers.

In the last discussion, we were assuming that the difference between production costs allows the existence of a Nash Price Equilibrium with Duopoly Property. It is also interesting to investigate the firm interaction in the case where such an equilibrium does not exist. From Theorem 7, we know that the Nash Price Equilibrium with Duopoly Property does not exist if, and only if,

$$\Delta c \geq 2t_A l + t_B l \text{ or } \Delta c \leq -2t_B l - t_A l \ .$$

Lema 3. *Consider that $\Delta c \geq 2t_A l + t_B l$. Then the Duopoly Zone only intersects the Regions A_{32} and A_{33}. If $\Delta c \leq -2t_B l - t_A l$, then the Duopoly Zone only intersects the Regions A_{23} and A_{33}.*

Proof. Let $\Delta c \geq 2t_A l + t_B l$. Then the boundary B_h is empty and the Duopoly Zone is limited by the boundary B_h. Under the considered condition, we obtain that

$$c_B \geq c_A + 2t_A l + t_B l \ .$$

Therefore, the Duopoly Zone can only intersect the Regions A_{31}, A_{32} and A_{33}. Consider the intersection point between the boundary $p_A = c_B - t_A l$ of the Region A_{31} and the boundary B_s

of the Duopoly Zone. This point is given by $(c_B - t_A l, c_B)$. Hence, $(c_B - t_A l, c_B)$ is the only point of Region A_{31} that could belong to to the Duopoly Zone. However, $(c_B - t_A l, c_B)$ does not have Duopoly Property and $(c_B - t_A l, c_B)$ does not belong to the Duopoly Zone.
By symmetry the second part of the lema follows. □

The above lema, allows us to search for equilibrium points of the Best-Response Dynamics under the considered conditions.

Theorem 10. *Let $\Delta c \geq 2t_A l + t_B l$. There exists a Boundary Nash Price Equilibrium given by*

$$P_1^* = (c_B - t_A l, c_B)$$

that belongs to the common boundary between the Regions A_{31} and A_{32}.
Let $\Delta c \leq -2t_B l - t_A l$. There exists a Boundary Nash Price Equilibrium given by

$$P_2^* = (c_A l, c_A - t_B l)$$

that belongs to the common boundary between the Regions A_{13} and A_{23}.

Proof. Consider $\Delta c \geq 2t_A l + t_B l$. By lema 3 the Duopoly Zone only intersects the Regions A_{32} and A_{33}. The intersection point $P^I = (p_A^I, c_B)$ between the boundaries B_h and B_s of the Duopoly Zone is given by the equation

$$c_B = p_A^I + t_A l \ .$$

Hence,

$$P^I = P_1^* = (c_B - t_A l, c_B) \ .$$

This implies that P_1^* lies in the common boundary between the Regions A_{31} and A_{32}. Since the Best-Response Dynamics vector field is continuous we can use the expression of the Best-Response Dynamics in the Region A_{32} to show that P_1^* is an equilibrium point. We obtain that in P_1^* the components dp_A/dt and dp_B/dt of the Best-Response Dynamics are zero and P_1^* is an equilibrium point. The second part of the theorem follows by a symmetrical argument. □

We note that neither P_1^* or P_2^* have Duopoly Property. In the price configuration P_1^* the firm A is supplying the entire market. On the other hand, in the price configuration P_2^* the firm B is the firm who supplies the entire market.
The next result classifies the equilibrium points P_1^* and P_2^* in terms of stability.

Theorem 11. *The equilibrium points P_1^* and P_2^* of the Best-Response Dynamics are asymptotically stable. Furthermore,*

- *P_1^* attracts every solution of the Best-Response Dynamics with initial conditions inside the Region A_{32}*
- *P_2^* attracts every solution of the Best-Response Dynamics with initial conditions inside the Region A_{23}.*

Proof. Let P_1^* be an equilibrium point for the Best-Response Dynamics. By lema 3 and having into account that the Best-Response Dynamics vector field is continuous, the matrix of the system is given by

$$A = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & -1 \end{pmatrix} . \tag{4.12}$$

Computing the eigenvalues of A, we obtain that $\lambda_1 = -1 - 1/\sqrt{2}$ and $\lambda_2 = -1 + 1/\sqrt{2}$. Hence, both eigenvalues of A are negative and P_1^* is asymptotically stable. In fact, since the Best-Response Dynamics is affine the point P_1^* attracts every solutions of the vector field with initial conditions inside the Region A_{32}. A symmetrical allows to prove the claims of the theorem regarding P_2^*. □

Theorem 12. *The equilibrium points P_1^* and P_2^* of the Best-Response Dynamics are global attractors within the Duopoly Zone. Furthermore, the solutions of the Best-Response Dynamics never leave the Duopoly Zone in finite time.*

Proof. Consider that P_1^* exists. The theorem 11 shows that P_1^* attracts every solution of the Best-Response Dynamics with initial conditions inside the region A_{32}. To prove that P_1^* is a global attractor within the Duopoly Zone is only necessary to show that the solution of the Best-Response Dynamics, with initial condition inside the Region A_{33}, enters in the Region A_{32} at some point. Consider the Best-Response Dynamics in the Region A_{33}. Consider the inner product given by

$$(\frac{dp_A}{dt}, \frac{dp_B}{dt}).(-1, -1) . \tag{4.13}$$

With some computations, we obtain that the above inner product is constant an equal to

$$(t_A + t_B)l \tag{4.14}$$

Hence, the inner product in equation 4.13 is always positive. Therefore, the Best-Response Dynamics solutions, with initial conditions in the Region A_{33}, enter the Region A_{32} for some positive t. We recall, that by theorem 2 a solution of the Best-Response Dynamics never drops below the line $p_B = c_B$. Therefore, P_1^* is a global attractor within the Duopoly Zone.

To conclude the prove of the theorem it is necessary to show that the solutions of the Best-Response Dynamics, with initial conditions inside the Regions A_{32} or A_{33}, don`t cross the boundaries B_s and B_i of the Duopoly Zone. This can be done by computing the inner product between the vector $(-1, 1)$ and the Best-Response Dynamics vector field along the boundaries B_s and B_i in the Regions A_{32} and A_{33}.

Region A_{32}:

The inner product between the vector $(-1, 1)$ and the Best-Response Dynamics vector field along B_s is given by

$$(0, \frac{-p_B + c_B}{2}) = \frac{-p_B + c_B}{2} \ . \tag{4.15}$$

Hence, the above inner product is equal to zero when $p_B = c_B$ and below zero otherwise. Therefore, the Best-Response Dynamics vector field point towards the inside of the Duopoly Zone. The inner product between the vector $(-1, 1)$ and the Best-Response Dynamics vector field along B_i is given by

$$(-(t_A + t_B)l, \frac{-p_B + c_B + t_A l + t_B l}{2}) = \frac{-p_B + c_B + 3t_A l + 3t_B l}{2} \ . \tag{4.16}$$

The maximum value of p_B in Region A_{32} is $c_A + 2t_A l + t_B l$. Therefore, the minimum of the above inner product is

$$\Delta c + 2t_B l + t_A l \ .$$

If P_1^* exists, then $\Delta c \geq 2t_A l + t_B l$. Hence, the inner product, given by equation 4.16 is greater than zero and the Best-Response Dynamics vector field points towards the inside of the Duopoly Zone.

Region A_{33}:

The inner product between the vector $(-1, 1)$ and the Best-Response Dynamics vector field along B_s is given by

$$(0, -(t_A + t_B)l).(-1, 1)) = -(t_A + t_B)l \ . \tag{4.17}$$

Hence, the Best-Response Dynamics vector field points towards the inside of the Duopoly Zone. The inner product between the vector $(-1, 1)$ and the Best-Response Dynamics vector field along B_i is given by

$$(-(t_A + t_B)l, 0).(-1, 1) = (t_A + t_B)l \ . \tag{4.18}$$

Hence, the Best-Response Dynamics vector field points towards the inside of the Duopoly Zone. The above discussion shows that a solution of the Best-Response Dynamics, with initial conditions inside the Duopoly Zone, converges to P_1^* and never leaves in finite time the Duopoly Zone. □

The above discussion shows, that when a interior Nash Price Equilibrium does not exist, then one of the firms ends supplying the entire market.

In the next chapter, we will introduce a different dynamics in the second sub game of the Hotelling model. Our goal, is to see, into what extend, the conclusions draw for the Best-Response Dynamics hold for a new dynamics.

Chapter 5

Myopic Dynamics

In this chapter we introduce a dynamics called **Myopic Dynamics**, given by

$$\begin{cases} \frac{dp_A}{dt} &= (p_A - c_A)\frac{\partial \pi_A}{\partial p_A} \\ \frac{dp_B}{dt} &= (p_B - c_B)\frac{\partial \pi_B}{\partial p_B} \ . \end{cases} \tag{5.1}$$

The idea behind this dynamics is fairly simple. In first place, a firm never has the incentive to drop below their respective production costs. With this in mind, we introduce the components $(p_A - c_A)$, $(p_B - c_B)$ in equation 5.1. The terms $\partial\pi_A/\partial p_A$ and $\partial\pi_B/\partial p_B$ have the function of characterizing the impact, on the firm`s profit, caused by their own prices. The idea is the following: if a positive fluctuation in the price causes a decrease in the profit of a firm then the firm lowers its price in an attempt to increase market share; On other hand, if a positive fluctuation in the prices leads to an increase in the profit of the firm then the firm raises its price, not minding with the loss of market share.

We call this dynamics Myopic Dynamics, in the sense that, a firm only looks to the instantaneous variation of its profit. In the Hotelling model, we consider the Myopic Dynamics to be valid in the closure of the Duopoly Zone.

In chapter 3 we introduced the Duopoly Property Condition,

$$-2t_B l - t_A l < \Delta c < 2t_A l + t_B l \ .$$

Under the Duopoly Property Condition the indifferent consumer is located inside the Duopoly Zone so the firms are sharing the market. In addition, we prove in the last chapter, that under this condition, the Nash Price Equilibrium exists and it is given by

$$(p_A^*, p_B^*) = \frac{1}{3}(2c_A + c_B + (t_A + 2t_B)l, c_A + 2c_B + (2t_A + t_B)l) \ . \tag{5.2}$$

The next proposition links the Myopic Dynamics and the equilibrium point in equation 5.2.

Proposition 1. *Under Duopoly Property Condition, the Myopic Dynamics has the following equilibrium points:*

- *a corner equilibrium point* (c_A, c_B).
- *two boundary equilibrium points* $(c_A, \frac{c_A+c_B+t_Al}{2})$ *and* $(\frac{c_A+c_B+t_Bl}{2}, c_B)$.
- *an interior equilibrium point that is the Nash Price Equilibrium.*

Proof. Solving for the fixed points $(\frac{dp_A}{dt}, \frac{dp_B}{dt}) = (0,0)$, we obtain

$$(p_A - c_A) = 0 \vee \frac{p_B - 2p_A + c_A + t_Bl}{t_A + t_B} = 0$$

and

$$(p_B - c_B) = 0 \vee \frac{p_A - 2p_B + c_B + t_Al}{t_A + t_B} = 0 \ .$$

Therefore, the differential equation has three equilibrium points located on the edges of the duopoly zone. Namely, (c_A, c_B), $(c_A, \frac{c_A+c_B+t_Al}{2})$ and $(\frac{c_A+c_B+t_Bl}{2}, c_B)$. The interior equilibrium point is given by

$$\begin{cases} \frac{p_B-2p_A+c_A+t_Bl}{t_A+t_B} & = 0 \\ \frac{p_A-2p_B+c_B+t_Al}{t_A+t_B} & = 0 \ . \end{cases}$$

Solving the system we obtain

$$(p_A^*, p_B^*) = \frac{1}{3}(2c_A + c_B + (t_A + 2t_B)l, c_A + 2c_B + (2t_A + t_B)l) \ .$$

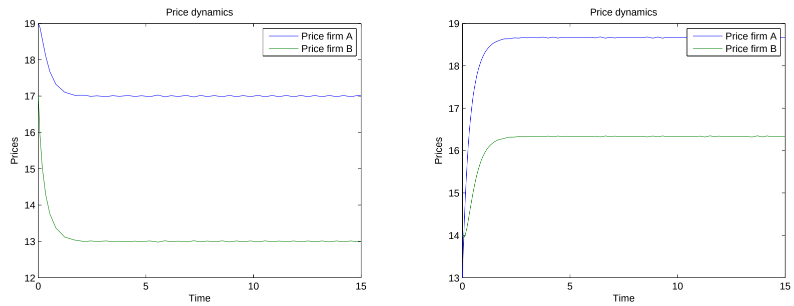
□

There are two economical questions that we want to answer:

- Is the Nash Price Equilibrium asymptotically stable?
- Can a firm, under the Myopic Dynamics, lose its market share ?

We present two simulations for the price evolution with two different set`s of parameters.

In both cases, shown in figure 5.1, the price trajectory converges to the Nash Price Equilibrium.



(a) Symmetric Hotelling (b) Non-Symmetric Hotelling

Figure 5.1: Myopic Dynamics simulations.

5.1 Stability of the Nash Price Equilibrium

We are going to show that the Nash Price Equilibrium is asymptotically stable using a Lyapunov function. We note, however, that the asymptotically stability, could be shown, by computing the eigenvalues of the linearization of system 5.1, in a neighborhood of the Nash Price Equilibrium. We opted by the traditional method to be consistent with the exposition made in chapter 6, in which, we apply this method to a more complex problem. Consider the change of coordinates given by the transformation

$$(x, y) = (p_A - c_A, p_B - c_B) \ . \tag{5.3}$$

Remembering that $c_B - c_A$ is represented by Δc, we write equation 5.1 in the new coordinate system,

$$\begin{cases} \frac{dx}{dt} & = x(\frac{y-2x+t_B l+\Delta c}{t_A+t_B}) \\ \frac{dy}{dt} & = y(\frac{x-2y+t_A l-\Delta c}{t_A+t_B}) \ . \end{cases} \tag{5.4}$$

In chapter 2, we introduced a special class of differential equations called Lotka-Volterra type equations. For two populations and an interaction matrix given by $A = (a_{ij})$, the general form of this equations is given by

$$\begin{cases} \frac{dx_1}{dt} & = x_1(r_1 + a_{11}x_1 + a_{12}x_2) \\ \frac{dx_2}{dt} & = x_2(r_2 + a_{21}x_1 + a_{22}x_2) \ . \end{cases} \tag{5.5}$$

Hence, equation 5.4 is a Lotka-Volterra differential equation with an interaction matrix A given by

$$A = \begin{pmatrix} \frac{-2}{t_A+t_B} & \frac{1}{t_A+t_B} \\ \frac{1}{t_A+t_B} & \frac{-2}{t_A+t_B} \end{pmatrix} . \tag{5.6}$$

This identification allow us to use the result, presented in chapter 2, regarding this type of equations.

Theorem 13. *The Nash Price Equilibrium is an asymptotically stable equilibrium point for the Myopic Dynamics.*

Proof. According with, theorem 3 [4] from chapter 2, if exists a positive diagonal matrix C such that $CA + A^TC$ is negative definite, then the feasible equilibrium point of a Lotka-Volterra equation with interaction matrix A is asymptotically stable. After the appropriated coordinate change and under the Duopoly Property Condition, the Nash Price Equilibrium is feasible since all of entries are positive. Consider the positive diagonal matrix C given by

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Computing $CA + A^TC$, we obtain

$$CA + A^TC = 2A = \begin{pmatrix} \frac{-4}{t_A+t_B} & \frac{2}{t_A+t_B} \\ \frac{2}{t_A+t_B} & \frac{-4}{t_A+t_B} \end{pmatrix}$$

The above matrix has two negative eigenvalues, namely, $\lambda_1 = \frac{-2}{t_A+t_B}$ and $\lambda_2 = \frac{-6}{t_A+t_B}$. Therefore is negative definite. This concludes the prove and the Nash Price Equilibrium is asymptotically stable. □

5.2 Invariance of the Duopoly Zone

At this point, we know that there are a set of initial prices in a neighborhood of the Nash Price Equilibrium, such that, the price trajectories converge towards the Nash equilibrium point. The second question we want to answer is, considering a initial price configuration inside the Duopoly Zone, will any of the firms have an incentive to cut off the other firm and supply the whole market? This question will be answered, by finding under which conditions, in terms of the production costs and the transportation costs, is the duopoly zone a forward invariant set. As a first approach to this question, we address the dynamic behavior of the Myopic Dynamics along the boundaries B_h and B_v of the Duopoly Zone. Having into account the considerations made in chapter 2, we note that B_h and B_v can be empty sets. Therefore, there are three cases that we must address:

- Case 1: where $-t_B l < \Delta c < t_A l$, which implies that both B_v and B_h are non-empty.
- Case 2: where $\Delta c \leq -t_B l$, which implies that B_h is empty.
- Case 3: where $t_A l \leq \Delta c$, which implies that B_v is empty.

The next theorem addresses Case 1.

Theorem 14. *If $-t_B l < \Delta c < t_A l$, the solutions of Myopic Dynamics with initial conditions inside the sets $B_v - (c_A, c_B)$ and $B_h - (c_A, c_B)$ converge, respectively, to the following equilibrium points:*

1. $P_1^* = (c_A, \frac{c_A+c_B+t_A l}{2})$ *and the stable manifold is B_v.*
2. $P_2^* = (\frac{c_A+c_B+t_B l}{2}, c_B)$ *and the stable manifold is B_h.*

The figure 5.2 shows a graphic representation of the claims made in the above theorem.

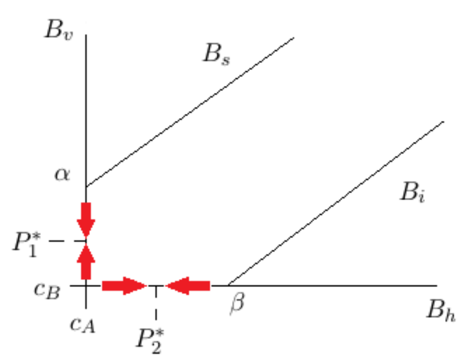


Figure 5.2: Stability along B_h and B_v in Case 1

Proof. If $-t_B l < \Delta c < t_A l$, the sets $B_h - (c_A, c_B)$ and $B_v - (c_A, c_B)$ are both non empty. Computing the Myopic Dynamics along the boundary B_v, we obtain

$$\begin{cases} \frac{dp_A}{dt} &= 0 \\ \frac{dp_B}{dt} &= (p_B - c_B)\frac{c_A - 2p_B + c_B + t_A l}{t_A + t_B}\ . \end{cases} \tag{5.7}$$

We note that, because dp_A/dt is null, the solutions of the Myopic Dynamics with initial conditions on the line $p_A = c_A$ do not leave this line. The derivative dp_B/dt is a concave parabola with zeros $p_B = c_B$ and $p_B = \frac{c_A+c_B+t_A l}{2}$. Since, we are assuming that $\Delta c < t_A l$ and having into account that $\alpha = (c_A, c_A + t_A l)$ is the interception point between the boundaries B_v and B_s, we obtain that

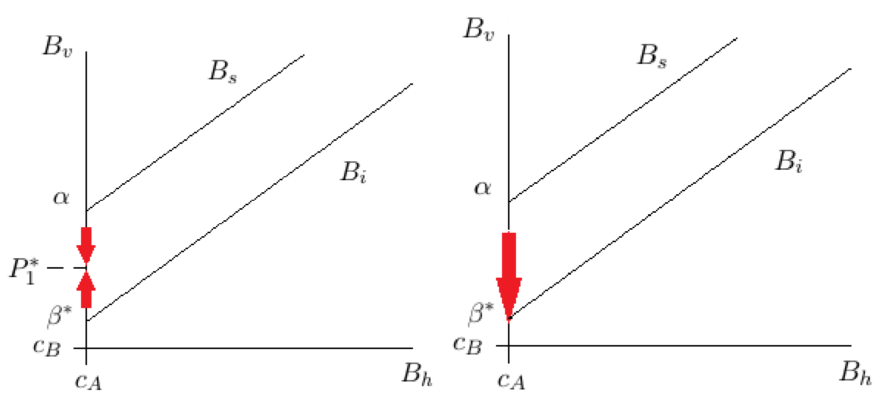
$$c_B < \frac{c_A + c_B + t_A l}{2} < c_A + t_A l \ . \tag{5.8}$$

Hence, dp_B/dt has:

- positive sign in the segment $]c_B, \frac{c_A+c_B+t_A l}{2}[$.
- negative sign in the segment $]\frac{c_A+c_B+t_A l}{2}, c_A + t_A l]$.

Therefore, the solution of the Myopic Dynamics converges to P_1^*. Applying a similar argument regarding the boundary B_h, we obtain that the solution of the Myopic Dynamics converges to P_2^*. □

We now move to Case 2. In this case the set B_h is empty, therefore we only address the behavior of the Myopic Dynamics for initial conditions on the boundary B_v.



(a) For $-2t_B l - t_A l < \Delta c \leq -t_B l$ (b) For $\Delta c < -2t_B l - t_A l$

Figure 5.3: Stability along B_v in Case 2

Theorem 15. *If $-2t_B l - t_A l < \Delta c \leq -t_B l$, the solutions of Myopic Dynamics, with initial conditions inside the set B_v, converge to the equilibrium point*

$$P_1^* = (c_A, \frac{c_A + c_B + t_A l}{2}) \ ,$$

and the stable manifold is B_v. If $\Delta c < -2t_Bl - t_Al$, the solutions of the Myopic Dynamics, with initial conditions in the set B_v, leave the Duopoly Zone.

The figure 5.3 shows a graphic scheme of the claims made in the last theorem.

Proof. First suppose that $\Delta c < -2t_Bl - t_Al$. From the computation of the Myopic Dynamics along the line $p_A = c_A$, we know that P_1^* and (c_A, c_B) are the only two possible equilibrium points, to which a solution, with initial condition in B_v can converge. To prove the claim made in the theorem it is only necessary to note that neither P_1^* or (c_A, c_B) has Duopoly Property.
Now, suppose that $-2t_Bl - t_Al < \Delta c \leq -t_Bl$. Consider the point $\beta^* = (c_A, c_A - t_Bl)$, that is the intersection point between $p_A = c_A$ and the boundary B_i. Again, from the computation of the Myopic Dynamics along the line $p_A = c_A$ we know that dp_B/dt is a concave parabola with zeros $p_B = c_B$ and $p_B = \frac{c_A+c_B+t_Al}{2}$. Under the assumed condition we obtain

$$c_B \leq c_A - t_Bl < \frac{c_A + c_B + t_Al}{2} < c_A + t_Al \; . \tag{5.9}$$

Hence, $\frac{dp_B}{dt}$ has:

- positive sign in the segment $[c_A - t_Bl, \frac{c_A+c_B+t_Al}{2}[$.
- negative sign in the segment $]\frac{c_A+c_B+t_Al}{2}, c_A + t_Al]$.

Therefore, in this case, the solution of the Myopic Dynamics converges to P_1^*. □

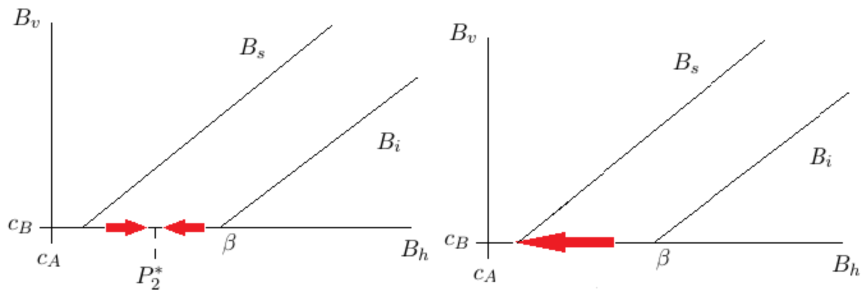
We observe, that if $\Delta c < -2t_Bl - t_Al$, then the solutions of Myopic Dynamics can leave the Duopoly Zone. This shows, that in general, the Duopoly Zone is not a forward invariant set for the Myopic Dynamics. In Case 3, we have a symmetric result to theorem 15.

Theorem 16. *If $t_Al \leq \Delta c < 2t_Al + t_Bl$, the solutions of the Myopic Dynamics, with initial conditions inside the set B_h, converge to the equilibrium point*

$$P_2^* = (\frac{c_A + c_B + t_Bl}{2}, c_B) \; ,$$

and the stable manifold is B_h. If $\Delta c > 2t_Al + t_Bl$, the solutions of Myopic the Dynamics, with initial conditions in the set B_h, leave the Duopoly Zone.

The proof of the above theorem is similar to theorem`s 15 proof. The next figure shows the graphic representation of the claims made in the last theorem. Again, we observe that the Duopoly Zone



(a) For $t_A l \leq \Delta c < 2t_A l + t_B l$ (b) For $\Delta c > 2t_A l + t_B l$

Figure 5.4: Stability along B_h in Case 3

might not be a forward invariant set for the Myopic Dynamics. This implies, that under certain conditions, a firm may drive the other firm out of the market. However, it is possible to encounter conditions, linking production costs and transportation costs, that make the Duopoly Zone a forward invariant set.

Theorem 17. *The Duopoly Zone is a forward invariant set for the Myopic Dynamics if, and only if, the production costs satisfy*

$$\frac{-3t_B l - t_A l}{2} \leq \Delta c \leq \frac{3t_A l + t_B l}{2} \ . \quad (5.10)$$

Proof. To prove that the Duopoly Zone is a forward invariant set for the Myopic Dynamics, we have to show that the vector field along the segments B_s and B_i points inwards. The vector field along B_s points inwards if, and only if, his projection on the vector $(1, -1)$ is greater or equal to zero, that is

$$(\frac{dp_A}{dt}, \frac{dp_B}{dt}).(1, -1) \geq 0 \ . \quad (5.11)$$

On other hand, the vector field along B_i points inwards if, and only if, his projection on the vector $(1, -1)$ is less or equal to zero, that is

$$(\frac{dp_A}{dt}, \frac{dp_B}{dt}).(1, -1) \leq 0 \ . \quad (5.12)$$

To simplify the computations, we apply the coordinate change presented in equation 5.3. In this new coordinate system the sets B_s and B_i become, respectively:

- $B_s^c = \{(x, y) : y = x + t_A l - \Delta c$ and $x \geq 0\}$.

- $B_i^c = \{(x,y) : x = y + t_B l + \Delta c$ and $y \geq 0\}$.

In first place, we focus in the set B_s^c. Computing the Myopic Dynamics along B_s^c we obtain

$$\begin{cases} \frac{dx}{dt} & = \frac{x(-x+t_A l+t_B l)}{t_A+t_B} \\ \frac{dy}{dt} & = \frac{-(x+t_A l-\Delta c)^2}{t_A+t_B} \end{cases} . \tag{5.13}$$

Computing $(dx/dt, dy/dt).(1,-1)$ we obtain

$$l(x) \equiv (\frac{dx}{dt}, \frac{dy}{dt}).(1,-1) = x(3t_A l + t_B l - 2\Delta c) + (t_A l - \Delta c)^2 . \tag{5.14}$$

Hence, $l(x) \geq 0$ for all $x \geq 0$ if, and only if, the slope is equal or greater than zero, that is

$$\Delta c \leq \frac{3t_A l + t_A l}{2} . \tag{5.15}$$

Proceeding in the same way for the inferior boundary, we obtain that the Myopic Dynamics vector field computed along B_i^c is given by

$$\begin{cases} \frac{dx}{dt} & = \frac{-(y+t_B l+\Delta c)^2}{t_A+t_B} \\ \frac{dy}{dt} & = \frac{y(-y+t_A l+t_B l)}{t_A+t_B} \end{cases} . \tag{5.16}$$

Computing $(dp_A/dt, dp_B/dt).(1,-1)$ we obtain

$$l_2(y) \equiv (\frac{dx}{dt}, \frac{dy}{dt}).(1,-1) = y(3t_B l + t_A l + 2\Delta c) - (t_B l + \Delta c)^2 . \tag{5.17}$$

Hence, $l_2(y) \leq 0$ for all $y \geq 0$ if, and only if, the slope is equal or lesser than zero, that is

$$\Delta c \leq \frac{3t_B l + t_A l}{2} . \tag{5.18}$$

Combining the conditions 5.15 and 5.18, we obtain that the Myopic Dynamics vector field along the sets B_s and B_i points inwards if, and only if,

$$\frac{-3t_B l - t_A l}{2} \leq \Delta c \leq \frac{3t_A l + t_B l}{2} . \tag{5.19}$$

Hence, we know that a solution for Myopic Dynamics, in the conditions of the theorem, can only leave the Duopoly Zone by the boundaries B_v and B_h. However, by theorems 15 and 16, this happens if and only if $\Delta c > 2t_A l + t_B l$ or $\Delta c < -2t_B l - t_A l$. Since the set of solutions of the inequalities 5.19, $\Delta c > 2t_A l + t_B l$ and $\Delta c < -2t_B l - t_A l$ are disjoint, the proof of the theorem follows. □

Note that, the Duopoly Property Condition is not sufficient to guarantee that the Duopoly Zone is forward invariant. Therefore, it is possible to exist a Nash Price Equilibrium, with Duopoly Property, without the Duopoly Zone being a forward invariant set. This implies that the Nash Price Equilibrium with Duopoly Property is not always achievable. Hence, under the Myopic Dynamics, one firm may have the incentive to supply the entire market. In this sense, the proof of theorem 17 brings interesting consequences from the economic point of view. If the Duopoly Zone is not a forward invariant set, then, only one of the firms is at risk of being driven off from the market. If $\Delta c > \frac{3t_A l + t_B l}{2}$, then firm B is the only firm at risk of be driven out of the market. This occurs because a solution of Myopic Dynamics can only leave the Duopoly Zone by the boundary B_s. On the other hand, if $\Delta < \frac{-3t_B l - t_A l}{2}$ firm A is the only firm at risk of being driven off the market. In this case a solution of Myopic Dynamics can only leave the Duopoly Zone by the boundary B_i.

5.3 Myopic Dynamics vs Best-Response Dynamics

The use of the Best-Response Dynamics or the Myopic Dynamics leads to the same conclusion about the stability of the Nash Price Equilibrium, that is, the Nash equilibrium point is asymptotically stable. The forward invariance of the Duopoly Zone, however, is affected by the choice of dynamics. In the Myopic Dynamics, it is possible to exist a Nash Price Equilibrium, with Duopoly Property, without the Duopoly Zone being forward invariant. This implies, that the Nash Price Equilibrium might not be achievable. This occurs, because in the Myopic Dynamics, the firm only have into consideration the instantaneous variation of their profits. In opposition, with the Best-Response Dynamics, if a Nash Price Equilibrium with Duopoly Property exists then the Duopoly Zone is forward invariant. This conclusion implies, that under the Best-Response Dynamics the market is much more stable.

Chapter 6

Hotelling Network

In this chapter, we consider the extension of the Hotelling model to a network of consumers and firms. This model was introduced by A. A. Pinto and T. Parreira in [2].

6.1 Model Overview

The Hotelling network consists of a group of roads or edges, denoted by R, where the consumers are uniformly distributed. Each road with vertices i and j, denoted by R_{ij}, has length l_{ij}. A group of firms F is distributed along the network and sell the same commodity. Similarly to the Hotelling model, each firm F_i sells the commodity at price p_i and supports a **unitary production cost** c_i. The consumers, in addition to the price of the commodity, have to support a **transportation cost** t_i depending of the firm F_i where each buys the commodity. Therefore, a consumer located at a point x in the network, who decides to buy at firm F_i spends

$$E(x; i, P) = p_i + t_i d(x, y_i) \ , \tag{6.1}$$

where $P = (p_1, \ldots, p_i, \ldots)$ is a price configuration and $d(x, y_i)$ represents the distance between the position of the consumer x and the position y_i of firm F_i in the network.
Given a price configuration P, the consumer will choose to buy at the firm F_i that minimizes its expenditure

$$v(x, P) = argmin_{i \in V} E(x; i, P) \ . \tag{6.2}$$

Again, in this model, it is assumed that each consumer buys one unity of commodity for each unity of time.

The **market** of each firm F_i is the set of all the consumers who minimize their expenditures by opting to buy at firm F_i,

$$M(i, P) = \{x : v(x, P) = i\} .$$

The market size $S_i(P)$ of firm F_i is given by the Lebesgue measure of the set $M(i, P)$. Therefore, the profit of each firm F_i, is given by

$$\pi_i(P, c_i) = (p_i - c_i)S_i(P) . \tag{6.3}$$

In this work, we will assume that every firm is located at one vertex of the network, that is, each firm F_i is located at the vertex i of the network. The **local firms** of a consumer, located in the road R_{ij}, are the firms F_i and F_j.

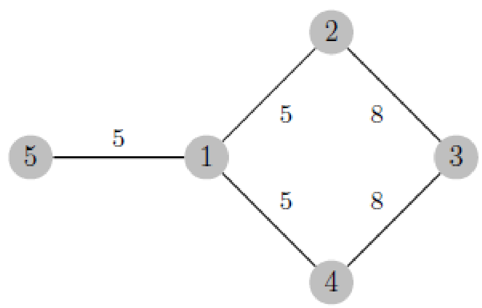


Figure 6.1: Example of Hotelling Network

Definition 13. A price configuration P, determines a **Local Market Structure** if every consumer buys from one of his local firms.

A price strategy P determines a Local Market Structure if, and only if, for every road $R_{i,j}$ there is one consumer located at a point $x_{i,j} \in R_{i,j}$ who is **indifferent** to the local firm from which he is going to buy the commodity, that is

$$E(x; i, P) = E(x; j, P) . \tag{6.4}$$

Lema 4. *A price configuration determines a Local Market Structure for every road R_{ij} if, and only if, the following inequality is satisfied,*

$$-t_j l_{ij} < p_j - p_i < -t_i l_{ij} . \tag{6.5}$$

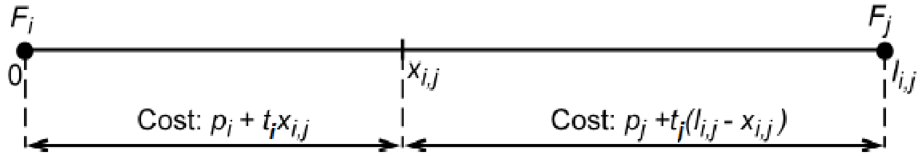


Figure 6.2: Indifferent Consumer in Hotelling Network

Proof. A price strategy P determines a Local Market Structure if, and only if, for every road $R_{i,j}$ there is one consumer located at a point $x_{i,j} \in R_{i,j}$, such that

$$p_i + t_i x_{ij} = p_j + t_j (l_{ij} - x_{ij}) \ . \tag{6.6}$$

Solving for x_{ij}, we obtain that

$$x_{ij} = \frac{p_j - p_i + t_j l_{ij}}{t_i + t_j} \ . \tag{6.7}$$

Since x_{ij} satisfies $0 < x_{ij} < l_{ij}$, we obtain that

$$-t_j l_{ij} < p_j - p_i < t_i l_{ij} \ . \tag{6.8}$$

□

From now on, we refer to the inequality given by equation 6.5 as the **Local Market Structure Condition**. Assuming that $|F| = n$, we observe that the Local Market Structure Condition defines a zone in $\mathbb{R}^n$ similar to the Duopoly Zone defined in $\mathbb{R}^2$ by the Duopoly Property. We denote this zone by **Local Market Structure Zone**.

Inside of the Local Market Structure Zone, the location of the indifferent consumer, x_{ij}, defines the size of the market, in the road R_{ij}, that belongs to firm F_i and firm F_j. This markets are given by x_{ij} and $l_{ij} - x_{ij}$, respectively.

Let V_i denote the vertices j such that exists a road R_{ij}. Inside of Local Market Structure Zone, every road as an indifferent consumer, therefore, the size of the market, $S_i(P)$, owned by the firm F_i, is given by

$$\sum_{j \in V_i} x_{ij} \ . \tag{6.9}$$

This allows us to define the profit function of each firm F_i,

$$\pi_i(P, c_i) = (p_i - c_i) \sum_{j \in V_i} x_{ij} \ . \tag{6.10}$$

The next definition introduces an important condition to the existence of Nash Price Equilibrium. Consider the following notation:

- c_M (resp. c_m) denotes the maximum (resp. minimum) of the production costs of all the firms in F.

- t_M (resp. t_m) denotes the maximum (resp. minimum) of the transportation costs for all the firms in F.

- k_M (resp. k_m) denotes the maximum (resp. minimum) degree of all the vertices in the network.

- Δc, Δt and Δk represent, respectively, $c_M - c_m$, $t_M - t_m$ and $k_M - k_m$.

Definition 14. An Hotelling network satisfies **Strong Bounded Length and Costs Condition** (SBLCC) if, and only if,

$$\Delta c + \frac{l_M t_M^2}{t_m} - l_m t_m + \Delta t l_M \leq \frac{(2 l_m t_m^2 - \Delta c t_m)^2}{4 t_M^2 l_M k_M (t_m + t_M)} \ . \qquad (6.11)$$

It was proven, by A. A. Pinto and T. Parreira (see [12]), that if a Hotelling network satisfies SLBCC, then a Nash Price Equilibrium exists inside the Local Market Structure Zone. In addition, this equilibrium point is unique. In the same work, A. A. Pinto and T. Parreira showed that the Nash Price Equilibrium point is determined by the system

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = 0 \\ \quad \dots \\ \frac{\partial \pi_i}{\partial p_i} = 0 \\ \quad \dots \end{cases} \qquad (6.12)$$

We denote the Nash Price Equilibrium by P^*. In the next section, we will study the stability of the Nash Price Equilibrium assuming that the firms adjust their prices according with the Myopic Dynamics.

6.2 Myopic Dynamics in the Hotelling Network

Following the same reasoning presented in chapter 3, the Myopic Dynamics for the Hotelling network is generally given by

$$\begin{cases} \frac{dp_1}{dt} = (p_1 - c_1)\frac{\partial \pi_1}{\partial p_1} \\ \dots \\ \frac{dp_i}{dt} = (p_i - c_i)\frac{\partial \pi_i}{\partial p_i} \\ \dots \end{cases} \tag{6.13}$$

Again, each firm F_i only has into consideration the instantaneous variation of its profit π_i caused by its own price p_i. Since the Nash Price Equilibrium P^* is determined by system 6.12, we obtain that P^* is an equilibrium point for the Myopic Dynamics.

Computing $\partial\pi_i/\partial p_i$ for every firm F_i, we obtain

$$\frac{\partial \pi_i}{\partial p_i} = \sum_{j\in V_i} \frac{p_j - p_i + t_j l_{ij}}{t_i + t_j} - (p_i - c_i)\sum_{j\in V_i}(t_i + t_j)^{-1} \ . \tag{6.14}$$

Consider the coordinate change given by the system

$$\begin{cases} x_1 = (p_1 - c_1) \\ \dots \\ x_i = (p_i - c_i) \\ \dots \end{cases} \tag{6.15}$$

The derivative $\partial\pi_i/\partial p_i$ can be written as

$$\sum_{j\in V_i} \frac{x_j}{t_i + t_j} - \frac{2|V_i|x_i}{t_i + t_j} + \sum_{j\in V_i} \frac{c_j - c_i}{t_i + t_j} \ . \tag{6.16}$$

Hence, every equation of the Myopic Dynamics can be written has

$$\frac{\partial \pi_i}{\partial p_i} = x_i(\sum_{j\in V_i} \frac{x_j}{t_i + t_j} - \frac{2|V_i|x_i}{t_i + t_j} + \sum_{j\in V_i} \frac{c_j - c_i}{t_i + t_j}) \ . \tag{6.17}$$

The system of equations 6.17 is of Lotka-Volterra type. The interaction matrix $A = (a_{ij})$ of equation 6.17 is given by

$$a_{ij} = \begin{cases} -\frac{2|V_i|}{t_i + t_j} & \text{if } i = j \\ \frac{1}{t_i + t_j} & \text{if } j \in V_i \\ 0 & \text{if } j \notin V_i \end{cases} \tag{6.18}$$

We observe, that matrix A is symmetric and therefore non-singular. Let us denote by P_c^*, the Nash Price Equilibrium P^* written in the coordinate system given by system 6.15. Now, we are in conditions to utilize theorem 3, presented in chapter 2, to address the stability of the Nash Price Equilibrium P^*.

Theorem 18. *Let P^* be a Nash Price Equilibrium, such that, P_c^* is a feasible equilibrium point for equation 6.17. Then P^* is asymptotically stable.*

The above result generalizes the theorem, obtained in chapter 5, regarding the stability of the Nash Price Equilibrium in the Hotelling Model, under the Myopic Dynamics.

Proof. Consider the identity matrix $I \in \mathbb{R}^n \times \mathbb{R}^n$. Since $A^T = A$ we obtain

$$C = IA + A^T I = 2A \tag{6.19}$$

Hence, every ij-entry of matrix C is given by

$$c_{ij} = \begin{cases} -\frac{4|v_i|}{t_i+t_j} & \text{if } i = j \\ \frac{2}{t_i+t_j} & \text{if } j \in V_i \\ 0 & \text{if } j \notin V_i \end{cases} \tag{6.20}$$

The matrix C is symmetric, which implies that all of its eigenvalues are real. By the Gershgorin theorem, presented in chapter two, every eigenvalue lies in at least one of the discs with center a_{ii} and radius $R_i = \frac{2|V_i|}{t_i+t_j}$. For every i, we obtain that

$$-\frac{4|V_i|}{t_i + t_j} + R_i < 0. \tag{6.21}$$

Therefore, all eigenvalues of C are negative and C is negative definite, which implies that P_c^* is asymptotically stable. Since the coordinate change, given by system 6.15, does not affect the stability of a equilibrium point, we obtain that the Nash Price Equilibrium P^* is asymptotically stable.

□

Chapter 7

Conclusions and Future Work

7.1 Conclusions

In this work, we introduced a dynamical factor in the price sub game of the Hotelling model. In chapter 4, using the Best-Response Dynamics, we showed that, under Duopoly Property Condition, the Nash Price Equilibrium is an asymptotically stable equilibrium point. Economically, this implies the existence of a set of initial prices, from which, the Best-Response Dynamics converge towards the Nash Price Equilibrium. We introduced the definition of Duopoly Zone, that is, the set of prices at which both firms have a non-empty market share. Under the Best-Response Dynamics we concluded, that the Duopoly Zone is forward invariant when a Nash Price Equilibrium with Duopoly Property exists. Hence, non of the firms is at risk of having a non-empty market share.

In chapter 5, we introduced the Myopic Dynamics, in which, the firms only have into consideration the instantaneous variation of their profits. Under the Myopic Dynamics, we were able to replicate the result above obtained with the Best-Response Dynamics. Again, the Nash Price Equilibrium is asymptotically stable. However, when we extend our analyses to the Duopoly Zone the results were quite different. Under the Myopic Dynamics the Duopoly Zone might not be forward invariant. Economically, this conclusion implies that one firm may end with an empty market share. We were able to find, however, a condition, linking transportation costs and production costs, that implies that the Duopoly Zone is forward invariant, and consequently, non of the firms will end with an empty-market share.

In chapter 6, we analyzed the stability of the Nash Price Equilibrium for the Hotelling Network introduced by A. A. Pinto and T. Parreira [2]. Under conditions that guarantee the existence of a Nash Price Equilibrium, we conclude that, under Myopic Dynamics, this equilibrium point is asymptotically stable.

7.2 Future Work

In his article from 1979 [5], D`Aspremont noted that the location sub game has a major effect on the indifferent consumer computation. In fact, this article redefined the conclusions, previous made by Hotelling, regarding the best strategy in the location game.
There are important open questions, regarding the effects of the locations chosen by the firms , on the stability of the Nash Price Equilibrium and the Duopoly Zone:

- What effect has the location game on the Duopoly Zone?
- Will the Nash Price Equilibrium still be asymptotically stable for the Myopic Dynamics and the Best-Response Dynamics?
- What happens to the price trajectory with initial condition in the Duopoly Zone?

Regarding the Hotelling Network model there are even more question to address, for example:

- Is the Nash Price Equilibrium asymptotically stable for the Best-Response Dynamics?
- Is the Duopoly Zone a forward invariant set?
- Which are the localization effects on the dynamical conclusions?

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