Renato Borges de Araújo de Moura Soeiro

Influence of individual decisions in competitive market policies

Department of Mathematics

Faculty of Sciences, University of Porto

September 2012

Renato Borges de Araújo de Moura Soeiro

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Thesis submitted to the Faculty of Sciences of the University of Porto, to obtain the degree of Master of Science in Engineering Mathematics, under the supervision of professor Alberto Adrego Pinto.

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Abstract

We observed that in a decision model, where individuals have gains in their utilities by making the same decisions as the other individuals of the same group, the pure Nash equilibria are cohesive. However if there are frictions among individuals of the same group, then there will be disparate Nash equilibria where the group is disrupted and different elements will make different decisions. Finally, we did a full analysis of a resort-tourist game that might become a paradigm to understand comercial interactions between individuals that have to choose among different offers of public or private services and care about the other individuals choices.

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Introduction

The main goal in Planned Behavior or Reasoned Action theories, as developed in the works of Ajzen (see [1]) and Baker (see [2]), is to understand and predict the way individuals turn intentions into behaviors. Almeida-Cruz-Ferreira-Pinto (see [3, 4, 5]) developed a game theoretical model for reasoned action, inspired by the works of J. Cownley and M. Wooders (see [6]). Here, we study the game theoretical model presented in [3] that we call the decision model.

In the decision model, the individuals will have to make decisions according to their preferences. The preferences have the interesting feature of taking into account not only how much the individuals like or dislike a certain decision, but also the other individuals' decisions. We consider that individuals with the same utility functions belong to a same group. We say that a group is cohesive if every individual has a gain in his utility when other individuals of the same group make the same decision as his. Almeida et al. [3] proved that all the individuals of a cohesive group make the same decision at a pure Nash equilibria. We extended this concept and so we say that a pure strategy is cohesive if all the individuals belonging to a same group will make the same decision. Hence, by Almeida et al. [3], if all groups are cohesive then all pure Nash are cohesive strategies. In the first chapter we find the cohesive thresholds that characterize the space of all parameters where the cohesive strategies are Nash equilibria (see [7]). Mousa et al. [8] show, for a model with two groups and two decisions, the existence of disparate Nash equilibria, where individuals in a same group can make different decisions at certain Nash equilibria. Here we extend these results to the general decision model discussed in this work. We present sufficient and necessary conditions that guarantee the existence of disparate Nash equilibria.

In the second chapter we do a first discussion of a resort-tourist game with two stages. In the first stage the resorts decide their prices and in the second stage the tourists (individuals) choose their favorite resort. Hence the second stage game is similar to the game discussed in the first chapter. We do a full discussion of the mixed Nash equilibria in this case. Finally, we study and fully characterize the Nash equilibria and subgame-perfect Nash equilibria for the resort-tourist game, when resorts use pure strategies. We link the equilibria with the relevant parameters of the model. These analysis will be the basis for future studies of the impacts of different policies of investment and publicity under uncertainties.

Chapter 1

Decision Model: general case.

The *decision model* has N types $\mathbf{T} = \{t_1, t_2, \ldots, t_N\}$ of individuals. Let $\mathbf{I}_t = \{1, \ldots, n_t\}$ be the set of all individuals with type $t \in \mathbf{T}$ and set $\mathbf{I} = \bigsqcup_{t=1}^{N} \mathbf{I}_t$. The individual $i \in \mathbf{I}$ has to make a decision $d \in \mathbf{D} = \{d_1, \ldots, d_M\}$.

The *type function* $t : \mathbf{I} \to \mathbf{T}$ associates to each individual i his type $t(i) \in \mathbf{T}$. Hence $t^{-1}(t)$ determines the group of all individuals i with type $t = t(i)$. We describe the pure decision of the individuals by a *strategy map* $S : \mathbf{I} \to \mathbf{D}$ that associates to each individual $i \in \mathbf{I}$ his decision $S(i) \in \mathbf{D}$. Let the *preference decision coordinates* ω_t^d indicate how much an individual i with type $t(i)$ likes or dislikes, to make decision $d = S(i)$. The preference decision coordinates indicate for each type of individuals the decision that the individuals prefer, i.e. the taste type of the individuals (see [3, 12, 6, 5, 10]). Hence, the *preference decision matrix* $\Omega = \Omega(S) \in \mathbb{R}^{NM}$ is given by

$$\Omega = \begin{pmatrix} \omega_1^1 & \ldots & \omega_1^M \\ \vdots & \ddots & \vdots \\ \omega_N^1 & \ldots & \omega_N^M \end{pmatrix} .$$

Let the *preference neighbors coordinates* $\alpha_{tt'}^d$ indicate how much an individual with type t who decides d likes or dislikes that an individual with type t' makes decision d. The preference neighbors coordinates indicate, for each type of individuals who make decision d, whom they prefer, or do not prefer, to share that decision with, i.e. the crowding type of the individuals (see [3, 12, 6, 5, 10]).

Given a strategy S, let $L(S)$ be the *strategic decision matrix* whose coordinates $l_t^d = l_t^d(S)$ indicate the number of individuals with type t who make decision d

$$L(S) = \begin{pmatrix} l_1^1 & \dots & l_1^M \\ \vdots & \ddots & \vdots \\ l_N^1 & \dots & l_N^M \end{pmatrix} .$$

Let $\mathbf{S}$ be the space of all strategies S. Let $\mathbf{L} = \{L(S) \in \mathbb{R}^{NM} : S \in \mathbf{S}\}$ be the set of all strategic decision matrices $L(S)$. The *auxiliar utility function*

$$\underline{U} : \mathbf{T} \times \mathbf{D} \times \mathbf{L} \to \mathbb{R}$$

is given by

$$\underline{U}(t; d, L) = \omega_t^d - \alpha_{tt}^d + \sum_{t'=1}^{N} \alpha_{tt'}^d l_{t'}^d .$$

The *utility function* $U : \mathbf{I} \times \mathbf{S} \to \mathbb{R}$ is

$$U(i; S) = \underline{U}(t(i); S(i); L(S)) .$$

We observe that, if two individuals i_1 and i_2 with the same type $t(i) = t(i')$ make the same decision $d = S(i) = S(i')$, then their utilities $U(i; S)$ and $U(i; S)$ are equal, i.e.

$$U(i; S) = \underline{U}(t; d, L(S)) = U(j; S).$$

Given a strategy S^*, for every $i \in \mathbf{I}$ and $d \in \mathbf{D} \setminus \{S(i)\}$, we define the strategy $S^*_{i \to d}$ by $S^*_{i \to d}(i) = d$ and $S^*_{i \to d}(j) = S^*(j)$, for every $j \in \mathbf{I} \setminus \{i\}$.

A strategy $S^* : \mathbf{I} \times \mathbf{S} \to \mathbf{D}$ is a *(pure) Nash equilibrium* if

$$U(i; S^*) \geq U(i; S^*_{i \to d})$$

for every $i \in \mathbf{I}$ and $d \in \mathbf{D} \setminus \{S(i)\}$. The *Nash domain* $\mathbf{N}(S)$ of a strategy $S \in \mathbf{S}$ is the set of all preference decision matrices $\Omega \in \mathbb{R}^{NM}$ with the property that S is a Nash equilibrium.

We define the *relative decision preference coordinate* $x(t; d, d')$ of an individual i with type $t = t(i)$ by

$$x(t; d, d') = \omega_t^d - \omega_t^{d'}$$

for every $d, d' \in \mathbf{D}$ with $d \neq d'$.

Given a strategy S, for every individual $i \in \mathbf{I}$ with type $t = t(i)$ and for every $d \in \mathbf{D} \setminus \{S(i)\}$, the *preference threshold* $T(i \to d)$ of the relative decision preference coordinate $x(t; d, S(i))$ is defined by

$$T(i \to d) = T(t(i); S(i), d) = -\alpha_{tt}^{S(i)} + \sum_{t'=1}^{N} \left(\alpha_{tt'}^{S(i)} l_{t'}^{S(i)} - \alpha_{tt'}^d l_{t'}^d \right) . \qquad (1.0.1)$$

Remark 1 (Transitive and reflexive thresholds properties)**.** *For every* $i_1, i_2, i_3 \in \boldsymbol{I}$ *satisfying* $t(i_1) = t(i_2) = t(i_3)$ *and* $S(i_1) \neq S(i_2)$*,* $S(i_2) \neq S(i_3)$ *and* $S(i_3) \neq S(i_1)$*, the following identities hold:*

$$T(i_1 \to S(i_2)) + T(i_2 \to S(i_3)) - T(i_1 \to S(i_3)) = -\alpha_{tt}^{S(i_2)} \quad (1.0.2)$$

$$T(i_1 \to S(i_2)) + T(i_2 \to S(i_1)) = -\left(\alpha_{tt}^{S(i_1)} + \alpha_{tt}^{S(i_2)}\right) \quad (1.0.3)$$

1.1 Cohesive Nash equilibria

A *cohesive strategy*[1] $S : \mathbf{I} \to \mathbf{D}$ is a strategy in which all individuals with the same type prefer to make the same decision, i.e. for every $t \in T$, $S(t^{-1}(t))$ is a singleton. We note that there are NM cohesive strategies.

Theorem 1. *A cohesive strategy S is a Nash equilibrium if, and only if,*

$$x(t(i); d, S(i)) \leq T(i \to d) \tag{1.1.1}$$

for every individual $i \in \mathbf{I}$ and every decision $d \in \mathbf{D} \setminus \{S(i)\}$.

Therefore, the Nash domain $\mathbf{N}(S)$ of a cohesive strategy S is non-empty and coincides with the half-hyper-plan consisting of all the preference decision matrices in the space $\mathbb{R}^{NM}$ satisfying inequalities (1.1.1).

Proof. A cohesive strategy S is a Nash equilibrium if, and only if, for every individual $i \in \mathbf{I}$ and every decision $d \in \mathbf{D} \setminus \{S(i)\}$,

$$U(i; S) \geq U(i; S_{i \to d}).$$

Let $c = S(i)$ and $t = t(i)$ and note that $l_t^c = n_t$ and $l_t^d = 0$. Substituting the values in the utilities, we obtain

$$\omega_t^c + \alpha_{tt}^c (n_t - 1) + \sum_{\substack{t'=1 \\ t' \neq t}}^{N} \alpha_{tt'}^c l_{t'}^c \geq \omega_t^d + \sum_{\substack{t'=1 \\ t' \neq t}}^{N} \alpha_{tt'}^d l_{t'}^d \ .$$

Rearranging the terms of the last inequality, we get

$$\omega_t^d - \omega_t^c \leq -\alpha_{tt}^c - \sum_{t'=1}^{N} \left(\alpha_{tt'}^d l_{t'}^d - \alpha_{tt'}^c l_{t'}^c \right) \ .$$

□

[1] or equivalently *herd strategy*

1.2 Disparate Nash equilibria

In this section, we study the existence of equilibria where individuals of the same type might be making different decisions.

We say that one type $t \in \mathbf{T}$ of individuals is *splitted* among different decisions, if there is at least one individual i with type $t(i) = t \in \mathbf{T}$ deciding $d = S(i)$ and another individual i' with the same type $t(i') = t$ not deciding d, meaning $S(i') \neq d$.

Definition 1. *A* disparate strategy *is a strategy with at least one type of individuals splitted among different decisions.*

Therefore, the herd effect is broken for a disparate strategy at least for one type of individuals. We note that a pure strategy is either a cohesive or a disparate strategy.

The set $\mathbf{D}_t(S)$ of *splitted decisions* of a strategy S is

$$\mathbf{D}_t(S) = \{\{d_1, d_2\} : \quad S(i) = d_1 \neq d_2 = S(j) \quad \text{with} \quad t(i) = t(j) = t\}.$$

Given a strategy S, for every individual $i \in \mathbf{I}$ and for every decision $d \in \mathbf{D} \setminus \{S(i)\}$, we define the interval $I(i \to d) = I(t(i); S(i), d)$ as follows:

1. if $\{S(i), d\} \notin \mathbf{D}_{t(i)}(S)$ then
$$I(i \to d) = (-\infty, T(i \to d)]$$
2. if $\{S(i), d\} \in \mathbf{D}_{t(i)}(S)$ then there is $j \in \mathbf{I}$ with $t(j) = t$ such that $S(j) = d$ and
$$I(i \to S(j)) = [-T(i \to S(j)), T(j \to S(i))]$$

Therefore, given a strategy S, for all individuals $i, j \in \mathbf{I}$ with the same type $t(i) = t(j) = t$ and $S(i) \neq S(j)$ the center of the interval $I(i \to S(j))$ is

$$c(i \to S(j)) = c(t(i); S(i), d) = \frac{T(j \to S(i)) - T(i \to S(j))}{2}.$$

Moreover, by Remark 1, the length of the interval $I(i \to S(j))$ is

$$|I(i \to S(j))| = -(\alpha_{tt}^{S(i)} + \alpha_{tt}^{S(j)}).$$

Hence,

$$I(i \to S(j)) = \left[\frac{\alpha_{tt}^{S(i)} + \alpha_{tt}^{S(j)}}{2} + c(i \to S(j)), \frac{-(\alpha_{tt}^{S(i)} + \alpha_{tt}^{S(j)})}{2} + c(i \to S(j)) \right].$$

Theorem 2. *A pure stategy S is a Nash equilibrium if, and only if, for every individual $i \in \boldsymbol{I}$ and for every decision $d \in \boldsymbol{D} \setminus \{S(i)\}$*

$$x(t(i); S(i), d) \in I(i \to d). \tag{1.2.1}$$

The set $\mathbf{D}_t$ of *negative relative influences* is

$$\mathbf{D}_t = \{\{d_1, d_2\} \subset \mathbf{D} : \alpha_{tt}^{d_1} + \alpha_{tt}^{d_2} \leq 0\}.$$

We say that a strategy S satisfies the *disparate property* if, for every type $t \in \mathbf{T}$,

$$\mathbf{D}_t(S) \subset \mathbf{D}_t.$$

Hence, putting together equality (1.0.3) and the inequality characterizing the set of negative relative influences, we get the following corollary.

Corollary 1. *If a disparate strategy $S \in \boldsymbol{S}$ is a Nash equilibrium then S satisfies the disparate property.*

Proof. The disparate strategy $S : \mathbf{I} \times \mathbf{S} \to \mathbf{D}$ is a Nash equilibrium if, and only if,

$$U(i; S) \geq U(i; S_{i \to d})$$

for every $i \in \mathbf{I}$ and $d \in D \setminus \{S(i)\}$. Letting $l_t^d = l_t^d(S)$ and $c = S(i)$ and $t = t(i)$, we get

$$\omega_t^c - \alpha_{tt}^c + \sum_{t'=1}^{N} \alpha_{tt'}^c l_{t'}^c \geq \omega_t^d + \sum_{t'=1}^{N} \alpha_{tt'}^d l_{t'}^d \ .$$

Rearranging the terms, the previous inequality is equivalent to

$$x(t; c, d) \geq \alpha_{tt}^c - \sum_{t'=1}^{N} \left(\alpha_{tt'}^c l_{t'}^c - \alpha_{tt'}^d l_{t'}^d\right) \ .$$

Hence, $U(i; S) \geq U(i; S_{i \to d})$ if, and only if,

$$x(t; c, d) \geq -T(i \to d). \tag{1.2.2}$$

If $\{c, d\} \notin \mathbf{D}_t(S)$ then (1.2.2) is equivalent to (1.2.1). On the other hand, if $\{c, d\} \in \mathbf{D}_t(S)$ then there is $j \in \mathbf{I}$ with $t = t(j)$ such that $S(j) = d$. Thus, similarly, we have that $U(j; S) \geq U(j; S_{j \to c})$ is equivalent to

$$x(t; d, c) \geq -T(j \to c)$$

Noting that $x(t; c, d) = -x(t; d, c)$, we get

$$T(j \to c) \leq x(t; c, d) \leq -T(i \to d)$$

that is equivalent to (1.2.1). □

For every $t \in \mathbf{T}$, let

$$d_t^* = \arg \max_{\{S(i) \in \mathbf{D} : t(i) = t\}} \{\alpha_{tt}^d\} \quad \text{and} \quad \alpha_t^* = \min\{0, \alpha_{tt}^{d_t^*}\}. \tag{1.2.3}$$

Let $i^* \in \mathbf{I}$ be such that $S(i^*) = d_t^*$. Given a strategy S, for every $t \in \mathbf{T}$ and for every individual $i \in \mathbf{I}_t$ with $\{S(i), d_t^*\} \in D_t(S)$, we define

$$J(t; S(i), d_t^*) = \left[\frac{\alpha_{tt}^{S(i)} + \alpha_{tt}^{d_t^*} - \alpha_t^*}{2} + c(i \to d_t^*)), \frac{-(\alpha_{tt}^{S(i)} + \alpha_{tt}^{d_t^*} - \alpha_t^*)}{2} + c(i \to d_t^*) \right].$$

and

$$g(t; S(i^*), S(i)) = \left[\frac{\alpha_{tt}^{d_t^*} + \alpha_{tt}^{S(i)} - \alpha_t^*}{2} + c(i^* \to S(i))), \frac{-(\alpha_{tt}^{d_t^*} + \alpha_{tt}^{S(i)} - \alpha_t^*)}{2} + c(i^* \to S(i)) \right].$$

Theorem 3. *Let S be a disparate strategy. If for every individual $i \in \boldsymbol{I}$*

$$x(t(i); S(i), S(i^*)) \in J(t(i); S(i), d_{t(i)}^*),$$

then S is a Nash equilibrium.

Proof. For every $t \in \mathbf{T}$ with $D_t(S) \neq \emptyset$, and for every individuals $i, j \in \mathbf{I}_t$ with $\{S(i), S(j)\} \in D_t(S)$, letting $c = S(i), d = S(j)$, we have that

$$x(t; c, d) = x(t; c, d_t^*) + x(t; d_t^*, d) \tag{1.2.4}$$

Let i^* be the individual with type t such that $S(i^*) = d_t^*$, where d_t^* is as in (1.2.3). Since $x(t; c, d_t^*) \in J(t; c, d_t^*)$ and $x(t; d_t^*, d) \in g(t; d_t^*, d)$, consider the deviations from the centers of the respective intervals, given by

$$\epsilon_c = |x(t; c, d_t^*) - c(i \to d_t^*)| \qquad \text{and} \qquad \epsilon_d = |x(t; d_t^*, d) - c(i^* \to d)| \,.$$

This deviatons are majorated

$$\epsilon_c \leq \left| \frac{\alpha_{tt}^c + \alpha_{tt}^{d_t^*} - \alpha_t^*}{2} \right| \qquad \text{and} \qquad \epsilon_d \leq \left| \frac{\alpha_{tt}^d + \alpha_{tt}^{d_t^*} - \alpha_t^*}{2} \right|$$

Thus, we obtain

$$\epsilon_c + \epsilon_d \leq \left| \frac{\alpha_{tt}^d + \alpha_{tt}^d}{2} + \alpha_{tt}^{d_t^*} - \alpha_t^* \right| \leq \left| \frac{\alpha_{tt}^d + \alpha_{tt}^d}{2} \right|.$$

We now note that

$$c(i \to d_t^*) + c(i^* \to d) = c(i \to d).$$

Therefore using (1.2.4)

$$x(t; c, d) = c(i \to d) + \epsilon_c + \epsilon_d$$

implying

$$x(t; c, d) \in I(i \to d).$$

Hence, by Theorem 2, S is a Nash equilibrium. □

Chapter 2

Tourists and Resorts Game

We consider a two stage game where there are two types of players: tourists and resorts. In the first stage resorts decide simultaneously their prices, and in the second stage tourists observe prices, and choose one of the resorts. As a clarifying example we take a group of $n \in \mathbb{N}$ tourists having to choose between spending holidays in a Beach or in a Mountain resort. We denote the set of players by $\mathcal{I} = \mathbf{R} \cup \mathbf{T}$, where the set of resorts is $\mathbf{R} = \{B, M\}$ and the set of tourists is $\mathbf{T} = \{1, \ldots, n\}$.

The tourists set of actions is the resorts. After a pair of prices has been set on the first stage of the game, the tourists choice will depend on their relative preferences and the influence they have on each other.

The parameters appearing in the tourists' utility function and the utility function itself follow from the model presented in the first chapter. Here we consider just one type of tourists facing a dichotomous decision, and we add the new feature of prices entering the utility function. Thus, for a given resort $R \in \mathbf{R}$, the parameters influencing a tourist decision are:

θ_R - the price set by resort R;

ω_R - how much a tourist likes or dislikes resort R

α_R - how much a tourist likes to be with other tourist in resort R;

The parameters regarding tourists preferences and influences are common knowledge for all players. The outcome of the game is the pair of prices and the tourists allocation.

We characterize game equilibria using the notions of Nash equilibrium and subgame-perfect Nash equilibrium. We allow for two different cases: tourists reaction to price changes depends on which resort is changing its price; tourists reaction depends only on the price difference[1]. In the latter case, if the average influence $(\alpha_B + \alpha_M)$ is positive then the subgame-perfect Nash equilibria are either monopolies or competitive equilibria where resorts have zero profits. When there are negative average influences, non-monopolistic and non-zero profit subgame-perfect Nash equilibria exist. We find the prices and preferences for which these equilibria occur.

[1]A different interpretation may be that resorts have different understandings of how the market will respond if they deviate.

2.1 Tourists Nash allocations

We describe the tourists decisions by a strategy map $S : \{1, \ldots, n\} \to [0, 1]$ that assignes to each individual i his choice probability. Namely, an individual chooses resort B with probability $S(i) = p_i^B$ and resort M with probability $p_i^M = 1 - S(i)$. The set of all tourists strategies is

$$\mathbf{S}(T) = \left\{(p_1^B, p_2^B, \ldots, p_n^B) : 0 \leq p_i^B \leq 1 \qquad \text{and} \qquad p_i^B + p_i^M = 1\right\}.$$

Given the tourists strategy $S(\theta_B, \theta_M) = (p_1^B, \ldots, p_n^B) \in \mathbf{S}(T)$, we define the *beach market share* S_B and the *mountain market share* S_M by

$$S_B(\theta_B, \theta_M) = \sum_{i=0}^{n} p_i^B \qquad \text{and} \qquad S_M(\theta_1, \theta_2) = \sum_{i=0}^{n} (1 - p_i^B) = N - S_B(\theta_B, \theta_M).$$

Let $S_B(i) = S(i)$ and $S_M(i) = 1 - S(i)$. The utility $U_i : \mathbf{S}(T) \to \mathbb{R}$ of each individual i is

$$U_i(S) = \sum_{R \in \{B,M\}} p_i^R(-\theta_R + \omega_R + \alpha_R(S_R(\theta_B, \theta_M) - p_i^R)).$$

Let $A = \alpha_B + \alpha_M$. We say the tourists *like to meet each other* in the same resort if $A > 0$, and that the tourists *do not like to meet each other* in the same resort if $A \leq 0$.

The *free price relative preference* for the tourists is $x_F = \omega_B - \omega_M \in \mathbb{R}$. The *price relative preference* for the tourists is

$$x = \theta_M - \theta_B + x_F \in \mathbb{R}. \tag{2.1.1}$$

Let the *relative preference thresholds* be $T(l) = -Al + \alpha_M(n - 1)$ and let the *auxiliar relative preference thresholds* be $T'(l) = T(l - 1)$ with respect to the relative price preference x.

An *l-strategy* $S \in \mathbf{S}(T)$ is a strategy (i) with l individuals i opting to choose resort B with probability $S(i) = 1$ and (ii) with $n - l$ individuals i opting to choose resort M with probability $1 - S(i) = 1$. Hence, $S_B = l$. The pure Nash equilibria are the union of all l-strategies that are Nash equilibria. We note that if one l-strategy S is a Nash equilibrium for some price relative preference x, then all l-strategies are Nash equilibria for the same x. The following two lemmas follow from the theorems 1 and 2 in chapter 1.

Lemma 1 (Pure Nash equilibria for tourists that like to meet each other)**.** *Let* $A > 0$.

(i) An l*-strategy is a Nash equilibrium if, and only if,* $l = 0$ *or* $l = n$.

(ii) If a 0*-strategy is a Nash equilibrium then* $x \in (-\infty, T(0)]$.

(iii) If an n*-strategy is a Nash equilibrium then* $x \in [T'(n), +\infty)$.

Proof. Since $A > 0$ the set of negative relative influences is empty (chapter 1, page 12). The proof then follows from corollary 1 and theorem 1. □

When $A > 0$, we have $T'(n) < T(0)$. Hence, if $x \in [T'(n), T(0)]$ then a pure Nash equilibrium can be either a 0-strategy or an n-strategy.

Lemma 2 (Pure Nash equilibria for tourists that do not like to meet each other)**.** *Let* $A \leq 0$.

(i) A 0*-strategy is a Nash equilibrium if, and only if,* $x \in (-\infty, T(0)]$.

(ii) An n*-strategy is a Nash equilibrium if, and only if,* $x \in [T'(n), +\infty)$.

(iii) An l*-strategy is a Nash equilibrium if, and only if,* $x \in [T'(l), T(l)]$.

Proof. Follows from theorem 2. □

We note that $T(l) = T'(l + 1)$. Hence, when $A \leq 0$, if $x \in \mathbb{R} \setminus \cup_{l=0}^{n-1}\{T(l)\}$ then there is a unique pure tourist strategy that is a Nash equilibrium; if $x = T(l)$ then a pure Nash equilibrium can be either an l-strategy or an $l + 1$-strategy.

An (l, k) *strategy* $S \in \mathbf{S}(T)$ is a strategy (i) with l individuals i opting to choose resort B with probability $S(i) = 1$, (ii) with $n - (l + k)$ individuals i opting to choose resort M with probability $1 - S(i) = 1$, and (iii) with $k \geq 1$ individuals i opting to choose resort B with some probability $0 \leq S(i) = p_S \leq 1$. We call p_S the (l, k) *probability* of the strategy S. Hence, $S_B = l + kp_S$. A *strict* (l, k) *strategy* $S \in \mathbf{S}(T)$ is an (l, k) strategy with (l, k)-probability $0 < p_S < 1$. Hence, l and $l + k$ strategies are contained in the (l, k) strategies but not in the strict (l, k) strategies. We note that if one (l, k) strategy S is a Nash equilibrium for some price relative preference x then all (l, k) strategies are Nash equilibria for the same x. Let $\mathbf{N}(T; x)$ be the set of all tourists' strategies that are Nash equilibria for some price relative preference x. The results below characterize the mixed Nash equilibria, and a more general version is proved in [8].

Lemma 3 (Mixed Nash equilibria for tourists that like to meet each other)**.** *Let $A > 0$.*

(i) A strict mixed strategy is a Nash equilibria if, and only if, the strategy is of the type $(l, k) = (0, n)$ and $x \in (T'(n), T(0))$.

(ii) Furthermore, the triple $(0, n; x)$ uniquely determines

$$S_B = l + \frac{k(x - T(0))}{-A(n-1)} \ , \ p_S = \frac{x - T(0)}{-A(n-1)} \ \textit{and} \ S_B - p_S = \frac{x - T(0)}{-A}.$$

Theorem 4 (Mixed Nash equilibria for tourists that do like to meet each other)**.** *Let $A \leq 0$.*

(i) The mixed Nash equilibria are the union of all (l, k)-strategies that are Nash equilibria.

(ii) An (l, k) strategy is a mixed Nash equilibria if, and only if, $x \in [T(l), T'(l + k)]$.

(iii) Furthermore, the triple $(l, k; x)$ uniquely determines

$$S_B = l + \frac{k(x - T(l))}{-A(k-1)} \ , \ p_S = \frac{x - T(l)}{-A(k-1)} \ \textit{and} \ S_B - p_S = \frac{x - T(0)}{-A}.$$

The results above concerning Nash equilibria are now summarized, characterizing the possible second stage Nash market shares for resorts.

The *horizontal market share fibers* are

$$\mathcal{H}_0^B = \{(x, 0) : x \leq T(0)\} \quad ; \qquad \mathcal{H}_n^B = \{(x, n) : x \geq T'(n)\}$$

$$\mathcal{H}_l^B = \{(x, l) : T'(l) \leq x \leq T(l)\}.$$

The *global horizontal market share fiber* is $\mathcal{H}_B = \bigcup_l^n \mathcal{H}_l^B$. The *vertical market share fibers* are

$$\mathcal{V}_l^B = \{(T(l), y) : l \leq y \leq l + 1\}.$$

The *global vertical market share fiber* is $\mathcal{V}_B = \bigcup_l^n \mathcal{V}_l^B$. Let $y_{l,k}$ be the straight-line given by

$$y_{l,k}(x) = l + \frac{k(x - T(l))}{-A(k-1)}.$$

The *oblique market share fibers* are

$$\mathcal{O}_{l,k}^B = \{(x, y_{l,k}(x)) : T(l) \leq x \leq T'(l + k)\}.$$

The *global oblique market share fiber* is $\mathcal{O}_B = \bigcup_{l,k \le n} \mathcal{O}^B_{l,k}$. If $A \le 0$ the *beach market share fiber* $\mathcal{F}_B$ is

$$\mathcal{F}^A_B = \mathcal{H}_B \cup \mathcal{V}_B \cup \mathcal{O}_B.$$

If $A > 0$ the *beach market share fiber* $\mathcal{F}_B$ is

$$\mathcal{F}^A_B = \mathcal{H}^B_0 \cup \mathcal{H}^B_n \cup \mathcal{O}^B_{0,n}.$$

The *mountain market share fiber* $\mathcal{F}_M$ is the set of all points $(x, y) \in \mathbb{R}^2$ with the property that $(x, n - y) \in \mathcal{F}_B$. In figure 2.1 it is depicted the market share fiber for the case where $A > 0$. In figure 2.2 it is depicted the market share fiber for the case where $A < 0$ and there are $n = 6$ tourists.

Corollary 2 (Geometry of the mixed Nash equilibria)**.** *Let $A \le 0$. There is a well defined correspondence between the Nash strategies of the beach Nash domain and the points on the beach market share fiber*

$$S(\theta_B, \theta_M) \in \boldsymbol{N}(T; x) \Leftrightarrow (x, S_B(\theta_B, \theta_M)) \in \mathcal{F}_B$$

with the following properties:

(i) An l tourist strategy $S(\theta_B, \theta_M)$ is Nash if, and only if, $(x, S_B(\theta_B, \theta_M)) \in \mathcal{H}^B_l$.

(ii) An $(l, 1)$ tourist strategy $S(\theta_B, \theta_M)$ is Nash if, and only if, $(x, S_B(\theta_B, \theta_M)) \in \mathcal{V}^B_l$.

(iii) An (l, k) tourist strategy $S(\theta_B, \theta_M)$ is Nash if, and only if, $(x, S_B(\theta_B, \theta_M)) \in \mathcal{O}^B_{l,k}$ with $k \ge 2$.

The oblique line segment $O^B_{l,k}$ starts at the corner $(T(l), y_{l,k}(T(l)))$ formed by the right end of the horizontal line segment H^B_l and the bottom end of the vertical line segment V^B_l. The oblique line segment $O^B_{l,k}$ ends at the corner $(T'(l+k), y_{l,k}(T'(l+k)))$ formed by the left end of the horizontal line segment H^B_{l+k} and the top end of the vertical line segment V^B_{l+k}. Every oblique line segment $O^B_{l,k}$ crosses the interior of the horizontal line segments H^B_{l+j} and the interior of the vertical line segments V^B_{l+j} with $1 \le j \le k - 1$. If $l < l'$ and $l + k < l' + k'$ then the oblique line segment $O^B_{l,k}$ and $O^B_{l',k'}$ do not cross each other and $O^B_{l,k}$ is on the left hand side of $O^B_{l',k'}$, i.e. $y^{-1}_{l,k}(y) \le y^{-1}_{l',k'}(y)$ for every $l \le y \le l + k$.

Lemma 4 (Geometry of the beach market share fiber)**.** *Let $A \le 0$. If $l' \le l$ and $k' \le k$ then the oblique line segments $O^B_{l,k}$ and $O^B_{l',k'}$ cross at the point*

$$(x(l, k; l', k'), y(l, k; l', k')) \in [T(l), T'(l' + k')] \times [l, l' + k']$$

given by

$$x(l,k;l',k') = \frac{-A(k-1)(l-l')}{(k'-k)} + T(l) = \frac{-A(k'-1)(l-l')}{(k'-k)} + T(l')$$

and

$$y(l,k;l',k') = l + \frac{k(l-l')}{(k'-k)} = l' + \frac{k'(l-l')}{(k'-k)}$$

$$y(l,k;l',k') = l + k + \frac{k(l+k-l'-k')}{(k'-k)} = l' + k' + \frac{k'(l+k-l'-k')}{(k'-k)}.$$

Before the crossing point, $O^B_{l,k}$ is on the right hand side of $O^B_{l',k'}$, and after the crossing point, $O^B_{l,k}$ is on the left hand side of $O^B_{l',k'}$.

Proof. Note that if a crossing point of two oblique line segments occurs, then $A \neq 0$ and for those segments $k > 1$. The crossing point is determined by

$$l + \frac{k(x-T(l))}{-A(k-1)} = y_{l,k}(x) = y_{l',k'}(x) = l' + \frac{k'(x-T(l'))}{-A(k'-1)}.$$

Hence,

$$-A(k-1)(k'-1)l + (k'-1)k(x-T(l)) = -A(k-1)(k'-1)l' + (k-1)k'(x-T(l')).$$

Thus,

$$(k'-k)x = A(k-1)(k'-1)(l-l') + (k'-1)kT(l) - (k-1)k'T(l').$$

Since,

$$T(l) = -Al + \alpha_M(n-1)$$

we get

$$(k'-k)x = -A(k'-1)l + A(k-1)l' + (k'-k)\alpha_M(n-1).$$

Therefore,

$$x = \frac{-A(k'-1)l + A(k-1)l'}{(k'-k)} + T(0)$$

or, equivalently,

$$x = \frac{-A(k-1)(l-l')}{(k'-k)} + T(l)$$

Hence,

$$y_{l,k}(x) = l + \frac{k(x-T(l))}{-A(k-1)} = l + \frac{k(l-l')}{(k'-k)}$$

or, equivalently,

$$y(l,k;l',k') = l + k + \frac{k(l+k-l'-k')}{(k'-k)} = l' + k' + \frac{k'(l+k-l'-k')}{(k'-k)}.$$

□

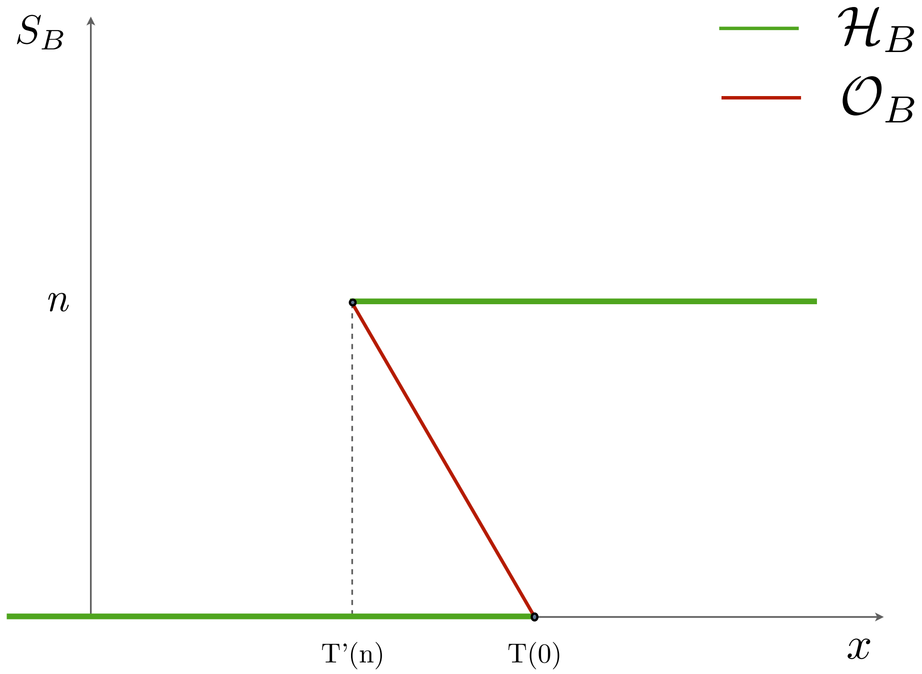

Figure 2.1: The beach market share fiber for the case when $A > 0$.

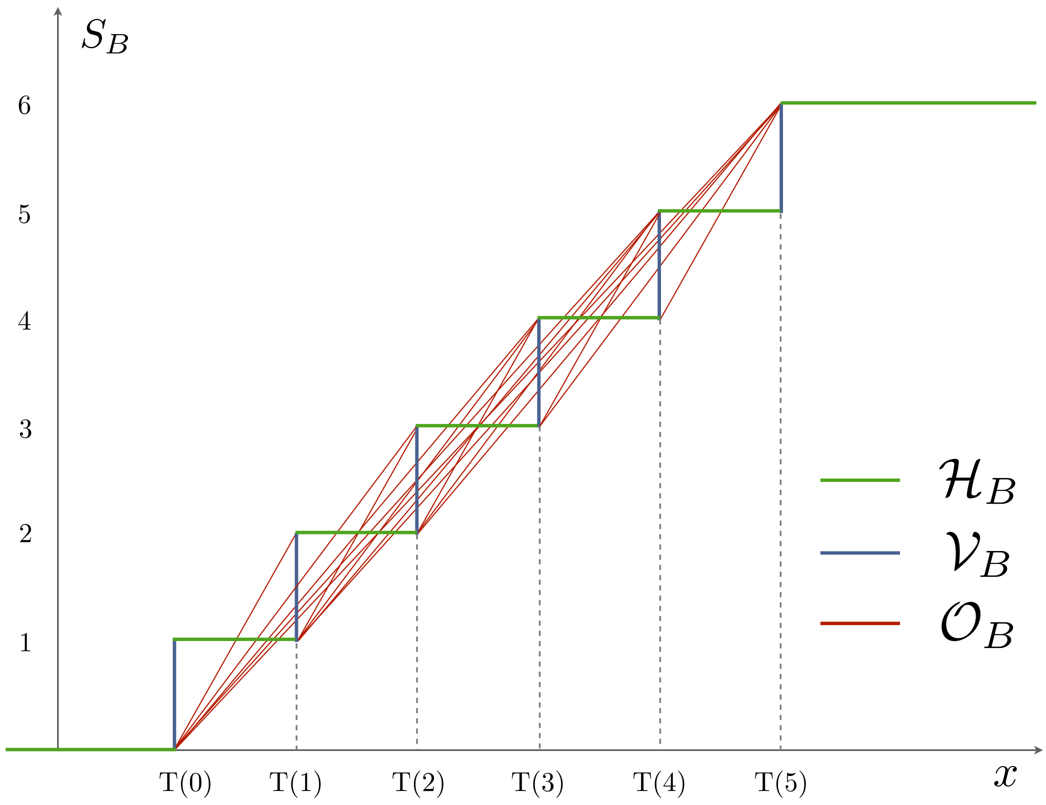

Figure 2.2: The beach market share fiber for the case when $A < 0$ and there are $n = 6$ tourists.

2.2 Resorts profit

In this section we analyze some consequences for resorts, arising from the existence of multiple Nash equilibria of the second stage game for the same pair of prices.

The *profit* $\Pi_B : \mathbf{S}(R) \to \mathbb{R}$ of the beach resort is

$$\Pi_B(\theta_B; S(\theta_B, \theta_M)) = \theta_B S_B(\theta_B, \theta_M).$$

The *profit* $\Pi_M : \mathbf{S}(R) \to \mathbb{R}$ of the mountain resort is

$$\Pi_M(\theta_B; S(\theta_B, \theta_M)) = \theta_M S_M(\theta_B, \theta_M).$$

The case $A < 0$ is particularly relevant in this setting, since the existence of multiple Nash equilibria in the second stage, for the same pair of prices, poses a problem as to which should be the strategy chosen by resorts on the first stage, given a relative price preference x. Namely we can ask: if the resorts do not know what equilibria will occur, is it possible to know what are the resorts' best responses and what are the resorts strategies leading to game equilibria? Before we address this question we will study the set of second stage equilibria $\mathbf{N}(T, x)$, given a pair of prices and the free price relative preferences x_F. Recall that $x = x_F - \theta_B + \theta_M$. Throughout this section we will assume $A < 0$.

By the properties of the fiber space we know that for every Nash equilibrium $S \in \mathbf{N}(T, x)$ there is a unique p_S associated to each (l, k) strategy. Therefore we have an order of the Nash equilibria in this set associated to their probability, $p_{min} \leq \cdots \leq p_S \leq \cdots \leq p_{Max}$. The profit of resorts will depend on the market share fiber equilibrium that will be chosen by the tourists.

By theorem 4

$$S_B - p_S = (x - T(0))/(-A)$$

and so $S_B - p_S$ does not depend upon the tourists strategy but only on the relative price preference x. Let the tourists strategy free fibers TSF_B and TSF_M of resorts B and M, respectively, be given by

$$TSF_B = \left\{ \left(x, \frac{x - T(0)}{-A} \right) : x \in [T(0), T'(n)] \right\} \tag{2.2.1}$$

$$TSF_M = \left\{ \left(x, \frac{T'(n) - x}{-A} \right) : x \in [T(0), T'(n)] \right\} \tag{2.2.2}$$

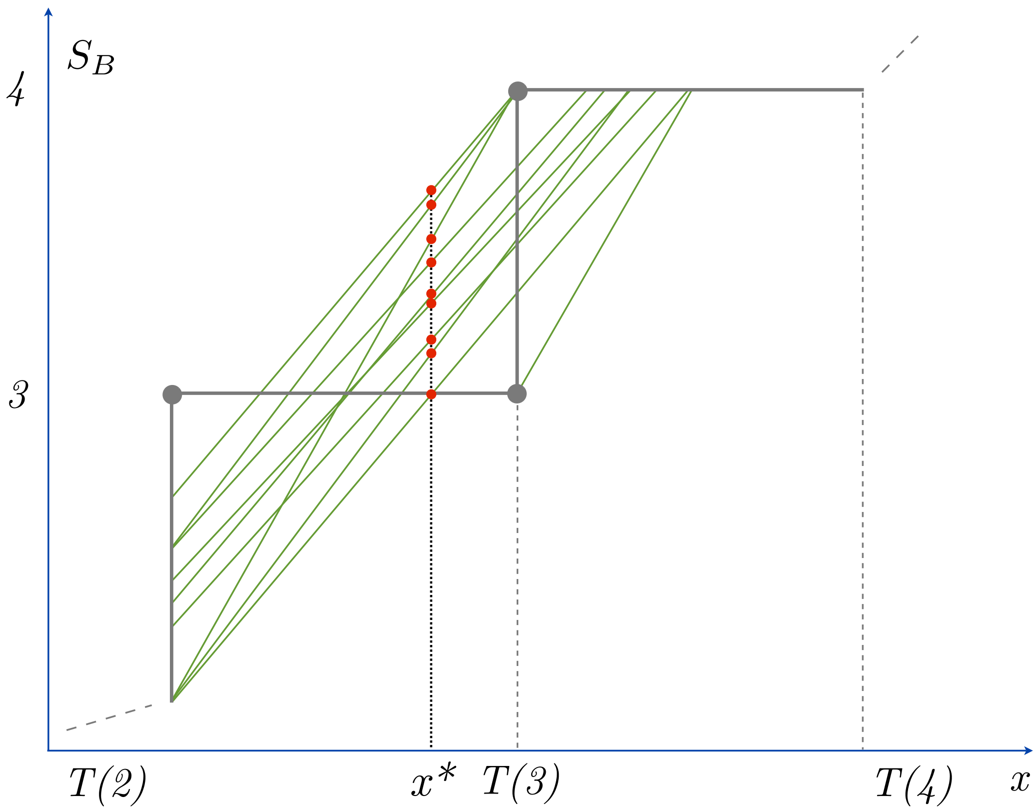

Figure 2.3: A zooming of the fiber space for $A < 0$, ilustrating the multiplicity of equilibria in the set $\mathbf{N}(T, x^*)$.

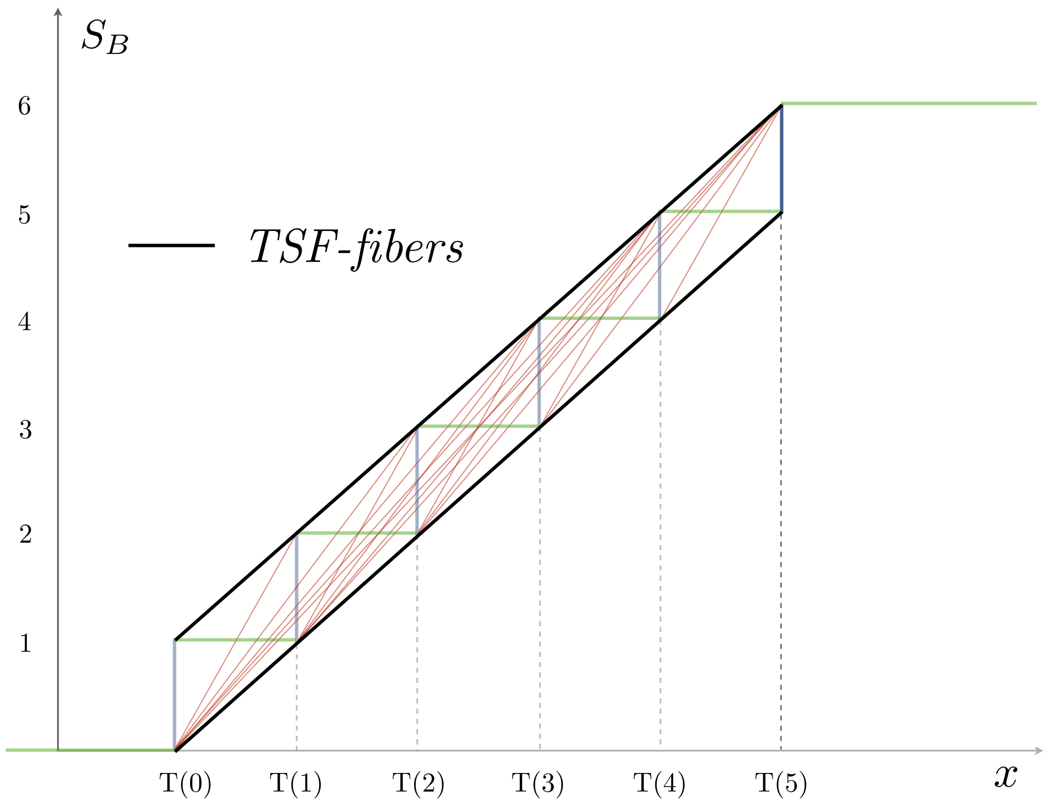

Figure 2.4: The TSF-fibers.

Let the *tourists strategy free* profit ($TSF - profit$) of resort B and M be

$$\underline{\Pi}_B(\theta_B, \theta_M; x) = \theta_B \cdot (S_B - p_S) = \theta_B \left(\frac{x - T(0)}{-A}\right) \tag{2.2.3}$$

$$\underline{\Pi}_M(\theta_B, \theta_M; x) = \theta_M \cdot (S_M - (1 - p_S)) = \theta_M \left(n - 1 - \frac{x - T(0)}{-A}\right) \tag{2.2.4}$$

We observe that the TSF-profit function does not depend upon the tourists strategy, but only on the relative price preference x. For a given tourists strategy, with strategy probability p_S, the resorts' profits are

$$\Pi_B(\theta_B; S(\theta_B, \theta_M)) = \underline{\Pi}_B(\theta_B, \theta_M; x) + \theta_B p_S. \tag{2.2.5}$$

$$\Pi_M(\theta_r; S(\theta_B, \theta_M)) = \underline{\Pi}_M(\theta_B, \theta_M; x) + \theta_M(1 - p_S). \tag{2.2.6}$$

Hence, the order of the equilibria associated with their strategy probability induces the same order on the profits of the set $\mathbf{N}(T, x)$ given by

$$\underline{\Pi}_B(\theta_B, \theta_M; x) \leq \Pi_B(p_{min}) \leq \cdots \leq \Pi_B(p_S) \leq \cdots \leq \Pi_B(p_{Max}) \leq \underline{\Pi}_B(\theta_B, \theta_M; x) + \theta_B \tag{2.2.7}$$

The profits $\underline{\Pi}_B(\theta_B, \theta_M; x)$ and $\underline{\Pi}_M(\theta_B, \theta_M; x)$ associated to the TSF fibers give the lower and upper bound in relation 2.2.7.

Definition 2. *A* TSF Nash equilibrium *is the pair of prices* (θ_B, θ_M) *that are Nash equilibrium with respect to the* TSF*-profit.*

Lemma 5. *A TSF Nash equilibrium is characterized by*

(i) the TSF *best response price* θ_B^* *of resort* B

$$\theta_B^*(\theta_M; x_F) = \frac{\theta_M + x_F + T(0)}{2}.$$

(ii) the TSF *best response price* θ_M^* *of resort* M

$$\theta_M^*(\theta_B; x_F) = \frac{-A(n-1) + \theta_B + T(0) - x_F}{2}$$

(iii) the TSF *Nash equilibrium prices are*

$$\theta_B^* = \frac{x_F + T(n-1) + 2T(0)}{3}$$

$$\theta_M^* = \frac{T(0) + 2T(n-1) - x_F}{3}$$

Proof. Using equations 2.2.3 and 2.2.4 and $x = x_F - \theta_B + \theta_M$, *TSF*-profit functions become

$$\underline{\Pi}_B(\theta_B, \theta_M; x) = (-A)^{-1}\theta_B\left(-\theta_B + \theta_M + x_F - T(0)\right) \tag{2.2.8}$$
$$\underline{\Pi}_M(\theta_r; S(\theta_B, \theta_M)) = \theta_M\left(n - 1 - \frac{-\theta_B + \theta_M + x_F - T(0)}{-A}\right). \tag{2.2.9}$$

For resort B we have

$$\underline{\Pi}'_B = (-A)^{-1}(-\theta_B^2 + \theta_B(\theta_M + x_F - T(0))$$

therefore the best response is

$$\theta_B^* = \frac{\theta_M + x_F + T(0)}{2}.$$

For resort M

$$\underline{\Pi}'_M = \left(n - 1 - \frac{-\theta_B + x_F - T(0)}{-A}\right) - 2\theta_M(-A)^{-1}$$

hence the best response is

$$\theta_M^* = \frac{-A(n-1) + \theta_B + T(0) - x_F}{2}$$

Using both responses we obtain the equilibrium. □

As depicted in figure 2.5, let

$$m(x^*) = \lfloor S_B - p_S \rfloor = \left\lfloor \frac{x^* - T(0)}{-A} \right\rfloor \in \{0, \ldots, n\};$$

and $q(x^*) = \frac{x^* - T(0)}{-A} - m(x^*) \in [0, 1]$. We call $m(x^*)$ the *closest threshold index* and $q(x^*)$ the *threshold distance.* This location parameters allow us to characterize, given x^* the set $\mathbf{N}(T, x^*)$.

Theorem 5 (Regularity paradox)**.** *Let $A < 0$ and $q(x^*) \in [0, 1)$. Consider the Nash equilibria $\boldsymbol{N}(T, x^*)$ and the resort's B profit restricted to these tourists Nash equilibria.*

(i) If $q(x^*) \geq 1/(m(x^*)+1)$ then the tourists Nash equilibrium that yields the highest profit for resort B is the mixed $(0, m(x^*)+1)$ strategy;

(ii) If $q(x^*) \leq 1/(m(x^*)+1)$ then the tourists Nash equilibrium that yields the highest profit for resort B is the pure $m(x^*)$ strategy;

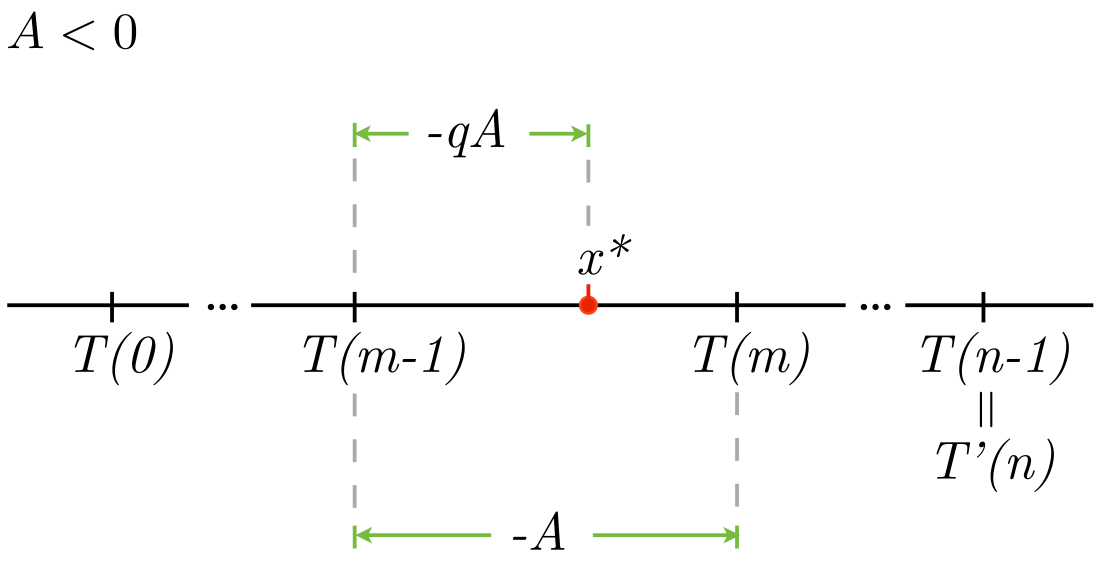

Figure 2.5: Geometric ilustration of $m(x^*)$ and $q(x^*)$.

(iii) If $q(x^*) \leq (n - m(x^*))/(n - m(x^*) + 1)$ *then the tourists Nash equilibrium that yields the lowest profit for resort B is the mixed* $(m(x^*) - 1, n - m(x^*) + 1)$ *strategy;*

(iv) If $q(x^*) \geq (n - m(x^*))/(n - m(x^*) + 1)$ *then the tourists Nash equilibrium that yields the lowest profit for resort B is the pure* $m(x^*)$ *strategy.*

Remark 2. *Observe that for every* $x \in \mathbb{R}$

$$1/(m(x) + 1) \leq (n - m(x))/(n - m(x) + 1).$$

Thus, no contradiction arises from lemma 5.

Proof. Given x^*, let $m = m(x^*)$ and $q = q(x^*)$. Recall that $T(l) = -Al + \alpha_M \cdot (n - 1)$, we note that

$$T(m - 1) - qA = x^* = T(l) - A(k - 1)p_S$$

For each (l, k) strategy with $k \neq 1$ and $S \in \mathbf{N}(T, x^*)$, using the above relation, we get

$$p_S = \frac{m - l - 1 + q}{k - 1}. \tag{2.2.10}$$

With $p_S \in (0, 1)$ and $k + l \leq n$. Hence, $l \in \{0, \ldots, m - 1\}$ and $k \in \{m - l + 1, \ldots, n - l\}$.

If $k = 1$ then $q = 0 \vee q = 1$, and any probability p_S results in equilibrium, hence the higher profit is associated to $p_S = 1$.

Suppose $k > 1$. By equation 2.2.10, the maximum profit of a Nash equilibrium $S \in \mathbf{N}(T, x)$ is an (l, k) strategy that maximizes $p_S = (m - l - 1 + q)/(k - 1) < 1$. The minimum k is $m - l + 1$. Therefore,

$$l^* = \arg\max \left\{ \frac{m - 1 - l + q}{m - l + 1 - 1} \right\}.$$

Hence $l^* = 0$ and $k^* = m + 1$.
Using a similar argument, the equilibrium with smallest profit is given by

$$l = \arg\min \left\{ \frac{m - 1 - l + q}{n - l - 1} \right\}.$$

Hence $l = m - 1$, $k = n - m + 1$.

Thus we have found the strict mixed equilibrium yielding the highest and lowest profits. Now we need to compare them with the profits associated to the pure equilibrium in $\mathbf{N}(T, x^*)$, when $l = m$ and $k = 0$ or $l = m - 1$, $k = 1$ and $p_S = 1$.

Recall that $S_B = l + kp_S$, hence we have that the strict mixed equilibrium for the $(0, m + 1)$ strategy still yields the highest profit when

$$(m + 1)p_S > m$$

Using equation (2.2.10) for p_S, we get the following condition

$$q > 1/(m + 1)$$

The strict mixed equilibrium for the $(m - 1, n - m + 1)$ strategy still yields the lowest profit when

$$m - 1 + (n - m + 1)p_S < m$$

Using again equation (2.2.10), we get the following condition

$$q < (n - m)/(n - m + 1)$$

Which concludes the proof. □

2.3 Resorts Nash prices

A *resort-tourist strategy* $((\theta'_B, \theta'_M); S(\theta_B, \theta_M))$ is (i) a price strategy $\theta'_B \geq 0$ for the beach resort and a price strategy $\theta'_M \geq 0$ for the mountain resort, and (ii) a strategy $S(\theta_B, \theta_M) \in \mathbf{S}(T)$ for the tourists for all admissible resorts' prices. Let $\mathbf{S}(R)$ be the set of all resort-tourist strategies. The *profit* $\Pi_B : \mathbf{S}(R) \to \mathbb{R}$ of the beach resort is

$$\Pi_B(\theta'_B; S(\theta'_B, \theta'_M)) = \theta'_B S_B(\theta'_B, \theta'_M).$$

The *profit* $\Pi_M : \mathbf{S}(R) \to \mathbb{R}$ of the mountain resort is

$$\Pi_M(\theta'_B; S(\theta'_B, \theta'_M)) = \theta'_M S_M(\theta'_B, \theta'_M).$$

The *beach resort best response price* θ^*_B to the price θ_M of the mountain resort is

$$\theta^*_B(\theta_M) = \arg\max_{\theta_B}\{\Pi_B(\theta_B; S(\theta_B, \theta_M))\}. \tag{2.3.1}$$

The *mountain resort best response price* θ^*_M to the price θ_B of the beach resort is

$$\theta^*_M(\theta_B) = \arg\max_{\theta_M}\{\Pi_M(\theta_B; S(\theta_B, \theta_M))\}. \tag{2.3.2}$$

A resort-tourist strategy $((\theta^*_B, \theta^*_M); S(\theta_B, \theta_M))$ is a *Nash resort-tourist equilibrium* if

$$S(\theta^*_B, \theta^*_M) \in \mathbf{N}(T, x^*)$$

and

$$\theta^*_B(\theta^*_M) = \theta^*_B \qquad \text{and} \qquad \theta^*_M(\theta^*_B) = \theta^*_M,$$

where $x^* = \theta^*_B - \theta^*_M + x_F$ and x_F is the free price relative preference. Let

$$\mathbf{N}(R - T; x_F)$$

be the set of all Nash resort-tourist equilibria.

If $S(\theta^*_B, \theta^*_M) \in \mathbf{N}(T; x)$ then $((\theta^*_B, \theta^*_M), S(\theta^*_B, \theta^*_M))$ is an *equilibrium price path strategy.* The *deviation from the equilibrium price path* of resort B is a strategy $S(\theta_B, \theta^*_M)$ for every $\theta_B \in [0, +\infty) \setminus \{\theta^*_B\}$. Let $I_B(\theta^*_B) = (-\infty, x^* + \theta^*_B] \setminus \{x^*\}$. The *deviation* of the price relative preference with respect to resort B is

$$x_B = \theta^*_B - \theta_B + x^*, \tag{2.3.3}$$

for $x_B \in I_B(\theta^*_B)$. We note that if $x_B > x^* + \theta^*_B$, the price $\theta_B < 0$ is not admissible as a strategy for the resort B. The *deviation from the equilibrium path* of resort M is a strategy $S(\theta_B^*, \theta_M)$ for every $\theta_M \in [0, +\infty) \setminus \{\theta_M^*\}$. Let $I_M(\theta_B^*) = [x^* - \theta_M^*, +\infty) \setminus \{x^*\}$. The *deviation* of the price relative preference with respect to resort M is

$$x_M = \theta_M - \theta_M^* + x^*$$

for $x_M \in I_M(\theta_M^*)$. We note that if $x_M < x^* - \theta_M^*$, the price $\theta_M < 0$ is not admissible as a strategy for the resort M.

Let $I_B = \{(x_B; \theta_B^*, \theta_M^*) : x_B \in I_B(\theta_B^*), \theta_B^* \geq 0, \theta_M^* \geq 0\}$, the *isoprofit market share of the beach resort* $h_B : I_B \to \mathbb{R}_0^+$ is given by

$$h_B(x_B; \theta_B^*, \theta_M^*) = \frac{\theta_B^* S_B(\theta_B^*, \theta_M^*)}{\theta_B^* + x^* - x_B} \geq 0.$$

We observe that $h_B(x^*; \theta_B^*, \theta_M^*) = S_B(\theta_B^*, \theta_M^*)$. Furthermore,

$$h_B'(x^*; \theta_B^*, \theta_M^*) = \frac{S_B(\theta_B^*, \theta_M^*)}{\theta_B^*} \geq 0. \qquad \text{and} \qquad h_B''(x^*; \theta_B^*, \theta_M^*) = \frac{2S_B(\theta_B^*, \theta_M^*)}{(\theta_B^*)^2} \geq 0. \tag{2.3.4}$$

Let $I_M = \{(x_M; \theta_B^*, \theta_M^*) : x_M \in I_M(\theta_M^*), \theta_B^* \geq 0, \theta_M^* \geq 0\}$, the *isoprofit market share of the mountain resort* $h_M : I_M \to \mathbb{R}_0^+$ is given by

$$h_M(x_M; \theta_B^*, \theta_M^*) = \frac{\theta_M^* S_M(\theta_B^*, \theta_M^*)}{\theta_M^* + x_M - x^*} \geq 0.$$

We observe that $h_M(x^*; \theta_B^*, \theta_M^*) = S_M(\theta_B^*, \theta_M^*)$. Furthermore,

$$h_M'(x^*; \theta_B^*, \theta_M^*) = -\frac{S_M(\theta_B^*, \theta_M^*)}{\theta_M^*} \qquad \text{and} \qquad h_M''(x^*; \theta_B^*, \theta_M^*) = \frac{2S_M(\theta_B^*, \theta_M^*)}{(\theta_M^*)^2}. \tag{2.3.5}$$

Theorem 6 (Geometric resort best response)**.** *The price θ_B^* of the resort B is the best response to the price θ_M^* of the resort M if, and only if, for every $x_B \in I_B(\theta_B^*)$,*

$$S_B(x^* - x_B + \theta_B^*, \theta_M^*) \leq h_B(x_B; \theta_B^*, \theta_M^*).$$

The price θ_M^* of the resort M is the best response to the price θ_B^* of the resort B if, and only if, for every $x_M \in I_M(\theta_M^*)$,

$$S_M(\theta_B^*, x_M - x^* + \theta_M^*) \leq h_M(x_M; \theta_B^*, \theta_M^*). \tag{2.3.6}$$

Hence, the price θ_B^* of the resort B is the best response to the price θ_M^* of the resort M if, and only if, the graph of S_B is below the graph of h_B. Similarly, the price θ_M^* of the resort M is the best response to the price θ_B^* of the resort B if, and only if, the graph of S_M is below the graph of h_M.

Proof. The price θ_B^* of the resort B is the best response to the price θ_M^* of the resort M if, and only if,

$$\Pi_B(\theta_B; S(\theta_B, \theta_M^*)) = \theta_B S_B(\theta_B, \theta_M^*) \leq \theta_B^* S_B(\theta_B^*, \theta_M^*) = \Pi_B(\theta_B; S(\theta_B, \theta_M^*)). \quad (2.3.7)$$

This is equivalent to

$$S_B(\theta_B, \theta_M^*) \leq \frac{\theta_B^* S_B(\theta_B^*, \theta_M^*)}{\theta_B}.$$

By equality (2.3.3), $x_B + \theta_B = \theta_B^* + x^*$. Hence 2.3.7 is equivalent to

$$S_B(\theta_B, \theta_M^*) \leq \frac{\theta_B^* S_B(\theta_B^*, \theta_M^*)}{\theta_B^* + x^* - x_B} = h_B(x_B; \theta_B^*, \theta_M^*)$$

The proof of inequality (2.3.6) follows similarly. □

2.3.1 Nash equilibria

Given an equilibrium price path $(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*))$, an *out of equilibrium price tourist* ***N*** *strategy* $S^*(\theta_B, \theta_M)$ is a tourist strategy with the following properties:

(i) $S^*(\theta_B^*, \theta_M^*) = S(\theta_B^*, \theta_M^*)$;

(ii) for every $x_B \in I_B(\theta_B^*)$,

$$S_B^*(x^* - x_B + \theta_B^*, \theta_M^*) \leq h_B(x_B; \theta_B^*, \theta_M^*);$$

(iii) for every $x_M \in I_M(\theta_M^*)$,

$$S_M^*(\theta_B^*, x_M - x^* + \theta_M^*) \leq h_M(x_M; \theta_B^*, \theta_M^*).$$

Let $\mathbf{O}(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F)$ be the set of all out of equilibrium price tourist **N** strategies. Let $S^*(\theta_B, \theta_M)$ be the following strategy:

(i) $S^*(\theta_B^*, \theta_M^*) = S(\theta_B^*, \theta_M^*)$;

(ii) for every $x_B \in I_B(\theta_B^*)$, $S_B^*(x^* - x_B + \theta_B^*, \theta_M^*) = 0$;

(iii) for every $x_M \in I_M(\theta_M^*)$, $S_M^*(\theta_B^*, x_M - x^* + \theta_M^*) = n$.

Hence $S^*(\theta_B, \theta_M) \in \mathbf{O}(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F)$, and so

$$\mathbf{O}(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F) \neq \emptyset.$$

Using Theorem 6, we obtain the following remark.

Remark 3 (Geometric resort Nash equilibria $\mathbf{N}(R-T,x_F)$)**.** *A resort-tourist strategy* $(\theta_B^*,\theta_M^*,S(\theta_B,\theta_M))\in \boldsymbol{N}(R-T,x_F)$ *if, and only if,*

1. $(\theta_B^*,\theta_M^*,S(\theta_B^*,\theta_M^*))$ *is an equilibrium price path; and*

2. $S\in \boldsymbol{O}(\theta_B^*,\theta_M^*,S(\theta_B^*,\theta_M^*);x_F)$.

Furthermore,

$$\boldsymbol{O}(\theta_B^*,\theta_M^*,S(\theta_B^*,\theta_M^*);x_F)\neq\emptyset.$$

A *resort-tourist deviation impartial strategy* $((\theta'_B,\theta'_M);S^+(\theta_B,\theta_M))\in\mathbf{S}(R)$ is a resort-tourist strategy with the property that

$$S^+(\theta_B,\theta_M)=S(\theta_B-\theta_M).$$

Let $\mathbf{S}^+(R)\subset\mathbf{S}(R)$ be the set of all resort-tourist deviation impartial strategies. Let

$$\mathbf{N}^+(R-T;x_F)=\mathbf{S}^+(R)\cap\mathbf{N}(R-T;x_F).$$

A $\mathbf{N}^+$ equilibrium price path $(\theta_B^*,\theta_M^*,S(\theta_B^*-\theta_M^*))$ is an equilibrium price with the property that

$$\frac{S_B(\theta_B^*-\theta_M^*)}{\theta_B^*}=\frac{S_M(\theta_B^*-\theta_M^*)}{\theta_M^*}=\frac{n}{\theta_B^*+\theta_M^*}. \tag{2.3.8}$$

Furthermore,

$$\frac{\Pi_B(\theta_B^*-\theta_M^*)}{(\theta_B^*)^2}=\frac{\Pi_M(\theta_B^*-\theta_M^*)}{(\theta_M^*)^2}=\frac{n}{\theta_B^*+\theta_M^*}.$$

The *auxiliar isoprofit market share of the mountain resort* $\underline{h}_M:I_M\to\mathbb{R}$ is given by

$$\underline{h}_M(x_M;\theta_B^*,\theta_M^*)=n-h_M(x_M;\theta_B^*,\theta_M^*)=\frac{n(x_M-x^*)+\theta_M^*S_B(\theta_B^*,\theta_M^*)}{\theta_M^*+x_M-x^*}.$$

We observe that $\underline{h}_M(x^*;\theta_B^*,\theta_M^*)=S_B(\theta_B^*,\theta_M^*)$. Furthermore,

$$\underline{h}'_M(x^*;\theta_B^*,\theta_M^*)=\frac{S_M(\theta_B^*,\theta_M^*)}{\theta_M^*}\geq 0\qquad\text{and}\qquad \underline{h}''_M(x^*;\theta_B^*,\theta_M^*)=-\frac{2S_M(\theta_B^*,\theta_M^*)}{(\theta_M^*)^2}\leq 0. \tag{2.3.9}$$

We note that equality (2.3.8) is equivalent to

$$h'_B(x^*,\theta_B^*,\theta_M^*)=\underline{h}'_M(x^*,\theta_B^*,\theta_M^*).$$

Given an equilibrium price path $(\theta_B^*,\theta_M^*,S(\theta_B^*-\theta_M^*))$, an *out of equilibrium price tourist* $\boldsymbol{N}^+$ *strategy* S^* is a tourist strategy with the following properties:

(i)

$$S^*(\theta_B^* - \theta_M^*) = S(\theta_B^* - \theta_M^*);$$

(ii) for every $x_B \in K = I_B(\theta_B^*) \cap I_M(\theta_M^*)$,

$$\underline{h}_M(x_B; \theta_B^*, \theta_M^*) \leq S_B(x^* - x_B + \theta_B^* - \theta_M^*) \leq h_B(x_B; \theta_B^*, \theta_M^*);$$

(iii) for every $x_M \in I_M(\theta_M^*) \setminus K$,

$$\underline{h}_M(x_M; \theta_B^*, \theta_M^*) \leq S_B(\theta_B^*, x_M - x^* + \theta_M^*).$$

(iv) for every $x_B \in I_B(\theta_B^*) \setminus K$,

$$S_B(x^* - x_B + \theta_B^* - \theta_M^*) \leq h_B(x_B; \theta_B^*, \theta_M^*);$$

Let $\mathbf{O}^+(\theta_B^*, \theta_M^*, S(\theta_B^* - \theta_M^*); x_F)$ be the set of all out of equilibrium price tourist Nash $\mathbf{N}^+$ strategies

Theorem 7 (Geometric resort Nash equilibria $\mathbf{N}^+(R - T; x_F)$)**.** *A resort-tourist strategy $(\theta_B^*, \theta_M^*, S(\theta_B, \theta_M)) \in \boldsymbol{N}^+(R - T; x_F)$ if, and only if,*

1. *$(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*))$ is an $\boldsymbol{N}^+$ equilibrium price path; and*
2. *$S \in \boldsymbol{O}^+(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F)$.*

Furthermore,

$$\boldsymbol{O}^+(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F) \neq \emptyset.$$

Proof. By theorem 6, if $(\theta_B^*, \theta_M^*, S(\theta_B, \theta_M)) \in \mathbf{N}^+(R-T; x_F)$ then for every $x_B \in K = I_B(\theta_B^*) \cap I_M(\theta_M^*)$, we have

$$\underline{h}_M(x_B; \theta_B^*, \theta_M^*) \leq S_B(x^* - x_B + \theta_B^* - \theta_M^*) \leq h_B(x_B; \theta_B^*, \theta_M^*). \qquad (2.3.10)$$

We note that

$$\underline{h}_M(x^*; \theta_B^*, \theta_M^*) = h_B(x^*; \theta_B^*, \theta_M^*) = S_B(\theta_B^* - \theta_M^*).$$

If

$$\underline{h}'_M(x^*; \theta_B^*, \theta_M^*) \neq h'_B(x^*; \theta_B^*, \theta_M^*)$$

then

$$\underline{h}_M(x_B; \theta_B^*, \theta_M^*) > h_B(x_B; \theta_B^*, \theta_M^*)$$

either for $x_B < x^*$ or for $x_B > x^*$. Hence, there is no strategy $S_B(x^* - x_B + \theta_B^* - \theta_M^*)$ satisfying (2.3.10). Therefore,

$$h'_B(x^*; \theta_B^*, \theta_M^*) = \underline{h}'_M(x^*; \theta_B^*, \theta_M^*)$$

is a necessary condition. Hence,

$$S_B(\theta_B^* - \theta_M^*)(\theta_B^*)^{-1} = S_M(\theta_B^* - \theta_M^*)(\theta_M^*)^{-1}.$$

Therefore, $\theta_M^* S_B(\theta_B^* - \theta_M^*) = \theta_B^*(n - S_B(\theta_B^* - \theta_M^*))$. Thus,

$$S_B(\theta_B^* - \theta_M^*) = \frac{\theta_B^* n}{\theta_B^* + \theta_M^*}.$$

Hence, $(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*))$ is an $\mathbf{N}^+$ equilibrium price path. Since $\underline{h}''_M(x^*; \theta_B^*, \theta_M^*) < 0$ and $h''_B(x^*; \theta_B^*, \theta_M^*) = S_B(\theta_B^* - \theta_M^*) > 0$, we obtain

$$\mathbf{O}^+(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F) \neq \emptyset.$$

Therefore, this theorem follows from theorem 6. □

In figure 2.6 is depicted a geometric ilustration of theorem 7.

Fix $S_B^* \in [l, l+k]$. Assume that

$$S_B^* = l + \frac{k(\theta_B^* - \theta_M^* + x_F - T(l))}{-A(k-1)}.$$

Let $S_M^* = n - S_B^*$.

Lemma 6. *If $(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*))$ is an $\mathbf{N}^+$ equilibrium price path then*

$$\theta_B^* = S_B^* \theta_M^* (S_M^*)^{-1} \quad \text{and} \quad \theta_M^* = S_M^* \theta_B^* (S_B^*)^{-1}.$$

Furthermore,

$$x_F = -A(k-1)p_S - \theta_B^* + \theta_M^* + T(l).$$

1. *If $S_M^* \leq N/2$ then $x_F \leq -A(k-1)p_S + T(l)$.*
2. *If $S_M^* \geq N/2$ then $x_F \geq -A(k-1)p_S + T(l)$.*

Proof. Since $S_B^* = \theta_B^* n(\theta_B^* + \theta_M^*)^{-1}$, we get $\theta_B^* = S_B^* \theta_M^* (S_M^*)^{-1}$ and $\theta_M^* = S_M^* \theta_B^* (S_B^*)^{-1}$. By theorem 4,

$$S_B^* = l + \frac{k(\theta_B^* - \theta_M^* + x_F - T(l))}{-A(k-1)}.$$

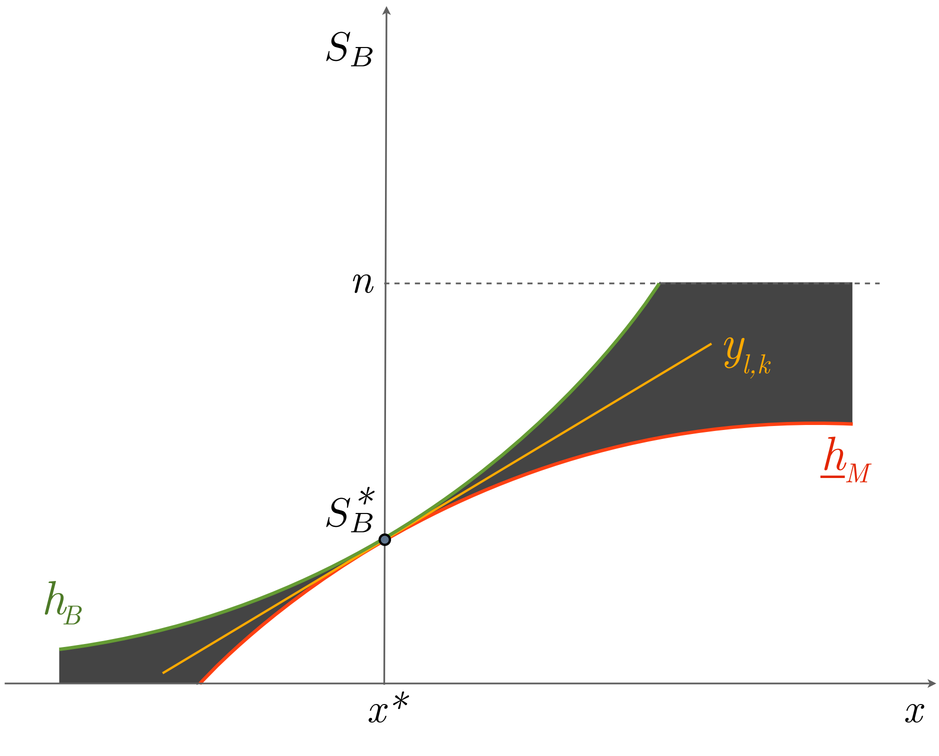

Figure 2.6: Geometric ilustration of theorem 7

Therefore,

$$x_F = -A(k-1)(S_B^* - l)k^{-1} - \theta_B^* + \theta_M^* + T(l). \tag{2.3.11}$$

Puting together (2.3.11) and $\theta_M^* = S_M^* \theta_B^* (S_B^*)^{-1}$,

$$(1 - S_M^*(S_B^*)^{-1})\theta_B^* = -A(k-1)(S_B^* - l)k^{-1} + T(l) - x_F.$$

If $S_M^* \leq N/2$ then $x_F \leq -A(k-1)(S_B^* - l)k^{-1} + T(l)$. If $S_M^* \geq N/2$ then $x_F \geq -A(k-1)(S_B^* - l)k^{-1} + T(l)$. □

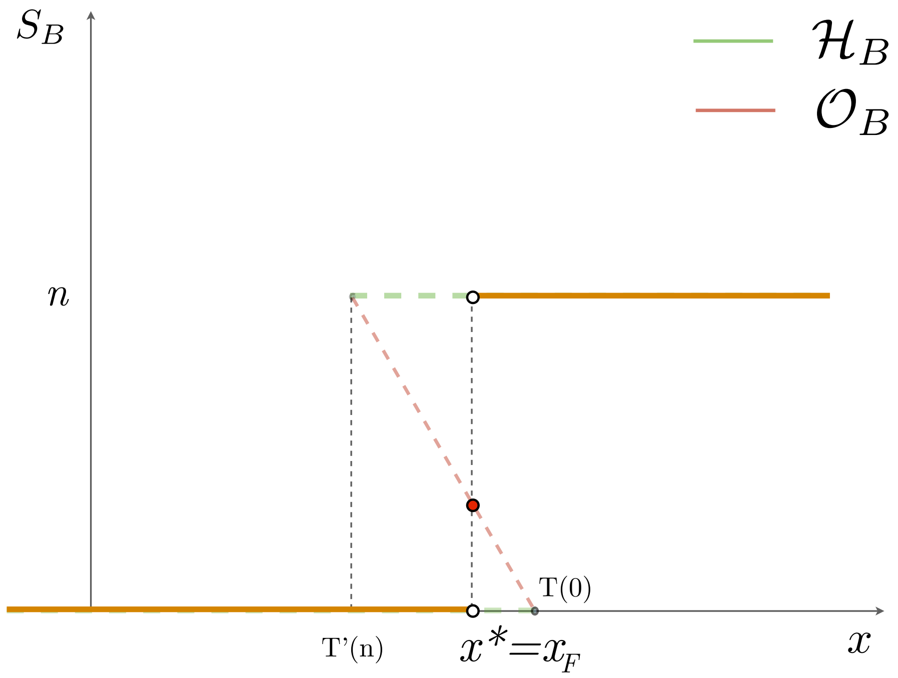

Figure 2.7: The competitive equilibrium when $A > 0$.

2.3.2 Subgame-perfect Nash equilibria

A *subgame-perfect Nash resort-tourist equilibrium* $((\theta_B^*, \theta_M^*); S(\theta_B, \theta_M))$ is a Nash resort-tourist equilibrium with the property that $S(\theta_B, \theta_M) \in \mathbf{N}(T)$ for every resort price strategy (θ_B, θ_M). Let $\mathbf{PN}(R; x_F)$ be the set of all subgame-perfect Nash resort-tourist equilibria.

Throughout we will prove the following results:

Lemma 7. *Let $A > 0$. A subgame-perfect equilibrium for tourists impartial strategies is either a monopoly or a competitive equilibrium where resorts have zero profits.*

Theorem 8. *Let $A \leq 0$. A subgame-perfect equilibrium for tourists impartial strategies is*

(i) a monopoly if $x_F \notin [T(0) + A(n-1), T(n) - A(n-1)]$;

(ii) a competitive equilibrium if $x_F \in [T(0) + A(n-1), T(n) - A(n-1)]$.

Furthermore we will characterize preferences and prices for each equilibria.

Let $A \leq 0$. The *deviation from the equilibrium path* of resort B is a strategy $S(\theta_B, \theta_M^*)$ for every $\theta_B \in [0, x^* + \theta_B^* - T(0)) \setminus \{\theta_B^*\}$. Let

$$J_B^A(\theta_B^*) = [T(0), x^* + \theta_B^*] \setminus \{x^*\}.$$

The *deviation* tourists price relative preference with respect of resort B is

$$x_B = \theta_B^* - \theta_B + x^*$$

for $x_B \in J_B^A(\theta_B^*)$. We note that if $x_B \leq T(0)$, there is only one Nash equilibrium strategy S and $S_B = 0$, so there are no tourists choosing resort B at a Nash equilibrium.

Similarly, if $A > 0$, we define

$$J_B^A(\theta_B^*) = [T(n), x^* + \theta_B^*] \setminus \{x^*\}.$$

We note that if $x_B < T(n)$, there is only one Nash equilibrium strategy S and $S_B = 0$, so there are no tourists choosing resort B at a Nash equilibrium.

Let $A \leq 0$. The *deviation from the equilibrium path* of resort M is a strategy $S(\theta_B^*, \theta_M)$ for every $\theta_M \in [0, T'(n) + \theta_M^* - x^*) \setminus \{\theta_M^*\}$. Let

$$J_M^A(\theta_B^*) = [x^* - \theta_M^*, T'(n)] \setminus \{x^*\}.$$

The *deviation from the equilibrium path of resort M* tourists relative preference is

$$x_M = \theta_M - \theta_M^* + x^*$$

for $x_M \in J_M^A(\theta_M^*)$. We note that if $x_M \geq T'(n)$ there is only one Nash equilibrium strategy S and $S_M = 0$, so there are no tourists choosing resort M at a Nash equilibrium.

Similarly, if $A > 0$, we define

$$J_M^A(\theta_B^*) = [x^* - \theta_M^*, T'(0)] \setminus \{\theta_M^*\}.$$

We note that if $x_B > T'(0)$, there is only one Nash equilibrium strategy S and $S_B = 0$, so there are no tourists choosing resort B at a Nash equilibrium.

An *out of equilibrium price* ***PN*** *tourist strategy* S^* is a tourist strategy with the following properties: (i) for every $x_B \in J_B^A(\theta_B^*)$, $S(\theta_B^*, \theta_M^*)$ is a Nash equilibrium and

$$\mathcal{F}_B^A \cap (\{x_B\} \times [0, h_B(x_B; \theta_B^*, \theta_M^*)]) \neq \emptyset;$$

(ii) for every $x_M \in J_M^A(\theta_M^*)$, $S(\theta_B^*, \theta_M^*)$ is a Nash equilibrium and

$$\mathcal{F}_M^A \cap (\{x_M\} \times [0, h_M(x_M; \theta_B^*, \theta_M^*)]) \neq \emptyset.$$

Let

$$\mathbf{OP}(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F)$$

be the set of all out of equilibrium price tourist **PN** strategies.

Theorems 6 and 2 imply the following result.

Remark 4 (Geometric resort subgame-perfect Nash equilibria $\mathbf{PN}(R-T; x_F)$)**.** *A resort-tourist strategy* $(\theta_B^*, \theta_M^*, S(\theta_B, \theta_M)) \in \boldsymbol{PN}(R-T; x_F)$ *if, and only if,*

1. $(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*))$ *is a equilibrium price path; and*

2. $S \in \boldsymbol{OP}(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F)$.

$$\boldsymbol{OP}(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F) \neq \emptyset.$$

A **PCN** equilibrium price path $(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*))$ is an equilibrium price with the following property: There are $l' \leq l$ and $k' \leq k$ such that $S_B(\theta_B^*, \theta_M^*) = y_{l,k}(x^*) = y_{l',k'}(x^*)$,

$$\frac{k}{-A(k-1)} \leq \frac{S_B(\theta_B^*, \theta_M^*)}{\theta_B^*} \leq \frac{k'}{-A(k'-1)}$$

and

$$\frac{k}{-A(k-1)} \leq \frac{S_M(\theta_B^*, \theta_M^*)}{\theta_M^*} \leq \frac{k'}{-A(k'-1)}.$$

One of the fibers has to be obliquous but the other one can be horizontal (k=0). We note that the market share fibers $O_{l,k}^B$ and $O_{l',k'}^B$ can be the same.

An *out of equilibrium price* ***PCN*** *tourist strategy* S^* is a continuous out of equilibrium price **PN** tourist strategy. Let $\mathbf{OPC}(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F)$ be the set of all out of equilibrium price tourist **PCN** strategies. If $A > 0$ then there is only one obliquous fiber $O_{0,n}^B$ and $\mathbf{OPC}(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F) = \emptyset$

Theorem 9 (Geometric resort subgame-perfect Nash equilibria $\mathbf{PCN}(R-T;x_F)$)**.** *Let $A \leq 0$. A resort-tourist strategy $(\theta_B^*, \theta_M^*, S(\theta_B, \theta_M)) \in \boldsymbol{PCN}(R-T;x_F)$ if, and only if,*

1. *$(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*))$ is a **PCN** equilibrium price path; and*
2. *$S \in \boldsymbol{OPC}(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F)$.*

Furthermore,

$$\boldsymbol{OPC}(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F) \neq \emptyset.$$

Proof. If a resort-tourist strategy $(\theta_B^*, \theta_M^*, S(\theta_B, \theta_M)) \in \mathbf{PCN}(R-T;x_F)$ the tourists strategy has to belong to the market share fiber $\mathcal{F}_B$, i.e.

$$\mathcal{F}_B \cap (\{x_B\} \times [0, h_B(x_B; \theta_B^*, \theta_M^*)]) \neq \emptyset.$$

We note that

$$\frac{k}{-A(k-1)} \leq \frac{S_B(\theta_B^*, \theta_M^*)}{\theta_B^*} \leq \frac{k'}{-A(k'-1)}$$

is equivalent to

$$\frac{k}{-A(k-1)} \leq h'_B(x_B; \theta_B^*, \theta_M^*) \leq \frac{k'}{-A(k'-1)}.$$

Hence, the Nash tourists strategies in the fiber $\mathcal{O}^B_{l',k'}$ with higher slope, for $x_B \leq x_B^*$, and the Nash tourists strategies in the fiber $\mathcal{O}^B_{l,k}$ with smaller slope, for $x_B \geq x_B^*$ give lower profit to resort B than $S(\theta_B^*, \theta_M^*)$. Similarly, the Nash tourists strategies in the fiber $\mathcal{O}^B_{l,k}$, for $x_M \leq x_M^*$, and the Nash tourists strategies in the fiber $\mathcal{O}^B_{l',k'}$, for $x_M \geq x_M^*$ give lower profit to resort M than $S(\theta_B^*, \theta_M^*)$. Hence, $\mathbf{OPC}(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F) \neq \emptyset$. The other equivalences follow from Theorem 6. □

Let

$$E = \frac{l'k - k'l}{k - k'}.$$

Remark 5. *If $(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*))$ is a **PCN** equilibrium price path and $(l,k) \neq (l',k')$ then*

$$S(\theta_B^*, \theta_M^*) = l + \frac{k(l-l')}{(k'-k)} = l' + \frac{k'(l-l')}{(k'-k)}$$

occurs for the crossing point *of the tourists relative preference $x^* = x(l,k;l',k') \in [T(l), T'(l+k)]$ given by*

$$x^* = \frac{-A(k-1)(l-l')}{(k'-k)} - T(l) = \frac{-A(k'-1)(l-l')}{(k'-k)} - T(l')$$

$$\frac{-A(k'-1)E}{k'} \leq \theta_B^* \leq \frac{-A(k-1)E}{k}$$
$$\frac{-A(k'-1)(n-E)}{k'} \leq \theta_M^* \leq \frac{-A(k-1)(n-E)}{k}$$

Since $\theta_M^* - \theta_B^* = x_F - x^*$, the above remark gives a lower and upper bound for x_F

$$\frac{-A(k'-1)n}{k'} + \frac{-AE(k+k')}{kk'} \leq x_F - x^* \leq \frac{-A(k-1)n}{k} + \frac{-AE(k+k')}{kk'}.$$

Furthermore, if $(l,k) = (l',k')$ then

$$x_F = \frac{-A(k-1)n}{k} + x^*.$$

Let

$$\mathbf{PN}^+(R;x_F) = \mathbf{S}^+(R) \cap \mathbf{PN}(R;x_F)$$

be the set of all resort-tourist deviation impartial strategies that are subgame-perfect Nash equilibria.

A $\mathbf{PN}^+$ equilibrium price path $(\theta_B^*, \theta_M^*, S(\theta_B^* - \theta_M^*))$ is an equilibrium price with the following property for some $l \geq 0$ and $k \geq 1$: $S_B(\theta_B^*, \theta_M^*) = y_{l,k}(x^*)$,

$$\frac{S_B(\theta_B^*, \theta_M^*)}{\theta_B^*} = \frac{S_M(\theta_B^*, \theta_M^*)}{\theta_M^*} = \frac{n}{\theta_B^* + \theta_M^*} = \frac{k}{-A(k-1)}.$$

Furthermore,

$$\frac{\Pi_B(\theta_B^* - \theta_M^*)}{(\theta_B^*)^2} = \frac{\Pi_M(\theta_B^* - \theta_M^*)}{(\theta_M^*)^2} = \frac{n}{\theta_B^* + \theta_M^*} = \frac{k}{-A(k-1)}.$$

An *out of equilibrium price tourist* $\boldsymbol{PN}^+$ *strategy* S^* is Nash equilibria tourist strategy with the following properties:

(i) $S^*(\theta_B^* - \theta_M^*) = S(\theta_B^* - \theta_M^*)$;

(ii) for every $x_B \in K_1 = J_B^A(\theta_B^*) \cap J_M^A(\theta_M^*)$,

$$\underline{h}_M(x_B;\theta_B^*,\theta_M^*) \leq S_B(x^* - x_B + \theta_B^* - \theta_M^*) \leq h_B(x_B;\theta_B^*,\theta_M^*);$$

(iii) for every $x_M \in J_M^A(\theta_M^*) \setminus K_1$,

$$\underline{h}_M(x_M;\theta_B^*,\theta_M^*) \leq S_B(\theta_B^*, x_M - x^* + \theta_M^*).$$

(iv) for every $x_B \in J_B^A(\theta_B^*) \setminus K_1$,

$$S_B(x^* - x_B + \theta_B^* - \theta_M^*) \leq h_B(x_B; \theta_B^*, \theta_M^*);$$

Let $\mathbf{OP}^+(\theta_B^*, \theta_M^*, S(\theta_B^* - \theta_M^*); x_F)$ be the set of all out of equilibrium price tourist Nash $\mathbf{PN}^+$ strategies.

Theorem 10 (Geometric resort subgame-perfect Nash equilibria $\mathbf{PN}^+(R - T; x_F)$)**.** *A resort-tourist strategy $(\theta_B^*, \theta_M^*, S(\theta_B, \theta_M)) \in \boldsymbol{PN}^+(R - T; x_F)$ if, and only if,*

1. $(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*))$ *is a* $\boldsymbol{PN}^+$ *equilibrium price path; and*
2. $S \in \boldsymbol{OP}^+(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F)$.

Furthermore,

$$\boldsymbol{OP}^+(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F) \neq \emptyset.$$

Proof. Similarly to theorem 7, $\mathbf{O}^+(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F) \neq \emptyset$ if, and only if,

$$h'_B(x^*; \theta_B^*, \theta_M^*) = \underline{h}'_M(x^*; \theta_B^*, \theta_M^*)$$

Furthermore,

$$h'_B(x^*; \theta_B^*, \theta_M^*) = \frac{k}{-A(k-1)}.$$

Hence,

$$\frac{S_B(\theta_B^*, \theta_M^*)}{\theta_B^*} = \frac{S_M(\theta_B^*, \theta_M^*)}{\theta_M^*} = \frac{n}{\theta_B^* + \theta_M^*} = \frac{k}{-A(k-1)}. \tag{2.3.12}$$

An example of an out of equilibrium price tourist strategy consists in the following strategy: for $x - x^* \geq 0$ follow the segment lines of the fiber at each crossing whose derivative of h increases but the less possible; for $x^* - x \geq 0$ follow the segment lines of the fiber at each crossing whose derivative of h decreases but the less possible. □

Let $\sigma_1 = (\theta_B^*, \theta_M^*, S(\theta_B^* - \theta_M^*))$ be a $\mathbf{PN}^+$ equilibrium price path. Let the σ_1 out of equilibrium price tourist Nash $\mathbf{PN}^+$ strategy be given by

(i) $S(\theta_B^* - \theta_M^*) = n$;

(ii) if $x \leq x^*$ then $S^*(\theta_B^* - \theta_M) = n$;

(iii) if $x > x^*$ then $S^*(\theta_B - \theta_M^*) \leq h_B(x_B; \theta_B^*, \theta_M^*)$.

Let $\sigma_2 = (\theta_B^*, \theta_M^*, S(\theta_B^* - \theta_M^*))$ be a $\mathbf{PN}^+$ equilibrium price path. Let the σ_2 out of equilibrium price tourist Nash $\mathbf{PN}^+$ strategy be given by

(i) $S(\theta_B^* - \theta_M^*) = n$;

(ii) if $x < x^*$ then $S(\theta_B^* - \theta_M) = 0$;

(iii) if $x > x^*$ then $S(\theta_B - \theta_M^*) \geq \underline{h}_M(x_B; \theta_B^*, \theta_M^*)$.

Let $\sigma_3(s^*) = (\theta_B^*, \theta_M^*, S(\theta_B^* - \theta_M^*))$ be a $\mathbf{PN}^+$ equilibrium price path. Let the $\sigma_3(s^*)$ out of equilibrium price tourist Nash $\mathbf{PN}^+$ strategy be given by

(i) $S(\theta_B^* - \theta_M^*) = s^*$;

(ii) if $x_F < x^*$ then $S(\theta_B^* - \theta_M) = 0$;

(iii) if $x_F > x^*$ then $S(\theta_B - \theta_M^*) = n$.

Remark 6. *Let $A > 0$. We have*

(i) σ_1 is a monopoly for resort B if $\theta_B = x_F - x^*$ and $\theta_M = 0$;

(ii) σ_2 is a monopoly for resort M if $\theta_M = x^* - x_F$ and $\theta_B = 0$;

(iii) for every $s^* \in [0, n]$, $\sigma_3(s^*)$ is a competive equilibrium where $\theta_B^* = \theta_M^* = 0$.

Furthermore,

$$\boldsymbol{OP}^+(\theta_B^*, \theta_M^*, S(\theta_B^*, \theta_M^*); x_F) = \{\sigma_1, \sigma_2, \sigma_3(s^*) : s^* \in [0, n]\}.$$

Remark 7. *Let $A < 0$ and let $S_{pure} = (\theta_B^*, \theta_M^*, S(\theta_B - \theta_M))$ be a tourist-resort strategy such that the out of equilibrium price tourist Nash $\boldsymbol{PN}^+$ strategies $S(\theta_B - \theta_M)$ consist in pure strategies that are Nash equilibria.*

(i) Let $x_F \leq T(0) + A(n-1)$. If $\theta_M^* = T(0) - x_F$ and $\theta_B^* = 0$, then the resort-tourist strategy S_{pure} is a $\boldsymbol{PN}^+$ Nash equilibrium.

(ii) Let $T(0) + A(n-1) \leq x_F \leq T(n) - A(n-1)$. The resort-tourist strategies S_{pure} are not $\boldsymbol{PN}^+$ Nash equilibria.

(iii) Let $x_F \geq T(n) - A(n-1)$. *If* $\theta_M^* = 0$ *and* $\theta_B^* = x_F - T(n)$, *then the resort-tourist strategy* S_{pure} *is a* $\boldsymbol{PN}^+$ *Nash equilibrium.*

Let $c(l,k) = T(0) + A(n - k^{-1}(l+1))$.

Theorem 11. *Let* $A \leq 0$ *and* $k > 1$. *If a resort-tourist strategy* $(\theta_B^*, \theta_M^*, S(\theta_B, \theta_M)) \in \boldsymbol{PN}^+(R-T; x_F)$ *then*

$$c(l,k) \leq x_F \leq c(l,k) - 3An(k-1)k^{-1}$$

Furthermore,

(i)

$$\theta_B^* = \frac{-A(k-1)(n+l) + k(x_F - T(l))}{3k};$$

(ii)

$$\theta_M^* = \frac{-A(k-1)(2n-l) + k(T(l) - x_F)}{3k};$$

(iii)

$$S_B(\theta_B^*, \theta_M^*) = \frac{n + l + k(x_F - T(l))}{3};$$

Proof. Since

$$S_B = l + \frac{k(x^* - T(l))}{-A(k-1)}$$

Using equation (2.3.12) we get

$$\frac{-A(k-1)l + k(\theta_B^* - \theta_M^* + (x_F - T(l)))}{-A(k-1)} = \frac{k}{-A(k-1)} \cdot \theta_B^*.$$

Therefore,

$$\theta_B^* = \frac{-A(k-1)l + k(x_F - T(l) + \theta_M^*)}{2k}$$

On the other hand, again by (2.3.12),

$$\theta_B^* + \theta_M^* = \frac{-A(k-1)n}{k}.$$

Hence, θ_B^* and θ_M^* are as presented above. Since

$$S_B(\theta_B^*, \theta_M^*) = \frac{\theta_B^* k}{-A(k-1)}$$

we get $S_B(\theta_B^*, \theta_M^*)$ as presented above.

The restricions $\theta_B^* \geq 0$ and $\theta_M^* \geq 0$ lead to

$$c(l,k) \leq x_F \leq c(l,k) - 3An(k-1)k^{-1}$$

□

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