OPTIMAL TIMING FOR MARKET ENTRY

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ABSTRACT

This paper considers the optimal market entry timing of a firm facing price uncertainty and investment irreversibility. When the entry decision is made, the firm has to pay the necessary investment costs and from then onwards will receive the expected future cash-flows. The total expected income of the investment is given by the sum over time of the expected discounted future cash-flows. For the investment to be worthwhile this value must be significantly above, and not just above, the investment cost. Therefore, we address investment decisions where one must decide when it is the best time to make a commitment, losing the option to wait for a better market opportunity.
We developed a model that provides us with a rule that specifies under which conditions we should enter the market. In addition, the methodology developed also provides an estimate on how long we should wait before entering the market.

KEYWORDS. Investment Timing, Markov Processes, Real Options, Dynamic Programming

1. Introduction

Deciding on the timing for market entry is a strategic question for enterprise management. This type of decisions involve trade-offs between commitment and flexibility under uncertainty. Entry decisions involve a monetary commitment, i.e. necessary investment costs, a loss of flexibility, since the postponement option is lost, and an increased exposure to market uncertainty, such as prices or demands.

Research on real options has been contributing on how to determine the timing of entry decisions since it, explicitly, takes into account the value of waiting (e.g., Ingersoll and Ross, 1992; McDonald and Siegel, 1986; Trigeorgis, 1991). The fundamental conclusion from the Real Options theory is that, under uncertainty, deferring sunk commitments (i.e. irreversible investment costs) may increase the expected value of the investment. If the value of waiting is larger than the benefits of commitment, then delaying entry increases firm value.

Consider a natural resources, e.g. oil, exploration firm that holds a long term exploration leasehold. Net Present Value (NPV) calculations that result from marketing forecasts of price may or may not indicate to be profitable to explore the aforementioned natural resources, given the price uncertainty. In some cases, it is more reasonable to postpone the initiation of the exploration in order to learn more about market conditions or to wait for better market conditions. Similar investment decisions are made in other type of applications, e.g. production of a new product (Pennings and Lint, 2000), disposal of nuclear waste (Loubergé et al., 2002), amongst others.

The NPV valuation procedure does not take into account managerial flexibility and thus it, typically, undervalues projects since it is static in nature. The flexibility to adapt decisions to changes in market conditions can be used to increase the value of investment by improving its upside potential and limiting the downside losses in relation to initial expectations under passive management.

This paper considers the optimal market entry timing of a firm facing price uncertainty and investment irreversibility. Our problem is an optimal stopping problem, comparable to the definition of the optimal exercise policy for a perpetual American call option. Thus, our model fits into the real options approach to investment under uncertainty (see Dixit and Pindyck, 1994; McDonald and Siegel, 1986). An investment project is an option which may be exercised, or not. Even though the project may have positive present value, i.e. computed as the present value of the sum of the future cash flows net of investment costs, given current information on future cash flows, it may be optimal to postpone it, in order to keep alive the option to make better decisions in the future. In financial theory, such a problem may be solved using risk-neutral valuation. In our case, we can also do so since it is possible to replicate the underlying random variables using options and futures on oil or shares in oil companies, assuming oil to be our natural resource.

In this work, we formulated the market entry problem in terms of call option exercise timing. The market uncertainty is represented by the commodity market price uncertainty which is modeled by a Markov decision process. The exploration can start at any time within a temporal window, typically large enough for the exploration to take place. Once the investment decision has been taken, then an investment cost is incurred and a series of expected cash-flows are to be received. This value must be compared with the value of the alternative decision, i.e. wait and make the decision later that is given by the expected income at all other possible future decision timings.

We developed a model that provides us with a rule that specifies under which conditions we should enter that market, that is, the results obtained specify the commodity price threshold value for which we should start the commodity exploration. This decision rule draws attention to the opportunity cost (i.e., the loss of call option value) associated with moving from positions of flexibility to sunk commitments. In addition, for a given current commodity price we can also find out an estimate for how long we should wait before starting the commodity exploration.

2. Markov Chains and Potentials

Consider a stochastic process $\{X_n : n \in IN\}$ taking values in a countable state space *E*. This process is a Markov chain if it satisfies

$$\text{Prob}\left(X_{n+1} = j \middle| X_0, \ldots, X_n\right) = \text{Prob}\left(X_{n+1} = j \middle| X_n\right), \text{ for all } j \in E,\ n \in IN.$$

We will restrict our attention to the time-homogeneous case, i.e. the case in which the transition probabilities are independent of *n*. The transition probabilities will be denoted by

$$P(i, j) = \text{Prob}\{X_{n+1} = j | X_n = i\} \quad i, j \in E,$$

and the transition matrix by *P*, having as entries $P(i, j) \quad i, j \in E$. This matrix has the properties that :a) all entries are non-negative; and b) row sums are equal to one. Generally matrices satisfying these properties are called Markov Matrices.

The *m*-step transition matrix is given by the *m*th power of *P*. And it obviously satisfies the semi-group property that $P^{m+n} = P^m P^n$ which written for a particular entry (*i*,*j*) is known as the Chapman-Kolmogorov Equation,

$$P^{m+n}(i, j) = \sum_{k \in E} P^m(i,k) P^n(k, j).$$

2.1 Potentials

Consider the process $\{X_n, n \in IN\}$ with transition matrix *P*. Suppose that the process is in state *i*, and that at time *n* a reward $g(i)$ is given. Furthermore, suppose that all future rewards are discounted with a rate $\alpha \in [0,1]$, then the present worth of the rewards received at times 0,1,2,.. will be $g(X_0), \alpha(X_1), \alpha^2 g(X_2), \ldots$.

We will call *α-potential of the function g* to the expected value of the total discounted return, which starting at *i* is

$$R^\alpha g(i) = E_i\left[\sum_{n=0}^{\infty} \alpha^n g(X_n)\right], \quad i \in E.$$

The potential R^α can be thought as an operator acting on a function or as a matrix multiplying a vector, $\underline{g}$ being the vector whose component $[\underline{g}]_i = g(i)$,

$$R^\alpha g(i) = \left[R^\alpha \cdot \underline{g}\right]_i = \sum_{j \in E} R^\alpha(i, j)\, g(j),$$

where the matrix R^α is given by

$$R^\alpha = \sum_{n=0}^{\infty} \alpha^n P^n,$$

and it can be computed by the use of the following results:

Proposition 1

If $\alpha \in [0,1)$ and *g* is a non-negative function bounded on *E*, $r = R^\alpha g$ is the unique solution of the system of equations

$$(I - \alpha P) r = \underline{g}.$$

Justification

$$R^\alpha g = g + \alpha P g + \alpha^2 P^2 g + \ldots,$$

multiplying by αP we have

$$\alpha P\left(R^\alpha g\right) = \alpha P g + \alpha^2 P^2 g + \alpha^3 P^3 g + \ldots = R^\alpha g - g,$$

yielding the result.□

Proposition 2

For a stopping time T and a non-negative function g

$$R^{\alpha} g(i) = E_i\left[\sum_{n=0}^{T-1} \alpha^n g(X_n)\right] + E_i\left[\alpha^T R^{\alpha} g(X_T)\right].$$

For the proof of these propositions see Çinlar [1975].

3. Optimal Stopping

In the optimal stopping problem we are faced with two possible actions that we can generally call "to stop" and "to continue". The action "to stop" may be taken only once, and the discussion concerns when it should be taken in order to optimise a certain objective function.

Having a stochastic process $\{X_t\}$ we can associate a cost $c(X_t)$ to the decision "to continue" and a reward $g(X_t)$ to the action "to stop".The objective function will typically be to maximise an expectation of a functional of $c(\cdot)$ and $g(\cdot)$.

An example of an application is statistical inference, where the experimenter should decide when the increase in information contained in further data will outweigh the cost of collecting it. Another example, the one in which we are interested in here, is the irreversible investment problem where we have to decide when the expected profit of investing immediately will outweigh the value of waiting for further information about the future prospect of investment.

For this latter case the objective can be to maximise the expected discounted value of a function of the state at the time of stopping, over all possible stopping times.

3.1 Optimal stopping of a Discrete Time Markov Process.

Consider the discrete time stochastic process $\{X_n, n \in IN\}$ with transition matrix P. Consider also that we get a reward $g(X_n)$ if we take the action "to stop" at time n. At each instant of time we can take one of two decisions, to stop, or to continue.

We define $V_n(X_n)$, the value function, as the value we get for choosing the best decisions at time n and afterwards.

Consider first the finite horizon case where $n \leq N$, using a simple dynamic programming argument we can say that at time N, $V_N(X_N) = g(X_N)$, and that for n=1,...,N-1

$$V_n(X_n) = \max\left\{g(X_n),\ E_{X_n} V_{n+1}(X_{n+1})\right\}.$$

And we say that we stop if

$$g(X_n) \geq E_{X_n} V_{n+1}(X_{n+1}),$$

and we continue if

$$g(X_n) < E_{X_n} V_{n+1}(X_{n+1}).$$

For an infinite time horizon, the value function, which can be thought of as $V(X) = \lim_{n\to\infty} V_n(X)$, is simply defined by

$$V(X_n) = \max\left\{g(X_n); E_{X_n} V(X_{n+1})\right\}.$$

For Markov processes with transition probability matrix P, we can define the operator T as

$$TV(X_n) = E\left[V(X_{n+1}) \middle| X_n = i\right] = \sum_j P(i,j) V(j),$$

and hence

$$V(X) = \max\left\{g(X),\ TV(X)\right\}, \qquad (1)$$

which is known as the Wald-Bellman equation and can be computed by the usual dynamic programming algorithms.

The usual way to formulate the optimal stopping time problem is defining the value function as

$$V(X) = \sup_{\tau} E_X\left[g(X_\tau)\right],$$

where the supreme is taken over all possible stopping times of the Markov process $\{X_T\}$, because at each time *n*, the decision to stop or to continue must be made on the basis of the history of the process available at that time.

An important concept is the concept of α-excessive functions that we will now define.

Definition

Let *f* be a finite-valued function defined on *E*, the state space of a Markov chain with transition matrix *P*, and let α be a number in [0,1]. The function *f* is said to be α-excessive if

$$f(X) \geq 0 \qquad \forall_{X \in E},$$

$$f(X) \geq \alpha\, T\, f(X) \qquad \forall_{X \in E}.$$

If *f* is 1-excessive it is simply called excessive.

It follows that the value function defined by (1) can be characterised by the following theorem

Theorem

The value function *V* is the minimal excessive function greater than or equal to the gain function *g*.

Justification

From (1) we have

$$V(X) \geq g(X) \qquad \forall_{X \in E},$$
$$V(X) \geq TV(X) \qquad \forall_{X \in E}, \tag{2}$$

and clearly $V(X) \geq 0$.

And we can also, from (1), say that at least one of these constraints is always active (i.e. satisfied as an equality). Hence *V* is the minimal function satisfying these constraints (for a rigorous proof see Shiryayev [1978] chapter 2).

As *T* is a linear operator, this provides a computational way to compute *V* by linear programming simply by minimising $\sum_{j \in E} V(j)$ subject to the mentioned constraints.

Having defined the value function, for all $x \in E$ we can easily determine the *stopping set*, *S*, the set of states for which it is optimal to stop.

This set is defined as

$$S = \{x \in E : V(x) = g(x)\}.$$

A typical representation of $V(x)$ and $g(x)$ is

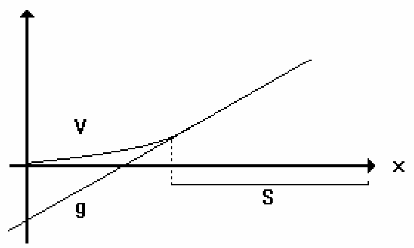

FIGURE 1.

The optimal stopping time τ, is the time of the first visit to the set S,

$$\tau = \inf\{t \geq 0 : V(X_t) = g(X_t)\}.$$

3.2. ε-**Optimal Stopping Times**

In the case of infinite state-spaces, it is possible that an optimal stopping time does not exist. However, for any defined $\varepsilon > 0$, there is a strategy that yields an expected payoff of at least $V(x) - \varepsilon$ for all $x \in E$. Such strategies are called ε-optimal.

Theorem

Let τ_ε be the first time of visit to the set

$$S_\varepsilon = \{x \in E: g(x) \geq V(x) - \varepsilon\}.$$

Then

$$E_x\left[g\left(X_{\tau_\varepsilon}\right)\right] \geq V(x) - \varepsilon.$$

For a proof see Çinlar [1975].

3.3 Discounting

In many applications, particularly for the problem we are interested in, a reward received at a future time *T*, has to be valued at the present by multiplying by discounting factor α^T. Where $\alpha = (1+r)^{-1}$, *r* being the interest earned in alternative investments per unit time. Hence, the reward being given as usual by $g(x)$, the optimal stopping problem can be formulated as

$$V(X) = \sup_\tau E_X\left[\alpha^\tau g\left(X_\tau\right)\right], \quad X \in E,$$

where the supreme is taken over all possible stopping times of the Markov process $\{X_T\}$.

The equivalent Wald-Bellman equation is

$$V(x) = \max\{g(x), \alpha TV(x)\}.$$

Hence, *V* is the minimal α-excessive function which is greater or equal to *g*.

For any ε>0, the ε-optimal stopping set is

$$S_\varepsilon = \{x \in E: g(x) \geq V(x) - \varepsilon\},$$

and the ε-optimal stopping time is the first visit time to this set.

It can be noted that if the gain function is α-excessive, then the minimal α-excessive function majoring *g* is obviously *g*. Hence, the optimal stopping time is $\tau = 0$. Moreover, an α-potential of a non-negative function is α-excessive because

$$R^\alpha g = g + \alpha Pg + \alpha^2 P^2 g + ... \geq \alpha Pg + \alpha^2 P^2 g + ... =$$
$$= \alpha P\left(g + \alpha Pg + ...\right) = \alpha P\left(R^\alpha g\right).$$

In the irreversible investment problem, if the fixed sunk cost is zero, then the gain function is an α-potential, and hence we would have the trivial solution $\tau = 0$.

4. Decision Model

4.1 Problem formulation

The problem is to decide the best instant of time to invest in the extraction of a resource given its actual price and a stochastic model of the price evolution. It is assumed that the investment can be done immediately, once decided, and the corresponding income start on the next time instant.

The prices are considered to evolve according to

$$p_{k+1} = (1+w_k)\, p_k, \tag{3}$$

where w_k values are uncorrelated, belonging to a finite ordered set of values $\Omega=\{\Omega_1,...,\Omega_N\}$ distributed according to $F(\Omega_i)=\text{Prob}(w_k \leq \Omega_i)$ and corresponding mean $Ew_k=m$ and density $f(\Omega_i)=\text{Prob}(w_k=\Omega_i)$.

Our decision is the time instant to invest that maximises the expected discounted profit

$$\max_k \left\{ E\left(\sum_{i=k+1}^{\infty}(1+r)^{-i} p_i\right) - (1+r)^{-k} I \middle| p_0, \ldots, p_k \right\},$$

where *I* is the fixed cost of investment and *r* the interest rate.

Alternatively the decision at each time instant is to invest now or wait at least one more unit time

max $J_0(p_0)$, where

$$J_k(p_k) = \max\left\{ E_{p_k}[\sum_{i=k+1}^{\infty}(1+r)^{-i} p_i) - (1+r)^{-k} I,\ E_{p_k}(J_{k+1}(p_{k+1})] \right\},$$

(invest) (wait)

or using the value function at current prices

$V_K(p_k)=(1+r)^k J_k(p_k)$,

$$V_k(p_k) = \max\left\{ E_{p_k}(\sum_{i=1}^{\infty}(1+r)^{-i} p_i) - I, (1+r)^{-1} E_{p_k}(V_{k+1}(p_{k+1})) \right\}.$$

(invest) (wait)

For the infinite time horizon case, the case in which we are interested, $V_k(p)=V(p)$ for all k, and so the value function V satisfies

$$V(p_k) = \max\left\{ E_{p_k}(\sum_{i=1}^{\infty}(1+r)^{-i} p_i) - I, (1+r)^{-1} E_{p_k}(V(p_{k+1})) \right\}$$

(invest) (wait)

which falls within the Optimal Stopping Problems.

4.2 Solution method

Defining $h(p)$ as the expected return of investing now, i.e. the sum of the discounted incomes from now till infinity

$$h(p_k)=E_{p_k}(\sum_{i=1}^{\infty}(1+r)^{-i} p_{i+k}) - I = \sum_{i=1}^{\infty}(\frac{1+m}{1+r})^{-i} p_k - I = \frac{1+m}{r-m} p_k - I, \qquad \text{if } r>m$$

and $\upsilon(p)$ as the expected return if we wait at least one unit time

$$\upsilon(p_k) = (1+r)^{-1} E_{p_k}(V_{k+1}(p_{k+1})).$$

At time k our decision will be to invest if $h(p_k)>\upsilon(p_k)$, to wait if $h(p_k)<\upsilon(p_k)$ and either decision is optimal if $h(p_k)=\upsilon(p_k)$. If in this last case we choose to invest, our decision rule will be to invest if and only if $h(p_k)\geq\upsilon(p_k)$, i.e.

iff $\dfrac{1+m}{r-m} p_k - I \geq \upsilon(p_k)$,

iff $p_k \geq \dfrac{r-m}{1+m}(\upsilon(p_k)+I)$,

iff $p_k \geq p^*$ where p^* satisfies $p^* = \dfrac{r-m}{1+m}(\upsilon(p^*)+I)$.

A graphical interpretation would be

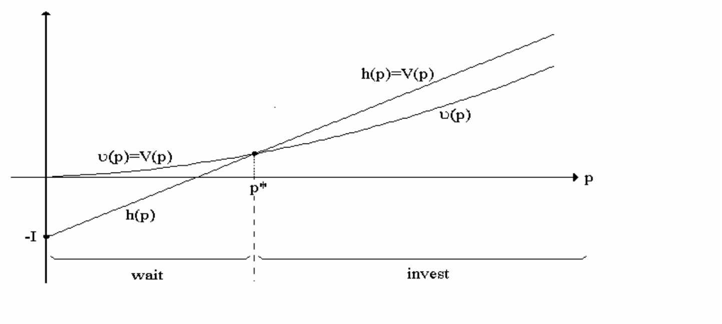

Figure 1.

Note that for p greater than p^* (inside the stopping region), $h(p)$ is greater than $\upsilon(p)$ because we are in fact loosing an opportunity if we do not make the correct decision even if it is only for one instant of time. The value function V coincides with $\upsilon(p)$ for p less or equal to p^* and coincides with $h(p)$ for p greater or equal to p^*.

To achieve the solution it remains only to develop $\upsilon(p)$.

4.2.*1* Determination of υ*(p)*

We know that $\upsilon(p_k)$ satisfies

$$\upsilon(0) = 0$$

$$\upsilon(p^*) = h(p^*)$$

$$\upsilon(p) = (1+r)^{-1}\, E\,\{\, V\,[(1+w)p]\,\} = (1+r)^{-1}\, E\{\max\{\, h[(1+w)p], \upsilon[(1+w)p]\}\}.$$

In order to develop it further we will consider separately the cases when $(1+w)p>p^*$ and when $(1+w)p\le p^*$.

a) Case (1+w)p>p*⇔w>**Erro!** - 1

Let $N1$ be such that Ω_{N1}=min{$\Omega_i \in \Omega$:Ω_i>**Erro!** - 1},

i.e. $N1$ is the index of the first term in Ω that satisfies the condition of being in this case.

$$\upsilon^a(p)=(1+r)^{-1} E\,\{\, h[(1+w)p] \mid w>\Omega_{N1}\,\}=$$

=$(1+r)^{-1}$ (**Erro!** E[(1+w)p | w>Ω_{N1}]-I)=

=$(1+r)^{-1}$ (**Erro!** $\dfrac{\sum_{i=N1}^{N}(1+\Omega_i)\, f(\Omega_i)}{\sum_{i=N1}^{N} f(\Omega_i)}$ p -I).

And the probability of being in this case is

$\sum_{i=N1}^{N} f(\Omega_i)$=1-Prob(w≤**Erro!** - 1)=1-F(**Erro!** - 1).

b) Case (1+w)$p\le p^*$⇔$w\le$**Erro!** - 1

Let $N2$ be such that Ω_{N2}=max{$\Omega_i \in \Omega$:$\Omega_i \le$**Erro!** - 1}.

i.e. $N2$ is the index of the last term in Ω that satisfies the condition of being in this case.

$$\upsilon^b(p)=(1+r)^{-1} E\,\{\,\upsilon[(1+w)p] \mid w\le\Omega_{N1}\,\}=$$

$$=(1+r)\text{-}1\ \frac{\sum_{i=1}^{N2}\upsilon\big[(1+\Omega_i)p\big] f(\Omega_i)}{\sum_{i=1}^{N2} f(\Omega_i)}.$$

And the probability of being in this case

$\sum_{i=1}^{N2} f(\Omega_i)$=Prob(w≤**Erro!** - 1)=F(**Erro!** - 1).

Finally, using the Bayes rule, $\upsilon(p)$ is given by

(1+r) υ (p)=[1-F(**Erro!** - 1)] υ^a(p)+F(**Erro!** - 1) υ^b(p),

(1+r) υ(p)= **Erro!** $\sum_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)$p-I[1-F(**Erro!** - 1)]+$\sum_{i=1}^{N2}\upsilon\big[(1+\Omega_i)p\big] f(\Omega_i)$.

4.2.2 Determination of the Optimal Stopping time

Recalling that the original problem was

$$\max_k \left\{ E\left[\sum_{i=k+1}^{\infty} (1+r)^{-i} p_i) - (1+r)^{-k} I \middle| p_0,\dots,p_k \right] \right\}.$$

In the case that at the present time k, the decision obtained is to wait, we may wish to know (given the present data $p_0,\dots,p_k$) how much time we should wait in order to invest. The answer is given by

$$\tau=\min\{t\geq 0 : E[h(p_{k+t}) \mid p_0,\dots,p_k] \geq E[\upsilon(p_{k+t}) \mid p_k]\}=$$
$$=\min\{t\geq 0 : E_{pk}[h(p_{k+t})] \geq E_{pk}[\upsilon(p_{k+t})]\},$$

which, as we have seen, is equivalent to

$$\tau=\min\{t\geq 0 : E[p_{k+t} \mid p_0,\dots,p_k] \geq p^* \}.$$

As $E(p_{k+t} \mid p_0,\dots,p_k)= E(p_{k+t} \mid p_k)=(1+m)^t p_k$, we get

$$\tau=\min\{t\geq 0 : (1+m)^t p_k \geq p^* \},$$

and so $\tau=\min\{t\geq 0 : t \geq \frac{\ln\left(p^*/p_k\right)}{\ln(1+m)} \}$.

Hence for each value of p_k, the corresponding stopping time can be directly determined as $\tau = \frac{\ln\left(p^*/p_k\right)}{\ln(1+m)}$.

4.3 Algorithm

An algorithm to compute the solution to this problem could be the following:

```
1. Set iteration index K=1 and initial guess for p*1
2. Initialise υ(p) as straight lines
       for p=0...p*1
              υ(p)=Erro!p,
       for p=p*1...Pmax
              υ(p)=h(p).
3. Update estimate of υ(p)
  for p=0..Pmax
```

$\upsilon(p)=(1+r)^{-1}\{$ **Erro!** $\sum_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)p - I[1-F($**Erro!**$-I)]+\sum_{i=1}^{N2}\upsilon[(1+\Omega_i)p]f(\Omega_i)\}$.

```
4. Stop condition
      If
```
$\max_p |\upsilon(p)-\upsilon_{OLD}(p)| < \varepsilon$ then STOP.

```
5. Update estimate of p*
       p*k+1=min{p:υ(p)=h(p)}.
6. k=k+1; GOTO 4.
```

4.4 Special Case

In the special case where the prices are monotonically increasing (i.e. Prob(w>0)=1) we have that F(**Erro! -** 1)=0 for $p=p^*$, and so the expression for $\upsilon(p)$ simplifies to

$\upsilon(p^*)=(1+r)^{-1}\{$ **Erro!** $\sum_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)p^*-I\}$,

and as $\upsilon(p^*)=h(p^*)$ and $\sum_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)=1+m$,

$$(1+r)^{-1}\left[\frac{(1+m)^2}{r-m}p^* - I\right] = \frac{1+m}{r-m}p^* - I.$$

Finally, after some algebra we get

p^*=**Erro!**.

So p^*can be determined explicitly in closed form for this special case. In the general case this can be used as an initial guess for p^* in the previous algorithm.

4.5 Examples

To illustrate the method and the proposed algorithm, consider the following examples:

Example *1*

interest rate: r=0.6 and investment cost: I=20,

Sample space of w: Ω={-0.5, -0.2, 0.1, 0.4, 0.7, 1},

$f(\Omega_i)$=1/6 i=1,...,6 (Uniformly distributed).

Starting with p^*=9.6 (calculated as suggested in the special case) it converges to its final value p^*=10.7 at the 2nd iteration, and the function $\upsilon(p)$ converges after 13 iterations.

The results obtained were:

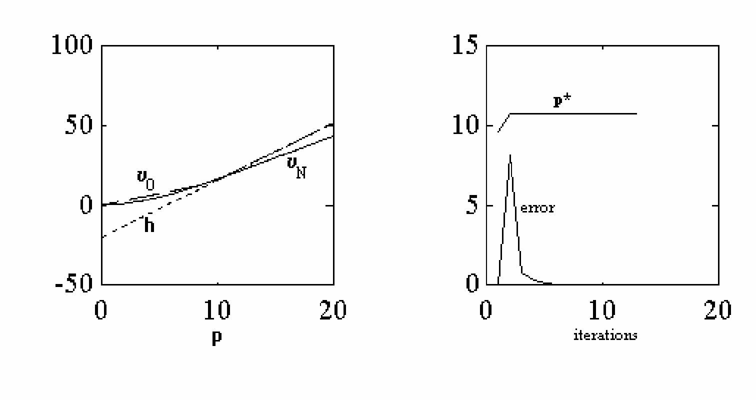

FIGURE 2.

```
Iteration: 1     Pstar: 9.6     Error: 0
Iteration: 3     Pstar: 10.7    Error: 0.768
Iteration: 5     Pstar: 10.7    Error: 0.16
Iteration: 7     Pstar: 10.7    Error: 0.0373
Iteration: 9     Pstar: 10.7    Error: 0.00916
Iteration: 11     Pstar: 10.7    Error: 0.00291
Iteration: 13     Pstar: 10.7    Error: 0.00104
```

Example 2

interest rate: r=0.9, and investment cost: I=20,

Sample space of w: Ω={0.1, 0.2,..., 1},

$f(\Omega_i)$=1/9 i=1,...,9 (Uniformly distributed).

This example falls within the special case because all the elements of the sample space are positive numbers. Hence, p^* can be computed explicitly in closed form giving p^*=11.5.

Starting with an initial guess of p^*=16 we can check that it converges rapidly to its correct value of 11.5.

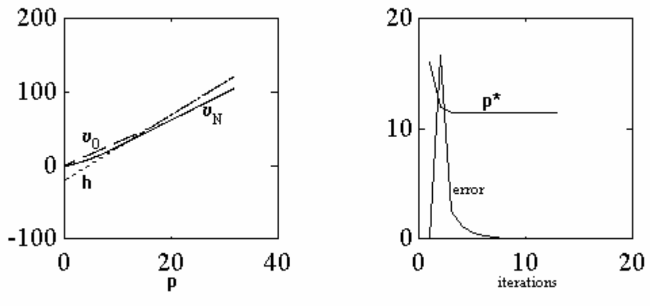

FIGURE 3.

```
Iteration: 1    Pstar: 16   Error: 0
Iteration: 3    Pstar: 11.5   Error: 2.542
Iteration: 5    Pstar: 11.5   Error: 0.601
Iteration: 7    Pstar: 11.5   Error: 0.151
Iteration: 9    Pstar: 11.5   Error: 0.0386
Iteration: 11    Pstar: 11.5   Error: 0.00953
Iteration: 13    Pstar: 11.5   Error: 0.00215
Pstar: 11.5
```

4.6 Reformulating as a finite state Markov chain model

As the evolution of the prices considered before is a process with independent increments, it satisfies the Markov property.
In order to get a finite state-space (the previous case had not a finite state-space unless the $(1+\Omega_i)$'s were multiples of each other) we will first apply logarithms to both sides of equation (1) getting

$$\log P_{k+1} = \log P_k + \log(1+w_k).$$

Defining

$$X_k = \log P_k,$$

we have now a countable state-space for the process $\{X_k, k \in IN\}$. If we now set upper and lower bounds for the state-space, $X_{\min}$ and $X_{\max}$, defining $x_1 = X_{\min}$ and $x_n = X_{\max}$, we get a finite state-space $\mathcal{X} = \{x_1, x_2, ..., x_n\}$. Naturally, by clipping the state-space in this way, we will get a different process, but in a real problem, the majority of the possible prices will certainly be within a bounded interval with high probability and so the difference will not be significant.
In such case, the new gain function $h(x)$ would be defined as

$$h(x) = E\left[\sum_{i=1}^{\infty}(1+r)^{-i}e^{X_i}\right] - I,$$

which is the $(1+r)$-potential of the exponential function minus the fixed cost of investment *I*.
Defining

$$\underline{h} = \left[h(x_1),...,h(x_n)\right]^T,$$

$$\underline{f} = \left[\exp(x_1),...,\exp(x_n)\right]^T,$$

$$\underline{I} = I*[1,1,...,1]^T,$$

and I_n as the identity matrix of dimension *n*. By *proposition* 1 of section 2

$$\underline{h} = \left(I_n - (1+r)Q\right)\underline{f} - \underline{I}.$$

Now the problem,

$$V(X_0) = \max_k E\left[(1+r)^{-k} h(X_k)\right],$$

would be solved simply for this finite state-space case, by calculating *V* for all $x \in \mathcal{X}$, such that *V* satisfies

$$V(x) = \max\{h(x), TV(x)\},$$

where

$$TV(X_k) = (1+r)^{-1}EV(X_{k+1}) = (1+r)^{-1}\sum_{x_j \in \mathcal{X}} Q(X_k, x_j)V(x_j).$$

Alternative solution method 1
This function *V* can be calculated iteratively as

$$\begin{cases} V_0(x) = h(x) \\ V_{m+1}(x) = \max\{h(x), TV_m(x)\}, \end{cases}$$

and $V_m \to V$ as $m \to \infty$.

Alternative solution method 2

We have seen in section 3 that *V*, the value function, is the minimum $(1+r)$-excessive function that majorises *h*. Hence it can be computed by linear programming as

$$\begin{aligned} \min \quad & \sum_{x \in X} V(x) \\ s.t. \quad & V(x) \geq h(x) \\ & V(x) \geq (1+r)^{-1} TV(x) \\ & V(x) \geq 0. \end{aligned}$$

These alternative solution methods will not be implemented here in this work. But a more general case of the Markov jump processes developed in another work (Fontes and Fontes 2007) include a comparative implementation of the three solution methods discussed here applied to the case of Markov Jump Processes.

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