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Oligopolies in private spectrum commons: analysis and regulatory implications

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OLIGOPOLIES IN PRIVATE SPECTRUM COMMONS: ANALYSIS AND REGULATORY IMPLICATIONS

by

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A hair perhaps divides the false and the true.
- Omar Khayyam

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ABSTRACT

In an effort to make more spectrum available, recent initiatives by the FCC let mobile providers offer spot service of their licensed spectrum to secondary users, hence paving the way to dynamic secondary spectrum markets. This dissertation investigates secondary spectrum markets under different regulatory regimes by identifying profitability conditions and possible competitive outcomes in an oligopoly model. We consider pricing in a market where multiple providers compete for secondary demand.

First, we analyze the market outcomes when providers adopt a coordinated access policy, where, besides pricing, a provider can elect to apply admission control on secondary users based on the state of its network. We next consider a competition when providers implement an uncoordinated access policy (*i.e.,* no admission control). Through our analysis, we identify profitability conditions and fundamental price thresholds, including *break-even* and *market sharing* prices. We prove that regardless of the specific form of the secondary demand function, competition under coordinated access always leads to a price war outcome. In contrast, under uncoordinated access, market sharing becomes a viable market outcome if the intervals of

prices for which the providers are willing to share the market overlap.

We then turn our attention to how a network provider use carrier (spectrum) aggregation in order to lower its break-even price and gain an edge over its competition. To this end, we determine the optimal (minimum) level of carrier aggregation that a smaller provider needs. Under a *quality-driven (QD)* regime, we establish an efficient way of numerically calculating the optimal carrier aggregation and derive scaling laws. We extend the results to delay-related metrics and show their applications to profitable pricing in secondary spectrum markets.

Finally, we consider the problem of profitability over a spatial topology, where identifying system behavior suffers from the curse of dimensionality. Hence, we propose an approximation model that captures system behavior to the first-order and provide an expression to calculate the break-even price at each network location and provide simulation results for accuracy comparison. All of our results hold for general forms of demand, thus avoid restricting assumptions of customer preferences and the valuation of the spectrum.

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List of Abbreviations

AP .... Access Point
BE .... Break-even
EFPA . Erlang Fixed Point Approximation
FCC ... Federal Communications Commission
MS .... Market Sharing
NE .... Nash Equilibrium
PCAST President's Council of Advisors on Science and Technology
QD .... Quality Driven

Chapter 1

Introduction and Motivation

When the first AM radios were built, surely no one envisioned that the spectrum on which these electromagnetic waves propagate would become a scarce resource that would auction for values exceeding a billion dollars in today's market. Yet, it is a reality of the modern information age where mobile phones, tablets and computers are a part of our daily lives, all of which heavily use the same wireless spectrum. As a highlight to the value of wireless spectrum today, the highly publicized auction of the 700 MHz band brought in $19.52 billion alone to the federal government (Federal Communications Commission, 2008).

With the proliferation of smart devices that drive the demand for data, wireless spectrum is more valuable than ever. There is a need to rethink the way in which the wireless spectrum is being managed. Under the legacy regulatory framework license holders (*e.g.*, network providers) can deliver only predetermined services and cannot transfer spectrum access rights in any form. Removing the existing inefficiencies in the way spectrum is allocated to license holders is one possible solution to increase its availability.

To address spectrum scarcity issues, reallocation of government held spectrum is pursued all around the globe,(e.g., switching from analog to digital broadcast in the US and utilizing TV White Spaces). In July 2012, the President's Council of Advisors on Science and Technology (PCAST) published a report identifying the need for better spectrum management methods and proposed spectrum sharing as the end goal in envisioning better spectrum utilization (President's Council of Advisors on Science and Technology, 2012).

As a step in this direction, recent initiatives by governmental agencies extend the reach of spectrum management policies that license holders (*e.g.*, network providers) are entitled to pursue (Akyildiz et al., 2006; Bae et al., 2008; Bykowsky, 2003; Bykowsky et al., 2010; Chapin and Lehr, 2007; Mayo and Wallsten, 2010). In recent years, the U.S. Federal Communications Commission (FCC) has made stringent efforts to clear spectrum bands and reallocate them for more efficient use. The main goal of this dissertation is to provide insight into the possible outcomes of these initiatives as well as studying their economic feasibility and ability to foster secondary spectrum market. We are specifically motivated by two recent rulings: i) *private commons* (Federal Communications Commission, 2003) and ii) *reserved spectrum in auctions* (Federal Communications Commission, 2004), which are explained in further detail in Chapter 2.

Realizing the potential of spectrum markets entails a number of challenges for a spectrum provider. One such challenge concerns strategic pricing of secondary spectrum access in the face of uncertainty of the demand function at every advertised price: Providing secondary access at a charge returns an immediate revenue for the provider, but it also incurs an opportunity cost due to lost primary revenue as spectrum is fundamentally a finite resource. The balance between these two effects determines the profitability of secondary spectrum provisioning, and it may possibly depend not only on the secondary price but also on the secondary demand. The relationship between secondary price and demand, however, is difficult to characterize explicitly and it may also be time-varying. Therefore obtaining results that hold for general forms of demand provides applicable and realistic insight to any price analysis.

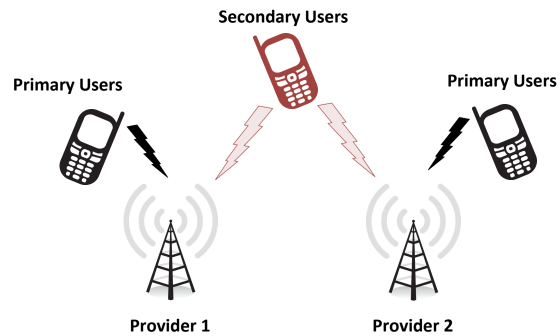


Figure 1·1: Illustration of provider competition over the secondary users in a private commons setting.

This issue is further aggravated in competitive situations in which multiple spectrum providers compete for the same pool of secondary demand. In such situations a provider may opt to beat the price of competitors thereby winning entire secondary market, or may opt to match competitors' price thereby serving part of the market but at a higher price. It is not readily clear which alternative is favorable, especially under the alluded uncertainty in the price-demand relationship.

Finally, the provider faces the decision to implement admission control on the secondary demand. In a coordinated access implementation the provider grants admission to secondary demand only under certain favorable conditions. Alternatively, the provider can opt for an uncoordinated access regime where no distinction between primary and secondary access requests is made, except for pricing. The implementation of either policy will have an impact on the pricing strategy and the market outcome.

As one can conclude from the discussion above, the problem can quickly take a complex form with several layers. The main goal of this dissertation is to provide insight into profitable pricing of secondary spectrum access and the possible competition outcomes of different access policies in markets that involve multiple providers as illustrated in Fig 1·1.

In Chapters 3 and 4 of this dissertation we consider a single geographic location where the providers are aiming to maximize their revenues collected form the secondary users who access the network spectrum opportunistically. We consider a game theoretic setting and identify equilibrium prices in term of Nash equilibria. In particular we seek to answer the question whether a single provider will win the entire secondary market or several providers will choose to share the market.

In Chapter 3, we consider an optimal coordinated access policy implemented by the participating providers, which is a occupancy based policy that admits secondary access requests if the network is serving a number of users less than certain threshold value at the time and rejects it otherwise. We adopt a model that explicitly captures the random nature of spectrum access requests of both primary and secondary users. Our conclusions make no assumptions on the secondary demand function, and therefore hold for arbitrary price-demand relations.

First, we prove the existence of a *break-even* price for each provider, which guarantees profitability as long as the provider sets its price above or equal to this value. We explicitly characterizes the break-even price, which is independent of the parameters of other providers and possesses the fundamental property of being *insensitive* to the specific shape of the demand function of secondary users. The analysis further reveals that the break-even price directly relates to the fraction of lost primary users (in the absence of secondary users), which can be expressed using the well-studied Erlang-B function. The break-even price therefore inherits all the mathematical properties of that function.

Our next contribution is to show that, under an optimal coordinated access policy, market sharing between providers is not an equilibrium outcome. Thus, a provider that employs optimal coordinated access opts to beat the price of its competitors, leading to a price war. Furthermore this property holds irrespective of the specific relation between the secondary price and the secondary demand. We formally establish the dominating strategy of each provider and list all possible market outcomes (i.e., Nash equilibria), which this price war can lead to. We demonstrate that the provider with the lowest break-even price wins the market. If multiple providers have the same break-even price, they are coerced into an equilibrium in which no provider makes a profit.

In Chapter 4, we revisit the same model except one where providers implement uncoordinated access policies instead of coordinated access.

Under uncoordinated access, we show that market equilibria may be drastically different than those under coordinated access. Through a numerical study, we illustrate that depending on the shape of the secondary demand function, market equilibria may reflect a situation wherein providers share the market by matching each other's price strictly above their respective break-even prices.

Theoretical analysis of the uncoordinated case is more complex because results are highly dependent on the specific shape of the demand function. Therefore, we relax the stochastic nature of traffic and assume a fluid model. Under this fluid relaxation, we obtain results once again for *general* demand functions satisfying mild technical conditions.

We prove the existence of a unique *break-even* price p^{BE} for any given secondary demand, for which we provide an explicit expression. However, unlike the coordinated access case, the break-even price depends on the specific secondary demand function.

Unique to the uncoordinated access setting, we derive another unique threshold price, called *market sharing price* p^{MS}, below which a provider finds it desirable to share secondary demand with another provider (i.e., its revenue increases). We demonstrate that the market sharing price is strictly greater than the break-even

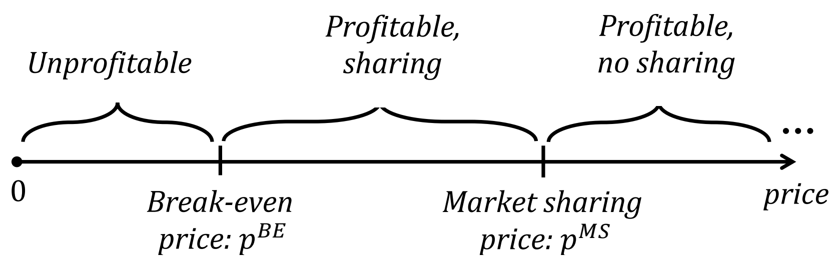


Figure 1·2: Illustration of profitability and market sharing price intervals under uncoordinated access

price, regardless of the demand function. This leads to the conclusion that there always exists a price interval in which a network provider would choose to reduce its secondary demand and maintain profitability as illustrated in Fig. 1·2.

Next, under the same fluid model, we analyze a duopoly competition where network providers make pricing decisions to maximize their revenues. We formally establish the best response strategy of each provider and use them to identify the two possible market outcomes in the form of Nash equilibria: i) if the market sharing intervals overlap, then the providers end up sharing the market; ii) if the market sharing intervals do not overlap, then the provider with the lower break-even price captures the entire market, which reflects the result of a price war. The equilibria prices under the first case are possibly much higher than the break-even prices of each provider, while under the second case the equilibrium price is slightly less than the higher break-even price.

Having investigated the secondary spectrum markets from a pricing point of view, in Chapter 5 we turn our attention to the possibility of providers sizing their capacity in an effort to lower their break-even and market sharing prices. Thus, we consider a market where providers have the freedom to acquire larger bandwidths in an effort to drive their costs down. We are particularly motivated by the FCC ruling that decided to set aside 30 MHz of spectrum for service providers that hold less than a third of the spectrum in a specific market (Federal Communications Commission, 2014a; Federal Communications Commission, 2014b).

We determine the optimal (minimum) level of carrier aggregation that a smaller provider needs to bring its quality of service in line with a larger provider operating in the same market. Furthermore, we aim to provide insight into the relationships between the optimal level of carrier aggregation and fundamental network parameters, such as the traffic load and capacity.

Towards this end we make several contributions. First, we propose an asymptotically exact approximation of the Erlang-B blocking formula under a *quality-driven (QD)* regime that holds for large traffic and network capacities (Borst et al., 2004). Since the Erlang-B formula does not easily yield itself to mathematical analysis, the QD formula is useful to provide more explicit insight into the impact of network parameters.

Using the QD formula, we identify the optimal carrier aggregation decision for the smaller provider through which the market outcome becomes favorable. We provide an efficient method for numerically calculating the optimal level of carrier aggregation. We also derive scaling laws on optimal carrier aggregation with respect to the *scaling factor*, *i.e.* the ratio of the capacity of the larger provider to that of the smaller provider, and establish a sub-linear relationship. We prove that while the level of carrier aggregation needed increases with the scaling factor, it decreases when the initial traffic load of the providers gets higher.

We extend our results to delay-related metrics (*i.e.*, based on the Erlang-C formula) and discuss the application of our results to the profitable pricing of secondary users in a dynamic spectrum sharing scenario as discussed in Chapters 3 and 4.

As a final contribution of this dissertation, we consider the secondary spectrum markets under a larger network topology subject to spatial interference between access points (APs). We seek to expand the profitable pricing analysis presented in Chapters 3 and 4 where we establish the break-even price for a single AP.

In Chapter 6, we identify necessary pricing of secondary access to ensure profitability in a network consisting of multiple access points (APs) some of which are in interference with each other. In our calculations we adopt a first-order approximation model based on the Erlang fixed point model that explicitly captures the random nature of spectrum access requests of both primary and secondary users and is asymptotically optimal in a system where the demand and service capacities are proportionally scaled up. Once again our conclusions make no assumptions on the secondary demand function, and therefore hold for arbitrary price-demand relations.

First, we show that in a network where individual APs experience a low level of blocking and interference, it is possible to decouple the steady state behavior of APs by using a first-order fixed point approximation formula. This approximation allows us to characterize the opportunity cost incurred at each AP due to the admission of secondary users to the network separately from its neighboring APs.

Our next contribution is to characterize and provide an expression to calculate the *break-even* price for at each AP. This break-even price is the lower limit to the price values for which a provider's profitability is guaranteed. The chapter characterizes the break-even price (under the first-order approximation), which is independent of the parameters of other providers and is *insensitive* to the specific shape of the demand function of secondary users.

In summary, in this dissertation:

- We investigate price thresholds and market outcomes in secondary spectrum markets both for coordinated and uncoordinated access. We show that uncoordinated access might lead to market sharing in contrast to coordinated access that always results in a price war..

- We investigate how a network provider can utilize carrier aggregation in order to gain a competitive edge over its competitors and provide scaling laws.

- We consider the profitable admission secondary demand in a network with a spatial topology subject to interference. We provide a first order refinement of the Erlang fixed point approximation to decouple interference between the neighboring cells and use it to determine profitable pricing of secondary access in the network.

The rest of this dissertation is organized as follows. We start by providing background on regulatory initiatives and surveying related work in Chapter 2. In Chapter 3 we consider the implementation of a coordinated access policy in a private commons setting and present the resulting market equilibrium. We determine the optimal coordinated access policy and derive the break-even price for each provider. Next, in Chapter 4 we turn our attention to the implementation of uncoordinated access in private commons and revisit the profitability conditions and the market outcomes. In Chapter 5, we determine the impact of reserving spectrum for smaller providers and how this can be implemented using carrier aggregation. We consider the problem of optimal carrier aggregation and investigate its relationship with respect to network and market parameters. In Chapter 6 we revisit the question of profitability of secondary spectrum access in a network topology under spatial interference constraints. We conclude the dissertation in Chapter 7.

Chapter 2

Background and Related Work

In this chapter, we first provide a detailed overview of FCC policies relevant to this dissertation. Then, we survey related work on competition and spectrum pricing for secondary markets and highlight the differing contributions of our research. The related work is categorized as following: In Section 2.2 we provide a literature survey on work that consider provider competition in private commons. In Section 2.3, we discuss related work on carrier aggregation and many-server approximations, specifically the Quality-Driven (QD) approximation in two separate subsections. In Section 2.4, we discuss related work that consider larger network topologies consisting of multiple access points subject to interference and their profitability.

2.1 Survey of FCC Policies

Private Commons. The FCC introduced a spectrum access policy model known as *Private Commons*, which is deemed both "commercially viable and technologically feasible" (Federal Communications Commission, 2003; Buddhikot, 2007). This new model supports spectrum transactions, where ownership of spectrum remains with the license holder providing service to its primary users, but this provider may also provide spectrum access to secondary users for a fee. As pointed out in FCC's report on secondary spectrum markets, private commons is a framework of spectrum ownership where license holders can grant secondary access to their spectrum band at their own discretion. Control of secondary access in private commons can be implemented in several different ways (Federal Communications Commission, 2004). In particular, access to the spectrum by secondary users may be *coordinated* by the provider, via signals that determine when or how such access is allowed (Buddhikot, 2007). A notable coordinated policy is the so-called *threshold (reservation)* policy, whereby secondary spectrum access is permitted as long as the number of channels occupied in a given spectrum band is below a certain threshold. Threshold (reservation) policies have the distinct advantage of requiring only the current number of customers present in the system to make an admission decision. Theoretical properties of the threshold policy, including optimality in certain settings, have been extensively studied in the literature (*cf.* (Miller, 1969; Key, 1990; Ramjee et al., 1997; Mutlu et al., 2009; Mutlu et al., 2010) and references therein). Access to a band may also be *uncoordinated*, in which case primary and secondary users share access to the band on an equal basis, in a way similar to ISM bands (Buddhikot, 2007). Uncoordinated access policies have the advantage that they are simple to implement with no to little extra operating costs and it can be argued that they are the most egalitarian access policy. The access policy is arguably the most commonly used and important (besides price) way a service provider can influence customer behavior within its network.

Private commons bear significant potential to increase spectrum utilization since cellular networks are generally over-provisioned to cope with short-term spikes in their loads. For instance, a measurement based study of close to 20,000 GSM base stations deployed in Germany indicates that the majority of base station in crowded areas, such as city centers, remain under-loaded by its contracted users at all times (Michalopoulou et al., 2011). Another study conducted in the Commonwealth of Virginia indicates that the US market is no exception to the case with maximum network occupancy levels around 45% (Shared Spectrum Company,). A measurement based study by Kone et. al. (Kone et al., 2012) indicates that conservative policies that minimize interference to primary users (such as one proposed by Jung and Liu (Jung and Liu, 2012)) result in spectrum inefficiencies, where only 20-30% of the available spectrum is extracted for secondary use. Such studies suggest that providing spot-on service to secondary users could increase spectrum utilization levels and thus translate into increased revenue rates.

To capture the entire spectrum of the provider competition one needs to take into account a global perceptive where the best interests of firms and customers are taken into account while trying to move the system to a desirable point from a regulatory perspective. Furthermore, since radio spectrum has a spatial component as well as a temporal once, one needs to consider the competition at a variety of locations, each of which might exhibit unique characteristics, such as the different telecommunications patterns observed in intense traffic load situations at dense urban centers such as New York and Chicago and sparsely populated rural areas where spectrum is under a light load.

Spectrum Reservation. To preserve the competitive landscape of the wireless industry, the FCC has decided to set aside 30 MHz of spectrum for service providers that hold less than a third of the spectrum in a specific market (Federal Communications Commission, 2014a; Federal Communications Commission, 2004). With the 600 MHz spectrum auction on the horizon, this ruling is poised to have a significant impact on the industry (Wall Street Journal, 2014). The ruling has already caused some controversy in the market as it restricts the amount of spectrum larger providers have access to (Reuters, 2014), though some public interest groups are asking for it to be increased to 40 MHz (Fierce Wireless, 2015).

The policy ruling is facilitated by a central feature of LTE-Advanced networks (as defined in 3GPP Release 10 and beyond) called *carrier aggregation* (Iwamura et al., 2010; Wang et al., 2010; Yuan et al., 2010; Doyle et al., 2012). Carrier aggregation allows service providers to aggregate contiguous or non-contiguous component carriers up to 100 MHz total bandwidth. This significantly improves the performance of the network compared to LTE specifications defined in Release 8 (3GPP, 2012), where the maximum supported bandwidth is 20 MHz.

A significant challenge associated with the ruling is to identify how much additional spectrum a smaller provider needs to improve its service to the level of a larger provider, which initially holds a competitive advantage in the market due to economies of scale. If this criterion is met, the spectrum reservation policy effectively fosters a competitive market. Otherwise, the policy inherently risks wasting highly valuable spectrum. We study this question in Chapter 5.

The problems considered in this dissertation exist in the wider literature of cognitive radios and dynamic spectrum sharing technologies. The literature includes proposed methods of utilizing TV white spaces to the deployment of small cells in an effort to better utilize the existing spectrum, all enabled by the emergence of cognitive radios (Nekovee, 2009),(Chandrasekhar et al., 2008). While some papers investigate these problems from a technical point of view there also exists a large literature concerning the economic analysis of the implementation of said technologies. Within the proposed methods of spectrum sharing and the different analytical approaches, our contributions sit in the techno-economic analysis of secondary spectrum markets, implemented in the form of private commons, where both the traffic characteristics of the resulting wireless networks and their economic interpretation through profitability analysis and equilibrium concepts are being investigated. We present the related work in the subsequent subsections.

2.2 Competition in Private Commons

Network providers in spectrum markets may face competition at two different levels. The first level consists of competition between secondary network providers to lease spectrum from a primary provider (or the government) that holds a spectrum license. The second level of competition arises after the said leasing of the spectrum, and is among providers, which hold a license or lease, competing to offer their services to the end-users.

Many papers in the literature consider the first level of competition, while our research is positioned to addresses the second one. For instance, in the works by Jagannathan (Jagannathan et al., 2012), Kasbekar (Kasbekar and Sarkar, 2012a), Duan (Duan et al., 2010), Ren (Ren et al., 2011), Niyato and Hossain (Niyato and Hossain, 2008), Sengupta and Chatterjee (Sengupta and Chatterjee, 2009) and Xing (Xing et al., 2007), game theoretic approaches to spectrum auctioning and leasing are analyzed. The set-up of all these papers (*i.e.*, competition between providers to lease spectrum) is different from what we consider (*i.e.*, competition between providers to lure users).

Several papers study the problem of ensuring profitability in secondary spectrum markets. Niyato and Hossain (Niyato and Hossain, 2008) derives market equilibria pricing by taking into consideration the demand and supply dynamics of spectrum auctions. However, the model uses a very specific secondary demand based on the utility from owning the spectrum and how much it costs to lease the spectrum. On the other hand, our results hold for general demand functions. Drawing conclusions under general demand functions generally requires a more elaborate analysis, as illustrated by several papers (Allon and Gurvich, 2010; Andrews et al., 2013; Besbes and Zeevi, 2009). Also, secondary users have the option to lease parts of their spectrum from different spectrum owners.

On the end-user side, Alanyali et al. (Alanyali et al., 2011) establishes a pricing policy which guarantees profitability for the network provider as long as a demand is generated. However, this paper assumes a monopolistic framework, while ours considers an oligopolistic one. Furthermore, (Alanyali et al., 2011) considers a multi-cell setting with a single frequency band in each cell, while in the related chapters we focus on an isolated cell offering multiple frequency bands.

Mutlu et al. (Mutlu et al., 2009) also consider a monopolistic framework and derive an optimal coordinated access policy under which revenue from secondary users is maximized. The results of that paper show that a threshold policy is optimal for coordinated access in an isolated cell, assuming that a provider advertises a fixed price (i.e., the price does not depend on the instantaneous channel occupancy). These results are leveraged for the analysis in our work.

In a work by Ileri et al. (Ileri et al., 2005), a comprehensive model including both the auction and the end-user sides of the competition is studied. Different from our work, this model focuses on the auctioning side of the competition where the revenue generated by secondary users is used to compensate for the costs of auctioning. In our model, we assume that providers own spectrum and need only to consider the revenue brought in by the primary and secondary users.

The works by Maille and Tuffin (Maille and Tuffin, 2010) and Maille et al. (Maille et al., 2011) use a model where both the auction side and the service side of the competition are considered. The work in (Maille and Tuffin, 2010) specifically focuses on the competition between two different but substitute technologies while (Maille et al., 2011) models a three level competition, where spectrum owners, lessees and users each make their own separate decisions. These decisions include the use of different technologies. In our model, we assume that providers offer the same type of services and therefore cannot influence the secondary users' preferences beside the price advertised. A related work by Ren et al. (Ren et al., 2011) studies and compares the market outcome achieved by respectively enforcing cooperation or competition among providers. While such external interventions might be useful in analyzing hypothetical outcomes, our model refrains from such enforcements as it aims to characterize the outcomes of a natural competition. In a work by Kim et al. (Kim et al., 2011), competition between two providers is analyzed where network pre-emption allows for primary users to evict secondary users from the system. Unlike ours, the network model is not a finite capacity multichannel network but rather a spatial distribution of channels that turn on and off, and the analysis relies on an approximation. In one recent study by Korcak et. al. (Korcak et al., 2012), the possibility of collusion between several wireless network providers is considered. This collusion is based on a coalition game model. In contrast, in our model, network providers do not communicate with each other about their intentions (i.e., it is a non-cooperative game). Thus, the possibility of market sharing between the providers is purely a result of market dynamics.

A paper by Fortetsanakis et. al. (Fortetsanakis et al., 2012) considers the second level of competition, where providers offer what the authors call the *Flex Service*. The simulation based results indicate that the welfare of the market increases through the use of a central database which collects information about pricing and quality of service. This work relies on explicit demand and utility functions. Our results hold without making such assumptions.

None of the previous work surveyed here considers competition among network providers implementing either coordinated or uncoordinated access and facing secondary demand governed by a *general* demand function. The characterization of the market equilibrium and demonstration of a price war won by the provider(s) with the lowest break-even price as well as the possibility of market sharing equilibria under uncoordinated access policies are unique contributions of this work.

2.3 Carrier Aggregation in LTE-Advanced

In this section, we survey previous work on many-server approximations of queuing systems and on carrier aggregation.

2.3.1 Many-Server Approximations

The many-server approximation that forms the basis of the QD regime was first introduced in Iglehart's work (Iglehart, 1965). The paper considers a setting where the arrival rate and the number of servers both become very large and the ratio of the arrival rate to the service rate (*i.e.*, the traffic load) is a constant that is strictly smaller than one. Under proper statistical assumptions, the process describing the evolution of the queue occupancy converges to a Ornstein-Uhlenbeck diffusion process. Halfin and Whitt (Halfin and Whitt, 1981) provide another many-server approximation that characterizes queues in a *quality-and-efficiency driven* (QED) regime, *i.e.*, where the arrival rate and the number of servers both become very large and the traffic load approaches one. The work of Zeltyn and Mandelbaum (Zeltyn, 2004) provides an overview of different types of many-server approximations and is useful as a general reference. In our work, we utilize the QD regime approximations that allow us to analyze the quality of service experienced by voice calls and data flows in cellular networks.

Scaling laws in wireless and wired networks have been studied in various contexts (Bolcskei et al., 2006; Gupta and Kumar, 2000; Xie and Kumar, 2004; Ozgur et al., 2007). The work of Bolcskei *et al.*, for example, focuses on the gains realized by increasing the number of antennas in a MIMO relay network. This work falls under the broad category of papers that analyze the dimensioning of telecommunications networks. Such papers are crucial in providing a better understanding of the relationship between resource allocation and system performance, allowing policy makers to look past the current state of the market. Another example is the work by Xie and Kumar (Xie and Kumar, 2004), where this time the focus is on the capacity achieved by cooperation between the nodes of a wireless network. The scaling question has been raised once again but this time for ad hoc networks by Ozgur and Leveque (Ozgur et al., 2007).

2.3.2 Carrier Aggregation

Carrier aggregation has been gaining significant attention since it has been introduced in 3GPP Release 10 on LTE-Advanced in 2011. Several papers in the literature explain practical considerations to achieve desired performance levels in networks, such as deployment options, implementation frameworks, and challenges in the physical layer (Iwamura et al., 2010; Wang et al., 2010; Yuan et al., 2010). The work by Shen *et al.* (Shen et al., 2012) provides an overview on all layers, while also underlining the interest of several major U.S. providers in the technology. Alotaibi and Sirbu provide a comprehensive cost benefit analysis of spectrum aggregation in (Alotaibi and Sirbu, 2011) and how it impacts network performance in (Alotaibi and Sirbu, 2015). A recent paper by Doyle *et al.* (Doyle et al., 2012) introduces an interesting application of carrier aggregation. The authors consider the possible uses of carrier aggregation in a dynamic spectrum access, such as dynamically aggregating carriers to address coverage or congestion issues. They also propose a regulatory framework that supports this enhanced form of carrier aggregation.

Fungibility of the aggregated spectrum is considered in (Weiss et al., 2012), where the authors seek to identify whether all spectrum bands provide the same performance. For example, low frequency spectrum, such as the 600 MHz band considered in Chapter 5, is generally viewed as more desirable than higher frequencies because of its propagation properties.

Considering the impact of spectrum reservation for smaller providers on the competitiveness of a wireless market is beyond the scope of the previous work surveyed here. The identification of the optimal carrier aggregation and the scaling laws provided thereunto, as well as simple methods of calculating it, are the unique contributions of this chapter.

2.4 Analysis of Multicell Networks under Spatial Interference

Modeling multi-cell networks with interference has been vastly studied in the literature. Perhaps the best known work in the area of modeling large networks is the seminal work of Kelly (Kelly, 1991; Kelly, 1986) where the fixed point methods are used to study loss networks with different routing schemes. Kelly established the so-called *Erlang fixed-point approximation* as an asymptotically exact (with respect to network size) and unique solution to obtain the loss probabilities in larger networks. Al Daoud et al. (Al Daoud et al., 2010) make use of this fixed point approximation to identify the price admission decision in a wireless network setting with interferences. That work, however, considers optimal pricing where the network provider leases a part of its network to a third party instead of identifying the minimum profitable price under the coexistence of both types of user in the same network which is our focus.

Several papers study the problem of pricing spectrum in a spatial network. The work Alanyali et. al. (Alanyali et al., 2011), which was also discussed in section 2.2, establish a pricing policy in a multi-cell setting which guarantees profitability for the network provider as long as a demand is generated. However, (Alanyali et al., 2011) considers a single frequency band in each cell, while our focus is on cells each offering multiple frequency bands. Kasberkar and Sarkar (Kasbekar and Sarkar, 2012b) study the pricing competition between network providers where spatial reuse of available bandwidth is taken into consideration. The model resembles that in (Alanyali et al., 2011), as they are mainly concerned with conflict graphs which determine whether spectrum can be used in neighboring cells. It, however, does not consider the question of profitable pricing and the possibility of co-existence in neighboring cells under limited interference.

In section 2.2 we mentioned that Mutlu et al. (Mutlu et al., 2009) derive an optimal coordinated access policy under which revenue from secondary users is maximized and showed that, for an isolated cell, a threshold policy was optimal. The results of this work are leveraged for the analysis of multiple cells as well. Specifically, by decoupling the spatial interference effects of the network, the optimal policy for an isolated cell becomes the optimal policy for the decoupled network as well.

None of the previous work surveyed here considers the profitable pricing of secondary users in a multi-cell network with interference, where the provider is facing secondary demand governed by a *general* demand function. The use of the fixed point methods to decouple the cells in the network and thus obtaining the break-even price under the secondary demand are the unique contributions of this dissertation on the subject.

Chapter 3

Coordinated Access

In this chapter we consider the profitable pricing of secondary access and competition between network providers under the implementation of coordinated access by the participants. We first provide a detailed description of the model under consideration. Next, we establish the optimal coordinated access policy and identify profitability conditions. We then demonstrate that the best response behavior of coordinated access leads to a price war, which gives the market outcome.

3.1 Network Model

In this section we introduce the network and market models considered and the accompanying notation. For convenience of exposition we present here a model with two providers, and later extend it to an arbitrary number of competing providers: Each provider $i = 1, 2$ has a finite number of channels C_i, and a dedicated primary-user base whose traffic generation rate (i.e., the average number of requests per unit time) is represented with $\lambda_i > 0$. For each primary user serviced, provider i collects a reward of K_i units.

The providers compete for an additional secondary demand, which is raised through offering secondary service at a fixed access price for the duration of a contract period. The contract period is long enough (relative to inter-arrival and holding

Results presented in this chapter appear in part in (Kavurmacioglu et al., 2012a),(Kavurmacioglu et al., 2012b),(Kavurmacioglu et al., 2014a)

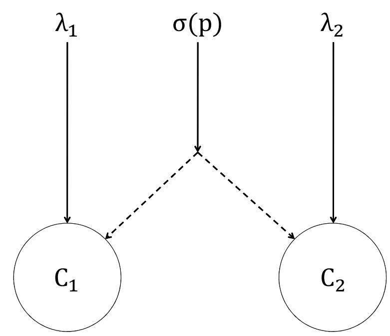


Figure 3·1: Market model: Two providers each with a capacity $C_i,\ i = 1, 2$, and a dedicated (primary) demand λ_i compete for secondary demand.

times of calls) to allow an equilibrium analysis. In the course of the contract period, neither the pricing nor the users preferences change.

If provider i charges p_i units per secondary access then the intensity of secondary demand is $\sigma(p_i)$. Here $\sigma(\cdot)$ is the well-known *demand function* and it is assumed to be continuous and non-increasing. We denote the maximum value of the secondary demand by $\sigma_{max} = \sigma(0)$.

We shall assume that each demand type (primary and secondary) consists of a random sequence of request arrivals that occur according to independent Poisson processes. We also assume that, if granted, each request holds a single channel for a random duration that is generally distributed with unit mean, independently of other requests and arrival times. We shall assume that the channel holding statistics are identical for primary and secondary requests. Such an assumption is valid when both types of traffic are generated by similar applications.

The general form of aggregate secondary demand $\sigma(p)$ captures the heterogeneity of customer preferences. Indeed, the demand function implicitly represents the fraction of users (user types) that find each price value acceptable. The generality of the demand function allows consideration of different user types. The separation between primary and secondary users and the random nature of service times capture additional levels of heterogeneity in our model.

Secondary demand is assumed to be attracted to the provider charging the lowest price. This behavior can be explained by *price aversion*, a concept employed in marketing management (Tellis and Gaeth, 1990). When both providers charge the same price, the resulting secondary demand splits between the two providers according to a static probability vector $[\alpha_1, \alpha_2]$ such that $\alpha_1 + \alpha_2 = 1$ and $\alpha_1, \alpha_2 > 0$. Namely, each provider i receives a secondary demand of volume $\alpha_i \sigma(p_i)$ every time market prices are equal.

Each provider i also has the choice of admitting or rejecting secondary requests according to an access policy, which we denote by A_i. We assume that actions taken by A_i depend only on the number of each class of users (primary and secondary) in the system. Thus, A_i belongs to the class of *occupancy-based* policies, the performance of which are insensitive to the call length distribution except through the mean (Mutlu et al., 2010). Hence, without loss of generality, we can assume exponentially distributed service times for the purpose of analysis in the rest of this chapter.

Since providers have a finite number of channels to provide service with, they cannot accommodate new requests if all of the channels are occupied. This results in some requests being blocked. We define $B_{i,j}(\lambda_i, \sigma, A_i)$ as the blocking probability for class j users ($j = 1$ for primary and 2 for secondary) when secondary demand is σ and the access policy is A_i.

The goal of each provider is to maximize the total revenue collected. The revenue rate of provider i when it services secondary demand of σ units is given by:

$$W_i(p_i, \sigma, A_i) = (1 - B_{i,2}(\lambda_i, \sigma, A_i))\sigma p_i + (1 - B_{i,1}(\lambda_i, \sigma, A_i))\lambda_i K_i. \tag{3.1}$$

Here the first and the second terms are respectively the revenue generated by primary and secondary requests that are admitted by the provider. Each term represents the expected long time rates per unit time.

Since the secondary demand a provider receives depends on prices of both providers, so does the revenue of the provider. We define the *reward* $R_i(p_i, p_{-i})$ of provider i as its revenue when provider i and its competitor $-i$ charge secondary access p_i and p_{-i} units respectively. Namely,

$$R_i(p_i, p_{-i}) = \begin{cases} W_i(p_i, \sigma(p_i), A_i) & \text{if } p_i < p_{-i} \\ W_i(p_i, \alpha_i \sigma(p_i), A_i) & \text{if } p_i = p_{-i} \\ W_i(p_i, 0, A_i) & \text{if } p_i > p_{-i}. \end{cases} \tag{3.2}$$

Hence the reward is affected by the amount of secondary demand provider i captures through the relationship between its own price p_i and the price of the other provider p_{-i}. Once the prices determine the secondary demand for each provider, the rewards are further shaped by the providers' access policies. Each provider has full information on its own network parameters and can observe the prices advertised by its competitors.

3.2 Optimal Coordinated Access Policy and Profitability

For a given secondary demand σ and secondary price p, let $A^*(p, \sigma)$ denote a coordinated access policy that maximizes the revenue rate for a provider (for analyses in which we consider a single provider, we will drop index i from our notation for the sake of simplicity). We refer to $A^*(p, \sigma)$ as the *optimal coordinated access policy.* We represent the resulting maximal revenue $W^*(p, \sigma)$ as follows:

$$W^*(p, \sigma) = W(p, \sigma, A^*(p, \sigma)) = \max_A W(p, \sigma, A). \tag{3.3}$$

One can formulate the provider's optimization problem using a Markov decision process (MDP), where the state is the total number of users in the network. Note that primary and secondary users have identical channel holding statistics, hence once admitted to the network they are indistinguishable. At every state, the provider needs to make a decision whether to admit or reject a secondary user arrival in order to maximize its expected revenue. MDPs can be solved with dynamic programming (DP) techniques (Bertsekas, 1976). Under the given assumptions, it is well-known that the coordinated access policy that yields the optimal solution to our DP problem is a threshold (reservation) policy: Secondary users are admitted by a provider when the channel occupancy of the provider is below a threshold $T \geq 0$ and they are blocked otherwise (Key, 1990; Miller, 1969; Mutlu et al., 2010; Ramjee et al., 1997). The optimal threshold value depends on all parameters of the provider including intensity of the secondary demand. We let the notation $A = T$ correspond to the implementation of a threshold policy with the specific threshold value being equal to T.

In the competitive setting considered in this chapter it will be important to identify conditions under which an optimal policy $A^*(p, \sigma)$ ever accepts a secondary request. Under such conditions the secondary price-demand pair (p, σ) yields profit relative to serving primary demand only; in turn (p, σ) represents an economically viable situation for a provider. The issue is closely related with the opportunity cost of accepting a secondary request: On the one hand such a request brings an immediate revenue of p, on the other hand it may cause rejecting future requests, possibly with higher immediate revenue, due to the channel that it holds temporally. To identify the profitability of admitting a secondary user, we utilize a policy improvement technique based on (Alanyali et al., 2011; Key, 1990). Specifically, we identify a price condition for which there exists a policy that yields a better revenue than a policy that flatly rejects all secondary arrivals. This determines the sign of the balance in the trade-off when making a control decision to admit a secondary user or not. We state our main result on this profitability condition in the following theorem:

Theorem 3.2.1 *For $\sigma > 0$ there exists a break-even price p^{BE} given by:*

$$p^{BE} = KE(\lambda, C), \tag{3.4}$$

where $E(\lambda, C) = \dfrac{\lambda^C/C!}{\sum_{k=0}^{C} \lambda^k/k!}$ *is the Erlang-B formula. such that:*

(a) $W^*(p,\sigma) > W^*(p,0)$ *if* $p > p^{BE}$,

(b) $W^*(p,\sigma) = W^*(p,0)$ *if* $p \leq p^{BE}$.

Proof. In order to calculate for which prices it is profitable to admit secondary users, we model the optimization problem as an MDP. Thus, we set up an infinite horizon average cost dynamic programming problem and identify the prices at which the optimal policy allows for the admission of secondary users into the network at some states. To do so, we take the total number of users in the network (i.e., occupancy) denoted by y as the state of the system, $\bar{J}$ as the *time-average reward* and $h(y)$ as the *differential reward function* (Bertsekas, 1976). $\bar{J}$ can be interpreted as the average reward collected from incoming arrivals over a period of time which length goes to infinity, whereas the differential reward function $h(y)$ characterizes the expected difference when we start the process from a particular state y instead of an arbitrary state y' which we take as the reference such that $h(y') = 0$. In our case, and without any loss of generality, we set $y' = 0$.

We uniformize the process with the maximum possible transition rate out of any state, which we denote by $\nu \triangleq \lambda + \sigma + C$. Since the service rate is the same for both primary and secondary users, they are indistinguishable once in the system. Following this observation, at state $\{y : 0 \leq y \leq C\}$ a user (either primary or secondary) will leave the system with probability $\frac{y}{\nu}$. With probability $\frac{\lambda}{\nu}$ a primary user will arrive, with probability $\frac{\sigma}{\nu}$ a secondary user will arrive, and with probability $\frac{C-y}{\nu}$ the state will remain the same (i.e., nothing happens). Note that an arrival of either kind to a full network is not admitted and thus no reward is collected. Then the well established Bellman equations for the average reward problem can be formulated as follows:

$$\begin{aligned}\bar{J} + h^*(y) = \frac{1}{\nu}\{ \; & yh^*(y-1) + (C-y)h^*(y) \\ & + \lambda(K + h^*(y+1)) \\ & + \sigma \max\left(p + h^*(y+1), h^*(y)\right)\}, \end{aligned} \tag{3.5}$$

for $0 < y \leq C - 1$. The last term on the right hand side of the equation reflects the admission choice to be made, that is either admit an incoming secondary user and collect a reward of p while increment the state or reject the arrival and preserve the state.

We also consider the two special cases, first when the network is full:

$$\bar{J} + h^*(C) = \frac{1}{\nu}\{Ch^*(C-1) + (\lambda + \sigma)h^*(C)\},$$

and next when the network is empty:

$$\bar{J} + h^*(0) = \frac{1}{\nu}\{Ch^*(0) + \lambda(K + h^*(1)) + \sigma \max\left(p + h^*(1), h^*(0)\right)\}.$$

Let us define the *lock-out* policy as an access policy where all secondary users are rejected, regardless of network occupancy. We will approach this pricing decision problem by determining when the lock-out policy on secondary users stops being optimal. Assuming a lock-out policy, which we denote by the use of the superscript LO, Eq. (3.5) reduces to:

$$\bar{J} + h^{LO}(y) = \frac{1}{\nu}\{\ yh^{LO}(y-1) + (C-y)h^{LO}(y) \\ + \lambda(K + h^{LO}(y+1)) + \sigma h^{LO}(y)\}. \tag{3.6}$$

From the last argument of Eq. (3.5), it is clear that when the state of the network is y, a lock-out policy is optimal if and only if $\max\left(p + h^{LO}(y+1), h^{LO}(y)\right) = h^{LO}(y)$ or:

$$p \leq h^{LO}(y) - h^{LO}(y+1). \tag{3.7}$$

Therefore if $p \geq h^{LO}(y) - h^{LO}(y+1)$, a lock-out policy is no longer optimal, which is equivalent to starting to admit some secondary users. We shall next obtain an analytical expression of the quantity $H(y) \triangleq h^{LO}(y) - h^{LO}(y+1)$.

Writing Eq. (3.6) for every state y and taking the difference between every two consecutive states yields the following set of equations:

$$\begin{array}{ccc} (\lambda+1)H(0) & = & \lambda H(1) \\ \vdots & \vdots & \vdots \\ (\lambda+y)H(y-1) & = & \lambda H(y) + (y-1)H(y-2) \\ \vdots & \vdots & \vdots \\ (\lambda+C)H(C-1) & = & \lambda K + (C-1)H(C-2) \end{array}$$

The solution to this set of equations is

$$H(y) = K\frac{E(\lambda, C)}{E(\lambda, y)} \quad \text{for } 0 \leq y \leq C-1. \tag{3.8}$$

Since we are specifically interested in finding the price p at which it is optimal to admit at least one secondary user into the network, through Eq. (3.7), we know that this price must be greater than or equal to

$$\min_{0 \leq y \leq C-1} H(y) = \min_{0 \leq y \leq C-1} \left(h^{LO}(y) - h^{LO}(y+1)\right).$$

By observing how Eq. (3.8) changes with respect to y, one can come to the conclusion that $H(y)$ is increasing in y, the minimum value such a price p can take is:

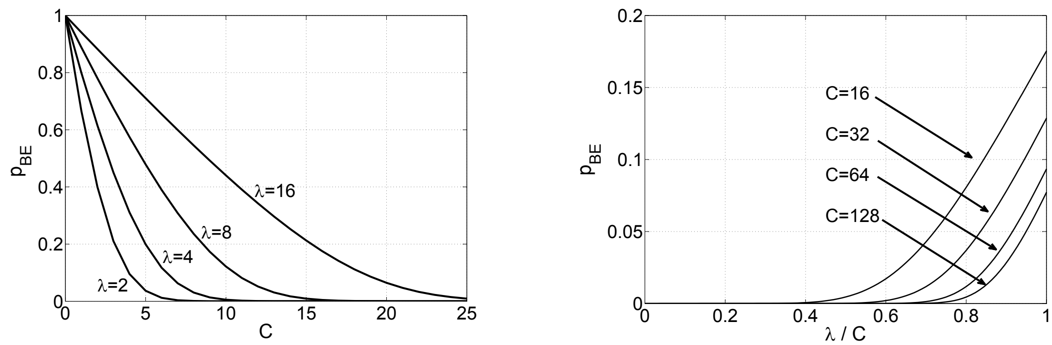
$$p^{BE} \triangleq H(0) = K\frac{E(\lambda, C)}{E(\lambda, 0)} = KE(\lambda, C). \tag{3.9}$$

Therefore, as long as the price is greater than p^{BE}, there exists at least one state y (i.e., when the system is empty) at which admitting secondary customers yields a better revenue rate than the revenue rate under a lock-out policy. □

Theorem 3.2.1(a) states that if the price exceeds p^{BE} then serving secondary demand yields strictly higher revenue for a provider than not serving it. Conversely, part (b) of the theorem states that secondary demand does not lead to any revenue improvement otherwise, implying that rejecting the entire secondary demand is optimal for such prices. In effect, at p^{BE} the immediate revenue balances the opportunity cost of a secondary request. We therefore coin p^{BE} as the *break-even price* of a provider.

It is striking that the break-even price expression (3.9) does not depend on the secondary demand. Namely, any price above p^{BE} strictly improves the revenue of a provider regardless of how much secondary demand it generates. This result can be intuitively understood as follows: The size of secondary demand does not play a role in profitability, for any positive secondary demand can be thinned down arbitrarily by the coordinated access policy. We have shown that at the break-even price the lock-out policy stops being optimal, which is equivalent to stating that secondary access is profitable when the network is empty. Since the profitability of the first admitted secondary user depends on a network where there are no other secondary users, secondary demand does not affect the break-even price.

Figures 3·2(a) and 3·2(b) illustrate how the normalized break-even price (i.e., p^{BE}/K) changes with respect to relevant network parameters, namely the system



(a) Break-even price p^{BE} with respect to network capacity C for different primary arrival rates λ and primary price $K = 1$.

(b) Break-even price p^{BE} with respect to network load λ/C for different network capacities C and primary price $K = 1$.

Figure 3·2: Behavior of break-even price as a function of network parameters.

capacity C and the network load λ/C. The normalized price is given by the Erlang-B function, which has been well studied in teletraffic theory. In particular upper and lower bounds are obtained in (Harel, 1988; Janssen et al., 2008), and it is demonstrated in (Jagers and Doorn, 1986) that for a given arrival load λ, the Erlang-B function (hence, the break-even price) is a convex function of the capacity C, as can be observed from Figure 3·2(a). It is also worth noting that as the network capacity increases, the value of the break-even price at the critical load where $\lambda = C$ decreases as demonstrated in Figure 3·2(b).

Figure 3·2(b) shows that for an over-provisioned network (in which primary load λ is below the capacity C by a significant margin), the break-even price is substantially lower than the primary price. We observe that for $C = 16$ the normalized break-even price is negligible compared to the primary price for network loads below 40%, a number close to the network utilization measurements reported in (Shared Spectrum Company,). As the network capacity increases, it takes even higher network loads to observe the slightest increase in the break-even price, almost as high as 80% when the capacity is increased to $C = 128$. This result suggests that, in an over-provisioned network, spectrum sharing at secondary prices that are low relative to primary reward would result in net profit, regardless of the secondary demand.

3.3 Best Response under Coordinated Access

In competing for and serving secondary demand, a provider's action consists of an advertised price for secondary access and an access policy to coordinate secondary access. For any price, and for any demand the price raises, each provider's revenue is highest under optimal coordination. Hence optimal coordination is a dominating choice uniformly for all situations. In this section we will assume all providers implement optimal coordinated access. With this assumption each provider's strategic action reduces to a pricing decision.

In the next theorem, we state that the best response of a provider is to set its price slightly lower than the competition in order to capture all of the secondary demand rather than sharing the secondary demand at that price. This can be formalized as follows:

Theorem 3.3.1 *If $p > p^{BE}$, for any given $\alpha \in [0, 1]$ there exists a price $p' \in (p^{BE}, p)$ such that:*

$$W^*(p', \sigma(p')) > W^*(p, \alpha\sigma(p)). \tag{3.10}$$

Before we prove Theorem 3.3.1, it is beneficial to establish the strictly dominated strategies for both providers under optimal coordinated access. This allows for the characterization of provider i's best response strategy for any price it's competitor chooses. To do so we introduce two lemmas. In the first lemma for two given secondary demand values of σ_1 and σ_2 such that $\sigma_1 > \sigma_2$, we will demonstrate that the revenue rate when facing higher secondary demand σ_1 is never less than the revenue rate when facing lower secondary demand σ_2 (i.e., $W^*(p_i, \sigma_1) \geq W^*(p_i, \sigma_2)$).

Lemma 3.3.1 *Let $p > 0$. For any σ_1, σ_2 such that $\sigma_1 > \sigma_2$:*

$$W^*(p, \sigma_1) \geq W^*(p, \sigma_2).$$

Proof. Consider the optimal access policy $A^*(p, \sigma_2)$ which yields a revenue rate of $W^*(p, \sigma_2)$ for demand σ_2. Now consider a policy $\hat{A}(p, \sigma_1)$ for demand σ_1, which does a random thinning of the secondary demand and brings it to σ_2 (*i.e.,* $\hat{A}(p, \sigma_1)$ accepts each arrival with probability σ_2/σ_1. Note that the thinned arrival process is still Poisson (Key, 1990). Once the secondary demand is reduced, access policy $A^*(p, \sigma_2)$ is implemented. Hence $\hat{A}(p, \sigma_1)$ and $A^*(p, \sigma_2)$ generate the same revenue rate, *i.e.,* $W(p, \sigma_1, \hat{A}) = W^*(p, \sigma_2)$.

Since by definition $A^*(p, \sigma_1)$ is the optimal coordinated access policy when secondary demand is σ_1, we know that it does not generate a revenue less than the revenue generated by the policy we have just described. We can formulate this conclusion as:

$$W^*(p, \sigma_1) \geq W(p, \sigma_1, \hat{A}) = W^*(p, \sigma_2). \tag{3.11}$$

■

In the previous lemma, we have demonstrated that an increase in secondary demand does not result in a decrease in the revenue rate of a provider. In the second lemma we will build on the previous lemma to show that when the price is set above the break-even price, an increase in secondary demand translates into strict increase in the revenue rate.

Lemma 3.3.2 *Let $p > p^{BE}$. For any σ_1, σ_2 such that $\sigma_1 > \sigma_2$: $W^*(p, \sigma_1) > W^*(p, \sigma_2)$.*

Proof. We know that when the price is greater than the break-even price (*i.e.,* $p > p^{BE}$), an optimal admission policy will never choose the threshold value $T = 0$.

Since in this lemma we only consider such prices, we can formalize this result through the formulation

$$\max_{0 \le T \le C} W(p, \sigma, T) = \max_{1 \le T \le C} W(p, \sigma, T).$$

In Mutlu et al.'s work (Mutlu et al., 2009), it is shown that for a fixed admission threshold value $T > 0$, $W(p, \sigma, T)$ is a *unimodal* function with respect to σ for any p. In other words, $W(p, \sigma, T)$ is strictly increasing until is reaches a certain maximum and strictly decreasing afterwards. We define the value of σ at which $W(p, \sigma, T)$ attains its maximum value over the interval $[0, \sigma_{max}]$ for an admission control policy with fixed threshold T by:

$$\overline{\sigma}_T = \operatorname*{argmax}_{\sigma \in [0, \sigma_{max}]} W(p, \sigma, T). \tag{3.12}$$

We define d to be the minimum of the distances between any two *distinct* maxima of $W(p, \sigma, T)$ for different values of T so that

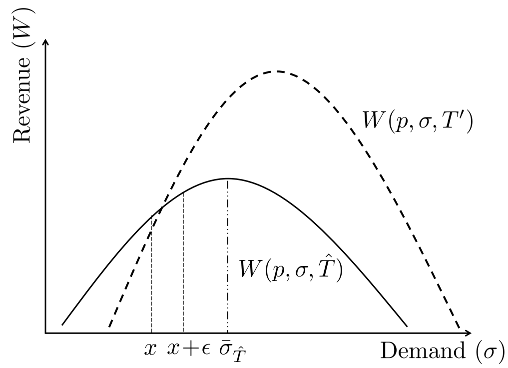
$$d = \inf_{m,n \in 1,2,\ldots,C} |\overline{\sigma}_m - \overline{\sigma}_n|, \qquad \overline{\sigma}_m \ne \overline{\sigma}_n.$$

Since there are a finite number of possible threshold policies, the infimum is achieved (*i.e.,* inf $=$ min) and $d > 0$. Having defined the minimum distance between distinct maxima of two different threshold revenue functions, we will prove the lemma by first showing that

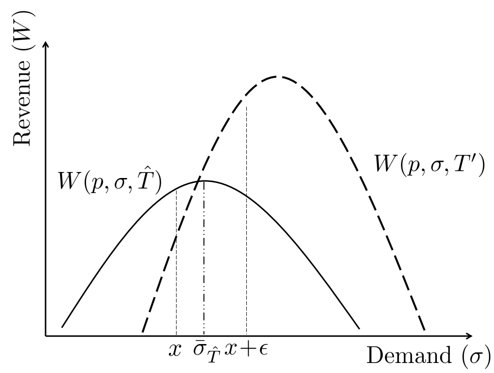
$$W^*(p, x + \epsilon) > W^*(p, x), \quad \forall x \in [0, \sigma_{max}] \text{ and } \epsilon < d,$$

where ϵ is a constant the value of which does not depend on the secondary demand x.

It should be noted that the value of ϵ does not depend on x. From the way ϵ has been chosen, there can be at most one distinct maximum over the interval $[x, x + \epsilon]$.



(a) Case 1: $\overline{\sigma}_{\hat{T}} \notin [x, x+\epsilon)$



(b) Case 2: $\overline{\sigma}_{\hat{T}} \in [x, x+\epsilon)$

Figure 3·3: Illustration of the two cases considered in the proof of Lemma 3.3.2

In the rest of this proof, let $\hat{T}$ denote the optimal threshold value at x (if there are more than one we can choose any). We will complete our proof by distinguishing between two cases, as illustrated in Fig. 3·3:

Case 1: $\overline{\sigma}_{\hat{T}} \notin [x, x+\epsilon)$.

Given the unimodality of $W(p, \sigma, \hat{T})$, this function must be either decreasing or increasing with respect to σ in the interval $[x, x+\epsilon]$. Furthermore, it must also be true that $x < \overline{\sigma}_{\hat{T}}$. Otherwise, if $x \geq \overline{\sigma}_{\hat{T}}$, through the way we have defined $\overline{\sigma}_{\hat{T}}$ in Eq. (3.12) we would have $W^*(p, x) = W(p, x, \hat{T}) < W(p, \overline{\sigma}_{\hat{T}}, \hat{T})$, which is a contradiction to

Lemma 3.3.1, which we can rewrite in the following form:

$$W^*(p, x) \geq W(p, \overline{\sigma}_{\hat{T}}, \hat{T}), \quad \forall x \geq \overline{\sigma}_{\hat{T}}.$$

Thus $W(p, \sigma, \hat{T})$ cannot be decreasing but must be increasing in σ over the interval $[x, x + \epsilon)$. By definition of optimality:

$$W^*(p, x + \epsilon) \geq W(p, x + \epsilon, \hat{T}) > W(p, x, \hat{T}) = W^*(p, x).$$

Case 2: $\overline{\sigma}_{\hat{T}} \in [x, x + \epsilon)$.

$W(p, \sigma, \hat{T})$ attains its maximum value over $[x, x+\epsilon)$ at $\overline{\sigma}_{\hat{T}}$. Given the unimodality of $W(p, \sigma, \hat{T})$ with respect to σ, the revenue function must be increasing on the interval $[x, \overline{\sigma}_{\hat{T}}]$. Next we show that the revenue must remain increasing over $[\overline{\sigma}_{\hat{T}}, x + \epsilon)$ for at least one other fixed threshold policy, which we prove by contradiction. Suppose that at $\sigma = \overline{\sigma}_{\hat{T}}$ there exists no threshold policy under which the revenue rate is both increasing and greater than or equal to $W(p, \sigma, \hat{T})$. Then, the revenue function under the optimal policy must be decreasing right after $\overline{\sigma}_{\hat{T}}$ as it is continuous in σ (see also proof of Theorem 3.3.1). This contradicts Lemma 3.3.1. Hence, there must exist at least one other threshold policy $A = T'$ such that the revenue rate under this new threshold value $W(p, \overline{\sigma}_{\hat{T}}, T')$ is increasing and satisfies $W(p, \overline{\sigma}_{\hat{T}}, T') \geq W(p, \overline{\sigma}_{\hat{T}}, \hat{T})$. Since the interval $[x, x+\epsilon)$ contains at most one distinct maximum, $W(p, x, T')$ must remain increasing over the interval $[\overline{\sigma}_{\hat{T}}, x + \epsilon)$. Then we can conclude

$$\begin{aligned} W^*(p, x + \epsilon) &\geq W(p, x + \epsilon, T') > W(p, \overline{\sigma}_{\hat{T}}, T') \\ &\geq W(p, \overline{\sigma}_{\hat{T}}, \hat{T}) > W(p, x, \hat{T}) = W^*(p, x). \end{aligned}$$

Having shown that $W^*(p, x + \epsilon) > W^*(p, x)$ for $\epsilon < d$ under both cases, we can finally proceed with making the connection between our proof and the lemma by first stating:

$$W^*(p,\sigma_1) > W^*(p,\sigma_1-\epsilon) > W^*(p,\sigma_1-2\epsilon) > \ldots > W^*(p,\sigma_1-k\epsilon), \tag{3.13}$$

where k is the largest integer such that $\sigma_1 - k\epsilon > \sigma_2$. Since we can take any $\epsilon < d$, we can choose one final $\epsilon' = \sigma_1 - k\epsilon - \sigma_2 < d$, such that:

$$W^*(p,\sigma_1-k\epsilon) = W^*(p,\sigma_2+\epsilon') > W^*(p,\sigma_2). \tag{3.14}$$

Combining Eqs. (3.13) and (3.14) gets us:

$$W^*(p,\sigma_1) > W^*(p,\sigma_2).$$

□

Proceeding with the proof of our theorem, we show that as long as the price is lowered by less than a certain amount, the relationship established in the Lemma 3.3.2 can be extended to different prices such that $W^*(p',\sigma(p')) > W^*(p,\alpha\sigma(p))$ where $p' < p$.

Proof of Theorem 3.3.1 Through Lemma 3.3.2 we know that the following inequality holds:

$$W^*(p,\sigma(p)) > W^*(p,\alpha\sigma(p)). \tag{3.15}$$

For a fixed threshold value T, the revenue takes the form:

$$W(p,\sigma(p),T) = (1-B_2(\lambda,\sigma(p),T))\sigma(p)p + (1-B_1(\lambda,\sigma(p),T))\lambda K, \tag{3.16}$$

where

$$B_1(\lambda,\sigma(p),T) = \frac{\frac{(\lambda+\sigma(p))^T\lambda^{C-T}}{C!}}{\sum_{n=0}^{T-1}\frac{(\lambda+\sigma(p))^n}{n!} + (\lambda+\sigma(p))^T\sum_{n=T}^{C}\frac{\lambda^{n-T}}{n!}},$$

and

$$B_2(\lambda, \sigma(p), T) = \frac{(\lambda + \sigma(p))^T \sum_{n=T}^{C} \frac{\lambda^{n-T}}{n!}}{\sum_{n=0}^{T-1} \frac{(\lambda+\sigma(p))^n}{n!} + (\lambda + \sigma(p))^T \sum_{n=T}^{C} \frac{\lambda^{n-T}}{n!}},$$

the derivation of which is given in (Mutlu et al., 2009). Since the respective blocking probabilities of primary secondary users $B_1(\cdot)$ and $B_2(\cdot)$ are a function of p through $\sigma(p)$, which is assumed to be continuous in p, we conclude from Eq. (3.16) that $W(p, \sigma(p), T)$ is also continuous in p.

From the way we have defined the optimal access policy in Eq. (3.3), $W^*(p, \sigma(p))$ is also continuous in p as we consider a finite set of possible values which T can take (Kaczor and Nowak, 2001, pp. 11&135).

First let us assume that there exists a $\hat{p} \in (p^{BE}, p)$ such that

$$W^*(\hat{p}, \sigma(\hat{p})) \geq W^*(p, \sigma(p)).$$

Then it follows by Eq. (3.15) that

$$W^*(\hat{p}, \sigma(\hat{p})) > W^*(p, \alpha\sigma(p))$$

and p' can be set equal to $\hat{p}$. On the other hand, assume that there exists no such price $\hat{p} < p$ for which

$$W^*(\hat{p}, \sigma(\hat{p})) \geq W^*(p, \sigma(p)).$$

This implies that the revenue is monotonically increasing for all $\hat{p} < p$ such that:

$$W^*(\hat{p}, \sigma(\hat{p})) < W^*(p, \sigma(p)). \tag{3.17}$$

Then by continuity, the following can be stated for $W^*(p, \sigma(p))$: $\forall \epsilon > 0,\ \exists \delta(\epsilon, p) > 0$

s.t. if $|p - \hat{p}| < \delta$ then

$$|W^*(p, \sigma(p)) - W^*(\hat{p}, \sigma(\hat{p}))| < \epsilon.$$

Making use of Eq. (3.17) and our assumption that $\hat{p} < p$, we can remove the absolute value from the previous equation and simplify it to:

$$W^*(p, \sigma(p)) - W^*(\hat{p}, \sigma(\hat{p})) < \epsilon. \tag{3.18}$$

Taking $\epsilon = W^*(p, \sigma(p)) - W^*(p, \alpha\sigma(p))$ and cancelling the terms $W^*(p, \sigma(p))$ on both sides of the inequality (3.18) we obtain $-W^*(\hat{p}, \sigma(\hat{p})) < -W^*(p, \alpha\sigma(p))$. Multiplying both sides by -1, the equation finally takes the form $W^*(\hat{p}, \sigma(\hat{p})) > W^*(p, \alpha\sigma(p))$ and p' can be set equal to $\hat{p}$. ■

Theorem 3.3.1 states that if a provider profits at a given price, obtaining the entire secondary demand at that price is strictly more profitable than obtaining part of the demand at a slightly higher price. This property reflects an incentive for each provider to unilaterally deviate from offering the same price as its opponent, provided that the price is strictly above its break-even price. This best response dynamics is illustrated in Figure 3·4 and the resulting market equilibrium is formally analyzed in the next section.

3.4 Market Equilibrium

Having identified the best response of a network provider under coordinated access in Theorem 3.3.1 in the previous section, we now proceed to establish the market equilibrium. Given initial prices p_1 and p_2 such that $p_i > p_i^{BE}$, $i = 1, 2$, the two providers will lower their prices in turn. This process continues until the market price drops so low that the provider with the higher break-even price finds himself unable

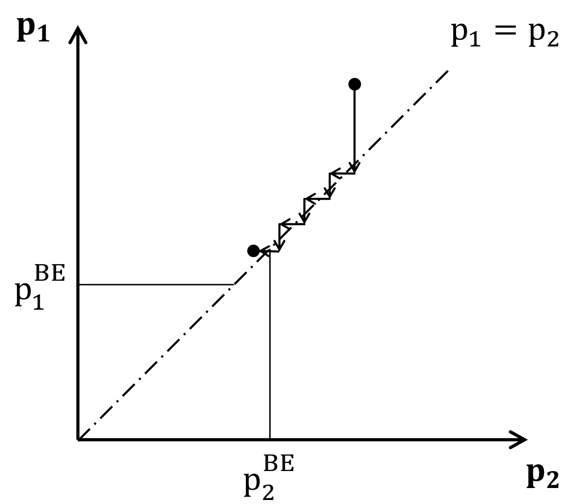


Figure 3·4: Representation of a price war as a result of best response dynamics under coordinated access.

to lower its price any further without incurring a net loss.

We start with a characterization for competitive equilibria in the considered scenario. However, before we do that, it is important to recall the formal definition of a Nash equilibrium.

Definition 3.4.1 *A pricing strategy profile* (p_1, p_2) *is a Nash equilibrium for rewards* $R_i(p_1, p_2)$ *if and only if*

$$R_1(p_1, p_2) = \max_p R_1(p, p_2)$$

and

$$R_2(p_1, p_2) = \max_p R_2(p_1, p).$$

Next, we provide a theorem which identifies possible market outcomes in terms of Nash equilibria.

The first part of the theorem is concerned with the case when one provider (without loss of generality provider 1) has strictly lower break-even price than the other provider. In the theorem, we show that the provider with the lower break-even price captures the entire market by pricing below its competitor's break-even price. However, when the price is continuous, it is impossible to provide an exact price that achieves this best response. Hence, following a well-known approach used in game theory to address this technicality (Osbourne, 2004, pages 64-67), we assume that each provider's price is a multiple of a sufficiently small discretization step ϵ.

Additionally, the exact value of the equilibrium price p_1 depends on where provider 1's revenue is maximized over the interval $[p_1^{BE}, p_2^{BE} - \epsilon]$. We formally define this maximum as the following:

$$\overline{W}_1^* = \max_{p \in [p_1^{BE}, p_2^{BE} - \epsilon]} W_1^*(p, \sigma(p)). \tag{3.19}$$

Note that the revenue may attain this maximum at several prices on the interval, which we denote by the following set:

$$\mathcal{P} = \mathop{argmax}_{p \in [p_1^{BE}, p_2^{BE} - \epsilon]} W_1^*(p, \sigma(p)). \tag{3.20}$$

The other provider is unable to underbid its competition in a profitable fashion. Hence it opts for any price that doesn't capture the secondary demand. In the equilibrium this price must also not give an incentive to the winner to deviate to a higher price. We next define the lowest price provider 2 can choose for which there exists an incentive for provider 1 to deviate from $\mathcal{P}$:

$$p^{max} \triangleq \mathop{\mathrm{arginf}}_{\hat{p} \geq p_2^{BE}} \left\{ \max_{p \in [p_1^{BE}, \hat{p}]} W_1^*(p, \sigma(p)) > \overline{W}_1^* \right\}. \tag{3.21}$$

If no such price exists, then we simply set $p^{max} = \infty$. Then this price effectively limits the price choice of provider 2 from above. Setting any price above p^{max} creates an incentive for provider 1 to deviate, thus disturbing the equilibrium. If provider 2 were to choose a price $p_2 > p^{max}$, then what follows is that provider 1 raises its price to this new maximizing price. However, provider 2 would then respond with underbidding provider 1 as a result of Theorem 3.3.1.

The second part of the theorem concerns the symmetric case when both providers have the same break-even price. In that case the unique Nash equilibrium outcome is defined by both providers charging their break-even prices, unable to capture the entire market due to profitability constraints.

Theorem 3.4.1 ***(Nash Equilibria)***

(a) If $p_1^{BE} < p_2^{BE}$ *then one or more Nash equilibria exist and have the strategy profile* (p_1, p_2) *where*

$$p_1 \in \mathcal{P}$$

$$p_2 \in (p_1, p^{max}),$$

where $\mathcal{P}$ *is as given by Eq. (3.20) and* p^{max} *by Eq. (3.21).*

(b) If $p_1^{BE} = p_2^{BE}$ *then there exists a unique Nash Equilibrium given by the strategy profile* (p_1, p_2) *such that*

$$p_1 = p_2 = p_1^{BE}.$$

Proof. We will prove the the two parts of Theorem 3.4.1 separately, first when $p_1^{BE} < p_2^{BE}$ and second when $p_1^{BE} = p_2^{BE}$. Under each case, we will demonstrate that the price pairs described in the theorem give the Nash equilibria by proving that neither provider $i = 1, 2$ can increase its reward $R_i(p_1, p_2)$ by employing any other strategy profile.

Part 1 - $p_1^{BE} < p_2^{BE}$ In a given Nash equilibrium the pricing strategy each provider chooses is given by:

$$p_1 \in \mathcal{P}$$

and

$$p_2 \in (p_1, p^{max}). \tag{3.22}$$

Under these strategies provider 1's reward is

$$R_1(p_1, p_2) = \overline{W}_1^* > W_1^*(p_1, 0),$$

where $\overline{W}_1^*$ is given by Eq. (3.19) and $W_1^*(p_1, 0)$ represents the base revenue rate provider 1 collects from the primary users in the absence of secondary users. Thus provider 1 collects a positive revenue from capturing the entire secondary market above its break-even price. On the other hand, provider 2 is unable to attract any secondary demand and faces the reward:

$$R_2(p_1, p_2) = W_2^*(p_2, 0).$$

We first analyze the possible increases in reward when provider 2 chooses other price strategies.

Suppose provider 2 chooses any price $p_2' < p_1$. Then provider 2 captures the secondary demand but since $p_2' < p_1 < p_2^{BE}$ this is a non-profitable price. Hence provider 2 choose to implement a lock-out policy which is reflected in the reward:

$$R_2(p_1, p_2') = W_2^*(p_2', \sigma(p_2')) = W_2^*(p_2, 0),$$

by Theorem 3.2.1. Therefore $R_2(p_1, p_2') = R_2(p_1, p_2)$.

Now suppose provider 2 chooses any price $p_2' \geq p^{max}$, which we have previously defined in Eq. (3.21). This action does not change the reward of provider 2 as it remains in a position where it capture no secondary demand. Hence, $R_1(p_1, p_2') = R_1(p_1, p_2)$.

Having proven provider 2 has no incentive to deviate, we shift our focus to provider 1.

If provider 1 chooses a price $p_1' > p_2$, this results in the loss of the secondary demand and its reward becomes $R_1(p_1', p_2) = W_1^*(p_1', 0) = W_1^*(p_1, 0) < R_1(p_1, p_2)$.

If provider 1 chooses a price $p_1' = p_2$, it shares the secondary demand with provider 2 and its reward becomes $R_1(p_1', p_2) = W_1^*(p_2, \alpha_1 \sigma(p_2))$. By Theorem 3.3.1 there exists an $\epsilon > 0$ such that:

$$W_1^*(p_2, \alpha_1 \sigma(p_2)) < W_1^*(p_2 - \epsilon, \sigma(p_2 - \epsilon)),$$

hence $R_1(p_1', p_2) < R_1(p_1, p_2)$.

If provider 1 chooses a price $p_2^{BE} \leq p_1' < p_2$, this implies through Eq. (3.22) that $p_1' < p^{max}$. By the definition of p^{max} in Eq. (3.21), for any price $p_1' < p^{max}$ we have:

$$W_1^*(p_1', \sigma(p_1')) \leq \overline{W}_1^*.$$

Hence $R_1(p_1', p_2) = W_1^*(p_1', \sigma(p_1')) < R_1(p_1, p_2)$.

If provider 1 chooses a price $p_1' < p_1^{BE}$, it serves secondary demand at a non-profitable price and hence faces the reward $R_1(p_1', p_2) = W_1^*(p_1', \sigma(p_1')) = W_1^*(p_1, 0) < R_1(p_1, p_2)$.

Finally, if provider 1 chooses a price $p_1' \in [p_1^{BE}, p_2^{BE} - \epsilon]$ but $p_1 \notin \mathcal{P}$, from the way $\mathcal{P}$ is defined, the new reward is $R_1(p_1', p_2) = W_1^*(p_1', \sigma(p_1')) < \overline{W}_1^*$. Therefore $R_1(p_1', p_2) < R_1(p_1, p_2)$.

Part 2 - $p_1^{BE} = p_2^{BE}$ Since both providers are identical, we will only consider provider 1. Also, for the sake of notational simplicity we will drop the index on the break-even price and denote it by p^{BE}. Provider 1, when at the Nash equilibrium, chooses the price strategy $p_1 = p^{BE}$ and faces the reward $R_1(p_1, p_2) = W_1^*(p_1, 0)$.

We fix provider 2's strategy to $p_2 = p^{BE}$ and demonstrate that provider 1's reward does not improve by choosing any other action pair.

If provider 1 chooses a pricing strategy $p_1' > p^{BE}$, it faces a reward $R_1(p_1', p_2) = W_1^*(p_1, 0) = R_1(p_1, p_2)$.

If provider 1 chooses any pricing strategy $p_1' < p^{BE}$, by definition of p^{BE} it faces a reward $R_1(p_1', p_2) = W_1^*(p_1, \sigma(p_1)) = W_1^*(p_1, 0) = R_1(p_1, p_2)$.

Because of provider symmetry, the same proof follows for player 2.

Therefore the only Nash equilibrium is given by the price pair $p_1^{BE} = p_2^{BE}$, from which uniqueness also follows since the break-even price of each provider is unique.

Having shown that under both cases Nash equilibria exist and can not be different from what is stated in Theorem 3.4.1, we conclude our proof. □

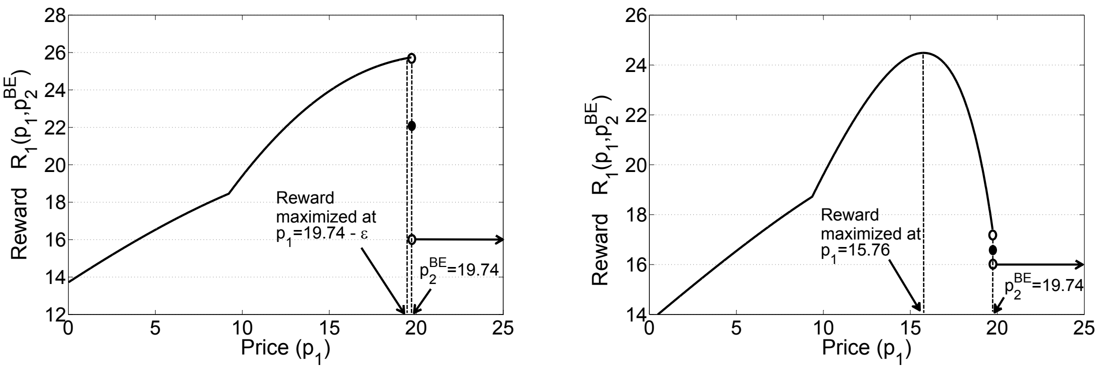
The following two examples aim to illustrate that qualitative differences in the placement of Nash equilibria are governed by the secondary demand function $\sigma(p)$. These examples are based on demand functions commonly used in the economics literature that are respectively exponentially and linearly decreasing with price (Talluri and Ryzin, 2004).

Example 3.4.1 Suppose that the secondary demand function follows a negative exponential demand $\sigma(p) = 10e^{-0.2p}$, which indicates sufficiently low price elasticity of demand so that the revenue rate remains increasing with price. We set the network parameters for both providers as:

$$(\lambda_1, C_1, K_1) = (1, 2, 20), \quad (\lambda_2, C_2, K_2) = (10, 5, 35),$$

which, through Eq. (3.9), yield $p_1^{BE} = 4.00, p_2^{BE} = 19.74$. Figure 3·5(a) demonstrates the low-elasticity property of provider 1's revenue rate function, $W_1^*(p_1, \sigma(p_1))$. The revenue rate is clearly maximized when the price is $p_1 = 19.74 - \epsilon$, at a price slightly below the other provider's break-even price.

Example 3.4.2 In this example we consider a linear demand function $\sigma(p) = 10 - 0.5p$. The network parameters and thus the break-even price are the same as in the previous example, which we omit. Under this new and faster decreasing demand



(a) Provider 1's reward maximized just below competitor's break-even price when secondary demand is $\sigma(p_1) = 10e^{-0.2p_1}$.

(b) Provider 1's reward maximized well below competitor's break-even price when secondary demand is $\sigma(p_1) = 10 - 0.5p_1$.

Figure 3·5: Different reward maximizing prices as provided in Examples 3.4.1 and 3.4.2.

function, we plot the revenue rate in Figure 3·5(b). The revenue rate achieves its maximum at $p_1 = 15.76$ after which it demonstrates high-elasticity and starts to decrease with price. This results in the revenue maximizing price being less than $p_2^{BE} = 19.74$. Therefore, facing such demand provider 1 would lower its price further below even though its competitor cannot match it without incurring a net loss, which demonstrates our result stated in Theorem 3.4.1(b).

Comparison with classical Bertrand duopoly. Theorem 3.4.1 essentially asserts that the equilibrium outcome of competition for secondary demand is a price war. Price wars are also typical outcomes in the classical Bertrand duopoly, hence it is worthwhile to put the two settings in perspective. In the Bertrand game, for a given price, both the revenue and the cost are linear functions of demand. In contrast, in the present setting neither revenue nor cost of secondary service are linear in secondary demand, primarily because both quantities rely heavily on blocking probabilities that are highly nonlinear in the demand. In addition, the Bertrand model precludes any capacity constraints and assumes that all demand can be satisfied, whereas the model of this chapter is centered on a fundamental limitation in capacity. Yet, interestingly, the equilibrium of the present game resembles (and, depending on the secondary demand function, may be identical to) the outcome of a Bertrand game in which marginal cost is constant and equal to the break-even price.

This similarity is a consequence of two nontrivial properties established in the present chapter: (i) insensitivity of break-even price against secondary demand, and (ii) Theorem 3.3.1, which indicates that having more secondary demand is always more favorable provided that secondary service is priced above break-even price. Both properties, however, rely on the assumption of optimal coordination of secondary access and does not necessarily extend to arbitrary access policies, as illustrated in the sequel.

Extension to multiple providers. Equilibrium strategy profiles given in Theorem 3.4.1 can be generalized to an arbitrary number of providers competing for the secondary demand, each with their own primary users, capacities and primary user rewards: Consider N such providers and let p_i^{BE} continue to represent the break-even price of provider i. Without any loss of generality, let us re-index the providers if necessary so that: $p_1^{BE} \leq p_2^{BE} \leq p_3^{BE} \leq \ldots \leq p_N^{BE}$.

Further we define $n = \max\{i : p_i^{BE} = p_1^{BE}\}$. Hence n is the number of providers that share the lowest break-even price. We generalize the two cases presented in Theorem 3.4.1:

- If $n > 1$ then any price profile $(p_1, p_2, \cdots, p_N)$ such that

$$p_i = p_1^{BE} \quad \text{for } i = 1, 2, \cdots, n$$

and

$$p_i > p_1^{BE} \quad \text{for } i = n+1, n+2, \cdots, N.$$

is a Nash equilibrium. In each such equilibrium providers $1, 2, \cdots, n$ service the secondary demand at their break-even prices thereby generating no additional profit. The secondary demand is split among these providers according to an arbitrary probability vector $[\alpha_1, \alpha_2, ..., \alpha_{n-1}, \alpha_n]$ where $\sum_i^n \alpha_i = 1, \alpha_i > 0$, which has no bearing on equilibrium prices. The remaining $N - n$ providers are not able to capture any secondary demand.

• If $n = 1$ then there is a single provider whose break-even price is lower than all the rest. In equilibrium this provider captures the entire secondary demand at a strictly profitable price, while the remaining $N - 1$ providers cannot serve any secondary demand. In particular Nash equilibria have the form:

$$p_1 \in \mathcal{P},$$

and

$$p_i \geq p_i^{BE} \quad \text{for } i = 2, \cdots, N,$$

and at least one provider $j \neq 1$ chooses a price such that $p_j < p^{max}$, where p^{max} is defined as in Eq. (3.21), so that there is no incentive for provider 1 to deviate from $\mathcal{P}$.

Quality of Service. QoS plays an important role wireless services. In this chapter, QoS is implicitly captured through the implementation of a coordinated access policy. Under this policy, the QoS experienced by primary users will naturally be higher than that experienced by secondary users, since the provider reserves a certain part of its network capacity for the exclusive use of primary users. A possible refinement of the model is through the introduction of penalties. Specifically, whenever a provider is unable to accommodate a service request of an incoming user, it would compensate the blocked user by paying a fee (or giving a discount). If the penalty is imposed only when primary users are blocked (secondary access is opportunistic, and therefore has no associated penalties when blocked), then our results still hold through a similar analysis.

3.5 Summary

In this chapter we analyzed the implementation of coordinated access, for which we demonstrated that the optimal access policy is of threshold type. We showed that each provider has a unique *break-even* price, above which profitability is guaranteed regardless of the secondary user demand response. We provided an explicit analytical formula for the break-even price, thus establishing its relationship with the system parameters (i.e., the primary load, primary reward, and system capacity). The break-even price of each provider is independent of the system parameters of other providers. Interestingly, the break-even price is, in general, significantly smaller than the primary reward. Thus, the break-even price is less than 1% of the primary reward if the primary load is below 68% and the number of channels C exceeds 32. Even at the critical load where the primary load is equal to the system capacity (i.e., $\lambda = C$), the break-even price remains below 20% of the primary reward for $C \geq 16$.

Next, using the notion of Nash equilibrium, we formalized the possible outcomes resulting from a non-cooperative game in which optimal coordinated access is implemented by two or more network providers. We explained how the best response dynamics of each provider reflect a price war, in which each provider is driven into advertising a price slightly below that of its competitors as long as this price is above the break-even price. This price war leads to a single provider (that with the lowest break-even price) capturing the entire secondary spectrum market. Although the demand function does not play a role in determining the identity of the winning provider, we showed that is does affects the revenue-maximizing price for that provider and the placement of the Nash equilibria. In the case of multiple providers sharing the same lowest break-even price, the game results in a market equilibrium in which none of the providers make profits.

Chapter 4

Uncoordinated Access

In this chapter we reconsider the question profitable pricing of secondary access and competition between network providers under an uncoordinated access setting. We first consider the implementation of such a policy under the stochastic nature of network traffic which was utilized in the previous chapter. Under the stochastic model we demonstrate that the break-even price and market outcomes, unlike in the coordinated access case, highly depend on the specific shape of the secondary demand function and therefore an analysis is difficult to conduct. We therefore present a fluid approximation to model network traffic. Using this approximation model, we establish profitability conditions and equilibrium outcomes. In addition to the break-even price, uncoordinated access results in a market-sharing price which is a measure of providers' willingness to share the secondary demand in equilibrium. We start the chapter with the uncoordinated access under stochastic traffic.

4.1 Uncoordinated Access with Stochastic Traffic

In this section we consider equilibrium regimes that arise when competing providers grant uncoordinated access to secondary demand. We shall argue that such equilibria can be drastically different than those under an optimal coordinated access.

Under uncoordinated access, a provider does not differentiate between primary and

Results presented in this chapter appear in part in (Kavurmacioglu et al., 2014a),(Kavurmacioglu et al., 2014b),(Kavurmacioglu et al., 2014c).

secondary users in granting spectrum access requests. In turn, both types of users experience the same blocking probability. This probability depends on the aggregate demand and system capacity, and can be computed using standard techniques in teletraffic. Namely, when provider i serves secondary demand σ, the two blocking probabilities are

$$B_{i,2}(\lambda_i, \sigma, A_i) = B_{i,1}(\lambda_i, \sigma, A_i) = E(\lambda_i + \sigma, C_i),$$

where $E(\lambda_i + \sigma, C)$ is the Erlang-B formula.

The revenue rate of provider i, when serving secondary demand σ by charging p_i per admitted request, is then given by

$$\hat{W}_i(p_i, \sigma) = (1 - E(\lambda_i + \sigma, C_i))\sigma p_i + (1 - E(\lambda_i + \sigma, C_i))\lambda_i K_i, \tag{4.1}$$

where the first term corresponds to the reward rate collected from secondary users that gain admission to the network, while the second term corresponds to the reward rate collected from the serviced primary users. (Here and in the rest of this section we will consistently use the symbol ˆ to indicate the quantities associated with uncoordinated access.) Once again, for analyses in which we consider a single provider, we will drop index i from our notation for the sake of simplicity.

4.1.1 Profitability

We recognize $\hat{W}(p, 0)$ as the revenue rate of a provider when it does not serve any secondary demand. Similar to the profitability conditions for the optimal coordinated access case stated in Theorem 3.2.1, note that

$$\hat{W}(p, \sigma(p)) \geq \hat{W}(p, 0) \tag{4.2}$$

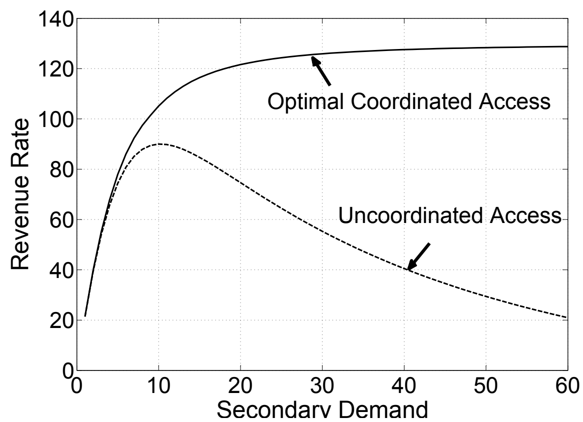


Figure 4·1: Revenue rates under optimal coordinated and uncoordinated access versus secondary demand - network parameters: $p_i = 30$, $\lambda_i = 13$, $C_i = 20$ and $K_i = 50$.

if and only if $p \geq \hat{p}^{BE}$, where $\hat{p}^{BE}$ satisfies:

$$\hat{p}^{BE} = \frac{(E(\lambda + \sigma(\hat{p}^{BE}), C) - E(\lambda, C))\lambda K}{(1 - E(\lambda + \sigma(\hat{p}^{BE}), C))\sigma(\hat{p}^{BE})}. \tag{4.3}$$

Hence the provider incurs loss and has no incentive to serve the secondary demand at a price below $\hat{p}^{BE}$. In turn $\hat{p}^{BE}$ is the break-even price of a provider under *uncoordinated access*.

It is instructive to compare the break-even prices under uncoordinated access and optimal coordinated access. Firstly, $\hat{p}^{BE} \geq p^{BE}$ because if the optimal admission policy does not yield positive profit from secondary demand then neither does any other policy. For typical parameters this inequality is strict. Consequently, providers need to charge a higher price to secondary users in order to avoid a net loss, which results in the tendency to bid higher prices under uncoordinated access. Secondly, in contrast to p^{BE}, the break-even price $\hat{p}^{BE}$ for uncoordinated access is given by an implicit equation that depends on the secondary demand $\sigma(p)$.

4.1.2 Market Sharing

While we established in Lemma 3.3.1 that market sharing is not favorable under optimal coordinated access, these results do not necessarily extend to a case when uncoordinated access is implemented. As a matter of fact, under an uncoordinated access implementation, whether the revenue rate increases or decreases by sharing secondary demand depends on another critical parameter we shall establish later.

Before we get into our analysis, it is insightful to compare how the revenue rates $\hat{W}(p,\sigma)$ and $W^*(p,\sigma)$ behave under uncoordinated and optimal coordinated access strategies. Figure 4·1 illustrates the two revenue rates for a range of secondary demand σ ,when all other parameters are fixed. When plotting both revenue rates, the secondary price p is chosen above both break-even prices so that the optimal revenue rate $W^*(p,\sigma)$ is strictly increasing in σ. As a by-product of optimality, $W^*(p,\sigma) \geq \hat{W}(p,\sigma)$ under all circumstances.

However, $\hat{W}(p,\sigma)$ has an important qualitative difference relative to its optimal counterpart: It increases for a range of secondary demand σ and decreases afterwards. This happens because for small σ, secondary users enhance revenue by using the leftover capacity from primary users, but as σ increases secondary access occurs at an increasing expense of primary access and that leads to a decline in revenue if primary users are more valuable (that is, if $p < K$). This property opens the possibility that $\hat{W}(p,\alpha\sigma) > \hat{W}(p,\sigma)$, in which case a provider has incentive to share secondary demand at prices higher than break-even. Consequently, it has a profound impact on the outcome of a competitive setting.

To formalize this intuition let us define p^{MS} as the solution to the following:

$$p = \frac{(E(\lambda+\sigma(p),C) - E(\lambda+\alpha\sigma(p),C))\lambda K}{(1-E(\lambda+\sigma(p),C))\,\sigma(p) - (1-E(\lambda+\alpha\sigma(p)))\,\alpha\sigma(p)}. \tag{4.4}$$

It can be verified directly from Eq. (4.1) that:

$$\hat{W}(p, \alpha\sigma(p)) \begin{cases} > \hat{W}(p, \sigma(p)) \text{ for } p < p^{MS} \\ \leq \hat{W}(p, \sigma(p)) \text{ for } p \geq p^{MS}. \end{cases} \tag{4.5}$$

Thus, only up to the price value p^{MS}, any provider would benefit from a reduction in its secondary demand. The price p^{MS} can be interpreted as a *market sharing threshold* for the provider: Any price above this threshold renders secondary demand too valuable to share and warrants a price war. Below this threshold, the provider has an incentive to remain at an equilibrium that reflects market sharing, provided the price satisfies the initial profitability condition in Eq. (4.2), which translates to being above the break-even price $\hat{p}^{BE}$.

4.1.3 Profitable Sharing Interval

In this section, we seek to determine the relationship between the maximum market sharing price p^{MS} and the break-even price $\hat{p}^{BE}$. In particular, if one can show that one price is always greater than the other, this can greatly simplify the results by ruling out or strictly establishing a preference to share the secondary market before making a negative profit. We present our results in the next theorem for the special case of fixed demand (we later present numerical evidence that similar results should hold for elastic demand):

Lemma 4.1.1 *For a fixed secondary demand such that $\sigma(p) = \sigma$, the following price relationships always hold under an uncoordinated access policy:*

$$\hat{p}^{BE} \leq p^{MS} < K. \tag{4.6}$$

Proof.

a) First, we prove $\hat{p}^{BE} \leq p^{MS}$. Through Eqs. (4.3) and (4.4) we can rewrite this inequality in the following form:

$$\frac{1 - E(\lambda + \sigma(p), C)}{E(\lambda + \sigma(p), C) - E(\lambda, C)} \cdot \frac{E(\lambda + \sigma(p), C) - E(\lambda + \alpha\sigma(p), C)}{1 - E(\lambda + \sigma(p), C) - \alpha - \alpha E(\lambda + \alpha\sigma(p), C)} \geq 1. \quad (4.7)$$

Next, using the well-known recursive equation of the Erlang-B formula (Krishnan, Sep):

$$E(\lambda, C) = \frac{\lambda E(\lambda, C-1)}{C + \lambda E(\lambda, C-1)},$$

we can expand Eq. (4.7) and after some algebra and regrouping of the terms, we can show that the inequality $\hat{p}^{BE} \leq p^{MS}$ is equivalent to demonstrating that:

$$\alpha(\lambda + \sigma)E(\lambda + \sigma, C-1) + (1-\alpha)\lambda E(\lambda, C-1) \geq (\lambda + \alpha\sigma)E(\lambda + \alpha\sigma, C-1). \quad (4.8)$$

Define $g_{C-1}(\lambda) = \lambda \cdot E(\lambda, C-1)$, which represents the traffic loss rate when the arrival process is Poisson with rate λ. For Eq. (4.8) to hold we need:

$$\alpha g_{C-1}(\lambda + \sigma) + (1-\alpha) g_{C-1}(\lambda) \geq g_{C-1}(\lambda + \alpha\sigma). \quad (4.9)$$

Once can observe that Eq. (4.9) is by definition the convexity condition on the traffic loss as a function of the arrival rate, which is proven in (Krishnan, Sep). Therefore, $\hat{p}^{BE} \leq p^{MS}$.

b) We now show that the market sharing price is always less than the primary reward, *i.e.,* $p^{MS} < K$. Recalling Eq. (4.4), this is equivalent to the following:

$$\frac{(E(\lambda + \sigma, C) - E(\lambda + \alpha\sigma, C_i))\lambda}{(1 - E(\lambda + \sigma, C))\,\sigma - (1 - E(\lambda + \alpha\sigma))\,\alpha\sigma} < 1.$$

After some rearrangement of the terms and substituting $g_C(\lambda)$ for $\lambda \cdot E(\lambda, C)$, the inequality takes the form:

$$g_C(\lambda+\sigma) - g_C(\lambda+\alpha\sigma) < \sigma(1-\alpha).$$

Upon careful observation, this inequality condition holds if one can show that:

$$g'_C(\lambda) = \frac{dg_C(\lambda)}{d\lambda} < 1. \tag{4.10}$$

In the paper (Krishnan, Sep), it has been demonstrated that $g'_C(\lambda) \leq 1$ for $C \geq 0$. The equality condition stems from the fact that the induction proof starts from $C = 0$, for which $g_C(\lambda) = \lambda$ and hence $g'_C(\lambda) = 1$. If one would start the induction from $C = 1$, using the following recursive formulation of $g_C(\lambda)$

$$g_C(\lambda) = \frac{\lambda g_{C-1}(\lambda)}{C + \lambda g_{C-1}}, \tag{4.11}$$

provided in (Krishnan, Sep), one can show that $g_1(\lambda) = \lambda^2/1+\lambda$. Taking the derivative with respect to λ,

$$g'_1(\lambda) = \frac{\lambda^2+2\lambda}{\lambda^2+2\lambda+1} < 1. \tag{4.12}$$

Then following the same steps as in (Krishnan, Sep) one can show that $g'_C(\lambda) < 1$ for $C \geq 1$, which establishes Eq. (4.10). □

Lemma 4.1.1 establishes a fundamental relationship between the break-even and market sharing prices and the primary reward K, thus proving the existence of a profitable market sharing price interval. This interval plays a critical role in defining the market outcomes, as we shall demonstrate in the next section.

4.1.4 Equilibrium

Competitive equilibria under uncoordinated access can now be determined depending on the critical price values $\hat{p}_i^{BE}$ and p_i^{MS} of all providers i. Figure 4·2 illustrates a particular placement of these parameters for two providers. In the illustrated setting,

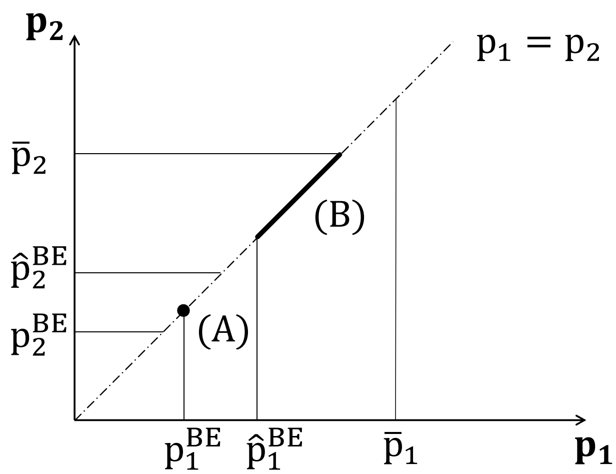


Figure 4·2: Point (A) represents the Nash equilibrium under optimal coordinated access, and the continuum of points in (B) is the set of Nash equilibria under uncoordinated access.

the market sharing intervals $[\hat{p}_1^{BE}, p_1^{MS}]$ and $[\hat{p}_2^{BE}, p_2^{MS}]$ have a non-empty intersection; therefore there exist common price values that are above break-even values and acceptable for market sharing for both providers. In turn, there is a continuum of equilibria strictly above the break-even prices.

We conclude this section with a numerical example concerning a symmetric setting.

Example 4.1.1 We consider two network providers with identical parameters: Primary arrival rate $\lambda_i = 13$, capacity $C_i = 20$, and revenue collected per serviced primary user $K_i = 50$. We continue to assume inelastic secondary demand whose value is chosen to be $\sigma = 20$. We assume that secondary demand splits equally in the case of equal prices, that is, $\alpha_1 = \alpha_2 = 0.5$.

The break-even price for coordinated access is computed as 0.91; hence by Theorem 3.4.1 the unique price equilibrium under coordinated access is $p_1 = p_2 = 0.91$ and no provider profits from secondary demand.

The break-even price for uncoordinated access is $\hat{p}_i^{BE} = 23.46$ and the market sharing threshold is $p_i^{MS} = 34.11$. Hence, any price profile (p, p) where p lies in the interval $[23.46, 34.11]$ constitutes a competitive equilibrium. For example, if provider

Access Policy	Equilibrium price	Equilibrium profit
Coordinated	$p_1 = p_2 = 0.91$	$P_1 = P_2 = 0$
Uncoordinated	$23.46 \leq \hat{p}_1 = \hat{p}_2 \leq 34.11$	$0 \leq \hat{P}_1 = \hat{P}_2 \leq 121.54$

Table 4.1: Equilibrium prices and resulting profits for the setting considered in Example 4.1.1.

$-i$ adopts the secondary price $p_{-i} = 30$ then

$$R_i(p_1, p_2) = \begin{cases} \hat{W}_i(p_i, \sigma) = 74.66 & \text{if } p_i = 29.99 \\ \hat{W}_i(p_i, 0.5\sigma) = 90.01 & \text{if } p_i = 30 \\ \hat{W}_i(p_i, 0) = 0 & \text{if } p_i > 30, \end{cases}$$

In particular $p_i = 30$ is the best response of provider i; and so the price profile $(30, 30)$ is a Nash equilibrium. A comparison of possible equilibria and associated profits under both access strategies is given in Table 4.1. It is worth noting that in the coordinated access policy, the price war drives the profits of both provider to zero by lowering the prices to the break-even price, which is the same for each provider. On the other hand, uncoordinated access gives a range of prices yielding positive profits in the sharing interval. Note that profit from primary users is not included in either case.

Interestingly, an uncoordinated access policy, which is sub-optimal to implement for a provider in isolation, results in competitive equilibria in which all providers are strictly better off than resorting to their optimal individual policies.

Example 4.1.2 This time, we consider an elastic demand to demonstrate that our results extend beyond inelastic secondary demand. Once again there are two network providers with identical parameters: Primary arrival rate $\lambda_i = 30$, capacity $C_i = 50$, and revenue collected per serviced primary user $K_i = 50$. We assume a secondary demand that is exponentially decreasing with the price $\sigma(p) = 80e^{-0.02p}$. We assume that secondary demand splits equally in the case of equal prices, that is, $\alpha_1 = \alpha_2 = 0.5$.

The break-even price for coordinated access is computed as 0.01; hence by Theorem 3.4.1 the unique price equilibrium under coordinated access is $p_1 = p_2 = 0.01$ and no provider profits from secondary demand.

The break-even price for uncoordinated access is $\hat{p}_i^{BE} = 20.06$ and the market sharing threshold is $p_i^{MS} = 33.39$. Hence, any price profile (p, p) where p lies in the interval $[20.06, 33.39]$ constitutes a competitive equilibrium. The same arguments discussed within Example 4.1.1 also apply here.

4.2 Fluid Approximation Model

In the previous section we have shown that the analysis of profitability and market equilibrium under stochastic traffic is highly dependent on the shape of the secondary demand function. In this section we relax the stochastic nature of the traffic and assume a fluid approximation model. We start with the description of the fluid model.

Consider two spectrum providers, where each provider i has a capacity C_i and a primary demand of volume λ_i, which generates a revenue of K_i units per service. These providers compete for a stream of secondary demand, whose volume depends of their pricing of secondary service as illustrated in Figure 4·3[1]. We assume a traffic model where if provider i receives a total demand of volume λ_i, then it can accommodate the volume $\min(C_i, \lambda_i)$. The excess demand $\max(\lambda_i - C_i, 0)$ does not generate any revenue for the provider.

The total demand for provider i consists of its primary demand λ_i and, depending on its pricing and the pricing of its competitor, a secondary demand σ_i. We shall assume that the two demand types access the capacity in an uncoordinated fashion, as suggested by documentation on private commons (Buddhikot, 2007)[2]. In this context, primary users could be viewed as high paying legacy users rather than users with higher priority. Specifically, the two types of demand share capacity *on equal*

[1]In order to keep the model generic, we shall not adopt a particular choice of units for capacity and demand at this point. Rather, we provide a discussion of possible choices at the end of this section.

[2]While the model considered in this chapter is applicable in Private Commons, it does not necessarily represent the only way to implement it.

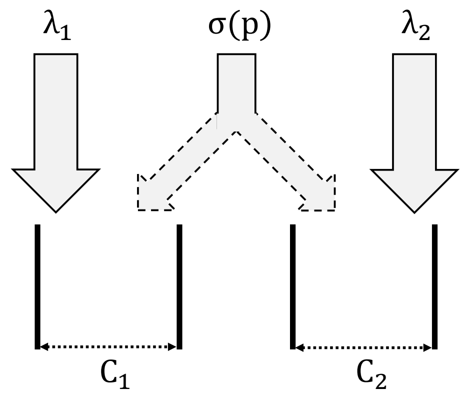


Figure 4·3: Market model: Two providers $i = 1, 2$, each with capacity C_i and fixed primary demand λ_i, compete for secondary demand stipulated by a general function of price p, $\sigma(p)$.

basis, such that if the demand of provider i is composed of two types with respective volumes λ_i and σ_i, then the overflow volume of *each type* is proportional to the intensity of demand of that type. That is, in view of our previous assumption, a fraction $\min\left(1, \frac{C_i}{\lambda_i+\sigma_i}\right)$ of each type of demand is actually accommodated. The steady-state primary and secondary demands, λ_i and σ_i, and the overflow assumption are consistent with fluid models. Such models have widely been used in the literature to characterize network traffic at the flow level (Kelly and Williams, 2004).

We denote the price that provider i charges per unit of serviced secondary demand by p_i. The volume of the secondary demand is assumed to be determined by the minimum price $\min(p_1, p_2)$ stipulated by the two providers. Specifically, the volume of secondary demand is $\sigma(\min(p_1, p_2))$, where $\sigma(\cdot)$ is the *demand function*. We make the mild assumption that this function is differentiable and non-increasing ($\frac{\partial}{\partial p}\sigma(p) \leq 0$). We shall also assume that there exists a positive demand when the service is offered for free ($\sigma(0) > 0$) and the demand eventually becomes zero as the price becomes arbitrarily high ($\lim_{p\to\infty} \sigma(p) = 0$).

It is assumed that the secondary demand is attracted to the provider that charges the lowest price. This behavior can be explained by *price aversion*, a concept employed in marketing management (Tellis and Gaeth, 1990). In the case when both providers charge the same price, the resulting secondary demand splits between the two providers according to an arbitrary but fixed probability vector $\boldsymbol{\alpha} = [\alpha_1, \alpha_2]$ such that $\alpha_1 + \alpha_2 = 1$ and $\alpha_i > 0$, $i = 1, 2$. Namely, in that case, each provider i receives a secondary demand of volume $\alpha_i \sigma(p_i)$. We will relax this assumption in Section 4.2.3, where instead of being randomly assigned the secondary demand will be split between the providers according to the accommodation levels.

If provider i receives a secondary demand of volume $\sigma(p_i)$, its overall revenue is given by:

$$W_i(p_i, \sigma(p_i)) \triangleq \; p_i \sigma(p_i) \min \left(1, \frac{C_i}{\lambda_i + \sigma(p_i)} \right) + K_i \lambda_i \min \left(1, \frac{C_i}{\lambda_i + \sigma(p_i)} \right). \quad (4.13)$$

In this case, the secondary *profit* (i.e., increment in revenue from secondary access) of the provider is:

$$\Pi_i(p_i, \sigma(p_i)) \triangleq W_i(p_i, \sigma(p_i)) - W_i(0, 0). \quad (4.14)$$

Since the secondary demand that a provider receives depends on the prices of both providers, so does the profit of the provider. We define the *reward* $R_i(p_i, p_{-i})$ of provider i as its profit when it charges secondary access p_i and its competitor charges p_{-i} units. Namely,

$$R_i(p_i, p_{-i}) \triangleq \begin{cases} \Pi_i(p_i, \sigma(p_i)) & \text{if } p_i < p_{-i} \\ \Pi_i(p_i, \alpha_i \sigma(p_i)) & \text{if } p_i = p_{-i} \\ \Pi_i(p_i, 0) & \text{if } p_i > p_{-i}. \end{cases}$$

In the interest of space, the discussion of the fluid model in this dissertation is limited to the case when each provider's network is underloaded prior to inclusion of any secondary demand, that is $\lambda_i < C_i$, but can be overloaded for low enough prices, that is $\lambda_i + \sigma(0) > C_i$. Though the omitted cases warrant their own respective analyses, those are arguably less challenging and practical. For instance, consider the case when the maximum possible total demand does not exceed the network capacity (i.e., $\lambda_i + \sigma(0) \leq C_i$). Since we assume that the secondary demand is non-increasing with price p_i advertised by the provider, the maximum secondary demand is given by $\sigma(0)$. If the maximum secondary demand generated by a zero price value does not exceed provider i's capacity such that $\lambda_i + \sigma(0) \leq C_i$ then it is also true that $\lambda_i + \sigma(p_i) \leq C_i$ for all price values $p_i \geq 0$. Then the network can accommodate the entire secondary demand at any price without losing any revenue collected from primary traffic.

On the other hand, if the primary demand already exceeds the capacity (i.e., $\lambda_i \geq C_i$), the network provider is already having difficulties in serving the primary traffic and is incurring an opportunity cost of the unserviced traffic volume. Then the revenue brought in by secondary demand would need to match or exceed the revenue per serviced primary demand (i.e., $p_i \geq K_i$) in order to have the network provider profitably replace some of its original traffic.

Discussion We provide next a possible interpretation of our model. The service capacity C_i can represent the number of sub-carriers in an OFDM modulation scheme used in LTE or the number of radio channels[3] available for assignment for voice or data traffic in common 3G standards (Paul et al., 2011). The steady primary and secondary demands, λ_i and $\sigma(p)$, and the overflow assumption are consistent with fluid models (Anick et al., 1982). Such models have widely been used in the literature to characterize network traffic at the flow level (Fred et al., 2001; Kelly and Williams, 2004; Hassidim et al., 2013). This assumption is substantiated by traffic

[3]This radio channel refers to any radio resource allocated to the user such as code, frequency or time slot.

measurements in cellular networks, which show that mean arrival rates do not show significant variations over the course of an hour (Paul et al., 2011; Willkomm et al., 2008). Obviously, specific values of λ_i and $\sigma(p)$ depend on the hour of the day or day of the week.

4.2.1 Characteristic Prices and Market Sharing Interval

In this section we present two characteristic prices and demand-invariant price relationships in a secondary spectrum markets. This section focuses on the viewpoint of a single provider. Therefore for simplicity, we omit the use of index i from of our notation throughout this section.

We define the break-even price $p^{BE}(\alpha)$ as the price at which the profit of a provider is zero when it attracts a fraction $0 < \alpha \leq 1$ of the total demand, namely $\Pi(p^{BE}, \alpha\sigma(p^{BE})) = 0$. We start off by providing a formal definition of a break-even price:

Definition 4.2.1 ***(Break-Even Price)*** *A price* $p^{BE}(\alpha) \geq 0$ *is called a break-even price if it satisfies the following conditions:*

$$\Pi(p^{BE}, \alpha\sigma(p^{BE})) = 0 \qquad \textit{and} \qquad \alpha\sigma(p^{BE}) > 0.$$

Note that the latter condition in the above definition is to rule out any price that does not generate any secondary demand.

We next define the *market sharing price* $p^{MS}(\alpha)$, that asserts whether a provider finds it desirable to share the secondary demand or not. Specifically, let

$$\Delta W(p) \triangleq W(p, \alpha\sigma(p)) - W(p, \sigma(p)).$$

Definition 4.2.2 ***(Market Sharing Price)*** *A price* $p^{MS}(\alpha) \geq 0$ *is called a market* *sharing price if the following is true:*

$$\Delta W(p) = 0 \qquad \textit{for } p = p^{MS}(\alpha).$$

These two prices characterize two important incentives for a network provider. We will show that the break-even price determines provider profitability, where any price set greater is guaranteed to result in a positive profit. We will also establish that, analogous to the relationship between the break-even price and provider profitability, a provider finds it undesirable to share the secondary demand at prices above the market sharing price, whereas the opposite is true for prices below the market sharing price. Having defined the break-even and market sharing prices, we can proceed with stating our main results in the following theorem:

Theorem 4.2.1 ***(Market Sharing Interval)*** *For any secondary demand function, satisfying the assumptions described in Section 4.2 and for all values of* $\alpha : 0 < \alpha \leq 1$*, there exists a price interval*

$$(\mathcal{P}) \equiv (p^{BE}(\alpha), p^{MS}(\alpha)),$$

such that for all $p \in (\mathcal{P})$*:*

1. $\Pi(p, \sigma(p)) > 0$,
2. $\Pi(p, \alpha\sigma(p)) > \Pi(p, \sigma(p))$.

Theorem 4.2.1 states that no matter the specific shape of a secondary demand function, the existence of the price interval $(\mathcal{P})$ at which a network provider is profitable and finds it preferable to share the secondary demand is guaranteed. In order to prove Theorem 4.2.1 we will first provide formulations for break-even and market sharing prices in Sections 4.2.1 and 4.2.1 respectively. Afterwards, we bring the proof of Theorem 4.2.1 in Section 4.2.1.

Profitability and Break-Even Price

In this section we seek to analyze a provider's profit and the resulting break-even price. Our result applies both to the cases when a network provider serves the entire secondary demand (i.e., $\alpha = 1$) and when it shares the market with another provider (i.e, $\alpha < 1$).

Since a break-even price is a measure of a provider's competitive ability in a price war, characterizing this price is important. The following lemma restricts the price interval on which a break-even price when the provider captures the entire secondary demand (*i.e.,* monopoly) lies:

Lemma 4.2.1 *For a given α such that $\lambda + \alpha\sigma(0) > C$, there exists a price $\bar{p}^{\alpha}$, which is the minimum price that satisfies $\alpha\sigma(p) = C - \lambda$. Then, any break-even price $p^{BE}(\alpha)$ satisfies the following inequality:*

1. *$p^{BE}(\alpha) \leq \bar{p}^{\alpha}$ for any demand function $\sigma(p)$.*

2. $$\lambda + \alpha\sigma(p^{BE}(\alpha)) \geq C. \tag{4.15}$$

Proof. The existence of $\bar{p}^{\alpha}$ follows from the assumption $\lambda + \alpha\sigma(0) > C$ and that the demand is non-increasing with the limit $\lim_{p\to\infty} \sigma(p) = 0$.

(1) Let p' be such a price that $\alpha\sigma(p') + \lambda \leq C$. Since we know that secondary demand is non-increasing in p it also follows that p' must satisfy the following inequality: $p' \geq \bar{p}^{\alpha}$. We know that setting price equal to p' results in a non-negative profit since by Eq.'s (4.13) and (4.14) we have that:

$$\Pi(p', \alpha\sigma(p')) = \alpha\sigma(p')p' \geq 0.$$

Given that any price greater than or equal $\bar{p}^{\alpha}$ yields a non-negative profit for a provider, we can conclude that $\bar{p}^{\alpha}$ is an upper bound on the break-even price $p^{BE}(\alpha)$

(i.e., $p^{BE}(\alpha) \leq \bar{p}^{\alpha}$).

(2) From part 1 of our proof we know that:

$$p^{BE}(\alpha) \leq \bar{p}^{\alpha}.$$

Then, through our assumption that the secondary demand is non-increasing in p, the following is also true:

$$\alpha\sigma(p^{BE}(\alpha)) \geq \alpha\sigma(\bar{p}^{\alpha}). \tag{4.16}$$

Thus, from Eq. (4.16) and the definition of $\bar{p}^{\alpha}$ we obtain: $\lambda + \alpha\sigma(p^{BE}(\alpha)) \geq C$. □

An intuitive explanation to Lemma 4.2.1 is that for all prices p such that $\lambda + \alpha\sigma(p) < C$, the overflow of either type of demand is zero. Thus, there is no associated penalty with serving additional secondary demand. However, once the excess demand becomes positive, a provider observes a trade-off between the revenue brought in by the secondary demand versus the potential revenue lost from the unserviced primary demand. The break-even price reflects the price at which both sides of this trade-off are equal.

Lemma 4.2.1 demonstrates that for all such values of α, including the monopolistic case when $\alpha = 1$, we can limit our analysis to those prices that satisfy (4.15). At these prices the fraction of both types of demand being accommodated is $C/(\lambda + \alpha\sigma(p))$. Then, we can remove the min operators from Eq. (4.13) and simplify Eq. (4.14) for the profit as follows:

$$\Pi(p, \alpha\sigma(p)) = \alpha\sigma(p)p \cdot \frac{C}{\lambda + \alpha\sigma(p)} + \lambda K \left(\frac{C}{\lambda + \alpha\sigma(p)} - 1 \right). \tag{4.17}$$

The following theorem, leveraging our previous results from Lemma 4.2.1 and Eq. (4.17), provides an equation that allows the computation of the break-even price $p^{BE}(\alpha)$ for the aforementioned values of α. The theorem also establishes the uniqueness of this price and the region of profitable prices.

Theorem 4.2.2 ***(Break-Even Price)***

1. *For a given* $0 < \alpha \leq 1$*, such that* $\lambda + \alpha\sigma(0) > C$*:*
   (a) *A break-even price* $p^{BE}(\alpha)$ *is a solution to the following equation*[4]*:*

$$p = \frac{(\alpha\sigma(p) + \lambda - C)\lambda K}{C\alpha\sigma(p)}. \tag{4.18}$$

   (b) *The break-even price* $p^{BE}(\alpha)$ *is unique.*

   (c) *The profit of a provider is such that:*

$$\begin{aligned} \Pi(p, \alpha\sigma(p)) > 0 \qquad & \text{if } p > p^{BE}(\alpha) \\ \Pi(p, \alpha\sigma(p)) < 0 \qquad & \text{if } p < p^{BE}(\alpha). \end{aligned}$$

2. *For a given* $0 < \alpha < 1$*, such that* $\lambda + \alpha\sigma(0) \leq C$*, the break-even price* $p^{BE}(\alpha)$ *is 0.*

Proof. (1) (a) We know that at a break-even price the profit is given by Eq. (4.17). In order to ensure $\Pi(p, \alpha\sigma(p)) = 0$, it can be verified through simple algebra that a price p needs to satisfy the following equation:

$$p = \frac{(\alpha\sigma(p) + \lambda - C)\lambda K}{C\alpha\sigma(p)}.$$

Furthermore, we know that at price $p^{BE}(\alpha)$, secondary demand will be positive by combining inequality (4.16) and the fact that $\lambda < C$:

$$\sigma(p^{BE}(\alpha)) \geq \sigma(\bar{p}) = C - \lambda > 0.$$

(b) We will proceed by demonstrating that the left hand side of Eq. (4.18) is strictly increasing with respect to p and the right hand side is non-increasing with

[4]This implicit equation can be solved with well-established fixed point iterations, such as Newton's Method.

respect to p, hence meaning that this equality only holds at a single value of p. Since the left hand side of Eq. (4.18) is p itself, we only need to prove that the right hand side is non-increasing. Under the assumption that $\sigma(p)$ is a differentiable and non-increasing function of p, taking the derivative of the right hand side with respect to p yields:

$$\begin{aligned}\frac{\partial}{\partial p}\left(\frac{(\alpha\sigma(p)+\lambda-C)\lambda K}{C\alpha\sigma(p)}\right) &= \left(\frac{1}{\alpha\sigma(p)} - \frac{\lambda+\alpha\sigma(p)-C}{\alpha^2\sigma^2(p)}\right)\alpha\sigma'(p)\left(\frac{\lambda K}{C}\right) \\ &= \left(\frac{C-\lambda}{\alpha^2\sigma^2(p)}\right)\alpha\sigma'(p)\left(\frac{\lambda K}{C}\right) \leq 0. \qquad (4.19)\end{aligned}$$

Eq. (4.19) holds because $\lambda < C$ and $\sigma'(p^{BE}(\alpha)) \leq 0$.

We also know that the lhs of Eq. (4.18) is continuous in p, which follows from the differentiability of the secondary demand $\sigma(p)$. Therefore, there can only be at most one solution for $p^{BE}(\alpha)$ that satisfies Eq. (4.18).

(c) From Eq. (4.17), it can verified that in order for $\Pi(p, \alpha\sigma(p)) > 0$ to hold, p needs to satisfy the following inequality:

$$p > \frac{(\alpha\sigma(p)+\lambda-C)\lambda K}{C\alpha\sigma(p)}.$$

In part (b) of our proof, we have demonstrated that the right hand side of Eq. (4.18) is non-increasing with respect to p. Therefore for $p' > p^{BE}(\alpha)$:

$$\frac{(\alpha\sigma(p^{BE}(\alpha))+\lambda-C)\lambda K}{C\alpha\sigma(p^{BE}(\alpha))} \geq \frac{(\alpha\sigma(p')+\lambda-C)\lambda K}{C\alpha\sigma(p')}.$$

Then, since $p^{BE}(\alpha)$ is the only value that satisfies Eq. (4.18),

$$p' > p^{BE}(\alpha) = \frac{(\alpha\sigma(p^{BE}(\alpha))+\lambda-C)\lambda K}{C\alpha\sigma(p^{BE}(\alpha))} \geq \frac{(\alpha\sigma(p')+\lambda-C)\lambda K}{C\alpha\sigma('p)}.$$

To show that $\Pi(p', \alpha\sigma(p')) < 0$ when $p' < p^{BE}(\alpha)$, the same argument follows in the reverse direction.

(2) For a given $0 < \alpha < 1$, such that $\lambda + \alpha\sigma(0) \leq C$, Eq. (4.14) simplifies to the following:

$$\Pi(p, \alpha\sigma(p)) = p\alpha\sigma(p).$$

Since $\sigma(0) > 0$ by assumption, the only price that satisfies both equations provided Definition 4.2.1 is $p = 0$. □

In the next lemma, we establish a useful bound on the break-even price $p^{BE}(1)$.

Lemma 4.2.2 *The break-even price when not sharing the secondary demand (*i.e., $\alpha = 1$*) is strictly smaller than the revenue generated by primary demand:*

$$p^{BE}(1) < K.$$

Proof. We can check this claim by taking a look at the right hand side of Eq. (4.18):

$$\frac{(\sigma(p) + \lambda - C)\,\lambda K}{C\sigma(p)}.$$

In order for the claim to hold, we need $(\sigma(p) + \lambda - C)\,\lambda < C\sigma(p)$, which can be rewritten as: $\lambda(\lambda + \sigma(p)) < C(\lambda + \sigma(p))$. This is true under our initial assumption $\lambda < C$. □

In general, there is no explicit expression for the break-even price for general demand functions. However, it allows us to characterize two distinct price regimes by identifying whether or not a price p generates a profit for the provider for any amount of secondary demand. We next provide an example with a simple demand function, where obtaining an explicit expression is rather straightforward.

Example 4.2.1 *We illustrate the relationship between the break-even price when* $\alpha = 1$ *(i.e., one provider captures the entire secondary demand) and network parameters under a constant elasticity secondary demand function,* $\sigma(p) = \frac{\sigma_0}{p}$*, where* $\sigma_0 > 0$ *is a constant.*

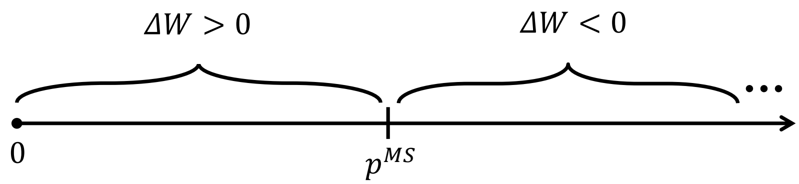


Figure 4·4: Market sharing revenue change regions in Theorem 4.2.3 with respect to the market sharing price $p^{MS}(\alpha)$.

Under this given demand we can simplify Eq. (4.18) and obtain the following explicit formula:

$$p^{BE}(1) = \frac{\sigma_0 \lambda K}{C\sigma_0 + \lambda K(C - \lambda)}. \tag{4.20}$$

We have effectively formulated and characterized the unique break-even price that determines a network provider's profitability. However, profitability alone is not enough to determine a market outcome. As was explained in the network model section, matching prices affects the reward a provider faces in a non-linear fashion. In the next section, we take into account the results of a provider choosing to share the market.

Market Sharing

We now turn our attention to the effects market sharing has on a provider's revenue. In the next theorem, we present our result on how market sharing affects a provider's profit. The theorem establishes the existence and uniqueness of the market sharing price $p^{MS}(\alpha)$ and provides an implicit equation to compute it. It also states that increased profit is achieved if and only if $p < p^{MS}(\alpha)$.

Theorem 4.2.3 ***(Market Sharing Price)*** *For any network provider there exists a unique market sharing price $p^{MS}(\alpha)$, which satisfies the following:*

1. If $\lambda + \alpha\sigma(K) \leq C$*,* $p^{MS}(\alpha)$ *is the solution to:*

$$p = \frac{(\lambda + \sigma(p) - C)\,\lambda K}{(C - \alpha(\lambda + \sigma(p)))\,\sigma(p)}. \tag{4.21}$$

2. If $\lambda + \alpha\sigma(K) > C$*,*

$$p^{MS}(\alpha) = K. \tag{4.22}$$

and for any given $p^{MS}(\alpha)$ *the following is true:*

$$\Delta W(p) > 0 \qquad \textit{for } p < p^{MS}(\alpha), \tag{4.23}$$

$$\Delta W(p) < 0 \qquad \textit{for } p > p^{MS}(\alpha). \tag{4.24}$$

Before we prove Theorem 4.2.3, we first establish several useful results that will later facilitate our proof. Since the general revenue function of a provider involves min operators, we need to make use of some auxiliary prices that will simplify the expressions of $W(p, \sigma(p))$ and $W(p, \alpha\sigma(p))$. In Lemma 4.2.1 we had already defined $\bar{p}^{\alpha}$ to be an auxiliary price that satisfies the equality $\lambda + \alpha\sigma(\bar{p}^{\alpha}) = C$. In this section we provide another such auxiliary price to simplify our analysis. We let $\bar{p}$ denote the price that satisfies the following equation:

$$\lambda + \sigma(\bar{p}) = C. \tag{4.25}$$

Since we assume that the secondary demand $\sigma(p)$ is non-increasing in p for all $0 < \alpha < 1$ it follows that $\bar{p}^{\alpha} < \bar{p}$, which is illustrated for a generic demand function in Figure 4·5.

By defining these prices we have effectively divided prices into three separate regions, i.e. $[0, \bar{p}^{\alpha})$, $[\bar{p}^{\alpha}, \bar{p})$, $[\bar{p}, \infty)$, in each of which we have a simplified revenue function. Now, we can start our analysis on how the revenue changes depending on which region a given price value p falls in.

a) We first consider the price region $\{p : p \geq \bar{p}\}$. Note that the price inequality corresponds to when the total demand under price p does not exceed the provider's service capacity. In the following lemma we establish that in this region, it is never optimal for a provider to choose market sharing.

Lemma 4.2.3 *Assume* $p \geq \bar{p}$, *then*

$$\Delta W(p) < 0. \tag{4.26}$$

Proof. Note that our assumption $p \geq \bar{p}$ is equivalent to stating that $\lambda + \sigma(p) < C$. Since $\bar{p} > \bar{p}^{\alpha}$, it is also true that $p > \bar{p}^{\alpha}$. Then, the total arrival under market sharing is also less than provider i capacity (i.e., $\lambda + \alpha\sigma(p) < C$). Simplifying Eq. (4.13) under these assumptions, we get:

$$\begin{aligned}
\Delta W &= W(p, \alpha\sigma(p)) - W(p, \sigma(p)) \\
&= \Big(\alpha\sigma(p)p + \lambda K\Big) - \Big(\sigma(p)p + \lambda K\Big) \\
&= \alpha\sigma(p)p - \sigma(p)p < 0.
\end{aligned}$$

Therefore we conclude that Eq. (4.26) holds. □

b) Next, we cover the price region $\{p : p < \bar{p}^{\alpha}\}$. Since price values need to be non-negative, we do not consider the case $\bar{p}^{\alpha} = 0$. In this price interval, there are two cases two consider. If the value of K happens to be in this region, then the revenue change is positive for price values below K and negative for price values above K. If K does not fall in this price interval, then the revenue change is always positive and thus a provider will always find it desirable to share the market. We formalize these results in the following lemma:

Lemma 4.2.4 *Assume* $\bar{p}^{\alpha} > 0$ *and* $p < \bar{p}^{\alpha}$, *then*

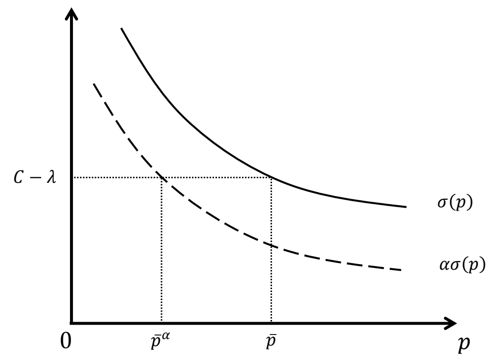


Figure 4·5: An illustration of the prices $\bar{p}$ and $\bar{p}^\alpha$ under a generic secondary demand.

(a) If $\bar{p}^\alpha \geq K$*:*

$$\Delta W(p) > 0 \qquad \text{if } p < K; \tag{4.27}$$

$$\Delta W(p) = 0 \qquad \text{if } p = K; \tag{4.28}$$

$$\Delta W(p) < 0 \qquad \text{if } p > K. \tag{4.29}$$

(b) If $\bar{p}^\alpha < K$*:*

$$\Delta W(p) > 0 \quad \forall p < \bar{p}^\alpha. \tag{4.30}$$

Proof. Note that our assumption $p < \bar{p}^\alpha$ is equivalent to stating that:

$$\lambda + \alpha\sigma(p) \geq C.$$

Since $\bar{p}^\alpha < \bar{p}$, it must also be true that $p < \bar{p}$. Then the combined demand without market sharing is greater than the provider's capacity (i.e., $\lambda + \sigma(p) \geq C$). Simplifying Eq. (4.13) under these assumptions, we obtain:

$$\begin{aligned}\Delta W &= W(p, \alpha\sigma(p)) - W(p, \sigma(p)) \\ &= \frac{\alpha\sigma(p)pC}{\alpha\sigma(p) + \lambda} + \frac{\lambda KC}{\alpha\sigma(p) + \lambda} - \frac{\sigma(p)pC}{\sigma(p) + \lambda} - \frac{\lambda KC}{\sigma(p) + \lambda}.\end{aligned}$$

After rearrangement we get:

$$\Delta W = \frac{(1-\alpha)\sigma(p)\lambda C}{(\alpha\sigma(p)+\lambda)(\sigma(p)+\lambda)}(K-p). \tag{4.31}$$

Eq. (4.31) only takes on the value zero when $p = K$. Additionally for price values $p < K$, ΔW is positive and for $p > K$, ΔW is negative. □

Remark 4.2.1 Lemma 4.2.4 considers prices for which the reduced secondary demand, when combined with the primary demand, exceeds the capacity of a provider. In that case, this provider can increase its revenue at prices up to K if $\bar{p}^{\alpha} \geq K$ or all prices p if $\bar{p}^{\alpha} < K$, by choosing to share the market with another provider. On the other hand, if $\bar{p}^{\alpha} \geq K$, choosing to share the market decreases the revenue at prices greater than K.

c) Finally, we cover the price region between the regions covered in parts a) and b), such that $\{p : \bar{p}^{\alpha} \leq p < \bar{p}\}$. Note that these are price values such that the combined demand of the primary and secondary types exceed the service capacity without market sharing and do not exceed the service capacity with market sharing. Once again, similar to the previous case, the revenue change depends on the relationship between K and how this price interval is defined. If $K \geq \bar{p}^{\alpha}$, then the market sharing price lies on this interval and the revenue change is negative for price values above and positive for price values below. Otherwise, the revenue change is always in the negative direction and market sharing is not desirable. We present the following lemma in this light:

Lemma 4.2.5 *Assume $\bar{p}^{\alpha} \leq p < \bar{p}$. Then,*

1) If $\bar{p}^{\alpha} \leq K$:

$$\begin{aligned} \Delta W(p) &> 0 \qquad \text{if } p < p^{MS}(\alpha); \\ \Delta W(p) &\leq 0 \qquad \text{if } p \geq p^{MS}(\alpha), \end{aligned} \tag{4.32}$$

where $p^{MS}(\alpha)$ denotes the solution to the following equation:

$$p = \frac{(\lambda + \sigma(p) - C)\, \lambda K}{(C - \alpha(\lambda + \sigma(p)))\, \sigma(p)}. \tag{4.33}$$

2) If $\bar{p}^{\alpha} > K$:

$$\Delta W(p) < 0. \tag{4.34}$$

Proof. Note that $\bar{p}^{\alpha} \leq p < \bar{p}$ is equivalent to stating that $\lambda + \sigma(p) > C$ and $\lambda + \alpha\sigma(p) \leq C$. Under these conditions the revenue change function becomes:

$$\Delta W(p) = \Big(\alpha\sigma(p)p + \lambda K\Big) - \Big(\sigma(p)p\frac{C}{\lambda + \sigma(p)} + \lambda K\frac{C}{\lambda + \sigma(p)}\Big).$$

Regrouping yields:

$$\Delta W(p) = \Big(\frac{\alpha(\lambda + \sigma(p)) - C}{\lambda + \sigma(p)}\sigma(p)p\Big) + \Big(\frac{\lambda + \sigma(p) - C}{\lambda + \sigma(p)}\lambda K\Big).$$

(1) Noting that $\alpha(\lambda + \sigma(p)) < C$, it can be verified that $\Delta W(p) = 0$ is satisfied by the solution of the following implicit equation:

$$p = \frac{(\lambda + \sigma(p) - C)\lambda K}{(C - \alpha(\lambda + \sigma(p)))\sigma(p)}. \tag{4.35}$$

Furthermore, one can check that Eq. (4.35) is satisfied by a unique price p. Multiplying both sides of Eq. (4.35) with the first term in the denominator we obtain:

$$(C - \alpha(\lambda + \sigma(p)))p = \frac{(\lambda + \sigma(p) - C)\lambda K}{\sigma(p)}. \tag{4.36}$$

Taking the derivative of the left hand side of Eq. (4.36) we get:

$$\frac{\partial}{\partial p}(C - \alpha(\lambda + \sigma(p)))p = (C - \alpha(\lambda + \sigma(p))) - \alpha\sigma'(p)p > 0.$$

Taking the derivative of the right hand of Eq. (4.36) side yields:

$$\frac{(\lambda+\sigma(p)-C)\lambda K}{\sigma(p)} = \left(\frac{1}{\sigma(p)} - \frac{\lambda+\sigma(p)-C}{\sigma^2(p)}\right)\sigma'(p)\lambda K = \left(\frac{C-\lambda}{\sigma^2(p)}\right)\sigma'(p)\lambda K \leq 0.$$

One side of the equation is strictly increasing with p, while the other is non-increasing in p. Since both sides are continuous in p, we conclude that equality (4.35) holds for a unique value of p.

If $\bar{p}^{\alpha} \leq K$, Lemma 4.2.4 states that $\Delta W(p) > 0$ for price values $p < \bar{p}^{\alpha}$. By Lemma 4.2.3 we have $\Delta W(p) < 0$ for $p \geq \bar{p}$. Therefore, it must be that $p^{MS}(\alpha) \in [\bar{p}^{\alpha}, \bar{p})$. Since $\Delta W(p) = 0$ only when $p = p^{MS}(\alpha)$, by continuity of revenue it follows that $\Delta W(p) > 0$ for all $p < p^{MS}(\alpha)$ and $\Delta W(p) < 0$ for all $p > p^{MS}(\alpha)$.

(2) In the previous part of our proof we have demonstrated that on the price interval $[\bar{p}^{\alpha}, \bar{p}]$, the only possible price that sets $\Delta W(p) = 0$ is given by:

$$p = \frac{(\lambda+\sigma(p)-C)\lambda K}{(C-\alpha(\lambda+\sigma(p)))\sigma(p)}. \tag{4.37}$$

We will show that if $\bar{p}^{\alpha} > K$, the solution to Eq. (4.37) lies outside the price interval $[\bar{p}^{\alpha}, \bar{p})$. Let p^* denote a particular solution to Eq. (4.37). Assume $p^* \in [\bar{p}^{\alpha}, \bar{p})$, which means that $p^* > K$. Taking the ratio of $\frac{p^*}{K}$ and substituting the right hand side of Eq. (4.37) for p^* yields:

$$\frac{(\lambda+\sigma(p^*)-C)\lambda}{(C-\alpha(\lambda+\sigma(p^*)))\sigma(p^*)} > 1. \tag{4.38}$$

After some rearrangement we get:

$$\lambda(\lambda+\sigma(p^*)) > C(\lambda+\sigma(p^*)) - \alpha\sigma(p^*)(\lambda+\sigma(p^*)),$$

$$\lambda > C - \alpha\sigma(p^*). \tag{4.39}$$

which is a contradiction to our initial assumption $p^* \in [\bar{p}^\alpha, \bar{p}]$. Therefore, no value of p yields $\Delta W(p) = 0$ on the price interval $[\bar{p}^\alpha, \bar{p}]$. Additionally, since $\bar{p}^\alpha > K$, Lemma 4.2.4 states that $\Delta W(\bar{p}^\alpha) < 0$. Due to the continuity of the revenue $W(p)$ and the fact that there are no zero crossings in this interval, it must also be true that $\Delta W(p) < 0$ for $p \in [\bar{p}^\alpha, \bar{p}]$. □

Having analyzed how the revenue changes under market sharing for the three price intervals we have defined, we can finally move on to proving Theorem 4.2.3:

Proof of Theorem 4.2.3

1. *Case 1* - $\lambda + \alpha\sigma(K) \leq C$: By the way we have defined $\bar{p}^\alpha$, this case is equivalent to stating that $\bar{p}^\alpha \leq K$. Then by Lemma 4.2.3 and Lemma 4.2.4 we have that:

$$\Delta W(p) < 0 \qquad \text{for } p \geq \bar{p}, \tag{4.40}$$

$$\Delta W(p) > 0 \qquad \text{for } p < \bar{p}^\alpha. \tag{4.41}$$

   Therefore the market sharing price $p^{MS}(\alpha)$ must lie on the price interval $[\bar{p}^\alpha, \bar{p})$. Lemma 4.2.5 states that $p^{MS}(\alpha)$ satisfies Eq. (4.33) such that $\Delta W(p^{MS}(\alpha)) = 0$ and for $p \in [\bar{p}^\alpha, \bar{p})$:

$$\Delta W(p) < 0 \qquad \text{if } p > p^{MS}(\alpha), \tag{4.42}$$

$$\Delta W(p) > 0 \qquad \text{if } p < p^{MS}(\alpha). \tag{4.43}$$

   Combining Eq.'s (4.40) to (4.43) we obtain the results stated in the proposition.

2. *Case 2*- $\lambda + \alpha\sigma(K) > C$: By the way we have defined $\bar{p}^\alpha$, this case is equivalent to stating that $\bar{p}^\alpha > K$. Then by Lemma 4.2.3 and Lemma 4.2.5 we have that:

$$\Delta W(p) < 0 \text{ for } p \geq \bar{p}^\alpha. \tag{4.44}$$

Therefore the market sharing price $p^{MS}(\alpha)$ must belong to the price interval $[0, \bar{p}^{\alpha})$. Lemma 4.2.4 states that the revenue change is equal to zero when $p = K$, therefore we conclude that the market sharing price $p^{MS}(\alpha) = K$. Additionally, we have that:

$$\Delta W(p) < 0 \qquad \text{if } p > p^{MS}(\alpha), \tag{4.45}$$

$$\Delta W(p) > 0 \qquad \text{if } p < p^{MS}(\alpha). \tag{4.46}$$

Combining Eq.'s (4.44) to (4.46) we obtain the results stated in the theorem.
□

Theorem 4.2.3 yields a rather non-straightforward result such that for any network provider there exists a unique price which acts as a threshold value: market sharing at all prices greater than this threshold results in a profit decrease, while at prices below this threshold the network provider is guaranteed a profit increase by decreasing its secondary demand. In this way, it serves a similar function to that of the break-even price: It further divides the price ranges into two regimes but this time by identifying when serving the reduced secondary demand generates more profit that serving the full demand.

In the next lemma, we establish an upper bound on the market sharing price, similar to what we did in Lemma 4.2.2.

Lemma 4.2.6 *The market sharing price is less than or equal to the revenue generated by primary demand:*

$$p^{MS}(\alpha) \leq K.$$

Proof. The inequality holds when $p_i^{MS}(\alpha) = K_i$. When $p_i^{MS}(\alpha)$ is given by the solution to the implicit equation in Eq. (4.33), we prove it by showing the following:

$$p_i^{MS} = \frac{(\lambda_i + \sigma(p_i^{MS}(\alpha)) - C_i)\lambda_i K_i}{(C_i - \alpha_i(\lambda_i + \sigma(p_i^{MS}(\alpha))))\sigma(p_i^{MS}(\alpha))} \leq K_i.$$

After some simple algebra and regrouping of terms we get:

$$\lambda_i \le C_i - \alpha_i \sigma(p_i^{MS}(\alpha)).$$

Note that if $p_i^{MS}(\alpha)$ is given by the solution to the implicit equation in Eq. (4.33), by Lemma 4.2.5 it also follows that $p_i^{MS}(\alpha) \in [\bar{p}_i^{\alpha}, \bar{p}_i)$. From the way we have defined $\bar{p}_i^{\alpha}$ in Lemma 4.2.1 we conclude that $\lambda + \alpha\sigma(p_i^{MS}(\alpha)) \le C$. □

Example 4.2.2 We illustrate the relationship between the market sharing price and network parameters under the same constant elasticity secondary demand function we used before, $\sigma(p) = \sigma_0/p$.

Under this given demand and assuming $\sigma_0 \le (C-\lambda)K/\alpha$ such that $\lambda + \alpha\sigma(K) \le C$, we obtain from Eq. (4.21) the following explicit formula for the market sharing price:

$$p^{MS}(\alpha) = \frac{\lambda K \sigma_0 + \alpha \sigma_0^2}{\sigma_0(C - \alpha\lambda) + \lambda K(C - \lambda)}. \tag{4.47}$$

If $\sigma_0 > (C - \lambda)K/\alpha$, then $p^{MS}(\alpha) = K$ by Eq. (4.22).

Theorems 4.2.2 and 4.2.3 provide implicit equations for the break-even price $p^{BE}(\alpha)$ and market sharing price $p^{MS}(\alpha)$ that depend on the demand function $\sigma(p)$. Strikingly, one can show through careful analysis that the ratio of $p^{BE}(\alpha)$ to $p^{MS}(\alpha)$ is strictly smaller than 1 for any demand function.

Proof of Theorem 4.2.1

(1) If $p^{BE}(\alpha) > 0$:

(a) Assume $p^{MS}(\alpha) = K$. In Lemma 4.2.2 we have established that $p^{BE}(1) < K$. Let us rearrange Eq. (4.18) as follows:

$$p = \frac{\lambda K}{C} - \frac{\lambda K(C - \lambda)}{C\alpha\sigma(p)}$$

One can observe that the right hand side of Eq. (4.18) is increasing with α since

$\lambda < C$. Therefore, the solution to the implicit equation that yields the break-even price is increasing with α. Hence, we have

$$p^{BE}(\alpha) < p^{BE}(1) \quad \forall \alpha \in [0, 1) \tag{4.48}$$

Combining Eq. (4.48) with the results of Lemma 4.2.2 we conclude that $p^{BE}(\alpha) < K$.

(b) Assume $p^{MS}(\alpha)$ is given by the solution to the implicit equation in Eq. (4.33). We will prove the inequality by contradiction. Assume:

$$p^{MS}(\alpha) \leq p^{BE}(1).$$

Since secondary demand is non-increasing in p it follows that $\sigma(p^{MS}(\alpha)) \geq \sigma(p^{BE}(1))$. Taking the ratio between Eq. (4.18) and Eq. (4.33) yields:

$$\frac{p^{BE}(1)}{p^{MS}(\alpha)} = \frac{(\sigma(p^{BE}(1)) + \lambda - C)\sigma(p^{MS}(\alpha))}{(\sigma(p^{MS}(\alpha)) + \lambda - C)\sigma(p^{BE}(1))} \cdot \frac{(C - \alpha(\lambda + \sigma(p^{MS}(\alpha)))}{C}. \tag{4.49}$$

The first fraction in Eq. (4.49) is less than or equal to 1 while the second is strictly less than 1. Furthermore, we know that both fractions must be positive since $\sigma(p^{BE}(1)) \geq C - \lambda$ by Lemma 4.2.1 and $p^{MS}(\alpha) < \bar{p}$. We have:

$$p^{BE}(1) < p^{MS}(\alpha), \tag{4.50}$$

which contradicts our initial assumption that $p^{MS}(\alpha) \leq p^{BE}(1)$. Hence, it must be true that $p^{MS}(\alpha) > p^{BE}(1)$. In Eq. (4.48) we have established that $p^{BE}(\alpha) < p^{BE}(1)$ for $\alpha < 1$. Hence it is true for all values of $\alpha \in (0, 1]$ that $p^{MS}(\alpha) > p^{BE}(\alpha)$.

(2) If $p^{BE}(\alpha) = 0$, we can show that the market sharing price is strictly greater than zero in both cases. $p^{MS}(\alpha) = K$ is self-explanatory and by Eq. (4.33), we conclude $p^{MS}(\alpha) > 0$ as $\sigma(p^{MS}(\alpha)) + \lambda - C > 0$.

4.2.2 Duopoly Competition

In the previous sections we have identified a provider's competitive ability in a price war through establishing the break-even price and its incentive to share the market through the market sharing price. However, spectrum markets do not consist of a single provider, but rather several providers competing with each other. Therefore, our previous results, while being important, are not enough to determine the outcome of a secondary spectrum market. In this section, we consider the simplest oligopoly possible, a duopoly where two providers compete to enhance their profits by first capturing and then serving the secondary demand. To identify a market equilibrium, we utilize the concept of *Nash equilibrium* (NE) from game theory. Since NE are classically determined by *best response* functions, we will first seek to establish the best response dynamics of provider i to a fixed competitor price p_{-i}, where the notation $-i$ signifies the competing provider.

Definition 4.2.3 ***(Best Response)*** *Given two providers, provider* i*'s best response to competitor's pricing decision* p_{-i} *is the payoff maximizing strategy such that:*

$$p_i^{BR}(p_{-i}) = \arg\max_{p_i} \; R_i(p_i, p_{-i}). \tag{4.51}$$

Definition 4.2.4 ***(Nash Equilibrium)*** *A pricing strategy profile* (p_1^*, p_2^*) *is a Nash equilibrium (NE) if and only if both prices are a best response to each other such that:*

$$p_1^* = p_1^{BR}(p_2^*) \quad \textit{and} \quad p_2^* = p_2^{BR}(p_1^*). \tag{4.52}$$

Facing a competitor price p_{-i}, the strategies available to provider i consist of either matching this price and sharing the secondary demand or not matching it and trying to capture all of the secondary demand. While setting the price below or above the competitor's price follows a rather straightforward approach, the case of matching the competitor's price requires a more detailed analysis due to the discontinuity in the profit function. The next lemma states that if it is possible to increase the profit by capturing all of the secondary demand $\sigma(p_i)$ at a certain price p_i, then it is also desirable to capture the secondary demand at a slightly lower price $p_i' < p_i$. We will then utilize this result in establishing provider i's best response for prices $p_i > p_i^{MS}(\alpha)$.

Lemma 4.2.7 *For any p_i such that $\Delta W_i(p_i) < 0$ holds, there exists a price p_i' such that $p_i^{MS}(\alpha) < p_i' < p_i$ and*

$$W_i(p_i', \sigma(p_i')) > W_i(p_i, \alpha_i \sigma(p_i)). \tag{4.53}$$

Proof. Since we know that $W_i(x, \sigma(x))$ is differentiable in x, we can always a pick a price $q_i < p_i$ such that on the interval $[q_i, p_i)$, the function $W_i(x, \sigma(x))$ is either monotonically increasing, constant or monotonically decreasing with respect to x[5].

We break our proof into two cases:

(1) Assume that for a given q_i such that $q_i < p_i$, the following is true for any $\hat{p}_i \in [q_i, p_i)$:

$$W_i(\hat{p}_i, \sigma(\hat{p}_i)) \geq W_i(p_i, \sigma(p_i)).$$

Then it follows by our assumption $\Delta W_i(p_i) < 0$ that $W_i(\hat{p}_i, \sigma(\hat{p}_i)) > W_i(p_i, \alpha_i \sigma(p_i))$, and $p_i' = \hat{p}_i$.

(2) Assume for a given q_i such that $q_i < p_i$, the following is true for all $\hat{p}_i \in [q_i, p_i)$:

$$W_i(\hat{p}_i, \sigma(\hat{p}_i)) < W_i(p_i, \sigma(p_i)). \tag{4.54}$$

Then by the definition of continuity, the following can be stated for $W_i(p_i, \sigma(p_i))$: $\forall \epsilon > 0,\ \exists \delta(\epsilon, p_i) > 0$ s.t. if $|p_i - \hat{p}_i| < \delta$ then

$$|W_i(p_i, \sigma(p_i)) - W_i(\hat{p}_i, \sigma(\hat{p}_i))| < \epsilon.$$

[5] It should be noted that differentiability is not a necessary condition for this statement; local monotonocity of $W_i(x, \sigma(x))$ would suffice. However, as we need differentiability elsewhere in the chapter, we simply use it here as well.

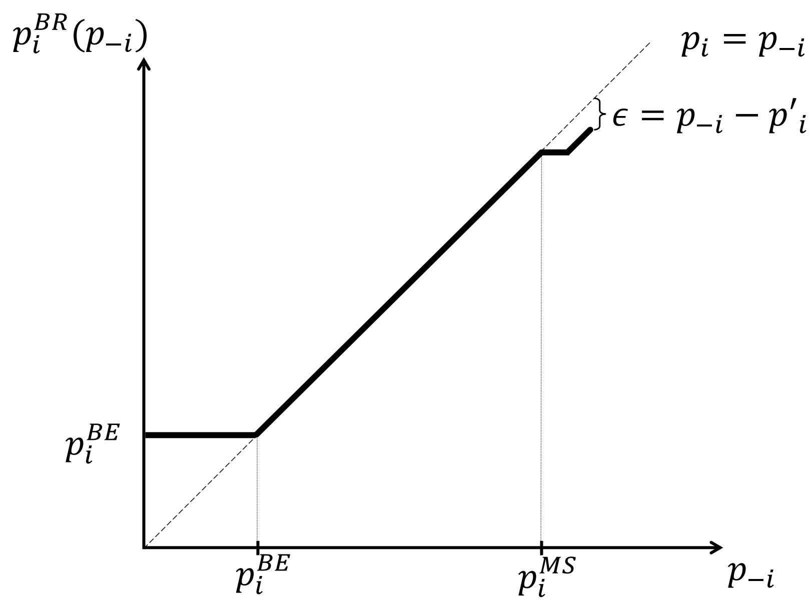


Figure 4·6: Best response of network provider i: a) When the competitor price is below p_i^{BE}, the provider sets its price to p_i^{BE}; b) When the competitor price is within the market sharing interval, the provider matches the price; c) When the competitor price is above p_i^{MS}, the provider sets its price slightly lower.

Making use of Eq. (4.54) and our assumption that $\hat{p}_i \in [q_i, p_i)$, we can remove the absolute value from the previous equation and simplify it to:

$$W_i(p_i, \sigma(p_i)) - W_i(\hat{p}_i, \sigma(\hat{p}_i)) < \epsilon. \tag{4.55}$$

Taking $\epsilon = W_i(p_i, \sigma(p_i)) - W_i(p_i, \alpha_i \sigma(p_i))$ and cancelling the terms $W_i(p_i, \sigma(p_i))$ on both sides of the inequality (4.55) we obtain:

$$-W_i(\hat{p}_i, \sigma(\hat{p}_i)) < -W_i(p_i, \alpha_i \sigma(p_i)),$$

$$W_i(\hat{p}_i, \sigma(\hat{p}_i)) > W_i(p_i, \alpha_i \sigma(p_i)),$$

and $p_i' = \hat{p}$. $\square$

The next theorem presents provider i's best response, which we shall utilize later to determine NE.

Theorem 4.2.4 ***(Best Response)*** *Provider i best response to its competing provider's pricing decision p_{-i} is:*

$$p_i^{BR}(p_{-i}) = \begin{cases} p_i^m(p_{-i}) & \text{for } p_{-i} > p_i^{MS}(\alpha) \\ p_{-i} & \text{for } p_i^{BE}(\alpha_i) \leq p_{-i} \leq p_i^{MS}(\alpha) \\ p_i^{BE} & \text{for } p_{-i} < p_i^{BE}(\alpha_i), \end{cases}$$

where $p_i^m(p_{-i}) < p_{-i}$ satisfies Eq. (4.53) in Lemma 4.2.7 to the optimality such that

$$W_i(p_i^m, \sigma(p_i^m)) = \max_{p_i \in (p_i^{MS}(\alpha), p_{-i})} W_i(p_i, \sigma(p_i)). \tag{4.56}$$

Remark 4.2.2 *The exact value of $p_i^m(p_{-i})$ depends on where the revenue is being maximized over the interval $(p_i^{MS}(\alpha), p_{-i})$. If the revenue is monotonically increasing up until p_{-i}, we can simplify Eq. (4.56) to the following:*

$$p_i^m(p_{-i}) = p_{-i} - \epsilon,$$

where ϵ is a sufficiently small discretization step, which is used when working with continuous prices. This assumption is a well-known approach used in game theory (Osbourne, 2004) because otherwise, a best response does not technically exist. On the other hand, it is possible that provider i's revenue attains a maximum at a lower price point, in which case $p_i^m(p_{-i})$ is as given in Eq. (4.56) and its exact value depends on the price elasticity of secondary demand.

Proof. We will consider each price condition described in Theorem 4.2.3 separately.

(1) In the first price condition, such that $\Delta W_i(p_{-i}) < 0$, provider i can either choose to match, lower or increase its price. Lowering the price such that $p_i' < p_{-i}$ is clearly better than price matching ($p_i = p_{-i}$) since we have demonstrated in Lemma 4.2.7 that:

$$W_i(p_i', \sigma(p_i')) > W_i(p_{-i}, \alpha_i \sigma(p_{-i})).$$

Lowering the price to p_i' is also better than increasing the price to $p_i > p_{-i}$ since the following is true:

$$W_i(p_i', \sigma(p_i')) > 0 = W_i(p_i, p_{-i}), \quad \text{for all } p_i > p_{-i}.$$

Hence, lowering the price to p_i' is the best response of provider i.

(2) In the second competitor price condition such that $p_i^{BE}(\alpha_i) \leq p_{-i} \leq p_i^{MS}(\alpha)$, we know that the following holds:

$$\Delta W_i(p_{-i}) = W_i(p_{-i}, \alpha_i \sigma(p_{-i})) - W_i(p_{-i}, \sigma(p_{-i})) \geq 0. \tag{4.57}$$

Selecting a price above the competitor's price such that $p_i > p_{-i}$ does not attract any secondary demand and therefore yields a profit of zero. Thus matching p_{-i} is better than increasing the price to $p_i > p_{-i}$:

$$\Pi_i(p_i, 0) = 0 \leq \Pi_i(p_{-i}, \alpha_i \sigma(p_{-i})), \quad \forall p_i > p_{-i}.$$

Next, we compare matching the price at p_{-i} to lowering the price to any price $\{p_i : p_i < p_{-i}\}$. We seek to find the price that maximizes the revenue function $W_i(p_i, \sigma(p_i))$ on the interval $[0, p_{-i}]$. We know from Lemmas 4.2.4 and 4.2.5 that $p_i^{MS}(\alpha) < \bar{p}$. Hence any price p on the interval $[0, p_{-i}]$ where $p_{-i} < p_i^{MS}(\alpha)$ satisfies $\lambda_i + \sigma(p) > C_i$. Simplifying Eq. (4.13) and by taking the derivative with respect to p_i we can show that:

$$\begin{aligned}
\frac{\partial}{\partial p_i} W_i(p_i, \sigma(p_i)) =& \frac{\partial}{\partial p_i} \left(\sigma(p_i) p_i \frac{C_i}{\lambda_i + \sigma(p_i)} + \lambda_i K_i \frac{C_i}{\lambda_i + \sigma(p_i)} \right) \\
&= (\sigma(p_i) + \sigma'(p_i) p_i) \frac{C_i}{\lambda_i + \sigma(p_i)} - \sigma'(p_i) \sigma(p_i) p_i \frac{C_i}{(\lambda_i + \sigma(p_i))^2} \\
&\quad - \sigma'(p_i) \lambda_i K_i \frac{C_i}{(\lambda_i + \sigma(p_i))^2}.
\end{aligned}$$

Regrouping the terms yields:

$$\frac{\partial}{\partial p_i} W_i(p_i, \sigma(p_i)) = \sigma(p_i) \frac{C_i}{\lambda_i + \sigma(p_i)} + \lambda_i C_i \sigma'(p_i) \frac{p_i - K_i}{(\lambda_i + \sigma(p_i))^2} > 0,$$

for $p_i \leq K_i$ since $\sigma'(p_i) \leq 0$.

We also know from Lemma 4.2.6 that $p_i^{MS}(\alpha) \leq K_i$. Therefore, the revenue maximizing price (which is also profit maximizing) is given by $p_i = p_{-i}$ such that for all $0 \leq p_i \leq p_{-i}$:

$$W_i(p_{-i}, \sigma(p_{-i})) \geq W_i(p_i, \sigma(p_i)).$$

By Equation (4.57), it follows that for all $p_i^{BE}(\alpha_i) \leq p_i \leq p_{-i}$, which demonstrates that matching the price at p_{-i} is better than lowering it to any $p_i < p_{-i}$:

$$W_i(p_{-i}, \alpha_i \sigma(p_{-i}) \geq W_i(p_i, \sigma(p_i)).$$

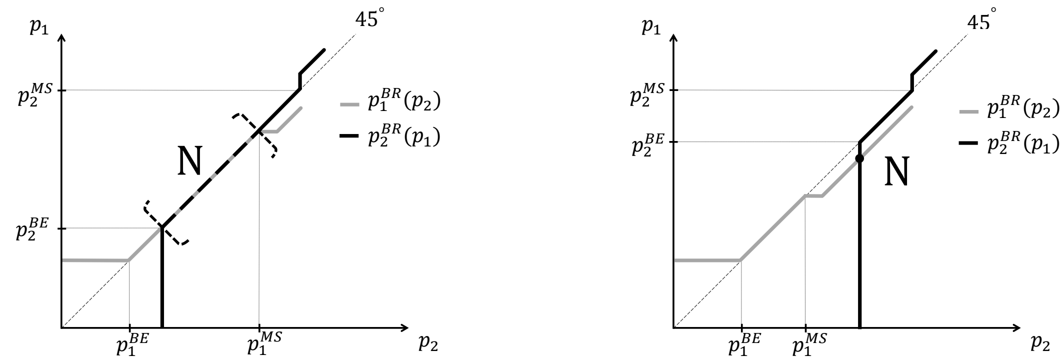
Hence, we conclude that $p_i^{BR}(p_{-i}) = p_i$.

(3) Lastly, we consider the case when $p_{-i} < p_i^{BE}$. Fortunately, this case can be quickly analyzed through the definition of the break-even price. If provider i chooses to match or lower its price by definition of the break-even price we have that:

$$\Pi_i(p_i, p_{-i}) < 0, \quad \text{for all } p_i \leq p_{-i}.$$

At any price $p_i > p_{-i}$ provider i profit will be zero. However setting the price to p_i^{BE} prevents the other provider from increasing its price further, thus we establish it as the best response. □

Theorem 4.2.4 establishes that for any network provider, a price interval, in which market sharing is the best response, is guaranteed to exist. Above this price interval, a provider will lower its price below the competitor's price, as in a typical price



(a) The placement of NE on the best response curves when market sharing intervals overlap, corresponding to part 1 of Theorem 4.2.5, where $p_1^{BE} < p_2^{BE}$ and $p_1^{MS} < p_2^{MS}$.

(b) The placement of NE on the best response curves when market sharing intervals do not overlap, corresponding to part 2 of Theorem 4.2.5, where $p_1^{MS} < p_2^{BE}$.

Figure 4·7: Illustration of the two possible types of market outcomes.

war. Below this price interval, profitability conditions from Section 4.2.1 are violated. While this interval is guaranteed to exist, whether the market equilibrium is established in this interval warrants further analysis. In the next theorem, we determine the different market outcomes by providing the resulting NE from the best response functions of the two providers.

Theorem 4.2.5 ***(Nash Equilibrium)*** *In a market with two network providers, a pricing strategy profile* (p_1^*, p_2^*) *is a NE such that:*

1. *If* $\max(p_1^{BE}(\alpha_1), p_2^{BE}(\alpha_2)) \leq \min(p_1^{MS}, p_2^{MS})$*, then* $p_1^* = p_2^*$*, and for* $i = 1, 2$

$$p_i^* \in [\max(p_1^{BE}(\alpha_1), p_2^{BE}(\alpha_2)), \min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2))].$$

2. *If* $\max(p_1^{BE}(\alpha_1), p_2^{BE}(\alpha_2)) > \min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_1))$ *and without loss of generality* $p_2^{BE}(\alpha_2) < p_1^{BE}(\alpha_1)$

$$p_1^* = p_1^{BE}(\alpha_1) \text{ and } p_2^* = p_2^m(p_1^{BE}(\alpha_1)).$$

where $p_i^m(p_{-i}) < p_{-i}$ *is defined as in Theorem 4.2.4.*

Proof. (1) Without loss of generality, assume that $p_1^{BE}(\alpha_1) < p_2^{BE}(\alpha_2)$. Now suppose $p_1^{MS}(\alpha_1) > p_2^{MS}(\alpha_2)$, such that we have the following relationship between the break-even and market sharing prices:

$$p_1^{BE}(\alpha_1) < p_2^{BE}(\alpha_2) < p_2^{MS}(\alpha_2) < p_1^{MS}(\alpha_1).$$

We will establish NE by determining when $p_1^* = p_1^{BR}(p_2^{BR}(p_1^*))$. In order to do so we first give provider 2's best response:

$$p_2^{BR}(p_1^*) = \begin{cases} p_2^{BE}(\alpha_2) & \text{for } p_1^* < p_2^{BE}(\alpha_2) \\ p_1^* & \text{for } p_2^{BE}(\alpha_2) \le p_1^* \le p_2^{MS}(\alpha_2) \\ p_2^m(p_1^*) & \text{for } p_1^* > p_2^{MS}(\alpha_2), \end{cases} \tag{4.58}$$

where $p_2^m(p_1^*)$ satisies Eq. (4.56) in Theorem 4.2.4.

We can now formulate provider 1's best response to provider 2's best response:

$$p_1^{BR}(p_2^{BR}(p_1^*)) = \begin{cases} p_1^{BE}(\alpha_1) & \text{for } p_1^* < p_1^{BE}(\alpha_1) \\ p_2^{BE}(\alpha_2) & \text{for } p_1^{BE}(\alpha_1) \le p_1^* < p_2^{BE}(\alpha_2) \\ p_1^* & \text{for } p_2^{BE}(\alpha_2) \le p_1^* \le p_2^{MS}(\alpha_2) \\ p_2^m(p_1^*) & \text{for } p_2^{MS}(\alpha_2) < p_1^* \le p_1^{MS}(\alpha_1) \\ p_1^m(p_2^*) & \text{for } p_1^* > p_1^{MS}(\alpha_1). \end{cases} \tag{4.59}$$

Therefore the only price interval where

$$p_1^* = p_1^{BR}(p_2^{BR}(p_1^*))$$

can be satisfied is $[p_2^{BE}(\alpha_2), p_2^{MS}(\alpha_2)]$ and from Eq. (4.58) in this interval we have that $p_2^* = p_1^*$, hence giving us the NE. The other case where $p_1^{MS}(\alpha_1) \le p_2^{MS}(\alpha_2)$ can be proven following the same argument.

(2) Suppose that $p_1^{BE}(\alpha_1) > p_2^{MS}(\alpha_2)$. Then, by Theorem 4.2.1, we also know the following relationship between the break-even and market sharing prices of both providers:

$$p_2^{BE}(\alpha_2) < p_2^{MS}(\alpha_2) < p_1^{BE}(\alpha_1) < p_1^{MS}(\alpha_1).$$

This time we will establish NE by determining when $p_2^* = p_2^{BR}(p_1^{BR}(p_2^*))$. In order to do so we first give provider 1's best response:

$$p_1^{BR}(p_2^*) = \begin{cases} p_1^{BE}(\alpha_1) & \text{for } p_1^* < p_1^{BE}(\alpha_1) \\ p_2^* & \text{for } p_1^{BE}(\alpha_1) \leq p_2^* \leq p_1^{MS}(\alpha_1) \\ p_1^m(p_2^*) & \text{for } p_2^* > p_1^{MS}(\alpha_1). \end{cases} \tag{4.60}$$

We can now formulate provider 2's best response to provider 1's best response:

$$p_2^{BR}(p_1^{BR}(p_2^*)) = \begin{cases} p_2^{BE}(\alpha_2) & \text{for } p_2^* < p_2^{BE}(\alpha_2) \\ p_1^{BE}(\alpha_1) & \text{for } p_2^{BE}(\alpha_2) \leq p_2^* \leq p_2^{MS}(\alpha_2) \\ p_2^m(p_1^{BE}(\alpha_1)) & \text{for } p_2^{MS}(\alpha_2) < p_2^* < p_1^{BE}(\alpha_1) \\ p_2^m(p_2^*) & \text{for } p_1^{BE}(\alpha_1) \leq p_2^* \leq p_1^{MS}(\alpha_1) \\ p_2^m(p_1^m(p_2^*)) & \text{for } p_2^* > p_1^{MS}(\alpha_1). \end{cases} \tag{4.61}$$

A careful look yields the result that the only time $p_2^* = p_2^{BR}(p_1^{BR}(p_2^*))$ is possible when $p_2^* = p_2^m(p_1^{BE}(\alpha_1))$, given in the third pricing interval in Eq. (4.61). From Eq. (4.60) we have that

$$p_1^* = p_1^{BR}(p_2^m(p_1^{BE}(\alpha_1))) = p_1^{BE}(\alpha_1), \tag{4.62}$$

thus completing the pricing strategy profile of the only NE possible in this case. Note that since $p_1^{BE}(\alpha_1) > p_2^{MS}(\alpha_2)$, it follows from Theorem 4.2.1 that $p_1^{BE}(\alpha_1) > p_2^{BE}(1)$.

Since ϵ can be chosen arbitrarily small, we can extend this result to $p_1^{BE}(\alpha_1) - \epsilon > p_2^{BE}(1)$, therefore provider 2 is profitable at this NE as a monopoly. We can use the same argument to prove the case when $p_1^{BE}(\alpha_1) \leq p_2^{MS}(\alpha_2)$. □

Next, we discuss the implications of Theorem 4.2.5 and provide examples that illustrate our results:

Interpretation of the NE. As stated in Theorem 4.2.5, the exact price profiles that give the NE depend on the relationship between the market sharing intervals of the two providers. If two price intervals overlap, as illustrated in part (a) of Fig. 4·7, any equal price pair in that interval will give us a NE. As a result, two providers share the market and set their prices at a value above their respective break-even prices but always less than the smaller of the two market sharing prices, a value which is guaranteed to be no greater than K_i, the primary reward collected by provider i.

On the other hand, if the market sharing price intervals of the two providers do not intersect, as illustrated in part (b) of Fig. 4·7, the market outcome is the same as the result of a price war, where the provider with the lower break-even price captures all of the secondary demand by pricing slightly below its competitor's break-even price. The losing provider cannot match this price without making a negative profit. In this case, even though both providers find it desirable to go into market sharing as the prices approach their break-even prices, the gap between the two market sharing intervals does not allow them to converge to a market sharing point.

Examples. In the following two examples, we seek to illustrate different market outcomes depending on the placement of the market sharing intervals on the price line. In the first example, we will use a constant elasticity demand function as in our previous examples. In the second example, we will use an exponentially decreasing demand to illustrate the fact that our results hold over general demand functions. Both types of demand functions are commonly used in the economics literature (Talluri and Ryzin, 2004).

Example 4.2.3 Suppose the secondary demand is given by $\sigma(p) = 20/p$. We consider two network providers whose parameters are:

$$(\lambda_1, C_1, K_1, \alpha_1) = (6, 10, 4, 0.5) \quad \text{and} \quad (\lambda_2, C_2, K_2, \alpha_2) = (5, 10, 4, 0.5).$$

Given these parameters it follows that $\lambda_i + \alpha_i\sigma(0) > C_i$ and $\lambda_i + \alpha_i\sigma(K) < C_i$ for $i = 1, 2$. Under these conditions, by making use of the explicit formulas provided in Eqs. (4.20) (substituting $\sigma(p)$ with $\alpha\sigma(p)$) and (4.47), we obtain the following break-even and market sharing prices of both providers:

$$\begin{aligned} p_1^{BE}(0.5) = 1.225, \quad p_1^{MS}(0.5) = 2.881, \\ p_2^{BE}(0.5) = 1.000, \quad p_2^{MS}(0.5) = 2.400. \end{aligned}$$

Clearly $p_1^{BE}(0.5) > p_2^{BE}(0.5)$ and $p_2^{MS}(0.5) < p_1^{MS}(0.5)$. Furthermore, it is also true that $p_1^{BE}(0.5) < p_2^{MS}(0.5)$. Therefore, both providers' market sharing price intervals are overlapping. Then, part one of Theorem 4.2.5 states that all NE price profiles (p_1^*, p_2^*) have the form: $p_1^* = p_2^* \in$ and lie in the price interval $[1.225, 2.400]$.

Example 4.2.4 In this example we consider an exponentially decreasing secondary demand given by $\sigma(p) = 20e^{-0.2p}$. This time we consider two similarly loaded providers with significantly different primary rewards. We choose the network parameters of these providers as such:

$$(\lambda_1, C_1, K_1, \alpha_1) = (6, 10, 6, 0.5) \quad \text{and} \quad (\lambda_2, C_2, K_2, \alpha_2) = (8, 10, 14, 0.5).$$

Notice that this time provider 2 has a higher primary demand and a higher associated reward collected. Once again, network parameters and the secondary demand satisfy $\lambda_i + \alpha_i\sigma(0) > C_i$ and $\lambda_i + \alpha_i\sigma(K) < C_i$ for $i = 1, 2$. Solving for the Eq. (4.18) in Theorem 4.2.2 and Eq. (4.21) in Theorem 4.2.3, we obtain the following break-even and market sharing prices of both providers:

$$p_1^{BE}(0.5) = 2.098, \quad p_1^{MS}(0.5) = 4.984,$$
$$p_2^{BE}(0.5) = 5.050, \quad p_2^{MS}(0.5) = 9.241.$$

Clearly $p_1^{BE}(0.5) < p_2^{BE}(0.5)$ and $p_2^{MS}(0.5) > p_1^{MS}(0.5)$. However, this time $p_2^{BE}(0.5) > p_1^{MS}(0.5)$. Therefore, the market sharing price intervals of the two providers do not intersect. As a result, these two providers will go into a price war and provider 1, having the lower break-even price will be the winner. In this light, part 2 of Theorem 4.2.5 states that the NE is given by$p_1^* = 5.050 - \epsilon$ and $p_2^* = 5.050$.

Best Response Dynamics. While Theorem 4.2.5 states that the NE exist and gives the pricing profiles of such, depending on the initial conditions one might never reach that equilibrium if best response dynamics change the prices in a different direction. In our case, the convergence to the NE is guaranteed from the way best response dynamics work. In both cases, for any price above the described NE prices, the best response dynamics lowers the price as each provider tries to capture the secondary demand by setting their price lower than the competitor's. For any prices below the NE, since this yields a negative profit for at least one provider, the best response dynamics now work to increase the prices to the break-even price of each provider, which in turn fall in the range of the NE given by Theorem 4.2.5.

Payoff Dominant Strategy Refinement. In part (1) of Theorem 4.2.5 we identified a price range in which all possible NE could lie. While all price pairs are viable NE , it is desirable to be able to characterize the market outcome through a single price pair. A possible refinement of the case when facing multiple NE is through the consideration of Payoff Dominant Strategy (PDS) equilibrium:

Definition 4.2.5 *Let $\mathcal{S}$ denote the set of price pairs $\{(p_1^*, p_2^*) : p_1^* = p_2^*\}$ that give the NE in part (1) of Theorem 4.2.5. Then, the PDS equilibrium (p_1^D, p_2^D) is a NE with* *the following refinement condition:*

$$R_i(p_1^D, p_2^D) = \max_{(p_1,p_2)\in\mathcal{S}} R_i(p_1, p_2) \quad \textit{for} \quad i = 1, 2,$$

In other words, when multiple NE are present, a PDS yields the maximum possible payoff for both providers (Straub, 1995). Using this condition we can identify the PDS equilibrium $(p_1^D, p_2^D) \in \mathcal{S}$. Since the prices in $\mathcal{S}$ are equal, we know from Eq. (3.2) that the payoff is equal to the profit under reduced demand. $(R_i(p_1, p_2) = \Pi_i(p_i, \alpha_i\sigma(p_i)))$. If $\sigma'(p) < 0$, let $\hat{p}$ denote the solution to:

$$p = -\sigma(p)/\sigma'(p). \tag{4.63}$$

Otherwise we set $\hat{p} = \infty$. Note that Eq. (4.63) corresponds to the price elasticity of demand. Through careful analysis, we can state the following:

Theorem 4.2.6 *For relatively inelastic demand such that* $\hat{p} > \max(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2))$*, there exists a unique PDS equilibrium* (p_1^D, p_2^D) *given by:*

$$p_1^D = p_2^D = \min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2)) \tag{4.64}$$

Proof. Combining Lemmas 4.2.4, 4.2.5 and Theorem 4.2.3 we know the following:

(i) If p_i^{MS} is given by Eq. (4.21), then $\bar{p}_i^{\alpha_i} \leq p_i^{MS}(\alpha_i) < \bar{p}_i$.

(ii) If $p_i^{MS}(\alpha_i) = K_i$, then $p_i^{MS} < \bar{p}_i^{\alpha_i}$.

Therefore, we need to consider two different formulations of the profit $\Pi_i(p_i, \alpha_i\sigma(p_i))$. One can observe from Eq. (4.14) that:

$$\frac{\partial}{\partial p_i} W_i(p_i, \alpha_i\sigma(p_i)) = \frac{\partial}{\partial p_i} \Pi_i(p_i, \alpha_i\sigma(p_i)).$$

Therefore, we will use the derivative of revenue with respect to price in our calculations instead of profit.

Case 1 - $p_i < \bar{p}_i^{\alpha_i}$

The price condition is equivalent to stating that $\lambda_i + \alpha_i\sigma(p) > C_i$. Simplifying Eq. (4.13) and by taking the derivative with respect to p_i we can show that:

$$\begin{aligned}\frac{\partial}{\partial p_i}W_i(p_i,\alpha_i\sigma(p_i)) =&\frac{\partial}{\partial p_i}\left(\alpha_i\sigma(p_i)p_i\frac{C_i}{\lambda_i+\alpha_i\sigma(p_i)}+\lambda_iK_i\frac{C_i}{\lambda_i+\alpha_i\sigma(p_i)}\right)\\ &= \alpha_i(\sigma(p_i)+\sigma'(p_i)p_i)\frac{C_i}{\lambda_i+\alpha_i\sigma(p_i)} - \alpha_i\sigma(p_i)p_i\frac{C_i\alpha_i\sigma'(p_i)}{(\lambda_i+\alpha_i\sigma(p_i))^2}\\ &\quad - \lambda_iK_i\frac{C_i\alpha_i\sigma'(p_i)}{(\lambda_i+\alpha_i\sigma(p_i))^2}.\end{aligned}$$

Regrouping the terms yields:

$$\frac{\partial}{\partial p_i}W_i(p_i,\alpha_i\sigma(p_i)) = \alpha_i\sigma(p_i)\frac{C_i}{\lambda_i+\alpha_i\sigma(p_i)} + \lambda_iC_i\alpha_i\sigma'(p_i)\frac{p_i-K_i}{(\lambda_i+\alpha_i\sigma(p_i))^2} > 0, \quad (4.65)$$

for $p_i \le K_i$ since $\sigma'(p_i) \le 0$.

Case 2 - $p_i \ge \bar{p}_i^{\alpha_i}$

Simplifying Eq. (4.13) and by taking the derivative with respect to p_i we can show that:

$$\begin{aligned}\frac{\partial}{\partial p_i}W_i(p_i,\alpha_i\sigma(p_i)) =&\frac{\partial}{\partial p_i}\left(\alpha_i\sigma(p_i)p_i+\lambda_iK_i\right)\\ &= \alpha_i\left(\sigma(p_i)+\sigma'(p_i)p_i\right).\end{aligned}$$

If $\sigma'(p_i) = 0$, then $\frac{\partial}{\partial p_i}W_i(p_i,\alpha_i\sigma(p_i)) > 0$ for all $p_i \ge \bar{p}_i^{\alpha_i}$. On the other hand, if $\sigma'(p_i) < 0$ we have the following:

$$\frac{\partial}{\partial p_i}W_i(p_i,\alpha_i\sigma(p_i))\begin{cases} >0 & \text{if } p_i<\hat{p}\\ =0 & \text{if } p_i=\hat{p}\\ <0 & \text{if } p_i>\hat{p},\end{cases} \quad (4.66)$$

where $\hat{p}$ denotes the solution to:

$$p = -\sigma(p)/\sigma'(p).$$

Note that $\hat{p}$ is the same for both providers. Now we consider the cases $(p_1^{MS}(\alpha_1) = K_1, p_2^{MS}(\alpha_2) = K_2)$, $(p_i^{MS}(\alpha_i) = K_i, p_{-i}^{MS} = g_{-i}(p_{-i}))$ and $(p_1^{MS} = g_1(p_1), p_2^{MS} = g_2(p_2))$ separately, where $g_i(p_i)$ represents the right hand side of Eq. (4.21).

1. Assume $p_1^{MS}(\alpha_1) = K_1, p_2^{MS}(\alpha_2) = K_2$. Recalling condition (ii) in the beginning of our proof, we have:

$$p_i^{MS}(\alpha_i) < \bar{p}_i^{\alpha_i}, \quad \text{for} \quad i = 1, 2. \tag{4.67}$$

   From Eq. (4.65) we know that for $p < \bar{p}_i^{\alpha_i}$ the profit is increasing on the interval $[0, K_i]$. Therefore, both providers obtain their maximum revenue rates at their respective market sharing prices. Then, the PDS equilibrium is:

$$p_1^D = p_2^D = \min(K_1, K_2) = \min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2)). \tag{4.68}$$

2. Assume $p_1^{MS}(\alpha_1) = K_1, p_2^{MS}(\alpha_2) = g_2(p_2)$. From Eq. (4.65) we know that provider 1's payoff is maximized at K_1. Recalling condition (i), we have $p_2^{MS}(\alpha_2) > \bar{p}_2^{\alpha_2}$. Then from Eq. (4.66) we know that provider 2's payoff is increasing until $\hat{p}$. Since we assume that $\hat{p} > p_2^{MS}(\alpha_2)$, and we consider price strategy profiles that are upper bounded by $\min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2))$, the PDS equilibrium is given by:

$$p_1^D = p_2^D = \min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2)). \tag{4.69}$$

3. $p_1^{MS}(\alpha_1) = g_1(p_1), p_2^{MS}(\alpha_2) = g_2(p_2)$. From Eq. (4.66) we conclude that both providers' profits are increasing until $\hat{p}$. Once again recalling our assumption that

$$\hat{p} > \max(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2))$$

and the upper bound $\min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2))$ on $\mathcal{S}$, we conclude that the PDS equilibrium is given by:

$$p_1^D = p_2^D = \min(p_1^{MS}(\alpha_1), p_2^{MS}(\alpha_2)). \tag{4.70}$$

□

4.2.3 Quality of Service Extension

In our model we have made the assumption that secondary users always choose the lowest price advertised and when the prices are the same arriving secondary traffic randomly choose a provider. While we have argued that *price aversion* might be a possible explanation for choosing the lower price, Quality of Service (QoS) might also have an impact on the customer's decision process. In this subsection we extend our model to take QoS into consideration.

We consider a simple QoS performance metric: the acceptance rate of the incoming traffic. Then we extend our model as follows: When both providers charge the same price and the secondary demand at this price is sufficiently large that the total demand in the market exceeds the total capacity (i.e., $\lambda_1 + \lambda_2 + \sigma(p) > C_1 + C_2$), secondary demand is split between the two providers according to a vector $\boldsymbol{\alpha} = [\alpha_1, \alpha_2]$ such that $\alpha_1 + \alpha_2 = 1$, $\alpha_1, \alpha_2 > 0$ and satisfying the following equality:

$$\frac{C_1}{\lambda_1 + \alpha_1\sigma(p)} = \frac{C_2}{\lambda_2 + \alpha_2\sigma(p)}. \tag{4.71}$$

Namely, instead of randomly choosing a provider, secondary demand distributes itself in a fashion that the accommodation level it faces is homogeneous across both providers. In the case of two providers, let $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$. Then we can obtain an explicit expression for α:

$$\alpha = \frac{C_1(\lambda_2 + \sigma(p)) - C_2\lambda_1}{(C_1 + C_2)\sigma(p)}. \tag{4.72}$$

and substituting Eq. (4.72) for α in $\alpha\sigma(p)$ we obtain:

$$\alpha\sigma(p) = \beta_1 + \gamma_1\sigma(p), \tag{4.73}$$

where $\beta_i = \frac{C_i\lambda_{-i} - C_{-i}\lambda_i}{C_i + C_{-i}}$ and $\gamma_i = \frac{C_i}{C_i + C_{-i}}$ for $i = 1, 2$.

Under this new model, the previous results stated in our theorems still hold. Since we consider all values of $\alpha \in (0, 1]$, we can simply replace $\alpha\sigma(p)$ in our equations with Eq. (4.73). We illustrate this result in the following example and extend our analysis afterwards.

Example 4.2.5 Suppose the secondary demand is given by $\sigma(p) = 30e^{-10p}$. We consider two network providers whose parameters are:

$$(\lambda_1, C_1, K_1) = (10, 20, 1) \quad \text{and} \quad (\lambda_2, C_2, K_2) = (8, 10, 1).$$

Suppose that secondary demand is split between the providers in a way that satisfies Eq. (4.71). Then, using Eq. (4.73), we have the following reduced demand functions:

$$\alpha_1\sigma(p) = 2\sigma(p)/3 + 2 \quad \text{and} \quad \alpha_2\sigma(p) = \sigma(p)/3 - 2.$$

Observe that $\alpha_1\sigma(p) + \alpha_2\sigma(p) = \sigma(p)$. We can check that $\lambda_i + \alpha_i\sigma(0) > C_i$ and $\lambda_i + \alpha_i\sigma(K) < C_i$ for $i = 1, 2$. Under these conditions, we need to use Eq. (4.18) for calculating the break-even price and (4.21) for the market sharing price for both providers. Doing the necessary calculations, one finds:

$$p_1^{BE} = 0.184, \quad p_1^{MS} = 0.529,$$
$$p_2^{BE} = 0.317, \quad p_2^{MS} = 0.417.$$

Clearly $p_1^{BE} < p_2^{BE}$ and $p_1^{MS} > p_2^{MS}$. Thus, providers 2's market sharing interval is a subset of provider 1's. Then, part one of Theorem 4.2.5 states that all NE price profiles (p_1^*, p_2^*) have the form $p_1^* = p_2^*$ and lie in the price interval $[0.162, 0.235]$.

Previous results under Quality of Service (QoS) Extension. Here we revisit our previous proofs for each theorem to demonstrate that they still hold under the QoS extension we have provided.

Before we begin our proof of Theorem 4.2.2 we need to revisit the two prices we have created before: $\bar{p}$ and $\bar{p}^{\alpha}$. By definition $\bar{p}$ is the same, while substituting Eq. (4.72) for α we get the new following relationship:

$$\lambda_1 + \alpha_1 \sigma(\bar{p}^{\alpha}) = C_1 \Longleftrightarrow \lambda_1 + \lambda_2 + \sigma(\bar{p}^{\alpha}) = C_1 + C_2. \tag{4.74}$$

Therefore, our previous result $p^{BE}(\alpha) \leq \bar{p}^{\alpha}$ from Lemma 4.2.1 is equivalent to the following:

$$\lambda_1 + \lambda_2 + \sigma(p^{BE}(\alpha)) \geq C_1 + C_2 \tag{4.75}$$

Proof of Theorem 4.2.2 revisited

Parts 1 & 3 of the proof remain unchanged. The following is a revision of part 2 in our proof:

(2) Let us rearrange the terms in Eq. (4.18) and reintroduce index i to get the following:

$$\frac{C_i}{\lambda_i K} p - 1 = \frac{\lambda_i - C_i}{\alpha_i \sigma(p)}. \tag{4.76}$$

We will proceed by demonstrating that the left hand side of Eq. (4.76) is strictly increasing with respect to p and the right hand side is non-increasing with respect to p, hence meaning that this equality only holds at a single value of p. Since the left hand side of Eq. (4.18) linearly increasing in p, we only need to prove that the right hand side is non-increasing. Under the assumption that $\sigma(p)$ is a differentiable and non-increasing function of p, substituting

$$\alpha_i \sigma(p) = \beta_i + \gamma_i \sigma(p), \tag{4.77}$$

and taking the derivative of the right hand side with respect to p yields:

$$\frac{\partial}{\partial p}\left(\frac{\lambda_i - C_i}{\beta_i + \gamma_i \sigma(p)}\right) = \gamma_i \frac{C_i - \lambda_i}{(\beta_i + \gamma_i \sigma(p))^2}\sigma'(p) \leq 0. \tag{4.78}$$

Eq. (4.78) holds because $\lambda_i < C_i$ and $\sigma'(p) \leq 0$. Therefore, there can only be at most one solution for $p^{BE}(\alpha)$ that satisfies Eq. (4.18). □

Before we revisit the proof of Theorem 4.2.3, we need to revisit the three lemmas used in its proof. Observe that under the QoS extension, Lemmas 4.2.3 and 4.2.5 remain unchanged. However, we need to revisit the price values where $p : p < \bar{p}^{\alpha}$, and revise the corresponding Lemma 4.2.4 as follows:

Lemma 4.2.8 *Assume $\bar{p}^{\alpha} > 0$ and $p < \bar{p}^{\alpha}$, then*

1. *If $\bar{p}^{\alpha} \geq K$:*

$$\Delta W_i(p) > 0 \qquad \text{if } p < K; \tag{4.79}$$

$$\Delta W_i(p) = 0 \qquad \text{if } p = K; \tag{4.80}$$

$$\Delta W_i(p) < 0 \qquad \text{if } p > K. \tag{4.81}$$

2. *If $\bar{p}^{\alpha} < K$:*

$$\Delta W(p) > 0 \qquad \forall p < \bar{p}^{\alpha}. \tag{4.82}$$

Proof. Note that our assumption $p < \bar{p}^{\alpha}$ is equivalent to stating that:

$$\lambda_1 + \lambda_2 + \sigma(p) > C_1 + C_2$$

Since $\bar{p}^{\alpha} < \bar{p}$, it must also be true that $p < \bar{p}$. Then the combined demand without market sharing is greater than the provider's capacity (i.e., $\lambda_i + \sigma(p) \geq C_i$). Simplifying Eq. (4.13) under these assumptions, we obtain:

$$\begin{aligned}\Delta W_i &= W_i(p, \alpha_i \sigma(p)) - W_i(p, \sigma(p)) \\ &= \frac{\alpha_i \sigma(p) p C_i}{\alpha_i \sigma(p) + \lambda_i} + \frac{\lambda_i K C_i}{\alpha_i \sigma(p) + \lambda_i} - \frac{\sigma(p) p C_i}{\sigma(p) + \lambda_i} - \frac{\lambda_i K C_i}{\sigma(p) + \lambda_i}.\end{aligned}$$

After rearrangement and substituting $\alpha_i\sigma(p)$ with $\beta_i + \gamma_i\sigma(p)$ we get:

$$\Delta W_i = \frac{(\beta_{-i} + \gamma_{-i})\sigma(p))\sigma(p)\lambda_i C_i}{((\beta_i + \gamma_i)\sigma(p) + \lambda)(\sigma(p) + \lambda)}(K - p). \tag{4.83}$$

Eq. (4.83) only takes on the value zero when $p = K$. Additionally for price values $p < K$, ΔW_i is positive and for $p > K$, ΔW_i is negative. □

Proof of Theorem 4.2.3 revisited.

From the way we have defined the new distribution vector $\boldsymbol{\alpha}$ in Eq. (4.72), our model does not extend to prices are greater than $\bar{p}^{\alpha}$ as the total demand does not exceed the total market capacity, and can be fully accommodated. Therefore, we keep our assumption of the random splitting of the secondary demand for these price values.

We have observed that lemmas 4.2.3 and 4.2.5 remain unchanged while the results of Lemma 4.2.8 and Lemma 4.2.4 are equivalent. As the proof of Theorem 4.2.3 follows from these three lemmas, we conclude it holds under the QoS extension as well. □

Having demonstrated that the main results stated in Theorems 4.2.2 and 4.2.3 hold under the extended model, we revisit Theorem 4.2.1.

Proof of Theorem 4.2.1 revisited.

The only part of the proof we need to revisit is for prices $p < \bar{p}^{\alpha}$. If the market sharing and the break-even prices are in the price range $[0, \bar{p}^{\alpha})$, then from Lemma 4.2.4 we conclude that $p^{MS}(\alpha) = K$. Further, by substituting the right hand side of Eq. (4.72) for α in Eq. (4.18), we can demonstrate that $p^{BE}(\alpha)$ is given by the solution to the following equation:

$$p = \frac{(\lambda_1 + \lambda_2 + \sigma(p) - C_1 - C_2)\,\lambda_1 K}{C_1(\lambda_2 + \sigma(p)) - C_2\lambda_1}.$$

Then we need to demonstrate that:

$$\frac{(\lambda_1 + \lambda_2 + \sigma(p) - C_1 - C_2)\,\lambda_1 K}{C_1(\lambda_2 + \sigma(p)) - C_2\lambda_1} < K.$$

After rearranging and collecting the terms we obtain the following:

$$\lambda_1(\lambda_1 + \lambda_2 + \sigma(p)) - C_2\lambda_1 < C_1(\lambda_1 + \lambda_2 + \sigma(p)) - C_2\lambda_1,$$

which is true for since $\lambda_1 < C_1$ in our initial assumptions. □

The results stated in the duopoly competition, once the break-even and market sharing prices are determined, do not depend on the specific value α_i takes. The results stated Lemma 4.2.7 depends of continuity of the price p_i and the sign of the revenue change $\Delta W_i(p_i)$. The proof Theorem 4.2.4 builds on Lemma 4.2.7 and utilities the revenue rate without sharing $(W_i(p_i, \sigma(p_i)))$. The proof of Theorem 4.2.5 is based on the game theoretic interpretation of the results stated in Theorem 4.2.4. All of these results hold as long as α_i takes on a value in the interval $[0, 1)$, which our extended model does not violate.

4.3 Summary

In this chapter, we showed that the market dynamics fundamentally differ when providers implement uncoordinated access. We demonstrated that the break-even price is no longer insensitive to the secondary demand and market sharing becomes a possible best response. It is worth noting that even though a provider i might find it desirable to share the market, it would still go into a price war for price values higher than its market sharing price p_i^{MS}, thus preventing convergence to an arbitrarily high price for secondary access. The possible market outcomes under an uncoordinated access policy become complex when the number of providers increases, but deserve further study, since they may result in a larger number of providers joining the market and higher revenues than possible under an optimal coordinated access policy.

To address this complexity, we next focused on an uncoordinated access regime for secondary spectrum detailed under private commons using a demand overflow model. Similar to the previous part, using the notions of best response and Nash equilibrium, we show the emergence of two markedly different possible market outcomes, depending on the secondary demand function $\sigma(p)$ and the network parameters of each provider (i.e, the service capacity C, primary demand λ, and primary reward K).

If the market sharing price intervals of the two providers intersect, as described in part one of Theorem 4.2.5, then the providers converge to a price profile where they will share the market. All prices falling between the maximum break-even price and minimum market sharing price among the two providers are possible Nash equilibria. On the other hand, if the market sharing price intervals do not intersect, as described in part two of Theorem 4.2.5, then the Nash equilibrium reflects a price war wherein the winning provider sets its price slightly below the break-even price of its competitor and gets all the profit.

Since market outcomes are determined by break-even and market sharing prices, we carefully analyzed these two crucial parameters. We demonstrated existence and uniqueness of these prices for each provider, under general demand functions. We further provided implicit formulas to compute both of these prices as a function of the system parameters.

Chapter 5

Carrier Aggregation

We now switch our focus from the pricing aspect of secondary spectrum markets and consider the impact of how network capacity impacts the market equilibrium presented in previous chapters. So far, our assumption has been that the primary user traffic and the capacity with which a provider has been serving that customer pool is fixed. This led to the formulation of different price thresholds which had a direct impact on the outcome of the competition. Particularly, the break-even price under a coordinated access regime is directly linked to the Erlang-B formula and this price also determines who wins the price war between the providers. Therefore, increasing network capacity is one strategy a network provider might follow in an effort to gain an edge over its competition. In this chapter, instead of specifically consider secondary spectrum markets we will first present a general framework for network dimensioning and later demonstrate how it can be utilized to answer questions related to secondary markets. We start with the description of the model:

5.1 Network Model

In this section, we introduce the network model considered and the accompanying notation. We consider a small provider with a finite capacity $C > 1$, which consists of the number of carriers in the spectrum owned by the provider. For example, in an

Results presented in this chapter appear in part in (Kavurmacioglu and Starobinski, 2015a),(Kavurmacioglu and Starobinski, 2015b).

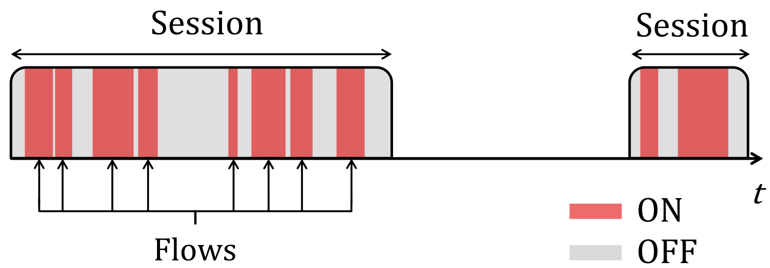


Figure 5·1: Illustration of sessions and flows. Each session consists of one or more flows separated by idle periods.

LTE network configuration, these carriers could be interpreted as the resource blocks.

To realistically model network traffic, such as web browsing and streaming applications, we assume that the user demand consists of a sequence of independent *session* arrivals that follow a Poisson process with rate $\lambda > 0$ (Bonald, 2006b). A session consists of a combination of arbitrarily distributed and possibly correlated *flows*, generated by the same user or application. Each session consists of "on" and "off" periods within, where an "on" period means that a flow is generating traffic. Figure 5·1 provides an illustration of sessions and flows. Without loss of generality, we assume that the *total* "on" time within an individual session follows a general probability distribution and has a mean equal to one, independently of other requests and arrival times. Each flow has a peak rate that corresponds to the capacity of a carrier. If an arriving flow finds all the carriers busy, it is lost, but the rest of the session proceeds as normal. Note that standard voice calls are a special case of this model, for which a session consists of a single flow.

Under the above statistical assumptions, the probability that a flow is lost (blocked), is given by the Erlang-B formula (Bonald, 2006a)(p.279):

$$E(\lambda, C) = \frac{\lambda^C / C!}{\sum_{k=0}^{C} \lambda^k / k!}. \tag{5.1}$$

The above formula is insensitive to all traffic characteristics, except for the mean number of session arrivals per time unit λ.

The provider finds itself in the same competitive spectrum market as a larger network provider that has similar network parameters, but scaled by a multiplicative factor $n > 1$ (*i.e.*, its session arrival rate is λn and capacity Cn). We refer to the parameter n as the *scaling factor*.

The objective of the smaller provider is to meet the quality of service (QoS) of the larger provider, given by its Erlang blocking probability formula. This can be achieved through making use of the spectrum set aside and implementing carrier aggregation. Our goal is to identify the optimal level of carrier aggregation and investigate how it changes with the network parameters λ and C and the scaling factor n.

5.2 Quality-Driven Approximation of Erlang-B Formula

The Erlang-B formula given by Eq. (5.1) does not easily yield itself to analysis due to the summand and the factorial functions. Therefore, we seek an approximation of the Erlang-B formula that is more tractable. One such approximation is obtained through the consideration of a *quality driven* (QD) regime, characterized by $C \to \infty$, $\lambda \to \infty$ and the following relationship:

$$C = \lambda(1 + \gamma), \tag{5.2}$$

where $\gamma > 0$ is a constant representing the *service grade*. In a QD regime, the provider positions itself in terms of capacity with respect to its load so that it offers a high quality service (*e.g.,* low probability of blocking or waiting).

The approximation that we will obtain under the QD regime works well for large values of C. Moreover, the approximation is asymptotically exact since the underlying stochastic process, when properly normalized, weakly converges to an Ornstein-Uhlenbeck diffusion process as $C \to \infty$ (Iglehart, 1965). Before we establish the QD approximation to the Erlang-B formula, it is beneficial to recall the following fundamental inequality of the logarithm function:

$$x - 1 \geq \ln(x) \geq 1 - \frac{1}{x}, \quad x > 0,$$

which we can rewrite as:

$$x \geq \ln(1 + x) \geq \frac{x}{1 + x}, \quad x > 0. \tag{5.3}$$

Under the QD regime we propose the following asymptotically exact approximation to the Erlang-B, which we will use in the rest of the chapter:

Lemma 5.2.1 *Under a QD regime such that $C = \lambda(1 + \gamma)$, the Erlang-B formula satisfies:*

$$\lim_{\lambda \to \infty} \frac{E(\lambda, C)}{\left(\sqrt{2\Pi C}(1+\gamma)^C e^{-\lambda\gamma}\right)^{-1}} = 1.$$

Proof. We first establish a relationship between the delay probability formula (Erlang-C) given by:

$$E_c(\lambda, C) = \frac{\frac{\lambda^C}{C!}\frac{C}{C-\lambda}}{\sum_{k=0}^{C-1}\frac{\lambda^k}{k!} + \frac{\lambda^C}{C!}\frac{C}{C-\lambda}},$$

and the Erlang-B formula. From the relationship between Erlang-B and Erlang-C provided in (Zeng, 2003), it can be shown that:

$$E(\lambda, C) = \frac{(1 - \rho)E_c(\lambda, C)}{1 - \rho E_c(\lambda, C)}, \tag{5.4}$$

where $\rho = \lambda/C = \frac{1}{1+\gamma}$ in a QD regime. Using the results provided in Section 16 of (Zeltyn, 2004) for the analysis of queuing systems in the QD regime we obtain:

$$E_c(\lambda, C) = \frac{e^{\lambda\gamma} + o(1/\lambda)}{\sqrt{2\Pi C}\gamma(1+\gamma)^{C-1} + e^{\lambda\gamma} + o(1/\lambda)}. \tag{5.5}$$

Substituting Eq. (5.5) for $E_c(\lambda, C)$ and Eq. (5.2) for C into Eq. (5.4) we get:

$$E(\lambda, C) = \frac{1 + o(1/\lambda)}{g(\lambda,\gamma) + 1 + o(1/\lambda)},$$

where $g(\lambda,\gamma) = \sqrt{2\pi\lambda(1+\gamma)}\left((1+\gamma)^{(1+\gamma)}e^{-\gamma}\right)^{\lambda}$.

Now we will show that $g(\lambda,\gamma)$ is the dominating term in the denominator as λ gets large. Observe that $(1+\gamma)^{(1+\gamma)} \geq e^{\gamma}$ since taking the natural log of both sides we obtain:

$$\begin{aligned}(1+\gamma)\ln(1+\gamma) &\geq \gamma \\ \ln(1+\gamma) &\geq \frac{\gamma}{1+\gamma},\end{aligned}$$

which we know to be true from Eq. (5.3). Therefore $g(\lambda,\gamma)$ gets arbitrarily large with λ. We conclude that:

$$\lim_{\lambda\to\infty} \frac{1}{g(\lambda,\gamma)} = 0.$$

Hence:

$$\lim_{\lambda\to\infty} \frac{E(\lambda, C)}{g(\lambda,\gamma)^{-1}} = \lim_{\lambda\to\infty} \frac{\dfrac{1 + o(1/\lambda)}{g(\lambda,\gamma) + 1 + o(1/\lambda)}}{\left(g(\lambda,\gamma)\right)^{-1}} = 1.$$

Finally, we obtain $g(\lambda,\gamma) = \sqrt{2\pi C}(1+\gamma)^{C}e^{-\lambda\gamma}$ through Eq. (5.2). □

Lemma 5.2.1 states that the Erlang-B formula can be approximated by (and is asymptotically equal to) the following expression, which we refer to as the *QD for-*

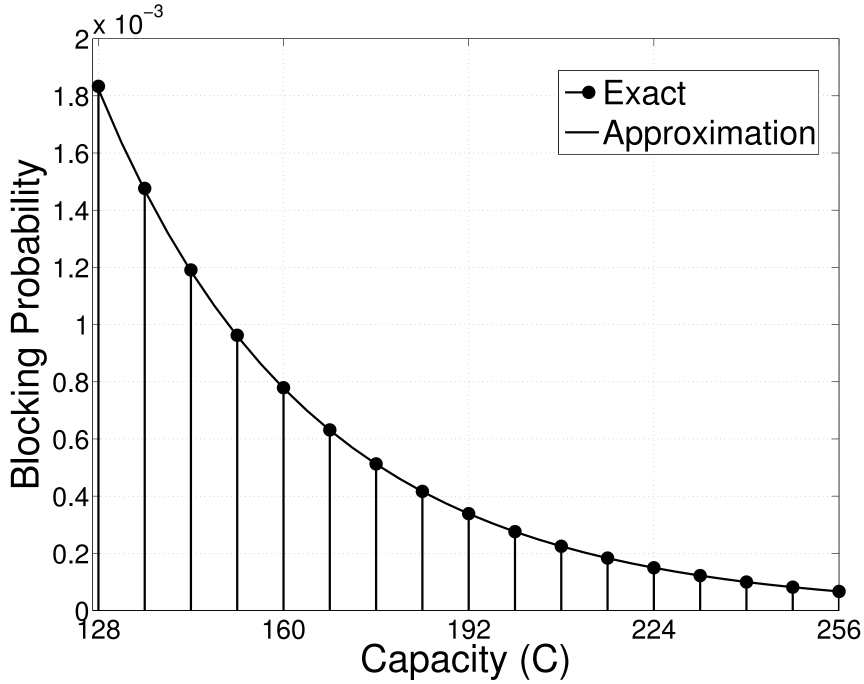


Figure 5·2: QD Approximation with $C = (1+\gamma)\lambda$ and $\gamma = 0.25$. The stem plot is the Erlang-B formula given by Eq. (5.1) while the line plot is the QD formula given by Eq. (5.6) .

mula:

$$E(\lambda, C) \simeq \left(\sqrt{2\pi C}(1+\gamma)^C e^{-\lambda\gamma}\right)^{-1}. \tag{5.6}$$

Figure 5·2 compares the Erlang-B and QD formulas, for carrier capacities typical to an LTE network (Telesystem Innovations, 2010). Clearly, the values obtained are almost indistinguishable. All of the results presented in the rest of this chapter are based the QD formula. Numerical examples will be provided to confirm their accuracy.

5.3 Optimal Carrier Aggregation

In this subsection, we define the problem of optimal carrier aggregation and provide numerical methods on calculating the level needed. Smith and Whitt (Smith and

Whitt, 1981) show that the Erlang-B formula is *upwards scalable*, that is:

$$E(\lambda, C) > E(\lambda n, Cn). \tag{5.7}$$

Thus, flows in a larger network experience a smaller blocking probability than that in a smaller network operating under a similar traffic load $\rho = \lambda/C$. This result is not surprising to teletraffic engineers, who know that combining two networks into a larger network results in better performance due to statistical multiplexing.

Therefore, when two providers experience similar loads but differ in network sizes in terms of the number of carriers they each possess, the larger provider initially provides an improved service to its users. Hence the smaller provider is inherently at a disadvantage in a competitive spectrum market.

We now turn our attention to the possibility of the smaller provider increasing its capacity by carrier aggregation. This way, the smaller provider can decrease the blocking probability experienced by its users. Let $\psi^*(n)$ denote the *minimum (optimal) level of carrier aggregation* the smaller provider needs to increase its network capacity to a size that achieves the same blocking performance as the larger provider, namely $E(\lambda, C\psi^*(n)) = E(\lambda n, Cn)$[1]. Formally:

$$\psi^*(n) \triangleq \min\{\psi : E(\lambda, C\psi) \leq E(\lambda n, Cn)\}. \tag{5.8}$$

Using the QD formula given by Eq. (5.6), we get:

$$E(\lambda, C\psi) \simeq \left(\sqrt{2\pi C\psi}(1+\gamma')^{C\psi} e^{-\lambda\gamma'}\right)^{-1}, \tag{5.9}$$

$$E(\lambda n, Cn) \simeq \left(\sqrt{2\pi Cn}(1+\gamma)^{Cn} e^{-\lambda n\gamma}\right)^{-1}, \tag{5.10}$$

[1] While $C\psi$ must be an integer value when using Eq. (5.1), there exist continuous relaxations of the Erlang-B formula (Jagerman, 1974). Furthermore, as the capacity tends to infinity in a QD regime, ψ can be treated as continuous.

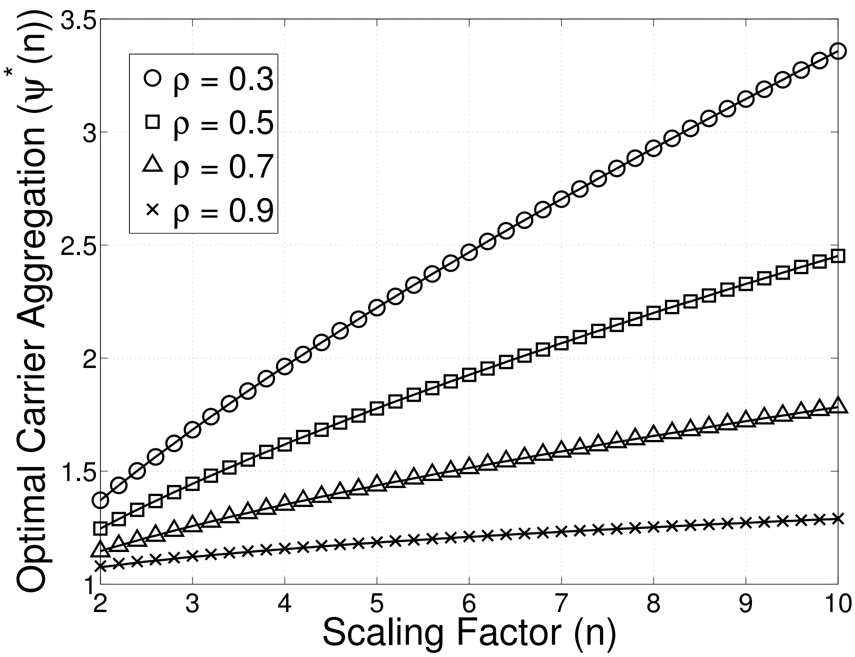


Figure 5·3: Optimal level of carrier aggregation $\psi^*(n)$ of the smaller provider with respect to scaling factor n for different initial traffic loads $\rho = \lambda/C$. Solid lines are exact, markers are QD approximation, and $C = 50$.

where $C\psi = \lambda(1+\gamma')$ and hence $(1+\gamma') = \psi(1+\gamma)$. Then we can rewrite Eq. (5.8) as:

$$\psi^*(n) \triangleq \min\left\{\psi : \sqrt{\frac{\psi}{n}}\left(\frac{(1+\gamma)^C}{e^{\lambda\gamma}}\right)^{\psi-n}\frac{\psi^{C\psi}}{e^{\lambda(\psi-1)}} \geq 1\right\}. \tag{5.11}$$

As the left hand side of the inequality in Eq. (5.11) is increasing in ψ, equivalently $\psi^*(n)$ is the solution of:

$$\sqrt{\frac{\psi}{n}}\left(\frac{(1+\gamma)^C}{e^{\lambda\gamma}}\right)^{\psi-n} e^{\lambda}\left(\frac{\psi^C}{e^{\lambda}}\right)^{\psi} = 1. \tag{5.12}$$

Eq. (5.12) provides a fast way of numerically calculating the optimal level of carrier aggregation needed, which can be achieved using a binary search procedure as the left hand side is increasing in ψ. In Figure 5·3 we illustrate the calculated values of the optimal level of carrier aggregation using the QD formula and the exact Erlang-B formula. One can observe that the calculations based on the QD formula work well: even at a capacity as low as 50 carriers, the maximum percentage error[2] between the QD approximation and the actual Erlang-B calculation is 0.5714%, which occurs when $\rho = 0.9$.

5.4 Structural Properties of Optimal Carrier Aggregation

In this section, we derive structural properties of optimal carrier aggregation. Specifically, we analyze the asymptotic behavior of the optimal carrier aggregation with respect to the scaling factor n. We also show that the amount of carrier aggregation needed diminishes when the initial traffic load at which the providers operate is higher.

5.4.1 Scaling Laws

From Eq. (5.7), it follows that the difference between the blocking probabilities of the two providers increases with the scaling factor n. Thus the disadvantaged provider needs to aggregate more carriers as n gets larger. We next provide asymptotic lower and upper bounds on the optimal level of carrier aggregation as a function of the scaling factor n:

Theorem 5.4.1 ***(Capacity Scaling Law)*** *Consider two providers differing by a scaling factor of n. Then the optimal level of carrier aggregation with respect to the scaling factor as $n \to \infty$ satisfies:*

1. $\psi^*(n) = o\left(\frac{n}{\log(n)}\right)$
2. $\psi^*(n) = \omega\left(n^{\alpha}\right)$, *for any constant $\alpha < 1$,*

where $o(\cdot)$ and $\omega(\cdot)$ are standard asymptotic notations respectively representing strict upper and lower asymptotic limiting behavior of the functions within the parentheses.

---

[2] Calculated by $\left|\frac{\psi^{QD} - \psi^{Erlang}}{\psi^{Erlang}}\right| \cdot 100$, where ψ^{QD} is given by Eq. (5.6) and ψ^{Erlang} is given by Eq. (5.1).

Proof of Theorem 5.4.1. Making use of Eq. (5.2), let us rearrange Eq. (5.12) to obtain:

$$\sqrt{\psi} a^{\lambda\psi} \psi^{\lambda(1+\gamma)\psi} = \sqrt{n} a^{\lambda n} e^{\lambda(\psi-1)},$$

where $a = \dfrac{(1+\gamma)^{(1+\gamma)}}{e^{\gamma}}$. Taking the $\log_a(\cdot)$ (which we will simply denote by $\log(\cdot)$ to alleviate the notation) of both sides and dividing by λ yields:

$$\frac{\log(\psi)}{2\lambda} + (1+\gamma)\psi\log(\psi) + \psi = \frac{\log(n)}{2\lambda} + n + (\psi - 1)\log(e). \tag{5.13}$$

We will prove the upper and lower bounds separately.

1. Let us assume that $\psi = \frac{n}{\log_a(n)}$. We will check the upper bound by substituting for ψ in Eq. (5.13) and showing that as $n \to \infty$, the left hand side is strictly greater than the right hand side. We get:

$$\frac{1}{2\lambda}\log\left(\frac{n}{\log(n)}\right) + \frac{n(1+\gamma)}{\log(n)}\log\left(\frac{n}{\log(n)}\right) + \frac{n}{\log(n)} > \frac{\log(n)}{2\lambda} + n + \frac{n\log(e)}{\log(n)} - \log(e).$$

Canceling the common terms and rearranging, we can rewrite this relationship as:

$$\gamma\log(n) + 1 + \frac{\log(e)\log(n)}{n} > \left(1 + \gamma + \frac{\log(n)}{2\lambda n}\right)\log(\log(n)) + \log(e),$$

which is true for sufficiently large n (as $\log(n) = o(n)$). Thus, we have demonstrated that when $\psi = \frac{n}{\log(n)}$ and n is sufficiently large, the left hand side of Eq. (5.12) is strictly greater than one. Since the left hand side of (5.12) is increasing in ψ, we conclude that there must be some $\psi' < \psi = \frac{n}{\log(n)}$ that satisfies Eq. (5.12).

2. Assume that $\psi = n^{\alpha}$, $\alpha < 1$. We will demonstrate that substituting for ψ in Eq. (5.13) results in the left hand side being strictly smaller than the right hand side

as $n \to \infty$. We get:

$$\frac{\alpha \log(n)}{2\lambda} + \alpha(1+\gamma)n^\alpha \log(n) + n^\alpha < n + \frac{\log(n)}{2\lambda} + \log(e)(n^\alpha - 1).$$

Dividing by n^α and collecting and rearranging the terms we have:

$$\alpha(1+\gamma)\log(n) + 1 + \frac{\log(e)}{n^\alpha} < n^{1-\alpha} + \frac{(1-\alpha)\log(n)}{2\lambda n^\alpha} + \log(e),$$

which, since $n = \omega(\log(n))$, holds as n gets large. Therefore, when $\psi = n^\alpha, \alpha < 1$, the left hand side of Eq. (5.12) is smaller than one. Hence, there must be another $\psi' > \psi = n^\alpha$ that satisfies Eq. (5.12). □

Theorem 5.4.1 states that $n/\log(n)$ and n^α are asymptotic upper and lower bounds on $\psi^*(n)$ respectively. Therefore as the scaling factor increases, the level of optimal carrier aggregation scales sub-linearly but also asymptotically approaches (though never achieves) a linear relationship. This behavior can be observed in Figure 5·3.

5.4.2 Traffic Load

Having provided scaling laws on optimal carrier aggregation with respect to the scaling factor n, we now turn our attention to the scaling with respect to the traffic load.

In the next theorem, we state that the optimal level of carrier aggregation needed by the smaller provider is lower in a market where both providers experience a high initial traffic load. Therefore, in high load markets it is easier for a smaller provider to aggregate spectrum in order to compete.

Theorem 5.4.2 ***(Traffic Load Scaling Law)*** *Let ρ_j denote the traffic load in a market j, which consist of two providers that differ in size by a scale of n such that:*

$$\rho_j = \frac{\lambda_j n}{Cn} = \frac{\lambda_j}{C} \text{ for } j = 1, 2.$$

Further define $\psi_j^*(n)$ to be the optimal level of carrier aggregation for the smaller *provider in the market characterized by load ρ_j. Then, for two given traffic loads, such that $\rho_1 > \rho_2$,*

$$\psi_1^*(n) < \psi_2^*(n).$$

The next two lemmas give inequalities that we will use in the proof of our theorem.

Lemma 5.4.1 *For $\gamma > 0$ and $n > 1$:*

$$(1 + n\gamma)\ln(1 + n\gamma) > n(1 + \gamma)\ln(1 + \gamma).$$

Proof. We start by exponentiating both sides of the equation to obtain:

$$(1 + n\gamma)^{(1+n\gamma)} > (1 + \gamma)^{n(1+\gamma)}.$$

Observe that the lhs is equal to the rhs when $n = 1$. Taking the derivative of both sides of the equation with respect to n we get:

$$\frac{\partial(1 + n\gamma)^{(1+n\gamma)}}{\partial n} = (1 + n\gamma)^{(1+n\gamma)}\gamma(\ln(1 + n\gamma) + 1)$$
$$\frac{\partial(1 + \gamma)^{n(1+\gamma)}}{\partial n} = (1 + \gamma)^{n(1+\gamma)}(1 + \gamma)\ln(1 + \gamma).$$

Using the inequality provided in Eq. (5.3) we can show that:

$$\gamma\ln(1 + n\gamma) + \gamma > \gamma\ln(1 + \gamma) + \gamma \geq (1 + \gamma)\ln(1 + \gamma).$$

Hence if $(1 + n\gamma)^{(1+n\gamma)} \geq (1 + \gamma)^{n(1+\gamma)}$,

$$\frac{\partial(1 + n\gamma)^{(1+n\gamma)}}{\partial n} > \frac{\partial(1 + \gamma)^{n(1+\gamma)}}{\partial n}.$$

Now we will proceed with a contradiction argument. Let $n' > 1$ denote the minimum n which once again satisfies such that:

$$(1 + n'\gamma)^{(1+n'\gamma)} = (1 + \gamma)^{n'(1+\gamma)}.$$

By continuity, there must exist an $n^* < n'$ such that

$$\left.\frac{\partial(1+n\gamma)^{(1+n\gamma)}}{\partial n}\right|_{n^*} < \left.\frac{\partial(1+\gamma)^{n(1+\gamma)}}{\partial n}\right|_{n^*}.$$

However, by the way we have defined n', it must be that:

$$(1+n^*\gamma)^{(1+n^*\gamma)} > (1+\gamma)^{n^*(1+\gamma)}, \tag{5.14}$$

which implies:

$$\left.\frac{\partial(1+n\gamma)^{(1+n\gamma)}}{\partial n}\right|_{n^*} > \left.\frac{\partial(1+\gamma)^{n(1+\gamma)}}{\partial n}\right|_{n^*}.$$

□

Lemma 5.4.2 *For $\gamma > 0$ and $n \geq 1$:*

$$\psi^*(n) < \rho + (1-\rho)n.$$

Proof. Substituting $\rho = \lambda/C = 1/(1+\gamma)$ from Eq. (5.2), this inequality is equivalent to:

$$\psi^*(n) < \frac{1+n\gamma}{1+\gamma}. \tag{5.15}$$

We will proceed to prove this inequality by showing that if $\psi = \frac{1+n\gamma}{1+\gamma}$, the lhs of Eq. (5.12) is strictly greater than the rhs. Substituting ψ into the Eq. (5.12) we obtain:

$$\sqrt{\frac{1+n\gamma}{n+n\gamma}}(1+\gamma)^{-\lambda(n-1)}\left(\frac{1+n\gamma}{1+\gamma}\right)^{\lambda(1+n\gamma)} > 1.$$

Taking the $\ln(\cdot)$ of both sides, $\lambda \to \infty$ and rearranging the terms, one gets the equivalent condition:

$$(1+n\gamma)\ln(1+n\gamma) > n(1+\gamma)\ln(1+\gamma),$$

which we demonstrated in Lemma 5.4.1. By continuity there must be a $\psi' < \frac{1+n\gamma_1}{1+\gamma_1}$

that satisfies Eq.(5.12). □

Proof of Theorem 5.4.2. Assume that the different loads are caused by different arrival rates such that $\lambda_1 > \lambda_2$ while the capacity is kept constant at C. Then we have:

$$C = \lambda_1(1+\gamma_1) = \lambda_2(1+\gamma_2). \tag{5.16}$$

It immediately follows that $(1+\gamma_1) < (1+\gamma_2)$. Using Eq. (5.12), the following need to be satisfied in optimality:

$$\sqrt{\frac{\psi}{n}}\left(\frac{(1+\gamma_1)^C}{e^{\lambda_1\gamma_1}}\right)^{\psi-n}\frac{\psi^{C\psi}}{e^{\lambda_1(\psi-1)}} = 1, \tag{5.17}$$

$$\sqrt{\frac{\psi}{n}}\left(\frac{(1+\gamma_2)^C}{e^{\lambda_2\gamma_2}}\right)^{\psi-n}\frac{\psi^{C\psi}}{e^{\lambda_2(\psi-1)}} = 1. \tag{5.18}$$

Suppose $\psi_2^*(n) = \psi$ and satisfies Eq. (5.18). Then we need to show that the left hand side of Eq. (5.17) is strictly greater than one when $\psi_1^*(n) = \psi$.

Let us rewrite the left hand side of Eq. (5.18) as the following:

$$\sqrt{\frac{\psi}{n}}\left(\frac{(1+\gamma_2)^C}{e^{\lambda_2\gamma_2}}\right)^{\psi-n}\frac{\psi^{C\psi}}{e^{\lambda_2(\psi-1)}} = \sqrt{\frac{\psi}{n}}\left(\frac{(1+\gamma_1)^C}{e^{\lambda_1\gamma_1}}\right)^{\psi-n}\frac{\psi^{C\psi}}{e^{\lambda_1(\psi-1)}}\frac{(\lambda_1/\lambda_2)^{C(\psi-n)}}{e^{(\lambda_1-\lambda_2)(1-n)}}.$$

Now we will demonstrate that:

$$e^{(\lambda_1-\lambda_2)(n-1)} < (\lambda_1/\lambda_2)^{C(n-\psi)}. \tag{5.19}$$

Start by taking the $\ln(\cdot)$ of both sides of (5.19) to get:

$$(\lambda_1-\lambda_2)(n-1) < C(n-\psi)\ln(\lambda_1/\lambda_2).$$

Since $\ln(\lambda_1/\lambda_2) \geq \frac{\lambda_1-\lambda_2}{\lambda_1}$ by inequality (5.3) and $C = \lambda_1(1+\gamma_1)$:

$$C(n-\psi)\ln(\lambda_1/\lambda_2) \geq (1+\gamma_1)(n-\psi)(\lambda_1-\lambda_2). \tag{5.20}$$

From Lemma 5.4.2 we have that $\psi < \rho + (1-\rho)n$, which by substituting $\rho = 1/(1+\gamma_1)$ and rearranging the terms, can be rewritten as:

$$(1+\gamma_1)(n-\psi) > (n-1). \tag{5.21}$$

Combining Eqs. (5.20) and (5.21), we get to the inequality in (5.19). Then we can claim that:

$$\sqrt{\frac{\psi}{n}}\left(\frac{(1+\gamma_1)^C}{e^{\lambda_1\gamma_1}}\right)^{\psi-n}\frac{\psi^{C\psi}}{e^{\lambda_1(\psi-1)}} > \sqrt{\frac{\psi}{n}}\left(\frac{(1+\gamma_2)^C}{e^{\lambda_2\gamma_2}}\right)^{\psi-n}\frac{\psi^{C\psi}}{e^{\lambda_2(\psi-1)}} = 1.$$

Therefore, by continuity and the fact that the left hand side of (5.17) is increasing in ψ, there must be another

$$\psi' < \psi = \psi_2^*(n)$$

that satisfies Eq. (5.17). Hence, $\psi_1^*(n) < \psi_2^*(n)$. □

Theorem 5.4.2 states that the level of carrier aggregation needed to provide a service level that can compete with the larger provider in the market is higher (lower) under a low (high) traffic load, which is also illustrated in Figure 5·3. This implies that the marginal benefit of aggregating spectrum is higher when the providers are operating under a higher load.

5.5 General Bounds

In this section, we seek to establish an upper bound that holds for all possible values of the scaling factor n. We will first establish that optimal carrier aggregation $\psi^*(n)$ is concave in n:

Lemma 5.5.1 ***(Concavity)*** *For $1 \leq n_1 < n_2$,*

$$\psi'^*(n_1) < \psi'^*(n_2).$$

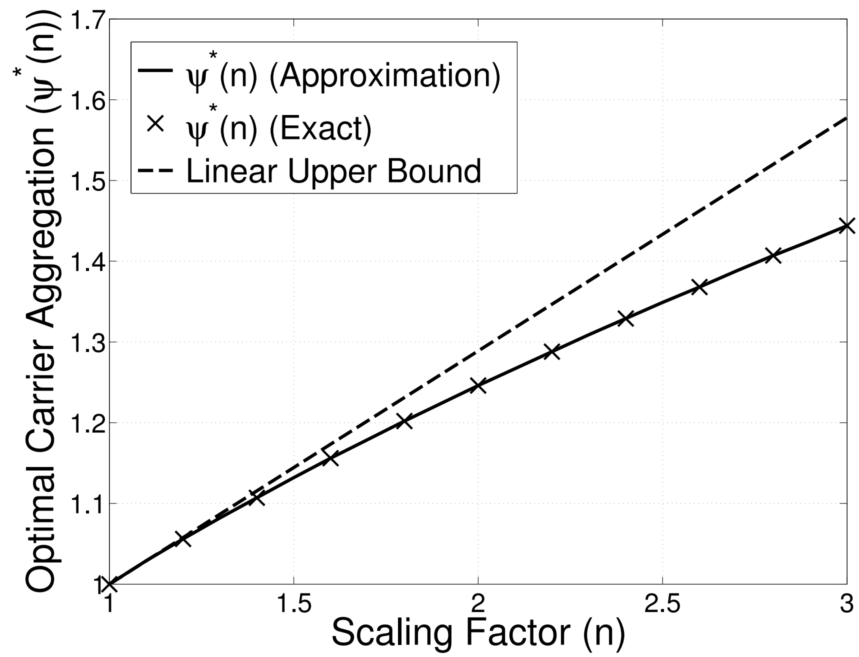


Figure 5·4: Linear upper bound on the optimal level of carrier aggregation $\psi^*(n)$ provided in Theorem 5.5.1, with $C = 50$ and $\rho = 0.5$.

Proof. Let $n_1 = n$ and $n_2 = n + x$, where $x \in \mathbb{R}^+$. We know that $\psi^*(n)$ satisfies Eq. (5.12). Taking the derivative of the both sides of Eq. (5.12) and rearranging the terms one can obtain the following:

$$\psi'^*(n) = \frac{\frac{1}{2n} + C\ln(1+\gamma) - \lambda\gamma}{\frac{1}{2\psi^*(n)} + C\ln(1+\gamma) + C\ln(\psi^*(n))}. \tag{5.22}$$

Note that the derivative depends on the exact value of $\psi^*(n)$. In order to show that

$$\psi'^*(n) < \psi'^*(n+x),$$

we must know something about $\psi^*(n)$ and $\psi^*(n+x)$. One can observe through Eq. (5.12) that when n is incremented by a constant $x > 0$, $\psi^*(n)$ increases by an amount smaller than x such that:

$$\psi^*(n+x) < \psi^*(n) + x.$$

Let $\psi^*(n+x) = \psi^*(n) + \delta$ where $\delta \in (0, x)$. Further define $\xi = C\ln(1+\gamma)$. Then we can make the following statement:

$$\psi'^*(n+x) = \frac{\frac{1}{2n+2x} + \xi - \lambda\gamma}{\frac{1}{2\psi^*(n+x)} + \xi + C\ln(\psi^*(n+x))}$$
$$= \frac{\frac{1}{2n+2x} + \xi - \lambda\gamma}{\frac{1}{2\psi^*(n)+2\delta} + \xi + C\ln(\psi^*(n)+\delta)}.$$

Using the expressions we have obtained for $\psi'^*(n)$ and $\psi'^*(n+x)$, we need to show:

$$\frac{\frac{1}{2n} + \xi - \lambda\gamma}{\frac{1}{2\psi^*(n)} + \xi + C\ln(\psi^*(n))} > \frac{\frac{1}{2n+2x} + \xi - \lambda\gamma}{\frac{1}{2\psi^*(n)+2\delta} + \xi + C\ln(\psi^*(n)+\delta)}. \tag{5.23}$$

Rearranging the terms in Eq. (5.23) we obtain:

$$\frac{\frac{x}{2n(n+x)} + \frac{1}{2n+2x} + \xi - \lambda\gamma}{\frac{\delta}{2\psi^*(n)(\psi^*(n)+\delta)} + \frac{1}{2\psi^*(n)+2\delta} + b + C\ln(\psi^*(n))} > \frac{\frac{1}{2n+2x} + \xi - \lambda\gamma}{\frac{1}{2\psi^*(n)+2\delta} + b + C\ln(\psi^*(n)+\delta)}. \tag{5.24}$$

Since $\frac{x}{2n(n+x)} > 0$ if we can show that:

$$\frac{\frac{1}{2n+2x} + \xi - \lambda\gamma}{\frac{\delta}{2\psi^*(n)(\psi^*(n)+\delta)} + \frac{1}{2\psi^*(n)+2\delta} + b + C\ln(\psi^*(n))} > \frac{\frac{1}{2n+2x} + \xi - \lambda\gamma}{\frac{1}{2\psi^*(n)+2\delta} + b + C\ln(\psi^*(n)+\delta)},$$

Eq. (5.24) also holds. We can rewrite the last inequality in the following fashion:

$$C\ln(\frac{\psi^*(n)+\delta}{\psi^*(n)}) > \frac{\delta}{2\psi^*(n)(\psi^*(n)+\delta)}. \tag{5.25}$$

By Eq. (5.3),

$$C\ln\left(\frac{\psi^*(n)+\delta}{\psi^*(n)}\right) \geq \frac{C\delta}{\psi^*(n)+\delta}. \tag{5.26}$$

Furthermore,

$$\frac{C\delta}{\psi^*(n)+\delta} > \frac{\delta}{2\psi^*(n)(\psi^*(n)+\delta)} \tag{5.27}$$

since $C \geq 1$ and $\psi^*(n) \geq 1$. Combining Eqs. (5.26) and (5.27), we obtain Eq. (5.25), thus confirming Eq. (5.23). □

Given the derivative of $\psi^*(n)$ is decreasing in n, we next establish the tightest possible linear upper bound on $\psi^*(n)$:

Theorem 5.5.1 ***(Linear Upper Bound)*** *For $\gamma > 0$ and $n \geq 1$:*

$$\psi^*(n) \leq (1 - f(\rho)) + f(\rho)n, \tag{5.28}$$

where $f(\rho) = 1 - \dfrac{1-\rho}{\frac{1}{2C} + \ln(1/\rho)}$.

Proof of Theorem 5.5.1. We will start our proof by providing a linear function of the form $g(n) = a + bn$, where a and b are constants, that is equal to $\psi^*(n)$ when $n = 1$ and has the same derivative at that point. From the way we defined $\psi^*(n)$ in Eq. (5.8), it follows that $\psi^*(1) = 1$. Then $g(n) = (1 - b) + bn$ in order to satisfy this inequality.

Taking the derivative of the both sides of Eq. (5.12) and rearranging the terms one can obtain the following:

$$\psi'^*(n) = \frac{\frac{1}{2n} + C\ln(1+\gamma) - \lambda\gamma}{\frac{1}{2\psi^*(n)} + C\ln(1+\gamma) + C\ln(\psi^*(n))}. \tag{5.29}$$

Note that Eq. (5.29) depends on the exact value of $\psi^*(n)$. Evaluating this expression at $\psi^*(1) = 1$ yields:

$$\left.\frac{d\psi^*(n)}{dn}\right|_{n=1,\psi^*(1)=1} = \frac{\frac{1}{2} + C\ln(1+\gamma) - \lambda\gamma}{\frac{1}{2} + C\ln(1+\gamma)}. \tag{5.30}$$

Rearranging the terms in Eq. (5.30) and substituting ρ for $\frac{1}{1+\gamma}$, we obtain:

$$\left.\frac{d\psi^*(n)}{dn}\right|_{n=1,\psi^*(1)=1} = 1 - \frac{1-\rho}{\frac{1}{2C} + \ln(1/\rho)}.$$

Then $b = 1 - \dfrac{1-\rho}{\frac{1}{2C} + \ln(1/\rho)}$ and

$$g(n) = \frac{1-\rho}{\frac{1}{2C} + \ln(1/\rho)} + \left(1 - \frac{1-\rho}{\frac{1}{2C} + \ln(1/\rho)}\right) n.$$

Now we will show that $g(n) \geq \psi^*(n)$ for $n \geq 1$. Observe that $g(1) = \psi^*(1)$. In Lemma 5.5.1 we have established that the derivative of optimal carrier aggregation with respect to the scaling factor n is decreasing in n. Then we can state that

$$\frac{dg(n)}{dn} \geq \frac{d\psi^*(n)}{dn} \text{ for any } n \geq 1.$$

Let $h(n) = g(n) - \psi^*(n)$. Taking the derivative with respect to n we get:

$$\frac{dh(n)}{dn} = \frac{dg(n)}{dn} - \frac{d\psi^*(n)}{dn} \geq 0.$$

By the mean value theorem there exists an n_0 such that:

$$\frac{dh(n_0)}{dn} = \frac{h(n) - h(1)}{n-1} = \frac{g(n) - \psi^*(n)}{n-1} \geq 0.$$

Since $n \geq 1$ we conclude that $g(n) \geq \psi^*(n)$. □

Theorem 5.5.1 provides a way to quickly calculate an upper bound on the optimal carrier aggregation, which is rather tight for small values of the scaling factor n as illustrated in Figure 5·4. However, since $\psi^*(n)$ is concave, as the scaling factor increases, the linear upper bound diverges from the actual value. The strength of the linear upper bound that we provide lies in its ability to provide simple insight on the impact of network parameters on optimal carrier aggregation.

As a possible solution to the divergence of the linear upper bound, one could seek to obtain a piece-wise linear upper bound expression on $\psi^*(n)$ by using the results provided in Lemma 5.5.1 and Theorem 5.5.1. Starting at $\psi^*(1) = 1$, one can use the linear bound provided in Eq. (5.28) to approximate the value of $\psi^*(n)$ at a larger value of n, which can then be used to obtain the derivative $\psi'(n)$ provided in Eq. (5.29). The derivative value can then be assumed to be the linear slope of $\psi^*(n)$, and the calculation procedure starts over.

Algorithm 1 Piecewise Upper Bound Calculation

```
procedure BOUND(ψ, Scale, StepSize)
    Initialize: ψ ← 1, n ← 1
    Set counter: State ← 1
    EvaluationPoints ← Floor(Scale/StepSize)
    while EvaluationPoints ≥ State do
        n ← n + StepSize
        ψ ← (1 − f(ρ)) + f(ρ)n
        f(ρ) ← ψ'*(n)|_ψ
        State ← State + 1
    return Bound
```

Next, we propose a simple algorithmic procedure to compute a piece-wise linear bound on $\psi^*(n)$ (see Algorithm 1). The algorithm takes as input the scaling factor n, referred to as *Scale*, as well as the step size, referred to as *StepSize*, that defines the distance between points at which the slope of the bound is recalculated. The procedure starts from the known point of $\psi^*(1) = 1$ and uses the linear bound established to calculate the bound on ψ^* at every evaluation point determined by the step size until the target scaling factor is reached.

Using Algorithm 1, if the step size is chosen small enough, the bound on $\psi^*(n)$ will approach the real value. Therefore, one can obtain a relatively tight piecewise linear upper bound on $\psi^*(n)$, which is illustrated in Figure 5·5 for several different traffic loads, with a step size of 1.

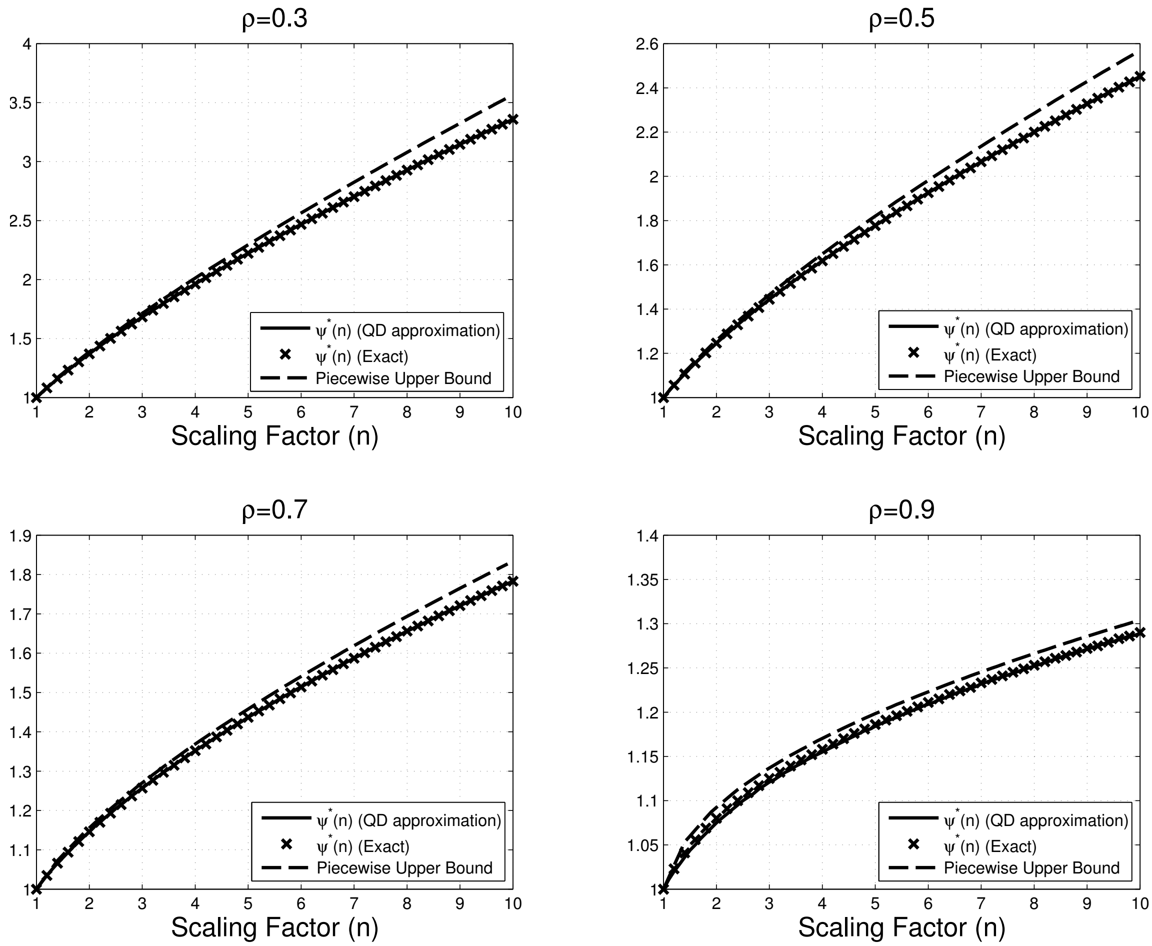


Figure 5·5: $C = 50$ - Piecewise linear upper bounds on the optimal level of carrier aggregation $\psi^*(n)$ obtained by the initial linear upper bound provided in Theorem 5.5.1, the slope of which is then adjusted at integer values of the scaling factor n using the derivative expression provided in Eq. (5.29).

5.6 Numerical Examples

In this section, we provide numerical examples, where we calculate how much spectrum needs to be aggregated to preserve competition in different markets.

Consider two providers in a spectrum market with network parameters given as follows:

$$(\lambda_1, C_1) = (90, 150) \text{ and } (\lambda_2, C_2) = (60, 100).$$

Bandwidth	1.4 MHz	3 MHz	5 MHz	10 MHz	15 MHz	20 MHz
Resource Blocks	6	15	25	50	75	100

Table 5.1: LTE bandwidth configurations and associated number of resource blocks as specified in 3GPP Release 8 (3GPP, 2012).

The capacity numbers provided here are in line with the spectrum holdings of Verizon and T-Mobile in the New York City area, which respectively are 30 MHz and 20 MHz (translated into the number of resource blocks from Table 5.1), according to the FCC's Spectrum Dashboard (Federal Communications Commission,). In this example, the scaling factor is $n = 150/100 = 1.5$ and both providers are in a moderately loaded market with $\rho = 90/150 = 60/100 = 0.60$. Using Equation (5.12), we obtain the carrier aggregation needed by the smaller provider: $\psi^*(1.5) = 1.102$.

This result tells us that in order to provide the same level of service as the larger provider, the smaller provider needs to increase its capacity at least by 1.102 times its current value. Therefore, $\lceil 100 \times 0.102 \rceil = 11$ additional carriers are needed to bring the smaller provider's service level in line with that of the larger provider. Taking a single carrier to be a resource block in an LTE deployment, the smallest LTE bandwidth that matches this requirement has a bandwidth of 3 MHz from Table 5.1. This is the amount of spectrum that the smaller provider needs to aggregate in order to guarantee its ability to compete with the larger provider.

Next, we consider two different markets: (i) a market where the spectrum holdings of the providers have the same scaling factor but the traffic load ρ is higher and (ii) a market where there is an increase in the scaling factor n while the traffic load ρ is the same.

(i) Consider a market where the scaling factor is $n = 1.5$ while the traffic load of the market increases to $\rho = 0.9$. The parameters of the two providers are now as follows:

$$(\lambda_1, C_1) = (135, 150) \text{ and } (\lambda_2, C_2) = (90, 100).$$

Under the new load, the carrier aggregation needed by the smaller provider becomes $\psi^*(1.5) = 1.037$.

Thus, $\lceil 100 \times 0.037 \rceil = 4$ additional carriers are needed by the smaller provider, fewer than the number of carriers calculated before and in line with Theorem 5.4.2. Under the same LTE scenario considered previously, Table 5.1 indicates that aggregating a minimum of 1.4 MHz of spectrum in the market with a higher traffic load is enough to achieve the same goal.

(*ii*) This time, we consider a market where the scaling factor is increased to $n = 6$ but the traffic load is the same as the first market (*i.e.*, $\rho = 0.6$). The parameters of the providers are given as follows:

$$(\lambda_1, C_1) = (90, 150) \text{ and } (\lambda_2, C_2) = (15, 25).$$

These numbers are in line with the spectrum holdings of Verizon and T-Mobile in Logan County, IL, which respectively are 30 MHz and 5 MHz (translated into the number of resource blocks from Table 5.1), according the FCC's Spectrum Dashboard (Federal Communications Commission,). In this case, the carrier aggregation needed by the smaller provider is $\psi^*(6) = 1.719$.

This time, the smaller provider needs an additional $\lceil 100 \times 0.719 \rceil = 72$ carriers. Notice that the increase in the total capacity needed is smaller than the increase in the scaling factor since:

$$\psi^*(6)/\psi^*(1.5) = 1.559 < 6/1.5 = 4.$$

Under the same LTE scenario considered previously, Table 5.1 indicates that aggregating a minimum of 15 MHz of spectrum is needed to achieve the same goal, as the scaling factor increases to 6.

5.7 Extension to Delay Systems

In Section 5.2, we presented a QD regime approximation of the Erlang-B formula through Lemma 5.2.1. The assumption was that if all the carriers are busy upon the arrival of a flow, then the flow is lost. This is referred to as a *loss* system.

Our results can easily be extended to a *delay* system. In such a system, all active flows share the entire network capacity. If the number of flows exceeds C, then the flows can still be transmitted but at a rate below their peak rate. In that case, the flows will experience congestion and additional delay. The probability that an arrival flow experiences congestion is given by the Erlang-C formula:

$$E_c(\lambda, C) = \frac{\frac{\lambda^C}{C!}\frac{C}{C-\lambda}}{\sum_{k=0}^{C-1}\frac{\lambda^k}{k!} + \frac{\lambda^C}{C!}\frac{C}{C-\lambda}}.$$

This equation holds for the same general traffic model as presented in Section 5.1 (Bonald and Roberts, 2012).

Using the results of (Zeltyn, 2004) for the analysis of queuing systems in the QD regime we have:

$$E_c(\lambda, C) \simeq \left(\sqrt{2\pi C}\gamma(1+\gamma)^{C-1}e^{\lambda\gamma}\right)^{-1}. \tag{5.31}$$

Through following similar steps as in Section 5.3 and replacing the QD formula of Erlang-B with Eq. (5.31), it is possible to show that the optimal carrier aggregation in a delay system is given by:

$$\psi_c^*(n) = \min\left\{\psi : \sqrt{\frac{\psi}{n}}\left(\frac{(1+\gamma)^C}{e^{\lambda\gamma}}\right)^{\psi-n}\frac{\psi^{C\psi}}{e^{\lambda(\psi-1)}}\left(1+\frac{\psi-1}{\psi\gamma}\right) \geq 1\right\}. \tag{5.32}$$

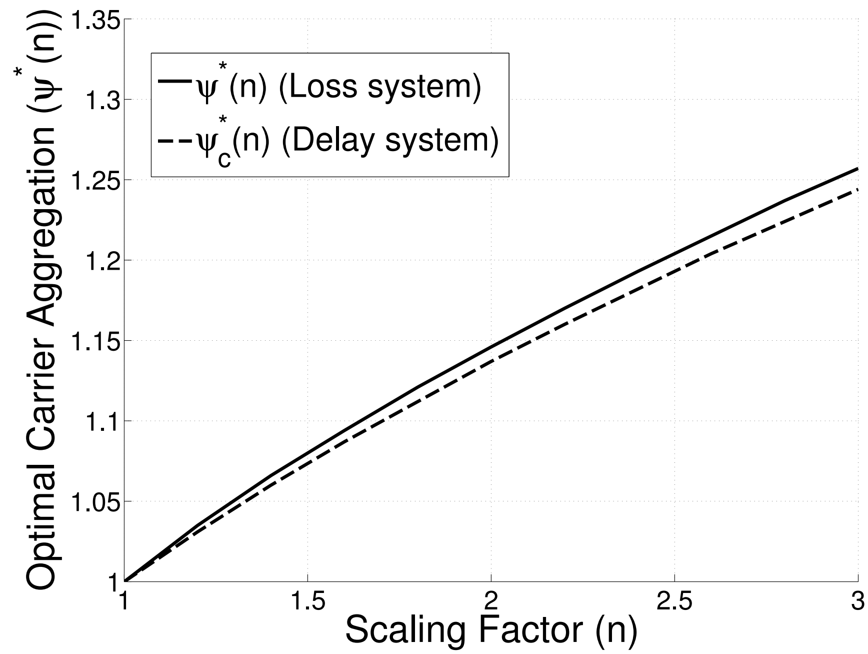


Figure 5·6: Comparison of optimal carrier aggregation under loss and delay systems, with $C = 50$ and $\rho = 0.7$.

As the left hand side of the inequality in Eq. (5.11) is increasing in ψ, equivalently $\psi_c^*(n)$ is the solution of:

$$\sqrt{\frac{\psi}{n}}\left(\frac{(1+\gamma)^C}{e^{\lambda\gamma}}\right)^{\psi-n}\frac{\psi^{C\psi}}{e^{\lambda(\psi-1)}}\left(1+\frac{\psi-1}{\psi\gamma}\right)=1. \tag{5.33}$$

Note that Eq. (5.33) is the same as Eq. (5.12) except for the $1+\frac{\psi-1}{\psi\gamma}$ term at the end. Since $1+\frac{\psi-1}{\psi\gamma} > 1$ for $\psi > 1$, one quickly concludes that the left hand side of Eq. (5.33) is always strictly greater than the left hand side of Eq. (5.12). In other words, for the same system parameters, the level of optimal carrier aggregation under the Erlang-C delay model is always smaller than that under the Erlang-B loss model:

$$\psi_c^*(n) < \psi^*(n)$$

Figure 5·6 illustrates this relationship. By replacing Eq. (5.12) with Eq. (5.33) and following a similar analysis, the same structural properties given in Theorems 5.4.1

and 5.4.2 also hold for the QD Erlang-C formula. To give an example, we revisit Theorem 5.4.1 here:

Theorem 5.7.1 ***(Erlang-C Capacity Scaling Law)*** *Consider two providers differing by a scaling factor of n. Then the optimal level of carrier aggregation with respect to the scaling factor n satisfies:*

1. $\psi_c^*(n) = o\left(\frac{n}{\log(n)}\right)$
2. $\psi_c^*(n) = \omega\left(n^{\alpha}\right)$, *for any* $\alpha < 1$,

where $o(\cdot)$ and $\omega(\cdot)$ respectively represent strict upper and lower asymptotic limiting behavior on the function within the parentheses.

Proof of Theorem 5.7.1. Let us rearrange Eq. (5.33) to obtain:

$$\sqrt{\frac{\psi}{n}}\left(\frac{(1+\gamma)^C}{e^{\lambda\gamma}}\right)^{\psi-n}\frac{\psi^{C\psi}}{e^{\lambda(\psi-1)}} = \frac{\psi\gamma}{\psi(1+\gamma)-1}. \tag{5.34}$$

We will prove that $n/log(n)$ and n^{α} are still asymptotic upper and lower bounds by showing that the term $\frac{\psi\gamma}{\psi(1+\gamma)-1}$ converges to a constant and thus does not affect the asymptotic relationships.

1. Assume that $\psi = \frac{n}{\log(n)}$. Then replacing ψ in the right hand side term of Eq. (5.34) we obtain:

$$\frac{\psi\gamma}{\psi(1+\gamma)-1} = \frac{\frac{n\gamma}{\log(n)}}{\frac{n}{\log(n)}(\gamma+1)-1},$$

and:

$$\lim_{n\to\infty}\frac{\frac{n\gamma}{\log(n)}}{\frac{n}{\log(n)}(\gamma+1)-1} = \frac{\gamma}{1+\gamma} = (1-\rho).$$

2. This time, assume that $\psi = n^{\alpha}, \alpha < 1$. Then replacing ψ in the right hand side term of Eq. (5.34) we obtain:

$$\frac{\psi\gamma}{\psi(1+\gamma)-1} = \frac{n^{\alpha}\gamma}{n^{\alpha}(1+\gamma)-1},$$

and

$$\lim_{n\to\infty} \frac{n^\alpha \gamma}{n^\alpha(1+\gamma)-1} = \frac{\gamma}{1+\gamma} = (1-\rho).$$

□

5.8 Applications to Secondary Spectrum Markets

We next discuss how our results on carrier aggregation apply to pricing games in secondary spectrum markets. In Chapter 3, we identified the minimum (break-even) price at which it is profitable for a provider implementing a coordinated access policy to start admitting secondary users. The *break-even price* p^{BE} is directly linked to the Erlang-B formula:

$$p^{BE} = KE(\lambda, C). \tag{5.35}$$

The break-even price plays a critical role in determining the Nash equilibrium of a game where two providers compete in prices to attract secondary demand. Without loss of generality, suppose that the break-even price of provider 1 is lower than that of provider 2. Recall that Theorem 3.4.1 provided in Chapter 3 states the competition results in a price war that is won by provider 1 (*i.e.*, provider 1 captures the entire market). One concludes that the outcome of the pricing game is directly related to the break-even prices, which in turn relate to the Erlang-B formula.

Hence, the level of optimal carrier aggregation acts as an identifier of necessary network provisioning to obtain a competitive price advantage in a secondary spectrum market. All of our previous results, such as the structural properties with respect to scaling factors and traffic loads and the established general bounds can be readily applied to the question of how to strategically implement carrier aggregation in a secondary spectrum market.

5.9 Summary

In this chapter of the dissertation, we investigated the impact adding additional capacity to a network provider through carrier aggregation. We accomplished this by providing computational methods, scaling laws, and bounds on the optimal carrier aggregation. Under a QD regime, we derived an approximation of the Erlang-B formula. This approximation is highly accurate as long as the number of carriers is large enough (*e.g.*, above 50) and the spectrum utilization does not approach 100% (*e.g.*, 90% or below), an assumption that is consistent with measurement studies (Valenta et al., 2010).

Using the QD formula, we investigated optimal carrier aggregation by proving two scaling laws: (i) with respect to the scaling factor n and (ii) with respect to the traffic load. Specifically, we obtained sub-linear (though close to linear) asymptotic upper and lower bounds in the form $\psi^*(n) = o\left(n/\log(n)\right)$ and $\psi^*(n) = \omega\left(n^\alpha\right)$ for any $\alpha < 1$. Then, we demonstrated that if the traffic load under which each provider operates increases, then the level of carrier aggregation required is reduced. This result indicates that the marginal benefit of carrier aggregation in a heavily loaded network is higher than that in a lightly loaded network.

Next, we derived an upper bound on $\psi^*(n)$ that applies to any value of n and is provably the tightest possible. This upper bound explicitly relates to the network parameters and can provide regulators and market players with useful guidelines. We also provided a method of improving it to a piece-wise linear bound by iteratively approximating $\psi'(n)$.

We explained how the results derived for loss systems, based on the Erlang-B formula, extend to delay systems based on the Erlang-C formula. We proved that for the same network parameters, the optimal level of carrier aggregation in a delay system is always smaller than in a loss system. Finally, we provided a relationship between the profitable pricing of users in secondary spectrum markets and the Erlang-B formula for which our results apply. Hence, the results on optimal carrier aggregation presented in this chapter are directly applicable to pricing strategies in secondary spectrum markets, where providers can aggregate spectrum to lower their prices in a possible price war.

Chapter 6

Networks with Spatial Interferences

In the previous chapters we have limited our analysis to a single access point (AP) in a wireless landscape. However, in reality wireless networks consist of many APs covering different geographical areas and depending of their distance some of these APs will be subject to interference. Therefore, to accurately model wireless network behavior it is important to consider the spatial aspect of wireless spectrum. In this chapter we extend on our previous analysis of profitability for a single AP presented in Chapters 3 and 4 to a network consisting of many APs and experiences interference. We describe the spatial model being considered and propose a decoupling method to characterize the stochastic behavior of APs under interference. Using this approximated model we then identify the profitability conditions for admitting secondary users into a network under the coordinated access policy described in Chapter 3.

6.1 Spatial Model

In this section we introduce the network model considered and the accompanying notation. Consider a cellular provider whose network is given by the graph $G = (I, E)$ where $|I| = n$. Each vertex $i \in I$ represents a cell which has a dedicated primary-user base whose traffic arrivals form a Poisson process with rate $\lambda_i > 0$ (i.e., the average number of requests per unit time), and a finite number of spectrum bands C_i on which these arrivals are serviced. For each served primary user, provider i collects a

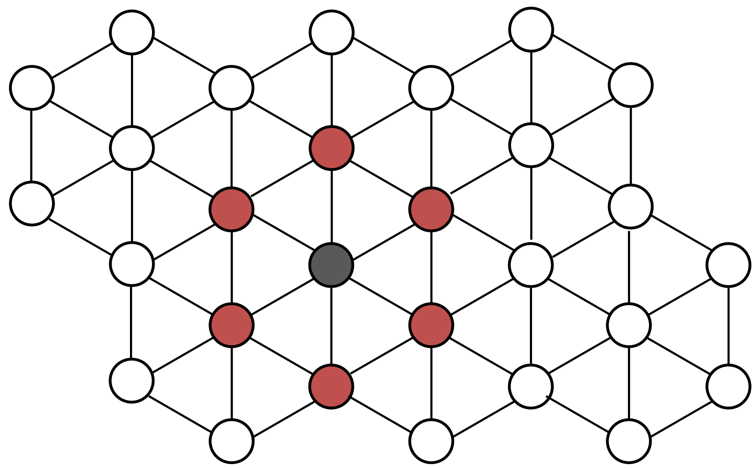


Figure 6·1: A Hexagonal Lattice Topology Network. Cells that are in the interference neighborhood of the black cell are colored red.

reward of K units. The primary users occupy a single unit of capacity for a duration that is generally distributed with mean μ, which we assume to be 1, without loss of generality, independently of other requests and arrival times.

In addition to the primary arrivals, the provider at a single cell i faces a possible independent secondary arrival process with rate σ_i that is also Poisson. The intensity of secondary arrivals is a function of the price p charged per access, denoted by $\sigma(p)$. Here $\sigma(\cdot)$ is the *demand function*, which is assumed to be continuous and non-increasing. We assume that the secondary users have the same channel holding statistics as the primary users. This assumption is valid when both types of traffic are generated by similar applications.

Let $x_i \in \{0, 1, 2, \ldots, C_i\}$ denote the occupancy level of cell i. Each cell i has a certain number of neighboring cells denoted by the set $N(i) = \{j \mid (j, i) \in E\}$, which consists of the cells that share an edge with cell i. Let w_{ij} denote the level of interference of a call arrival at cell i with cell j, such that in the duration of the call

w_{ij} units of resources are also occupied at cell j. We assume that the interference is symmetric such that $w_{ij} = w_{ji}$. Self interference is allowed such that $w_{ii} = 1$. A network occupancy is feasible if for all cells $i \in \mathcal{I}$:

$$x_i + \sum_{j \in N(i)} w_{ji} x_j \leq C_i. \tag{6.1}$$

Figure 6·2 illustrates the model being considered on a hexagonal topology common to cellular networks.

We define $\boldsymbol{\lambda}$, $\boldsymbol{C}$, and $\boldsymbol{x}$ as the corresponding vectors containing the individual parameters of all cells in the network G. Further, let $\boldsymbol{w_i} = [w_{i1}, w_{i2}, ..., w_{iI}]$ denote the interference vector of cell i in the network.

The provider uses an admission policy to decides whether to admit or reject a secondary arrival to its network, which we denote by A_i. We assume that A_i depends only on the number of each class of users (primary and secondary) in the system and belongs to the class of *occupancy-based* policies. The performance of occupancy-based policies are insensitive to the call length distribution except through the mean (Mutlu et al., 2010). Hence, without loss of generality, we can assume exponentially distributed service times for the purpose of analysis in the rest of the chapter.

Since network cells have a finite number of channels, they cannot accommodate new requests if all of the channels are occupied. This results in some requests being blocked. We define $B_{i,k}(\boldsymbol{\lambda}, \sigma_i, A_i)$ to be the blocking probability for class k users ($k = 1$ for primary and 2 for secondary) when secondary demand given by σ_i is admitted at cell i and the access policy implemented for secondary users in cell is A_i.

The goal of the provider is to maximize the total revenue collected. The revenue rate of cell i when the network services secondary demand of σ_i units at cell i is given by:

$$R_{ii}(p, \sigma_i, A_j) = (1 - B_{i,2}(\boldsymbol{\lambda}, \sigma_i, A_i))\sigma_i p + (1 - B_{i,1}(\boldsymbol{\lambda}, \sigma_i, A_i))\lambda_i K$$

whereas the revenue rate of cell $j \neq i$ is given by:

$$R_{ij}(p, \sigma_i, A_j) = (1 - B_{j,1}(\boldsymbol{\lambda}, \sigma_i, A_i))\lambda_j K.$$

For cell i, the first and the second terms are respectively the revenue generated by secondary and primary requests being serviced. For cells other than i, the only revenue is that collected from the primary users. Each term represents the expected long time rates per unit time. The total reveue rate of the network is given by the summation of the revenue rates over all cells in the network G:

$$R(p, \sigma_i, A_i) = (1 - B_{i,2}(\boldsymbol{\lambda}, \sigma_i, A_i))\sigma_i p + \sum_{j \in I}(1 - B_{j,1}(\boldsymbol{\lambda}, \sigma_i, A_i))\lambda_j K. \tag{6.2}$$

6.2 First-order Erlang Fixed Point Approximation

In this section we will consider a network in the absence of secondary users and provide a decomposition model that effectively removes interference between the cell while still capturing its impact of interference on the network. We start by providing a quick explanation of Kelly's work on loss network where he introduced the use of Erlang Fixed Point Approximation (EFPA) to approximate the blocking probabilities (Kelly, 1991; Kelly, 1986).

For a single cell the probability that an arriving call will be blocked is given by the well-established Erlang-B formula:

$$E(\lambda, C) = \frac{\frac{\lambda^C}{C!}}{\sum_{k=0}^{C} \frac{\lambda^k}{k!}}. \tag{6.3}$$

In a network setting, however, where a call arrival uses resources in multiple cells, the Erlang-B formula is no longer valid. Kelly in (Kelly, 1986) provides the well-studied Erlang Fixed Point Approximation formula that yields the loss probabilities in a network. The approximation assumes that each cell blocking is independent from the rest of the network and the call arrivals are thinned accordingly. Al Daoud et. al. provide the EFPA formula in (Al Daoud et al., 2007), which is given by the following set of equations:

$$b_i = E\left((1-b_i)^{-1}\sum_{j\in I} w_{ij}\lambda_j \prod_{k\in I}(1-b_k)^{w_{jk}}, C_i\right) \quad \text{for } i = 1,2,3,\cdots,n. \tag{6.4}$$

If a call arrival to cell i is blocked with probability b_i and assuming that such events are independent, the effective load at cell i is given by $\sum_j w_{ij}\lambda_j \prod_k (1-b_k)^{w_{jk}}$. Eq. (6.4) states that the blocking probability at cell i is consistent with the load it faces. The set of equations (6.4) has a unique solution (Kelly, 1991); hence the approximation is well defined.

The loss probability of a call arriving in cell i is simply the probability of not being able to accommodate the call at all cells which are being affected by its arrival. This probability is given by the following:

$$B_i(\boldsymbol{\lambda}, \boldsymbol{C}) = 1 - \prod_{j\in I}(1-b_j)^{w_{ij}}. \tag{6.5}$$

Kelly has shown in (Kelly, 1991) that when both the arrival rates and capacities at the network are increased in line with the other, *i.e.,*

$$\lim_{n\to\infty}\frac{1}{n}\lambda_i(n) = \lambda \text{ and } \lim_{n\to\infty}\frac{1}{n}C_i(n) = C.$$

the blocking probabilities calculated by the approximation in Eq. (6.5) converge to

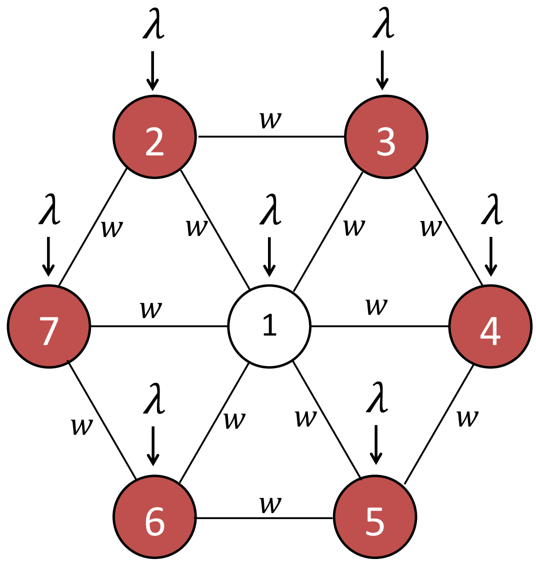


Figure 6·2: A symmetric hexagonal network - Red cells are in the interference neighborhood of the white cell, where a call arrival to the white cell will also occupy w units capacity in the red cells.

their exact values since it has been demonstrated in (Kelly, 1991) that:

$$1 - B_i(\boldsymbol{\lambda}, \boldsymbol{C}) = \prod_{j \in I} (1 - b_j)^{w_{ij}} + o(n^{-1/2}). \tag{6.6}$$

For small interference values w_{ij}, making use of the binomial series expansion, we can make the following approximation:

$$1 - B_i(\boldsymbol{\lambda}, \boldsymbol{C}) = \prod_{j \in I} (1 - b_j)^{w_{ij}} \approx \prod_{j \in I} (1 - w_{ij} b_j). \tag{6.7}$$

Now we consider only the first iteration of Eq. (6.4) in an effort to remove the coupling between the cells in the network, which then allows us to treat each cell as an independent queue. For relatively low blocking regimes such that almost all of the arrivals to the network are accepted, Eq. (6.4) is well approximated by:

$$\hat{b}_i = E\left(\sum_{j \in I} w_{ij}\lambda_j, C_i\right) = E\left(\lambda_i + \sum_{j \in N(i)} w_{ij}\lambda_j, C_i\right) \tag{6.8}$$

Therefore it is as if the total arrival rate at cell i is the sum of its own arrival rate and its neighbors' arrival rates multiplied by the respective probability of interference at each neighbor cell. Let

$$\lambda'_i = \lambda_i + \lambda_{N(i)}, \tag{6.9}$$

where λ'_i is the *effective load* at cell i and $\lambda_{N(i)} = \sum_{j \in N(i)} w_{ij}\lambda_j$ is the effective interfering arrival rate from the neighboring cells. Hence, the blocking at cell i in the first-order approximation is given by the Erlang-B formula under the effective load:

$$\hat{b}_i = E(\lambda'_i, C_i). \tag{6.10}$$

In the next lemma we will show that as the network is scaled up with the load (λ/C) kept constant, the Erlang-B formula tends to zero.

Lemma 6.2.1 *For* $\lambda < C$,

$$\lim_{n\to\infty} E(\lambda n, Cn) = 0. \tag{6.11}$$

Proof. Smith and Whitt's work (Smith and Whitt, 1981) (page 54) has shown that we can express the Erlang-B formula in the following form:

$$E(\lambda n, Cn)^{-1} = \int_0^\infty e^{-x}\left(\frac{n + x/\lambda}{n}\right)^{Cn} dx. \tag{6.12}$$

To determine the limit, we first need to determine:

$$\lim_{n\to\infty}\left(\frac{n + x/\lambda}{n}\right)^{Cn}. \tag{6.13}$$

We can rewrite Eq. (6.13) in the following form:

$$\lim_{n\to\infty}\left(\frac{n+x/\lambda}{n}\right)^{Cn} = e^{\lim_{n\to\infty} Cn\log\left(\frac{n+x/\lambda}{n}\right)}.$$

Taking $t = 1/n$, we obtain:

$$\lim_{n\to\infty} Cn\log\left(\frac{n+x/\lambda}{n}\right) = \lim_{t\to 0} C\frac{\log(1+tx/\lambda)}{t}.$$

Using L'Hospital's rule:

$$\begin{aligned}\lim_{t\to 0} C\frac{\log(1+tx/\lambda)}{t} &= \lim_{t\to 0} C\frac{\frac{d\log(1+tx/\lambda)}{dt}}{\frac{dt}{dt}}\\ &= \lim_{t\to 0} C\frac{x/\lambda}{1+tx/\lambda} = Cx/\lambda.\end{aligned}$$

Now we can take the limit:

$$\begin{aligned}\lim_{n\to\infty} E(\lambda n, Cn)^{-1} &= \lim_{n\to\infty}\int_0^\infty e^{-x}\left(\frac{n+x/\lambda}{n}\right)^{Cn} dx\\ &= \int_0^\infty e^{-x}e^{Cx/\lambda}dx = \infty\end{aligned}$$

for $\lambda < C$ which implies $\lim_{n\to\infty} E(\lambda n, Cn) = 0$. □

Now we present our first theorem, which establishes that the first-order approximation to the EFPA is also asymptotically exact as the cell arrivals and capacities increase proportionally.

Theorem 6.2.1 *Let* $\boldsymbol{b} = (b_1, b_2, ..., b_I)$ *denote the solution to EFPA given in Eq. (6.4). Then in a network where* $\sum_{j\in I} w_{ij}\lambda_j < C_i$ *for all* i*:*

w=0.05, λ=8	Cell 1	Cell 2	Cell 3	Cell 4	Cell 5	Cell 6	Cell 7
B_i (simulated)	2.9319e-3	9.8786e-4	9.7598e-4	9.8269e-4	9.8703e-4	9.7383e-41	9.8934e-4
$\hat{B}_i$ (approximated)	2.9823e-3	9.9993e-4	9.9993e-4	9.9993e-4	9.9993e-4	9.9993e-4	9.9993e-4
$\hat{b}_i$ (approximated)	2.7477e-3	7.8410e-4	7.8410e-4	7.8410e-4	7.8410e-4	7.8410e-4	7.8410e-4
Error (%)	1.72	1.22	2.45	1.75	1.31	2.68	1.07
w=0.15, λ=5	**Cell 1**	**Cell 2**	**Cell 3**	**Cell 4**	**Cell 5**	**Cell 6**	**Cell 7**
B_i (simulated)	1.1394e-3	2.2178e-4	2.2042e-4	2.2218e-4	2.2131e-4	2.2196e-4	2.2113e-4
$\hat{B}_i$ (approximated)	1.1461e-3	2.2674e-4	2.2674e-4	2.2674e-4	2.2674e-4	2.2674e-4	2.2674e-4
$\hat{b}_i$ (approximated)	1.1039e-3	4.6991e-5	4.6991e-5	4.6991e-5	4.6991e-5	4.6991e-5	4.6991e-5
Error (%)	0.59	2.23	2.86	2.05	2.45	2.15	2.53

Table 6.1: Blocking probabilities of a symmetric hexagonal network given in Figure 6·2 with two different configurations. B_i is the blocking probability obtained from the simulation of the exact system. $\hat{B}_i$ and $\hat{b}_i$ represent the approximations made in Eq.'s (6.5) and (6.10), respectively. The error percentages are between the simulated and approximated call blocking probabilities (B_i and $\hat{B}_i$). All cells have a capacity of $C = 20$. The interference weights are the same and $w_{ij} = w$ for all i,j.

$$\lim_{n\to\infty} \frac{E\left((1-b_i)^{-1}\sum_{j\in I} w_{ij}\lambda_j n \prod_{k\in I}(1-b_k)^{w_{jk}}, C_i n\right)}{E\left(\sum_{j\in I} w_{ij}\lambda_j n, C_i n\right)} = 1 \tag{6.14}$$

Proof. In the proof of Lemma 6.2.1 we have stated that the Erlang-B formula can be expressed in the following form:

$$E(\lambda n, Cn)^{-1} = \int_0^\infty e^{-x}\left(\frac{n + x/\lambda}{n}\right)^{Cn} dx.$$

Rewriting the ratio in Eq. (6.14) using the integral form we get:

$$\lim_{n\to\infty} \frac{\int_0^\infty e^{-x}\left(\frac{n+x/\sum_{j\in I} w_{ij}\lambda_j}{n}\right)^{C_i n} dx}{\int_0^\infty e^{-x}\left(\frac{n+x/(1-b_i)^{-1}\sum_{j\in I} w_{ij}\lambda_j \prod_{k\in I}(1-b_k)^{w_{jk}}}{n}\right)^{C_i n} dx}. \tag{6.15}$$

Using Lebesgue's dominated convergence theorem (Rudin, 1964, pp. 321), we can take the limit inside the integral, to show the ratio of these two expressions is equal to one, it is sufficient to show that:

$$\lim_{n\to\infty} \frac{\left(\frac{n+x/\sum_{j\in I} w_{ij}\lambda_j}{n}\right)^{C_i n}}{\left(\frac{n+x/(1-b_i)^{-1}\sum_{j\in I} w_{ij}\lambda_j \prod_{k\in I}(1-b_k)^{w_{jk}}}{n}\right)^{C_i n}} = 1, \tag{6.16}$$

Now assume that $\lim_{n\to\infty} b_i = 0$ for all i. Taking the limit of Eq. (6.16) then yields:

$$\frac{e^{C_i x/\sum_{j\in I} w_{ij}\lambda_j}}{e^{C_i x/\sum_{j\in I} w_{ij}\lambda_j}} = 1.$$

Finally, we need to check our assumption that $\lim_{n\to\infty} b_i = 0$. We consider the right hand side of Eq. (6.4). Using the integral notation provided in Eq. (6.12), if the assumption holds, we have just shown that in the limit it is given by the expression:

$$\left(\int_0^\infty e^{x(C_i/\sum_{j\in I} w_{ij}\lambda_j - 1)} dx\right)^{-1}$$

which goes to zero for $\sum_{j\in I} w_{ij}\lambda_j < C_i$ following from the results in Lemma 6.2.1. Hence our initial assumption holds and is a solution to the EFPA. Since we know that the solution of the EFPA is unique, there can be no other solution that satisfies Eq. (6.4). □

Theorem 6.2.1 states that as the network size increases, the blocking probabilities calculated by the EFPA and the first-order calculation proposed in Eq. (6.8) are the same in the limit. Therefore one can conclude that for large networks, the blocking probability of an individual cell can be well approximated by modeling it as a single cell in isolation as long as the load increase due to its neighbors are accounted for. In Table 6.1 we provide the results of a computer simulation of a hexagonal network topology shown in Figure 6·2. Even at low values of λ and C, the approximation is

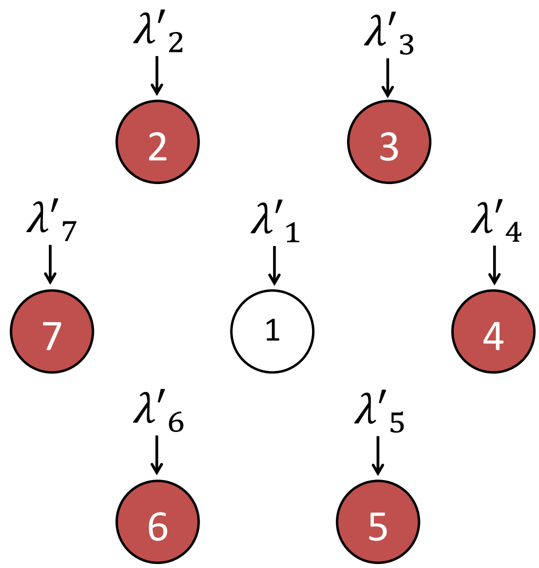


Figure 6·3: The auxillary model to of the symmetric hexagonal network provided in Figure 6·2

very close to the simulation results. Naturally, one can expect the approximation to be worse at these values if the interference given by w_{ij} had been higher. However, for relatively low levels of interference one can conclude that the EFPA is sufficiently close to the real value and hence in the rest of the chapter we shall assume that $\hat{b}_j = b_j$.

6.3 Profitability of Secondary Demand

In this section we seek to establish profitability conditions on secondary demand where admitting secondary users into the network at a single cell site results in an improved revenue for the network provider.

For a given secondary demand σ admitted at cell i for secondary price p, let $A_i^*(p, \sigma)$ denote an access policy that maximizes the revenue rate for a provider. We refer to $A_i^*(p, \sigma)$ as the *optimal coordinated access policy*. We represent the resulting maximal revenue $R_i^*(p, \sigma)$ as follows:

$$R_i^*(p, \sigma) = R_i(p, \sigma, A_i^*(p, \sigma)) = \max_{A_i} \; R_i(p, \sigma, A_i). \tag{6.17}$$

Now, consider a policy that admits secondary users only when the occupancy (*i.e.,* the number of users being served) in a node drops below a constant $T_i \in [0, C_i]$, which we term the admisssion *threshold*. We denote the implementation of a threshold policy with the notation $A_i(\sigma, p) = T_i$. Let T_i^* denote the optimal threshold value that maximizes the revenue of the provider such that:

$$T_i^* = \underset{T_i=0,\ldots,C_i}{\arg\max} \, R_i(p, \sigma, T_i) \tag{6.18}$$

The optimal threshold value depends on all parameters of the provider including intensity of the secondary demand. It is well-known that the optimal threshold policy is also the optimal admission policy for a network consisting of a single AP (Key, 1990; Miller, 1969; Ramjee et al., 1997).

It is important for a network provider to identify conditions it makes sense from a revenue perspective to ever accept a secondary request at a single cell i. If the secondary price-demand pair (p, σ) yields profit relative to serving primary demand only, (p, σ) represents an economically viable situation for a provider. The issue is closely related with the opportunity cost of accepting a secondary request: On the one hand such a request brings an immediate revenue of p, on the other hand it may cause rejecting future requests, possibly with higher immediate revenue, due to the channel that it holds temporally.

Definition 6.3.1 Let $p_i^{BE}(A_i)$ denote the break-even price that determines the profitability of the admission of secondary users in node i under policy A_i such that:

$$\begin{aligned} R_i(p,\sigma,A_i) &\geq R_i(p,0,A_i) \quad \textit{if} \quad p \geq p_i^{BE}(A_i) \\ R_i(p,\sigma,A_i) &< R_i(p,0,A_i) \quad \textit{if} \quad p < p_i^{BE}(A_i) \end{aligned}$$

Following from our definition, $p_i^{BE}(A_i^*)$ denotes the break-even price under the optimal admission policy, which we name the *general optimal break-even price*; whereas $p_i^{BE}(T_i^*)$ denotes the break-even price under the optimal threshold policy, which we name the *threshold break-even price*. It has been previously demonstrated that the optimal break-even price is insensitive to the shape or the intensity of the secondary demand (Al Daoud et al., 2012).

In Al Daoud et. al.'s work (Al Daoud et al., 2012) it has been established that $p_i^{BE}(A_i^*)$ does not depend on the value of the secondary demand σ. Hence the value of the optimal break-even price does not change whether we set $\sigma = \infty$ or a constant σ'. We will next derive an expression for the break-even price under threshold policies and infinite secondary demand, assuming this type of policies are optimal. Once we establish this price, we will show that this gives us the tightest upper bound among the threshold break-even prices on the general optimal break-even price if the optimal policy is not of threshold type.

Assume that the optimal admission policy is of threshold type such that $A_i^* = T_i^*$. Since the optimal admission policy is insensitive to secondary demand, we will identify the break-even price by setting $\sigma = \infty$ and thus considering a network where all states below the threshold value are always occupied by secondary users. Specifically, we identify a price condition that yields a better revenue under any threshold policy than the revenue generated by a policy that flatly rejects all secondary arrivals. This determines the sign of the balance in the trade-off when making a control decision to admit a secondary user or not. We state our main result on this profitability condition in the following theorem:

Theorem 6.3.1 *The break-even price $p_i^{BE}(A_i^*)$ is given by the following expression:*

$$p_i^{BE}(A_i^*) = \min_T \frac{K}{\pi_o(T)\prod_{j\in N(i)}(1-w_{ij}\tilde{b}_j)} \Bigg(\frac{\lambda_i}{\lambda_i'}\Big(\prod_{j\in I}(1-w_{ij}b_j) - (1-\pi_o(C))\prod_{j\in N(i)}(1-w_{ij}\tilde{b}_j)\Big) + \sum_{k\in N(i)}\Big(\frac{\lambda_k}{\lambda_i'}\prod_{j\in I}(1-w_{kj}b_j) - (1-w_{ki}\pi_o(C))\prod_{j\in N(i)}(1-w_{kj}\tilde{b}_j)\Big)\Bigg). \quad (6.19)$$

where

$$\pi_o(k) = \begin{cases} \frac{(\lambda_i')^k/k!}{G_o} & \text{for } k = T+1, T+2, \ldots, C_i \\ 0 & \text{otherwise,} \end{cases}$$

$$\hat{b}_i = E(\lambda_i', C_i),$$
$$\tilde{b}_j = E(\lambda_j' + w_{ij}(T+1)\pi_{io}(T+1), C_j),$$

and $G_o = \sum_{k=T+1}^{C_i} \frac{(\lambda_i')^k}{k!}$ is a normalizing constant, λ_i' is the effective load given in Eq. (6.9).

Proof. Let T be the maximum state of cell i at which secondary arrivals are admitted. If we set $\sigma = \infty$, then $T+1$ channels at cell i will always be occupied, since upon a departure at occupancy level $T+1$ a new secondary request is immediately admitted to the system. Therefore, the state space of cell i reduces to $T+1, T+2, ..., C_i$. The occupancy process of cell i under such a policy is illustrated in Figure 6·4 and has the following equilibrium distribution:

$$\pi_o = \begin{cases} \frac{(\lambda_i')^k/k!}{G_{io}} & \text{for } k = T+1, T+2, \ldots, C_i \\ 0 & \text{otherwise,} \end{cases} \quad (6.20)$$

where $G_o = \sum_{k=T+1}^{C_i} \frac{(\lambda_i')^k}{k!}$ is a normalizing constant.

Before considering the impact on a neighboring cell j, we first calculate the effec-

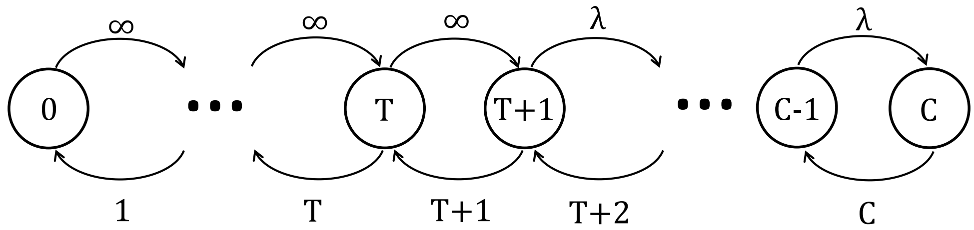


Figure 6·4: Occupancy process of cell i when the secondary demand is set to ∞ and the threshold value is T. $T+1$ channels are always occupied by secondary users.

tive secondary demand that is the result of the threshold policy in cell i. Secondary demand is generated when a secondary arrival is admitted immediately after a departure from cell i where $T+1$ channels are busy. Since secondary users have unit holding time, $T+1$ secondary arrivals are served on average in a unit time, which then translates into an effective secondary demand with a rate of $(T+1)\pi_o(T+1)$ at cell i. The effective load at the neighboring cell j is now given by $\lambda'_j + w_{ij}(T+1)\pi_o(T+1)$ and the blocking becomes:

$$\tilde{b}_j = E(\lambda'_j + w_{ij}(T+1)\pi_o(T+1), C_j). \tag{6.21}$$

The secondary revenue generated at cell i is given by $\sigma_{eff} \cdot p = (T+1)\pi_{io}(T+1) \cdot p$. On the other hand, a primary request is admitted when less than C_i channels are occupied, the revenue rate of which is then given by $\lambda_i K(1 - \pi_{io}(C_i))$ at cell i and $\lambda_j K(1 - \pi_{jo}(C_j))$ for all $j \in N(i)$. These revenue rates are then subjected to the blocking of the neighboring nodes. The policy is profitable if the revenue rate we formulated exceeds the revenue rate in the absence of secondary users, *i.e.,*

$$(T+1)\pi_o(T+1)\ p \prod_{j\in N(i)}(1-w_{ij}\tilde{b}_j) + \lambda_i K(1-\pi_o(C)) \prod_{j\in N(i)}(1-w_{ij}\tilde{b}_j) \tag{6.22}$$
$$+ \sum_{k\in N(i)} \lambda_k K(1-w_{ki}\pi_o(C)) \prod_{j\in N(i)}(1-w_{kj}\tilde{b}_j) >$$
$$\lambda_i K \prod_{j\in I}(1-w_{ij}b_j) + \sum_{k\in N(i)} \lambda_k K \prod_{j\in I}(1-w_{kj}b_j).$$

Algebraic manipulation of this inequality yields:

$$p > \frac{K}{\pi_o(T)\prod_{j\in N(i)}(1-w_{ij}\tilde{b}_j)} \left(\frac{\lambda_i}{\lambda_i'} \left(\prod_{j\in I}(1-w_{ij}b_j) - (1-\pi_o(C)) \prod_{j\in N(i)}(1-w_{ij}\tilde{b}_j) \right) \right.$$
$$\left. + \sum_{k\in N(i)} \left(\frac{\lambda_k}{\lambda_i'} \prod_{j\in I}(1-w_{kj}b_j) - (1-w_{ki}\pi_o(C)) \prod_{j\in N(i)}(1-w_{kj}\tilde{b}_j) \right) \right). \tag{6.23}$$

The value of T that minimizes the right hand side of Eq. (6.23) yields the form in Eq. (6.19). □

Now consider the case when the optimal policy is not threshold type, *i.e.,* $A_i^*(p,\sigma) \neq T_i^*$. Then the property of being insensitive to secondary demand does not apply for the threshold type policy we have just considered and under which we derived the break-even price. To avoid confusion, let us expand our notation and let $T_i^*(\sigma)$ denote the optimal threshold under secondary demand σ. Thus the break-even price derived in Theorem 6.3.1 becomes $p_i^{BE}(A_i^*) = p_i^{BE}(T_i^*(\infty))$. In the following theorem, we will establish that this break-even price gives us the tightest upper bound that can be established on the optimal break-even price among all threshold type policies regardless of the secondary demand.

Theorem 6.3.2 *The threshold break-even price established under infinite secondary demand is a lower bound on all other threshold break-even prices such that for any* *secondary demand* $\sigma' < \infty$*:*

$$p_i^{BE}(T_i^*(\infty)) \leq p_i^{BE}(T_i^*(\sigma'))$$

Proof. We will demonstrate the inequality in the theorem by establishing that setting the price equal to the threshold break-even price under a finite secondary demand yields a positive revenue under infinite demand, *i.e.*,

$$R_i(p_i^{BE}(T_i^*(\sigma')), \infty, T_i^*(\infty)) \geq R_i(p_i^{BE}(T_i^*(\sigma')), \sigma', T_i^*(\sigma')).$$

Consider a policy $\hat{A}(p, \infty)$ that does a random thinning of the secondary demand and brings it to σ' by setting an initial threshold value. It then implements the threshold policy $T_i^*(\sigma')$. Hence the policy $\hat{A}(p, \infty)$ generates the same revenue as the policy $T_i^*(\sigma')$:

$$R_i(p, \infty, \hat{A}(p, \infty)) = R_i(p, \sigma', T_i^*(\sigma')). \tag{6.24}$$

Since $T_i^*(\infty)$ represents the optimal threshold policy when the secondary demand is infinite, we have:

$$R_i(p, \infty, T_i^*(\infty)) \geq R_i(p, \infty, \hat{A}(p, \infty)). \tag{6.25}$$

Combining Eq.'s (6.24) and (6.25) gives us the desired inequality:

$$R_i(p, \infty, T_i^*(\infty)) \geq R_i(p, \sigma', T_i^*(\sigma')). \tag{6.26}$$

□

6.3.1 Numerical Results

In this section we present numerical results on the break-even price expression we provided in Theorem 6.3.1. Although the expression we provided in Eq. (6.19) does not state which threshold value T minimizes the break-even price, based on previous

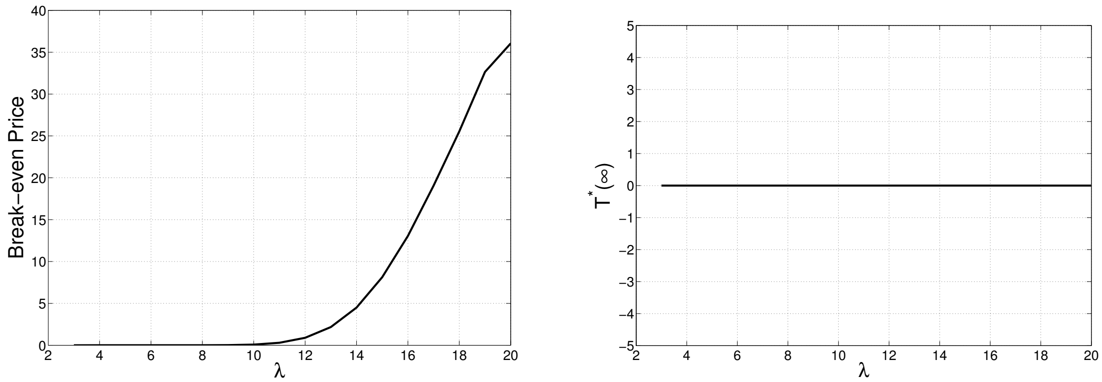


Figure 6·5: The break-even price and the optimal threshold value that minimizes it as provided in Eq. (6.19) for a network with 7 cells, each with a capacity of $C = 30$ and the interferences are as shown in Figure 6·2 with $w = 0.1$. (K=100)

results obtained in Chapter 3 and numerical analysis, we conjecture that the optimal threshold value is zero. Figure 6·5 illustrates this where we calculate the optimal threshold value over a range of primary arrival rates that maintain low blocking within the network.

In other words, the break-even price derived in Theorem 6.3.1 is obtained when the network provider admits secondary users into the network only when the cell at which they are admitted into the network is empty. This has the following intuitive explanation: The impact secondary users have on the network is two-fold. First, they occupy a channel at the cell where they are being admitted. Secondly, they increase the likelihood that the call arrivals in the neighborhood of cell i will be blocked. Both of these effects result in an opportunity cost for the provider in terms of the primary revenue lost. Due to the stochastic nature of the call arrival departure process at each cell, this risk is minimized when cell i is completely empty.

We then turn our attention to the accuracy of approximation of the break-even price. In order to do so we have simulated the revenue rate of the network with a hexagonal topology as illustrated in Figure 6·2 under a lock-out (no secondary

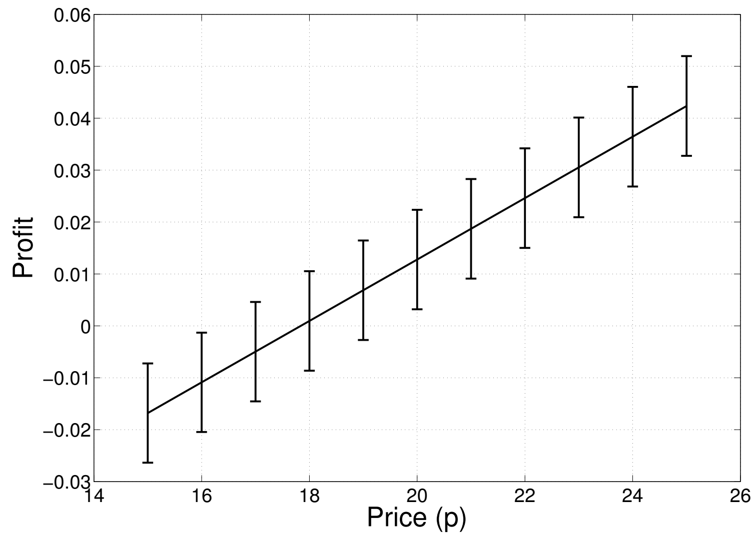


Figure 6·6: The simulated profit of a network provider shown in Figure 6·2 for different prices. The error bars indicate 95% confidence intervals. (w=0.05, C=5, λ=3, K=100). The results are obtained after 10 million primary arrivals at each cells and over 60 simulations.

admission) and the $T = 0$ threshold policy. Figure 6·6 illustrates the increase in profits when secondary users are admitted to an empty network. One can observe from the figure that the break-even price is approximately given by $p^{BE} = 18$, with a possible range of $[16.2, 19.5]$ due to the confidence interval provided. Using Eq. (6.19) we obtain $p^{BE} = 20.4$ which indicates a 10% error between the simulated result and our approximation. A possible explanation for this discrepancy is that for the network values of $\lambda = 3$ and $C = 5$ there is significant amount of blocking in the network where our first order approximation is not very accurate. While it would be desirable to simulate the same network with low blocking values, simulating the network becomes more time intensive in order to stay within the same confidence intervals. This because the difference in revenues are essentially generated whenever the network is full or the network is empty both of which become very rare events as the capacity is increased.

6.4 Summary

In this chapter we have presented a framework to determine the profitable admission of secondary users in a multicell network that is subject to spatial interference. We have first provided a method to decouple the behavior in the individual cells from the rest of the network using a first order approximation to Erlang fixed point methods. Then, using the provided decoupling we have identified profitability conditions of secondary user admission at a single cell and provided the corresponding price expression in terms of the network parameters.

Chapter 7

Concluding Remarks and Future Directions

In the dissertation, we have investigated the profitability and the competitive behavior of oligopolies in secondary spectrum markets, which can be formed as a result of dynamic spectrum sharing in wireless communications. We have been motivated by the initiatives of the FCC in this direction, encouraging the formation of such markets as a possible way to improve spectrum utilization compared to the current situation.

In Chapters 3 and 4 of this dissertation we have focused on the pricing of secondary users in a possible secondary market modeled after FCC's private commons. We have investigated competitive behavior and the market outcomes it leads to between the market participants in an attempt to collect revenues from secondary users at a profitable price.

In Chapter 3, we focused on the first of the two proposed regimes for secondary spectrum access, namely coordinated access under private commons. Such an investigation can help provide important guidance to a firm's strategic decision process, by explicitly determining the parameters on which market success depends. To accomplish this goal, we formulated the problem as a non-cooperative game, in which providers with finite network capacities are making strategic pricing and access control decisions with respect to secondary users.

We analyzed the implementation of coordinated access, for which we demonstrated that the optimal access policy is of threshold type. We showed that each provider has a unique *break-even* price, above which profitability is guaranteed regardless of the secondary user demand response. Next, using the notion of Nash equilibrium, we formalized the possible outcomes resulting from a non-cooperative game in which optimal coordinated access is implemented by two or more network providers. We explained how the best response dynamics of each provider reflect a price war, which leads to a single provider (that with the lowest break-even price) capturing the entire secondary spectrum market.

In Chapter 4, we considered the same private commons competition however this time under an uncoordinated access policy. Under uncoordinated access, the break-even price is no longer insensitive to the secondary demand and market sharing between providers becomes a possible best response. The possible market outcomes under an uncoordinated access policy are highly dependent on the specific shape of the secondary demand function, the analysis of which can become very complex. To address this complexity, we next focused on an uncoordinated access regime for secondary spectrum detailed under private commons using a demand overflow model.

Once again using the notions of best response and Nash equilibrium, we show the emergence of two markedly different possible market outcomes, depending on the secondary demand function $\sigma(p)$ and the network parameters of each provider. We established the existence market sharing price intervals and showed that if they intersect, then the providers converge to a price profile where they will share the market. On the other hand, if the market sharing price intervals do not intersect, the Nash equilibrium reflects a price war wherein the winning provider sets its price slightly below the break-even price of its competitor and gets all the profit.

Next, in Chapter 5 we shifted our focus from the pricing and competition in private commons to the question of aggregating more spectrum in an effort to improve a network providers ability to compete in secondary spectrum markets. Motivated by the recent FCC ruling regarding reserved spectrum in auctions and added capability of carrier aggregation by LTE-Advanced, we sought to investigate the possibility of a single small provider increasing its spectrum holdings to be able to compete with larger providers.

First, we derived a many server approximation for the Erlang-B formula under a quality driven (QD) regime. Using this QD approximation, we identified an optimal level of carrier aggregation. We investigated the scaling behavior of optimal carrier aggregation two-fold: (i) with respect to the scaling factor n and (ii) with respect to the traffic load. We obtained sub-linear (though close to linear) asymptotic upper and lower bounds. Then, we demonstrated that if the traffic load under which each provider operates increases, then the level of carrier aggregation required is reduced. This result indicates that the marginal benefit of carrier aggregation in a heavily loaded network is higher than that in a lightly loaded network. Finally, we provided a relationship between the profitable pricing of users in secondary spectrum markets and the Erlang-B formula for which our results apply. Hence, the results on optimal carrier aggregation presented in the chapter are directly applicable to pricing strategies in secondary spectrum markets, where providers can aggregate spectrum to lower their prices in a possible price war.

Finally, in Chapter 6, we reconsidered the problem of profitable pricing of secondary users in a spatial topology, in contrast to the single cell setting of in previous chapters. We presented a framework to determine the profitable admission of secondary users in a multicell network that is subject to spatial interference. We used the Erlang fixed point approximation (EFPA), which works well for low blocking regimes, to obtain a better understanding of the total revenue a multicell network generates. Through the EFPA we decoupled the blocking and revenue generation of each cell from the rest of the network. Then, using the provided decoupling we identified profitability conditions of secondary user admission at a single cell and provided the corresponding price expression in terms of the network parameters.

In our analysis, we have made some simplifying assumptions that can be gradually removed to include more general topologies and traffic patterns in the future works. Specifically, the EFPA works well in network that experience low levels of blocking. While it can be expected that commercially deployed networks will be operating in or near this level of performance, the cases where usage peaks should also be considered separately. Also, we only considered the existence of secondary users in a single cell in the network and provided a specific break-even price for this type of admission. However, in reality secondary users can be expected to be present at all cell sites at different density levels. Admitting secondary users at multiple locations is a larger problem that deserves its own investigation. However, we also note that the break-even price provided in Chapter 6 considers when should the first secondary user be admitted to the network and hence should be applicable when further admission scenarios are considered.

In summary, with this dissertation we shed light into regulatory impact of the formation of healthy oligopolies in secondary spectrum markets. We accomplished this by considering the nature of provider competition in private commons under coordinated and uncoordinated access and its subtle outcomes. Future work could focus on extending the model by taking into account spillover, quality of service and spatial distribution factors in bringing the competition closer to reality and analyzing the impact of each layer of additional complexity on the market outcomes.

A self-evident line of future work is the price competition analysis of the multicell network setting given profitable pricing conditions. For example, once the break-even price for every cell is calculated using the methodology provided in Chapter 6, where should the provider choose to open its spectrum for secondary access? If this provider is facing competition from one or more providers who also have cell cites at the same geographic area, what would be the market equilibrium? Would every provider choose to isolate their secondary service areas in an effort to prevent price wars and maximize profits or would they choose to compete over cell sites that are particularly profitable or desirable? Many questions such as these remain open and provide rich directions for future work.

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