TOPICS IN SPECTRAL THEORY OF DIFFERENTIAL OPERATORS

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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ABSTRACT

This dissertation is devoted to two eigenvalue counting problems: Determining the asymptotic behavior of large eigenvalues of self-adjoint extensions of partial differential operators, and computing the number of negative eigenvalues for bounded from below operators with compact resolvents.

In the first part of this thesis we derive a Weyl-type asymptotic formula and a bound for the eigenvalue counting function for the Krein–von Neumann extension of differential operators on open bounded subsets of $\mathbb{R}^n$.

In the second part of this thesis we obtain a formula relating the Maslov index, a topological invariant counting the signed number of conjugate points of paths of Lagrangian planes in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, and the Morse index, the number of negative eigenvalues, for the second order differential operators with domains of definition contained in $H^1(\Omega)$ for open bounded subsets $\Omega \subset \mathbb{R}^n$.

Chapter 1

Introduction

This dissertation is devoted to two eigenvalue counting problems: Determining the asymptotic behavior of large eigenvalues of self-adjoint extensions of partial differential operators, and computing the number of negative eigenvalues for operators with compact resolvents bounded from below.

The first part of this thesis concerns Weyl-type asymptotic formulas and global estimates for eigenvalue counting functions. More precisely, we derive a Weyl-type asymptotic formula and a uniform bound for the eigenvalue counting function for the Krein–von Neumann extension of higher-order elliptic differential operators on *arbitrary* open bounded sets in $\mathbb{R}^n$. It is worth noting that the Krein–von Neumann extension is the smallest non-negative self-adjoint extension (in the sense of forms) of a positive symmetric operator.

The main contribution of our work in this direction can be succinctly summarized as follows: Given an *arbitrary* open bounded set $\Omega \subset \mathbb{R}^n$, we consider the Krein–von Neumann, $A_{K,\Omega,2m}(a,b,q)$, and Friedrichs, $A_{F,\Omega,2m}(a,b,q)$, realizations of the differential expression

$$\tau_{2m}(a,b,q) := \left(\sum_{j,k=1}^{n} (-i\partial_j - b_j(x)) a_{j,k}(x) (-i\partial_k - b_k(x)) + q(x) \right)^m, \quad m \in \mathbb{N}, \; x \in \mathbb{R}^n,$$

where the coefficients $(a - I_n), b, q$ are sufficiently smooth with compact support containing $\overline{\Omega}$. Denoting the number of positive eigenvalues of $A_{K,\Omega,2m}(a,b,q)$ not exceeding λ by $N(\lambda; A_{K,\Omega,2m}(a,b,q))$, we derived in [8], [74], the following global bounds

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) \leq C v_n (2\pi)^{-n} \bigg(1 + \frac{2m}{2m+n}\bigg)^{n/(2m)} \lambda^{n/(2m)}, \quad \lambda > 0,$$

$$N(\lambda; A_{F,\Omega,2m}(a,b,q)) \leq C v_n (2\pi)^{-n} \bigg(1 + \frac{2m}{n}\bigg)^{n/(2m)} \lambda^{n/(2m)}, \quad \lambda > 0,$$

where $C = C(a,b,q,\Omega) > 0$ (with $C(I_n, 0, 0, \Omega) = |\Omega|$, the volume of Ω). Moreover, we obtain the Weyl-type asymptotic formula,

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) \underset{\lambda \to \infty}{=} \frac{v_n}{(2\pi)^n} \bigg(\int_\Omega d^n x \, (\det a(x))^{-1/2} \bigg) \lambda^{n/(2m)} + o\big(\lambda^{n/(2m)}\big).$$

The second part of this dissertation addresses the evaluation of the Morse index via the Maslov index. Motivated by applications in stability theory for traveling waves and other patterns of multi-dimensional nonlinear partial differential equations, in [98], [115], [116] we obtained relations between the Morse index, the counting of the number of unstable eigenvalues, and the Maslov index, an invariant from symplectic geometry. This was achieved by counting the signed number of conjugate points for families of elliptic self-adjoint operators on Lipschitz domains obtained by linearization of the PDE about a particular pattern of interest.

The Maslov index (cf., e.g., [5], [6], [83], [105], [124], [142], [143], [153], [155], [164]) has been proven to be instrumental in counting eigenvalues of differential operators [50], [51], [52], [91], [92], [97], [98], [115], [116], [117]. Various classical results from the spectral theory of ordinary and partial differential operators can be placed in the framework of abstract theorems relating the eigenvalue counting function and the Maslov index. Among such results are: the Sturm oscillation theorem for ordinary differential operators, Arnold's generalization of Sturm-type theorems for systems of ordinary differential equations [6], Courant's nodal domain theorem, the Morse index formula derived by Smale [155], Friedlander's index formula for Dirichlet and Neumann Schrödinger operators on bounded domains [69], etc.

In this dissertation we obtain (cf. [98], [115], [116]) relations between the Morse index of self-adjoint differential operators whose domains are contained in $H^1(\Omega)$ and the Maslov index of paths of Lagrangian planes in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ defined by means of Dirichlet and Neumann traces of the weak solutions of the respective eigenvalue problems. In addition, we show that the spectral flow of a one-parameter family of such operators is equal to the Maslov index of a certain path of Lagrangian planes.

We will now describe our main results relating the Maslov and Morse indices in more detail. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain, $m \in \mathbb{N}$, and let the functions

$$A : t \mapsto A^t \in \mathbb{C}^{m\times m},\ B : t \mapsto B^t \in \mathbb{C}^m, q : t \mapsto q^t \in \mathbb{R},\ t \in [\alpha, \beta], \tag{1.1}$$

satisfy the following assumptions:

$$A \in C([\alpha, \beta], L^\infty(\Omega, \mathbb{C}^{m\times m})),\ A^t \text{ is a self-adjoint matrix for all } t \in [\alpha, \beta], \tag{1.2}$$

$$B \in C([\alpha, \beta], L^\infty(\Omega, \mathbb{C}^m)),\ q \in C([\alpha, \beta], L^\infty(\Omega, \mathbb{R})). \tag{1.3}$$

Let us consider the family $\{\mathcal{L}^t\}_{t=\alpha}^{\beta}$ of formally self-adjoint differential expressions,

$$\mathcal{L}^t := -\text{div}A^t\nabla + B^t\nabla - \nabla \cdot \overline{B^t} + q^t,\ t \in [\alpha, \beta]. \tag{1.4}$$

The associated family of symmetric operators acting in $L^2(\Omega)$ is given by

$$L^t u := \mathcal{L}^t u,\ u \in \text{dom}(L^t) := C_0^\infty(\Omega),\ t \in [\alpha, \beta]. \tag{1.5}$$

Each operator L^t is closable in $L^2(\Omega)$, its closure is denoted by $\mathcal{L}^t_{min}$, $t \in [\alpha, \beta]$.

First, we establish a natural one-to-one correspondence between the self-adjoint extensions of $\mathcal{L}^t_{min}$ and the Lagrangian planes in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. The Lagrangian plane $\mathcal{G}_{\mathscr{D}_t}$ associated to a self-adjoint extension $\mathcal{L}^t_{\mathscr{D}_t}$ of $\mathcal{L}^t_{min}$ with the domain $\mathscr{D}_t \subset H^1(\Omega)$ is given by the formula

$$\mathcal{G}_{\mathscr{D}_t} = \overline{\operatorname{tr}_{\mathcal{L}^t}(\mathscr{D}_t)}^{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)},\ t \in [\alpha, \beta], \tag{1.6}$$

where $\operatorname{tr}_{\mathcal{L}^t}$ is a trace map defined below in (3.18) by means of the differential expression $\mathcal{L}^t$ from (1.4). For example, the plane corresponding to the Dirichlet Laplacian is given by $\{0\} \times H^{-1/2}(\partial\Omega)$, to the Neumann Laplacian is given by $H^{1/2}(\partial\Omega) \times \{0\}$, and to the Robin Laplacian is given by

$$\{(f, -\Theta f) : f \in H^{1/2}(\partial\Omega)\}, \Theta \in \mathcal{B}_\infty\big(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\big).$$

Second, we define the set of traces of the weak solutions of the corresponding homogeneous PDE with no boundary conditions by the formula

$$\mathcal{K}_{\lambda,t} := \operatorname{tr}_{\mathcal{L}^t}\left(\{u \in H^1(\Omega) : \mathcal{L}^t u - \lambda u = 0\}\right). \tag{1.7}$$

We show that this plane is Lagrangian, and recast the eigenvalue problem

$$\mathcal{L}^t u - \lambda u = 0,\ u \in \mathscr{D}_t, \tag{1.8}$$

in terms of the intersection of the Lagrangian planes $\mathcal{K}_{\lambda,t}$ and $\mathcal{G}_{\mathscr{D}_t}$. Namely, we prove that

$$\dim\ker(\mathcal{L}_{\mathscr{D}_t} - \lambda) = \dim\big(\mathcal{K}_{\lambda,t} \cap \mathcal{G}_{\mathscr{D}_t}\big),\ \lambda \in \mathbb{R},\ t \in [\alpha, \beta]. \tag{1.9}$$

Formula (1.9) provides a link between the eigenvalues of $\mathcal{L}^t_{\mathscr{D}_t}$ and the conjugate points of the paths formed by the Lagrangian planes $\mathcal{K}_{\lambda,t}, \mathcal{G}_t$ in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. With this at hand we show the principal result of our work, cf. Theorem 3.18,

$$\mathrm{Mor}(\mathcal{L}^{\alpha}_{\mathscr{D}_\alpha}) - \mathrm{Mor}(\mathcal{L}^{\beta}_{\mathscr{D}_\beta}) = \mathrm{Mas}\big((\mathcal{K}_{0,t}, \mathcal{G}_{\mathscr{D}_t})|_{t\in[\alpha,\beta]}\big), \tag{1.10}$$

where the Morse index $\mathrm{Mor}(\mathcal{L}_{\mathscr{D}_\alpha})$ is defined as the number of negative eigenvalues of the operator $\mathcal{L}_{\mathscr{D}_\alpha}$, and $\mathrm{Mas}\big((\mathcal{K}_{0,t}, \mathcal{G}_{\mathscr{D}_t})|_{t\in[\alpha,\beta]}\big)$ denotes the Maslov index of the paths $\{\mathcal{K}_t\}_{t=\alpha}^{\beta}, \{\mathcal{G}_{\mathscr{D}_t}\}_{t=\alpha}^{\beta}$ defined in Section 2.1.

The left-hand side of (1.10) can be viewed as the spectral flow through zero of the eigenvalues of the operator family $\{\mathcal{L}_{\mathscr{D}_t}\}_{t=\alpha}^{\beta}$. A slight generalization of (1.10) recovers the above mentioned relations between the Maslov index and the spectral flow obtained in [29], [36] (in case of second order operators), and between the Maslov index and the Morse index obtained in [50, 51, 57, 97, 98, 116].

In addition, we establish a connection of the Lagrangian description of the self-adjoint extensions of symmetric operators as in [3, 29], and the theory of abstract boundary triples as in [30, 31, 82, 108]. In this abstract setting, we show that the resolvent convergence of the self-adjoint extensions of a given symmetric operator is equivalent to convergence of the corresponding Lagrangian planes, cf. Theorem 3.35. In the more concrete setting of PDE traces, we deduce the resolvent convergence of perturbed Robin Laplacians from convergence of the associated Lagrangian planes using a new for this topic tool provided by the Krein-type resolvent formula, cf. Proposition 3.10. Furthermore, we obtain formulas relating the Morse and Maslov indices in abstract settings, assuming the existence of a family of perturbations and a family of boundary triples. We demonstrate how to apply these formulas for the matrix one- and multidimensional Schrödinger operators.

Chapter 2

A Bound for the Eigenvalue Counting Function For Krein–von Neumann and Friedrichs Extensions

2.1 Introduction

We briefly recall some background material: Suppose S is a densely defined, symmetric, closed operator with nonzero deficiency indices in a separable complex Hilbert space $\mathcal{H}$ that satisfies

$$S \geq \varepsilon I_{\mathcal{H}} \text{ for some } \varepsilon > 0. \tag{2.1}$$

Then, according to M. Krein's celebrated 1947 paper [112], among all nonnegative self-adjoint extensions of S, there exist two distinguished ones, S_F, the Friedrichs extension of S, and S_K, the Krein–von Neumann extension of S, which are, respectively, the largest and smallest such extension (in the sense of quadratic forms). In particular, a nonnegative self-adjoint operator $\widetilde{S}$ in $\mathcal{H}$ is a self-adjoint extension of S if and only if $\widetilde{S}$ satisfies

$$S_K \leq \widetilde{S} \leq S_F \tag{2.2}$$

(again, in the sense of quadratic forms).

An abstract version of [86, Proposition 1], presented in [10], describing the following intimate connection between the nonzero eigenvalues of S_K, and a suitable abstract buckling problem, can be summarized as follows:

$$\text{There exists } 0 \neq v_\lambda \in \operatorname{dom}(S_K) \text{ satisfying } S_K v_\lambda = \lambda v_\lambda, \quad \lambda \neq 0, \tag{2.3}$$

if and only if

$$\text{there exists a } 0 \neq u_\lambda \in \operatorname{dom}(S^*S) \text{ such that } S^*Su_\lambda = \lambda S u_\lambda, \tag{2.4}$$

and the solutions v_λ of (2.3) are in one-to-one correspondence with the solutions u_λ of (2.4) given by the pair of formulas

$$u_\lambda = (S_F)^{-1} S_K v_\lambda, \quad v_\lambda = \lambda^{-1} S u_\lambda. \tag{2.5}$$

As briefly recalled in Section 2.2, (2.4) represents an abstract buckling problem. The latter has been the key in all attempts to date in proving Weyl-type asymptotics for eigenvalues of S_K when S represents an elliptic partial differential operator in $L^2(\Omega)$. In fact, it is convenient to go one step further and replace the abstract buckling eigenvalue problem (2.4) by the variational formulation,

$$\begin{gathered} \text{there exists } u_\lambda \in \operatorname{dom}(S)\backslash\{0\} \text{ such that} \\ \mathfrak{a}(w, u_\lambda) = \lambda\, \mathfrak{b}(w, u_\lambda) \text{ for all } w \in \operatorname{dom}(S), \end{gathered} \tag{2.6}$$

where the symmetric forms $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathcal{H}$ are defined by

$$\mathfrak{a}(f,g) := (Sf, Sg)_{\mathcal{H}}, \quad f, g \in \operatorname{dom}(\mathfrak{a}) := \operatorname{dom}(S), \tag{2.7}$$

$$\mathfrak{b}(f,g) := (f, Sg)_{\mathcal{H}}, \quad f, g \in \operatorname{dom}(\mathfrak{b}) := \operatorname{dom}(S). \tag{2.8}$$

In our present context, the role of the symmetric operator S will be played by the closed, strictly positive operator in $L^2(\Omega)$,

$$A_{\Omega,2m}(a,b,q)f = \tau_{2m}(a,b,q)f, \quad f \in \operatorname{dom}(A_{\Omega,2m}(a,b,q)) := W_0^{2m,2}(\Omega), \tag{2.9}$$

where the differential expression $\tau_{2m}(a,b,q)$ is of the type,

$$\tau_{2m}(a,b,q) := \bigg(\sum_{j,k=1}^{n}(-i\partial_j - b_j)a_{j,k}(-i\partial_k - b_k) + q\bigg)^m, \quad m \in \mathbb{N}, \tag{2.10}$$

under the assumption that $\emptyset \neq \Omega \subset \mathbb{R}^n$ is open and bounded and under sufficient smoothness hypotheses on the coefficients a, b, q (cf. Hypothesis 2.11 (i)). The Krein–von Neumann and Friedrichs extensions of $A_{\Omega,2m}$ will then be denoted by $A_{K,\Omega,2m}(a,b,q)$ and $A_{F,\Omega,2m}(a,b,q)$, respectively.

Since $A_{K,\Omega,2m}(a,b,q)$ has purely discrete spectrum in $(0,\infty)$ bounded away from zero by $\varepsilon > 0$, let $\{\lambda_{K,\Omega,j}\}_{j\in\mathbb{N}} \subset (0,\infty)$ be the strictly positive eigenvalues of $A_{K,\Omega,2m}(a,b,q)$ enumerated in nondecreasing order, counting multiplicity, and let

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) := \#\{j \in \mathbb{N} \,|\, 0 < \lambda_{K,\Omega,j} < \lambda\}, \quad \lambda > 0, \tag{2.11}$$

be the eigenvalue distribution function for $A_{K,\Omega,2m}(a,b,q)$ (which takes into account only strictly positive eigenvalues of $A_{K,\Omega,2m}(a,b,q)$); $N(\,\cdot\,; A_{K,\Omega,2m}(a,b,q))$ is the principal object of this note. Similarly, $N(\lambda; A_{F,\Omega,2m}(a,b,q))$, $\lambda > 0$, denotes the eigenvalue counting function for $A_{F,\Omega,2m}(a,b,q)$.

For convenience of the reader, we recall the basic abstract facts on the Friedrichs extension, S_F and the Krein–von Neumann extension S_K of a strictly positive, closed, symmetric operator S in a complex, separable Hilbert space $\mathcal{H}$ and describe the intimate link between the Krein–von Neumann extension and the underlying abstract buckling problem in Section 2.2. Section 2.3 focuses on basic domain and spectral properties of the operators, $\widetilde{A}_{2m}(a,b,q)$, $A_{\Omega,2m}(a,b,q)$, $A_{K,\Omega,2m}(a,b,q)$, and $A_{F,\Omega,2m}(a,b,q)$, $m \in \mathbb{N}$, and their associated quadratic forms, on open, bounded subsets $\Omega \subset \mathbb{R}^n$ (without imposing any constraints on $\partial\Omega$). In our principal Section 2.4

we derive the bounds

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) \leq \frac{v_n}{(2\pi)^n}\left(1+\frac{2m}{2m+n}\right)^{n/(2m)} \sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)}\, \lambda^{n/(2m)}, \quad \lambda > 0, \tag{2.12}$$

and

$$N(\lambda; A_{F,\Omega,2m}(a,b,q)) \leq \frac{v_n}{(2\pi)^n}\left(1+\frac{2m}{n}\right)^{n/(2m)} \sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)}\, \lambda^{n/(2m)}, \quad \lambda > 0, \tag{2.13}$$

where $v_n := \pi^{n/2}/\Gamma((n+2)/2)$ denotes the (Euclidean) volume of the unit ball in $\mathbb{R}^n$ ($\Gamma(\cdot)$ being the Gamma function), and $\phi(\,\cdot\,,\,\cdot\,)$ represent the suitably normalized generalized eigenfunctions of $\widetilde{A}_2(a,b,q)$ satisfying

$$\widetilde{A}_2(a,b,q)\phi(\,\cdot\,,\xi) = |\xi|^2\phi(\,\cdot\,,\xi), \quad \xi \in \mathbb{R}^n, \tag{2.14}$$

in the distributional sense (cf. Hypothesis 2.19). In particular, whenever the property

$$\sup_{(x,\xi)\in\Omega\times\mathbb{R}^n} |\phi(x,\xi)| < \infty \tag{2.15}$$

has been established, then

$$\sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)} \leq |\Omega| \sup_{(x,\xi)\in\Omega\times\mathbb{R}^n} \big(|\phi(x,\xi)|^2\big), \tag{2.16}$$

explicitly exhibits the volume dependence on Ω of the right-hand sides of (2.12) and (2.13), respectively (see also Section 2.5).

Our method of employing the eigenfunction transform (i.e., the distorted Fourier transform) associated with the variable coefficient operator $\widetilde{A}_{2m}(a,b,q)$ (replacing the standard Fourier transform in connection with the constant coefficient case in [74]) to derive the results (2.12) and (2.13) appears to be new under any assumptions on

$\partial\Omega$. A comparison of (2.12), (2.13) with the existing literature on eigenvalue counting function bounds will be provided in Remark 2.24.

We remark that the power law behavior $\lambda^{n/(2m)}$ coincides with the one in the known Weyl asymptotic behavior. This in itself is perhaps not surprising as it is *a priori* known that

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) \leq N(\lambda; A_{F,\Omega,2m}(a,b,q)), \quad \lambda > 0, \tag{2.17}$$

and $N(\lambda; A_{F,\Omega,2m}(a,b,q))$ is known to have the power law behavior $\lambda^{n/(2m)}$ (cf. [114] in the case $a = I_n$, $b = q = 0$, extending the corresponding result in [119] in the case $m = 1$). We emphasize that (2.17) is not in conflict with variational eigenvalue estimates since $N(\lambda; A_{K,\Omega,2m}(a,b,q))$ only counts the strictly positive eigenvalues of $A_{K,\Omega,2m}(a,b,q)$ less than $\lambda > 0$ and hence avoids taking into account the (generally, infinite-dimensional) null space of $A_{K,\Omega,2m}(a,b,q)$. Rather than relying on estimates for $N(\,\cdot\,; A_{F,\Omega,2m}(a,b,q))$ (cf., e.g., [18]–[24], [72], [73], [89], [90], [114], [119], [120], [128], [132], [146], [147], [149], [151], [168], typically for $a = I_n$, $b = 0$), we will use the one-to-one correspondence of nonzero eigenvalues of $A_{K,\Omega,2m}(a,b,q)$ with the eigenvalues of its underlying buckling problem (cf. (2.3)–(2.5)) and estimate the eigenvalue counting function for the latter. Section 2.5 illustrates the purely absolutely continuous spectrum and eigenfunction assumption we impose on $\widetilde{A}_{2m}(a,b,q)$ in $L^2(\mathbb{R}^n)$. Finally, Appendix A derives a crucial minimization result needed in the derivation of the bound (2.12), it also compares (2.12) with the abstract bound (2.17), given (2.13), and points out that the bound (2.12) is always superior to the abstract one guaranteed by combining (2.13) and (2.17).

In the special case $a = I_n$, $b = q = 0$, the bound (2.12) was derived in [74], while the bound (2.13) is due to [114] in this case.

Since Weyl asymptotic for $N(\,\cdot\,; A_{F,\Omega,2m}(a,b,q))$ is not considered in this thesis (with exception of Remark 2.25), we just refer to the monographs [118] and [152], and to [129], [130], but note that very detailed bibliographies on this subject appeared in [9] and [11]. At any rate, the best known result on Weyl asymptotics with remainder estimate for $N(\,\cdot\,; A_{K,\Omega,2m}(I_n,0,q))$ to date for bounded Lipschitz domains appears to be [15] (the case of quasi-convex domains having been discussed earlier in [9]). In contrast to Weyl asymptotics with remainder estimates, the estimates (2.12), (2.13) assume no regularity of $\partial\Omega$ at all.

We conclude this introduction by summarizing the notation used in this chapter. Throughout this chapter, the symbol $\mathcal{H}$ is reserved to denote a separable complex Hilbert space with $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second argument), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$. Next, let T be a linear operator mapping (a subspace of) a Banach space into another, with $\mathrm{dom}(T)$ and $\mathrm{ran}(T)$ denoting the domain and range of T. The closure of a closable operator S is denoted by $\overline{S}$. The kernel (null space) of T is denoted by $\ker(T)$. The spectrum, point spectrum (i.e., the set of eigenvalues), discrete spectrum, essential spectrum, and resolvent set of a closed linear operator in $\mathcal{H}$ will be denoted by $\sigma(\cdot)$, $\sigma_p(\cdot)$, $\sigma_d(\cdot)$, $\sigma_{ess}(\cdot)$, and $\rho(\cdot)$, respectively. The symbol s-lim abbreviates the limit in the strong (i.e., pointwise) operator topology (we also use this symbol to describe strong limits in $\mathcal{H}$).

The Banach spaces of bounded and compact linear operators on $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively. Similarly, the Schatten–von Neumann (trace) ideals will subsequently be denoted by $\mathcal{B}_p(\mathcal{H})$, $p \in (0,\infty)$. In addition, $U_1 \dotplus U_2$ denotes the direct sum of the subspaces U_1 and U_2 of a Banach space $\mathcal{X}$. Moreover, $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ denotes the continuous embedding of the Banach space $\mathcal{X}_1$ into the Banach space $\mathcal{X}_2$.

The symbol $L^2(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$ open, $n \in \mathbb{N}$, is a shorthand for $L^2(\Omega, d^n x)$, whenever the n-dimensional Lebesgue measure $d^n x$ is understood. For brevity, the identity operator in $L^2(\Omega)$ will typically be denoted by I_Ω. The symbol $\mathcal{D}(\Omega)$ is reserved for the set of test functions $C_0^\infty(\Omega)$ on Ω, equipped with the standard inductive limit topology, and $\mathcal{D}'(\Omega)$ represents its dual space, the set of distributions in Ω. The distributional pairing, compatible with the L^2-scalar product, $(\,\cdot\,,\,\cdot\,)_{L^2(\Omega)}$, is abbreviated by ${}_{\mathcal{D}'(\Omega)}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{D}(\Omega)}$. The (Euclidean) volume of Ω is denoted by $|\Omega|$.

The cardinality of a set M is abbreviated by $\#M$.

For each multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ (abbreviating $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$) we denote by $|\alpha| := \alpha_1 + \cdots + \alpha_n$ the length of α. In addition, we use the standard notations $\partial_j = (\partial/\partial x_j)$, $1 \leq j \leq n$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$, and $\Delta = \sum_{j=1}^n \partial_j^2$.

2.2 Basic Facts on the Krein–von Neumann extension and the Associated Abstract Buckling Problem

In this preparatory section we recall the basic facts on the Krein–von Neumann extension of a strictly positive operator S in a complex, separable Hilbert space $\mathcal{H}$ and its associated abstract buckling problem as discussed in [9, 10]. For an extensive survey of this circle of ideas and an exhaustive list of references as well as pertinent historical comments we refer to [11].

To set the stage throughout this section, we denote by S a linear, densely defined, symmetric (i.e., $S \subseteq S^*$), and closed operator in $\mathcal{H}$. We recall that S is called *nonnegative* provided $(f, Sf)_{\mathcal{H}} \geq 0$ for all $f \in \operatorname{dom}(S)$. The operator S is called *strictly positive*, if for some $\varepsilon > 0$ one has $(f, Sf)_{\mathcal{H}} \geq \varepsilon \|f\|_{\mathcal{H}}^2$ for all $f \in \operatorname{dom}(S)$; one then writes $S \geq \varepsilon I_{\mathcal{H}}$. Next, we recall that two nonnegative, self-adjoint operators A, B in $\mathcal{H}$ satisfy $A \leq B$ (in the sense of forms) if

$$\operatorname{dom}\left(B^{1/2}\right) \subset \operatorname{dom}\left(A^{1/2}\right) \tag{2.18}$$

and

$$\left(A^{1/2}f, A^{1/2}f\right)_{\mathcal{H}} \leq \left(B^{1/2}f, B^{1/2}f\right)_{\mathcal{H}}, \quad f \in \operatorname{dom}\left(B^{1/2}\right). \tag{2.19}$$

We also recall ([66, Section I.6], [103, Theorem VI.2.21]) that for A and B both self-adjoint and nonnegative in $\mathcal{H}$ one has

$$0 \leq A \leq B \text{ if and only if } (B + aI_{\mathcal{H}})^{-1} \leq (A + aI_{\mathcal{H}})^{-1} \text{ for all } a > 0. \tag{2.20}$$

Moreover, we note the useful fact that $\ker(A) = \ker(A^{1/2})$.

The following is a fundamental result in M. Krein's celebrated 1947 paper [112] (cf. also Theorems 2 and 5–7 in the English summary on page 492):

Theorem 2.1. *Assume that S is a densely defined, closed, nonnegative operator in $\mathcal{H}$. Then, among all nonnegative self-adjoint extensions of S, there exist two distinguished ones, S_K and S_F, which are, respectively, the smallest and largest such extension* (*in the sense of* (2.18)–(2.19)). *Furthermore, a nonnegative self-adjoint operator $\widetilde{S}$ in $\mathcal{H}$ is a self-adjoint extension of S if and only if $\widetilde{S}$ satisfies*

$$S_K \leq \widetilde{S} \leq S_F. \tag{2.21}$$

In particular, the fact that (2.21) *holds for all nonnegative self-adjoint extensions $\widetilde{S}$ of S determines S_K and S_F uniquely. In addition, if $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$, one* *has* $S_F \geq \varepsilon I_{\mathcal{H}}$*, and*

$$\operatorname{dom}(S_F) = \operatorname{dom}(S) \dotplus (S_F)^{-1} \ker(S^*), \tag{2.22}$$

$$\operatorname{dom}(S_K) = \operatorname{dom}(S) \dotplus \ker(S^*), \tag{2.23}$$

$$\begin{aligned} \operatorname{dom}(S^*) &= \operatorname{dom}(S) \dotplus (S_F)^{-1} \ker(S^*) \dotplus \ker(S^*) \\ &= \operatorname{dom}(S_F) \dotplus \ker(S^*), \end{aligned} \tag{2.24}$$

and

$$\ker(S_K) = \ker\big((S_K)^{1/2}\big) = \ker(S^*) = \operatorname{ran}(S)^{\perp}. \tag{2.25}$$

One calls S_K the *Krein–von Neumann extension* of S and S_F the *Friedrichs extension* of S. We also recall that

$$S_F = S^*|_{\operatorname{dom}(S^*)\cap\operatorname{dom}((S_F)^{1/2})}. \tag{2.26}$$

Furthermore, if $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$, then (2.23) implies

$$\ker(S_K) = \ker\big((S_K)^{1/2}\big) = \ker(S^*) = \operatorname{ran}(S)^{\perp}. \tag{2.27}$$

For abstract results regarding the parametrization of all nonnegative self-adjoint extensions of a given strictly positive, densely defined, symmetric operator we refer the reader to Krein [112], Višik [165], Birman [17], Grubb [84, 85], subsequent expositions due to Alonso and Simon [3], Faris [66, Sect. 15], and [87, Sect. 13.2], [88], [154, Ch. 13], and Derkach and Malamud [58], Malamud [122], see also [77, Theorem 9.2].

Let us collect a basic assumption which will be imposed in the rest of this section.

Hypothesis 2.2. *Suppose* S *is a densely defined, symmetric, closed operator with nonzero deficiency indices in* $\mathcal{H}$ *that satisfies* $S \geq \varepsilon I_{\mathcal{H}}$ *for some* $\varepsilon > 0$.

For subsequent purposes we note that under Hypothesis 2.2, one has

$$\dim\big(\ker(S^* - zI_{\mathcal{H}})\big) = \dim\big(\ker(S^*)\big), \quad z \in \mathbb{C}\backslash[\varepsilon, \infty). \tag{2.28}$$

We recall that two self-adjoint extensions S_1 and S_2 of S are called *relatively prime* (or *disjoint*) if $\operatorname{dom}(S_1) \cap \operatorname{dom}(S_2) = \operatorname{dom}(S)$. The following result will play a role later on (cf., e.g., [9, Lemma 2.8] for an elementary proof):

Lemma 2.3. *Assume Hypothesis* 2.2. *Then the Friedrichs extension S_F and the Krein–von Neumann extension S_K of S are relatively prime, that is,*

$$\operatorname{dom}(S_F) \cap \operatorname{dom}(S_K) = \operatorname{dom}(S). \tag{2.29}$$

Next, we consider a self-adjoint operator T in $\mathcal{H}$ which is bounded from below, that is, $T \geq \alpha I_{\mathcal{H}}$ for some $\alpha \in \mathbb{R}$. We denote by $\{E_T(\lambda)\}_{\lambda\in\mathbb{R}}$ the family of strongly right-continuous spectral projections of T, and introduce for $-\infty \leq a < b$, as usual,

$$E_T\big((a,b)\big) = E_T(b_-) - E_T(a) \quad \text{and} \quad E_T(b_-) = \operatorname*{s-lim}_{\varepsilon\downarrow 0} E_T(b-\varepsilon). \tag{2.30}$$

In addition, we set

$$\mu_{T,j} := \inf\big\{\lambda \in \mathbb{R}\,\big|\, \dim(\operatorname{ran}(E_T((-\infty,\lambda)))) \geq j\big\}, \quad j \in \mathbb{N}. \tag{2.31}$$

Then, for fixed $k \in \mathbb{N}$, either:

(i) $\mu_{T,k}$ is the kth eigenvalue of T counting multiplicity below the bottom of the essential spectrum, $\sigma_{ess}(T)$, of T,

or,

(ii) $\mu_{T,k}$ is the bottom of the essential spectrum of T,

$$\mu_{T,k} = \inf\big\{\lambda \in \mathbb{R}\,\big|\, \lambda \in \sigma_{ess}(T)\big\}, \tag{2.32}$$

and in that case $\mu_{T,k+\ell} = \mu_{T,k}$, $\ell \in \mathbb{N}$, and there are at most $k-1$ eigenvalues (counting multiplicity) of T below $\mu_{T,k}$.

We now record a basic result of M. Krein [112] with an extension due to Alonso and Simon [3] and some additional results recently derived in [10]. For this purpose we introduce the *reduced Krein–von Neumann operator* $\widehat{S}_K$ in the Hilbert space

$$\widehat{\mathcal{H}} := \big(\ker(S^*)\big)^{\perp} = \big(\ker(S_K)\big)^{\perp} \tag{2.33}$$

by

$$\widehat{S}_K := P_{(\ker(S_K))^{\perp}} S_K|_{(\ker(S_K))^{\perp}}, \quad \operatorname{dom}(\widehat{S}_K) = \operatorname{dom} S_K \cap \widehat{\mathcal{H}}, \tag{2.34}$$

where $P_{(\ker(S_K))^{\perp}}$ denotes the orthogonal projection onto $(\ker(S_K))^{\perp}$. One then obtains

$$\big(\widehat{S}_K\big)^{-1} = P_{(\ker(S_K))^{\perp}} (S_F)^{-1}|_{(\ker(S_K))^{\perp}}, \tag{2.35}$$

a relation due to Krein [112, Theorem 26] (see also [122, Corollary 5]).

Theorem 2.4. *Assume Hypothesis* 2.2. *Then*

$$\varepsilon \leq \mu_{S_F,j} \leq \mu_{\widehat{S}_K,j}, \quad j \in \mathbb{N}. \tag{2.36}$$

In particular, if the Friedrichs extension S_F of S has purely discrete spectrum, then, except possibly for $\lambda = 0$, the Krein–von Neumann extension S_K of S also has purely discrete spectrum in $(0,\infty)$, that is,

$$\sigma_{ess}(S_F) = \emptyset \text{ implies } \sigma_{ess}(S_K) \subseteq \{0\}. \tag{2.37}$$

In addition, if $p \in (0,\infty]$, then $(S_F - z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_p(\mathcal{H})$ for some $z_0 \in \mathbb{C}\backslash[\varepsilon,\infty)$ implies

$$(S_K - z I_{\mathcal{H}})^{-1}\big|_{(\ker(S_K))^{\perp}} \in \mathcal{B}_p\big(\widehat{\mathcal{H}}\big) \text{ for all } z \in \mathbb{C}\backslash[\varepsilon,\infty). \tag{2.38}$$

In fact, the $\ell^p(\mathbb{N})$-based trace ideal $\mathcal{B}_p(\mathcal{H})$ $\big($resp., $\mathcal{B}_p(\widehat{\mathcal{H}})\big)$ of $\mathcal{B}(\mathcal{H})$ $\big($resp., $\mathcal{B}(\widehat{\mathcal{H}})\big)$ can be replaced by any two-sided symmetrically normed ideal of $\mathcal{B}(\mathcal{H})$ $\big($resp., $\mathcal{B}(\widehat{\mathcal{H}})\big)$.

We note that (2.37) is a classical result of Krein [112]. Apparently, (2.36) in the context of infinite deficiency indices was first proven by Alonso and Simon [3] by a somewhat different method. The implication (2.38) was proved in [10].

Assuming that S_F has purely discrete spectrum, let $\{\lambda_{K,j}\}_{j\in\mathbb{N}} \subset (0,\infty)$ be the strictly positive eigenvalues of S_K enumerated in nondecreasing order, counting multiplicity, and let

$$N(\lambda; S_K) := \#\{j \in \mathbb{N} \,|\, 0 < \lambda_{K,j} < \lambda\}, \quad \lambda > 0, \tag{2.39}$$

be the eigenvalue distribution function for S_K. Similarly, let $\{\lambda_{F,j}\}_{j\in\mathbb{N}} \subset (0,\infty)$ denote the eigenvalues of S_F, again enumerated in nondecreasing order, counting multiplicity, and by

$$N(\lambda; S_F) := \#\{j \in \mathbb{N} \,|\, \lambda_{F,j} < \lambda\}, \quad \lambda > 0, \tag{2.40}$$

the corresponding eigenvalue counting function for S_F. Then inequality (2.36) implies

$$N(\lambda; S_K) \leq N(\lambda; S_F), \quad \lambda > 0. \tag{2.41}$$

In particular, any upper estimate for the eigenvalue counting function for the Friedrichs extension S_F, in turn, yields one for the Krein–von Neumann extension S_K (focusing on strictly positive eigenvalues of S_K according to (2.39)). While this is a viable approach to estimate the eigenvalue counting function (2.39) for S_K, we will proceed along a different route in Section 2.3 and directly exploit the one-to-one corrspondence between strictly positive eigenvalues of S_K and the eigenvalues of its underlying abstract buckling problem to be described next.

To discuss the abstract buckling problem naturally associated with the Krein–von Neumann extension as treated in [10], we start by introducing an abstract version of [86, Proposition 1] (see [10] for a proof):

Lemma 2.5. *Assume Hypothesis* 2.2 *and let* $\lambda \in \mathbb{C}\backslash\{0\}$. *Then there exists some* $f \in \mathrm{dom}(S_K)\backslash\{0\}$ *with*

$$S_K f = \lambda f, \tag{2.42}$$

if and only if there exists $w \in \mathrm{dom}(S^*S)\backslash\{0\}$ *such that*

$$S^*Sw = \lambda Sw. \tag{2.43}$$

In fact, the solutions f *of* (2.42) *are in one-to-one correspondence with the solutions* w *of* (2.43) *in the precise sense that*

$$w = (S_F)^{-1} S_K f, \tag{2.44}$$

$$f = \lambda^{-1} Sw. \tag{2.45}$$

Of course, since $S_K \geq 0$ is self-adjoint, any $\lambda \in \mathbb{C}\backslash\{0\}$ in (2.42) and (2.43) necessarily satisfies $\lambda \in (0, \infty)$.

It is the linear pencil eigenvalue problem $S^*Sw = \lambda Sw$ in (2.43) that we call the *abstract buckling problem* associated with the Krein–von Neumann extension S_K of S.

Next, we turn to a variational formulation of the correspondence between the inverse of the reduced Krein–von Neumann extension $\widehat{S}_K$ and the abstract buckling problem in terms of appropriate sesquilinear forms by following [109]–[111] in the elliptic PDE context. This will then lead to an even stronger connection between the

Krein–von Neumann extension S_K of S and the associated abstract buckling eigenvalue problem (2.43), culminating in the unitary equivalence result in Theorem 2.6 below.

Given the operator S, we introduce the following symmetric forms in $\mathcal{H}$,

$$\mathfrak{a}(f,g) := (Sf, Sg)_{\mathcal{H}}, \quad f,g \in \operatorname{dom}(\mathfrak{a}) := \operatorname{dom}(S), \tag{2.46}$$

$$\mathfrak{b}(f,g) := (f, Sg)_{\mathcal{H}}, \quad f,g \in \operatorname{dom}(\mathfrak{b}) := \operatorname{dom}(S). \tag{2.47}$$

Then S being densely defined and closed implies that the sesquilinear form $\mathfrak{a}$ shares these properties, while $S \geq \varepsilon I_{\mathcal{H}}$ from Hypothesis 2.2 implies that $\mathfrak{a}$ is bounded from below, specifically,

$$\mathfrak{a}(f,f) \geq \varepsilon^2 \|f\|_{\mathcal{H}}^2, \quad f \in \operatorname{dom}(S). \tag{2.48}$$

(Inequality (2.48) follows from the assumption $S \geq \varepsilon I_{\mathcal{H}}$ by estimating $(Sf, Sf)_{\mathcal{H}} = \big([(S-\varepsilon I_{\mathcal{H}}) + \varepsilon I_{\mathcal{H}}]f, [(S-\varepsilon I_{\mathcal{H}}) + \varepsilon I_{\mathcal{H}}]f\big)_{\mathcal{H}}$ from below.)

Thus, one can introduce the Hilbert space

$$\mathcal{W} := \big(\operatorname{dom}(S), (\,\cdot\,,\,\cdot\,)_{\mathcal{W}}\big), \tag{2.49}$$

with associated scalar product

$$(f,g)_{\mathcal{W}} := \mathfrak{a}(f,g) = (Sf, Sg)_{\mathcal{H}}, \quad f,g \in \operatorname{dom}(S). \tag{2.50}$$

In addition, we note that $\iota_{\mathcal{W}} : \mathcal{W} \hookrightarrow \mathcal{H}$, the embedding operator of $\mathcal{W}$ into $\mathcal{H}$, is continuous due to $S \geq \varepsilon I_{\mathcal{H}}$. Hence, a more precise notation would be writing

$$(w_1, w_2)_{\mathcal{W}} = \mathfrak{a}(\iota_{\mathcal{W}} w_1, \iota_{\mathcal{W}} w_2) = (S\iota_{\mathcal{W}} w_1, S\iota_{\mathcal{W}} w_2)_{\mathcal{H}}, \quad w_1, w_2 \in \mathcal{W}, \tag{2.51}$$

but in the interest of simplicity of notation we will omit the embedding operator $\iota_{\mathcal{W}}$ in the following.

With the sesquilinear forms $\mathfrak{a}$ and $\mathfrak{b}$ and the Hilbert space $\mathcal{W}$ as above, given $w_2 \in \mathcal{W}$, the map $\mathcal{W} \ni w_1 \mapsto (w_1, Sw_2)_{\mathcal{H}} \in \mathbb{C}$ is continuous. This allows us to define the operator Tw_2 as the unique element in $\mathcal{W}$ with the property that

$$(w_1, Tw_2)_{\mathcal{W}} = (w_1, Sw_2)_{\mathcal{H}} \text{ for all } w_1 \in \mathcal{W}. \tag{2.52}$$

This implies

$$\mathfrak{a}(w_1, Tw_2) = (w_1, Tw_2)_{\mathcal{W}} = (w_1, Sw_2)_{\mathcal{H}} = \mathfrak{b}(w_1, w_2) \tag{2.53}$$

for all $w_1, w_2 \in \mathcal{W}$. In addition, the operator T satisfies

$$0 \leq T = T^* \in \mathcal{B}(\mathcal{W}) \quad \text{and} \quad \|T\|_{\mathcal{B}(\mathcal{W})} \leq \varepsilon^{-1}. \tag{2.54}$$

We will call T the *abstract buckling problem operator* associated with the Krein–von Neumann extension S_K of S.

Next, recalling the notation $\widehat{\mathcal{H}} = \big(\ker(S^*)\big)^{\perp}$ (cf. (2.33)), we introduce the operator

$$\widehat{S} : \mathcal{W} \to \widehat{\mathcal{H}}, \quad w \mapsto Sw. \tag{2.55}$$

Clearly, $\operatorname{ran}\big(\widehat{S}\big) = \operatorname{ran}(S)$ and since $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$ and S is closed in $\mathcal{H}$, $\operatorname{ran}(S)$ is also closed, and hence coincides with $\big(\ker(S^*)\big)^{\perp}$. This yields

$$\operatorname{ran}\big(\widehat{S}\big) = \operatorname{ran}(S) = \widehat{\mathcal{H}}. \tag{2.56}$$

In fact, it follows that $\widehat{S} \in \mathcal{B}(\mathcal{W}, \widehat{\mathcal{H}})$ maps $\mathcal{W}$ unitarily onto $\widehat{\mathcal{H}}$ (cf. [10]).

Continuing, we briefly recall the polar decomposition of S,

$$S = U_S |S|, \tag{2.57}$$

where, with $\varepsilon > 0$ as in Hypothesis 2.2,

$$|S| = (S^* S)^{1/2} \geq \varepsilon I_{\mathcal{H}} \text{ and } U_S \in \mathcal{B}\big(\mathcal{H}, \widehat{\mathcal{H}}\big) \text{ unitary.} \tag{2.58}$$

Then the principal unitary equivalence result proved in [10] reads as follows:

Theorem 2.6. *Assume Hypothesis* 2.2. *Then the inverse of the reduced Krein–von Neumann extension* $\widehat{S}_K$ *in* $\widehat{\mathcal{H}}$ *and the abstract buckling problem operator* T *in* $\mathcal{W}$ *are unitarily equivalent. Specifically,*

$$\big(\widehat{S}_K\big)^{-1} = \widehat{S}T\big(\widehat{S}\,\big)^{-1}. \tag{2.59}$$

In particular, the nonzero eigenvalues of S_K *are reciprocals of the eigenvalues of* T*. Moreover, one has*

$$\big(\widehat{S}_K\big)^{-1} = U_S\big[|S|^{-1}S|S|^{-1}\big](U_S)^{-1}, \tag{2.60}$$

where $U_S \in \mathcal{B}\big(\mathcal{H}, \widehat{\mathcal{H}}\big)$ *is the unitary operator in the polar decomposition* (2.57) *of* S *and the operator* $|S|^{-1}S|S|^{-1} \in \mathcal{B}(\mathcal{H})$ *is self-adjoint and strictly positive in* $\mathcal{H}$*.*

We emphasize that the unitary equivalence in (2.59) is independent of any spectral assumptions on S_K (such as the spectrum of S_K consists of eigenvalues only) and applies to the restrictions of S_K to its pure point, absolutely continuous, and singularly continuous spectral subspaces, respectively.

Equation (2.60) is motivated by rewriting the abstract linear pencil buckling eigenvalue problem (2.43), $S^*Sw = \lambda Sw$, $\lambda \in \mathbb{C}\backslash\{0\}$, in the form

$$|S|^{-1}Sw = (S^*S)^{-1/2}Sw = \lambda^{-1}(S^*S)^{1/2}w = \lambda^{-1}|S|w \tag{2.61}$$

and hence in the form of a standard eigenvalue problem

$$|S|^{-1}S|S|^{-1}v = \lambda^{-1}v, \quad \lambda \in \mathbb{C}\backslash\{0\}, \quad v := |S|w. \tag{2.62}$$

Again, self-adjointness and strict positivity of $|S|^{-1}S|S|^{-1}$ imply $\lambda \in (0,\infty)$.

We continue this section with an elementary result (recently noted in [74]) that relates the nonzero eigenvalues of S_K directly with the sesquilinear forms $\mathfrak{a}$ and $\mathfrak{b}$:

Lemma 2.7. *Assume Hypothesis 2.2 and introduce*

$$\sigma_p(\mathfrak{a},\mathfrak{b}) := \big\{\lambda\in\mathbb{C}\,\big|\, \textit{there exists } g_\lambda\in\mathrm{dom}(S)\backslash\{0\} \\ \textit{such that } \mathfrak{a}(f,g_\lambda)=\lambda\,\mathfrak{b}(f,g_\lambda), \quad f\in\mathrm{dom}(S)\big\}. \tag{2.63}$$

Then

$$\sigma_p(\mathfrak{a},\mathfrak{b}) = \sigma_p(S_K)\backslash\{0\} \tag{2.64}$$

(*counting multiplicity*), *in particular*, $\sigma_p(\mathfrak{a},\mathfrak{b})\subset(0,\infty)$, *and* $g_\lambda\in\mathrm{dom}(S)\backslash\{0\}$ *in* (2.63) *actually satisfies*

$$g_\lambda\in\mathrm{dom}(S^*S), \quad S^*Sg_\lambda=\lambda Sg_\lambda. \tag{2.65}$$

In addition,

$$\lambda\in\sigma_p(\mathfrak{a},\mathfrak{b}) \ \textit{if and only if} \ \lambda^{-1}\in\sigma_p(T) \tag{2.66}$$

(*counting multiplicity*). *Finally,*

$$T\in\mathcal{B}_\infty(\mathcal{W}) \iff \big(\widehat{S}_K\big)^{-1}\in\mathcal{B}_\infty\big(\widehat{\mathcal{H}}\big) \iff \sigma_{ess}(S_K)\subseteq\{0\}, \tag{2.67}$$

and hence,

$$\sigma_p(\mathfrak{a},\mathfrak{b}) = \sigma(S_K)\backslash\{0\} = \sigma_d(S_K)\backslash\{0\} \tag{2.68}$$

if (2.67) *holds. In particular, if one of* S_F *or* $|S|$ *has purely discrete spectrum* (*i.e.,* $\sigma_{ess}(S_F)=\emptyset$ *or* $\sigma_{ess}(|S|)=\emptyset$), *then* (2.67) *and* (2.68) *hold.*

One notices that $f\in\mathrm{dom}(S)$ in the definition (2.63) of $\sigma_p(\mathfrak{a},\mathfrak{b})$ can be replaced by $f\in C(S)$ for any (operator) core $C(S)$ for S (equivalently, by any form core for the form $\mathfrak{a}$).

We conclude this section with three auxiliary facts to be used in the proof of Theorem 2.21 and start by recalling an elementary result noted in [74]:

Lemma 2.8. *Suppose that S is a densely defined, symmetric, closed operator in $\mathcal{H}$. Then $|S|$ and hence S is infinitesimally bounded with respect to S^*S, more precisely, one has*

$$\begin{aligned} \textit{for all } \varepsilon > 0,\ \|Sf\|_{\mathcal{B}(\mathcal{H})} = \||S|f\|_{\mathcal{B}(\mathcal{H})} \leq \varepsilon\|S^*Sf\|_{\mathcal{H}}^2 + (4\varepsilon)^{-1}\|f\|_{\mathcal{H}}^2, \\ f \in \operatorname{dom}(S^*S). \end{aligned} \tag{2.69}$$

In addition, S is relatively compact with respect to S^*S if $|S|$, or equivalently, S^*S, has compact resolvent. In particular,

$$\sigma_{ess}(S^*S - \lambda S) = \sigma_{ess}(S^*S), \quad \lambda \in \mathbb{R}. \tag{2.70}$$

Given a lower-semibounded, self-adjoint operator $T \geq c_T I_{\mathcal{H}}$ in $\mathcal{H}$, we denote by q_T its uniquely associated form, that is,

$$\mathfrak{q}_T(f, g) = \big(|T|^{1/2}f, \operatorname{sgn}(T)|T|^{1/2}g\big)_{\mathcal{H}}, \quad f, g \in \operatorname{dom}(\mathfrak{q}) = \operatorname{dom}\big(|T|^{1/2}\big), \tag{2.71}$$

and by $\{E_T(\lambda)\}_{\lambda\in\mathbb{R}}$ the family of spectral projections of T. We recall the following well-known variational characterization of dimensions of spectral projections $E_T([c_T, \mu))$, $\mu > c_T$.

Lemma 2.9. *Assume that $c_T I_{\mathcal{H}} \leq T$ is self-adjoint in $\mathcal{H}$ and $\mu > c_T$. Suppose that $\mathcal{F} \subset \operatorname{dom}\big(|T|^{1/2}\big)$ is a linear subspace such that*

$$\mathfrak{q}_T(f, f) < \mu\|f\|_{\mathcal{H}}^2, \quad f \in \mathcal{F}\backslash\{0\}. \tag{2.72}$$

Then,

$$\dim\big(\operatorname{ran}(E_T([c_T, \mu)))\big) = \sup_{\mathcal{F}\subset\operatorname{dom}(|T|^{1/2})} (\dim(\mathcal{F})). \tag{2.73}$$

We add the following elementary observation: Let $c \in \mathbb{R}$ and $B \geq cI_{\mathcal{H}}$ be a self-adjoint operator in $\mathcal{H}$, and introduce the sesquilinear form b in $\mathcal{H}$ associated with B

via

$$
\begin{aligned}
&b(u,v) = \big((B - cI_{\mathcal{H}})^{1/2}u, (B - cI_{\mathcal{H}})^{1/2}v\big)_{\mathcal{H}} + c(u,v)_{\mathcal{H}}, \\
&u, v \in \operatorname{dom}(b) = \operatorname{dom}\big(|B|^{1/2}\big).
\end{aligned} \tag{2.74}
$$

Given B and b, one introduces the Hilbert space $\mathcal{H}_b \subseteq \mathcal{H}$ by

$$
\begin{aligned}
&\mathcal{H}_b = \big(\operatorname{dom}\big(|B|^{1/2}\big), (\,\cdot\,,\,\cdot\,)_{\mathcal{H}_b}\big), \\
&(u,v)_{\mathcal{H}_b} = b(u,v) + (1-c)(u,v)_{\mathcal{H}} \\
&\qquad\quad = \big((B - cI_{\mathcal{H}})^{1/2}u, (B - cI_{\mathcal{H}})^{1/2}v\big)_{\mathcal{H}} + (u,v)_{\mathcal{H}} \\
&\qquad\quad = \big((B + (1-c)I_{\mathcal{H}})^{1/2}u, (B + (1-c)I_{\mathcal{H}})^{1/2}v\big)_{\mathcal{H}}.
\end{aligned} \tag{2.75}
$$

One observes that

$$
(B + (1-c)I_{\mathcal{H}})^{1/2} \colon \mathcal{H}_b \to \mathcal{H} \text{ is unitary.} \tag{2.76}
$$

Finally, we recall the following fact (cf., e.g., [76]).

Lemma 2.10. *Let $\mathcal{H}$, B, b, and $\mathcal{H}_b$ be as in* (2.74)–(2.76)*. Then B has purely discrete spectrum, that is, $\sigma_{ess}(B) = \emptyset$, if and only if $\mathcal{H}_b$ embeds compactly into $\mathcal{H}$.*

2.3 Preliminaries on a Class of Partial Differential Operators

In this section we set the stage for our principal results in Section 2.4 and introduce the class of even-order partial differential operators $\widetilde{A}_{2m}(a,b,q)$ in $L^2(\mathbb{R}^n)$ as well as $A_{\Omega,2m}(a,b,q)$ in $L^2(\Omega)$ (see (2.82) for the underlying differential expressions), with $\emptyset \neq \Omega \subset \mathbb{R}^n$ open and bounded (but otherwise arbitrary). In particular, we provide a detailed study of their domains and quadratic form domains, including spectral properties such as strict boundedness from below for the Friedrichs extension $A_{F,\Omega,2m}(a,b,q)$ of $A_{\Omega,2m}(a,b,q)$ in $L^2(\Omega)$, employing a diamagnetic inequality.

Hypothesis 2.11. (i) *Let $m \in \mathbb{N}$. Assume that*

$$b = (b_1, b_2, \dots, b_n) \in \big[W^{(2m-1),\infty}(\mathbb{R}^n)\big]^n, \quad b_j \;\; \textit{real-valued}, \;\; 1 \leq j \leq n, \tag{2.77}$$

$$0 \leq q \in W^{(2m-2),\infty}(\mathbb{R}^n). \tag{2.78}$$

Suppose $a := \{a_{j,k}\}_{1 \leq j,k \leq n}$ is a real symmetric matrix satisfying

$$a_{j,k} \in C^{(2m-1)}\big(\mathbb{R}^n\big) \cap L^\infty(\mathbb{R}^n), \quad 1 \leq j, k \leq n, \tag{2.79}$$

and with the property that there exists $\varepsilon_a > 0$ such that

$$\sum_{j,k=1}^{n} a_{j,k}(x) y_j y_k \geq \varepsilon_a |y|^2 \;\; \textit{for all} \;\; x \in \mathbb{R}^n, \;\; y = (y_1, \dots, y_n) \in \mathbb{R}^n. \tag{2.80}$$

(ii) *Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be open and bounded. In addition, assume that the $n \times n$ matrix-valued function a equals the identity I_n outside a ball $B_n(0; R_0)$ containing $\overline{\Omega}$, that is, there exists $R_0 > 0$ such that*

$$a(x) = I_n \;\; \textit{whenever} \;\; |x| \geq R_0, \;\; \textit{and} \;\; \overline{\Omega} \subset B_n(0; R_0). \tag{2.81}$$

For simplicity we introduced the ball $B_n(0; R_0)$ containing $\overline{\Omega}$ in Hypothesis 2.11 (ii), but for any fixed $\varepsilon > 0$, one can of course replace $B_n(0; R_0)$ by an open ε-neighborhood Ω_ε of $\overline{\Omega}$.

We will consider various closed (and self-adjoint) L^2-realizations of the differential expression

$$\tau_{2m}(a, b, q) := \Bigg(\sum_{j,k=1}^{n} (-i\partial_j - b_j(x)) a_{j,k}(x) (-i\partial_k - b_k(x)) + q(x) \Bigg)^m, \qquad m \in \mathbb{N}, \;\; x \in \mathbb{R}^n. \tag{2.82}$$

We note that Hypothesis 2.11 (i) was of course chosen with $\tau_{2m}(a, b, q)$ in mind. In some instances we only consider the special case $m = 1$, that is, $\tau_2(a, b, q)$, and then choosing the most general case $m = 1$ in Hypothesis 2.11 (i) will of course be sufficient. We will tacitly assume such a relaxation of hypotheses on the coefficients a, b, q without necessarily dwelling on this explicitly in every such instance.

In the following we find it convenient using auxiliary operators corresponding to the leading and the lower-order terms of the differential expression (2.82). To this end we first introduce the differential expression $\tau_{2m}(a) = \tau_{2m}(a, 0, 0)$,

$$\tau_{2m}(a) := \bigg(- \sum_{j,k=1}^{n} \partial_j a_{j,k}(x) \partial_k \bigg)^m, \quad m \in \mathbb{N}, \ x \in \mathbb{R}^n, \tag{2.83}$$

and the associated linear operator $\widetilde{T}_{2m}(a)$ in $L^2(\mathbb{R}^n)$ given by

$$\widetilde{T}_{2m}(a)u := \tau_{2m}(a)u, \quad u \in \operatorname{dom}\big(\widetilde{T}_{2m}(a)\big) := W^{2m,2}(\mathbb{R}^n). \tag{2.84}$$

Second, we observe that due to boundedness of the coefficients a, b, q (cf. (2.77)) and sufficiently many of their derivatives, one has

$$\begin{aligned} &\tau_{2m}(a, b, q)u = \tau_{2m}(a)u + \sum_{0 \le |\alpha| \le 2m-1} g_\alpha(a, b, q, x) \partial^\alpha u, \\ &\tau_{2m}(a, b, q)u \in L^2(\mathbb{R}^n), \quad u \in W^{2m,2}(\mathbb{R}^n), \end{aligned} \tag{2.85}$$

for some $g_\alpha(a, b, q, \,\cdot\,) \in L^\infty(\mathbb{R}^n)$, $0 \le |\alpha| \le 2m - 1$. The sum of the lower-order terms in (2.85) gives rise to a linear operator $\widetilde{S}_{2m-1}(a, b, q)$ in $L^2(\mathbb{R}^n)$,

$$\begin{aligned} &\widetilde{S}_{2m-1}(a, b, q)u := \sum_{0 \le |\alpha| \le 2m-1} g_\alpha(a, b, q, x) \partial^\alpha u, \\ &u \in \operatorname{dom}\big(\widetilde{S}_{2m-1}(a, b, q)\big) := W^{2m,2}(\mathbb{R}^n). \end{aligned} \tag{2.86}$$

Next, we introduce the operator $\widetilde{A}_{2m}(a, b, q)$ in $L^2(\mathbb{R}^n)$ by

$$\widetilde{A}_{2m}(a, b, q)u := \tau_{2m}(a, b, q)u, \quad u \in \operatorname{dom}\big(\widetilde{A}_{2m}(a, b, q)\big) := W^{2m,2}(\mathbb{R}^n), \tag{2.87}$$

and its restriction $\widetilde{A}_{0,2m}(a, b, q)$ to $C_0^\infty(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ via

$$\widetilde{A}_{0,2m}(a, b, q)u := \tau_{2m}(a, b, q)u, \quad u \in \operatorname{dom}\big(\widetilde{A}_{0,2m}(a, b, q)\big) := C_0^\infty(\mathbb{R}^n). \tag{2.88}$$

Making use of standard perturbation results, it is convenient to view the operator $\widetilde{A}_{2m}(a,b,q)$ as perturbation of $\widetilde{T}_{2m}(a)$ by $\widetilde{S}_{2m-1}(a,b,q)$ and state the following auxiliary fact.

Theorem 2.12. *Assume Hypothesis* 2.11 (*i*). *Then* $\widetilde{A}_{0,2m}(a,b,q)$ *is essentially self-adjoint in* $L^2(\mathbb{R}^n)$*, its closure equals* $\widetilde{A}_{2m}(a,b,q)$*, and hence,*

$$\widetilde{A}_{2m}(a,b,q) \geq 0. \tag{2.89}$$

In addition, the graph norm of $\widetilde{A}_{2m}(a,b,q)$ *is equivalent to the norm of the Sobolev space* $W^{2m,2}(\mathbb{R}^n)$*, that is, there exist finite constants* $0 < c < C$*, depending only on* a,b,q,m,n*, such that*

$$\begin{aligned} c\|u\|^2_{W^{2m,2}(\mathbb{R}^n)} \leq \big\|\widetilde{A}_{2m}(a,b,q)u\big\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)} \leq C\|u\|^2_{W^{2m,2}(\mathbb{R}^n)}, \\ u \in W^{2m,2}(\mathbb{R}^n). \end{aligned} \tag{2.90}$$

Proof. We introduce the minimal operator $\widetilde{T}_{0,2m}(a)$ in $L^2(\mathbb{R}^n)$ by

$$\widetilde{T}_{0,2m}(a)u := \tau_{2m}(a)u, \quad u \in \operatorname{dom}(\widetilde{T}_{0,2m}(a)) := C_0^\infty(\mathbb{R}^n), \tag{2.91}$$

and will show that it is essentially self-adjoint and that $\widetilde{T}_{2m}(a) = \big(\widetilde{T}_{0,2m}(a)\big)^*$; the operator $\widetilde{A}_{2m}(a,b,q)$ will then be considered as an infinitesimally bounded perturbation of $\widetilde{T}_{2m}(a)$.

Let $u \in L^2(\mathbb{R}^n) \cap W^{2m,2}_{loc}(\mathbb{R}^n)$ and $\tau_{2m}(a)u \in L^2(\mathbb{R}^n)$, then for arbitrary $v \in \operatorname{dom}\big(\widetilde{T}_{0,2m}(a)\big) = C_0^\infty(\mathbb{R}^n)$ one has

$$\begin{aligned} \big(u, \widetilde{T}_{0,2m}(a)v\big)_{L^2(\mathbb{R}^n)} &= (u, \tau_{2m}(a)v)_{L^2(\mathbb{R}^n)} \\ &= {}_{\mathcal{D}'(\mathbb{R}^n)}\langle \tau_{2m}(a)u, v\rangle_{\mathcal{D}(\mathbb{R}^n)} = (\tau_{2m}(a)u, v)_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{2.92}$$

hence $u \in \operatorname{dom}\big(\big(\widetilde{T}_{0,2m}(a)\big)^*\big)$ and $\big(\widetilde{T}_{0,2m}(a)\big)^* u = \tau_{2m}(a)u$, implying

$$\big\{u \in L^2(\mathbb{R}^n) \,\big|\, u \in W^{2m,2}_{loc}(\mathbb{R}^n),\, \tau_{2m}(a)u \in L^2(\mathbb{R}^n)\big\} \subseteq \operatorname{dom}\big(\big(\widetilde{T}_{0,2m}(a)\big)^*\big). \tag{2.93}$$

Using the interior regularity for elliptic differential operators, one obtains the converse inclusion: Indeed, if $u \in \operatorname{dom}\big(\big(\widetilde{T}_{0,2m}(a)\big)^*\big)$, then $u \in L^2(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ and for some $v \in L^2(\mathbb{R}^n)$ one has $\tau_{2m}(a)u = v$, implying $u \in W^{2m,2}_{loc}(\mathbb{R}^n)$ (see, e.g., [137, Theorem 1.3], see also [162]).

Our next objective is to show that $\operatorname{dom}\big(\big(\widetilde{T}_{0,2m}(a)\big)^*\big) = W^{2m,2}(\mathbb{R}^n)$. Let $\varphi_{R_0} \in C_0^\infty(\mathbb{R}^n)$ and $\varphi_{R_0}(x) = 1$, $x \in B_n(0; R_0)$, cf. (2.81). Since $u\varphi_{R_0} \in W^{2m,2}(\mathbb{R}^n)$ for any $u \in \operatorname{dom}\big(\big(\widetilde{T}_{0,2m}(a)\big)^*\big)$, in order to prove that $\operatorname{dom}\big(\big(\widetilde{T}_{0,2m}(a)\big)^*\big) \subseteq W^{2m,2}(\mathbb{R}^n)$ it suffices to obtain the inclusion $u(1-\varphi_{R_0}) \in W^{2m,2}(\mathbb{R}^n)$. This, in turn, will be guaranteed once we prove the following fact,

$$\operatorname{dom}\big(\big(\widetilde{T}_{0,2m}(a)\big)^*(1-\varphi_{R_0})\big) = \operatorname{dom}\big(H_0^m(1-\varphi_{R_0})\big). \tag{2.94}$$

Here the self-adjoint operator H_0 in $L^2(\mathbb{R}^n)$ is defined by

$$H_0 u = (-\Delta)u, \quad u \in \operatorname{dom}(H_0) = W^{2,2}(\mathbb{R}^n), \tag{2.95}$$

and hence

$$H_0^\alpha u = (-\Delta)^\alpha u, \quad u \in \operatorname{dom}\big(H_0^\alpha\big) = W^{2\alpha,2}(\mathbb{R}^n), \quad \alpha \in (0,\infty). \tag{2.96}$$

For $u \in \operatorname{dom}(H_0^m(1-\varphi_{R_0}))$, the expression $\big(\widetilde{T}_{0,2m}(a)\big)^*(1-\varphi_{R_0})u - H_0^m(1-\varphi_{R_0})u$ does not contain derivatives of u of order higher than $2m-1$, therefore, for any $\varepsilon > 0$ there exists some finite $k(\varepsilon) > 0$ such that

$$\begin{aligned} &\big\|\big(\widetilde{T}_{0,2m}(a)\big)^*(1-\varphi_{R_0})u - H_0^m(1-\varphi_{R_0})u\big\|^2_{L^2(\mathbb{R}^n)} \\ &\quad \le \varepsilon\big\|H_0^m(1-\varphi_{R_0})u\big\|^2_{L^2(\mathbb{R}^n)} + k(\varepsilon)\|u\|^2_{L^2(\mathbb{R}^n)}, \quad u \in \operatorname{dom}\big(H_0^m(1-\varphi_{R_0})\big). \end{aligned} \tag{2.97}$$

Combining (2.97) and [103, Theorem IV 1.1] one obtains equality of the domains in (2.94), and hence also $\operatorname{dom}((\widetilde{T}_{0,2m}(a))^*) \subseteq W^{2m,2}(\mathbb{R}^n)$. The opposite inclusion is clear from (2.93).

Next we will show that

$$(\widetilde{T}_{0,2m}(a))^* u = \tau_{2m}(a)u, \quad u \in \operatorname{dom}\big((\widetilde{T}_{0,2m}(a))^*\big) = W^{2m,2}(\mathbb{R}^n). \tag{2.98}$$

To this end, fix $v \in \operatorname{dom}\big(\widetilde{T}_{0,2m}(a)\big) = C_0^\infty(\mathbb{R}^n)$ and an arbitrary $u \in W^{2m,2}(\mathbb{R}^n)$. Then using the membership $\tau_{2m}(a)u \in L^2(\mathbb{R}^n)$, one obtains

$$\begin{aligned}
\big(u, \widetilde{T}_{0,2m}(a)v\big)_{L^2(\mathbb{R}^n)} &= (u, \tau_{2m}(a)v)_{L^2(\mathbb{R}^n)} \\
&= {}_{\mathcal{D}'(\mathbb{R}^n)}\langle \tau_{2m}(a)u, v\rangle_{\mathcal{D}(\mathbb{R}^n)} = (\tau_{2m}(a)u, v)_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{2.99}$$

and hence $\big(\widetilde{T}_{0,2m}(a)\big)^* u = \tau_{2m}(a)u$. The arbitrariness of u implies that $(\widetilde{T}_{0,2m}(a))^*$ is symmetric. Therefore $\widetilde{T}_{0,2m}(a)$ is essentially self-adjoint and thus $\widetilde{T}_{2m}(a) = \big(\widetilde{T}_{0,2m}(a)\big)^*$ is self-adjoint.

The proof thus far showed an important fact: The graph norms of the operators $\widetilde{T}_{2m}(a)$ and H_0^m, both defined on $W^{2m,2}(\mathbb{R}^n)$, are equivalent, that is, there exist finite constants $0 < c_1 < C_1$, depending only on the coefficients a, b, q, m, n, such that

$$\begin{aligned}
c_1\big[\big\|H_0^m u\big\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}\big] &\leq \big\|\widetilde{T}_{2m}(a)u\big\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)} \\
&\leq C_1\big[\big\|H_0^m u\big\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}\big], \quad u \in W^{2m,2}(\mathbb{R}^n).
\end{aligned} \tag{2.100}$$

In particular, the graph norm of $\widetilde{T}_{2m}(a)$ is equivalent to the norm of $W^{2m,2}(\mathbb{R}^n)$.

Finally we show that $\big(\widetilde{A}_{0,m}(a,b,q)\big)^*$ is symmetric, actually, self-adjoint, proving that $\widetilde{A}_{0,m}(a,b,q)$ is essentially self-adjoint. To this end, we recall the operator $\widetilde{S}_{2m-1}(a,b,q)$ in (2.86), corresponding to lower-order terms in the differential expression $\tau_{2m}(a,b,q)$. Since $\widetilde{S}_{2m-1}(a,b,q)$ has bounded coefficients and its order is at most $2m-1$, it is infinitesimally bounded with respect to the polyharmonic operator H_0^m. Thus, for any $\varepsilon > 0$ there exists some finite $k(\varepsilon) > 0$ such that

$$\big\|\widetilde{S}_{2m-1}(a,b,q)u\big\|^2_{L^2(\mathbb{R}^n)} \leq \varepsilon\|H_0^m u\|^2_{L^2(\mathbb{R}^n)} + k(\varepsilon)\|u\|^2_{L^2(\mathbb{R}^n)}, \quad u \in W^{2m,2}(\mathbb{R}^n). \tag{2.101}$$

Combining this inequality with the equivalence of the graph norms of $\widetilde{T}_{2m}(a)$ and H_0^m, one concludes that $\widetilde{S}_{2m-1}(a,b,q)$ is infinitesimally bounded with respect to $\widetilde{T}_{2m}(a)$. Hence, $\widetilde{A}_{0,2m}(a,b,q) = \widetilde{T}_{0,2m}(a) + \widetilde{S}_{2m-1}(a,b,q)$ is essentially self-adjoint, and $\operatorname{dom}((A_{0,m}(a,b,q))^*) = \operatorname{dom}\big(\widetilde{T}_{2m}(a)\big) = W^{2m,2}(\mathbb{R}^n)$. The fact (2.90) follows from [61, Proposition 7.2] and the fact that $\widetilde{A}_{2m}(a,b,q)$ and H_0^m have the common domain $W^{2m,2}(\mathbb{R}^n)$ and both are closed (in fact, self-adjoint). ∎

Lemma 2.13. *Assume Hypothesis* 2.11 *(i). Then for all* $\alpha \in (0,1]$,

$$\operatorname{dom}\big((\widetilde{A}_{2m}(a,b,q))^\alpha\big) = W^{2m\alpha,2}(\mathbb{R}^n), \tag{2.102}$$

and there exist finite constants $0 < c < C$ *depending only on* a,b,q,m,n, *such that*

$$\begin{aligned} c\|u\|^2_{W^{m,2}(\mathbb{R}^n)} \leq \big\|\widetilde{A}_{2m}(a,b,q)^{1/2}u\big\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)} \leq C\|u\|^2_{W^{m,2}(\mathbb{R}^n)}, \\ u \in W^{m,2}(\mathbb{R}^n), \end{aligned} \tag{2.103}$$

and hence,

$$\begin{aligned} c\|u\|^2_{W^{m,2}(\mathbb{R}^n)} \leq \big(u, \widetilde{A}_{2m}(a,b,q)u\big)_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)} \leq C\|u\|^2_{W^{m,2}(\mathbb{R}^n)}, \\ u \in W^{2m,2}(\mathbb{R}^n). \end{aligned} \tag{2.104}$$

Proof. We start with a well-known interpolation argument: Let S and T be closed operators in $\mathcal{H}$ satisfying $\operatorname{dom}(S) \supseteq \operatorname{dom}(T)$. Then S is relatively bounded with respect to T (cf., e.g., [61, Proposition III.7.2], [103, Remark IV.1.5]) and hence there exist finite constants $a > 0$ and $b > 0$ such that

$$\begin{aligned} \||S|f\|^2_{\mathcal{H}} = \|Sf\|^2_{\mathcal{H}} \leq a^2\|Tf\|^2_{\mathcal{H}} + b^2\|f\|^2_{\mathcal{H}} = a^2\||T|f\|^2_{\mathcal{H}} + b^2\|f\|^2_{\mathcal{H}} \\ = \big\|\big[a^2|T|^2 + b^2I_{\mathcal{H}}\big]^{1/2}f\big\|^2_{\mathcal{H}}, \quad f \in \operatorname{dom}(T) = \operatorname{dom}(|T|). \end{aligned} \tag{2.105}$$

Thus, applying the Loewner–Heinz inequality (cf., e.g., [100], [113, Theorem IV.1.11]), one infers that (see also [75])

$$\operatorname{dom}\big(|S|^\alpha\big) \supseteq \operatorname{dom}\big(\big(a^2|T|^2 + b^2I_{\mathcal{H}}\big)^{\alpha/2}\big) = \operatorname{dom}\big(|T|^\alpha\big), \quad \alpha \in (0,1]. \tag{2.106}$$

In particular, if $\operatorname{dom}(S)=\operatorname{dom}(T)$ one concludes that

$$\operatorname{dom}\big(|S|^{\alpha}\big)=\operatorname{dom}\big(|T|^{\alpha}\big), \quad \alpha\in(0,1]. \tag{2.107}$$

Identifying S with $\widetilde{A}_{2m}(a,b,q)$ and T with H_0^m, (2.96) and (2.107) prove (2.102).

Employing (2.102) with $\alpha=1/2$ one infers that

$$\begin{aligned} \big\|\widetilde{A}_{2m}(a,b,q)^{1/2}u\big\|^2_{L^2(\mathbb{R}^n)}+\|u\|^2_{L^2(\mathbb{R}^n)} \approx \big\|H_0^{m/2}u\big\|^2_{L^2(\mathbb{R}^n)}+\|u\|^2_{L^2(\mathbb{R}^n)},& \\ u\in W^{m,2}(\mathbb{R}^n).& \end{aligned} \tag{2.108}$$

Assuming, in addition, that $u\in W^{2m,2}(\mathbb{R}^n)$, the equivalence in (2.108) may be rewritten as

$$\big(u,\widetilde{A}_{2m}(a,b,q)u\big)_{L^2(\mathbb{R}^n)}+\|u\|^2_{L^2(\mathbb{R}^n)} \approx \big(u,H_0^m u\big)_{L^2(\mathbb{R}^n)}+\|u\|^2_{L^2(\mathbb{R}^n)}, \tag{2.109}$$

and together with the fact that the right-hand side of (2.109) is equivalent to the norm $\|\cdot\|^2_{W^{m,2}(\mathbb{R}^n)}$, one arrives at (2.104). ∎

Given Lemma 2.13, the sequilinear form $Q_{\widetilde{A}_{2m}(a,b,q)}$ in $L^2(\mathbb{R}^n)$ associated with $\widetilde{A}_{2m}(a,b,q)$ is given by

$$\begin{aligned} &Q_{\widetilde{A}_{2m}(a,b,q)}(u,v) := \big(\widetilde{A}_{2m}(a,b,q)^{1/2}u,\widetilde{A}_{2m}(a,b,q)^{1/2}v\big)_{L^2(\mathbb{R}^n)}, \\ &u,v\in\operatorname{dom}(Q_{\widetilde{A}_{2m}(a,b,q)})=\operatorname{dom}\big(\widetilde{A}_{2m}(a,b,q)^{1/2}\big)=W^{m,2}(\mathbb{R}^n), \end{aligned} \tag{2.110}$$

and we also introduce

$$Q_{H_0^m}(u,v) := \big(H_0^{m/2}u,H_0^{m/2}v\big)_{L^2(\mathbb{R}^n)}, \quad u,v\in\operatorname{dom}(Q_{H_0^m})=W^{m,2}(\mathbb{R}^n). \tag{2.111}$$

In addition, we will employ the explicit representation of the form $Q_{\widetilde{A}_{2m}(a,b,q)}$ in terms of $\widetilde{A}_{2m}(a,b,q)$,

$$Q_{\widetilde{A}_{2m}(a,b,q)}(u,v)=\begin{cases} (\tau_{2\ell}u,\tau_{2\ell}v)_{L^2(\mathbb{R}^n)}, \quad m=2\ell,\ \ell\in\mathbb{N}, \\ \sum_{j,k=1}^n((-i\partial_j-b_j)\tau_{2\ell}u,a_{j,k}(-i\partial_k-b_k)\tau_{2\ell}v)_{L^2(\mathbb{R}^n)} \\ +(\tau_{2\ell}u,q\,\tau_{2\ell}v)_{L^2(\mathbb{R}^n)}, \quad m=2\ell+1,\ \ell\in\mathbb{N}\cup\{0\}, \end{cases}$$

$$u, v \in W^{m,2}(\mathbb{R}^n). \tag{2.112}$$

Here, in obvious notation, $\tau_0 = 1$.

Assuming Hypothesis 2.11 (i), we introduce one of the main objects of our study, the symmetric operator $A_{\Omega,2m}(a,b,q)$ in $L^2(\Omega)$ by

$$A_{\Omega,2m}(a,b,q)f = \tau_{2m}(a,b,q)f, \quad f \in \operatorname{dom}(A_{\Omega,2m}(a,b,q)) = W_0^{2m,2}(\Omega), \tag{2.113}$$

and note that $\widetilde{A}_{2m}(a,b,q)$ formally represents its extended version in $L^2(\mathbb{R}^n)$. In addition, we introduce the associated minimal operator $A_{min,\Omega,2m}(a,b,q)$ in $L^2(\Omega)$ by

$$\begin{aligned} &A_{min,\Omega,2m}(a,b,q)f := \tau_{2m}(a,b,q)f, \\ &f \in \operatorname{dom}(A_{min,\Omega,2m}(a,b,q)) := C_0^\infty(\Omega). \end{aligned} \tag{2.114}$$

Clearly, $A_{min,\Omega,2m}(a,b,q)$ is symmetric (hence, closable) in $L^2(\Omega)$ (upon elementary integration by parts) and nonnegative,

$$A_{min,\Omega,2m}(a,b,q) \geq 0. \tag{2.115}$$

Theorem 2.14. *Assume Hypothesis* 2.11 (i)*. Then the closure of* $A_{min,\Omega,2m}(a,b,q)$ *in* $L^2(\Omega)$ *is given by* $A_{\Omega,2m}(a,b,q)$*,*

$$\overline{A_{min,\Omega,2m}(a,b,q)} = A_{\Omega,2m}(a,b,q). \tag{2.116}$$

In particular, $A_{\Omega,2m}(a,b,q)$ *is symmetric and nonnegative in* $L^2(\Omega)$*,*

$$A_{\Omega,2m}(a,b,q) \geq 0. \tag{2.117}$$

In addition, there exist finite constants $0 < c < C$*, depending only on* a,b,q,m,n*, such that*

$$\begin{aligned} c\|f\|^2_{W_0^{2m,2}(\Omega)} \leq \|A_{\Omega,2m}(a,b,q)f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} \leq C\|f\|^2_{W_0^{2m,2}(\Omega)}, \\ f \in W_0^{2m,2}(\Omega). \end{aligned} \tag{2.118}$$

Proof. Using (2.90) with $v \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}(v) \subset \Omega$ one concludes that the graph norm of $A_{\Omega,2m}(a,b,q)$ is equivalent to the norm of $\mathring{W}^{2m,2}(\Omega)$ on $C_0^\infty(\Omega)$. Therefore, $\operatorname{dom}\big(\overline{A_{min,\Omega,2m}(a,b,q)}\big) = \mathring{W}^{2m,2}(\Omega)$. In order to prove that $\overline{A_{min,\Omega,2m}(a,b,q)}f = \tau_{2m}(a,b,q)f$, we consider $\{f_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\Omega)$, $f,g \in L^2(\Omega)$, such that

$$\lim_{j\to\infty} \|f_j - f\|_{L^2(\Omega)} = 0 \text{ and } \lim_{j\to\infty} \big\|A_{min,\Omega,2m}(a,b,q)f_j - g\big\|_{L^2(\Omega)} = 0. \tag{2.119}$$

Since $A_{min,\Omega,2m}(a,b,q)$ is symmetric and hence closable in $L^2(\Omega)$, one infers that

$$f \in \operatorname{dom}\big(\overline{A_{min,\Omega,2m}(a,b,q)}\big) = W_0^{2m,2}(\Omega) \text{ and } \overline{A_{min,\Omega,2m}(a,b,q)}f = g. \tag{2.120}$$

Taking arbitrary $\psi \in C_0^\infty(\Omega)$, and recalling our notation for the distributional pairing ${}_{\mathcal{D}'(\Omega)}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{D}(\Omega)}$ (compatible with the scalar product $(\,\cdot\,,\,\cdot\,)_{L^2(\Omega)}$), one concludes that

$$\begin{aligned}
(g,\psi)_{L^2(\Omega)} &= {}_{\mathcal{D}'(\Omega)}\langle g,\psi\rangle_{\mathcal{D}(\Omega)} = \lim_{j\to\infty} {}_{\mathcal{D}'(\Omega)}\langle \tau_{2m}(a,b,q)f_j,\psi\rangle_{\mathcal{D}(\Omega)} \\
&= \lim_{j\to\infty} \int_\Omega \overline{f_j(x)}\,\big(\tau_{2m}(a,b,q)\psi\big)(x)\,d^nx = \int_\Omega \overline{f(x)}\,\big(\tau_{2m}(a,b,q)\psi\big)(x)\,d^nx \\
&= {}_{\mathcal{D}'(\Omega)}\langle \tau_{2m}(a,b,q)f,\psi\rangle_{\mathcal{D}(\Omega)},
\end{aligned} \tag{2.121}$$

implying $g = \tau_{2m}(a,b,q)f$ and hence, $\overline{A_{min,\Omega,2m}(a,b,q)}f = A_{\Omega,2m}(a,b,q)f$ implying (2.116). This also completes the proof of (2.118). Finally, being the closure of the symmetric operator $A_{min,\Omega,2m}(a,b,q)$, also $A_{\Omega,2m}(a,b,q)$ is symmetric in $L^2(\Omega)$ (cf., e.g., [169, Theorem 5.4 (b)]). ∎

Next, still assuming Hypothesis 2.11 (i), we introduce the form $Q_{A_{\Omega,2m}(a,b,q)}$ in $L^2(\Omega)$ generated by $A_{\Omega,2m}(a,b,q)$, via

$$\begin{aligned}
&Q_{A_{\Omega,2m}(a,b,q)}(f,g) := (f, A_{\Omega,2m}(a,b,q)g)_{L^2(\mathbb{R}^n)}, \\
&f,g \in \operatorname{dom}(Q_{A_{\Omega,2m}(a,b,q)}) := W_0^{2m,2}(\Omega).
\end{aligned} \tag{2.122}$$

Lemma 2.15. *Assume Hypothesis* 2.11 (*i*). *Then the form* $Q_{A_{\Omega,2m}(a,b,q)}$ *is closable and its closure in* $L^2(\Omega)$, *denoted by* $Q_{A_{F,\Omega,2m}(a,b,q)}$, *is the form uniquely associated to the Friedrichs extension* $A_{F,\Omega,2m}(a,b,q)$ *of* $A_{\Omega,2m}(a,b,q)$, *that is,*

$$\begin{aligned} &Q_{A_{F,\Omega,2m}(a,b,q)}(f,g) = \big(A_{F,\Omega,2m}(a,b,q)^{1/2}f, A_{F,\Omega,2m}(a,b,q)^{1/2}g\big)_{L^2(\Omega)}, \\ &f,g \in \operatorname{dom}\big(Q_{A_{F,\Omega,2m}(a,b,q)}\big) = \operatorname{dom}\big(A_{F,\Omega,2m}(a,b,q)^{1/2}\big) = W_0^{m,2}(\Omega). \end{aligned} \tag{2.123}$$

Proof. That $Q_{A_{\Omega,2m}(a,b,q)}$ is closable follows from abstract results relating sectorial (in particular, non-negative, symmetric) operators and their forms (cf., e.g., [61, Theorem IV.2.3], [103, Theorem VI.1.27], [139, Theorem X.23]). In order to prove (2.123), we fix $f \in W_0^{2m,2}(\Omega)$ and denote its extension by zero outside of Ω by $\widetilde{f}$. Then $\widetilde{f} \in W^{2m,2}(\mathbb{R}^n)$ and employing (2.104) with u replaced by $\widetilde{f}$, and using the fact that $\operatorname{supp}(\widetilde{u}) \subseteq \Omega$, one obtains

$$c\|f\|^2_{W_0^{m,2}(\Omega)} \leq (f, A_{\Omega,2m}(a,b,q)f)_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} \leq C\|f\|^2_{W_0^{m,2}(\Omega)}, \tag{2.124}$$

that is,

$$c\|f\|^2_{W_0^{m,2}(\Omega)} \leq Q_{A_{\Omega,2m}(a,b,q)}(f,f) + \|f\|^2_{L^2(\Omega)} \leq C\|f\|^2_{W_0^{m,2}(\Omega)}, \tag{2.125}$$

for some finite constants $0 < c < C$, independent of f, proving that the domain of the closure of the form $Q_{A_{\Omega,2m}(a,b,q)}$ equals $W_0^{m,2}(\Omega)$. Together with [103, Sect. VI.2.3] or [139, Theorem X.23], and the second representation theorem for forms (see, e.g., [61, Theorem IV.2.6, Theorem IV.2.8], [103, Theorem VI.2.23]), this proves (2.123).

∎

In Section 2.4 we will also use the following explicit representation of the form $Q_{A_{F,\Omega,2m}(a,b,q)}$,

$$Q_{A_{F,\Omega,2m}(a,b,q)}(f,g) = \begin{cases} (\tau_{2\ell}f, \tau_{2\ell}g)_{L^2(\Omega)}, \quad m = 2\ell,\ \ell \in \mathbb{N}, \\ \sum_{j,k=1}^n ((-i\partial_j - b_j)\tau_{2\ell}f, a_{j,k}(-i\partial_k - b_k)\tau_{2\ell}g)_{L^2(\Omega)} \\ +(\tau_{2\ell}f, q\,\tau_{2\ell}g)_{L^2(\Omega)}, \quad m = 2\ell+1,\ k \in \mathbb{N}\cup\{0\}, \end{cases}$$

$$f, g \in W_0^{m,2}(\Omega). \tag{2.126}$$

(Again, we use the convention $\tau_0 = 1$.)

Finally, we introduce the following symmetric form in $L^2(\Omega)$,

$$\begin{aligned} &\mathfrak{a}_{\Omega,4m,a,b,q}(f,g) := (A_{\Omega,2m}(a,b,q)f, A_{\Omega,2m}(a,b,q)g)_{L^2(\Omega)}, \\ &f, g \in \operatorname{dom}(\mathfrak{a}_{\Omega,4m,a,b,q}) := \operatorname{dom}(A_{\Omega,2m}(a,b,q)), \end{aligned} \tag{2.127}$$

and the Hilbert space

$$\begin{aligned} \mathcal{H}_{A_{\Omega,2m}(a,b,q)} :=& (\operatorname{dom}(A_{\Omega,2m}(a,b,q)), \mathfrak{a}_{\Omega,4m,a,b,q}(\,\cdot\,,\,\cdot\,)) \\ =& \big(W_0^{2m,2}(\Omega), \mathfrak{a}_{\Omega,4m,a,b,q}(\,\cdot\,,\,\cdot\,)\big), \end{aligned} \tag{2.128}$$

equipped with the scalar product $\mathfrak{a}_{\Omega,4m,a,b,q}(\,\cdot\,,\,\cdot\,)$.

Lemma 2.16. *Assume Hypothesis* 2.11 (*i*). *Then the Hilbert space $\mathcal{H}_{A_{\Omega,2m}(a,b,q)}$ embeds compactly into $L^2(\Omega)$.*

Proof. This is a consequence of the compact embedding of $\mathring{W}^{2m,2}(\Omega)$ into $L^2(\Omega)$ (see, e.g., [61, Theorem V.4.18]) and the inequalities (2.118). ∎

At this point we strengthen the lower bounds (2.115), (2.117):

Theorem 2.17. *Assume Hypothesis* 2.11 (*i*) *with $m = 1$. Then there exists $\varepsilon > 0$, depending only on a and Ω, such that $A_{min,\Omega,2}(a,b,q)$ defined as in* (2.114) *with $m = 1$ satisfies*

$$A_{min,\Omega,2}(a,b,q) \geq \varepsilon I_\Omega, \tag{2.129}$$

and hence,

$$A_{\Omega,2}(a,b,q) \geq \varepsilon I_\Omega \ \text{ and } \ A_{F,\Omega,2}(a,b,q) \geq \varepsilon I_\Omega. \tag{2.130}$$

Proof. It suffices to prove that there exists $\varepsilon > 0$ such that $A_{F,\Omega,2}(a,b,q) \geq \varepsilon I_\Omega$. Since $\operatorname{dom}\big(A_{F,\Omega,2}(a,b,q)^{1/2}\big) = W_0^{1,2}(\Omega)$ according to (2.123), one recalls that

$$f \in W_0^{1,2}(\Omega) \ \text{ implies } \ |f| \in W_0^{1,2}(\Omega) \tag{2.131}$$

(cf., e.g., [61, Corollary VI.2.4]), and that by [134, Proposition 4.4],

$$\partial_j |f| = \mathrm{Re}\big(\mathrm{sgn}\,\big(\overline{f}\big)(\partial_j f)\big) \text{ a.e.,} \quad f \in W_0^{1,2}(\Omega),\ 1 \le j \le n, \tag{2.132}$$

with

$$\mathrm{sgn}(g(x)) = \begin{cases} g(x)/|g(x)|, & \text{if } g(x) \neq 0, \\ 0, & \text{if } g(x) = 0. \end{cases} \tag{2.133}$$

Thus, $\nabla|f| = \mathrm{Re}\big(\mathrm{sgn}\,\big(\overline{f}\big)(\nabla f)\big)$, $f \in W_0^{1,2}(\Omega)$, and hence one obtains the diamagnetic inequality on Ω,

$$\begin{aligned} |\nabla|f|| \le \big|\mathrm{Re}\big(\mathrm{sgn}\,\big(\overline{f}\big)(\nabla f)\big)\big| = \big|\mathrm{Re}\big(\mathrm{sgn}\,\big(\overline{f}\big)((\nabla - ib)f)\big)\big| \le |(-i\nabla - b)f| \text{ a.e.,}& \\ f \in W_0^{1,2}(\Omega),& \end{aligned} \tag{2.134}$$

since b_j, $1 \le j \le n$, are real-valued, according to a device of Kato [102] and Simon [158] (see also [14, Theorem 4.5.1], [121, Theorem 7.21]). Hence, employing the min-max principle for the infimum of the spectrum of self-adjoint operators bounded from below one estimates,

$$\begin{aligned} \inf(\sigma(A_{F,\Omega,2}(a,b,q))) &= \inf_{f \in W_0^{1,2}(\Omega),\, \|f\|_{L^2(\Omega)}=1} Q_{A_{F,\Omega,2}(a,b,q)}(f,f) \\ &= \inf_{f \in W_0^{1,2}(\Omega),\, \|f\|_{L^2(\Omega)}=1} \big(A_{F,\Omega,2}(a,b,q)^{1/2} f, A_{F,\Omega,2}(a,b,q)^{1/2} f\big)_{L^2(\Omega)} \\ &= \inf_{f \in W_0^{1,2}(\Omega),\, \|f\|_{L^2(\Omega)}=1} \Bigg(\sum_{j,k=1}^n ((-i\partial_j - b_j)f, a_{j,k}(-i\partial_k - b_k)f)_{L^2(\Omega)} \\ &\qquad\qquad + (f, q\, f)_{L^2(\Omega)} \Bigg) \\ &\ge \varepsilon_a \inf_{f \in W_0^{1,2}(\Omega),\, \|f\|_{L^2(\Omega)}=1} ((-i\nabla - b)f, (-i\nabla - b)f)_{L^2(\Omega)^n} \\ &\ge \varepsilon_a \inf_{f \in W_0^{1,2}(\Omega),\, \|f\|_{L^2(\Omega)}=1} (|\nabla|f||, |\nabla|f||)_{L^2(\Omega)} \\ &= \varepsilon_a \inf_{f \in W_0^{1,2}(\Omega),\, \|f\|_{L^2(\Omega)}=1} (\nabla|f|, \nabla|f|)_{L^2(\Omega)^n} \\ &\ge \varepsilon_a \inf_{\varphi \in W_0^{1,2}(\Omega),\, \|\varphi\|_{L^2(\Omega)}=1} \big((\nabla\varphi, \nabla\varphi)_{L^2(\Omega)^n}\big) \end{aligned}$$

$$\geq \varepsilon_a \inf\big(\sigma\big(-\Delta_\Omega^D\big)\big)$$

$$= \varepsilon_a \varepsilon_\Omega =: \varepsilon, \tag{2.135}$$

using the fact that $-\Delta_\Omega^D \geq \varepsilon_\Omega I_\Omega$ for some $\varepsilon_\Omega > 0$, since Ω is bounded (see, for instance, [56, p. 31], or use domain monotonicity, [141, p. 270] together with the well-known strictly positive lower bounds for a ball or cube that encloses Ω). ∎

The result (2.130) holds under more general assumptions on the coeffcients a, b, q and also for certain boundary conditions other than Dirichlet, but the current setup suffices for our discussion in Section 2.4 (we intend to revisit this issue elsewhere).

Next, we note that as a consequence of Hypothesis 2.11 (i), also all higher-order powers $A_{\Omega,2m}(a,b,q) = A_{\Omega,2}(a,b,q)^m$, $m \in \mathbb{N}$, $m \geq 2$, of $A_{\Omega,2}(a,b,q)$ are strictly positive.

Lemma 2.18. *Assume Hypothesis* 2.11 (i). *Then there exists* $\varepsilon_m > 0$ *such that*

$$A_{\Omega,2m}(a,b,q) \geq \varepsilon_m I_\Omega, \quad m \in \mathbb{N}. \tag{2.136}$$

Proof. We employ induction with respect to $m \in \mathbb{N}$. The case $m = 1$ holds by Hypothesis 2.11 (i). Assume that the statement holds for all $k < m$ and fix any $0 \neq f \in \operatorname{dom}(A_{\Omega,2m}(a,b,q))$. We consider two cases:

(i) $m = 2\ell$, $\ell \in \mathbb{N}$. Then due to symmetry of $A_{\Omega,2}(a,b,q)^\ell$ one obtains

$$(f, A_{\Omega,2m}(a,b,q)f)_{L^2(\Omega)} = \big(f, A_{\Omega,2}(a,b,q)^{2\ell} f\big)_{L^2(\Omega)} = \big\|A_{\Omega,2}(a,b,q)^\ell f\big\|_{L^2(\Omega)}^2. \tag{2.137}$$

By the induction hypothesis, there exists $\varepsilon_\ell > 0$ such that, $A_{\Omega,2\ell}(a,b,q) \geq \varepsilon_\ell$, and hence

$$\varepsilon_\ell \|f\|_{L^2(\Omega)}^2 \leq (f, A_{\Omega,2\ell}(a,b,q)f)_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|A_{\Omega,2\ell}(a,b,q)f\|_{L^2(\Omega)}, \tag{2.138}$$

implying

$$(f, A_{\Omega,2m}(a,b,q)f)_{L^2(\Omega)} = \big\|A_{\Omega,2}(a,b,q)^\ell f\big\|^2_{L^2(\Omega)} \geq \varepsilon_\ell^2 \|f\|^2_{L^2(\Omega)} = \varepsilon_m \|f\|^2_{L^2(\Omega)}, \quad (2.139)$$

with $\varepsilon_m = \varepsilon_\ell^2$.

(ii) $m = 2\ell + 1$, $\ell \in \mathbb{N}$. Then by (2.129)

$$\begin{aligned} (f, A_{\Omega,2m}(a,b,q)f)_{L^2(\Omega)} &= \big(f, A_{\Omega,2}(a,b,q)^{2\ell+1} f\big)_{L^2(\Omega)} \\ &= \big(A_{\Omega,2}(a,b,q)^\ell f, A_{\Omega,2}(a,b,q) A_{\Omega,2}(a,b,q)^\ell f\big)_{L^2(\Omega)} \\ &\geq \varepsilon \big\|A_{\Omega,2}(a,b,q)^\ell f\big\|^2_{L^2(\Omega)} \geq \varepsilon\varepsilon_\ell^2 \|f\|^2_{L^2(\Omega)} = \varepsilon_m \|f\|^2_{L^2(\Omega)}, \end{aligned} \quad (2.140)$$

with $\varepsilon_m = \varepsilon\varepsilon_\ell^2$. ∎

2.4 An Upper Bound for the Eigenvalue Counting Function for the Krein–von Neumann and Friedrichs Extensions of Higher-Order Operators

In this section we derive an upper bound for the eigenvalue counting function for Krein–von Neumann extensions of higher-order differential operators on open, bounded, nonempty domains $\Omega \subset \mathbb{R}^n$. In particular, no assumptions on the boundary of Ω will be made.

In the following we denote by $A_{K,\Omega,2m}(a,b,q)$ and $A_{F,\Omega,2m}(a,b,q)$ the Krein–von Neumann and Friedrichs extensions of $A_{\Omega,2m}(a,b,q)$ in $L^2(\Omega)$. Since by Lemma 2.16, $\mathcal{H}_{A_{\Omega,2m}(a,b,q)}$ embeds compactly into $L^2(\Omega)$, $A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q)$ has purely discrete spectrum by Lemma 2.10. Equivalently, $A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q)$ has a compact resolvent, in particular,

$$[A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q)]^{-1} \in \mathcal{B}_\infty\big(L^2(\Omega)\big). \quad (2.141)$$

Consequenty, also

$$|A_{\Omega,2m}(a,b,q)|^{-1} = [A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q)]^{-1/2} \in \mathcal{B}_\infty\big(L^2(\Omega)\big), \tag{2.142}$$

implying

$$\big(\widehat{A}_{K,\Omega,2m}(a,b,q)\big)^{-1} \in \mathcal{B}_\infty\big(L^2(\Omega)\big) \tag{2.143}$$

by (2.60). Thus,

$$\sigma_{ess}(A_{K,\Omega,2m}(a,b,q)) \subseteq \{0\}. \tag{2.144}$$

We recall that the form $\mathfrak{a}_{\Omega,4m,a,b,q}(\,\cdot\,,\,\cdot\,)$ in $L^2(\Omega)$ associated with the operator $A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q)$ has been introduced in (2.127).

Let $\{\lambda_{K,\Omega,j}\}_{j\in\mathbb{N}} \subset (0,\infty)$ be the strictly positive eigenvalues of $A_{K,\Omega,2m}(a,b,q)$ enumerated in nondecreasing order, counting multiplicity, and let

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) := \#\{j \in \mathbb{N} \,|\, 0 < \lambda_{K,\Omega,j} < \lambda\}, \quad \lambda > 0, \tag{2.145}$$

be the eigenvalue distribution function for $A_{K,\Omega,2m}(a,b,q)$.

To derive an effective estimate for $N(\lambda; A_{K,\Omega,2m}(a,b,q))$ we need to introduce one more spectral hypothesis imposed on $\widetilde{A}_{2m}(a,b,q)$:

Hypothesis 2.19. *Assume Hypothesis* 2.11*.*

(i) Suppose there exists $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ *such that the operator*

$$(\mathbb{F}f)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)\overline{\phi(x,\xi)}\, d^n x, \quad \xi \in \mathbb{R}^n, \tag{2.146}$$

originally defined on functions $f \in L^2(\mathbb{R}^n)$ *with compact support, can be extended to a unitary operator in* $L^2(\mathbb{R}^n)$*, such that*

$$f \in W^{2,2}(\mathbb{R}^n; d^n x) \text{ if and only if } |\xi|^2(\mathbb{F}f)(\xi) \in L^2(\mathbb{R}^n; d^n\xi), \tag{2.147}$$

and

$$\widetilde{A}_2(a,b,q) = \mathbb{F}^{-1} M_{|\xi|^2} \mathbb{F}, \tag{2.148}$$

where $M_{|\xi|^2}$ *represents the maximally defined operator of multiplication by* $|\xi|^2$ *in* $L^2(\mathbb{R}^n; d^n\xi)$.

(*ii*) *In addition, suppose that*

$$\sup_{\xi \in \mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|_{L^2(\Omega)} < \infty. \tag{2.149}$$

Remark 2.20. (*i*) As becomes clear from Theorems 2.21 and 2.22 below, our primary concerns are the operators $A_{K,\Omega,2m}(a,b,q)$ and $A_{F,\Omega,2m}(a,b,q)$ in $L^2(\Omega)$, and hence we are primarily interested in the coefficients a, b, q on the open, bounded, but otherwise arbitrary, set Ω. However, since the existence of an eigenfunction expansion of a self-adjoint "continuation" of this pair of operators to all of $\mathbb{R}^n$, denoted by $\widetilde{A}_{2m}(a,b,q)$, is a crucial tool in our derivation of the bound on the corresponding eigenvalue counting functions of $A_{K,\Omega,2m}(a,b,q)$ and $A_{F,\Omega,2m}(a,b,q)$, the continuation of the coefficients a, b, q through a possibly highly nontrivial boundary $\partial\Omega$ of Ω becomes a nontrivial issue. To avoid intricate technicalities, we chose to simply assume a sufficiently smooth behavior of a, b, q throughout $\mathbb{R}^n$ in Hypothesis 2.11 (*i*).

(*ii*) Hypothesis 2.19 (*i*) implies that $\widetilde{A}_2(a,b,q)$ (and hence any of its powers) is spectrally purely absolutely continuous (i.e., its point and singular continuous spectra are empty), while Hypothesis 2.19 (*ii*) requires a uniform $L^2(\Omega)$-bound on $\phi(\,\cdot\,,\xi)$, $\xi \in \mathbb{R}^n$. In particular, $\phi(\,\cdot\,,\,\cdot\,)$ represent the suitably normalized generalized eigenfunctions of $\widetilde{A}_2(a,b,q)$ satisfying

$$\widetilde{A}_2(a,b,q)\phi(\,\cdot\,,\xi) = |\xi|^2\phi(\,\cdot\,,\xi), \quad \xi \in \mathbb{R}^n, \tag{2.150}$$

in the distributional sense. In the special Laplacian case $a = I_n$, $b = q = 0$, one obtains

$$\phi(x,\xi) = e^{i\xi\cdot x}, \quad \|\phi(\,\cdot\,,\xi)\|_{L^2(\Omega)}^2 = |\Omega|, \quad (x,\xi) \in \mathbb{R}^{2n}. \tag{2.151}$$

(iii) In the case $a = I_n$, and with the exception of a possible zero-energy resonance and/or eigenvalue of $\widetilde{A}_2(I_n, b, q)$ in $L^2(\mathbb{R}^n)$, we expect Hypothesis 2.19 to hold for $\widetilde{A}_2(I_n, b, q)$ under the regularity assumptions made on b, q in Hypothesis 2.11 (i) assuming in addition that b and q have sufficiently fast decay as $|x| \to \infty$ (e.g., if b, q have compact support). While we have not found the corresponding statement in the literature, and an attempt to prove it in full generality would be an independent project, we will illustrate in our final Section 2.5 explicit situations in which Hypothesis 2.19 holds for $a = I_n$. The case $a \neq I_n$, on the other hand, is much more involved due trapping/non-trapping issues which affect the existence of bounds of the type (2.217); we refer, for instance, to [40], [41], [55], [166], [167], and the literature therein.

(iv) We note from the outset, that a zero-energy resonance and/or eigenvalue of $\widetilde{A}_{2m}$ cannot be excluded even in the special case $a = I_n$, $b = 0$, and $q \in C_0^\infty(\mathbb{R}^n)$. However, the existence of such zero-energy resonances or eigenvalues is highly unstable with respect to small variations of a, b, q and hence their absence holds generically. In particular, by slightly varying $R_0 > 0$ in Hypothesis 2.11 (ii), or the ε-neighborhood Ω_ε of Ω mentioned after (2.81), or by slightly perturbing the coefficients a, b, or q outside $B_n(0; R_0)$, or outside Ω_ε, one can guarantee the absence of such zero-energy resonances and/or eigenvalues. Since we are primarily interested in the operators $A_{K,\Omega,2m}(a, b, q)$ and $A_{F,\Omega,2m}(a, b, q)$ in $L^2(\Omega)$, we can indeed freely choose the form of

a, b, q in an ε-neighborhood outside of Ω, especially, in a neighborhood of infinity. $\diamond$

With the standard notation

$$x_+ := \max(0, x), \quad x \in \mathbb{R}, \tag{2.152}$$

we have the following estimate for $N(\,\cdot\,; A_{K,\Omega,2m}(a, b, q))$ (extending the results in [114] where the special case $a = I_n$, $b = q = 0$ has been considered):

Theorem 2.21. *Assume Hypothesis* 2.19. *Then the following estimate holds:*

$$N(\lambda; A_{K,\Omega,2m}(a, b, q)) \leq \frac{v_n}{(2\pi)^n}\left(1 + \frac{2m}{2m+n}\right)^{n/(2m)} \sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)}\, \lambda^{n/(2m)}, \quad \text{for all } \lambda > 0, \tag{2.153}$$

where $v_n := \pi^{n/2}/\Gamma((n+2)/2)$ *denotes the* (*Euclidean*) *volume of the unit ball in* $\mathbb{R}^n$ ($\Gamma(\cdot)$ *being the Gamma function*), *and* $\phi(\,\cdot\,,\,\cdot\,)$ *represents the suitably normalized generalized eigenfunctions of* $\widetilde{A}_2(a, b, q)$ *satisfying* $\widetilde{A}_2(a, b, q)\phi(\,\cdot\,,\xi) = |\xi|^2\phi(\,\cdot\,,\xi)$ *in the distributional sense* (*cf. Hypothesis* 2.19).

Proof. Following our abstract Section 2.2, we introduce in addition to the symmetric form $\mathfrak{a}_{\Omega,4m,a,b,q}$ in $L^2(\Omega)$ (cf. (2.127)), the form

$$\begin{aligned} &\mathfrak{b}_{\Omega,2m,a,b,q}(f, g) := (f, A_{\Omega,2m}(a, b, q)g)_{L^2(\Omega)}, \\ &f, g \in \operatorname{dom}(\mathfrak{b}_{\Omega,2m,a,b,q}) := \operatorname{dom}(A_{\Omega,2m}(a, b, q)). \end{aligned} \tag{2.154}$$

By Lemma 2.7, particularly, by (2.66), one concludes that

$$\begin{aligned} N(\lambda; A_{K,\Omega,2m}(a, b, q)) \leq \max\big(\dim\big\{f \in \operatorname{dom}(A_{\Omega,2m}(a, b, q))\,\big| \\ \mathfrak{a}_{\Omega,4m,a,b,q}(f, f) - \lambda\, \mathfrak{b}_{\Omega,2m,a,b,q}(f, f) < 0\big\}\big). \end{aligned} \tag{2.155}$$

Here we also employed (2.68) and the fact that

$$\begin{aligned} &\mathfrak{a}_{\Omega,4m,a,b,q}(f_{K,\Omega,j}, f_{K,\Omega,j})) - \lambda\, \mathfrak{b}_{\Omega,2m,a,b,q}(f_{K,\Omega,j}, f_{K,\Omega,j}) \\ &\quad = (\lambda_{K,\Omega,j} - \lambda)\|f_{K,\Omega,j}\|^2_{L^2(\Omega)} < 0, \end{aligned} \tag{2.156}$$

where $f_{K,\Omega,j} \in \operatorname{dom}(A_{\Omega,2m}(a,b,q))\backslash\{0\}$ additionally satisfies

$$
\begin{aligned}
&f_{K,\Omega,j} \in \operatorname{dom}(A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q)) \text{ and} \\
&A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q) f_{K,\Omega,j} = \lambda_{K,\Omega,j}\, A_{\Omega,2m}(a,b,q) f_{K,\Omega,j}.
\end{aligned}
\tag{2.157}
$$

To further analyze (2.155) we now fix $\lambda \in (0,\infty)$ and introduce the auxiliary operator

$$
\begin{aligned}
&L_{\Omega,4m,\lambda}(a,b,q) := A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q) - \lambda\, A_{\Omega,2m}(a,b,q), \\
&\operatorname{dom}(L_{\Omega,4m,\lambda}(a,b,q)) := \operatorname{dom}(A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q)).
\end{aligned}
\tag{2.158}
$$

By Lemma 2.8, $L_{\Omega,4m,\lambda}(a,b,q)$ is self-adjoint, bounded from below, with purely discrete spectrum as its form domain satisfies (cf. (2.128))

$$
\operatorname{dom}\left(|L_{\Omega,4m,\lambda}(a,b,q)|^{1/2}\right) = \mathcal{H}_{A_{\Omega,2m}(a,b,q)}, \tag{2.159}
$$

and the latter embeds compactly into $L^2(\Omega)$ by Lemma 2.16 (cf. Lemma 2.10). We will study the auxiliary eigenvalue problem,

$$
L_{\Omega,4m,\lambda}(a,b,q)\varphi_j = \mu_j \varphi_j, \quad \varphi_j \in \operatorname{dom}(L_{\Omega,4m,\lambda}(a,b,q)), \tag{2.160}
$$

where $\{\varphi_j\}_{j\in\mathbb{N}}$ represents an orthonormal basis of eigenfunctions in $L^2(\Omega)$ and for simplicity of notation we repeat the eigenvalues μ_j of $L_{\Omega,4m,\lambda}(a,b,q)$ according to their multiplicity. Since $\varphi_j \in W_0^{2m,2}(\Omega)$, the zero-extension of φ_j to all of $\mathbb{R}^n$,

$$
\widetilde{\varphi}_j(x) := \begin{cases} \varphi_j(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n\backslash\Omega, \end{cases} \tag{2.161}
$$

satisfies

$$
\widetilde{\varphi}_j \in W^{2m,2}(\mathbb{R}^n), \quad \partial^\alpha \widetilde{\varphi}_j = \widetilde{\partial^\alpha \varphi_j}, \quad 0 \leq |\alpha| \leq 2m. \tag{2.162}
$$

Next, given $\mu > 0$, one estimates

$$
\begin{aligned}
\mu^{-1} \sum_{\substack{j\in\mathbb{N} \\ \mu_j<\mu}} (\mu - \mu_j) &\geq \mu^{-1} \sum_{\substack{j\in\mathbb{N}, \\ \mu_j<0,\, \mu_j<\mu}} (\mu - \mu_j) \geq \mu^{-1} \sum_{\substack{j\in\mathbb{N}, \\ \mu_j<0,\, \mu_j<\mu}} \mu \\
&= n_-(L_{\Omega,4m,\lambda}(a,b,q)),
\end{aligned}
\tag{2.163}
$$

where $n_-(L_{\Omega,4m,\lambda}(a,b,q))$ denotes the number of strictly negative eigenvalues of $L_{\Omega,4m,\lambda}(a,b,q)$. Combining, Lemma 2.9 and (2.155) one concludes that

$$\begin{aligned} N(\lambda; A_{K,\Omega,2m}(a,b,q)) &\leq \max\big(\dim\big\{f\in \mathrm{dom}(A_{min,\Omega,2m})\,\big| \\ &\qquad \mathfrak{a}_{\Omega,4m,a,b,q}(f,f)-\lambda\,\mathfrak{b}_{\Omega,2m,a,b,q}(f,f)<0\big\}\big) \qquad (2.164)\\ &= n_-(L_{\Omega,4m,\lambda}(a,b,q)) \leq \mu^{-1}\sum_{\substack{j\in\mathbb{N}\\ \mu_j<\mu}}(\mu-\mu_j) = \mu^{-1}\sum_{j\in\mathbb{N}}[\mu-\mu_j]_+, \quad \mu>0. \end{aligned}$$

Next, we focus on estimating the right-hand side of (2.164).

$$\begin{aligned} N(\lambda; A_{K,\Omega,2m}(a,b,q)) &\leq \mu^{-1}\sum_{j\in\mathbb{N}}(\mu-\mu_j)_+ = \mu^{-1}\sum_{j\in\mathbb{N}}\big[(\varphi_j,(\mu-\mu_j)\varphi_j)_{L^2(\Omega)}\big]_+ \\ &= \mu^{-1}\sum_{j\in\mathbb{N}}\Big[\mu\|\varphi_j\|^2_{L^2(\Omega)} - \|A_{\Omega,2m}(a,b,q)\varphi_j\|^2_{L^2(\Omega)} \\ &\qquad + \lambda(\varphi_j, A_{\Omega,2m}(a,b,q)\varphi_j)_{L^2(\Omega)}\Big]_+ \\ &= \mu^{-1}\sum_{j\in\mathbb{N}}\Big[\mu\|\widetilde{\varphi}_j\|^2_{L^2(\mathbb{R}^n)} - \big\|\widetilde{A}_{2m}(a,b,q)\widetilde{\varphi}_j\big\|^2_{L^2(\mathbb{R}^n)} \\ &\qquad + \lambda\big(\widetilde{\varphi}_j, \widetilde{A}_{2m}(a,b,q)\widetilde{\varphi}_j\big)_{L^2(\mathbb{R}^n)}\Big]_+ \\ &= \mu^{-1}\sum_{j\in\mathbb{N}}\Big[\int_{\mathbb{R}^n}\big[\mu-|\xi|^{4m}+\lambda|\xi|^{2m}\big]|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\,d^n\xi\Big]_+ \\ &\leq \mu^{-1}\sum_{j\in\mathbb{N}}\int_{\mathbb{R}^n}\big[\mu-|\xi|^{4m}+\lambda|\xi|^{2m}\big]_+|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\,d^n\xi \\ &\leq \mu^{-1}\int_{\mathbb{R}^n}\big[\mu-|\xi|^{4m}+\lambda|\xi|^{2m}\big]_+\sum_{j\in\mathbb{N}}|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\,d^n\xi. \qquad (2.165) \end{aligned}$$

Since Ω is bounded, $\widetilde{\varphi}_j$ has compact support and hence

$$(\mathbb{F}\widetilde{\varphi}_j)(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\widetilde{\varphi}_j(x)\overline{\phi(x,\xi)}\,d^nx, \qquad (2.166)$$

and

$$\begin{aligned} \sum_{j\in\mathbb{N}}|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2 &= (2\pi)^{-n}\sum_{j\in\mathbb{N}}\bigg|\int_{\mathbb{R}^n}\widetilde{\varphi}_j(x)\overline{\phi(x,\xi)}\,d^nx\bigg|^2 \\ &= (2\pi)^{-n}\sum_{j\in\mathbb{N}}\Big|\int_{\Omega}\varphi_j(x)\overline{\phi(x,\xi)}\,d^nx\Big|^2 = (2\pi)^{-n}\|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega,d^nx)}, \end{aligned} \qquad (2.167)$$

are well-defined. Combining (2.165) and (2.167) one arrives at

$$\begin{aligned}
N(\lambda; A_{K,\Omega,2m}(a,b,q)) &\leq \mu^{-1} \int_{\mathbb{R}^n} \big[\mu - |\xi|^{4m} + \lambda|\xi|^{2m}\big]_+ \sum_{j\in\mathbb{N}} |(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2 \, d^n\xi \\
&= (2\pi)^{-n}\mu^{-1} \int_{\mathbb{R}^n} \big[\mu - |\xi|^{4m} + \lambda|\xi|^{2m}\big]_+ \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)} \, d^n\xi \\
&\leq (2\pi)^{-n} \sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)} \mu^{-1} \int_{\mathbb{R}^n} \big[\mu - |\xi|^{4m} + \lambda|\xi|^{2m}\big]_+ \, d^n\xi. \qquad (2.168)
\end{aligned}$$

Introducing $\alpha = \lambda^{-2}\mu$, changing variables, $\xi = \lambda^{1/(2m)}\eta$, and taking the minimum with respect to $\alpha > 0$, proves the bound

$$\begin{aligned}
N(\lambda; A_{K,\Omega,2m}(a,b,q)) &\leq (2\pi)^{-n} \sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)} \\
&\quad \times \min_{\alpha>0} \bigg(\alpha^{-1} \int_{\mathbb{R}^n} \big[\alpha - |\eta|^{4m} + |\eta|^{2m}\big]_+ d^n\eta\bigg) \lambda^{n/(2m)}, \quad \lambda > 0.
\end{aligned} \qquad (2.169)$$

Explicitly computing the minimum over $\alpha > 0$ in (2.169) yields the result (2.153). This minimization step is carried out in detail in Appendix A. ∎

Next, we also derive an upper bound for the eigenvalue counting function of the Friedrichs extension $A_{F,\Omega,2m}(a,b,q)$ of $A_{\Omega,2m}(a,b,q)$.

Theorem 2.22. *Assume Hypothesis* 2.19. *Then the following estimate holds:*

$$N(\lambda; A_{F,\Omega,2m}(a,b,q)) \leq \frac{v_n}{(2\pi)^n} \bigg(1 + \frac{2m}{n}\bigg)^{n/(2m)} \sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)} \lambda^{n/(2m)}, \quad \text{for all } \lambda > 0, \qquad (2.170)$$

with $v_n := \pi^{n/2}/\Gamma((n+2)/2)$ *and* $\phi(\,\cdot\,,\,\cdot\,)$ *given as in Theorem* 2.21.

Proof. First, one notices that

$$\begin{aligned}
N(\lambda; A_{F,\Omega,2m}(a,b,q)) \leq \max\big(\dim\big\{f \in \operatorname{dom}(A_{F,\Omega,2m}(a,b,q)) \,\big|\, \\
(f, A_{F,\Omega,2m}(a,b,q)f)_{L^2(\Omega)} - \lambda\|f\|^2_{L^2(\Omega)} < 0\big\}\big),
\end{aligned} \qquad (2.171)$$

To further analyze the right–hand side of (2.171) fix $\lambda \in (0,\infty)$ and introduce the auxiliary operator

$$\begin{aligned} &K_{\Omega,2m,\lambda}(a,b,q) := A_{F,\Omega,2m}(a,b,q) - \lambda I_\Omega, \\ &\operatorname{dom}(K_{\Omega,2m,\lambda}(a,b,q)) := \operatorname{dom}(A_{F,\Omega,2m}(a,b,q)). \end{aligned} \tag{2.172}$$

We will study the eigenvalue problem,

$$K_{\Omega,2m,\lambda}(a,b,q)\varphi_j = \mu_j\varphi_j, \quad \varphi_j \in \operatorname{dom}(K_{\Omega,m,\lambda}(a,b,q)), \tag{2.173}$$

where $\{\varphi_j\}_{j\in\mathbb{N}}$ represents an orthonormal basis of eigenfunctions in $L^2(\Omega)$ and for simplicity of notation we repeat the eigenvalues μ_j of $K_{\Omega,2m,\lambda}(a,b,q)$ according to their multiplicity. Since $\varphi_j \in W_0^m(\Omega)$, their zero-extension to all of $\mathbb{R}^n$,

$$\widetilde{\varphi}_j(x) := \begin{cases} \varphi_j(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n\backslash\Omega, \end{cases} \tag{2.174}$$

satisfies

$$\widetilde{\varphi}_j \in W^m(\mathbb{R}^n), \quad \partial^\alpha\widetilde{\varphi}_j = \widetilde{\partial^\alpha\varphi_j}, \quad 0 \le |\alpha| \le m. \tag{2.175}$$

Next, given $\mu > 0$, one estimates

$$\begin{aligned} \mu^{-1}\sum_{\substack{j\in\mathbb{N}\\ \mu_j<\mu}}(\mu-\mu_j) &\ge \mu^{-1}\sum_{\substack{j\in\mathbb{N},\\ \mu_j<0,\,\mu_j<\mu}}(\mu-\mu_j) \ge \mu^{-1}\sum_{\substack{j\in\mathbb{N},\\ \mu_j<0,\,\mu_j<\mu}}\mu \\ &= n_-(K_{\Omega,2m,\lambda}(a,b,q)), \end{aligned} \tag{2.176}$$

where $n_-(K_{\Omega,2m,\lambda}(a,b,q))$ denotes the number of strictly negative eigenvalues of $K_{\Omega,2m,\lambda}(a,b,q)$. Then one has

$$\begin{aligned} &N(\lambda; A_{F,\Omega,2m}(a,b,q)) \\ &\quad \le \max\big(\dim\big\{f \in \operatorname{dom}(A_{F,\Omega,2m}(a,b,q))\,\big| \\ &\qquad (f, A_{F,\Omega,2m}(a,b,q)f)_{L^2(\Omega)} - \lambda\|f\|^2_{L^2(\Omega)} < 0\big\}\big) \\ &\quad = n_-(K_{\Omega,2m,\lambda}(a,b,q)) \le \mu^{-1}\sum_{\substack{j\in\mathbb{N}\\ \mu_j<\mu}}(\mu-\mu_j) = \mu^{-1}\sum_{j\in\mathbb{N}}[\mu-\mu_j]_+, \quad \mu > 0. \end{aligned} \tag{2.177}$$

To estimate the right-hand side of (2.177) we rewrite $(\psi_1, A_{F,\Omega,2m}(a,b,q)\psi_2)_{L^2(\Omega)}$ for $\psi_1,\psi_2 \in \operatorname{dom}(A_{F,\Omega,2m}(a,b,q))$, as follows

$$\begin{aligned}
(\psi_1, A_{F,\Omega,2m}(a,b,q)\psi_2)_{L^2(\Omega)} &= Q_{A_{F,\Omega,2m}(a,b,q)}(\psi_1,\psi_2) = Q_{\widetilde{A}_{2m}(a,b,q)}(\widetilde{\psi}_1,\widetilde{\psi}_2) \\
&= \Big(\big(\widetilde{A}_{2m}(a,b,q)\big)^{1/2}\widetilde{\psi}_1, \big(\widetilde{A}_{2m}(a,b,q)\big)^{1/2}\widetilde{\psi}_2\Big)_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{2.178}$$

the second equality in (2.178) following from representations (2.112), (2.126). Next, we focus on estimating the right-hand side of (2.177).

$$\begin{aligned}
N(\lambda; A_{F,\Omega,2m}(a,b,q)) &\le \mu^{-1}\sum_{j\in\mathbb{N}}(\mu-\mu_j)_+ = \mu^{-1}\sum_{j\in\mathbb{N}}\big[(\varphi_j,(\mu-\mu_j)\varphi_j)_{L^2(\Omega)}\big]_+ \\
&= \mu^{-1}\sum_{j\in\mathbb{N}}\Big[\mu\|\varphi_j\|^2_{L^2(\Omega)} + \lambda\|\varphi_j\|^2_{L^2(\Omega)} - (\varphi_j, A_{F,\Omega,2m}(a,b,q)\varphi_j)_{L^2(\Omega)}\Big]_+ \\
&= \mu^{-1}\sum_{j\in\mathbb{N}}\Big[\mu\|\mathbb{F}\widetilde{\varphi}_j\|^2_{L^2(\mathbb{R}^n)} + \lambda\|\mathbb{F}\widetilde{\varphi}_j\|^2_{L^2(\mathbb{R}^n)} - \||\xi|^m\mathbb{F}\widetilde{\varphi}_j\|^2_{L^2(\mathbb{R}^n)}\Big]_+ \\
&= \mu^{-1}\sum_{j\in\mathbb{N}}\bigg[\int_{\mathbb{R}^n}\big[\mu+\lambda-|\xi|^{2m}\big]|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\,d^n\xi\bigg]_+ \\
&\le \mu^{-1}\sum_{j\in\mathbb{N}}\int_{\mathbb{R}^n}\big[\mu+\lambda-|\xi|^{2m}\big]_+|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\,d^n\xi \\
&\le \mu^{-1}\int_{\mathbb{R}^n}\big[\mu+\lambda-|\xi|^{2m}\big]_+\sum_{j\in\mathbb{N}}|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\,d^n\xi.
\end{aligned} \tag{2.179}$$

Combining (2.167) and (2.179) one arrives at

$$\begin{aligned}
N(\lambda; A_{F,\Omega,2m}(a,b,q)) &\le \mu^{-1}\int_{\mathbb{R}^n}\big[\mu+\lambda-|\xi|^{2m}\big]_+\sum_{j\in\mathbb{N}}|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\,d^n\xi \\
&= (2\pi)^{-n}\mu^{-1}\int_{\mathbb{R}^n}\big[\mu+\lambda-|\xi|^{2m}\big]_+\|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)}\,d^n\xi \\
&\le (2\pi)^{-n}\sup_{\xi\in\mathbb{R}^n}\|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)}\mu^{-1}\int_{\mathbb{R}^n}\big[\mu+\lambda-|\xi|^{2m}\big]_+\,d^n\xi.
\end{aligned} \tag{2.180}$$

Introducing $\alpha = \lambda^{-1}\mu$, changing variables, $\xi = \lambda^{1/(2m)}\eta$, and taking the minimum with respect to $\alpha > 0$, proves the bound,

$$\begin{aligned}
N(\lambda; A_{F,\Omega,2m}(a,b,q)) &\le (2\pi)^{-n}\sup_{\xi\in\mathbb{R}^n}\|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)} \\
&\quad\times\min_{\alpha>0}\bigg(\alpha^{-1}\int_{\mathbb{R}^n}\big[\alpha+1-|\eta|^{2m}\big]_+d^n\eta\bigg)\lambda^{n/(2m)}, \quad \lambda>0.
\end{aligned} \tag{2.181}$$

Denoting

$$\mathcal{I}_F(\alpha) := \alpha^{-1} \int_{\mathbb{R}^n} \big[\alpha + 1 - |\eta|^{2m}\big]_+ d^n\eta, \tag{2.182}$$

one explicitly computes $\mathcal{I}_F(\alpha)$ and obtains

$$\mathcal{I}_F(\alpha) = \frac{2mv_n}{2m+n}\alpha^{-1}(\alpha+1)^{(2m+n)/(2m)}, \tag{2.183}$$

$$\mathcal{I}_F'(\alpha) = \frac{nv_n}{2m+n}(\alpha+1)^{n/(2m)}\alpha^{-2}\left(\alpha - \frac{2m}{n}\right), \tag{2.184}$$

$$\min_{\alpha>0}\big(\mathcal{I}_F(\alpha)\big) = \mathcal{I}_F(2m/n) = v_n\left(1+\frac{2m}{n}\right)^{n/(2m)}. \tag{2.185}$$

Equation (2.185) together with (2.181) yields (2.170). ∎

Remark 2.23. (i) One notes that whenever the property

$$\sup_{(x,\xi)\in\Omega\times\mathbb{R}^n} (|\phi(x,\xi)|) < \infty \tag{2.186}$$

has been established, then

$$\sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)} \leq |\Omega| \sup_{(x,\xi)\in\Omega\times\mathbb{R}^n} \big(|\phi(x,\xi)|^2\big), \tag{2.187}$$

explicitly exhibits the volume dependence on Ω of the right-hand sides of (2.153) and (2.170), respectively. We will briefly revisit this in Section 2.5.

(ii) Given two self-adjoint operators A, B in $\mathcal{H}$ bounded from below with purely discrete spectra such that $A \leq B$ in the sense of quadratic forms, then clearly $N(\lambda; B) \leq N(\lambda; A)$, $\lambda \in \mathbb{R}$; in addition, $N(\lambda; \alpha A) = N(\lambda/\alpha; A)$, $\alpha > 0$, $\lambda \in \mathbb{R}$. Thus, since a is real symmetric, the uniform ellipticity condition (2.80) implies $a \geq \varepsilon_a I_n$, and hence $A_{F,\Omega,2}(a,b,q) \geq \varepsilon_a A_{F,\Omega,2}(I_n,b,q)$ assuming $\varepsilon_a \in (0,1]$ without loss of generality. Combining this with (2.41) then yields

$$\begin{aligned} N(\lambda; A_{K,\Omega,2}(a,b,q)) \leq N(\lambda; A_{F,\Omega,2}(a,b,q)) &\leq N(\lambda; \varepsilon_a A_{F,\Omega,2}(I_n,b,q)) \\ &= N(\lambda/\varepsilon_a; A_{F,\Omega,2}(I_n,b,q)), \quad \lambda \in \mathbb{R}. \end{aligned} \tag{2.188}$$

Finally, we note that estimates of the type $N(\lambda; A) \leq c_A \lambda^\gamma$ for $A \geq 0$ yield lower bounds for the jth eigenvalue $\lambda_j(A)$ of the form $\lambda_j(A) \geq d_A j^{1/\gamma}$, clearly applicable in the context of (2.153) and (2.170). ⋄

Remark 2.24. As far as we know, employing the technique of the eigenfunction transform (i.e., the distorted Fourier transform) associated with the variable coefficient operator $\widetilde{A}_{2m}(a,b,q)$ (replacing the standard Fourier transform in connection with the constant coefficient case in [74]) to derive the results (2.12) and (2.13) is new.

On the other hand, the literature on eigenvalue counting function bounds in connection with arbitrary bounded open sets $\Omega \subset \mathbb{R}^n$ (or even open sets $\Omega \subset \mathbb{R}^n$ of finite Euclidean volume) is fairly extensive, originating with the seminal work by Birman–Solomyak, Rozenblum, and others. More specifically, starting around 1970, in this context of rough sets Ω, Birman and Solomyak pioneered the leading-order Weyl asymptotics and eigenvalue counting function estimates for generalized (linear pencil) eigenvalue problems of the form $Af = \lambda Bf$ for elliptic partial differential operators A of order n_A and lower-order differential operators B of order $n_B < n_A$ and obtained great generality of the coeffcients in A and B by systematically employing a variational formulation of this generalized eigenvalue problem. The boundary conditions employed are frequently of Dirichlet type, but Neumann and Robin boundary conditions are studied as well. In particular (focusing on the Dirichlet case only), the variational form of the problem associated with

$$\sum_{|\alpha|=|\beta|=m} D^\alpha\big(a_{\alpha,\beta}(x) D^\beta u\big)(x) = \lambda\, p(x)\, u(x), \quad u \in W_0^{m,2}(\Omega), \tag{2.189}$$

with special emphasis on the polyharmonic case, $(-\Delta)^m u = \lambda\, p\, u$, and extensions to the situation

$$\sum_{|\alpha|=|\beta|=m} D^{\alpha}\big(a_{\alpha,\beta}(x)D^{\beta}u\big)(x) = \lambda \sum_{0\leq|\gamma|,|\delta|\leq m} D^{\gamma}\big(b_{\gamma,\delta}(x)D^{\delta}u\big)(x), \quad |\gamma|+|\delta| = 2\ell,\, 0\leq \ell < m,\; u\in W_0^{m,2}(\Omega), \tag{2.190}$$

including the scenario where a, b are block matrices, or b is an appropriate (matrix-valued) measure, were studied in [18]–[24], [146]–[149], [150, Ch. 5]. In particular, the hypotheses on $a_{\alpha,\beta}$ are very general ($a \in L^1_{loc}(\Omega)^{m\times m}$, a positive definite a.e., $a^{-1} \in L^{\alpha}(\Omega)^{m\times m}$ for appropriate $\alpha \geq 1$) permitting a certain weak degeneracy of the ellipticity of the left-hand side in (2.189), (2.190). The case of the Friedrichs extension for $m = 1$ corresponding to $\tau_2(a, b, q)$ was treated in [127].

Thus, in the case $m = 1$, $p(\cdot) = 1$, and in some particular higher-order cases, where $m > 1$, in the context of $A_{F,\Omega,2m}(a, 0, 0)$ (i.e., $b = q = 0$), there is clearly some overlap of our result (2.170) with the above results concerning (2.189). The same applies to the magnetic field results in [127] in connection with $\tau_2(a, b, q)$. Similarly, considering the perturbed buckling problem in the form

$$(-\Delta)^{2m}u = \lambda\,(-\Delta)^m u, \quad u \in W_0^{2m,2}(\Omega), \tag{2.191}$$

there is of course some overlap between our result (2.153) (actually, the result in [74]) and the results concerning (2.190) with $m \in \mathbb{N}$, $a = I_n$, $b = q = 0$, but since lower-order terms are not explicitly included on the left-hand side of (2.190), a direct comparison is difficult. According to G. Rozenblum (private communication), the left-hand sides in (2.189), (2.190) can be extended to include also lower-order terms under appropriate hypotheses on the coefficients, but this seems not to have appeared explicitly in print.

Since we focused on the case of nonconstant coefficients throughout, we did not enter the vast literature on eigenvalue counting function estimates in connection with the Laplacian and its (fractional) powers. In this context we refer, for instance, to [68], [90], and the extensive literature cited therein. ◇

Although Weyl asymptotics itself is not the main objective of this thesis, we conclude this section with the following observation.

Remark 2.25. The Weyl asymptotics of $N(\,\cdot\,; A_{K,\Omega,2}(a,b,q))$ in [9, Sect. 8] in the case of quasi-convex domains and in [15] in the case of bounded Lipschitz domains derived an error bound of the form $O\big(\lambda^{(n-(1/2))/2}\big)$ as $\lambda \to \infty$. If one is only interested in the leading-order asymptotics results, combining the spectral equivalence of nonzero eigenvalues of $A_{K,\Omega,2m}(a,b,q))$ to the (generalized) buckling problem (cf. Lemma 2.5), with results by Kozlov [109]–[111], and taking into account that lower-order differential operator perturbations do not influence the leading-order asymptotics of $N(\,\cdot\,; A_{K,\Omega,2m}(a,b,q))$ (cf. [21, Lemmas 1.3, 1.4]) imply

$$\begin{aligned} &N(\lambda; A_{K,\Omega,2m}(a,b,q)) \\ &\quad \underset{\lambda\to\infty}{=} \frac{1}{n(2\pi)^n}\bigg(\int_\Omega d^n x \int_{|\xi|=1} d\omega_{n-1}(\xi)(\xi, a(x)\,\xi)_{\mathbb{R}^n}^{-\frac{n}{2}}\bigg)\,\lambda^{n/(2m)} + o\big(\lambda^{n/(2m)}\big) \\ &\quad \underset{\lambda\to\infty}{=} \frac{v_n}{(2\pi)^n}\bigg(\int_\Omega d^n x\,(\det a(x))^{-1/2}\bigg)\,\lambda^{n/(2m)} + o\big(\lambda^{n/(2m)}\big), \end{aligned} \tag{2.192}$$

for any bounded open set $\Omega \subset \mathbb{R}^n$. Here $d\omega_{n-1}$ denotes the surface measure on the unit sphere $S^{n-1} = \{\xi \in \mathbb{R}^n \,|\, |\xi| = 1\}$ in $\mathbb{R}^n$. Of course, the same leading-order asymptotics applies to $N(\,\cdot\,; A_{F,\Omega,2m}(a,b,q))$.

Since $N(\lambda; A) \underset{\lambda\to\infty}{=} c(A)\lambda^\alpha$ is equivalent to $\lambda_j(A) \underset{j\to\infty}{=} (j/c(A))^{1/\alpha}$, relation (2.192) yields the corresponding result for the eigenvalues of $A_{K,\Omega,2m}(a,b,q)$ and $A_{F,\Omega,2m}(a,b,q)$. ◇

2.5 Illustrations

To demonstrate why we expect Hypothesis 2.19 to hold under Hypothesis 2.11 alone in the case $a = I_n$ (with the obvious exception of zero-energy resonances and eigenvalues, which generically will be absent), we discuss three exceedingly complex scenarios in this section.

We start with the most elementary case which nevertheless served as the guiding motivation for this chapter:

Example 2.26. Let $a := I_n$, $n \in \mathbb{N}$, $b = q = 0$, then the operator $\mathbb{F}$ from Theorem 2.19 is the standard Fourier transform in $L^2(\mathbb{R}^n)$, and $\phi(\xi, x) = e^{i\xi\cdot x}$, $(\xi, x) \in \mathbb{R}^{2n}$. Thus, Hypothesis 2.19 obviously holds for $\widetilde{A}_2(I_n, 0, 0) = H_0$, and

$$\sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,, \xi)\|^2_{L^2(\Omega)} = |\Omega|. \tag{2.193}$$

In this rather special case the estimate for the eigenvalue counting function $N(\lambda; -\Delta_{K,\Omega})$ was previously obtained in [74], while that of $N(\lambda; -\Delta_{D,\Omega})$ was derived in [114].

Next, we turn to Schrödinger operators in $L^2(\mathbb{R}^n)$.

Example 2.27. Assume that $a = I_n$, $b = 0$, and $0 \le q \in L^\infty(\mathbb{R}^n)$, $\operatorname{supp}(q)$ compact. In addition, suppose that zero is neither an eigenvalue nor a resonance of $\widetilde{A}_2(I_n, b, q)$ (cf. [63]). Then Hypothesis 2.19 holds.

In addition, in the special case $n = 3$, there exists $C(q) \in (0, \infty)$ such that

$$\sup_{(x,\xi)\in\mathbb{R}^6} |\phi(x, \xi)| \le C(q). \tag{2.194}$$

Indeed, the absence of strictly positive eigenvalues of $\widetilde{A}_2(I_n, 0, q)$ was established by Kato [101] (see also [156]), and the existence of the distorted Fourier transform $\mathbb{F}$

and hence an eigenfunction transform was established by Ikebe [94, Theorem 5] for $n = 3$ and Thoe [163, Sect. 4] for $n \geq 4$, and Alsholm and Schmidt [4] for $n \geq 3$ (see also [140, Theorem XI.41], [141, Theorems XIII.33 and XIII.58], [144], [157, Sect. V.4]), implying, in particular, that

$$\begin{aligned} &\sigma\big(\widetilde{A}_2(I_n,0,q)\big) = \sigma_{ac}\big(\widetilde{A}_2(I_n,0,q)\big) = [0,\infty), \\ &\sigma_{sc}\big(\widetilde{A}_2(I_n,0,q)\big) = \sigma_p\big(\widetilde{A}_2(I_n,0,q)\big) \cap (0,\infty) = \emptyset. \end{aligned} \tag{2.195}$$

Moreover, it is shown in [94] and [163] that for all $R > 0$,

$$\sup_{\xi \in B_n(0;R),\, x \in \mathbb{R}^n} |\phi(x,\xi)| =: c(q,R) < \infty. \tag{2.196}$$

Thus we will focus on proving that

$$\sup_{\xi \in \mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|_{L^2(\Omega)} < \infty, \tag{2.197}$$

and in the special case $n = 3$ that for sufficiently large $R > 0$,

$$\sup_{\xi \in \mathbb{R}^3 \backslash B_3(0;R),\, x \in \mathbb{R}^3} |\phi(x,\xi)| =: C(q,R) < \infty. \tag{2.198}$$

Clearly, estimates (2.196) and (2.198) imply (2.194).

The distorted plane waves $\phi(\,\cdot\,,\,\cdot\,)$ can be chosen as one of $\phi_+(\,\cdot\,,\,\cdot\,)$ or $\phi_-(\,\cdot\,,\,\cdot\,)$, which are defined as solutions of the following Lippmann–Schwinger integral equation,

$$\phi_\pm(x,\xi) = e^{i\xi\cdot x} - \int_{\mathbb{R}^n} G_n\big(|\xi|^2 \pm i0; x, y\big) q(y) \phi_\pm(y,\xi)\, d^n y, \quad (x,\xi) \in \mathbb{R}^{2n}, \tag{2.199}$$

where

$$G_n(z;x,y) = \begin{cases} \frac{i}{4}\Big(\frac{2\pi|x-y|}{z^{1/2}}\Big)^{(2-n)/2} H^{(1)}_{(n-2)/2}\big(z^{1/2}|x-y|\big), & n \geqslant 2,\ z \in \mathbb{C}\backslash\{0\}, \\ \frac{-1}{2\pi}\ln(|x-y|), & n = 2,\ z = 0, \\ \frac{1}{(n-2)\omega_{n-1}}|x-y|^{2-n}, & n \geqslant 3,\ z = 0, \end{cases}$$
$$\operatorname{Im}\big(z^{1/2}\big) \geq 0,\ x, y \in \mathbb{R}^n,\ x \neq y, \tag{2.200}$$

represents the fundamental solution of the Helmholtz equation $(-\Delta - z)\psi(z;\,\cdot\,) = 0$ in $\mathbb{R}^n$, that is, the Green's function of the n-dimensional Laplacian, $n \in \mathbb{N}$, $n \geqslant 2$. Here $H_\nu^{(1)}(\,\cdot\,)$ denotes the Hankel function of the first kind with index $\nu \geq 0$ (cf. [2, Sect. 9.1]) and $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ ($\Gamma(\,\cdot\,)$ the Gamma function, cf. [2, Sect. 6.1]) represents the volume of the unit sphere S^{n-1} in $\mathbb{R}^n$. For simplicity we focus on $n \geq 3$ for the rest of this example, but note that the cases $n = 1, 2$ can be treated exactly along the same lines (see, e.g., the results in [25]–[47]).

Multiplying both sides of this equation by the weight $w > 0$ satisfying

$$\begin{aligned} & w \in C^\infty(\mathbb{R}^n), \quad 0 < w \leq 1, \quad w(x) := \begin{cases} 1, & 0 \leq |x| \leq R, \\ \exp(-|x|^2), & |x| \geq 2R, \end{cases} \\ & \Omega \subset B_n(0; R), \end{aligned} \tag{2.201}$$

for some $R > 0$, (2.199) can be written as follows

$$\begin{aligned} \Phi_\pm(x,\xi) = \Phi_0(x,\xi) - \int_{\mathbb{R}^n} w(x) G_n\big(|\xi|^2 \pm i0; x, y\big) w(y) \frac{q(y)}{w^2(y)} \Phi_\pm(y,\xi)\, d^3y, \\ (x,\xi) \in \mathbb{R}^{2n}, \end{aligned} \tag{2.202}$$

where

$$\Phi_\pm(x,\xi) := w(x)\phi_\pm(x,\xi), \quad \Phi_0(x,\xi) := w(x) e^{i\xi\cdot x}, \quad (x,\xi) \in \mathbb{R}^{2n}. \tag{2.203}$$

In this form (2.202) becomes an integral equation in $L^2(\mathbb{R}^n)$ since $\Phi_0(\,\cdot\,,\xi) \in L^2(\mathbb{R}^n)$. In fact, (2.202) will be viewed in $L^2(\mathbb{R}^n)$ as

$$\Phi_\pm(\,\cdot\,,\xi) = \Phi_0(\,\cdot\,,\xi) + K_\pm(\xi) M_{q/w^2} \Phi_\pm(\,\cdot\,,\xi), \quad \xi \in \mathbb{R}^n, \tag{2.204}$$

or equivalently, as

$$[I_{L^2(\mathbb{R}^n, d^n x)} - K_\pm(\xi) M_{q/w^2}] \Phi_\pm(\,\cdot\,,\xi) = \Phi_0(\,\cdot\,,\xi), \quad \xi \in \mathbb{R}^n, \tag{2.205}$$

where we introduced the Birman–Schwinger-type operator $K_\pm(\xi)$, $\xi \in \mathbb{R}^n$, in $L^2(\mathbb{R}^n)$,

$$K_\pm(\xi) \in \mathcal{B}\big(L^2(\mathbb{R}^n)\big),$$

$$(K_\pm(\xi)f)(x) := -\int_{\mathbb{R}^n} w(x) G_n\big(|\xi|^2 \pm i0; x, y\big) w(y) f(y,\xi)\, d^n y, \qquad (2.206)$$
$$f \in L^2(\mathbb{R}^n),\ (x,\xi) \in \mathbb{R}^{2n},$$

and the operator of multiplication by the function q/w^2, M_{q/w^2} in $L^2(\mathbb{R}^n)$,

$$M_{q/w^2} \in \mathcal{B}\big(L^2(\mathbb{R}^n)\big), \quad (M_{q/w^2} f)(x) := q(x) w(x)^{-2} f(x), \quad f \in L^2(\mathbb{R}^n),\ x \in \mathbb{R}^n. \qquad (2.207)$$

One recalls from [157, Sect. V.4] for $n = 3$ and [65] for $n \geq 3$ (the case $n = 2$ being analogous) that

$$\|K_\pm(\xi)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \underset{|\xi|\to\infty}{\longrightarrow} 0, \qquad (2.208)$$

and hence,

$$\begin{aligned}
&\|\Phi_\pm(\,\cdot\,,\xi) - \Phi_0(\,\cdot\,,\xi)\|_{L^2(\mathbb{R}^n)} \\
&\quad = \big\| \big(I_{L^2(\mathbb{R}^n)} - (I_{L^2(\mathbb{R}^n)} - K_\pm(\xi) M_{q/w^2}))^{-1}\big)\, \Phi_0(\,\cdot\,,\xi)\big\|_{L^2(\mathbb{R}^n)} \\
&\quad \leq \|w(\cdot)\|_{L^2(\mathbb{R}^n)} \big\| I_{L^2(\mathbb{R}^n)} - (I_{L^2(\mathbb{R}^n)} - K_\pm(\xi) M_{q/w^2}))^{-1}\big\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \\
&\quad \underset{|\xi|\to\infty}{=} o(1), \qquad (2.209)
\end{aligned}$$

implying,

$$\|\Phi_\pm(\,\cdot\,,\xi)\|_{L^2(\mathbb{R}^n)} \underset{|\xi|\to\infty}{=} O(1), \qquad (2.210)$$

and hence (2.197).

In the special case $n = 3$, where

$$G_3(z; x, y) = (4\pi|x-y|)^{-1} e^{iz^{1/2}|x-y|}, \quad \operatorname{Im}\big(z^{1/2}\big) \geq 0,\ x, y \in \mathbb{R}^3,\ x \neq y, \qquad (2.211)$$

one can easily go one step further: Using the Cauchy–Schwarz inequality, (2.210), and the fact that q has compact support, one estimates the second term in (2.202) as follows,

$$
\begin{aligned}
&\bigg| \int_{\mathbb{R}^3} \frac{w(x)e^{\pm i|\xi||x-y|}w(y)}{4\pi|x-y|} \frac{q(y)}{w^2(y)} \Phi_\pm(y,\xi)\, d^3y \bigg| \\
&\quad \leq (4\pi)^{-1} w(x) \int_{\operatorname{supp}(q)} \frac{w(y)}{|x-y|} \frac{q(y)}{w^2(y)} |\Phi_\pm(y,\xi)|\, d^3y \\
&\quad \leq (4\pi)^{-1} w(x) \|qw^{-2}\|_{L^\infty(\mathbb{R}^3)} \bigg(\int_{\operatorname{supp}(q)} \frac{w^2(y)}{|x-y|^2}\, d^ny \bigg)^{1/2} \|\Phi_\pm(\,\cdot\,,\xi)\|_{L^2(\mathbb{R}^3)} \\
&\quad \underset{|\xi|\to\infty}{=} w(x)O(1), \quad x \in \mathbb{R}^3,
\end{aligned} \tag{2.212}
$$

with the $O(1)$-term bounded uniformly in $(x,\xi) \in \mathbb{R}^6$. Combining (2.203), (2.204), and (2.212) one obtains

$$
\sup_{x\in\mathbb{R}^3} |\phi_\pm(x,\xi)| \underset{|\xi|\to\infty}{=} O(1), \tag{2.213}
$$

proving (2.196) since $\phi_\pm$ is continuous on $\mathbb{R}^6$ (see, e.g., [94, Sect. 4], [163, Sect. 3]). $\square$

Example 2.28. Assume that $n \in \mathbb{N}$, $a = I_n$, $b \in \big[W^{1,\infty}(\mathbb{R}^n)\big]^n$, $\operatorname{supp}(b)$ compact, $0 \leq q \in L^\infty(\mathbb{R}^n)$, $\operatorname{supp}(q)$ compact. In addition, suppose that zero is neither an eigenvalue nor a resonance of $\widetilde{A}_2(I_n,b,q)$ (cf. [63]). Then Hypothesis 2.19 holds.

We start verifying this claim by noting that under these assumptions on a, b, q, $\widetilde{A}_2(I_n,b,q)$ has empty singular continuous spectrum and no strictly positive eigenvalues, see, for instance, Erdogan, Goldberg, and Schlag [62], [63], Ikebe and Saitō [95], (see also, [7], [16], [64], [71], [107], [159]); in particular, the analog of (2.195) holds for $\widetilde{A}_2(I_n,b,q)$.

Next, we recall the unperturbed operator $H_0 := -\Delta$, $\operatorname{dom}(H_0) = W^{2,2}(\mathbb{R}^n)$, and introduce the first-order perturbation term,

$$
L_1 f = 2i \sum_{k=1}^n b_k \partial_k f + (i \operatorname{div}(b) + |b|^2 + q) f, \quad f \in \operatorname{dom}(L_1) = W^{1,2}(\mathbb{R}^n). \tag{2.214}
$$

We denote the distorted plane waves associated with $\widetilde{A}_2(I_n, b, q)$ by $\phi(\,\cdot\,,\,\cdot\,)$, and abbreviate

$$\phi_0(x,\xi) := e^{i\xi\cdot x}, \quad (x,\xi) \in \mathbb{R}^{2n}. \tag{2.215}$$

In the following we will show that

$$\sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|_{L^2(\Omega)} < \infty. \tag{2.216}$$

To this end, we employ [63, Theorem 1.2] (see also [62, Theorem 2]) with $\alpha = 0$, $\sigma = 1$ and infer

$$K := \sup_{|\xi|\geq 0} \Big(\langle|\xi|\rangle \big\|\langle\,\cdot\,\rangle^{-2}\big(\widetilde{A}_2(I_n,b,q) - (|\xi|^2 \pm i0)\big)^{-1}\langle\,\cdot\,\rangle^{-2}\big\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\Big) < \infty, \tag{2.217}$$

abbreviating $\langle\,\cdot\,\rangle := \big[1 + (\,\cdot\,)^2\big]^{1/2}$.

The distorted plane wave $\phi(\,\cdot\,,\,\cdot\,)$ can again be chosen as one of $\phi_+(\,\cdot\,,\,\cdot\,)$ or $\phi_-(\,\cdot\,,\,\cdot\,)$ and be decomposed in the form

$$\phi_\pm(x,\xi) = \phi_0(x,\xi) + \psi_\pm(x,\xi), \quad (x,\xi) \in \mathbb{R}^{2n}, \tag{2.218}$$

where

$$\psi_\pm(x,\xi) := -\Big(\big(\widetilde{A}_2(I_n,b,q) - (|\xi|^2 \pm i0)\big)^{-1}(L_1\phi_0)\Big)(x,\xi), \quad (x,\xi) \in \mathbb{R}^{2n}. \tag{2.219}$$

(In this context we recall that

$$\begin{aligned} |\xi|^2\phi_\pm(x,\xi) &= \big(\widetilde{A}_2(I_n,b,q)\phi_\pm\big)(x,\xi) \\ &= |\xi|^2\phi_0(x,\xi) + (L_1\phi_0)(x,\xi) + \big(\widetilde{A}_2(I_n,b,q)\psi_\pm\big)(x,\xi), \end{aligned} \tag{2.220}$$

or equivalently,

$$-(L_1\phi_0)(x,\xi) = \Big(\big(\widetilde{A}_2(I_n,b,q)\big) - |\xi|^2\big)\psi_\pm\Big)(x,\xi), \tag{2.221}$$

in the sense of distributions, illustrating (2.219).)

One then infers

$$
\begin{aligned}
&\|\psi_\pm(\,\cdot\,,\xi)\|_{L^2(\Omega)} \\
&\quad = \big\|\chi_\Omega\langle\,\cdot\,\rangle^2\langle\,\cdot\,\rangle^{-2}\big(\widetilde{A}_2(I_n,b,q)-(|\xi|^2\pm i0)\big)^{-1}\langle\,\cdot\,\rangle^{-2}\langle\,\cdot\,\rangle^2(L_1\phi_0)\big\|_{L^2(\mathbb{R}^n)} \\
&\quad \leq \big\|\chi_\Omega\langle\,\cdot\,\rangle^2\big\|_{L^\infty(\mathbb{R}^n)}\,\big\|\langle\,\cdot\,\rangle^{-2}\big(\widetilde{A}_2(I_n,b,q)-(|\xi|^2\pm i0)\big)^{-1}\langle\,\cdot\,\rangle^{-2}\big\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \\
&\qquad \times\big\|\langle\,\cdot\,\rangle^2(L_1\phi_0)\big\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{2.222}
$$

Employing (2.217), the fact that Ω is bounded, and that the coefficients of L_1 have compact support (cf. (2.214)), one concludes

$$
\begin{aligned}
&\big\|\langle\,\cdot\,\rangle^{-2}\big(\widetilde{A}_2(I_n,b,q)-(|\xi|^2\pm i0)\big)^{-1}\langle\,\cdot\,\rangle^{-2}\big\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\big\|\langle\,\cdot\,\rangle^2(L_1\phi_0)\big\|_{L^2(\mathbb{R}^n)} \\
&\quad \leq K\langle\,|\xi|\,\rangle^{-1}\big\|\langle\,\cdot\,\rangle^2\phi_0(\,\cdot\,,\xi)\big(-2b\cdot\xi+i\,\mathrm{div}(b)+|b|^2+q\big)\big\|_{L^2(\mathbb{R}^n)} \\
&\quad \leq 2K|\xi|\langle\,|\xi|\,\rangle^{-1}\big\|\langle\,\cdot\,\rangle^2 b\big\|_{[L^2(\mathbb{R}^n)]^n}+K\langle\,|\xi|\,\rangle^{-1}\big\|\langle\,\cdot\,\rangle^2\left(i\,\mathrm{div}(b)+|b|^2+q\right)\big\|_{L^2(\mathbb{R}^n)} \\
&\quad \underset{|\xi|\to\infty}{=} O(1).
\end{aligned} \tag{2.223}
$$

Combining (2.222) and (2.223) one obtains the required estimate (2.216). $\square$

Chapter 3

The Maslov index and the spectra of second order elliptic operators

3.1 Introduction

This chapter intertwines three major themes: (1) Relations between the spectral flow for a family of linear elliptic differential operators and the Maslov index of a path of Lagrangian planes formed by the abstract traces of solutions of respective homogeneous partial differential equations; (2) Relations between the Morse index and the Maslov index in the context of Lagrangian planes given by standard PDE traces of weak solutions of the homogeneous equations; (3) Relations between the self-adjoint extensions of abstract symmetric operators and the Lagrangian planes defined by means of boundary triples.

The first topic is motivated by the celebrated Atiyah–Patodi–Singer index theorem [12, 13], and goes back to the classical works [42, 133, 143]. In particular, great progress has been recently made in calculations of the spectral flow via the Maslov index, [29, 34, 35, 36, 106, 70, 153]. Here, the spectral flow is the net count of the eigenvalues of a family of self-adjoint differential operators that move through a given value of the spectral parameter, and the Maslov index is a topological invariant that counts the signed number of intersections of paths in the space of Lagrangian planes [5, 6, 83, 126]. The second topic has its roots in the classical Morse–Smale-type theorems, see [1, 28, 53, 48, 59, 131]. It is of great interest in stability theory for multidimensional patterns for reaction-diffusion equations, see [104, 57]. In recent years the relation between the Morse index (the number of unstable eigenvalues) and the Maslov index has attracted much attention, see [50, 51, 52, 57, 91, 92, 93, 97, 98, 116, 136]. These results can be viewed as a far reaching generalization of the classical Sturm Theorems for ODE's and systems of ODE's, cf. [28, 5, 6], Courant's nodal domain theorem [49], and more recent results in [69]. The third topic is originated in the Birman–Krein–Vishik theory of self-adjoint extensions of symmetric operators, see a modern exposition in [3, 86], and also in the theory of the abstract boundary triples, see [82, 108]. A critical series of results in this direction is that the self-adjoint extensions of a symmetric operator can be parametrized by Lagrangian planes in some abstract boundary spaces. We note that although the Lagrangian language is not used in [30, 38, 31, 82] one can easily see an equivalent Lagrangian reformulation of the results contained therein [135]. Summarizing, one can say that the connection between self-adjoint extensions and Lagrangian planes resulted in various formulas relating the spectral flow and the Maslov index of paths of Lagrangian planes formed by *strong traces* of solutions to elliptic PDE's, see [34, 35, 36]. In contrast, the Lagrangian planes considered in [57, 50, 51, 52] are formed by the *weak* traces of weak solutions to second order elliptic PDE's. Our main contribution is in tying together all three topics discussed above.

Our work is motivated by that of J. Deng and C. K. R. T. Jones [57] who proposed to compute the Morse index of the Schrödinger operator $L = -\Delta + V$ on a star-shaped domain Ω by scaling it, and counting negative eigenvalues of L via the conjugate points defined by intersections of certain paths of Lagrangian planes in the PDE boundary space $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. The paths are formed by the boundary data of the weak solutions to the equation $Lu - \lambda u = 0, \lambda \in \mathbb{R},\ u \in H^1(\Omega)$, and by the subspaces of $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ corresponding to the boundary conditions. This approach leads to a natural generalization of the Smale Theorem for Schrödinger operators with Robin-type boundary conditions, cf. [50]. Further advances of this idea appeared in subsequent works: Significantly more general domain variations are considered in [51, 52], the Schrödinger operators with non-separated boundary conditions are considered in [97, 98, 116, 91], the one dimensional Schrödinger operators defined on $\mathbb{R}$ are considered in [92, 93].

While all these papers deal with specific boundary value problems, most of them may be viewed through the prism of abstract theory of self-adjoint extensions of symmetric operators. In a different context, the work in the latter direction was initiated in the foundational paper [29], where the following setup was used: Let $S \subset S^*$ be a symmetric operator acting in a Hilbert space $\mathcal{H}$, and $\{V_t\}_{t=\alpha}^{\beta}$ be a continuous family of bounded self-adjoint operators acting in $\mathcal{H}$. Let us suppose that $S_{\mathscr{D}}$, $\operatorname{dom}(\mathcal{S}_{\mathscr{D}}) = \mathscr{D}$, is a self-adjoint extension of S having compact resolvent. Then $\Upsilon_t := \ker(S^* + V_t)/\operatorname{dom}(S),\ t \in [\alpha, \beta]$, and $\mathscr{D}/\operatorname{dom}(S)$ are Lagrangian planes in the quotient space $\mathcal{H}_S := \operatorname{dom}(S^*)/\operatorname{dom}(S)$ with respect to the natural symplectic form

$$\omega([x],[y]) = \langle S^*x, y\rangle_{\mathcal{H}} - \langle x, S^*y\rangle_{\mathcal{H}},\ [x],[y] \in \operatorname{dom}(S^*)/\operatorname{dom}(S). \tag{3.1}$$

It is shown in [29] that the spectral flow of $\{S_{\mathscr{D}} + V_t\}_{t=\alpha}^{\beta}$ is equal to the Maslov index of the path $\Upsilon_t, t \in [\alpha, \beta]$, with respect to the reference plane $\mathscr{D}/\operatorname{dom}(S)$. We notice that the operator S gives rise not only to the one-parameter family $\{S_{\mathscr{D}} + V_t\}_{t=\alpha}^{\beta}$ but also to the symplectic Hilbert space $\mathcal{H}_S$ itself. Therefore, the scheme is not suited for a parameter dependent family $\{S_t\}_{t=\alpha}^{\beta}$ in place of a single operator S. However, the subsequent manuscripts [34], [35], [36] suggest an elegant way out of this issue. Let us consider a family $\{S_t\}_{t=\alpha}^{\beta}$ of symmetric operators with a fixed domain, and fix an intermediate space $D_M \subset \mathcal{H}$ such that

$$\operatorname{dom}(S_t) \equiv \operatorname{dom}(S_\alpha) \subset D_M \subset \operatorname{dom}(S_t^*) \subset \mathcal{H},\ t \in [\alpha, \beta]. \tag{3.2}$$

We will now consider only those self-adjoint extensions of S_t whose domains are subsets of the *fixed* subspace D_M. Under these assumptions [36] proves the equality of the spectral flow for the family of self-adjoint extensions of S_t, and the Maslov index defined by means of the quotient space $D_M/\operatorname{dom}(S_\alpha)$.

The techniques developed in [36] cover elliptic operators of order $d \in \mathbb{N}$ with D_M being equal to the Sobolev space of degree d. In particular, letting $d = 2$ we consider now a family $\{S_t\}_{t=\alpha}^{\beta}$ of second order uniformly elliptic operators on a smooth domain $\Omega \subset \mathbb{R}^n$. Then [36] yields the equality between the spectral flow of the self-adjoint extensions of S_t with domains containing in $D_M = H^2(\Omega)$ and the Maslov index of the corresponding paths of Lagrangian planes.

The purpose of our work in this chapter is threefold. First, we will reduce the regularity assumption and consider the self-adjoint extensions of second order elliptic operators S_t with domains containing in $H^1(\Omega)$. To illuminate the importance of this improvement we recall that many differential operators of interest in mathematical physics, spectral geometry, and partial differential equations are defined via first order sesquilinear forms with the help of Lax–Milgram Theorem. This procedure *a priori* leads to self-adjoint operators with domains contained in $H^1(\Omega)$. The higher $H^2(\Omega)-$regularity is a subtle issue and depends not only on coefficients of the differential operators but also on geometric characteristics of $\partial\Omega$. Thus the assumption that the domains of self-adjoint extensions of S_t belongs to $H^1(\Omega)$ is quite natural. The main technical obstacle preventing from passing from H^2- to H^1- regularity is that the natural candidate for D_M from (3.2) is given by the subspace

$$\{u \in H^1(\Omega) : S_t u \in L^2(\Omega)\}, \tag{3.3}$$

and thus varies together with parameter t (if $H^1(\Omega)$ here is replaced by $H^2(\Omega)$ then $S_t u \in L^2(\Omega)$ holds automatically). To overcome this difficulty we map the family of subspaces (3.3) into a *fixed* Hilbert space $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ using the usual PDE trace map consisting of the Dirichlet and Neumann trace operators.

This brings us to the second goal of this chapter. We will replace the quotient space $H^1(\Omega)/H^2_0(\Omega)$ used in [34, 35, 36] by the more conventional PDE boundary space $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ of distributions on the boundary. We stress that the two-component trace map consisting of the Dirichlet and Neumann trace operators is not onto when considered as a map from $H^1(\Omega)$ into $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, cf. Proposition 3.11. Moreover, the quotient space $H^1(\Omega)/H^2_0(\Omega)$ is not symplectomorphic to the boundary space $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. Nevertheless, we show how one can bypass the quotient spaces and instead work directly in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. And, finally, our third goal is to analyze the variation of spectra of differential operators with respect to geometric deformations of the domain Ω.

Employing the approach outlined above we derive the spectral count formulas in a very general setting. In particular, it at once covers the following known cases: The Schrödinger operators with non-local Robin-type boundary conditions on star-shaped domains $\Omega \subset \mathbb{R}^n, n \geq 2$, as in [50], the Schrödinger operators with $\vec{\theta}-$periodic boundary conditions on the unit cell Q, as in [116], the second order elliptic operators with Dirichlet and Neumann boundary conditions defined by means of a one-parameter family of diffeomorophic domains $\{\Omega_t\}_{t=\alpha}^{\beta}$, as in [51, 52].

The chapter is organized as follows. Section 3.2 provides the one-to-one correspondence between the self-adjoint extensions of $\mathcal{L}_{min}$, with the domains contained in $H^1(\Omega)$, and Lagrangian planes in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. In Section 3.3 we derive a general formula relating the Maslov and Morse indices for the second order elliptic operators subject to self-adjoint boundary conditions on smooth domains. The applications of the general result are illustrated for three topics: the spectral flow formula, the spectral count in the context of geometric deformations, and the Smale-type theorem for Robin boundary conditions. In Section 3.4 we deal with the Maslov–Morse type formulas for the Schrödinger operators with matrix-valued potentials subject to self-adjoint boundary conditions on Lipschitz domains. In particular, we consider $\vec{\theta}-$periodic and non-local Robin-type boundary conditions. Finally, in Section 3.5 we discuss the abstract boundary triples [30, 31, 38, 82] in the context of the quotient spaces introduced in [29].

To conclude, we summarize the notation used in this chapter. The scalar product in a complex Hilbert space $\mathcal{H}$ is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. $\mathrm{Spec}(S)$, $\mathrm{Spec}_{ess}(S)$, $\mathrm{Spec}_d(S)$ denote the spectrum, the essential spectrum, the discrete spectrum of a closed operator S correspondingly. The number of negative eigenvalues of S is denoted by $\mathrm{Mor}(S)$. If

$\mathcal{G} \subset \mathcal{H}$ then $\overline{\mathcal{G}}^{\mathcal{H}}$ denotes the closure of $\mathcal{G}$ with respect to the norm of $\mathcal{H}$. The range of an operator $\mathcal{S}$ acting from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ is denoted by $\mathrm{ran}(\mathcal{S}) \subset \mathcal{Y}$, the kernel of $\mathcal{S}$ is denoted by $\ker(\mathcal{S}) \subset \mathcal{X}$. The space of bounded operators acting from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$, the space of compact operators is denoted by $\mathcal{B}_\infty(\mathcal{X}, \mathcal{Y})$. If $\Omega \subset \mathbb{R}^n$, then $\mathcal{D}(\Omega)$ denotes the space of test functions $C_0^\infty(\Omega)$ equipped with the standard inductive limit topology, $\mathcal{D}'(\Omega)$ denotes the dual space, the paring between $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ is denoted by ${}_{\mathcal{D}(\Omega)}\langle \cdot, \cdot \rangle_{\mathcal{D}'(\Omega)}$. Duality pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ is denoted by $\langle \cdot, \cdot \rangle_{-1/2}$.

3.2 Self-adjoint extensions and Lagrangian planes

In this section we focus on the one-to-one correspondence between self-adjoint extensions of second order elliptic operators on bounded domains in $\mathbb{R}^n$ and Lagrangian subspaces in $H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$.

3.2.1 Assumptions

In this subsection we state our main assumptions and recall some known facts regarding partial differential operators on Lipschitz domains.

Hypothesis 3.1. *Let $n \in \mathbb{N}, n \geq 2$ and assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain.*

To set the stage, we introduce a formally self-adjoint differential expression

$$\mathcal{L} := -\sum_{j,k=1}^{n} \partial_j A_{jk} \partial_k + \sum_{j=1}^{n} A_j \partial_j - \partial_j \overline{A_j}^{\top} + A, \tag{3.4}$$

where bar means complex conjugation, $\top$ means matrix transposition, and the coefficients satisfy the following standard assumptions, see, e.g., [125, Chapter 4].

Hypothesis 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and assume that*

$$A_{jk} = \left\{a_{pq}^{jk}\right\}_{p,q=1}^{m} \in L^\infty(\overline{\Omega}, \mathbb{C}^{m\times m}),\ A_{jk} = \overline{A_{jk}}^\top,\ 1 \le j,k \le n,$$

$$A_j = \left\{a_{pq}^{j}\right\}_{p,q=1}^{m} \in L^\infty(\overline{\Omega}, \mathbb{C}^{m\times m}), 1 \le j \le n,$$

$$A_{jk},\ A_j \text{ are Lipschitz functions on } \overline{\Omega},\ 1 \le j,k \le n,$$

$$A = \left\{a_{pq}\right\}_{p,q=1}^{m} \in L^\infty(\overline{\Omega}, \mathbb{C}^{m\times m}),\ A = \overline{A}^\top.$$

The differential expression $\mathcal{L}$ acts on a vector-valued function $u \in C^\infty(\Omega, \mathbb{C}^m)$ as follows

$$\big((\mathcal{L}u)(x)\big)_p = -\sum_{j,k=1}^{n}\sum_{q=1}^{m} \partial_j\{a_{pq}^{jk}(x)\partial_k u_q(x)\} + \sum_{j=1}^{n}\sum_{q=1}^{m} a_{pq}^{j}(x)\partial_j u_q(x) \tag{3.5}$$

$$-\sum_{q=1}^{m} \partial_j\{\overline{a}_{qp}^{j}(x)u_q(x)\} + \sum_{q=1}^{m} a_{pq}(x)u_q(x),\ \text{a.e. } x \in \Omega,\ 1 \le p \le m, \tag{3.6}$$

where $(v)_p$ denotes the $p-$th coordinate of a vector $v \in \mathbb{C}^m$. The sesquilinear form associated with $\mathcal{L}$ is given by

$$\begin{aligned}\mathfrak{l}[u,v] = &\sum_{j,k=1}^{n} \langle A_{jk}\partial_k u, \partial_j v\rangle_{L^2(\Omega,\mathbb{C}^m)} + \sum_{j=1}^{n}\langle A_j\partial_j u, v\rangle_{L^2(\Omega,\mathbb{C}^m)}\\ &+ \sum_{j=1}^{n}\langle u, A_j\partial_j v\rangle_{L^2(\Omega,\mathbb{C}^m)} + \langle Au, v\rangle_{L^2(\Omega,\mathbb{C}^m)},\ \ u,v \in H^1(\Omega,\mathbb{C}^m).\end{aligned} \tag{3.7}$$

We seek to establish a one-to-one correspondence between self-adjoint extensions of $\mathcal{L} : C_0^\infty(\Omega, \mathbb{C}^m) \subset L^2(\Omega, \mathbb{C}^m) \to L^2(\Omega, \mathbb{C}^m)$ and Lagrangian planes in $H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$ employing the second Green identity. To this end, we recall that by the standard trace theorem (cf., e.g., [125, Theorem 3.38]) the linear mapping

$$\gamma_D^0 : C(\overline{\Omega}, \mathbb{C}^m) \to C(\partial\Omega, \mathbb{C}^m),\quad \gamma_D^0 u = u_{|\partial\Omega}, \tag{3.8}$$

can be extended by continuity and considered as a linear bounded operator,

$$\gamma_D \in \mathcal{B}(H^s(\Omega, \mathbb{C}^m), H^{(s-\frac{1}{2})}(\partial\Omega, \mathbb{C}^m)),\ 1/2 < s < 3/2; \tag{3.9}$$

in addition (cf. [78, Lemma A.4]),

$$\gamma_{D} \in \mathcal{B}(H^{(3/2)+\varepsilon}(\Omega, \mathbb{C}^m), H^1(\partial\Omega, \mathbb{C}^m)),\ \varepsilon > 0. \tag{3.10}$$

The conormal derivative corresponding to the differential expression $\mathcal{L}$ is given by

$$\gamma_{N}^{\mathcal{L},2} u := \sum_{j,k=1}^{n} A^{jk}\nu_j \gamma_{D}(\partial_k u) + \sum_{j=1}^{n} \overline{A_j}^{\top} \nu_j \gamma_{D} u,\ \ u \in H^2(\Omega, \mathbb{C}^m), \tag{3.11}$$

with $\nu = (\nu_1, \cdots, \nu_n)$ denoting the outward unit normal on $\partial\Omega$. Setting $\varepsilon = 1/2$ in (3.10) and using (3.11), we introduce the trace map

$$\mathrm{tr}_{\mathcal{L},2} : \begin{cases} H^2(\Omega, \mathbb{C}^m) \to H^1(\partial\Omega, \mathbb{C}^m) \times L^2(\partial\Omega, \mathbb{C}^m), \\ \qquad u \mapsto (\gamma_{D} u, \gamma_{N}^{\mathcal{L},2}). \end{cases} \tag{3.12}$$

Also, we introduce the function space

$$D^s_{\mathcal{L}}(\Omega) := \{u \in H^s(\Omega, \mathbb{C}^m) : \mathcal{L}u \in L^2(\Omega, \mathbb{C}^m)\},\ \ s \geq 0, \tag{3.13}$$

equipped with the graph norm of $\mathcal{L}$,

$$\|u\|_{\mathcal{L},s} := \Big(\|u\|^2_{H^s(\Omega,\mathbb{C}^m)} + \|\mathcal{L}u\|^2_{L^2(\Omega,\mathbb{C}^m)}\Big)^{1/2}, \tag{3.14}$$

where $\mathcal{L}u$ should be understood in the sense of distributions. Next, we recall the extension of $\mathrm{tr}_{\mathcal{L},2}$ defined on $D^1_{\mathcal{L}}(\Omega)$ and the first and second Green identities.

Proposition 3.3. *[125, Lemma 4.3] Assume Hypothesis* 3.1*. Then the operator*

$$\gamma_{N}^{\mathcal{L},2} : H^2(\Omega, \mathbb{C}^m) \to L^2(\partial\Omega, \mathbb{C}^m),\ \ u \in H^2(\Omega, \mathbb{C}^m), \tag{3.15}$$

can be extended to a bounded, linear operator $\gamma_N^{\mathcal{L}} \in \mathcal{B}\big(\mathcal{D}^1_{\mathcal{L}}(\Omega), H^{-1/2}(\partial\Omega, \mathbb{C}^m)\big)$*. Moreover, the first Green identity holds, that is,*

$$\mathfrak{l}[u, v] = \langle \mathcal{L}u, v\rangle_{L^2(\Omega,\mathbb{C}^m)} + \langle \gamma_{N}^{\mathcal{L}} u, \gamma_{D} v\rangle_{-1/2}, \tag{3.16}$$

for all $u \in \mathcal{D}^1_{\mathcal{L}}(\Omega), v \in H^1(\Omega, \mathbb{C}^m)$*.*

Proposition 3.4. *[125, Theorem 4.4 (iii)] Assume Hypothesis* 3.1. *Then the second Green identity holds, that is,*

$$\langle \mathcal{L}u, v\rangle_{L^2(\Omega,\mathbb{C}^m)} - \langle u, \mathcal{L}v\rangle_{L^2(\Omega,\mathbb{C}^m)} = \overline{\langle \gamma_{N}^{\mathcal{L}} v, \gamma_{D} u\rangle}_{-1/2} - \langle \gamma_{N}^{\mathcal{L}} u, \gamma_{D} v\rangle_{-1/2}, \tag{3.17}$$

for all $u, v \in \mathcal{D}^1_{\mathcal{L}}(\Omega)$.

The trace operator

$$\mathrm{tr}_{\mathcal{L}} \in \mathcal{B}\big(\mathcal{D}^1_{\mathcal{L}}(\Omega), H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)\big),\ \mathrm{tr}_{\mathcal{L}} : u \mapsto (\gamma_{D} u, \gamma_{N}^{\mathcal{L}} u), \tag{3.18}$$

is compatible with (3.12). We will frequently use the fact that

$$\ker\{\mathrm{tr}_{\mathcal{L}}\} \cap \{u \in H^1(\Omega) : \mathcal{L}u = 0 \text{ in } (H^1_0(\Omega))^*\} = \{0\}, \tag{3.19}$$

which follows from the unique continuation principle, [96, Theorem 3.2.2]. Next we turn to the symmetric operator acting in $L^2(\Omega, \mathbb{C}^m)$ and associated with differential expression (3.4).

Proposition 3.5. *Assume Hypotheses* 3.1 *and* 3.2. *Then the linear operator defined by*

$$Lf := \mathcal{L}f,\ f \in \mathrm{dom}(L) := C_0^\infty(\Omega), \tag{3.20}$$

and considered in $L^2(\Omega, \mathbb{C}^m)$ *is closable. Its closure* $\mathcal{L}_{min}$ *is densely defined symmetric operator in* $L^2(\Omega, \mathbb{C}^m)$. *Moreover, the linear operator acting in* $L^2(\Omega, \mathbb{C}^m)$ *and given by*

$$\mathcal{L}_{max} u := \mathcal{L}u,\ u \in \mathrm{dom}(\mathcal{L}_{max}) := \{u \in L^2(\Omega, \mathbb{C}^m) : \mathcal{L}u \in L^2(\Omega, \mathbb{C}^m)\}, \tag{3.21}$$

(*where* $\mathcal{L}u$ *is defined is the sense of distributions*) *is adjoint to* $\mathcal{L}_{min}$, *i.e.,*

$$(\mathcal{L}_{min})^* = \mathcal{L}_{max}. \tag{3.22}$$

Proof. Using the second Green identity (3.17) with arbitrary $u, v \in C_0^\infty(\Omega)$ and noticing that $\mathrm{tr}_{\mathcal{L}}\, u = \mathrm{tr}_{\mathcal{L}}\, v = 0$ we arrive at

$$\langle \mathcal{L}u, v\rangle_{L^2(\Omega,\mathbb{C}^m)} = \langle u, \mathcal{L}v\rangle_{L^2(\Omega,\mathbb{C}^m)}, \text{ for all } u, v \in \mathrm{dom}(L). \tag{3.23}$$

Hence, $L \subset L^*$ that is L is symmetric in $L^2(\Omega, \mathbb{C}^m)$, consequently it is closable.

Next, we turn to (3.22). Let us show $(\mathcal{L}_{min})^* \subset \mathcal{L}_{max}$. Pick any $f \in \mathrm{dom}\,((\mathcal{L}_{min})^*)$, then $g = (\mathcal{L}_{min})^* f \in L^2(\Omega, \mathbb{C}^m)$, and for arbitrary $\psi \in C_0^\infty(\Omega) \subset \mathrm{dom}\,(\mathcal{L}_{min})$ one has

$$_{\mathcal{D}(\Omega)}\langle \psi, \mathcal{L}f\rangle_{\mathcal{D}'(\Omega)} = \langle \mathcal{L}\psi, f\rangle_{L^2(\Omega,\mathbb{C}^m)} = \langle \psi, g\rangle_{L^2(\Omega,\mathbb{C}^m)}. \tag{3.24}$$

Therefore, $g = \mathcal{L}f$ in distributional sense and $\mathcal{L}f \in L^2(\Omega, \mathbb{C}^m)$ as required. In order to show the opposite inclusion we notice that

$$\langle \mathcal{L}\varphi, g\rangle_{L^2(\Omega,\mathbb{C}^m)} = \langle \varphi, \mathcal{L}g\rangle_{L^2(\Omega,\mathbb{C}^m)}, \text{ for all } \varphi \in C_0^\infty(\Omega), g \in \mathrm{dom}(\mathcal{L}_{max}). \tag{3.25}$$

Whenever $f \in \mathrm{dom}(\mathcal{L}_{min})$, there exists a sequence $\{\varphi_\ell, \ell \geq 1\} \subset C_0^\infty(\Omega) = \mathrm{dom}(L)$, such that

$$\lim_{\ell\to\infty} \varphi_\ell = f \;\text{ and }\; \lim_{\ell\to\infty} \mathcal{L}\varphi_\ell = \mathcal{L}f \;\;(\text{in } L^2(\Omega, \mathbb{C}^m)). \tag{3.26}$$

Plugging φ_ℓ in (3.25) and passing to limit as $\ell \to \infty$, one obtains

$$\langle \mathcal{L}f, g\rangle_{L^2(\Omega,\mathbb{C}^m)} = \langle f, \mathcal{L}g\rangle_{L^2(\Omega,\mathbb{C}^m)}, \text{ for all } f \in \mathrm{dom}(\mathcal{L}_{min}),\ g \in \mathrm{dom}(\mathcal{L}_{max}). \tag{3.27}$$

Thus, $\mathcal{L}_{max} \subset (\mathcal{L}_{min})^*$, and the proof is completed. ∎

Hypothesis 3.6. *Assume Hypotheses* 3.1 *and* 3.2. *Suppose that the deficiency indices of $\mathcal{L}_{min}$ are equal, that is,*

$$\dim\ker(\mathcal{L}_{max} - \mathbf{i}) = \dim\ker(\mathcal{L}_{max} + \mathbf{i}),$$

where both indices could be infinite. In addition:

(i) assume that $\operatorname{ran}(\operatorname{tr}_{\mathcal{L}})$ *is dense in* $H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$*,*

(ii) assume that $\mathcal{D}^1_{\mathcal{L}}(\Omega)$ *is dense in* $\mathcal{D}^0_{\mathcal{L}}(\Omega)$*.*

Remark 3.7. In the sequel we consider two special cases, see Section 3.3 and 3.4 respectively. In the first case, the coefficients A_{jk}, A_j, A of the uniformly elliptic operator $\mathcal{L}$ from (3.4) are scalar-valued and defined on domains with smooth boundary, cf. Hypothesis 3.16 below. In this scenario both parts of Hypothesis 3.6 are satisfied. Indeed, by [84, Proposition 2.1], [31, Section 4.3], one has

$$\operatorname{ran}(\operatorname{tr}_{\mathcal{L},2}) = H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega),$$

and the right-hand side is dense in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. Furthermore, by [84, Theorem 3.2], $H^2(\Omega)$ is dense in $\mathcal{D}^s_{\mathcal{L}}(\Omega), s < 2$, hence $D^1_{\mathcal{L}}(\Omega)$ is dense in $\mathcal{D}^0_{\mathcal{L}}(\Omega)$. In the second case, the coefficients are given by $A_{jk} = \delta_{jk} I_m$, where δ_{jk} denotes the Kronecker delta, $A_j = 0_n, 1 \leq j,k \leq n$, and $A = V$, that is, we deal with the Schrödinger operator $\mathcal{L} = -\Delta + V$ with a matrix potential, cf. Section 3.4. The domain Ω in this case is assumed to be Lipschitz. Then, using auxiliary spaces of distributions on $\partial\Omega$, cf. [78], we show in Proposition 3.23 that both *(i)* and *(ii)* from the Hypothesis 3.6 are verified. While a detailed analysis of Hypothesis 3.6 is of independent interest (cf., e.g., [39], [60], [80], [77], [78]) and barely touched upon in the present work, we stress that in all our applications the assumptions of Hypothesis 3.6 are satisfied.

3.2.2 The Lagrangian planes and the self-adjoint extensions of differential operators

Let us introduce the following complex symplectic bilinear form

$$\omega : H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m) \to \mathbb{C} \times \mathbb{C},$$
$$\omega((f_1, g_1), (f_2, g_2)) = \overline{\langle g_2, f_1\rangle}_{-1/2} - \langle g_1, f_2\rangle_{-1/2}, \tag{3.28}$$
$$(f_1, g_1), (f_2, g_2) \in H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m).$$

Then the second Green identity (3.17) reads as follows

$$\langle \mathcal{L}u, v\rangle_{L^2(\Omega,\mathbb{C}^m)} - \langle u, \mathcal{L}v\rangle_{L^2(\Omega,\mathbb{C}^m)} = \omega\left((\gamma_{_D} u, \gamma_{_N}^{\mathcal{L}} u), (\gamma_{_D} v, \gamma_{_N}^{\mathcal{L}} v)\right), \tag{3.29}$$

for all $u, v \in \mathcal{D}^1_{\mathcal{L}}(\Omega)$. We recall that the annihilator of

$$\mathcal{F} \subset H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$$

is defined by

$$\mathcal{F}^\circ := \{(f, g) \in H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)|$$
$$\omega((f, g), (\phi, \psi)) = 0, \text{ for all}\, (\phi, \psi) \in \mathcal{F}\}.$$

The subspace $\mathcal{F}$ is said to be isotropic if $\mathcal{F} \subset \mathcal{F}^\circ$, co-isotropic if $\mathcal{F}^\circ \subset \mathcal{F}$, $\mathcal{F}$ is called Lagrangian if it is simultaneously isotropic and co-isotropic. Furthermore, $\mathcal{F}$ is Lagrangian if and only if it is maximally isotropic, cf., e.g., [70].

The principal goal of this section is to identify self-adjoint extensions of $\mathcal{L}_{min}$, whose domains are subsets of $H^1(\Omega, \mathbb{C}^m)$, with the Lagrangian subspaces in $H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$ as described in the next theorem. We recall notation (3.18).

Theorem 3.8. *Assume Hypothesis* 3.6. *Then the self-adjoint extensions of* $\mathcal{L}_{min}$ *whose domains are contained in* $H^1(\Omega)$ *are in one-to-one correspondence with Lagrangian planes in* $H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$*, that is, the following two assertions hold.*

1. Let $\mathscr{D} \subset \mathcal{D}^1_{\mathcal{L}}(\Omega)$*, and let* $\mathcal{L}_{\mathscr{D}}$ *be the linear operator acting in* $L^2(\Omega, \mathbb{C}^m)$ *and given by the formula*

$$\mathcal{L}_{\mathscr{D}} f := \mathcal{L}_{max} f,\ \ f \in \operatorname{dom}(\mathcal{L}_{\mathscr{D}}) := \mathscr{D}. \tag{3.30}$$

If $\mathcal{L}_{\mathscr{D}}$ *is self-adjoint then the set*

$$\mathcal{G}_{\mathscr{D}} := \overline{\operatorname{tr}_{\mathcal{L}}(\mathscr{D})}^{H^{1/2}(\partial\Omega,\mathbb{C}^m)\times H^{-1/2}(\partial\Omega,\mathbb{C}^m)} \tag{3.31}$$

is a Lagrangian plane in $H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$ *with respect to the symplectic form* ω *defined in* (3.28)*.*

2. A Lagrangian plane $\mathcal{G} \subset H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$ *defines a self-adjoint extension of* $\mathcal{L}_{min}$*. Namely, the linear operator* $\mathcal{L}_{\operatorname{tr}_{\mathcal{L}}^{-1}(\mathcal{G})}$ *acting in* $L^2(\Omega)$ *and given by the formula*

$$\mathcal{L}_{\operatorname{tr}_{\mathcal{L}}^{-1}(\mathcal{G})} f := \mathcal{L}_{max} f,\ \ f \in \operatorname{dom}\left(\mathcal{L}_{\operatorname{tr}_{\mathcal{L}}^{-1}(\mathcal{G})}\right) := \operatorname{tr}_{\mathcal{L}}^{-1}(\mathcal{G}), \tag{3.32}$$

is essentially self-adjoint; here $\operatorname{tr}_{\mathcal{L}}^{-1}(\mathcal{G})$ *denotes the preimage of* $\mathcal{G}$*, that is,*

$$\operatorname{tr}_{\mathcal{L}}^{-1}(\mathcal{G}) := \{u \in \mathcal{D}_{\mathcal{L}}(\Omega)\ :\ \operatorname{tr}_{\mathcal{L}} u \in \mathcal{G}\}.$$

Proof. Part 1. In order to show that $\mathcal{G}_{\mathscr{D}}$ is isotropic we employ the second Green identity (3.17): For arbitrary pairs $(\gamma_D u, \gamma_N^{\mathcal{L}} u) \in \mathcal{G}_{\mathscr{D}}$, $(\gamma_D v, \gamma_N^{\mathcal{L}} v) \in \mathcal{G}_{\mathscr{D}}$ one has

$$\begin{aligned}\omega\left((\gamma_D u, \gamma_N^{\mathcal{L}} u), (\gamma_D v, \gamma_N^{\mathcal{L}} v)\right) &= \overline{\langle \gamma_N^{\mathcal{L}} v, \gamma_D u\rangle}_{-1/2} - \langle \gamma_N^{\mathcal{L}} u, \gamma_D v\rangle_{-1/2}\\ &= \langle \mathcal{L}u, v\rangle_{L^2(\Omega)} - \langle u, \mathcal{L}v\rangle_{L^2(\Omega)} = 0,\end{aligned} \tag{3.33}$$

where the latter equality follows since $\mathcal{L}_{\mathscr{D}}$ is symmetric. Next, we show maximality of the isotropic subspace $\mathcal{G}_{\mathscr{D}}$, that is, that

$$\mathcal{G}_{\mathscr{D}}^{\circ} \subset \mathcal{G}_{\mathscr{D}}. \tag{3.34}$$

To this end, we shall establish an intermediate inclusion

$$\mathcal{G}_{\mathscr{D}}^{\circ} \cap \operatorname{tr}_{\mathcal{L}}(\mathcal{D}_{\mathcal{L}}^{1}(\Omega)) \subset \mathcal{G}_{\mathscr{D}}. \tag{3.35}$$

Indeed, if $(f,g) \in \mathcal{G}_{\mathscr{D}}^{\circ} \cap \operatorname{tr}_{\mathcal{L}}\left(\mathcal{D}_{\mathcal{L}}^{1}(\Omega)\right)$ then

$$(f,g) = \left(\gamma_{D} u_0, \gamma_{N}^{\mathcal{L}} u_0\right), \text{ for some } u_0 \in \mathcal{D}_{\mathcal{L}}^{1}(\Omega), \tag{3.36}$$

and

$$\omega\left((\gamma_{D} u_0, \gamma_{N}^{\mathcal{L}} u_0), (\gamma_{D} v, \gamma_{N}^{\mathcal{L}} v)\right) = 0, \text{ for all } v \in \mathscr{D}. \tag{3.37}$$

On the other hand, using the second Green identity (3.17) with $u = u_0$ and $v \in \mathscr{D}$, one has

$$\langle \mathcal{L} u_0, v\rangle_{L^2(\Omega,\mathbb{C}^m)} - \langle u_0, \mathcal{L} v\rangle_{L^2(\Omega,\mathbb{C}^m)} = \omega\left((\gamma_{D} u_0, \gamma_{N}^{\mathcal{L}} u_0), (\gamma_{D} v, \gamma_{N}^{\mathcal{L}} v)\right). \tag{3.38}$$

Hence,

$$\langle \mathcal{L} u_0, v\rangle_{L^2(\Omega)} = \langle u_0, \mathcal{L} v\rangle_{L^2(\Omega)}, \text{ for all } v \in \mathscr{D}, \tag{3.39}$$

and therefore,

$$u_0 \in \operatorname{dom}\left((\mathcal{L}_{\mathscr{D}})^*\right) = \operatorname{dom}(\mathcal{L}_{\mathscr{D}}), \tag{3.40}$$

since $\mathcal{L}_{\mathscr{D}}$ is self-adjoint by the assumption. Finally, using (3.36) and inclusion (3.40), one infers $(f,g) \in \mathcal{G}_{\mathscr{D}}$ and completes the proof of assertion (2.138). Next we prove inclusion (3.34). Employing (2.138) and the standard properties of annihilator, we obtain

$$\mathcal{G}_{\mathscr{D}}^{\circ} \subset \overline{\mathcal{G}_{\mathscr{D}} + \{\operatorname{tr}_{\mathcal{L}}(\mathcal{D}_{\mathcal{L}}^{1}(\Omega))\}^{\circ}}^{H^{1/2}(\partial\Omega,\mathbb{C}^m)\times H^{-1/2}(\partial\Omega,\mathbb{C}^m)}. \tag{3.41}$$

But

$$\left\{\operatorname{tr}_{\mathcal{L}}(\mathcal{D}^1_{\mathcal{L}}(\Omega))\right\}^{\circ} = \{(0,0)\}, \tag{3.42}$$

since by the assumption (i) in Hypothesis 3.6 the set $\operatorname{tr}_{\mathcal{L}}(\mathcal{D}^1_{\mathcal{L}}(\Omega)))$ is dense in $H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$. Hence, the proof of part *1* is completed.

Part 2. Following [29, Section 3.1] we introduce the space of abstract boundary values $\mathcal{H}_{\mathcal{L}} := \operatorname{dom}(\mathcal{L}_{max})/\operatorname{dom}(\mathcal{L}_{min})$ equipped with the norm

$$\|[x]\|_{\mathcal{H}_{\mathcal{L}}} := \inf\{\|x+f\|_{\mathcal{L},0} : f \in \operatorname{dom}(\mathcal{L}_{min})\}, \tag{3.43}$$

where $[x]$ is the equivalence class of $x \in \operatorname{dom}(\mathcal{L}_{max})$ and $\|\cdot\|_{\mathcal{L},0}$ is the graph norm from (3.14). We define the symplectic form on $\mathcal{H}_{\mathcal{L}}$ by the formula

$$\begin{gathered}\widetilde{\omega}([x],[y]) := \langle \mathcal{L}_{max}x, y\rangle_{L^2(\Omega,\mathbb{C}^m)} - \langle x, \mathcal{L}_{max}y\rangle_{L^2(\Omega,\mathbb{C}^m)},\\ \text{for all } [x],[y] \in \operatorname{dom}(\mathcal{L}_{max})/\operatorname{dom}(\mathcal{L}_{min}).\end{gathered} \tag{3.44}$$

Now we are ready to proceed with the proof of *Part 2*. Let $\mathcal{D}_{\mathcal{G}} := \operatorname{tr}^{-1}(\mathcal{G})$. By [29, Lemma 3.3 (b)], it suffices to show that the closure of the subspace

$$[\mathcal{D}_{\mathcal{G}}] := \{[x] : x \in \mathcal{D}_{\mathcal{G}}\}, \tag{3.45}$$

is Lagrangian in $\mathcal{H}_{\mathcal{L}}$ with respect to $\widetilde{\omega}$. Denoting the annihilator of $[\mathcal{D}_{\mathcal{G}}]$ by $[\mathcal{D}_{\mathcal{G}}]^{\circ}$, we notice that

$$[\mathcal{D}_{\mathcal{G}}] \subset [\mathcal{D}_{\mathcal{G}}]^{\circ}, \tag{3.46}$$

hence, the subspace is isotropic. In order to show the maximality of the closure of $[\mathcal{D}_{\mathcal{G}}]$, we will show that $[\mathcal{D}_{\mathcal{G}}]^{\circ} \subset \overline{[\mathcal{D}_{\mathcal{G}}]}$. First, we will obtain an auxiliary inclusion

$$[\mathcal{D}_{\mathcal{G}}]^{\circ} \cap [\mathcal{D}^1_{\mathcal{L}}(\Omega)] \subset [\mathcal{D}_{\mathcal{G}}]. \tag{3.47}$$

Starting the proof of (3.47) we notice that, if $[u_0] \in [\mathcal{D}_{\mathcal{G}}]^{\circ} \cap [\mathcal{D}^1_{\mathcal{L}}(\Omega)]$ then

$$\widetilde{\omega}\left([u_0],[v]\right) = 0, \ \text{ for all } [v] \in [\mathcal{D}_{\mathcal{G}}], \tag{3.48}$$

that is,

$$\langle \mathcal{L}_{max} u_0, v\rangle_{L^2(\Omega,\mathbb{C}^m)} - \langle u_0, \mathcal{L}_{max} v\rangle_{L^2(\Omega,\mathbb{C}^m)} = 0, \text{ for all } [v] \in [\mathcal{D}_{\mathcal{G}}]. \tag{3.49}$$

Since $[u_0] \in [\mathcal{D}^1_{\mathcal{L}}(\Omega)]$, the trace map $\mathrm{tr}_{\mathcal{L}}$ is well defined on u_0 and $(\gamma_{_D} u_0, \gamma^{\mathcal{L}}_{_N} u_0) \in H^{1/2}(\partial\Omega,\mathbb{C}^m) \times H^{-1/2}(\partial\Omega,\mathbb{C}^m)$. Hence, by the second Green identity, one has

$$\begin{aligned} &\langle \mathcal{L}_{max} u_0, v\rangle_{L^2(\Omega)} - \langle u_0, \mathcal{L}_{max} v\rangle_{L^2(\Omega,\mathbb{C}^m)} \\ &= \overline{\langle \gamma^{\mathcal{L}}_{_N} v, \gamma_{_D} u_0\rangle}_{-1/2} - \langle \gamma^{\mathcal{L}}_{_N} u_0, \gamma_{_D} v\rangle_{-1/2} = 0, \end{aligned} \tag{3.50}$$

for all $[v] \in [\mathcal{D}_{\mathcal{G}}]$. Therefore

$$\big(\gamma_{_D} u_0, \gamma^{\mathcal{L}}_{_N} u_0\big) \in \big(\mathcal{G} \cap \mathrm{tr}_{\mathcal{L}}(\mathcal{D}^1_{\mathcal{L}}(\Omega))\big)^{\circ}. \tag{3.51}$$

We claim that $\big(\mathcal{G} \cap \mathrm{tr}_{\mathcal{L}}(\mathcal{D}^1_{\mathcal{L}}(\Omega))\big)^{\circ} = \mathcal{G}$. Indeed,

$$\big(\mathcal{G} \cap \mathrm{tr}_{\mathcal{L}}(\mathcal{D}^1_{\mathcal{L}}(\Omega))\big)^{\circ} = \overline{\mathcal{G} + \big(\mathrm{tr}_{\mathcal{L}}(\mathcal{D}^1_{\mathcal{L}}(\Omega))\big)^{\circ}} = \overline{\mathcal{G} + (0,0)} = \mathcal{G}, \tag{3.52}$$

where we used (3.42). Inclusion (3.51) together with (3.52) yield $(\gamma_{_D} u_0, \gamma^{\mathcal{L}}_{_N} u_0) \in \mathcal{G}$, which in turn implies $\mathrm{tr}^{-1}_{\mathcal{L}}(u_0) \in \mathcal{D}_{\mathcal{G}}$. Consequently (3.47) holds. Next, applying the annihilator operator $\circ$ to (3.47), one obtains

$$[\mathcal{D}_{\mathcal{G}}]^{\circ} \subset \overline{\overline{[\mathcal{D}_{\mathcal{G}}]} + [\mathcal{D}^1_{\mathcal{L}}(\Omega)]^{\circ}}. \tag{3.53}$$

Since $\mathcal{D}^1_{\mathcal{L}}(\Omega)$ is dense in $D^0_{\mathcal{L}}(\Omega)$ by Hypothesis 3.6 (*ii*), one has

$$\big[\mathcal{D}^1_{\mathcal{L}}(\Omega)\big]^{\circ} = \{[0]\}. \tag{3.54}$$

Combining (3.54) and (3.53) one obtains $[\mathcal{D}_{\mathcal{G}}]^{\circ} \subset \overline{[\mathcal{D}_{\mathcal{G}}]}$ and thus the subspace $[\mathcal{D}_{\mathcal{G}}]$ is Lagrangian as required. ∎

We illustrate Theorem 3.8 by describing the Lagrangian planes associated with the self-adjoint extensions of $\mathcal{L}_{min}$ obtained by two standard PDE constructions.

First, we consider the setup from [67, Chapter 7], cf. also [51]. Let $\mathcal{X}$ be a closed subspace in $H^1(\Omega, \mathbb{C}^m)$ and assume that $H^1_0(\Omega, \mathbb{C}^m) \subset \mathcal{X} \subset H^1(\Omega, \mathbb{C}^m)$. In addition, suppose that the form

$$\mathfrak{l} : L^2(\Omega, \mathbb{C}^m) \times L^2(\Omega, \mathbb{C}^m) \to \mathbb{C}, \operatorname{dom}(\mathfrak{l}) := \mathcal{X}, \tag{3.55}$$

is closed and bounded from below in $L^2(\Omega, \mathbb{C}^m)$. Then, by [61, Theorem 2.8], there exists a unique self-adjoint operator $\mathcal{L}_{\mathcal{X}}$ acting in $L^2(\Omega, \mathbb{C}^m)$ such that

$$\mathfrak{l}[u, v] = \langle \mathcal{L}_{\mathcal{X}} u, v\rangle_{L^2(\Omega,\mathbb{C}^m)} \text{ for all } u \in \operatorname{dom}(\mathcal{L}_{\mathcal{X}}), v \in \mathcal{X}. \tag{3.56}$$

The domain of $\mathcal{L}_{\mathcal{X}}$ is given by the formula

$$\begin{aligned}\operatorname{dom}(\mathcal{L}_{\mathcal{X}}) := \{u \in \mathcal{X} \ : & \exists w \in L^2(\Omega) \text{ such that} \\ & \langle w, v\rangle_{L^2(\Omega)} = \mathfrak{l}[w, v] \text{ for all } v \in \mathcal{X}\}.\end{aligned} \tag{3.57}$$

This construction of the self-adjoint operator $\mathcal{L}_{\mathcal{X}}$ based on a choice of the subspace $\mathcal{X}$ is quite standard [67, Chapter 7]. Theorem 3.8 offers an alternative construction based on a choice of the Lagrangian subspace $\mathcal{G}$. The two constructions are closely related due to the following fact.

Proposition 3.9. *Let*

$$\begin{aligned}\mathcal{G}_{\mathcal{X}} := \{(f, g) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) : \\ f \in \gamma_{_D}(\mathcal{X}), \langle g, \gamma_{_D} w\rangle_{-1/2} = 0 \textit{ for all } w \in \mathcal{X}\}.\end{aligned} \tag{3.58}$$

Then $\mathcal{G}_{\mathcal{X}}$ is a Lagrangian plane. Moreover, $\operatorname{tr}^{-1}_{\mathcal{L}}(\mathcal{G}_{\mathcal{X}})$ is a core of $\mathcal{L}_{\mathcal{X}}$.

Proof. The plane $\mathcal{G}_{\mathcal{X}}$ is Lagrangian by [51, Lemma 3.6]. It remains to show that $\operatorname{tr}_{\mathcal{L}}(\operatorname{dom}(\mathcal{L}_{\mathcal{X}})) \subset \mathcal{G}_{\mathcal{X}}$. By the first Green identity (3.16), for each $u \in \mathcal{D}^1_{\mathcal{L}}(\Omega)$, $v \in \mathcal{X}$ we have

$$\mathfrak{l}[u, v] = \langle \mathcal{L}u, v\rangle_{L^2(\Omega,\mathbb{C}^m)} + \langle \gamma^{\mathcal{L}}_{_N} u, \gamma_{_D} v\rangle_{-1/2}. \tag{3.59}$$

Combining (3.56) and (3.59) we obtain

$$\langle \gamma_{N}^{\mathcal{L}} u, \gamma_{D} v\rangle_{-1/2} = 0 \text{ for all } u \in \operatorname{dom}(\mathcal{L}_{\mathcal{X}}), v \in H^1(\Omega, \mathbb{C}^m). \tag{3.60}$$

Using $\operatorname{dom}(\mathcal{L}_{\mathcal{X}}) \subset \mathcal{X}$ and (3.60) we conclude that $(\gamma_{D} u, \gamma_{N}^{\mathcal{L}} u) \in \mathcal{G}_{\mathcal{X}}$ if $u \in \operatorname{dom}(\mathcal{L}_{\mathcal{X}})$, as required. ∎

Proposition 3.9 shows that the operator $\mathcal{L}_{\mathcal{X}}$, defined in (3.57), (3.56), is associated with the Lagrangian plane $\mathcal{G}_{\mathcal{X}}$ as indicated in Theorem 3.8. In particular, if $\mathcal{X} := H^1_0(\Omega, \mathbb{C}^m)$ then $\mathcal{G}_{\mathcal{X}} = \{0\} \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$ and $\mathcal{L}_{\mathcal{X}}$ is equipped with the Dirichlet boundary conditions. If $\mathcal{X} := H^1(\Omega, \mathbb{C}^m)$ then $\mathcal{G}_{\mathcal{X}} = H^{1/2}(\partial\Omega, \mathbb{C}^m) \times \{0\}$ and $\mathcal{L}_{\mathcal{X}}$ corresponds to the Neumann boundary conditions.

Second, we consider the Schrödinger operator with Robin-type boundary conditions in the context of Theorem 3.8. We will describe the Lagrangian plane corresponding to the Robin Laplacian perturbed by a bounded potential. Then using the Krein-type formula from [79] we will deduce the uniform resolvent convergence of Schrödinger operators with Robin-type boundary conditions from the convergence of the corresponding Lagrangian planes.

Let us recall from [79] the definition of the Robin Laplacian. Assume Hypothesis 3.1 and let a_Θ be a closed sesquilinear form in the Hilbert space $L^2(\partial\Omega)$ with domain $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, bounded from below by c_Θ. Let $\Theta \in \mathcal{B}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$ be a unique bounded, self-adjoint (with respect to duality paring between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$, cf. [78, (B5)-(B9)]) operator associated with the form a_Θ by means of the Lax–Milgram Theorem as discussed in [78, 79]. Then the operator

$$\begin{aligned} -\Delta_\Theta = -\Delta;\ \ \operatorname{dom}(-\Delta_\Theta) = \{&u \in H^1(\Omega) | \Delta u \in L^2(\Omega); \\ &\gamma_{N} u + \Theta \gamma_{D} u = 0 \text{ in } H^{-1/2}(\Omega)\}, \end{aligned} \tag{3.61}$$

is self-adjoint in $L^2(\Omega)$. Clearly, the Lagrangian space $\mathcal{G}_\Theta$ associated with the operator $-\Delta_\Theta$ by Theorem 3.8 is given by

$$\mathcal{G}_\Theta = \text{graph}(-\Theta) = \{(f, -\Theta f) | f \in H^{1/2}(\partial\Omega)\}. \tag{3.62}$$

Proposition 3.10. *Assume Hypothesis* 3.1*. Let*

$$\Theta_n \in \mathcal{B}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)), n \geq 0,$$

be a sequence of bounded, self-adjoint operators (*corresponding to closed, bounded from below sesquilinear forms a_{Θ_n} with domain $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$*)*, and denote by $-\Delta_{\Theta_n}, n \geq 0$, the corresponding sequence of self-adjoint Robin Laplacians defined as in* (3.61)*. Assume that $V \in L^\infty(\Omega)$ is a real-valued potential and let us denote $L_{\Theta_n} := -\Delta_{\Theta_n} + V$, $n \geq 0$. If*

$$\Theta_n \to \Theta_0 \text{ as } n \to \infty \text{ in } \mathcal{B}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)), \tag{3.63}$$

then

$$\left(L_{\Theta_n} - \mathbf{i} I_{L^2(\Omega)}\right)^{-1} \to \left(L_{\Theta_0} - \mathbf{i} I_{L^2(\Omega)}\right)^{-1}, \ n \to \infty, \tag{3.64}$$

in $\mathcal{B}(L^2(\Omega))$.

Proof. First, we recall from [79, Theorem 4.4] that the resolvents $\left(L_{\Theta_n} - \mathbf{i} I_{L^2(\Omega)}\right)^{-1}$, $n \geq 0$ originally considered in $\mathcal{B}(L^2(\Omega))$ can be extended to operators in $\mathcal{B}((H^1(\Omega))^*, (H^1(\Omega))$. We denote the extensions by

$$\widetilde{R}(\mathbf{i}, L_{\Theta_n}) := \left(\widetilde{L}_{\Theta_n} - \mathbf{i}\widetilde{I}_{L^2(\Omega)}\right)^{-1} \in \mathcal{B}((H^1(\Omega))^*, H^1(\Omega)), \ n \geq 0, \tag{3.65}$$

where $\widetilde{I}_{L^2(\Omega)}$ denotes the continuous inclusion map of $H^1(\Omega)$ into $(H^1(\Omega))^*$. Due to the following chain of continuous embeddings,

$$H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow (H^1(\Omega))^*, \tag{3.66}$$

one has,

$$\begin{aligned}&\left\|\left(L_{\Theta_n}-\mathbf{i}I_{L^2(\Omega)}\right)^{-1}-\left(L_{\Theta_0}-\mathbf{i}I_{L^2(\Omega)}\right)^{-1}\right\|_{\mathcal{B}(L^2(\Omega))}\\&\quad\leq\left\|\widetilde{R}(\mathbf{i},L_{\Theta_n})-\widetilde{R}(\mathbf{i},L_{\Theta_0})\right\|_{\mathcal{B}((H^1(\Omega))^*,H^1(\Omega))}.\end{aligned}\tag{3.67}$$

Hence, it suffices to show that the right-hand side of inequality (3.67) tends to 0 as $n\to\infty$. The proof of this statement relies on the Krein-type formula valid on $(H^1(\Omega))^*$, cf. [78, Lemma 5.2],

$$\widetilde{R}(\mathbf{i},L_{\Theta_n})=\widetilde{R}(\mathbf{i},L_{\Theta_0})+\widetilde{R}(\mathbf{i},L_{\Theta_n})\gamma_{_D}^*(\Theta_n-\Theta_0)\gamma_{_D}\widetilde{R}(\mathbf{i},L_{\Theta_0}),\ n\geq 1.\tag{3.68}$$

We will use (3.68) twice: first to obtain the uniform in n boundedness of $\widetilde{R}(\mathbf{i},L_{\Theta_n})$, that is,

$$\sup_{n\geq1}\left\|\widetilde{R}(\mathbf{i},L_{\Theta_n})\right\|_{\mathcal{B}((H^1(\Omega))^*,H^1(\Omega))}<\infty,\tag{3.69}$$

and, second, to prove the asserted convergence of the extended resolvents. The Krein-type formula (3.68) yields

$$\begin{aligned}&\|\widetilde{R}(\mathbf{i},L_{\Theta_n})\|_{\mathcal{B}((H^1(\Omega))^*,H^1(\Omega))}-\|\widetilde{R}(\mathbf{i},L_{\Theta_0})\|_{\mathcal{B}((H^1(\Omega))^*,H^1(\Omega))}\\&\quad\leq\|\widetilde{R}(\mathbf{i},L_{\Theta_n})\|_{\mathcal{B}((H^1(\Omega))^*,H^1(\Omega))}\times\|\widetilde{R}(\mathbf{i},L_{\Theta_0})\|_{\mathcal{B}((H^1(\Omega))^*,H^1(\Omega))}\\&\quad\times\|\gamma_{_D}\|^2_{\mathcal{B}(H^1(\Omega),H^{1/2}(\partial\Omega))}\|\Theta_n-\Theta_0\|_{\mathcal{B}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))}.\end{aligned}$$

Therefore, for n large enough, one has

$$\begin{aligned}&\|\widetilde{R}(\mathbf{i},L_{\Theta_n})\|_{\mathcal{B}((H^1(\Omega))^*,H^1(\Omega))}\leq\|\widetilde{R}(\mathbf{i},L_{\Theta_0})\|_{\mathcal{B}((H^1(\Omega))^*,H^1(\Omega))}\\&\quad\times\Big(1-\|\gamma_{_D}\|^2_{\mathcal{B}(H^1(\Omega),H^{1/2}(\partial\Omega))}\|\Theta_n-\Theta_0\|_{\mathcal{B}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))}\\&\qquad\times\|\widetilde{R}(\mathbf{i},L_{\Theta_0})\|_{\mathcal{B}((H^1(\Omega))^*,H^1(\Omega))}\Big)^{-1},\end{aligned}\tag{3.70}$$

since $\|\Theta_n-\Theta_0\|_{\mathcal{B}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))}\to 0$ as $n\to\infty$. The expression in the right-hand side of (3.70) is well defined and bounded, hence, (3.69) holds. Formula (3.68) implies

$$\begin{aligned}
&\|\widetilde{R}(\mathbf{i}, L_{\Theta_n}) - \widetilde{R}(\mathbf{i}, L_{\Theta_0})\|_{\mathcal{B}((H^1(\Omega))^*, H^1(\Omega))} \\
&\quad \leq \|\widetilde{R}(\mathbf{i}, L_{\Theta_n})\|_{\mathcal{B}((H^1(\Omega))^*, H^1(\Omega))} \times \|\widetilde{R}(\mathbf{i}, L_{\Theta_0})\|_{\mathcal{B}((H^1(\Omega))^*, H^1(\Omega))} \\
&\quad \times \|\gamma_{D}\|^2_{\mathcal{B}(H^1(\Omega), H^{1/2}(\partial\Omega))} \|\Theta_n - \Theta_0\|_{\mathcal{B}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}.
\end{aligned} \tag{3.71}$$

Finally, combining (3.63), (3.69), and (3.71), we obtain

$$\left\|\widetilde{R}(\mathbf{i}, L_{\Theta_n}) - \widetilde{R}(\mathbf{i}, L_{\Theta_0})\right\|_{\mathcal{B}((H^1(\Omega))^*, H^1(\Omega))} \to 0, \text{ as } n \to \infty, \tag{3.72}$$

and thereby complete the proof. ∎

We remark next that taking the closure in (3.31) is essential due to the next proposition showing that $\operatorname{tr}_{\mathcal{L}}$ is not onto in general. This amounts to the fact that $\operatorname{tr}_{\mathcal{L}}$ does not map domains of self-adjoint extensions into Lagrangian planes in $H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$, but only into their dense subsets.

Proposition 3.11. *Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be an open, bounded domain with smooth boundary. Then the map tr_Δ corresponding to the Laplacian,*

$$\operatorname{tr}_\Delta := (\gamma_{D}, \gamma_{N}) : \mathcal{D}^1_\Delta(\Omega) \to H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega), \tag{3.73}$$

is not surjective.

Proof. We prove the assertion by contradiction. Suppose that tr_Δ is surjective. Under this assumption one can show that $\mathcal{F} := \operatorname{tr}_\Delta(H^2(\Omega) \cap H^1_0(\Omega))$ is a Lagrangian plane in $H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$. Indeed, $\mathcal{F} \subset \mathcal{F}^\circ$ since

$$\begin{gathered}
\omega(\operatorname{tr}_\Delta u, \operatorname{tr}_\Delta v) = \overline{\langle \gamma_{N} v, \gamma_{D} u\rangle}_{-1/2} - \langle \gamma_{N} u, \gamma_{D} v\rangle_{-1/2} = 0, \\
\text{for all } u, v \in H^2(\Omega) \cap H^1_0(\Omega).
\end{gathered} \tag{3.74}$$

In order to prove $\mathcal{F}^\circ \subset \mathcal{F}$, let us fix an arbitrary $(f,g) \in \mathcal{F}^\circ$. Since tr_Δ is assumed to be surjective, there exists $v_0 \in \mathcal{D}^1_\Delta(\Omega)$ such that $\operatorname{tr}_\Delta(v_0) = (f,g)$. Furthermore,

$$\omega(\operatorname{tr}_\Delta u, (f,g)) = \omega\left(\operatorname{tr}_\Delta u, \operatorname{tr}_\Delta v_0\right) = 0, \text{ for all } u \in H^2(\Omega) \cap H^1_0(\Omega). \tag{3.75}$$

In addition, the second Green identity yields

$$\langle -\Delta u, v_0 \rangle_{L^2(\Omega)} - \langle u, -\Delta v_0 \rangle_{L^2(\Omega)} = \overline{\langle \gamma_{N} v_0, \gamma_{D} u \rangle}_{-1/2} - \langle \gamma_{N} u, \gamma_{D} v_0 \rangle_{-1/2}. \tag{3.76}$$

Combining (3.75) and (3.76), one obtains

$$\langle -\Delta u, v_0 \rangle_{L^2(\Omega)} - \langle u, -\Delta v_0 \rangle_{L^2(\Omega)} = 0, \text{for all } u \in H^2(\Omega) \cap H^1_0(\Omega). \tag{3.77}$$

Let us recall that $H^2(\Omega) \cap H^1_0(\Omega)$ is the domain of the Dirichlet Laplacian $-\Delta_D$, which is a self-adjoint operator in $L^2(\Omega)$. Therefore, (3.77) leads to $v_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, which in turn implies $\operatorname{tr}_\Delta v_0 = (f,g) \in \mathcal{F}$ and $\mathcal{F}^\circ \subset \mathcal{F}$. Finally, we arrive at $\mathcal{F} = \mathcal{F}^\circ$. On the other hand,

$$\mathcal{F} = \operatorname{tr}_\Delta \left(H^2(\Omega) \cap H^1_0(\Omega)\right) = \{0\} \times H^{1/2}(\partial\Omega). \tag{3.78}$$

The set $\{0\} \times H^{1/2}(\partial\Omega)$ is not closed in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, thus it is not Lagrangian. This contradiction completes the proof. ∎

3.2.3 The Maslov index

We will now recall from [34], [35], [36], [37] a definition of the Maslov index of a path of Lagrangian planes in a complex Hilbert space $\mathcal{X}$ relative to a reference plane. This will require some preliminaries. Let ω be a symplectic form on $\mathcal{X}$, i.e., we assume that $\omega : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is a sesquilinear, bounded, skew-symmetric, non-degenerate form. Then there exists a bounded operator $J : \mathcal{X} \to \mathcal{X}$, such that

$$\omega(u,v) = \langle Ju, v \rangle_{\mathcal{X}}, \quad u, v \in \mathcal{X}, \tag{3.79}$$

and

$$J^2 = -I_{\mathcal{X}}, J^* = -J. \tag{3.80}$$

Moreover, $\mathcal{X}$ admits an orthogonal decomposition into direct sum of the eigenspaces of the operator J, that is,

$$\mathcal{X} = \ker(J - \mathbf{i}I) \oplus \ker(J + \mathbf{i}I). \tag{3.81}$$

Therefore, the form $-\mathbf{i}\omega$ is positive definite on $\ker(J - \mathbf{i}I)$, the form $-\mathbf{i}\omega$ is negative definite on $\ker(J + \mathbf{i}I)$, and $\omega(u, v) = 0$ whenever $u \in \ker(J - \mathbf{i}I), v \in \ker(J + \mathbf{i}I)$. We denote the annihilator of a subset $\mathcal{F} \subset \mathcal{X}$ by

$$\mathcal{F}^\circ := \{u \in \mathcal{X} : \omega(u, v) = 0 \text{ for all } v \in \mathcal{F}\}. \tag{3.82}$$

The subspace $\mathcal{F}$ is called *Lagrangian* if $\mathcal{F} = \mathcal{F}^\circ$. The set of Lagrangian subspaces of $\mathcal{X}$ is denoted by

$$\Lambda(\mathcal{X}) := \{\mathcal{F} \subset \mathcal{X} : \mathcal{F} \text{ is Lagrangian in } \mathcal{X}\}. \tag{3.83}$$

Following [36, Lemma 3], we notice that every Lagrangian plane $\mathcal{F}$ can be uniquely represented as a graph of a bounded operator $U \in \mathcal{B}(\ker(J + \mathbf{i}I_{\mathcal{X}}), \ker(J - \mathbf{i}I_{\mathcal{X}}))$, i.e., one has

$$\mathcal{F} = \text{graph}(U) := \{y + Uy : y \in \ker(J + \mathbf{i}I_{\mathcal{X}})\}. \tag{3.84}$$

That is, $Uy \in \ker(J - \mathbf{i}I_{\mathcal{X}})$ is the unique vector satisfying $y + Uy \in \mathcal{F}$ for $y \in \ker(J + \mathbf{i}I_{\mathcal{X}})$. Moreover,

$$\omega(x, y) = -\omega(Ux, Uy), \; x, y \in \ker(J + \mathbf{i}I_{\mathcal{X}}). \tag{3.85}$$

The operator U is a unitary map acting between the Hilbert spaces $\ker(J+\mathbf{i}I_{\mathcal{X}})$ and $\ker(J-\mathbf{i}I_{\mathcal{X}})$. Indeed, for arbitrary $x, y \in \ker(J+\mathbf{i}I_{\mathcal{X}})$ one has

$$\begin{aligned}\langle x, y\rangle_{\mathcal{X}} &= \mathbf{i}\langle Jx, y\rangle_{\mathcal{X}} = \mathbf{i}\omega(x,y)\\ &= -\mathbf{i}\omega(Ux, Uy) = -\mathbf{i}\langle JUx, Uy\rangle_{\mathcal{X}} = \langle Ux, Uy\rangle_{\mathcal{X}}.\end{aligned} \tag{3.86}$$

A pair of Lagrangian planes $\mathcal{F}, \mathcal{Z}$ is called Fredholm pair if

$$\dim(\mathcal{F}\cap\mathcal{Z}) < \infty,\ \mathcal{F}+\mathcal{Z} \text{ is closed in } \mathcal{X}, \text{ and } \operatorname{codim}(\mathcal{F}+\mathcal{Z}) < \infty. \tag{3.87}$$

Let $\mathcal{F} = \operatorname{graph}(U)$ and $\mathcal{Z} = \operatorname{graph}(V)$ be Lagrangian planes in $\mathcal{X}$, then by [36, Lemma 2], the pair $(\mathcal{F}, \mathcal{Z})$ is Fredholm if and only if $UV^{-1} - I_{\mathcal{X}}$ is Fredholm operator in $\ker(J-\mathbf{i}I_{\mathcal{X}})$. Furthermore,

$$\dim(\mathcal{F}\cap\mathcal{Z}) = \dim\ker(UV^{-1} - I_{\mathcal{X}}). \tag{3.88}$$

Let us fix a Lagrangian plane

$$\mathcal{Z} \subset \mathcal{X}, \mathcal{Z} = \operatorname{graph}(V), \tag{3.89}$$

where $V \in \mathcal{B}(\ker(J+\mathbf{i}I_{\mathcal{X}}), \ker(J-\mathbf{i}I_{\mathcal{X}}))$ is a unitary operator. The Fredholm-Lagrangian-Grassmannian is the space

$$F\Lambda(\mathcal{Z}) := \{\mathcal{F}\subset\mathcal{X} : \mathcal{F} \text{ is Lagrangian, and the pair } (\mathcal{F},\mathcal{Z}) \text{ is Fredholm}\}, \tag{3.90}$$

equipped with metric

$$d(\mathcal{F}_1, \mathcal{F}_2) := \|P_{\mathcal{F}_1} - P_{\mathcal{F}_2}\|_{\mathcal{B}(\mathcal{H})},\ \mathcal{F}_1, \mathcal{F}_2 \in F\Lambda(\mathcal{Z}), \tag{3.91}$$

where $P_{\mathcal{F}}$ denotes the orthogonal projection onto $\mathcal{F}$. Let $\mathcal{I} = [a, b] \subset \mathbb{R}$ be a set of parameters. Let us fix a continuous path in $F\Lambda(\mathcal{Z})$

$$\Upsilon : \mathcal{I} \to F\Lambda(\mathcal{Z}),\quad \Upsilon(s) = \mathcal{F}_s,\ \Upsilon \in C(\mathcal{I}, F\Lambda(\mathcal{Z})), \tag{3.92}$$

and introduce the corresponding family of unitary operators U_s such that

$$\mathcal{F}_s = \text{graph}(U_s),\ s \in \mathcal{I},$$

$$\upsilon : \mathcal{I} \to \mathcal{B}(\ker(J + \mathbf{i}I_{\mathcal{X}}), \ker(J - \mathbf{i}I_{\mathcal{X}})),\ \upsilon(s) = U_s.$$

The following is proved in [35]:

$$\upsilon \in C(\mathcal{I}, \mathcal{B}(\ker(J + \mathbf{i}I_{\mathcal{X}}), \ker(J - \mathbf{i}I_{\mathcal{X}}))), \tag{3.93}$$

$$U_s V^{-1} \text{ is unitary in } \ker(J - \mathbf{i}I_{\mathcal{X}}),\ s \in \mathcal{I}, \tag{3.94}$$

$$U_s V^{-1} - I_{\mathcal{X}} \text{ is Fredholm in } \ker(J - \mathbf{i}I_{\mathcal{X}}),\ s \in \mathcal{I}, \tag{3.95}$$

$$\dim(\mathcal{F}_s \cap \mathcal{Z}) = \dim\ker(U_s V^{-1} - I_{\mathcal{X}}),\ s \in \mathcal{I}. \tag{3.96}$$

Utilizing (3.93)-(3.96) we will now define the Maslov index as the spectral flow through the point $1 \in \mathbb{C}$ of the family $\upsilon(s), s \in \mathcal{I}$. An illuminating discussion of the notion of the spectral flow of a family of closed operators through an admissible curve $\ell \subset \mathbb{C}$ can be found in [36, Appendix]. To proceed with the definition, note that due to (3.95) there exists a partition $a = s_0 < s_1 < \cdots < s_N = b$ of $[a, b]$ and positive numbers $\varepsilon_j \in (0, \pi)$ such that $\varepsilon^{\pm \mathbf{i}\varepsilon_j} \notin \text{Spec}(U_s V^{-1})$ if $s \in [s_{j-1}, s_j]$, for each $1 \leq j \leq N$, see [70, Lemma 3.1]. For any $\varepsilon > 0$ and $s \in [a, b]$ we let

$$k(s, \varepsilon) := \sum\nolimits_{0 \leq \varkappa \leq \varepsilon} \dim\ker(U_s V^{-1} - \varepsilon^{\mathbf{i}\varkappa}), \tag{3.97}$$

and define the Maslov index

$$\text{Mas}(\Upsilon, \mathcal{Z}) := \sum_{j=1}^{N} \left(k(s_j, \varepsilon_j) - k(s_{j-1}, \varepsilon_j)\right). \tag{3.98}$$

The number $\text{Mas}(\Upsilon, \mathcal{X})$ is well defined, i.e., it is independent on the choice of the partition s_j and ε_j (cf., [70, Proposition 3.3]).

Next we turn to the computation of the Maslov index via the crossing forms. Assume that $\Upsilon \in C^1(\mathcal{I}, F\Lambda(\mathcal{X}))$ and let $s_* \in \mathcal{I}$. There exists a neighbourhood $\mathcal{I}_0$ of s_* and a family $R_s \in C^1(\mathcal{I}_0, \mathcal{B}(\Upsilon(s_*), \Upsilon(s_*)^\perp))$, such that $\Upsilon(s) = \{u + R_s u \big| u \in \Upsilon(s_*)\}$, for $s \in \mathcal{I}_0$ see, e.g., [70] or [50, Lemma 3.8]. We will use the following terminology from [70, Definition 3.20].

Definition 3.12. Let $\mathcal{Z}$ be a Lagrangian subspace and $\Upsilon \in C^1(\mathcal{I}, F\Lambda(\mathcal{Z}))$.

(i) We call $s_* \in \mathcal{I}$ a conjugate point or crossing if $\Upsilon(s_*) \cap \mathcal{Z} \neq \{0\}$.

(ii) The finite dimentional form

$$\mathcal{Q}_{s_*,\mathcal{Z}}(u, v) := \frac{d}{ds}\omega(u, R_s v)\big|_{s=s_*} = \omega(u, \dot{R}_{s=s_*} v), \text{ for } u, v \in \Upsilon(s_*) \cap \mathcal{Z},$$

is called the crossing form at the crossing s_*.

(iii) The crossing s_* is called regular if the form $\mathcal{Q}_{s_*,\mathcal{Z}}$ is non-degenerate, positive if $\mathcal{Q}_{s_*,\mathcal{Z}}$ is positive definite, and negative if $\mathcal{Q}_{s_*,\mathcal{Z}}$ is negative definite.

The following result (cf., [35, Proposition 3.2.7] and Remark 3.14) provides an efficient tool for computing the Malsov index at regular crossings. We denote by n_+ and n_- the number of positive and negative squares of a form, the signature is defined by the formula sign $= n_+ - n_-$.

Theorem 3.13. *Let $\Upsilon \in C^1(\mathcal{I}, F\Lambda(\mathcal{Z}))$, and assume that all crossings are regular. Then the crossings are isolated, and one has*

$$\text{Mas}(\Upsilon, \mathcal{Z}) = -n_-(\mathcal{Q}_{a,\mathcal{Z}}) + \sum_{a<s<b} \text{sign}(\mathcal{Q}_{s,\mathcal{Z}}) + n_+(\mathcal{Q}_{b,\mathcal{Z}}). \tag{3.99}$$

We will now review the definition of the Maslov index for *two* paths with values in Lagrangian–Grassmannian $\Lambda(\mathcal{X})$, see [70, Section 3.5]. Let us fix $\Upsilon_1, \Upsilon_2 \in$

$C(\mathcal{I}, \Lambda(\mathcal{X}))$ and assume that $(\Upsilon_1(s), \Upsilon_2(s))$ is a Fredholm pair for all $s \in \mathcal{I}$. Let $\text{diag} := \{(p, p) : p \in \mathcal{X}\}$ denote the diagonal plane in $\mathcal{X} \oplus \mathcal{X}$. On $\mathcal{X} \oplus \mathcal{X}$ we define the symplectic form $\hat{\omega} := \omega \oplus (-\omega)$ with the complex structure $\widetilde{J} := J \oplus (-J)$, denoting the resulting space of Lagrangian planes by $\Lambda_{\hat{\omega}}(\mathcal{X} \oplus \mathcal{X})$. We consider the path $\widetilde{\Upsilon} := \Upsilon_1 \oplus \Upsilon_2 \in C(\mathcal{I}, \Lambda_{\hat{\omega}}(\mathcal{X} \oplus \mathcal{X}))$ and define the Maslov index of the two paths Υ_1, Υ_2 as $\text{Mas}(\Upsilon_1, \Upsilon_2) := \text{Mas}(\widetilde{\Upsilon}, \text{diag})$. If $\Upsilon_2(s) = \mathcal{Z}$ for all $s \in \mathcal{I}$, then $\text{Mas}(\Upsilon_1 \oplus \Upsilon_2, \text{diag}) = \text{Mas}(\Upsilon_1, \mathcal{Z})$.

Remark 3.14. We adopted definition (3.98) of the Maslov index as the spectral flow of $U_s V^{-1}$ through the point 1. Since $\varkappa$ in (3.97) is allowed to be equal to zero, the Maslov index defined in (3.98) counts the number of the eigenvalues of $U_s V^{-1}$ that leave the *closed* segment $\{e^{\mathbf{i}\varkappa} : \varkappa \in [0, \varepsilon]\}$ through 1 as parameter s varies from a to b. In comparison, the Maslov index defined in [35, Definition 2.1.1] counts the number of eigenvalues that leave the *open* segment $\{e^{\mathbf{i}\varkappa} : \varkappa \in (0, \varepsilon)\}$. This difference in definitions is reflected in the formula relating the Maslov index and the signature of the crossing form. In our case, the Maslov index at the left (respectively, right) regular endpoint crossing is equal to minus(respectively, plus) the number of negative (respectively, positive) directions of the crossing form. The Maslov index from [35, Proposition 3.2.7] is equal to the number of positive(respectively, minus the number of negative) directions. We find definition (3.98) more convenient as it permits to obtain a relation between the Maslov index of a certain path, and the Morse index of a family of self-adjoint operators without adding the dimension of subspace corresponding to the zero eigenvalue into the Morse index.

Remark 3.15. The starting point for the definition of the Maslov index given in

[29], [70] is a *real* Hilbert space $\mathcal{H}_{\mathbb{R}}$ equipped with a symplectic form. The Maslov index in [29], [70] is defined as the spectral flow (through -1) of a family of unitary operators (acting in an auxiliary complex space $\mathcal{H}_{\mathbb{C}}$) obtained via the Souriau map. While the assumption that $\mathcal{H}_{\mathbb{R}}$ is a *real* Hilbert space is not restrictive in many applications (cf., e.g., [50], [51], [52], [97], [98], [116]), it does prevent one from considering complex-valued boundary conditions (such as $\theta-$periodic, see below) without reduction to equivalent real-valued boundary conditions. Given the abstract nature of the eigenvalue problem for self-adjoint extensions of $\mathcal{L}$ (as in (3.4)), a reduction to the real Hilbert spaces (i.e., to the real boundary conditions) cannot be carried out explicitly. Instead, we choose to adopt the definition of the Maslov index in complex symplectic Hilbert spaces. As it was pointed out in [36, Corollary 2], there is a natural identification between the Maslov index in the real Hilbert space $\mathcal{H}_{\mathbb{R}}$ and the Maslov index in the complex Hilbert space $\mathcal{H}_{\mathbb{R}} \otimes \mathbb{C}$ (the complexification of $\mathcal{H}_{\mathbb{R}}$) defined as in (3.98).

3.3 The Maslov index for second order elliptic operators on smooth domains

The main result of this section concerns with an index formula for second order elliptic operators with scalar coefficients defined on a smooth domain $\Omega \subset \mathbb{R}^n$, see Theorem 3.18.

3.3.1 Weak solutions and their traces

In this subsection we reformulate the eigenvalue problems for elliptic operators in terms of Lagrangian subspaces formed by the traces of *weak* solutions of corresponding equations.

Hypothesis 3.16. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with smooth boundary. Let $\mathcal{I} := [\alpha, \beta]$, $-\infty < \alpha < \beta < +\infty$, be the interval of parameters. Assume that a^t, a^t_j, a^t_{jk} are contained in $C^\infty(\overline{\Omega})$ for all $t \in \mathcal{I}$. Suppose that*

$$a_{jk} : t \mapsto a^t_{jk},\ a_{jk} \in C^1(\mathcal{I}, L^\infty(\Omega)),\ a^t_{jk}(x) = \overline{a^t_{kj}(x)},\ 1 \leq j \leq n, x \in \overline{\Omega}, \tag{3.100}$$

$$a^t_{jk}(x)\xi_k\overline{\xi_j} \geq c\sum_{j=1}^{n} |\xi_j|^2 \text{ for all } x \in \overline{\Omega}, \xi = (\xi_j)_{j=1}^n \in \mathbb{C}^n, t \in \mathcal{I}; \text{for some } c > 0, \tag{3.101}$$

$$a_j : t \mapsto a^t_j,\ a_j \in C^1(\mathcal{I}, L^\infty(\Omega)),\ 1 \leq j \leq n, \tag{3.102}$$

$$a : t \mapsto a^t,\ a \in C^1(\mathcal{I}, L^\infty(\Omega)),\ a^t(x) \in \mathbb{R},\ x \in \Omega,\ t \in \mathcal{I}. \tag{3.103}$$

Given the families of the functions $\{a^t\}_{t=\alpha}^{\beta}$, $\{a^t_j\}_{t=\alpha}^{\beta}$, $\{a^t_{jk}\}_{t=\alpha}^{\beta}$ we now consider the family $\{\mathcal{L}^t\}_{t=\alpha}^{\beta}$ of the differential expressions

$$\mathcal{L}^t := -\sum_{j,k=1}^{n} \partial_j a^t_{jk}\partial_k + \sum_{j=1}^{n} a^t_j\partial_j - \partial_j\overline{a^t_j} + a^t,\ t \in \mathcal{I}, \tag{3.104}$$

which are formally self-adjoint. For $t \in \mathcal{I}$ the minimal operator corresponding to the differential expression $\mathcal{L}^t$ in $L^2(\Omega)$ is defined by the formula

$$\mathcal{L}^t_{min} f = \mathcal{L}^t f,\ f \in \operatorname{dom}(\mathcal{L}^t_{min}) := H^2_0(\Omega). \tag{3.105}$$

The operator $\mathcal{L}^t_{min}$ is a densely defined, bounded from bellow, symmetric operator. Its adjoint $\mathcal{L}^t_{max} := (\mathcal{L}^t_{min})^*$ is acting in $L^2(\Omega)$ and given by the formula

$$\mathcal{L}^t_{max} u := \mathcal{L}^t u,\ u \in \operatorname{dom}(\mathcal{L}^t_{max}) := \{u \in L^2(\Omega, \mathbb{C}^m) : \mathcal{L}^t u \in L^2(\Omega, \mathbb{C}^m)\}. \tag{3.106}$$

Given a family of self-adjoint extensions $\{\mathcal{L}^t_{\mathscr{D}_t}\}_{t=\alpha}^{\beta}$ of $\mathcal{L}^t_{min}$ with $\operatorname{dom}(\mathcal{L}^t_{\mathscr{D}_t}) = \mathscr{D}_t$, one has the chain of extensions

$$\mathcal{L}^t_{min} \subset \mathcal{L}^t_{\mathscr{D}_t} \subset \mathcal{L}^t_{max}. \tag{3.107}$$

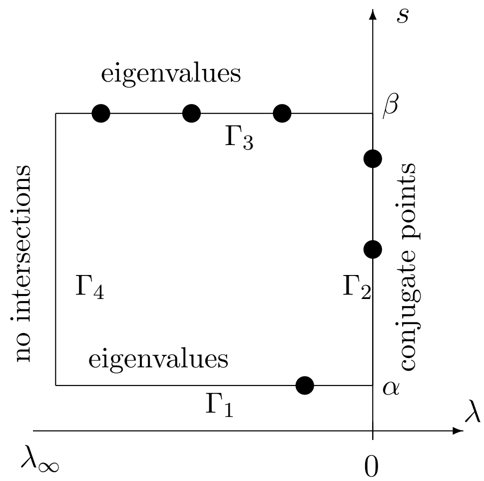

Figure 3.1:

As discussed in Remark 3.7, all assumptions of Hypothesis 3.6 are satisfied with $\mathcal{L}$ in (3.4) replaced by $\mathcal{L}^t$ from (3.104). Hence $\mathcal{L}^t_{min}$ fits the framework of Theorem 3.8 and its self-adjoint extensions are uniquely associated with Lagrangian planes in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ via Theorem 3.8.

Our objective is to relate the Morse indices of the operators $\mathcal{L}^{\beta}_{\mathscr{D}_\beta}$ and $\mathcal{L}^{\alpha}_{\mathscr{D}_\alpha}$ to the Maslov index of a certain path of Lagrangian planes in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ defined by the given one parameter family of self-adjoint operators $\{\mathcal{L}^t_{\mathscr{D}_t}\}_{t=\alpha}^{\beta}$. This will be achieved by utilizing homotopy invariance of the Maslov index. To this end we introduce a parametrization of the square loop in Figure 1,

$$\Sigma := \cup_{j=1}^4 \Sigma_j \to \Gamma = \cup_{j=1}^4 \Gamma_j,\ s \mapsto (\lambda(s), t(s)), \tag{3.108}$$

where $\Gamma_j,\ j = 1, \cdots, 4$ are the positively oriented sides of the boundary of the square $[\lambda_\infty, 0] \times [\alpha, \beta]$, the parameter set $\Sigma = \cup_{j=1}^4 \Sigma_j$ and $\lambda(\cdot)$, $t(\cdot)$ are defied as follows:

$$\lambda(s) = s,\ t(s) = \alpha,\ s \in \Sigma_1 := [\lambda_\infty, 0], \tag{3.109}$$

$$\lambda(s) = 0,\ t(s) = s + \alpha,\ s \in \Sigma_2 := [0, \beta - \alpha], \tag{3.110}$$

$$\lambda(s) = -s + \beta - \alpha,\ t(s) = \beta,\ s \in \Sigma_3 := [\beta - \alpha, \beta - \alpha - \lambda_\infty], \tag{3.111}$$

$$\lambda(s) = \lambda_\infty,\ t(s) = -s + 2\beta - \alpha - \lambda_\infty, \tag{3.112}$$

$$s \in \Sigma_4 := [\beta - \alpha - \lambda_\infty, 2(\beta - \alpha) - \lambda_\infty].$$

We now turn to the eigenvalue problem

$$\mathcal{L}^{t(s)}_{\mathscr{D}_{t(s)}} u = \lambda(s)u,\ \ u \neq 0,\ \ s \in \Sigma. \tag{3.113}$$

Recalling notation (3.18), for the family $\{\mathcal{L}^{t(s)}_{\mathscr{D}_t}\}_{s\in\Sigma}$ of the self-adjoint operators from (3.104)–(3.107) and the parametrization $t(\cdot)$, $\lambda(\cdot)$ from (3.108)–(3.112) we now define the following subspaces:

$$\mathcal{G}_{t(s)} := \overline{\operatorname{tr}_{\mathcal{L}^{t(s)}}(\mathscr{D}_{t(s)})},\ \mathcal{K}_{\lambda(s),t(s)} := \operatorname{tr}_{\mathcal{L}^{t(s)}}(\mathbf{K}_{\lambda(s),t(s)}), \tag{3.114}$$

$$\begin{aligned}\mathbf{K}_{\lambda(s),t(s)} := \Big\{ u \in H^1(\Omega) : \sum_{j,k=1}^n \langle a^{t(s)}_{jk}\partial_k u, \partial_j \varphi\rangle_{L^2(\Omega)} + \sum_{j=1}^n \langle a^{t(s)}_j \partial_j u, \varphi\rangle_{L^2(\Omega)} \\ + \sum_{j=1}^n \langle u, a^{t(s)}_j \partial_j \varphi\rangle_{L^2(\Omega)} + \langle a^{t(s)}u - \lambda(s)u, \varphi\rangle_{L^2(\Omega)} = 0,\ \varphi \in H^1_0(\Omega)\Big\},\ \ s \in \Sigma.\end{aligned}$$

The subspace $\mathbf{K}_{\lambda(s),t(s)}$ is the set of weak solutions to the equation $\mathcal{L}^{t(s)}u = \lambda(s)u$, the subspace $\mathcal{K}_{\lambda(s),t(s)}$ is the set of their traces, and $\mathcal{G}_{t(s)}$ is the subspace in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ that corresponds to $\mathscr{D}_{t(s)}$ as indicated in Theorem 3.8.

Our next Theorem 3.17 shows, in particular, that the existence of nontrivial solutions to (3.113) is equivalent to

$$\mathcal{G}_{t(s)} \cap \mathcal{K}_{\lambda(s),t(s)} \neq \{0\},\ \ s \in \Sigma. \tag{3.115}$$

Theorem 3.17 is an improvement of [51, Proposition 3.5], see also [50, Propositoin 4.10]. Proposition 3.5 in [29] provides an elegant proof of a related assertion in the context of strong solutions and abstract boundary traces. This result cannot be directly applied in the setting of the weak traces and weak solutions, however, we adopted the proof of [29, Proposition 3.5] in order to show part *ii)* in the following theorem. The novel part *ii)* of this theorem states that just the Fredholm property of the operator $\mathcal{L}^{t(s)}_{\lambda(s)} - \lambda_{(s)} I_{L^2(\Omega)}$ alone implies that the pair of subspaces $\mathcal{K}_{\lambda(s),t(s)}$ (*weak* traces of *weak* solutions) and $\mathcal{G}_{t(s)}$ is Fredholm. We note that assertion *iii)* in the next theorem was proved in [51, Proposition 3.5] (see also [50, Proposition 4.10]).

Theorem 3.17. *Assume Hypothesis* 3.16. *Let* $\mathscr{D}_t \subset \mathcal{D}^1_{\mathcal{L}^t}(\Omega)$, $t \in \mathcal{I} := [\alpha, \beta]$, *and assume that the linear operator* $\mathcal{L}^t_{\mathscr{D}_t}$ *acting in* $L^2(\Omega)$ *and given by*

$$\mathcal{L}^t_{\mathscr{D}_t} u := \mathcal{L}^t u, \; u \in \operatorname{dom}(\mathcal{L}^t_{\mathscr{D}_t}) := \mathscr{D}_t, \tag{3.116}$$

is self-adjoint for each $t \in \mathcal{I}$.

Then the following assertions hold:

i) if $s \in \Sigma$, *then* $\mathcal{K}_{\lambda(s),t(s)}$ *and* $\mathcal{G}_{t(s)}$ *are Lagrangian planes with respect to the symplectic form* (3.28),

ii) if $\operatorname{Spec}_{ess}\left(\mathcal{L}^{t(s)}_{\mathscr{D}_{t(s)}}\right) \cap (-\infty, 0] = \emptyset$ *then* $\left(\mathcal{K}_{\lambda(s),t(s)}, \mathcal{G}_{t(s)}\right)$ *is a Fredholm pair of Lagrangian planes in* $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, *moreover,*

$$\dim\left(\mathcal{K}_{\lambda(s),t(s)} \cap \mathcal{G}_{t(s)}\right) = \dim\ker\left(\mathcal{L}^{t(s)}_{\mathscr{D}_{t(s)}} - \lambda(s)\right), s \in \Sigma, \tag{3.117}$$

iii) the path $s \mapsto \mathcal{K}_{\lambda(s),t(s)}$ *on* $\Sigma = \cup_{j=1}^4 \Sigma_j$ *is continuous and is contained in the space*

$$C^1\big(\Sigma_k, \Lambda(H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega))\big), 1 \le k \le 4.$$

Proof. If $s \in \Sigma$ then by Theorem 3.8 the subset $\mathcal{G}_{t(s)} \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ is Lagrangian. The fact that $\mathcal{K}_{\lambda(s),t(s)}$, $s \in \Sigma$, is Lagrangian and part *iii)* were proved in [51, Proposition 3.5].

It remains to prove part *ii)*. Let $s \in \Sigma$ be fixed. In order to prove (3.117), we will firstly show an auxiliary result: The map

$$\operatorname{tr}_{\mathcal{L}^{t(s)}} : \ker\left(\mathcal{L}^{t(s)}_{\mathscr{D}_{t(s)}} - \lambda(s)\right) \to \mathcal{K}_{\lambda(s),t(s)} \cap \mathcal{G}_{t(s)}, \tag{3.118}$$

is one-to-one and onto. Indeed, it is injective since

$$\text{If } \operatorname{tr}_{\mathcal{L}^{t(s)}} u = 0 \text{ and } u \in \ker\left(\mathcal{L}^{t(s)}_{\mathscr{D}_{t(s)}} - \lambda(s)\right) \text{ then } u = 0,$$

due to the unique continuation principle (cf. [96, Theorem 3.2.2]). Next, we prove that (3.118) is surjective. To this end, let us fix an arbitrary $(\phi, \psi) \in \mathcal{K}_{\lambda(s),t(s)} \cap \mathcal{G}_{t(s)}$. Since $(\phi, \psi) \in \mathcal{K}_{\lambda(s),t(s)}$ there exists $u \in \mathcal{D}^1_{\mathcal{L}^{t(s)}}(\Omega)$ such that $\operatorname{tr}_{\mathcal{L}^{t(s)}} u = (\phi, \psi)$. It suffices to show that $u \in \mathscr{D}_{t(s)}$. Recall that

$$\operatorname{tr}_{\mathcal{L}^{t(s)}} u \in \mathcal{G}_{t(s)} = \overline{\operatorname{tr}_{\mathcal{L}^{t(s)}}\left(\mathscr{D}_{t(s)}\right)},$$

thus, there exists a sequence $u_n \in \mathscr{D}_{t(s)},\ n \geq 1$, such that

$$\operatorname{tr}_{\mathcal{L}^{t(s)}} u_n = \left(\gamma_{{}_D} u_n, \gamma_{{}_N}^{\mathcal{L}^{t(s)}} u_n\right) \to \operatorname{tr}_{\mathcal{L}^{t(s)}} u,\ n \to \infty,$$

in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. For arbitrary $v \in \mathscr{D}_{t(s)}$ and all $n \geq 1$, one has

$$\begin{aligned}
&\omega\left((\gamma_{{}_D} u_n, \gamma_{{}_N}^{\mathcal{L}^{t(s)}} u_n), (\gamma_{{}_D} v, \gamma_{{}_N}^{\mathcal{L}^{t(s)}} v)\right) \\
&\quad = \overline{\langle \gamma_{{}_N}^{\mathcal{L}^{t(s)}} v, \gamma_{{}_D} u_n \rangle}_{-1/2} - \langle \gamma_{{}_N}^{\mathcal{L}^{t(s)}} u_n, \gamma_{{}_D} v \rangle_{-1/2} \\
&\quad = \langle \mathcal{L}^{t(s)} u_n, v \rangle_{L^2(\Omega)} - \langle u_n, \mathcal{L}^{t(s)} v \rangle_{L^2(\Omega)} = 0,
\end{aligned} \tag{3.119}$$

since $\mathcal{L}^{t(s)}_{\mathscr{D}_{t(s)}}$ is self-adjoint. Passing to the limit in (3.119), one obtains

$$\omega\left((\gamma_{{}_D} u, \gamma_{{}_N}^{\mathcal{L}^{t(s)}} u), (\gamma_{{}_D} v, \gamma_{{}_N}^{\mathcal{L}^{t(s)}} v)\right) = 0, \text{ for all } v \in \mathscr{D}_{t(s)}. \tag{3.120}$$

By the second Green identity (3.17)

$$\omega\left((\gamma_{{}_D} u, \gamma_{{}_N}^{\mathcal{L}^{t(s)}} u), (\gamma_{{}_D} v, \gamma_{{}_N}^{\mathcal{L}^{t(s)}} v)\right) = \langle \mathcal{L}^{t(s)} u, v \rangle_{L^2(\Omega)} - \langle u, \mathcal{L}^{t(s)} v \rangle_{L^2(\Omega)}. \tag{3.121}$$

From (3.120) and (3.121) one infers

$$\langle \mathcal{L}^{t(s)}u, v\rangle_{L^2(\Omega)} - \langle u, \mathcal{L}^{t(s)}v\rangle_{L^2(\Omega)} = 0, \tag{3.122}$$

for all $v \in \mathscr{D}_{t(s)}$. Combining (3.122) and the fact that $\mathcal{L}^{t(s)}_{\mathscr{D}_{t(s)}}$ is self-adjoint we conclude that $u \in \mathscr{D}_{t(s)}$ and thus that map (3.118) is onto.

In order to show that the pair $\big(\mathcal{K}_{\lambda(s),t(s)}, \mathcal{G}_{t(s)}\big)$ is Fredholm we need to check the following assertions,

$$\dim\big(\mathcal{K}_{\lambda(s),t(s)} \cap \mathcal{G}_{t(s)}\big) < \infty \text{ and } \operatorname{codim}\big(\mathcal{K}_{\lambda(s),t(s)} + \mathcal{G}_{t(s)}\big) < \infty, \tag{3.123}$$

$$\mathcal{K}_{\lambda(s),t(s)} + \mathcal{G}_{t(s)} \text{ is closed in } H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega). \tag{3.124}$$

The first inequality in (3.123) follows from the fact that $\mathcal{L}^{t(s)}_{\mathscr{D}_{t(s)}} - \lambda(s)$ is a Fredholm operator and that map (3.118) is bijective. To show the second one, we observe that

$$\begin{aligned} &\operatorname{codim}\big(\mathcal{K}_{\lambda(s),t(s)} + \mathcal{G}_{t(s)}\big) = \dim\big(\mathcal{K}_{\lambda(s),t(s)} + \mathcal{G}_{t(s)}\big)^{\circ} \\ &= \dim\big((\mathcal{K}_{\lambda(s),t(s)})^{\circ} \cap (\mathcal{G}_{t(s)})^{\circ}\big) = \dim\big(\mathcal{K}_{\lambda(s),t(s)} \cap \mathcal{G}_{t(s)}\big) < \infty, \end{aligned} \tag{3.125}$$

because both $\mathcal{K}_{\lambda(s),t(s)}$ and $\mathcal{G}_{t(s)}$ are Lagrangian subspaces. Next we show (3.124). Let us notice that

$$\begin{aligned} \mathbf{K}_{\lambda(s),t(s)} + \mathscr{D}_{t(s)} = \big\{u \in \mathcal{D}^1_{\mathcal{L}^{t(s)}}(\Omega) : \mathcal{L}^{t(s)}u - \lambda(s)x = \mathcal{L}^{t(s)}v - \lambda(s)v, \\ \text{in } (H^1_0(\Omega))^* \text{for some } v \in \mathscr{D}_{t(s)}\big\} \end{aligned} \tag{3.126}$$

(a similar equality first appeared in [29, Proposition 3.5] in the context of strong kernel of $\mathcal{L}^{t(s)} - \lambda(s)$). Utilizing (3.126) and the fact that the operator $\mathcal{L}^{t(s)} - \lambda(s)$ is Fredholm we will show that $\mathbf{K}_{\lambda(s),t(s)} + \mathscr{D}_{t(s)}$ is closed in $\mathcal{D}^1_{\mathcal{L}^{t(s)}}(\Omega)$. Indeed, if

$$u_n \in \big(\mathbf{K}_{\lambda(s),t(s)} + \mathscr{D}_{t(s)}\big), n \geq 1, \text{ and } u_n \to u \text{ in } \mathcal{D}^1_{\mathcal{L}^{t(s)}}(\Omega),$$

then

$$\mathcal{L}^{t(s)}u_n - \lambda(s)u_n = \mathcal{L}^{t(s)}v_n - \lambda(s)v_n, \text{ for some } v_n \in \mathscr{D}_{t(s)},\ n \geq 1. \tag{3.127}$$

Since $\mathcal{L}^{t(s)} \in \mathcal{B}\left(\mathcal{D}^1_{\mathcal{L}^{t(s)}}(\Omega), L^2(\Omega)\right)$ then

$$\mathcal{L}^{t(s)}u_n - \lambda(s)u_n \to \mathcal{L}^{t(s)}u - \lambda(s)u,\ n \to \infty,\ \text{in } L^2(\Omega), \tag{3.128}$$

moreover, since the operator $\mathcal{L}^{t(s)}_{\mathscr{D}_{t(s)}} - \lambda(s)$ is Fredholm, one has

$$\mathcal{L}^{t(s)}v_n - \lambda(s)v_n \to \mathcal{L}^{t(s)}v - \lambda(s)v,\ n \to \infty,\ \text{in } L^2(\Omega), \tag{3.129}$$

for some $v \in \mathscr{D}_{t(s)}$. Combining (3.127), (3.128), (3.129), one obtains

$$\mathcal{L}^{t(s)}u - \lambda(s)u = \mathcal{L}^{t(s)}v - \lambda(s)v,$$

hence, $u \in \left(\mathbf{K}_{\lambda(s),t(s)} + \mathscr{D}_{t(s)}\right)$. Next, the linear operator

$$\operatorname{tr}_{\mathcal{L}^{t(s)}} : \left(\mathbf{K}_{\lambda(s),t(s)} + \mathscr{D}_{t(s)}\right) \to H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega),$$

acting from the Banach space $\left(\mathbf{K}_{\lambda(s),t(s)} + \mathscr{D}_{t(s)}\right)$ equipped with $\mathcal{D}^1_{\mathcal{L}^{t(s)}}(\Omega)-$norm to the Hilbert space $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, is bounded. Furthermore, its range

$$\operatorname{tr}_{\mathcal{L}^{t(s)}}\left(\mathbf{K}_{\lambda(s),t(s)} + \mathscr{D}_{t(s)}\right) = \left(\mathcal{K}_{\lambda(s),t(s)} + \operatorname{tr}_{\mathcal{L}^{t(s)}}(\mathscr{D}_{\mathcal{L}^{t(s)}})\right)$$

has finite codimension. Therefore, by [81, Corollary 2.3], the subset

$$\mathcal{K}_{\lambda(s),t(s)} + \operatorname{tr}_{\mathcal{L}^{t(s)}}(\mathscr{D}_{\mathcal{L}^{t(s)}}) \subset H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$$

is closed. Hence, $\left(\mathcal{K}_{\lambda(s),t(s)} + \mathcal{G}_{t(s)}\right)$ is also closed. ∎

3.3.2 The Maslov and Morse indices

We are ready to state the principal result of this section. In the following theorem we consider a one-parameter family of self-adjoint extensions of uniformly elliptic operators. One of our main assumptions is that each operator from this family is semibounded from below. This assumption is satisfied for all standard self-adjoint extensions such as the Dirichlet, Neumann, Robin, and periodic Laplace operators. However, it is not evident that all self-adjoint extensions of an elliptic operator are necessarily semibounded from below (cf. (3.130)). Next, we notice that the relations between the Maslov and Morse indices have been extensively studied by many authors cf., e.g., [29, Theorem 5.1], [57, Theorem 2.4, Theorem 2.5], [34, Theorem 1.5], [35, Theorem 4.5.4], [50, Theorem 1.3, Theorem 1.4], [51, Theorem 1], [91, Theorem 1.5]. The work in this direction was originated in [29], where the authors considered the Lagrangian planes formed by the abstract traces of strong solutions (i.e., by the abstract traces of the kernels of adjoint operators) assuming that the domain of the adjoint operator is fixed. Later this assumption was relaxed in a series of works [34, 35, 36, 37] by considering only those extensions whose domains are contained in a fixed subspace. We, on the other hand, consider the Lagrangian planes formed by the weak traces of weak solutions which allows us to reduce regularity assumptions for the domains of self-adjoint extensions.

Theorem 3.18. *Assume Hypothesis* 3.16 *and recall the differential expressions* (3.104)*. Let $\mathscr{D}_t \subset \mathcal{D}^1_{\mathcal{L}^t}(\Omega)$, $t \in \mathcal{I}$, and assume that the linear operator $\mathcal{L}^t_{\mathscr{D}_t}$ acting in $L^2(\Omega)$ and given by*

$$\mathcal{L}^t_{\mathscr{D}_t} u := \mathcal{L}^t u,\ \ u \in \operatorname{dom}\left(\mathcal{L}^t_{\mathscr{D}_t}\right) := \mathscr{D}_t,$$

is self-adjoint with the property

$$\operatorname{Spec}_{ess}\left(\mathcal{L}^t_{\mathscr{D}_t}\right) \cap (-\infty, 0] = \emptyset,\ \text{ for all } t \in \mathcal{I}.$$

Assume further that there exists $\lambda_\infty < 0$, such that

$$\ker\left(\mathcal{L}^t_{\mathscr{D}_t} - \lambda\right) = \{0\},\ \text{ for all } \lambda \leq \lambda_\infty, t \in \mathcal{I}. \tag{3.130}$$

Suppose, finally, that the path

$$t \mapsto \mathcal{G}_t := \overline{\mathrm{tr}_{\mathcal{L}^t}\left(\mathscr{D}_t\right)},\ t \in \mathcal{I},$$

is contained in $C\left(\mathcal{I}, \Lambda\left(H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)\right)\right)$.

Then

$$\mathrm{Mor}\left(\mathcal{L}^{\alpha}_{\mathscr{D}_\alpha}\right) - \mathrm{Mor}\left(\mathcal{L}^{\beta}_{\mathscr{D}_\beta}\right) = \mathrm{Mas}\left((\mathcal{K}_{0,t}, \mathcal{G}_t)|_{t\in\mathcal{I}}\right), \tag{3.131}$$

where the Lagrangian plane $\mathcal{K}_{0,t}$ *is defined by* (3.114).

Proof. We will compute the Maslov index of the path $s \mapsto (\mathcal{K}_{\lambda(s),t(s)}, \mathcal{G}_{t(s)})$ on each interval $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ parameterizing the respective sides of the boundary of the square $[\lambda_\infty, 0] \times [\alpha, \beta]$, see Figure 1, and use a catenation argument to determine the Maslov index on Σ. To this end we split the proof into four parts.

Step 1. In this step we show that

$$\mathrm{Mas}\left(\mathcal{K}_{\alpha,\lambda(s)}|_{s\in\Sigma_1}, \mathcal{G}_\alpha\right) = -\,\mathrm{Mor}\left(\mathcal{L}_\alpha\right). \tag{3.132}$$

The proof goes along the lines of the argument in [29], where a variant of (3.132) is established in the context of strong kernels, abstract trace maps, and fixed domains of the maximal operators. In order to obtain (3.132) in our setting, we intend to prove that each crossing on Σ_1 is negative (hence, non-degenerate), and use (3.99) to verify that geometric multiplicities of negative eigenvalues of $\mathcal{L}^{\alpha}_{\mathscr{D}_\alpha}$ add up to minus the Maslov index. Let $s_* \in (\lambda_\infty, 0)$ be a conjugate point, i.e. $\mathcal{K}_{\lambda(s_*),\alpha} \cap \mathcal{G}_\alpha \neq \{0\}$. By Theorem 3.17 part (*iii*) the map $s \mapsto \mathcal{K}_{\lambda(s),\alpha}$ is contained in $C^1\big((\lambda_\infty, 0), \Lambda(H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega))\big)$. Then there exists a small neighbourhood $\Sigma_{s_*} \subset (\lambda_\infty, 0)$ of s_* and a family of operators R_{s+s_*} so that

$$(s + s_*) \mapsto R_{(s+s_*)} \text{ in } C^1\big(\Sigma_{s_*}, \mathcal{B}(\mathcal{K}_{\lambda(s_*),\alpha}, (\mathcal{K}_{\lambda(s_*),\alpha})^{\perp})\big),\ R_{s_*} = 0, \tag{3.133}$$

and

$$\mathcal{K}_{\lambda(s),\alpha} = \{(\phi,\psi) + R_{s+s_*}(\phi,\psi)\big|(\phi,\psi) \in \mathcal{K}_{\lambda(s_*),\alpha}\} \text{ for all } \ (s+s_*) \in \Sigma_{s_*}, \qquad (3.134)$$

see [50, Lemma 3.8]. Let us fix $(\phi_0,\psi_0) \in \mathcal{K}_{\lambda(s_*),\alpha}$ and consider the family

$$(\phi_s,\psi_s) := (\phi_0,\psi_0) + R_{(s+s_*)}(\phi_0,\psi_0) \text{ with small } |s|.$$

Since $(\phi_s,\psi_s) \in \mathcal{K}_{\lambda(s),\alpha}$, by the unique continuation principle (cf. (3.19)), there exists a unique $u_s \in \mathbf{K}_{\lambda(s+s_*),\alpha} \subset \mathcal{D}^1_{\mathcal{L}^\alpha}(\Omega)$ such that

$$\operatorname{tr}_{\mathcal{L}^\alpha} u_s = (\phi_s,\psi_s) \text{ for small } |s|.$$

Next, using the second Green identity (3.17) and (3.28), we calculate:

$$\begin{aligned}
\omega\big((\phi_0,\psi_0), R_{(s+s_*)}(\phi_0,\psi_0)\big) &= \overline{\langle\psi_s,\phi_0\rangle}_{-1/2} - \langle\psi_0,\phi_s\rangle_{-1/2}\\
&= \langle\mathcal{L}^\alpha u_0, u_s - u_0\rangle_{L^2(\Omega)} - \langle u_0, \mathcal{L}^\alpha(u_s - u_0)\rangle_{L^2(\Omega)}\\
&= \langle(\mathcal{L}^\alpha - \lambda(s_*))u_0, u_s - u_0\rangle_{L^2(\Omega)} - \langle u_0, (\mathcal{L}^\alpha - \lambda(s_*))(u_s - u_0)\rangle_{L^2(\Omega)}\\
&= -\langle u_0, (\mathcal{L}^\alpha - \lambda(s_*))u_s\rangle_{L^2(\Omega)}\\
&= -\langle u_0, (\mathcal{L}^\alpha - \lambda(s+s_*))u_s\rangle_{L^2(\Omega)} + \langle u_0, (\lambda(s_*) - \lambda(s+s_*))u_s\rangle_{L^2(\Omega)}\\
&= -\langle u_0, s u_s\rangle_{L^2(\Omega)}.
\end{aligned}$$

The mapping $s \mapsto u_s \in H^1(\Omega)$ is continuous at 0, since, using the standard elliptic estimate in Lemma 3.19 given below,

$$\begin{aligned}
\|u_s - u_0\|_{H^1(\Omega)} &\le C\big\|\operatorname{tr}_{\mathcal{L}^\alpha}(u_s - u_0)\big\|_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)}\\
&= C\|(\phi_s - \phi_0, \psi_s - \psi_0)\|_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)},
\end{aligned} \qquad (3.135)$$

where $C > 0$ does not depend on s. We proceed by evaluating the crossing form from Definition 3.12 *(ii)*,

$$\mathcal{Q}_{s_*,\mathcal{G}_\alpha}\left((\phi_0,\psi_0),(\phi_0,\psi_0)\right) := \frac{d}{ds}\omega\big((\phi_0,\psi_0), R_{(s+s_*)}(\phi_0,\psi_0)\big)\big|_{s=0}$$

$$= \lim_{s\to 0} \frac{\omega\left((\phi_0,\psi_0), R_{(s+s_*)}(\phi_0,\psi_0)\right)}{s} = \lim_{s\to 0} \frac{-\langle u_0, s u_s\rangle_{L^2(\Omega)}}{s} = -\|u_0\|^2_{L^2(\Omega)}.$$

Therefore, the crossing form is negative definite at all conjugate points on $[\lambda_\infty, 0]$ and, using (3.99), one obtains

$$\text{Mas}\left(\mathcal{K}_{\lambda(s),\alpha}|_{s\in\Sigma_1}, \mathcal{G}_\alpha\right) = -n_-\left(\mathcal{Q}_{\lambda_\infty,\mathcal{G}_\alpha}\right) + \sum_{\substack{\lambda_\infty<s<0:\\ \mathcal{K}_{\lambda(s),\alpha}\cap\mathcal{G}_\alpha\neq\{0\}}} \text{sign}\ \mathcal{Q}_{s,\mathcal{G}_\alpha}$$
$$+ n_+(\mathcal{Q}_{0,\mathcal{G}_\alpha}) = -\sum_{\lambda_\infty\leq s<0} \dim\ker\left(\mathcal{L}^\alpha_{\mathscr{D}_\alpha} - \lambda(s)\right) = -\operatorname{Mor}\left(\mathcal{L}^\alpha_{\mathscr{D}_\alpha}\right), \tag{3.136}$$

where we employed $n_+\left(\mathcal{Q}_{0,\mathcal{G}_\alpha}\right) = 0$, and the fact that there are no crossings to the left of λ_∞.

Step 2. A similar computation can be carried out in case $s \in \Sigma_3$, leading to the analog of (3.132),

$$\text{Mas}\left(\mathcal{K}_{\alpha,\lambda(s)}|_{s\in\Sigma_3}, \mathcal{G}_\alpha\right) = \operatorname{Mor}\left(\mathcal{L}_\alpha\right). \tag{3.137}$$

Step 3. Since, by assumptions, $\ker(\mathcal{L}^t_{\mathscr{D}_t} - \lambda) = \{0\}$ for all $\lambda \leq \lambda_\infty, t \in \mathcal{I}$, there are no crossings on Σ_4, therefore, the Maslov index vanishes on this interval

$$\text{Mas}\left((\mathcal{K}_{t(s),\lambda_\infty}, \mathcal{G}_{t(s)})|_{s\in\Sigma_4}\right) = 0. \tag{3.138}$$

Step 4. In this step we will combine (3.132), (3.137), (3.138), and the homotopy invariance of the Maslov index to obtain (3.131). Since the curve Γ, cf., (3.108), can be continuously contracted to a point, one has

$$\text{Mas}\left((\mathcal{K}_{t(s),\lambda(s)}, \mathcal{G}_{t(s)})|_{s\in\Sigma}\right) = 0. \tag{3.139}$$

On the other hand, due to the catenation property of the Maslov index,

$$\text{Mas}\left((\mathcal{K}_{t(s),\lambda(s)}, \mathcal{G}_{t(s)})|_{s\in\Sigma}\right) = \text{Mas}\left((\mathcal{K}_{t(s),\lambda(s)}, \mathcal{G}_{t(s)})|_{s\in\Sigma_1}\right)$$
$$+ \text{Mas}\left((\mathcal{K}_{t(s),\lambda(s)}, \mathcal{G}_{t(s)})|_{s\in\Sigma_2}\right) + \text{Mas}\left((\mathcal{K}_{t(s),\lambda(s)}, \mathcal{G}_{t(s)})|_{s\in\Sigma_3}\right) \tag{3.140}$$
$$+ \text{Mas}\left((\mathcal{K}_{t(s),\lambda(s)}, \mathcal{G}_{t(s)})|_{s\in\Sigma_4}\right).$$

Combining (3.132), (3.137), (3.138), (3.139), (3.140), one obtains (3.131). ∎

Lemma 3.19. *Assume Hypothesis* 3.16*. Then there exists a positive constant* $C > 0$ *independent of* t *such that if* $u \in H^1(\Omega)$ *is a weak solutions to* $\mathcal{L}^t u = 0, t \in \mathcal{I}$ *then*

$$\|u\|_{H^1(\Omega)} \leq C \big\| \operatorname{tr}_{\mathcal{L}^t} u \big\|_{H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)}, \text{ for all } t \in \mathcal{I}. \tag{3.141}$$

Proof. Recall the function space (3.13). For arbitrary $u \in \mathcal{D}^1_{\mathcal{L}^s}(\Omega)$, $s \in \mathcal{I}$, one has

$$\mathfrak{l}^s[u,u] = \langle \mathcal{L}^s u, u \rangle_{L^2(\Omega)} + \langle \gamma_{{}_N}^{\mathcal{L}^s,1} u, \gamma_{{}_D} u \rangle_{H^{-1/2}(\partial\Omega)}, \tag{3.142}$$

where

$$\begin{aligned} \mathfrak{l}^s[u,v] = &\sum_{j,k=1}^{n} \langle a^s_{jk} \partial_k u, \partial_j v \rangle_{L^2(\Omega)} + \sum_{j=1}^{n} \langle a^s_j \partial_j u, v \rangle_{L^2(\Omega)} \\ &+ \sum_{j=1}^{n} \langle u, a^s_j \partial_j v \rangle_{L^2(\Omega)} + \langle a^s u, v \rangle_{L^2(\Omega)}, \ u, v \in H^1(\Omega), s \in \mathcal{I}. \end{aligned} \tag{3.143}$$

Our immediate objective is to show that the inequality

$$\|u\|^2_{H^1(\Omega)} \leq C \big(\|u\|^2_{L^2(\Omega)} + \|\mathcal{L}^s u\|^2_{L^2(\Omega)} + \| \operatorname{tr}_{\mathcal{L}^s} u\|^2_{H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)} \big) \tag{3.144}$$

holds for some $C > 0$ independent of s, and all $s \in \mathcal{I}$. To this end, we first notice that by the elliptic property (3.101), one has

$$\sum_{j,k=1}^{n} \langle a^s_{jk} \partial_k u, \partial_j u \rangle_{L^2(\Omega)} \geq c \|\nabla u\|^2_{L^2(\Omega)}. \tag{3.145}$$

Second, using (3.142) and (3.143) we obtain

$$\left| \sum_{j,k=1}^{n} \langle a^s_{jk} \partial_k u, \partial_j u \rangle_{L^2(\Omega)} \right| \leq \left| \sum_{j=1}^{n} \langle a^s_j \partial_j u, u \rangle_{L^2(\Omega)} \right| + \left| \sum_{j=1}^{n} \langle u, a^s_j \partial_j u \rangle_{L^2(\Omega)} \right| \tag{3.146}$$

$$+ \left| \langle a^s u, u \rangle_{L^2(\Omega)} \right| + \left| \langle \mathcal{L}^s u, u \rangle_{L^2(\Omega)} \right| + \left| \langle \gamma_{{}_N}^{\mathcal{L}^s,1} u, \gamma_{{}_D} u \rangle_{H^{-1/2}(\partial\Omega)} \right|. \tag{3.147}$$

Next, the Cauchy–Schwarz inequality together with (3.146), (3.147) yield

$$\left| \sum_{j,k=1}^{n} \langle a^s_{jk} \partial_k u, \partial_j u \rangle_{L^2(\Omega)} \right| \leq 2 \sup_{\substack{s \in [\alpha,\beta], x \in \overline{\Omega}, \\ 1 \leq j \leq n}} \|a^s_j(x)\|_{\mathbb{C}^m} \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \tag{3.148}$$

$$+\sup_{s\in[\alpha,\beta],x\in\overline{\Omega}}\|a^s(x)\|_{\mathbb{C}^{m\times m}}\|u\|^2_{L^2(\Omega)}+\|\mathcal{L}^s u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \tag{3.149}$$

$$+\|\gamma_N^{\mathcal{L}^s,1}u\|_{H^{-1/2}(\partial\Omega)}\|\gamma_D u\|_{H^{1/2}(\partial\Omega)}. \tag{3.150}$$

Finally, the inequalities

$$\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}\leq\frac{\|u\|^2_{L^2(\Omega)}}{2\varepsilon^2}+\frac{\varepsilon^2\|\nabla u\|^2_{L^2(\Omega)}}{2},\text{ with } \varepsilon>0 \text{ small enough,}$$
$$\|\mathcal{L}^s u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}\leq\frac{1}{2}\left(\|u\|^2_{L^2(\Omega)}+\|\mathcal{L}^s u\|^2_{L^2(\Omega)}\right),$$
$$\|\gamma_N^{\mathcal{L}^s,1}u\|_{H^{-1/2}(\partial\Omega)}\|\gamma_D u\|_{H^{1/2}(\partial\Omega)}\leq\frac{1}{2}\left(\|\gamma_N^{\mathcal{L}^s,1}u\|^2_{H^{-1/2}(\partial\Omega)}+\|\gamma_D u\|^2_{H^{1/2}(\partial\Omega)}\right),$$

together with (3.148)-(3.150) imply

$$\begin{aligned}\left|\sum_{j,k=1}^n\langle a^s_{jk}\partial_k u,\partial_j u\rangle_{L^2(\Omega)}\right|&\leq C_1\big(\|u\|^2_{L^2(\Omega)}+\|\mathcal{L}^s u\|^2_{L^2(\Omega)}\\ &+\|\gamma_N^{\mathcal{L}^s}u\|^2_{H^{-1/2}(\partial\Omega)}+\|\gamma_D u\|^2_{H^{1/2}(\partial\Omega)}\big)+\frac{c}{2}\|\nabla u\|^2_{L^2(\Omega)}, s\in\mathcal{I},\end{aligned} \tag{3.151}$$

where $0<C_1=C_1(a,a_j,n,\Omega)$, and $c>0$ is from (3.101). Combining (3.145) and (3.151), one infers (3.144).

We intend to derive from (3.144) yet a stronger inequality,

$$\|u\|^2_{H^1(\Omega)}\leq C\left(\|\mathcal{L}^s u\|^2_{L^2(\Omega)}+\|\operatorname{tr}_{\mathcal{L}^s}u\|^2_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)}\right),\ s\in\mathcal{I}, \tag{3.152}$$

which trivially implies (3.141). We prove (3.152) by contradiction: Assume that there exist

$$s_n\in\Sigma, u_n\in\mathcal{D}^1_{\mathcal{L}^{s_n}}(\Omega), n\geq 1,$$

such that

$$\|u_n\|^2_{H^1(\Omega)}>n\left(\|\mathcal{L}^{s_n}u_n\|^2_{L^2(\Omega)}+\|\operatorname{tr}_{\mathcal{L}^{s_n}}u_n\|^2_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)}\right),\ n\geq 1. \tag{3.153}$$

Without loss of generality we may assume that

$$s_n\to s_0, n\to\infty,\text{ and that } \|u_n\|_{L^2(\Omega)}=1, n\geq 1.$$

It follows from (3.144) and (3.153) that the sequence $\{u_n : n \geq 1\}$ is bounded in $H^1(\Omega)$, and therefore that

$$\|\mathcal{L}^{s_n} u_n\|_{L^2(\Omega)} \to 0 \text{ and } \| \operatorname{tr}_{\mathcal{L}^{s_n}} u_n\|_{H^{1/2}(\partial\Omega)\times H^{-1/2}(\partial\Omega)} \to 0, \text{ as } n \to \infty. \tag{3.154}$$

Passing to a subsequence if necessary, we have the weak convergence

$$u_n \rightharpoonup u_0, \text{ as } n \to \infty \text{ in } H^1(\Omega). \tag{3.155}$$

Since $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, we conclude that $u_n \to u_0$, $n \to \infty$ in $L^2(\Omega)$. We claim that

$$u_0 \in \mathcal{D}^1_{\mathcal{L}^{s_0}}(\Omega),\ \mathcal{L}^{s_0} u_0 = 0, \tag{3.156}$$

$$\operatorname{tr}_{\mathcal{L}^{s_0}} u_0 = 0. \tag{3.157}$$

Granted (3.156),(3.157), we notice that the unique continuation principle yields $u_0 = 0$, which in turn, contradicts the fact that $\|u_0\|_{L^2(\Omega)} = 1$, and finishes the proof of (3.152).

It remains to prove the claim. First, we prove (3.156). For arbitrary $\varphi \in C_0^\infty(\Omega)$ the second Green identity (3.17) yields

$$\langle u_n, \mathcal{L}^{s_n}\varphi\rangle_{L^2(\Omega)} = \langle \mathcal{L}^{s_n} u_n, \varphi\rangle_{L^2(\Omega)},\ n \geq 1. \tag{3.158}$$

On the other hand, since $\gamma_{{}_D}\varphi = 0, \gamma_{{}_N}^{\mathcal{L}^{s_n}}\varphi = 0$, the first Green identity (3.16) yields

$$\langle \mathcal{L}^{s_n} u_n, \varphi\rangle_{L^2(\Omega)} = \mathfrak{l}^{s_n}[u_n, \varphi],\ n \geq 1. \tag{3.159}$$

Furthermore, using the first limit in (3.154) and (3.155) we obtain

$$\mathfrak{l}^{s_n}[u_n, \varphi] \to \mathfrak{l}^{s_0}[u_0, \varphi] \text{ and } \langle \mathcal{L}^{s_n} u_n, \varphi\rangle_{L^2(\Omega)} \to 0, \text{ as } n \to \infty. \tag{3.160}$$

Combining (3.159) and (3.160) we obtain

$$0 = \mathfrak{l}^{s_0}[u_0, \varphi], \text{ for arbitrary } \varphi \in C_0^\infty(\Omega), \tag{3.161}$$

moreover, by the first Green identity

$$\mathfrak{l}^{s_0}[u_0, \varphi] = \langle u_0, \mathcal{L}^{s_0}\varphi\rangle_{L^2(\Omega)} \text{ for arbitrary } \varphi \in C_0^\infty(\Omega).$$

Hence, (3.156) holds.

It remains to check (3.157). First, the equality $\gamma_{_D} u_0 = 0$ holds since, by using the second limit in (3.154),

$$\gamma_{_D} u_n \rightharpoonup \gamma_{_D} u_0 \text{ and } \gamma_{_D} u_n \to 0, \text{as } n \to \infty \text{ in } H^{1/2}(\partial\Omega).$$

Next, by the first Green identity

$$\mathfrak{l}^{s_n}[u_n, f] = \langle \mathcal{L}^{s_n} u_n, f\rangle_{L^2(\Omega)} + \langle \gamma_{_N}^{\mathcal{L}^{s_n}} u_n, \gamma_{_D} f\rangle_{H^{-1/2}(\partial\Omega)}, \ n \geq 1, \tag{3.162}$$

for arbitrary $f \in H^1(\Omega)$. The left hand-side of (3.162) tends to $\mathfrak{l}^{s_0}[u_0, f]$ (due to the weak convergence of u_n), whereas by (3.154), the right hand-side converges to 0, as $n \to 0$, implying

$$\mathfrak{l}^{s_0}[u_0, f] = 0, \text{ for all } f \in H^1(\Omega). \tag{3.163}$$

The first Green identity and (3.156) yield

$$0 = \mathfrak{l}^{s_0}[u_0, f] = \langle \gamma_{_N}^{\mathcal{L}^{s_0}} u_0, \gamma_{_D} f\rangle_{H^{-1/2}(\partial\Omega)}, \text{ for all } f \in H^1(\Omega). \tag{3.164}$$

Finally, since $\gamma_{_D} : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is onto, (3.164) implies $\gamma_{_N}^{\mathcal{L}^{s_0}} u_0 = 0$ as required.

∎

In the remaining part of this section we illustrate several applications of the general formula (3.131), that is, we indicate how several known and some unknown results can be derived from this formula.

3.3.3 The spectral flow and the Maslov index

Assume hypotheses of Theorem 3.18. Then the spectral flow through $\lambda = 0$ of the one-parameter operator family $\left\{\mathcal{L}^t_{\mathscr{D}_t}\right\}_{t=\alpha}^{\beta}$ is defined as follows: There exists a partition $\alpha = t_0 < t_1 < \cdots < t_N = \beta$, and N intervals $[a_\ell, b_\ell]$, $a_\ell < 0 < b_\ell$, $1 \le \ell \le N$, such that

$$a_\ell, b_\ell \notin \operatorname{Spec}\left(\mathcal{L}^t_{\mathscr{D}_t}\right), \text{ for all } t \in [t_{\ell-1}, t_\ell],\ 1 \le \ell \le N. \tag{3.165}$$

The spectral flow through $\lambda = 0$ is defined by the formula

$$\operatorname{SpFlow}\left(\{\mathcal{L}^t_{\mathscr{D}_t}\}_{t=\alpha}^{\beta}\right) := \sum_{\ell=1}^{N} \sum_{a_\ell \le \lambda < 0} \left(\dim\ker\left(\mathcal{L}^{t_{\ell-1}}_{\mathscr{D}_{t_{\ell-1}}} - \lambda\right) - \dim\ker\left(\mathcal{L}^{t_\ell}_{\mathscr{D}_{t_\ell}} - \lambda\right)\right). \tag{3.166}$$

It can be shown that $\operatorname{SpFlow}\left(\left\{\mathcal{L}^t_{\mathscr{D}_t}\right\}_{t=a}^{b}\right)$ does not depend on the choice of partition of the interval $[\alpha, \beta]$ (cf., [35, Appendix]). In fact, since $\{\mathcal{L}^t_{\mathscr{D}_t}\}_{t=a}^{b}$ is uniformly bounded from below (with lower bound λ_∞), we can assume that $[\lambda_\infty, 0] \subset (a_\ell, b_\ell)$, $1 \le \ell \le N$. In this case (3.166) reads

$$\operatorname{SpFlow}\left(\left\{\mathcal{L}^t_{\mathscr{D}_t}\right\}_{t=\alpha}^{\beta}\right) = \sum_{\ell=1}^{N} \left(\operatorname{Mor}(\mathcal{L}^{t_{\ell-1}}_{\mathscr{D}_{t_{\ell-1}}}) - \operatorname{Mor}(\mathcal{L}^{t_\ell}_{\mathscr{D}_{t_\ell}})\right). \tag{3.167}$$

Combining (3.131) and (3.167), one obtains

$$\operatorname{SpFlow}\left(\left\{\mathcal{L}^t_{\mathscr{D}_t}\right\}_{t=\alpha}^{\beta}\right) = \operatorname{Mas}\left((\mathcal{K}_{0,t}, \mathcal{G}_t)|_{t\in\mathcal{I}}\right). \tag{3.168}$$

By rescaling, a similar formula holds for the spectral flow through any point $\lambda_0 \in \mathbb{R}$ with $\mathcal{K}_{0,t}$ replaced by $\mathcal{K}_{\lambda_0,t}$. Of course, relations between the spectral flow and the Maslov index of this type have been obtained in many important papers, cf., e.g., [34], [35], [36], [37] [42], [70], [106], [133], [142], [143], [153]. We stress, however, that in our case $\mathscr{D}_t \subset H^1(\Omega)$, $t \in [\alpha, \beta]$, and that we use the "usual" PDE trace operators as oppose to the abstract traces acting into the quotient spaces.

3.3.4 Spectra of elliptic operators on deformed domains and the Maslov index

In this subsection we revisit the main result, Theorem 1, from [51], and place it in the general framework of Theorem 3.18. Given a second order elliptic operator $\mathcal{L}$ on Ω and a one-parameter family of diffeomorphisms [51, Theorem 1] expresses the difference of Morse indices of $\mathcal{L}$ and its pullback in terms of the Maslov index.

Following [51], let $\Omega_0 \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, let $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n,\ t \in [0,1]$, be a one-parameter family of diffeomorphisms such that the mapping $t \mapsto \varphi_t$ is contained in $C^1([0,1], L^\infty(\Omega_0, \mathbb{R}^n))$, and $\varphi_0 = \mathrm{Id}_{\Omega_0}$. Let us denote

$$\Omega_t := \{\varphi_t(x) : x \in \Omega_0\},\ \Omega := \cup_{0\leq t\leq 1}\Omega_t.$$

Suppose that the coefficients of the second order differential operator satisfy

$$\mathcal{A} := \{a_{jk}\}_{1\leq j,k\leq n} \in C^\infty(\Omega, \mathbb{C}^{n\times n}), \mathcal{A} = \overline{\mathcal{A}^\top}, \tag{3.169}$$

$$a_{jk}(x)\xi_k\overline{\xi_j} \geq c\sum_{j=1}^n |\xi_j|^2,\ \text{for all } \xi = (\xi_j)_{j=1}^n \in \mathbb{C}^n, x \in \overline{\Omega}, \text{and some } c > 0, \tag{3.170}$$

$$b_j \in C^\infty(\Omega),\ 1 \leq j \leq n, B := (b_1, \cdots, b_n)^\top, \tag{3.171}$$

$$q \in C^\infty(\Omega), q(x) \in \mathbb{R},\ x \in \Omega, \tag{3.172}$$

and fix a subspace $\mathcal{X}_0$ such that

$$H_0^1(\Omega_0) \subset \mathcal{X}_0 \subset H^1(\Omega_0),\ \mathcal{X}_0 \text{ is a closed subset of } H^1(\Omega_0). \tag{3.173}$$

Using (3.169)-(3.172) we construct a family $\{\mathcal{L}^t_{\mathcal{X}_0}\}$ of operators in $L^2(\Omega_0)$ as follows. Let us define the one-parameter family of sesquilinear forms on $H^1(\Omega_t)$,

$$\begin{aligned}\mathfrak{l}^t[u,v] : = &\langle \mathcal{A}\nabla u, \nabla v\rangle_{L^2(\Omega_t)} + \langle B\nabla u, v\rangle_{L^2(\Omega_t)}\\ &+ \langle u, B\nabla v\rangle_{L^2(\Omega_t)} + \langle qu, v\rangle_{L^2(\Omega_t)},\ u, v \in H^1(\Omega_t), t \in [0,1].\end{aligned} \tag{3.174}$$

Changing variables in the right hand-side of (3.174), we arrive at

$$\begin{aligned}\widetilde{\mathfrak{l}}^t[\widetilde{u},\widetilde{v}] :&= \langle \mathcal{A}^t\nabla\widetilde{u},\nabla\widetilde{v}\rangle_{L^2(\Omega_0)} + \langle B^t\nabla\widetilde{u},\widetilde{v}\rangle_{L^2(\Omega_0)} \\ &\quad + \langle \widetilde{u}, B^t\nabla\widetilde{v}\rangle_{L^2(\Omega_0)} + \langle q^t\widetilde{u},\widetilde{v}\rangle_{L^2(\Omega_0)},\ \widetilde{u},\widetilde{v}\in H^1(\Omega_0), t\in[0,1],\end{aligned} \tag{3.175}$$

where the functions on Ω_0 satisfy

$$\mathcal{A}^t := \det(D\varphi_t)(D\varphi_t^\top)^{-1}[\mathcal{A}\circ\varphi_t](D\varphi_t)^{-1},\ B^t := \det(D\varphi_t)(D\varphi_t^\top)^{-1}[B\circ\varphi_t],$$

$$q^t := \det(D\varphi_t)q\circ\varphi_t,\ \widetilde{u} := u\circ\varphi_t,\ \widetilde{v} := v\circ\varphi_t,\ t\in[0,1].$$

If $t\in[0,1]$ then the form

$$\widetilde{\mathfrak{l}}^t : L^2(\Omega_0)\times L^2(\Omega_0)\to\mathbb{C},\ \operatorname{dom}(\widetilde{\mathfrak{l}}) := \mathcal{X}_0, \tag{3.176}$$

is closed and bounded from below. Hence, by [61, Theorem 2.8] there exists a unique self-adjoint operator $\mathcal{L}^t_{\mathcal{X}_0}$ acting in $L^2(\Omega_0)$, such that

$$\widetilde{\mathfrak{l}}^t[u,v] = \langle \mathcal{L}^t_{\mathcal{X}_0}u, v\rangle_{L^2(\Omega_0)} \text{ for all } u\in\operatorname{dom}(\mathcal{L}^t_{\mathcal{X}_0}),\ v\in\mathcal{X}_0. \tag{3.177}$$

Moreover, by [51, Lemma 4.1 and Proposition C.1] there exist positive constants C_1, C_2 such that

$$\widetilde{\mathfrak{l}}^t[f,f] \geq C_1\|f\|^2_{H^1(\Omega_0)} - C_2\|f\|^2_{L^2(\Omega_0)}. \tag{3.178}$$

Since the form domain of $\widetilde{\mathfrak{l}}^t$ is compactly embedded into $L^2(\Omega_0)$, the spectrum of $\mathcal{L}^t_{\mathcal{X}_0}$ is purely discrete. Since $\widetilde{\mathfrak{l}}^t, t\in[0,1]$, is uniformly bounded from below in $L^2(\Omega_0)$ there exists λ_∞ such that

$$\ker\left(\mathcal{L}^t_{\mathcal{X}_0} - \lambda\right) = \{0\} \text{ for all } \lambda\leq\lambda_\infty, t\in[0,1]. \tag{3.179}$$

We notice that $\mathcal{L}^t_{\mathcal{X}_0}$ is a self-adjoint extension of $\mathcal{L}^t_{min}$ given by the closure of

$$L^t u := -\operatorname{div}\mathcal{A}^t\nabla u + B^t\nabla u - \nabla\cdot\overline{B^t}u + q^t u, \operatorname{dom}(L^t) := C_0^\infty(\Omega_0). \tag{3.180}$$

Proposition 3.20. *Let $\mathcal{L}^t_{\mathcal{X}_0}$, $t \in [0,1]$ be the one-parameter family of self-adjoint operators defined by* (3.177)*. Then*

$$\begin{aligned}\overline{\mathrm{tr}_{\mathcal{L}^t}\left(\mathrm{dom}(\mathcal{L}^t_{\mathcal{X}_0})\right)} = \{(f,g) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) :\\ f \in \gamma_{{}_D}(\mathcal{X}_0), \langle g, \gamma_{{}_D} u\rangle_{-1/2} = 0 \text{ for all } u \in \mathcal{X}_0\},\ t \in [0,1]\end{aligned} \tag{3.181}$$

where the bar in the left-hand side denotes closure in $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. Hence, the right-hand side of (3.181) *is a Lagrangian plane.*

Proof. Let us fix $t \in [0,1]$. The right-hand side of (3.181) is isotropic. By Theorem 3.8, $\overline{\mathrm{tr}_{\mathcal{L}^t}(\mathrm{dom}(\mathcal{L}^t_{\mathcal{X}_0}))}$ is Lagrangian, hence, it suffices to show that $\mathrm{tr}_{\mathcal{L}^t}(\mathrm{dom}(\mathcal{L}^t_{\mathcal{X}_0}))$ is contained in the right-hand side of (3.181). The first Green identity yields

$$\widetilde{\mathfrak{l}}^t[u,v] = \langle \mathcal{L}^t u, v\rangle_{L^2(\Omega)} + \langle \gamma_{{}_N}^{\mathcal{L}^t} u, \gamma_{{}_D} v\rangle_{-1/2},\ u \in \mathcal{D}^1_{\mathcal{L}^t}(\Omega), v \in H^1(\Omega). \tag{3.182}$$

On the other hand,

$$\widetilde{\mathfrak{l}}^t[u,v] = \langle \mathcal{L}^t u, v\rangle_{L^2(\Omega_0)},\ \text{for all } u \in \mathrm{dom}(\mathcal{L}^t_{\mathcal{X}_0}), v \in \mathcal{X}_0. \tag{3.183}$$

Since $\mathrm{dom}(\mathcal{L}^t_{\mathcal{X}_0}) \subset \mathcal{D}^1_{\mathcal{L}^t}(\Omega)$ and $\mathcal{X}_0 \subset H^1(\Omega_0)$, one has

$$\langle \gamma_{{}_N}^{\mathcal{L}^t} u, \gamma_{{}_D} v\rangle_{-1/2} = 0,\ \text{for all } u \in \mathrm{dom}(\mathcal{L}^t_{\mathcal{X}_0}), v \in \mathcal{X}_0, \tag{3.184}$$

thus $(\gamma_{{}_D} u, \gamma_{{}_N}^{\mathcal{L}^t} u)$ is contained in the right-hand side of (3.181) whenever $u \in \mathrm{dom}(\mathcal{L}^t_{\mathcal{X}_0})$.

∎

The form $\widetilde{\mathfrak{l}}^1$ and the subspace $\mathcal{X}_0$ can be pulled back to Ω_1 (via $\varphi : \Omega_0 \to \Omega_1$), giving rise to a self-adjoint operator $\mathcal{L}^1_{\mathcal{X}_1}$ acting in $L^2(\Omega_1)$ and defined by

$$\mathfrak{l}^1[u,v] = \langle \mathcal{L}^1_{\mathcal{X}_1} u, v\rangle_{L^2(\Omega_1)},\ u \in \mathrm{dom}(\mathcal{L}^1_{\mathcal{X}_1}), v \in \mathcal{X}_1, \tag{3.185}$$

where $\mathcal{X}_1 := \{u \circ \varphi_1^{-1} : u \in \mathcal{X}_0\}$. Employing min-max type argument one can show that

$$\mathrm{Mor}(\mathcal{L}^1_{\mathcal{X}_1}) = \mathrm{Mor}(\mathcal{L}^1_{\mathcal{X}_0}). \tag{3.186}$$

Finally, let us introduce the path of Lagrangian planes in $H^{1/2}(\partial\Omega_0) \times H^{-1/2}(\partial\Omega_0)$, corresponding to the weak solutions by setting

$$\mathcal{K}_{0,t} := \mathrm{tr}_{\mathcal{L}^t}\{u \in H^1(\Omega_0) : \widetilde{\mathfrak{l}}^t[u,\psi] = 0, \text{ for all } \psi \in H^1_0(\Omega_0)\},\ t \in [0,1], \tag{3.187}$$

and the *constant* (cf., Proposition 3.20) path of Lagrangian planes corresponding to the boundary conditions

$$\mathcal{G}_t := \overline{\mathrm{tr}_{\mathcal{L}^t}(\mathrm{dom}(\mathcal{L}^t_{\mathcal{X}_0}))},\ t \in [0,1]. \tag{3.188}$$

Then employing Theorem 3.18, we arrive at the formula originally derived in [51, Theorem 1],

$$\mathrm{Mor}(\mathcal{L}^0_{\mathcal{X}_0}) - \mathrm{Mor}(\mathcal{L}^1_{\mathcal{X}_0}) = \mathrm{Mas}\big((\mathcal{K}_{0,t}, \mathcal{G}_t)|_{t\in[0,1]}\big), \tag{3.189}$$

and, using (3.186), at the formula

$$\mathrm{Mor}(\mathcal{L}^0_{\mathcal{X}_0}) - \mathrm{Mor}(\mathcal{L}^1_{\mathcal{X}_1}) = \mathrm{Mas}\big((\mathcal{K}_{0,t}, \mathcal{G}_t)|_{t\in[0,1]}\big). \tag{3.190}$$

3.3.5 Spectra of elliptic operators with Robin boundary conditions and the Maslov index

We will now derive the Smale-type formula (3.194) for second order differential operators subject to Robin boundary conditions, cf. [155, 164], and also [50, 51, 136]. Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded open set with smooth boundary. Let us fix coefficients $\mathcal{A}, B, q$ as in (3.169), (3.170), (3.171), (3.172), and define the differential expression

$$\mathcal{L} := -\mathrm{div}\mathcal{A}\nabla + B\nabla - \nabla \cdot \overline{B} + q. \tag{3.191}$$

If $\Theta \in \mathbb{R}$ is given then the linear operator $\mathcal{L}_\Theta$ acting in $L^2(\Omega)$ and defined by

$$\mathcal{L}_\Theta u := -\mathrm{div}(\mathcal{A}\nabla u) + B\nabla u - \nabla \cdot (\overline{B}u) + qu, \ u \in \mathrm{dom}(\mathcal{L}_\Theta), \tag{3.192}$$

$$\mathrm{dom}(\mathcal{L}_\Theta) := \{u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega), \ \gamma_{_N}^{\mathcal{L}} u + \Theta\gamma_{_D} u = 0\}, \tag{3.193}$$

is self-adjoint, moreover, its essential spectrum is empty, cf. [145, Proposition 2.3].

Proposition 3.21. *If $\Theta_1 < \Theta_2$, then for the operator $\mathcal{L}_\Theta$ from* (3.192) *one has*

$$\mathrm{Mor}(\mathcal{L}_{\Theta_1}) - \mathrm{Mor}(\mathcal{L}_{\Theta_2}) = \sum_{\Theta_1 \leq s \leq \Theta_2} \dim\ker(\mathcal{L}_s). \tag{3.194}$$

Proof. We will use (3.131) and show that all crossing corresponding to the variation of parameter $s \in [\Theta_1, \Theta_2]$ are sign-definite. Theorem 3.18 yields

$$\mathrm{Mor}(\mathcal{L}_{\Theta_1}) - \mathrm{Mor}(\mathcal{L}_{\Theta_2}) = \mathrm{Mas}\left((\mathcal{K}, \mathcal{G}_s)|_{s\in[\Theta_1,\Theta_2]}\right), \tag{3.195}$$

where

$$\mathcal{K} = \mathrm{tr}_{\mathcal{L}}\{u \in H^1(\Omega) : \langle \mathcal{A}\nabla u, \nabla\psi\rangle_{L^2(\Omega)} + \langle B\nabla u, \psi\rangle_{L^2(\Omega)} \tag{3.196}$$

$$+ \langle u, B\nabla\psi\rangle_{L^2(\Omega)} + \langle qu, \psi\rangle_{L^2(\Omega)} = 0, \ \text{for all } \psi \in H_0^1(\Omega)\}, \tag{3.197}$$

$$\mathcal{G}_s := \{(f, -sf) : f \in H^{1/2}(\partial\Omega)\} \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega), \ s \in [\Theta_1, \Theta_2] \tag{3.198}$$

(we notice that $\mathcal{K}$ does not depend on parameter s). Clearly $\mathcal{G}_s$ is Lagrangian for each $s \in \mathbb{R}$, moreover, the path

$$[\Theta_1, \Theta_2] \ni s \mapsto \mathcal{G}_s \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega), \tag{3.199}$$

is continuously differentiable. Let $s_* \in [\Theta_1, \Theta_2]$ be a crossing. Then, by [50, Lemma 3.8], there exists a neighbourhood $\Sigma_* \subset [\Theta_1, \Theta_2]$ containing s_* and a mapping

$$s \mapsto R_s \text{ in } C^1\left(\Sigma_*, \mathcal{B}(\mathcal{G}_{\theta_*}, \mathcal{G}_{\theta_*}^{\perp})\right), \tag{3.200}$$

such that

$$\mathcal{G}_s = \{(f,g) + R_s(f,g) : (f,g) \in \mathcal{G}_{\theta_*}\}, s \in \Sigma_*. \tag{3.201}$$

Next, pick any $(f_{s_*}, g_{s_*}) \in \mathcal{G}_{s_*} \cap \mathcal{K}$. Then $g_{s_*} = -s_* f_{s_*}$, $f_{s_*} \in H^{1/2}(\partial\Omega)$, and there exists $u_* \in \ker(\mathcal{L}_{s_*})$ such that

$$\operatorname{tr}_{\mathcal{L}} u_* = (f_{s_*}, g_{s_*}). \tag{3.202}$$

Moreover, there is a family $\{f_s\}_{s\in\Sigma_*} \subset H^{1/2}(\partial\Omega)$ such that, in view of (3.198), (3.201)

$$(f_{s_*}, g_{s_*}) + R_s(f_{s_*}, g_{s_*}) = (f_s, -sf_s), s \in \Sigma_*, \tag{3.203}$$

where the mapping $s \mapsto f_s$ is contained in $C^1(\Sigma_*, H^{1/2}(\partial\Omega))$. The derivative of f_s with respect to s evaluated at s_* is denoted by f'_{s_*} . We proceed by evaluating the Maslov crossing form at $\operatorname{tr}_{\mathcal{L}} u_* = (f_{s_*}, g_{s_*})$

$$\begin{aligned}
&\mathcal{Q}_{s_*,\mathcal{K}}(\operatorname{tr}_{\mathcal{L}} u_*, \operatorname{tr}_{\mathcal{L}} u_*) = \omega\left((f_{s_*}, g_{s_*}), \frac{d}{ds} R_s(f_{s_*}, g_{s_*})\right)|_{s=s_*} \\
&= \omega\big((f_{s_*}, -s_* f_{s_*}), (f'_{s_*}, -(f_{s_*} + s_* f'_{s_*}))\big) \\
&= -\overline{\langle f_{s_*} + s_* f'_{s_*}, f_{s_*}\rangle}_{L^2(\partial\Omega)} - \langle -s_* f_{s_*}, f'_{s_*}\rangle_{L^2(\partial\Omega)} = -\|f_{s_*}\|^2_{L^2(\partial\Omega)}.
\end{aligned}$$

Finally, we arrive at

$$\mathcal{Q}_{s_*,\mathcal{K}}(\operatorname{tr}_{\mathcal{L}} u_*, \operatorname{tr}_{\mathcal{L}} u_*) = -\|\gamma_D u_*\|^2_{L^2(\partial\Omega)} < 0. \tag{3.204}$$

Therefore, a calculation similar to (3.136) shows that

$$\begin{aligned}
&\operatorname{Mas}\big((\mathcal{K}, \mathcal{G}_s)|_{s\in[\Theta_1,\Theta_2]}\big) = -\operatorname{Mas}\big((\mathcal{G}_s, \mathcal{K})|_{s\in[\Theta_1,\Theta_2]}\big) \\
&= \sum_{\Theta_1 \le s \le \Theta_2} \dim(\mathcal{K} \cap \mathcal{G}_s) = \sum_{\Theta_1 \le s \le \Theta_2} \dim\ker(\mathcal{L}_s),
\end{aligned}$$

as asserted. ∎

3.4 The Maslov index for the Schrödinger operators on Lipschitz domains

In this section we establish relations between the Maslov and Morse indices, and, consequently, relations between the Maslov index and the spectral flow for Schrödinger operators with matrix valued potentials on Lipschitz domains. The general result will be applied to two specific types of boundary conditions: First, to the $\vec{\theta}-$periodic conditions on a cell $\Omega \subset \mathbb{R}^n$, and second to the Robin-type boundary conditions on star-shaped domains. Hypothesis 3.1 is imposed throughout this section.

3.4.1 A general result for the Schrödinger operators

First, we verify Hypothesis 3.6 in the present settings, that is, for the Schrödinger operator $\mathcal{L} = -\Delta + V$ with a bounded $m \times m$ matrix valued potential. Assuming Hypothesis 3.1 and denoting the outward pointing normal unit vector to $\partial\Omega$ by $\vec{\nu} = (\nu_1, \cdots, \nu_n)$, we recall from [77] two boundary spaces:

$$N^{1/2}(\partial\Omega, \mathbb{C}^m) := \{g \in L^2(\partial\Omega, \mathbb{C}^m) \,|\, g\nu_j \in H^{1/2}(\partial\Omega, \mathbb{C}^m),\, 1 \le j \le n\}, \tag{3.205}$$

equipped with the natural norm

$$\|g\|_{N^{1/2}(\partial\Omega,\mathbb{C}^m)} := \sum_{j=1}^{n} \|g\nu_j\|_{H^{1/2}(\partial\Omega,\mathbb{C}^m)}, \tag{3.206}$$

and

$$N^{3/2}(\partial\Omega) := \{g \in H^1(\partial\Omega) \,|\, \nabla_{\tan} g \in (H^{1/2}(\partial\Omega))^n\}, \tag{3.207}$$

equipped with the natural norm

$$\|g\|_{N^{3/2}(\partial\Omega)} := \|g\|_{L^2(\partial\Omega)} + \|\nabla_{\tan} g\|_{H^{1/2}(\partial\Omega)^n}. \tag{3.208}$$

Here, the tangential gradient operator $\nabla_{\tan} : H^1(\partial\Omega) \mapsto L^2(\partial\Omega)^n$ is defined as

$$f \mapsto \left(\sum_{k=1}^{n} \nu_k \frac{\partial f}{\partial \tau_{k,l}} \right)_{l=1}^{n},$$

and $\frac{\partial}{\partial \tau_{k,l}}$ is the tangential derivative, which is a bounded operator between $H^s(\partial\Omega)$ and $H^{s-1}(\partial\Omega),\ 0 \le s \le 1$, that extends the operator

$$\frac{\partial}{\partial \tau_{k,l}} : \psi \mapsto \nu_k (\partial_l \psi)\big|_{\partial\Omega} - \nu_l (\partial_k \psi)\big|_{\partial\Omega},$$

originally defined for C^1 function ψ in a neighbourhood of $\partial\Omega$.

Lemma 3.22 ([77], Lemma 6.3)**.** *Assume Hypothesis* 3.1*. Then the Neumann trace operator* $\gamma_{_N} u = \nu \cdot \nabla u|_{\partial\Omega}$*,* $u \in H^2(\Omega)$*, considered in the context*

$$\gamma_{_N} : H^2(\Omega) \cap H_0^1(\Omega) \to N^{1/2}(\partial\Omega), \tag{3.209}$$

is well-defined, bounded, onto, and with a bounded right-inverse. In addition, the null space of $\gamma_{_N}$ *in* (3.209) *is precisely* $H_0^2(\Omega)$*, the closure of* $C_0^\infty(\Omega)$ *in* $H^2(\Omega)$*.*

We will now show that both density assumptions in Hypothesis 3.6 are satisfied for the Schrödinger operators on Lipschitz domains. Since V is bounded it suffices to verify the assumptions for the Laplace operator. Let the function space

$$\mathcal{D}_\Delta^s(\Omega) := \{u \in H^s(\Omega, \mathbb{C}^m) : \Delta u \in L^2(\Omega, \mathbb{C}^m)\},\ s \ge 0, \tag{3.210}$$

be equipped with the natural norm

$$\|u\|_{\Delta,s} := \left(\|u\|^2_{H^s(\Omega,\mathbb{C}^m)} + \|\Delta u\|^2_{L^2(\Omega,\mathbb{C}^m)} \right)^{1/2},\ s \ge 0. \tag{3.211}$$

Let us denote

$$\mathrm{tr}_\Delta := (\gamma_{_D} u, \gamma_{_N} u)\,, \mathrm{tr}_\Delta \in \mathcal{B}\left(\mathcal{D}^1_\Delta(\Omega), H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)\right). \tag{3.212}$$

Proposition 3.23. *Assume Hypothesis* 3.1. *Then*

$$i)\ \operatorname{ran}(\operatorname{tr}_{\Delta,1}) \text{ is dense in } H^{1/2}(\partial\Omega,\mathbb{C}^m)\times H^{-1/2}(\partial\Omega,\mathbb{C}^m), \tag{3.213}$$

$$ii)\ \mathcal{D}^1_\Delta(\Omega) \text{ is dense in } \mathcal{D}^0_\Delta(\Omega). \tag{3.214}$$

Proof. First we prove part *i*). It suffices to show that

$$\big(\big\{(\gamma_{D}u,\gamma_{N}u) : u\in\mathcal{D}^1_\Delta(\Omega)\big\}\big)^\circ = \{(0,0)\}, \tag{3.215}$$

where the left-hand side denotes the annihilator with respect to the sympletic form (3.28). Pick an arbitrary

$$(\varphi,\psi)\in\big(\big\{(\gamma_{D}u,\gamma_{N}u) : u\in\mathcal{D}^1_\Delta(\Omega)\big\}\big)^\circ, \tag{3.216}$$

then

$$\overline{\langle\psi,\gamma_{D}f\rangle}_{-1/2}-\langle\gamma_{N}f,\varphi\rangle_{-1/2}=0,\ \text{ for all } f\in\mathcal{D}^1_\Delta(\Omega). \tag{3.217}$$

By Lemma 3.22, for arbitrary $g\in N^{1/2}(\partial\Omega,\mathbb{C}^m)$ there exists $F_g\in H^2(\Omega,\mathbb{C}^m)$ such that

$$\gamma_{D}F_g=0,\ \gamma_{N}F_g=g. \tag{3.218}$$

Using equation (3.217) with $f=F_g$, one obtains

$$\langle g,\varphi\rangle_{-1/2}=0,\ \text{ for all } g\in N^{1/2}(\partial\Omega,\mathbb{C}^m). \tag{3.219}$$

In addition, by [77, Corollary 6.12], we have

$$N^{1/2}(\partial\Omega,\mathbb{C}^m)\hookrightarrow L^2(\partial\Omega,\mathbb{C}^m)\hookrightarrow H^{-1/2}(\partial\Omega,\mathbb{C}^m), \tag{3.220}$$

where both inclusions are dense and continuous. Therefore, (3.219) can be extended by continuity to $H^{-1/2}(\partial\Omega,\mathbb{C}^m)$, and one has

$$\langle g,\varphi\rangle_{-1/2}=0,\ \text{ for all } g\in H^{-1/2}(\partial\Omega,\mathbb{C}^m), \tag{3.221}$$

hence, $\varphi = 0$. Combining (3.217) and (3.221), one obtains

$$\langle \psi, \gamma_D f \rangle_{-1/2} = 0, \text{ for all } f \in \mathcal{D}^1_\Delta(\Omega). \tag{3.222}$$

Recall from [77, Lemma 2.3] that γ_D considered in the context

$$\gamma_D : \mathcal{D}^{3/2}_\Delta(\Omega) \to H^1(\partial\Omega, \mathbb{C}^m), \tag{3.223}$$

is compatible with (3.9), bounded, has bounded right-right inverse (hence, onto). Then, for arbitrary $h \in H^1(\partial\Omega, \mathbb{C}^m)$ there exits $G_h \in \mathcal{D}^{3/2}_\Delta(\Omega) \subset \mathcal{D}^1_\Delta(\Omega)$, such that $\gamma_D G_h = h$. Let us set $f = G_h$ in (3.222) and obtain

$$\langle \psi, h \rangle_{-1/2} = 0, \text{ for all } h \in H^1(\partial\Omega, \mathbb{C}^m). \tag{3.224}$$

Since the inclusion

$$H^1(\partial\Omega, \mathbb{C}^m) \hookrightarrow H^{1/2}(\partial\Omega, \mathbb{C}^m) \tag{3.225}$$

is dense, (3.224) yields $\psi = 0$. Thus, $(\varphi, \psi) = (0, 0)$ and consequently part *i)* holds.

The second assertion follows from the fact that

$$C^\infty(\overline{\Omega}) \hookrightarrow \mathcal{D}^0_\Delta(\Omega), \tag{3.226}$$

densely, cf. [31]. ∎

Next, we turn to a Lagrangian formulation of eigenvalue problems for self-adjoint extensions of $-\Delta_{min}$,

$$-\Delta_{min} u := -\Delta u, u \in \operatorname{dom}(-\Delta_{min}) := H^2_0(\Omega). \tag{3.227}$$

Recall, that $(-\Delta_{min})^* = -\Delta_{max}$, where

$$-\Delta_{max} u := -\Delta u, \ u \in \operatorname{dom}(-\Delta_{max}) := \mathcal{D}^1_\Delta(\Omega). \tag{3.228}$$

The self-adjoint extension of $-\Delta_{min}$ with domain $\mathscr{D} \subset \mathcal{D}^1_\Delta(\Omega)$ is denoted by $-\Delta_{\mathscr{D}}$.

Hypothesis 3.24. *Let* $\Omega \subset \mathbb{R}^n, n \geq 2$ *be open, bounded, Lipschitz domain and assume that the mapping*

$$\mathcal{I} \ni t \mapsto V_t \in L^\infty(\Omega, \mathbb{C}^{m\times m}),\ V_t = \overline{V_t}^\top, t \in \mathcal{I},$$

is contained in $C^1(\mathcal{I}, L^\infty(\Omega, \mathbb{C}^{m\times m})),\ \mathcal{I} := [\alpha, \beta]$.

Moreover, let us assume that f *is a given function such that*

$$f : \mathcal{I} \to \mathbb{R},\ f \in C^1(\mathcal{I}),\ f(t) > 0,\ \partial_t f(t) \neq 0,\ t \in \mathcal{I}. \tag{3.229}$$

Let us denote by $K_{\lambda,t,f}$ the trace of the set of weak solutions to the eigenvalue problem $-\Delta u + Vu = \lambda u$, that is,

$$\begin{aligned} K_{\lambda,t,f} := \operatorname{tr}_\Delta \big\{ &u \in \mathcal{D}^1_\Delta(\Omega) : f(t)\langle \nabla u, \nabla\varphi\rangle_{L^2(\Omega,\mathbb{C}^m)} + \langle V_t u, \varphi\rangle_{L^2(\Omega,\mathbb{C}^m)} \\ &= \lambda \langle u, \varphi\rangle_{L^2(\Omega,\mathbb{C}^m)},\ \text{for all } \varphi \in H^1_0(\Omega, \mathbb{C}^m)\big\}, \lambda \in \mathbb{R}, t \in \mathcal{I}, \end{aligned} \tag{3.230}$$

where $\nabla u := [\nabla u_1, \cdots, \nabla u_m]^\top \in \mathbb{C}^{m\times n}$,

$$\langle \nabla u, \nabla v\rangle_{L^2(\Omega,\mathbb{C}^m)} := \sum_{i=1}^m \langle \nabla u_i, \nabla v_i\rangle_{[L^2(\Omega,\mathbb{C})]^n},$$

for given $u = (u_i)_{i=1}^m, v = (v_i)_{i=1}^m \in H^1(\Omega, \mathbb{C}^m)$.

Theorem 3.25. *Assume Hypotheses* 3.1 *and* 3.24. *Let* $\mathscr{D}_t \subset \mathcal{D}^1_\Delta(\Omega)$, $t \in \mathcal{I}$, *and assume that the linear operator* $\mathcal{L}^t_{\mathscr{D}_t} = -f(t)\Delta_{\mathscr{D}_t} + V_t$ *acting in* $L^2(\Omega, \mathbb{C}^m)$ *and given by*

$$\mathcal{L}^t_{\mathscr{D}_t} u := -f(t)\Delta u + V_t u,\ u \in \operatorname{dom}(\mathcal{L}^t_{\mathscr{D}_t}) := \mathscr{D}_t, \tag{3.231}$$

is self-adjoint with $\operatorname{Spec}_{ess}\left(\mathcal{L}^t_{\mathscr{D}_t}\right) \cap (-\infty, 0] = \emptyset,\ t \in \mathcal{I}$. *Assume that there exists* $\lambda_\infty < 0$, *such that*

$$\ker(\mathcal{L}^t_{\mathscr{D}_t} - \lambda) = \{0\} \textit{ for all } \lambda \leq \lambda_\infty, t \in \mathcal{I}.$$

Suppose that the path

$$t \mapsto \mathcal{G}_t := \overline{\mathrm{tr}_{\Delta,1}(\mathscr{D}_t)} \in H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m), \tag{3.232}$$

is contained in $C^1\left(\mathcal{I}, \Lambda\left(H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)\right)\right)$.

Then one has

$$\mathrm{Mor}\left(\mathcal{L}^{\alpha}_{\mathscr{D}_\alpha}\right) - \mathrm{Mor}\left(\mathcal{L}^{\beta}_{\mathscr{D}_\alpha}\right) = \mathrm{Mas}\left((K_{0,t,f}, \mathcal{G}_t)|_{t\in\mathcal{I}}\right). \tag{3.233}$$

The proof of Theorem 3.25 is similar to that of Theorem 3.18, and is omitted. We complete this section by illustrating applications of (3.233). We note that the Maslov index of the path $\left((K_{0,t,f}, \mathcal{G}_t)|_{t\in\mathcal{I}}\right)$ is equal to the spectral flow of $\{\mathcal{L}^t_{\mathscr{D}_t}\}_{t=\alpha}^{\beta}$, that is, the following formula holds:

$$\mathrm{SpFlow}\left(\{\mathcal{L}^t_{\mathscr{D}_t}\}_{t=\alpha}^{\beta}\right) = \mathrm{Mas}\left((K_{0,t,f}, \mathcal{G}_t)|_{t\in\mathcal{I}}\right). \tag{3.234}$$

3.4.2 Spectra of $\vec{\theta}$−periodic Schrödinger operators and the Maslov index

In this subsection we derive a relation between the Maslov and Morse indices for multidimensional $\vec{\theta}$−periodic Schrödinger operators as an application of (3.233).

Firstly, we define the self-adjoint extension of $-\Delta_{min}$ corresponding to the $\vec{\theta}$-periodic boundary conditions

$$u(x + a_j) = \varepsilon^{2\pi \mathbf{i}\theta_j} u(x),\ \frac{\partial u}{\partial \vec{\nu}}(x + a_j) = \varepsilon^{2\pi \mathbf{i}\theta_j} \frac{\partial u}{\partial \vec{\nu}}(x),\ x \in \partial Q_j^0,$$

where $\{a_1, \ldots a_n\} \subset \mathbb{R}^n$ are linearly independent vectors, $\vec{\theta} := (\theta_1, \ldots, \theta_n) \in [0, 1)^n$, and ∂Q_j^0 are the faces of the unit cell Q

$$Q := \{t_1 a_1 + \cdots + t_n a_n |\ 0 \le t_j \le 1, j \in \{1, \ldots, n\}\},$$

so that $\partial Q = \cup_{\ell=0}^{1} \cup_{j=1}^{n} \partial Q_j^\ell$ defined by

$$\partial Q_j^\ell := \{t_1 a_1 + \cdots + t_n a_n \in Q \big| \, t_j = \ell\}, \ j \in \{1, \ldots, n\}, \ \ell \in \{0, 1\}.$$

The n-tuple $\{a_1, \ldots a_n\} \subset \mathbb{R}^n$ is uniquely associated with an $n \times n$ matrix A by the condition $A a_j = 2\pi e_j$, where $\{e_j\}_{1 \le j \le n}$ is the standard basis in $\mathbb{C}^n$. For the matrix A just defined and $k \in \mathbb{Z}^n$ we denote

$$\zeta_k(x) := |Q|^{-1} \varepsilon^{\mathbf{i} A^\top (\vec{\theta} - k) \cdot x}, \ x \in Q. \tag{3.235}$$

Recalling that $\partial Q = \cup_{\ell=0}^{1} \cup_{j=1}^{n} \partial Q_j^\ell$, we define the Dirichlet trace operators corresponding to each face of Q as follows,

$$\gamma_{D, \partial Q_j^\ell} : H^2(Q, \mathbb{C}^m) \to L^2(\partial Q_j^\ell, \mathbb{C}^m),$$
$$\gamma_{D, \partial Q_j^\ell}(u) := (\gamma_D u)|_{\partial Q_j^\ell}, \quad 1 \le j \le n, \ \ell \in \{0, 1\}.$$

It follows that $\gamma_{D, \partial Q_j^\ell} \in \mathcal{B}\big(H^2(Q, \mathbb{C}^m), L^2(\partial Q_j^\ell; \mathbb{C}^m)\big)$ for $1 \le j \le n$ and $\ell \in \{0, 1\}$. The Neumann trace is given by

$$\gamma_{N, \partial Q_j^\ell} : H^2(Q, \mathbb{C}^m) \to L^2(\partial Q_j^\ell; \mathbb{C}^m),$$
$$\gamma_{N, \partial Q_j^\ell}(u) := \big(\gamma_D (\nabla u) \overrightarrow{\nu}\big)\big|_{\partial Q_j^\ell}, \quad 1 \le j \le n, \ \ell \in \{0, 1\},$$

where $\vec{\nu}$ is the outward pointing normal unit vector to ∂Q. The inclusion

$$\gamma_{N, \partial Q_j^\ell} \in \mathcal{B}\left(H^2(Q, \mathbb{C}^m), L^2\left(\partial Q_j^\ell; \mathbb{C}^{m \times n}\right)\right),$$

holds for all $1 \le j \le n, \ \ell \in \{0, 1\}$. For each $u \in H^2(\Omega; \mathbb{R}^{2m})$ we denote

$$u_j^\ell := \gamma_{D, \partial Q_j^\ell}(u), \quad \partial_\nu u_j^\ell := \gamma_{N, \partial Q_j^\ell}(u), \ 1 \le j \le n, \ \ell \in \{0, 1\}. \tag{3.236}$$

Let us also introduce the weighted translation operators

$$\begin{aligned} &\mathbb{M}_j \in \mathcal{B}\left(L^2(\partial Q_j^0; \mathbb{C}^m), L^2(\partial Q_j^1; \mathbb{C}^m)\right), \\ &(\mathbb{M}_j u)(x) = \varepsilon^{2\pi \mathbf{i} \theta_j} u(x - a_j) \text{ for a.a. } x \in \partial Q_j^1, \ 1 \le j \le n. \end{aligned} \tag{3.237}$$

Proposition 3.26. *Recall notation* (3.236), (3.237)*. Then the linear operator*

$$-\Delta_{\vec{\theta}}: \mathrm{dom}(-\Delta_{\vec{\theta}}) \subset L^2(Q,\mathbb{C}^m) \to L^2(Q,\mathbb{C}^m), \tag{3.238}$$

$$\mathrm{dom}(-\Delta_{\vec{\theta}}) := \big\{u \in H^2(Q,\mathbb{C}^m):\ u_j^1 = \mathbb{M}_j u_j^0, \partial_\nu u_j^1 = -\mathbb{M}_j \partial_\nu u_j^0, 1 \le j \le n\big\}, \tag{3.239}$$

$$-\Delta_{\vec{\theta}} u := -\Delta u,\ \ u \in \mathrm{dom}(-\Delta_{\vec{\theta}}) \tag{3.240}$$

is self-adjoint, moreover

$$-\Delta_{\min} \subset -\Delta_{\vec{\theta}} \subset -\Delta_{\max}.$$

In addition, $-\Delta_{\vec{\theta}}$ has compact resolvent, in particular, it has purely discrete spectrum. Finally, $\mathrm{Spec}(-\Delta_{\vec{\theta}}) = \big\{\|A^\top(\vec{\theta}-k)\|^2_{\mathbb{R}^n}\big\}_{k\in\mathbb{Z}^n}$.

Proof. Recall (3.235). Then the sequence of functions

$$\phi_{k,l}(x) := (0,\cdots,\underbrace{\zeta_k(x)}_{l-th\ \text{position}},\cdots,0)^\top, k\in\mathbb{Z}^n, 1\le l\le m, \tag{3.241}$$

form an orthonormal basis in $L^2(Q,\mathbb{C}^m)$. In addition, $\phi_{k,l} \in \mathrm{dom}(-\Delta_{\vec{\theta}})$, since by

$$A^\top(\vec{\theta}-k)\cdot a_j = (\vec{\theta}-k)\cdot Aa_j = 2\pi(\vec{\theta}-k)\cdot e_j = 2\pi(\theta_j - k_j),\ k\in\mathbb{Z}^n,\ 1\le j\le n, \tag{3.242}$$

one has

$$\begin{aligned} |Q|^{-1}\varepsilon^{\mathbf{i}A^\top(\vec{\theta}-k)\cdot(x+a_j)} &= \varepsilon^{2\pi\mathbf{i}\theta_j}|Q|^{-1}\varepsilon^{\mathbf{i}A^\top(\vec{\theta}-k)\cdot x},\\ \nu\cdot\nabla\Big(|Q|^{-1}\varepsilon^{\mathbf{i}A^\top(\vec{\theta}-k)\cdot(x+a_j)}\Big) &= \varepsilon^{2\pi\mathbf{i}\theta_j}\nu\cdot\nabla\Big(|Q|^{-1}\varepsilon^{\mathbf{i}A^\top(\vec{\theta}-k)\cdot x}\Big), \end{aligned} \tag{3.243}$$

that is

$$(\phi_{k,l})_j^1 = \mathbb{M}_j(\phi_{k,l})_j^0 \text{ and } \partial_\nu(\phi_{k,l})_j^1 = \mathbb{M}_j\partial_\nu(\phi_{k,l})_j^0, 1\le j\le n.$$

Furthermore,

$$-\Delta\phi_{k,l} = \|A^\top(\vec{\theta}-k)\|^2_{\mathbb{R}^n}\phi_{k,l}, k\in\mathbb{Z}^n, 1\le l\le m. \tag{3.244}$$

From these facts we infer (cf., [116] for details) that

$$\mathrm{span}\{\phi_{k,l}: k\in\mathbb{Z}^n,\ 1\le l\le m\}, \tag{3.245}$$

is a core of the operator $-\Delta_{\vec{\theta}}$. Hence, $-\Delta_{\vec{\theta}}$ is self-adjoint with domain (3.239), it has compact resolvent due to the fact that

$$\|A^{\top}(\vec{\theta}-k)\|^2_{\mathbb{R}^n} \to \infty, \text{ as } \|k\|_{\mathbb{C}^n} \to \infty, \tag{3.246}$$

cf. [116, Lemma 3.2]. ∎

Let $tQ := \{tx, x \in Q\}, t \in (0,1]$, and define

$$\begin{aligned}
&-\Delta^t_{\vec{\theta}} : \mathrm{dom}(-\Delta_{\vec{\theta}}) \subset L^2(tQ, \mathbb{C}^m) \to L^2(tQ, \mathbb{C}^m),\\
&\mathrm{dom}(-\Delta_{\vec{\theta}}) := \big\{u \in H^2(tQ, \mathbb{C}^m) : \ u^1_j = \mathbb{M}^t_j u^0_j, \partial_\nu u^1_j = -\mathbb{M}^t_j \partial_\nu u^0_j, 1 \le j \le n\big\},\\
&-\Delta^t_{\vec{\theta}} u := -\Delta u, \ u \in \mathrm{dom}(-\Delta^t_{\vec{\theta}}),
\end{aligned}$$

where $\mathbb{M}^t_j$ is the weighted translation operator acting from $L^2(\partial(tQ)^0_j; \mathbb{C}^m)$ to $L^2(\partial(tQ)^1_j; \mathbb{C}^m)$, by formula (3.237) with a_j replaced by ta_j. Assume that $V \in L^\infty(Q, \mathbb{C}^{m\times m})$, and denote

$$\begin{aligned}
K_{\lambda,t} := \mathrm{tr}_\Delta \Big\{u \in \mathcal{D}^1_\Delta(Q) : &\int_Q t^{-2}\langle \nabla u(x), \nabla\varphi(x)\rangle_{\mathbb{C}^{m\times n}}\\
&+ \langle V(tx)u(x), \varphi(x)\rangle_{\mathbb{C}^m} - \lambda\langle u, \varphi\rangle_{\mathbb{C}^m} d^n x = 0,\\
&\text{for all } \varphi \in H^1_0(Q, \mathbb{C}^m)\Big\}, \ \lambda \in \mathbb{R}, \ t \in \mathbb{R},
\end{aligned} \tag{3.247}$$

$$\mathcal{G}_{\vec{\theta}} := \overline{\mathrm{tr}_\Delta \left\{ \mathrm{dom}(-\Delta_{\vec{\theta}})\right\}}^{H^{1/2}(\partial\Omega,\mathbb{C}^m)\times H^{-1/2}(\partial\Omega,\mathbb{C}^m)},$$

where $\mathrm{tr}_\Delta u := (\gamma_D u, \gamma^\Delta_N u)$, cf. (3.11), (3.12).

Theorem 3.27. *If $V \in L^\infty(Q, \mathbb{C}^{m\times m})$ then for any $\tau \in (0,1]$, $\vec{\theta} \in [0,1)^n$, one has*

$$\mathrm{Mor}\left(-\Delta^\tau_{\vec{\theta}} + V|_{\tau Q}\right) - \mathrm{Mor}\left(-\Delta_{\vec{\theta}} + V\right) = \mathrm{Mas}\left((K_{0,t}, \mathcal{G}_{\vec{\theta}})|_{t\in[\tau,1]}\right), \tag{3.248}$$

where $K_{\lambda,t}$ and $\mathcal{G}_{\vec{\theta}}$ are defined in (3.247)*.*

If $\vec{\theta} \neq 0$, then

$$\mathrm{Mor}\left(-\Delta_{\vec{\theta}} + V\right) = -\,\mathrm{Mas}\left((K_{0,t}, \mathcal{G}_{\vec{\theta}})|_{t\in[\tau_0,1]}\right), \tag{3.249}$$

for small enough $\tau_0 > 0$.

If V *is continuous at* 0 *and* $V(0)$ *is invertible, then*

$$\mathrm{Mor}(V(0)) - \mathrm{Mor}\left(-\Delta_{\vec{0}} + V\right) = \mathrm{Mas}\left((K_{0,t}, \mathcal{G}_{\vec{0}})|_{t\in[\tau_0,1]}\right). \tag{3.250}$$

for small enough $\tau_0 > 0$.

Proof. Introducing the one-parameter family of self-adjoint operators acting in $L^2(Q, \mathbb{C}^m)$ by the formula

$$\mathcal{L}^t := -t^{-2}\Delta_{\vec{\theta}} + V(t\cdot),\ \mathrm{dom}(\mathcal{L}^t) := \mathrm{dom}(-\Delta_{\vec{\theta}}),\ t \in (0,1], \tag{3.251}$$

and using Theorem 3.25, we arrive at the relation

$$\mathrm{Mor}(\mathcal{L}^\tau) - \mathrm{Mor}(\mathcal{L}^1) = \mathrm{Mas}\left((K_{0,t}, \mathcal{G}_{\vec{\theta}})|_{t\in[\tau,1]}\right). \tag{3.252}$$

Notice that $\mathcal{L}^1 = -\Delta_{\vec{\theta}} + V$, and that

$$u \in \ker(\mathcal{L}^\tau) \text{ if and only if } u(\cdot/\tau) \in \ker(-\Delta_{\vec{\theta}}^\tau + V|_{\tau Q}). \tag{3.253}$$

Then

$$\mathrm{Mor}(\mathcal{L}^\tau) = \mathrm{Mor}(-\Delta_{\vec{\theta}}^\tau + V|_{\tau Q}). \tag{3.254}$$

Combining (3.252) and (3.254), we infer (3.248). By [116, Lemma 3.10, Proposition 3.13], we infer

$$\mathrm{Mor}(-\Delta_{\vec{\theta}}^\tau + V|_{\tau Q}) = 0, \text{whenever } \tau \text{ is small enough}, \tag{3.255}$$

$$\mathrm{Mor}(-\Delta_{\vec{0}}^\tau + V|_{\tau Q}) = \mathrm{Mor}(V(0)),\ \text{whenever } \tau \text{ is small enough}. \tag{3.256}$$

Equations (3.248), (3.255), (3.256) imply (3.249), (3.250). ∎

3.4.3 Spectra of Schrödinger operators on star-shaped domains

We now show how to use Theorem 3.25 to recover (under fewer hypotheses) the relations between the Morse and Maslov indices obtained in [50] and [57]. To set the stage we impose the following hypothesis.

Hypothesis 3.28. *Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be non-empty, open, bounded, star-shaped, Lipschitz domain. Let $\mathcal{G} \subset H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)$ be a Lagrangian plane with respect to symplectic form* (3.28)*. Assume that $V \in L^\infty(\Omega, \mathbb{C}^m)$, $m \in \mathbb{N}$.*

Without loss of generality we assume that Ω is centered at the origin. Let $\tau > 0$, $t \in [\tau, 1)$ and denote

$$\Omega_t := \{x \in \Omega : x = t'y, \text{ for } t' \in [0, t), \ y \in \partial\Omega\}. \tag{3.257}$$

The Dirichlet and Neumann trace operators considered in Ω_t are denoted by

$$\gamma_{D,t} \in \mathcal{B}(H^1(\Omega_t), H^{1/2}(\partial\Omega_t)), \ \gamma_{N,t} \in \mathcal{B}(\mathcal{D}^1_\Delta(\Omega_t), H^{-1/2}(\partial\Omega_t)),$$

$$\text{tr}_{\Delta,t} := (\gamma_{D,t}, \gamma_{N,t}) : \mathcal{D}^1_\Delta(\Omega_t) \to H^{1/2}(\partial\Omega_t) \times H^{-1/2}(\partial\Omega_t), \ t \in [\tau, 1).$$

The minimal and maximal Laplacians on Ω_t are denoted by $\Delta_{min,t}$ and $\Delta_{max,t}$. Following [50, Section 4.1] we introduce the scaling operators,

$$U_t : L^2(\Omega_t) \to L^2(\Omega), \ \ (U_t w)(x) := t^{n/2} w(tx), x \in \Omega,$$

$$U_t^\partial : L^2(\partial\Omega_t) \to L^2(\partial\Omega), \ \ (U_t^\partial h)(y) := t^{(n-1)/2} h(ty), y \in \partial\Omega,$$

$$U_{1/t}^\partial : L^2(\partial\Omega) \to L^2(\partial\Omega_t), \ \ (U_{1/t}^\partial f)(z) := t^{-(n-1)/2} h(t^{-1}z), z \in \partial\Omega_t. \tag{3.258}$$

Finally, we notice that $U_t \in \mathcal{B}(H^1(\Omega_t), H^1(\Omega))$, $U_t^\partial \in \mathcal{B}(H^{1/2}(\partial\Omega_t), H^{1/2}(\partial\Omega))$, and define $U_t^\partial : H^{-1/2}(\partial\Omega_t) \to H^{-1/2}(\partial\Omega)$ by

$$\langle U_t^\partial g, \phi\rangle_{-1/2} :=_{H^{-1/2}(\partial\Omega_t)} \langle g, U_{1/t}^\partial \phi\rangle_{H^{1/2}(\partial\Omega_t)}, \ \phi \in H^{1/2}(\partial\Omega). \tag{3.259}$$

It follows that the subset

$$\mathcal{G}_{\partial\Omega_t} := \left\{ \left(U^{\partial}_{1/t} f, U^{\partial}_{1/t} g\right) : (f,g) \in \mathcal{G} \right\} \subset H^{1/2}(\partial\Omega_t) \times H^{-1/2}(\partial\Omega_t), \tag{3.260}$$

is Lagrangian with respect to the natural symplectic form ω_t defined on $H^{1/2}(\partial\Omega_t) \times H^{-1/2}(\partial\Omega_t)$. Let $\mathcal{S}_{\Omega_t}$ denote the self-adjoint extension of $-\Delta_{min,t} + V|_{\Omega_t}$ associated with $\mathcal{G}_{\partial\Omega_t}$ via Theorem 3.8.

Hypothesis 3.29. *Assume that* $\operatorname{Spec}_{ess}(\mathcal{S}_{\Omega_t}) \cap (-\infty, 0] = \emptyset$, $t \in [\tau, 1)$, *and that there exists* $\lambda_\infty < 0$ *such that*

$$\operatorname{Spec}(\mathcal{S}_{\Omega_t}) \subset [\lambda_\infty, +\infty) \text{ for all } t \in [\tau, 1). \tag{3.261}$$

Proposition 3.30. *Assume Hypotheses* 3.28 *and* 3.29. *Then, for arbitrary* $\tau > 0$, *one has*

$$\operatorname{Mor}(\mathcal{S}_{\Omega_\tau}) - \operatorname{Mor}(\mathcal{S}_{\Omega_1}) = \operatorname{Mas}\left((\mathcal{K}_{0,t}, \mathcal{G}_t)|_{t\in[\tau,1]}\right), \tag{3.262}$$

where $\mathcal{K}_{0,t}$ *is defined by* (3.247) *with* $\lambda = 0$ *and* Q *replaced by* Ω, *and*

$$\mathcal{G}_t := \left\{ (f,g) \in H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m) : (f, t^{-1} g) \in \mathcal{G} \right\}, \; t \in [\tau, 1].$$

Proof. Clearly, the map $t \mapsto \mathcal{G}_t$, $t \in [\tau, 1]$ is contained in

$$C^1\left([\tau, 1], \Lambda\left(H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m)\right)\right).$$

Let $\mathcal{L}_t$ be the self-adjoint operator associated (via Theorem 3.8) with the differential expression

$$L_t = -t^{-2}\Delta + V(tx), x \in \Omega, \tag{3.263}$$

and the Lagrangian plane $\mathcal{G}_t$, $t \in [\tau, 1]$. By [50, Lemma 4.1],

$$w \in \ker\left(\mathcal{S}_{\Omega_t} - \lambda\right) \text{ if and only if } (U_t w) \in \ker\left(\mathcal{L}_t - \lambda\right), t \in [\tau, 1], \lambda \in \mathbb{R}. \tag{3.264}$$

Hence, $\mathrm{Mor}(\mathcal{S}_{\Omega_t}) = \mathrm{Mor}(\mathcal{L}_t), t \in [\tau, 1]$. The one-parameter family of self-adjoint operators $\mathcal{L}_t$ acting in $L^2(\Omega)$ together with the one-parameter family of the Lagrangian planes $\mathcal{G}_t, t \in [\tau, 1]$ satisfy hypotheses of Theorem 3.25, therefore

$$\mathrm{Mor}(\mathcal{L}_\tau) - \mathrm{Mor}(\mathcal{L}_1) = \mathrm{Mas}\left((\mathcal{K}_{0,t}, \mathcal{G}_t)|_{t\in[\tau,1]}\right). \tag{3.265}$$

Combining (3.265), $\mathcal{L}_1 = \mathcal{S}_{\Omega_1}$ and $\mathrm{Mor}(\mathcal{S}_{\Omega_\tau}) = \mathrm{Mor}(\mathcal{L}_\tau)$, we arrive at (3.262). ∎

Example 3.31. Assume Hypothesis 3.28. Let Θ be a given function satisfying

$$0 \le \Theta \in L^\infty(\partial\Omega, \mathbb{C}^{m\times m}),\ \Theta(x) = \overline{\Theta(x)}^\top, x \in \Omega.$$

The Lagrangian plane

$$\mathcal{G} := \left\{(f, g) \in H^{1/2}(\partial\Omega, \mathbb{C}^m) \times H^{-1/2}(\partial\Omega, \mathbb{C}^m) : \Theta f + g = 0\right\}, \tag{3.266}$$

gives rise to a one-parameter family of self-adjoint Schrödinger operators $\mathcal{S}_{\Omega_t}, t \in [\tau, 1]$ acting in $L^2(\Omega_t), t \in [\tau, 1], 0 < \tau < 1$ and given by

$$\mathcal{S}_{\Omega_t} u = -\Delta u + V|_{\Omega_t} u, u \in \mathrm{dom}(\mathcal{S}_{\Omega_t}),$$

$$\mathrm{dom}(\mathcal{S}_{\Omega_t}) = \{u \in \mathcal{D}^1_\Delta(\Omega_t) : \Theta\,(x/t)\,\gamma_{D,t} u(x) + \gamma_{N,t} u(x) = 0, x \in \partial\Omega_t\}.$$

By [78, Theorem 2.6], the operator $\mathcal{S}_{\Omega_t}$ is bounded from below and has compact resolvent. Hypothesis 3.29 is satisfied since Θ is bounded and nonnegative. Therefore, (3.262) holds in case of Schrödinger operators with Robin boundary conditions on star-shaped domains.

3.5 The abstract boundary value problems

In this section we elaborate on a natural relation between the theory of ordinary boundary triples originated in [38], [82], [108] and the theory of abstract boundary value spaces exploited in [29].

3.5.1 Lagrangian planes and self-adjoint extensions via the abstract boundary triples

We begin with several abstract results concerning the relations between the Morse and Maslov indices in the context of boundary triples. The following hypothesis is imposed throughout this section.

Hypothesis 3.32. *Let $\mathcal{H}, \mathfrak{H}$ be complex, separable Hilbert spaces. Assume that A is a densely defined, symmetric operator acting in $\mathcal{H}$. Assume that A has equal deficiency indices, that is,*

$$\dim\ker(A^* - \mathbf{i}) = \dim\ker(A^* + \mathbf{i}). \tag{3.267}$$

Definition 3.33 ([82])**.** Assume Hypothesis 3.32. Let $\Gamma_1, \Gamma_2 : \operatorname{dom}(A^*) \to \mathfrak{H}$ be linear maps. Then $(\mathfrak{H}, \Gamma_1, \Gamma_2)$ is said to be a boundary triple if the following assumptions are satisfied:

1) the abstract second Green identity holds, that is, for all $f, g \in \operatorname{dom}(A^*)$,

$$\langle A^* f, g\rangle_{\mathcal{H}} - \langle f, A^* g\rangle_{\mathcal{H}} = \langle \Gamma_1 f, \Gamma_2 g\rangle_{\mathfrak{H}} - \langle \Gamma_2 f, \Gamma_1 g\rangle_{\mathfrak{H}}; \tag{3.268}$$

2) the map $\operatorname{tr}_{\mathfrak{H}} := (\Gamma_1, \Gamma_2) : \operatorname{dom}(A^*) \to \mathfrak{H} \times \mathfrak{H}$ is onto, i.e., for arbitrary $(\varphi, \psi) \in \mathfrak{H} \times \mathfrak{H}$ there exists $u \in \operatorname{dom}(A^*)$, such that $\Gamma_1 u = \varphi,\ \Gamma_2 u = \psi$.

If Hypothesis 3.32 holds then there always exists a boundary triple associated to A, cf., [82]. Moreover,

$$\operatorname{tr}_{\mathfrak{H}} \in \mathcal{B}(\operatorname{dom}(A^*), \mathfrak{H} \times \mathfrak{H}) \text{ and } \ker(\operatorname{tr}_{\mathfrak{H}}) = \operatorname{dom}(A), \tag{3.269}$$

where $\operatorname{dom}(A^*)$ is viewed as a Hilbert space equipped with the graph norm of A^*

$$\|x\|_{A^*}^2 := \|x\|_{\mathcal{H}}^2 + \|A^* x\|_{\mathcal{H}}^2,\ x \in \operatorname{dom}(A^*). \tag{3.270}$$

The quotient space $\operatorname{dom}(A^*)/\operatorname{dom}(A)$ equipped with the bounded, non-degenerate, skew-symmetric form $\omega_{\mathcal{H}_A}$ defined by

$$\omega_{\mathcal{H}_A}([x],[y]) := \langle A^*x, y\rangle_{\mathcal{H}} - \langle x, A^*y\rangle_{\mathcal{H}}, [x],[y] \in \operatorname{dom}(A^*)/\operatorname{dom}(A), \tag{3.271}$$

is a symplectic Hilbert space with respect to the standard quotient norm induced by $\|\cdot\|_{A^*}$, where $[x]$ denotes the equivalence class of the vector $x \in \operatorname{dom}(A^*)$. The space $\operatorname{dom}(A^*)/\operatorname{dom}(A)$ was originally used in [29].

Proposition 3.34. *Let* $(\mathfrak{H}, \Gamma_1, \Gamma_2)$ *be a boundary triple. The map*

$$\widetilde{\operatorname{tr}}_{\mathfrak{H}} : \operatorname{dom}(A^*)/\operatorname{dom}(A) \to \mathfrak{H} \times \mathfrak{H},, \tag{3.272}$$

$$\operatorname{dom}(A^*)/\operatorname{dom}(A) \ni [x] \mapsto (\Gamma_1 x, \Gamma_2 x) \in \mathfrak{H} \times \mathfrak{H}, \tag{3.273}$$

is well defined, bounded, has bounded inverse, and

$$\omega_{\mathcal{H}_A}([x],[y]) = \omega_{\mathfrak{H}}\left(\widetilde{\operatorname{tr}}_{\mathfrak{H}}[x], \widetilde{\operatorname{tr}}_{\mathfrak{H}}[y]\right), \ [x],[y] \in \operatorname{dom}(A^*)/\operatorname{dom}(A), \tag{3.274}$$

where the symplectic form $\omega_{\mathfrak{H}}$ *is defied by*

$$\omega_{\mathfrak{H}}((f_1,g_1),(f_2,g_2)) := \langle f_1, g_2\rangle_{\mathfrak{H}} - \langle g_1, f_2\rangle_{\mathfrak{H}}, (f_k, g_k) \in \mathfrak{H} \times \mathfrak{H},\ k = 1,2. \tag{3.275}$$

That is, $\widetilde{\operatorname{tr}}_{\mathfrak{H}}$ *is a symplectomorphism of* $(\operatorname{dom}(A^*)/\operatorname{dom}(A), \omega_{\mathcal{H}})$ *onto* $(\mathfrak{H} \times \mathfrak{H}, \omega_{\mathfrak{H}})$.

Proof. Combining (3.269) and the fact that $\operatorname{tr}_{\mathfrak{H}}$ is onto, we infer that $\widetilde{\operatorname{tr}}_{\mathfrak{H}}$ is well defined, one-to-one, onto, and bounded. By the Open Mapping Theorem, $(\widetilde{\operatorname{tr}}_{\mathfrak{H}})^{-1} \in \mathcal{B}(\mathfrak{H} \times \mathfrak{H}, \operatorname{dom}(A^*)/\operatorname{dom}(A))$. The abstract second Green identity (3.268) yields (3.274). ∎

We will now provide a description of all self-adjoint extensions of A in terms of Lagrangian subspaces of $(\mathfrak{H} \times \mathfrak{H}, \omega_{\mathfrak{H}})$ (which is a consequence of the Lagrangian description via the abstract traces acting into the quotient space $\mathrm{dom}(A^*)/\mathrm{dom}(A)$ cf. [29, Lemma 3.3]), and prove the equivalence of the resolvent convergence of the extensions to the convergence of respective Lagrangian subspaces.

Theorem 3.35. *Assume Hypothesis* 3.32 *and let* $(\mathfrak{H}, \Gamma_1, \Gamma_2)$ *be a boundary triple, see, Definition* 3.33. *Then the self-adjoint extensions of* A *are in one-to-one correspondence with the Lagrangian planes in* $\mathfrak{H}\times\mathfrak{H}$, *that is the following two assertions hold.*

1. Let $\mathscr{D}\subset \mathrm{dom}(A^*)$, *and let* $A_{\mathscr{D}}$ *be an operator acting in* $\mathcal{H}$ *and given by*

$$A_{\mathscr{D}}u = A^*u,\ u\in \mathrm{dom}(A_{\mathscr{D}}) := \mathscr{D}. \tag{3.276}$$

If $A_{\mathscr{D}}$ *is self-adjoint, then the subspace*

$$\mathrm{tr}_{\mathfrak{H}}(\mathscr{D}) = \{(\Gamma_1 u, \Gamma_2 u) : u\in\mathscr{D}\}\subset \mathfrak{H}\times\mathfrak{H}, \tag{3.277}$$

is Lagrangian with respect to the symplectic form (3.275).

2. Conversely, if $\mathcal{G}\subset\mathfrak{H}\times\mathfrak{H}$ *is a Lagrangian subspace, then the operator* $A_{\mathrm{tr}_{\mathfrak{H}}^{-1}(\mathcal{G})}$ *acting in* $\mathcal{H}$ *and given by*

$$A_{\mathrm{tr}_{\mathfrak{H}}^{-1}(\mathcal{G})}u = A^*u,\ u\in \mathrm{dom}(A_{\mathrm{tr}_{\mathfrak{H}}^{-1}(\mathcal{G})}) := \mathrm{tr}_{\mathfrak{H}}^{-1}(\mathcal{G}), \tag{3.278}$$

is self-adjoint (here $\mathrm{tr}_{\mathfrak{H}}^{-1}(\mathcal{G}) := \{u : \mathrm{tr}_{\mathfrak{H}}\, u\in\mathcal{G}\}$ *denotes the preimage of the set* $\mathcal{G}$*).*

Moreover, let $A_n, n\geq 0$, *be a sequence of self-adjoint extensions of the operator* A *and let* $\mathcal{G}_n\subset\mathfrak{H}\times\mathfrak{H}, n\geq 0$, *be the corresponding sequence of Lagrangian planes such that* A_n *and* $\mathcal{G}_n$ *are related to each other as indicated in* (3.276), (3.277), (3.278). *Then*

$$R(\mathbf{i}, A_n)\to R(\mathbf{i}, A_0),\ n\to\infty,\quad \text{in } \mathcal{B}(\mathcal{H}), \tag{3.279}$$

(*here* $R(\mathbf{i}, A_n)$ *denotes the resolvent of* A_n) *if and only if*

$$\mathcal{G}_n \to \mathcal{G}_0,\ n \to \infty, \quad in\ \Lambda(\mathfrak{H} \times \mathfrak{H}). \tag{3.280}$$

Proof. Assume that $A_{\mathscr{D}}$ is self-adjoint. Then by Lemma 3.3 in [29], the subspace

$$[\mathscr{D}] := \{[x] : x \in \mathscr{D}\}, \tag{3.281}$$

is Lagrangian in $\operatorname{dom}(A^*)/\operatorname{dom}(A)$ with respect to the symplectic form $\omega_{\mathcal{H}_A}$, cf., (3.271). Since $\widetilde{\operatorname{tr}}_{\mathfrak{H}}$ is a symplectomorphism, $\widetilde{\operatorname{tr}}_{\mathfrak{H}}([\mathscr{D}])$ is a Lagrangian plane in $\mathfrak{H} \times \mathfrak{H}$ with respect to the form $\omega_{\mathfrak{H}}$, cf., (3.275). Furthermore,

$$\widetilde{\operatorname{tr}}_{\mathfrak{H}}([\mathscr{D}]) = \operatorname{tr}_{\mathfrak{H}}(\mathscr{D}), \tag{3.282}$$

hence, $\operatorname{tr}_{\mathfrak{H}}(\mathscr{D})$ is also Lagrangian.

Conversely, assume that $\mathcal{G}$ is Lagrangian in $\mathfrak{H} \times \mathfrak{H}$. Then, since $\ker(\operatorname{tr}_{\mathfrak{H}}) = \operatorname{dom}(A)$, one has

$$A \subset A_{\operatorname{tr}_{\mathfrak{H}}^{-1}(\mathcal{G})}. \tag{3.283}$$

By Proposition 3.34, $\widetilde{\operatorname{tr}}_{\mathfrak{H}}^{-1}(\mathcal{G})$ is Lagrangian in $\operatorname{dom}(A^*)/\operatorname{dom}(A)$. Since $\widetilde{\operatorname{tr}}_{\mathfrak{H}}^{-1}(\mathcal{G}) = [\operatorname{tr}_{\mathfrak{H}}^{-1}(\mathcal{G})]$ (we denote $[\operatorname{tr}_{\mathfrak{H}}^{-1}(\mathcal{G})] = \{[x] : x \in \operatorname{tr}_{\mathfrak{H}}^{-1}(\mathcal{G})\}$), by Lemma 3.3 in [29] the operator $A_{\operatorname{tr}_{\mathfrak{H}}^{-1}(\mathcal{G})}$ is self-adjoint in $\mathcal{H}$.

Next, we prove the last statement of the theorem. To this end, let us notice first that since

$$\|A_n x\|_{\mathcal{H}} = \|A^* x\|_{\mathcal{H}},\ x \in \operatorname{dom}(A_n), \tag{3.284}$$

the Hausdorff distance (cf., e.g., [103, Section IV.2]) between the graphs of A_n and A_0 with respect to the norm of $\mathcal{H} \times \mathcal{H}$ is equal to the Hausdorff distance between $\operatorname{dom}(A_n)$ and $\operatorname{dom}(A_0)$ with respect to the graph norm of A^*.

Combining convergence of the resolvents (3.279) and [103, Theorem IV-2.25], we obtain that $\operatorname{graph}(A_n) \to \operatorname{graph}(A_0),\ n \to \infty$, with respect to the Hausdorff distance in $\mathcal{H} \times \mathcal{H}$, hence, $\operatorname{dom}(A_n) \to \operatorname{dom}(A_0),\ n \to \infty$, with respect to the Hausdorff distance in $\operatorname{dom}(A^*)$ equipped with the graph norm of A^*. That is, $P_n \to P_0$ in $\mathcal{B}(\operatorname{dom}(A^*), \|\cdot\|_{A^*})$, where $P_n, n \geq 0$, denote the orthogonal projections on $\operatorname{dom}(A_n)$ in the Hilbert space $(\operatorname{dom}(A^*), \|\cdot\|_{A^*})$. Let P denote the orthogonal projection on $\operatorname{dom}(A)$, then, since $\operatorname{dom}(A) \subset \operatorname{dom}(A_n), n \geq 0$, the operator $Q_n := P_n - P$ is an orthogonal projection on $\operatorname{dom}(A_n) \ominus \operatorname{dom}(A), n \geq 0$ (with respect to A^*-graph inner product). Furthermore, since $\operatorname{dom}(A^*)/\operatorname{dom}(A)$ is isometrically isomorphic to $\operatorname{dom}(A^*) \ominus \operatorname{dom}(A)$, the orthogonal projections $Q_n, n \geq 0$, give rise to a sequence of projections $\widetilde{Q}_n, n \geq 0$ in $\mathcal{B}(\operatorname{dom}(A^*)/\operatorname{dom}(A))$ with $\widetilde{\operatorname{tr}}_{\mathfrak{H}}(\operatorname{ran}(\widetilde{Q}_n)) = \mathcal{G}_n$. Therefore $\mathcal{G}_n \to \mathcal{G}_0$.

Conversely, assume that $\mathcal{G}_n \to \mathcal{G}_0, n \to \infty$. Then using Proposition 3.34, we obtain a sequence of orthogonal projections $\widetilde{Q}_n, n \geq 0$, in $\operatorname{dom}(A^*)/\operatorname{dom}(A)$ with $\operatorname{ran}(\widetilde{Q}_n) = \widetilde{\operatorname{tr}}_{\mathfrak{H}}^{-1}(\mathcal{G}_n), n \geq 0$ and $\widetilde{Q}_n \to \widetilde{Q}_0, n \to \infty$ in $\mathcal{B}(\operatorname{dom}(A^*)/\operatorname{dom}(A))$. Then for a sequence of orthogonal projections $Q_n \in \mathcal{B}(\operatorname{dom}(A^*) \ominus \operatorname{dom}(A), \|\cdot\|_{A^*}), n \geq 0$, with $\operatorname{ran}(Q_n) = \operatorname{dom}(A_n) \ominus \operatorname{dom}(A), n \geq 0$, one has

$$Q_n \to Q_0,\ n \geq 0,\ \text{in } \mathcal{B}(\operatorname{dom}(A^*) \ominus \operatorname{dom}(A), \|\cdot\|_{A^*}).$$

Let P denote the orthogonal projection on $\operatorname{dom}(A)$, then

$$\operatorname{ran}(Q_n + P) = \operatorname{dom}(A_n), n \geq 0,$$

and

$$Q_n + P \to Q_0 + P,\ n \geq 0,\ \text{in } \mathcal{B}(\operatorname{dom}(A^*), \|\cdot\|_{A^*}).$$

Therefore, $\mathrm{dom}(A_n) \to \mathrm{dom}(A_0)$, $n \to \infty$, with respect to the Hausdorff distance in $\mathrm{dom}(A^*)$ equipped with the graph norm of A^*, and $\mathrm{graph}(A_n) \to \mathrm{graph}(A_0)$, $n \to \infty$, with respect to the Hausdorff distance in $\mathcal{H} \times \mathcal{H}$. Finally, [103, Theorem IV-2.25] yields (3.279). ∎

Remark 3.36. 1) The convergence $\mathcal{G}_n \to \mathcal{G}$, as $n \to \infty$ from the last statement of Theorem 3.35, in fact, yields that if $z \in \mathbb{C} \setminus \mathrm{Spec}(A_0)$ then $z \in \mathbb{C} \setminus \mathrm{Spec}(A_k)$ for all sufficiently large $k \in \mathbb{N}$.

2) We remark that in some instances the Krein-type resolvent formulas (cf., e.g., [78, 135]) can be used in order to deduce resolvent convergence from the convergence of the corresponding Lagrangian planes, see Proposition 3.10 above.

Next we turn to the Maslov index in the context of self-adjoint, Fredholm extensions of symmetric operators.

Hypothesis 3.37. *(1) Assume that a one-parameter family $t \mapsto V_t \in \mathcal{B}(\mathcal{H})$ is contained in $C^1([\alpha,\beta],\mathcal{B}(\mathcal{H}))$, $\alpha < \beta$, and $V_t^* = V_t$, $t \in [\alpha,\beta]$.*

(2) Assume Hypothesis 3.32 *and that* $\ker(A^* + V_t - \lambda) \cap \mathrm{dom}(A) = \{0\}$ *for all* $t \in [\alpha,\beta]$, *and* $\lambda \geq \lambda_\infty$ *for some* $\lambda_\infty < 0$.

(3) Assume that $(\mathfrak{H}, \Gamma_{1,t}, \Gamma_{2,t})$, $t \in [\alpha,\beta]$, is a one-parameter family of boundary triples associated with A such that the family $t \mapsto \mathrm{tr}_{\mathfrak{H},t} := (\Gamma_{1,t}, \Gamma_{2,t})$ is contained in $C^1\big([\alpha,\beta], \mathcal{B}(\mathrm{dom}(A^*), \mathfrak{H} \times \mathfrak{H})\big)$.

We remark that the second condition in Hypothesis 3.37 often holds in case of second order differential operators considered on bounded domains $\Omega \subset \mathbb{R}^n$ (and can be viewed as an abstract version of the unique continuation principle, cf. [96, Theorem 3.2.2]). The third condition is natural in the context of the geometric deformations of domain Ω and the corresponding change of variables in conormal derivative, cf. Subsections 3.3.4 and 3.4.3.

The following theorem is a corollary of results from [29] and Proposition 3.34, hence we will only sketch the proof.

Theorem 3.38. *Assume Hypotheses* 3.32 *and* 3.37. *Let* $t \mapsto \mathcal{G}_t$ *be a one-parameter family containing in* $C^1([\alpha,\beta], \Lambda(\mathfrak{H} \times \mathfrak{H}))$. *Let* $A_{\mathscr{D}_t}, t \in [\alpha,\beta]$, *denote the self-adjoint extension of the operator* A *with domain* $\operatorname{tr}_{\mathfrak{H},t}^{-1}(\mathcal{G}_t), t \in [\alpha,\beta]$. *Assume that* $A_{\mathscr{D}_t}, t \in [\alpha,\beta]$, *has compact resolvent and that*

$$\ker(A_{\mathscr{D}_t} + V_t - \lambda) = 0 \text{ for all } t \in [\alpha,\beta], \lambda < \lambda_\infty,$$

where λ_∞ *is defined in Hypothesis* 3.37 *(2). Then*

$$\operatorname{Mor}\left(A_{\mathscr{D}_\alpha} + V_\alpha\right) - \operatorname{Mor}\left(A_{\mathscr{D}_\beta} + V_\beta\right) = \operatorname{Mas}\left((\mathbb{K}_{0,t}, \mathcal{G}_t)|_{t\in[\alpha,\beta]}\right), \tag{3.285}$$

where $\mathbb{K}_{\lambda,t}$ *denotes the traces of the "strong" solutions of the equation* $A^*u + V_tu = \lambda u,\ u \in \operatorname{dom}(A^*)$, *that is,*

$$\mathbb{K}_{\lambda,t} := \operatorname{tr}_{\mathfrak{H},t}\left(\ker(A^* + V_t - \lambda)\right), t \in [\alpha,\beta], \lambda \in \mathbb{R}. \tag{3.286}$$

Proof. First, using parametrization (3.108)-(3.112) we introduce two loops with values in $\Lambda(\mathfrak{H} \times \mathfrak{H})$ by the formulas

$$\Sigma \ni s \mapsto \mathbb{K}_{\lambda(s),t(s)} \in \Lambda(\mathfrak{H} \times \mathfrak{H}), \tag{3.287}$$

$$\Sigma \ni s \mapsto \mathcal{G}_{t(s)} \in \Lambda(\mathfrak{H} \times \mathfrak{H}). \tag{3.288}$$

By [29, Theorem 3.9], the one-parameter family $\Sigma \ni s \mapsto \ker(A^* + V_{t(s)} - \lambda(s))/\operatorname{dom}(A)$ is continuous and contained in $C^1(\Sigma_k, \Lambda(\operatorname{dom}(A^*)/\operatorname{dom}(A)),\ 1 \le k \le 4$. That is, there exists a family of orthogonal projections $\Sigma \ni s \mapsto P_s \in \mathcal{B}\left(\operatorname{dom}(A^*)/\operatorname{dom}(A)\right)$ such that

$$\operatorname{ran}(P_s) = \ker(A^* + V_{t(s)} - \lambda(s))/\operatorname{dom}(A), \tag{3.289}$$

$$s \mapsto P_s \in C^1(\Sigma_k, \mathcal{B}(\operatorname{dom}(A^*)/\operatorname{dom}(A))),\ 1 \le k \le 4, \tag{3.290}$$

$$s \mapsto P_s \in C(\Sigma, \mathcal{B}(\operatorname{dom}(A^*)/\operatorname{dom}(A))). \tag{3.291}$$

Then $\Sigma \ni s \mapsto Q_s := \widetilde{\operatorname{tr}}_{\mathfrak{H},t(s)} P_s (\widetilde{\operatorname{tr}}_{\mathfrak{H},t(s)})^{-1} \in \mathcal{B}(\mathfrak{H} \times \mathfrak{H})$ is a family of bounded projections such that

$$\operatorname{ran}(Q_s) = \operatorname{tr}_{\mathfrak{H},t(s)}\left(\ker(A^* + V_{t(s)} - \lambda(s)\right), s \in \Sigma, \tag{3.292}$$

$$s \mapsto Q_s \in C^1(\Sigma_k, \mathcal{B}(\mathfrak{H} \times \mathfrak{H})),\ 1 \le k \le 4,\ s \mapsto Q_s \in C(\Sigma, \mathcal{B}(\mathfrak{H} \times \mathfrak{H})). \tag{3.293}$$

The projection Q_s may not be orthogonal, however, it can be replaced by orthogonal projection while preserving regularity as in (3.293).

Second, we observe that $\operatorname{Mas}\left((\mathbb{K}_{\lambda(s),t(s)}, \mathcal{G}_{t(s)})|_{s\in\Sigma}\right) = 0$ by the homotopy invariance of the Maslov index. On the other hand,

$$\begin{aligned} &\operatorname{Mas}\left((\mathbb{K}_{\lambda(s),t(s)}, \mathcal{G}_{t(s)})|_{s\in\Sigma}\right) = \\ &\quad + \operatorname{Mas}\left((\mathbb{K}_{\lambda(s),t(s)}, \mathcal{G}_{t(s)})|_{s\in\Sigma_1}\right) + \operatorname{Mas}\left((\mathbb{K}_{\lambda(s),t(s)}, \mathcal{G}_{t(s)})|_{s\in\Sigma_2}\right) \\ &\quad + \operatorname{Mas}\left((\mathbb{K}_{\lambda(s),t(s)}, \mathcal{G}_{t(s)})|_{s\in\Sigma_3}\right) + \operatorname{Mas}\left((\mathbb{K}_{\lambda(s),t(s)}, \mathcal{G}_{t(s)})_{s\in\Sigma_4}\right). \end{aligned} \tag{3.294}$$

Finally, proceeding as in the proof of Theorem 3.18 one can show that the crossings on Σ_1 are negative definite, the crossings on Σ_3 are positive definite, and that there are no crossings on Σ_4. Thus,

$$\begin{aligned} 0 = &- \sum_{\lambda_\infty < \lambda < 0} \dim\left(\ker(A_{\mathscr{D}_\alpha} + V_\alpha - \lambda)\right) \\ &+ \operatorname{Mas}\left((\mathbb{K}_{\lambda(s),t(s)}, \mathcal{G}_{t(s)})|_{s\in\Sigma_2}\right) + \sum_{\lambda_\infty < \lambda < 0} \dim(\ker(A_{\mathscr{D}_\beta} + V_\beta - \lambda)), \end{aligned} \tag{3.295}$$

as asserted in (3.286). ∎

We will now discuss several particular applications of the abstract results of this subsection.

3.5.2 Spectra of $\theta-$periodic Schrödinger operators on [0,1] and the Maslov index

The boundary triple technique employed in Theorem 3.38 is well suited for ordinary differential operators. Indeed, let

$$\mathcal{T}_{min} := -\partial_x^2,\ \mathrm{dom}(\mathcal{T}_{min}) := H_0^2[0,1], \mathcal{T}_{max} := (\mathcal{T}_{min})^*,$$

and recall from [82, Chapter 3] that

$$\mathcal{T}_{max} := -\partial_x^2,\ \mathrm{dom}(\mathcal{T}_{max}) := H^2[0,1].$$

The operator $\mathcal{T}_{min}$ admits a boundary triple

$$\begin{aligned}&\mathfrak{H} = \mathbb{C}^{2m},\ \Gamma_1 : H^2[0,1] \to \mathbb{C}^{2m}, \Gamma_1 u := (u(1), u(0))^\top,\\ &\Gamma_2 : H^2[0,1] \to \mathbb{C}^{2m}, \Gamma_2 u := (u'(1), -u'(0))^\top.\end{aligned} \tag{3.296}$$

Next we turn to a self-adjoint extension of $\mathcal{T}_{\min}$. For each fixed $\theta \in [0, 2\pi)$ the operator

$$(-\partial_x^2)_\theta : L^2([0,1], \mathbb{C}^m) \to L^2([0,1], \mathbb{C}^m),\ (-\partial_x^2)_\theta u := -u'',\ u \in \mathrm{dom}((-\partial_x^2)_\theta),$$

$$\mathrm{dom}((-\partial_x^2)_\theta) := \{u \in AC([0,1], \mathbb{C}^m) : u' \in AC[0,1], u'' \in L^2([0,1], \mathbb{C}^m),$$

$$u(1) = e^{\mathbf{i}\theta} u(0),\ u'(1) = e^{\mathbf{i}\theta} u'(0)\},$$

is self-adjoint with compact resolvent. Let $V \in L^\infty([0,1], \mathbb{C}^{m\times m}), V = \overline{V}^\top$, and denote $\mathcal{L}_\theta := (-\partial_x^2)_\theta + V$. Then $\mathcal{L}_\theta$ is also self-adjoint, has compact resolvent, and

$$\inf_{\theta\in[0,2\pi)} \min\{\lambda : \lambda \in \mathrm{Spec}(H_\theta)\} > -\infty. \tag{3.297}$$

Let us denote $\mathcal{G}_\theta := (\Gamma_1, \Gamma_2)(\mathrm{dom}(\mathcal{L}_\theta))$. Clearly, the map $\theta \mapsto \mathcal{G}_\theta$, where

$$\mathcal{G}_\theta = \{(e^{\mathbf{i}\theta} a, a, -e^{\mathbf{i}\theta} b, b) : a, b \in \mathbb{C}^m\}, \tag{3.298}$$

is contained in $C^1([0,\pi],\Lambda(\mathbb{C}^{2m}\times\mathbb{C}^{2m}))$. Hence, the one-parameter family $\mathcal{L}_\theta, \theta \in [0,\pi]$, together with boundary triple $(\mathbb{C}^{2m},\Gamma_1,\Gamma_2)$ satisfy hypotheses of Theorem 3.38. Then

$$\operatorname{Mor}(\mathcal{L}_{\theta_1}) - \operatorname{Mor}(\mathcal{L}_{\theta_2}) = \operatorname{Mas}\left((\mathbb{K},\mathcal{G}_\theta)|_{\theta\in[\theta_1,\theta_2]}\right), 0\le\theta_1<\theta_2\le\pi, \tag{3.299}$$

where

$$\mathbb{K} := \{(u(1),u(0),u'(1),-u'(0))^\top : -u''+Vu=0\} \subset \mathbb{C}^{4m}.$$

Remark 3.39. We stress that the result concerning equality of the spectral flow and the Maslov index for Sturm-Liouville operators on $[0,1]$ is obtained in full generality in [37, Theorem 0.4] . In particular, (3.299) can be alternatively derived using [37, Theorem 0.4]. The symplectic structure used in [37, Theorem 0.4] is determined by the first order system of ODE's equivalent to the eigenvalue problem for original Sturm-Liouville operator. In contrast, our symplectic structure is induced by the right-hand side of the Green's formula (3.268) and we do not need to rewrite the eigenvalue problem as the first order ODE. As a result we deal with Lagrangian planes that are symplectomorphic to their counterparts from [37].

3.5.3 Spectra of self-adjoint Schrödinger operators and the Maslov index

In this section we illustrate (3.285) in the context of the self-adjoint extensions of $-\Delta_{min}$, cf. (3.227), (3.228). Hypothesis 3.1 is assumed throughout this subsection. Let as recall the following two facts from [77]:

1) there exists a unique linear, bounded operator

$$\widehat{\gamma}_D : \mathcal{D}^1_\Delta(\Omega) \to (N^{1/2}(\partial\Omega))^*, \tag{3.300}$$

which is compatable with the Dirichlet trace γ_D, cf. (3.205);

2) there exists a unique linear, bounded operator

$$\widehat{\gamma}_N : \mathcal{D}^1_\Delta(\Omega) \to (N^{3/2}(\partial\Omega))^*, \tag{3.301}$$

which is compatable with the Neumann trace $\gamma_{\scriptscriptstyle N}$, cf. (3.206).

The Dirichlet-to-Neumann map $M_{D,N}$ is defined by

$$M_{D,N} : \begin{cases} (N^{1/2}(\partial\Omega))^* & \to (N^{3/2}(\partial\Omega))^* \\ \qquad\qquad f & \to -\widehat{\gamma}_N(u_D), \end{cases} \tag{3.302}$$

where u_D is the unique solution of the boundary value problem

$$-\Delta u = 0 \text{ in } \Omega, \quad u \in L^2(\Omega),\ \widehat{\gamma}_D u = f \text{ in } \partial\Omega. \tag{3.303}$$

Denoting $\tau_{\scriptscriptstyle N} u := \widehat{\gamma}_N u + M_{D,N}(\widehat{\gamma}_D u)$, one has

$$\begin{aligned} &(-\Delta u, v)_{L^2(\Omega)} - (u, -\Delta v)_{L^2(\Omega)} \\ &= \overline{{}_{N^{1/2}(\partial\Omega)}\langle \tau_{\scriptscriptstyle N} v, \widehat{\gamma}_D u\rangle_{(N^{1/2}(\partial\Omega))^*}} - {}_{N^{1/2}(\partial\Omega)}\, \langle \tau_{\scriptscriptstyle N} u, \widehat{\gamma}_D v\rangle_{(N^{1/2}(\partial\Omega))^*}, \end{aligned} \tag{3.304}$$

for every $u, v \in \operatorname{dom}(-\Delta_{\max})$, cf. [77]. In other words, $-\Delta_{min}$ admits the following boundary triple

$$\mathfrak{H} := N^{1/2}(\partial\Omega), \Gamma_1 := R^{-1}\widehat{\gamma}_D,\ \Gamma_2 := \tau_N, \tag{3.305}$$

where $R : N^{1/2}(\partial\Omega) \to (N^{1/2}(\partial\Omega))^*$ is the Riesz duality isomorphism. With this at hand, the following proposition is a corollary of Theorem 3.38.

Corollary 3.40. *Assume Hypothesis* 3.1. *Let* $\mathscr{D} \subset \mathcal{D}^1_\Delta(\Omega)$ *and assume that the operator* $-\Delta_{\mathscr{D}}$ *acting in* $L^2(\Omega, \mathbb{C}^m), m \in \mathbb{N}$, *and given by*

$$-\Delta_{\mathscr{D}} u = -\Delta u,\ u \in \operatorname{dom}(-\Delta_{\mathscr{D}}) := \mathscr{D}. \tag{3.306}$$

is self-adjoint, bounded from below and has compact resolvent. Let

$$t \mapsto V_t \in L^\infty(\Omega, \mathbb{C}^{m\times m}), V_t(x) = \overline{V_t(x)^\top}, x \in \Omega,$$

be a one-parameter family containing in $C^1([\alpha,\beta], L^\infty(\Omega, \mathbb{C}^m))$. *Then*

$$\mathrm{Mor}(-\Delta_{\mathscr{D}} + V_\alpha) - \mathrm{Mor}(-\Delta_{\mathscr{D}} + V_\beta) = \mathrm{Mas}\big((K_{0,t}, \mathcal{G})|_{t\in[\alpha,\beta]}\big), \tag{3.307}$$

where

$$K_{0,t} = (R^{-1}\widehat{\gamma}_D, \tau_{{}_N})\big(\{u \in \mathcal{D}^1_\Delta(\Omega) : -\Delta_{max} u + V_t u = 0\}\big),\ \mathcal{G} := (R^{-1}\widehat{\gamma}_D, \tau_{{}_N})(\mathscr{D}).$$

Appendix A

A Minimization Problem

In this appendix we carry out the explicit minimization in α for $\alpha > 0$ of the integral

$$\mathcal{I}_K(\alpha) := \alpha^{-1} \int_{\mathbb{R}^n} \left[\alpha - |\eta|^{4m} + |\eta|^{2m}\right]_+ d^n\eta. \tag{A.1}$$

Since the integral is only over the region of n-space where $\alpha - |\eta|^{4m} + |\eta|^{2m}$ is positive, and this function is radial, our problem immediately reduces to the minimization of α^{-1} times a radial integral in $r = |\eta|$. Since the function $r^{4m} - r^{2m} = r^{2m}(r^{2m} - 1)$ is negative on $0 < r < 1$ and is positive and increasing for $r > 1$, for $\alpha > 0$ the relation $\alpha = r^{4m} - r^{2m}$ implicitly determines a unique value $r_\alpha > 1$, with r_α^{2m} given explicitly by

$$r_\alpha^{2m} = \frac{1}{2} + \left(\alpha + \frac{1}{4}\right)^{1/2}. \tag{A.2}$$

It is clear that the value of r_α is a strictly increasing function of α and runs from 1 to ∞ as α runs from 0 to ∞.

By the reductions mentioned above, one obtains

$$\mathcal{I}_K(\alpha) = n v_n \alpha^{-1} \int_0^{r_\alpha} [\alpha + r^{2m} - r^{4m}]\, r^{n-1}\, dr, \tag{A.3}$$

where v_n is the volume of the ball of unit radius in $\mathbb{R}^n$ as mentioned with (2.153). Since the v_n here is included explicitly in (2.153), to prove (2.153), in what remains we will show that the function $f_{n,m}(\alpha)$ defined by

$$f_{n,m}(\alpha) := n\alpha^{-1}\int_0^{r_\alpha} [\alpha + r^{2m} - r^{4m}]\, r^{n-1}\, dr \tag{A.4}$$

has minimum given by

$$\widetilde{f}_{n,m} := \Big(1 + \frac{2m}{n+2m}\Big)^{n/(2m)}, \quad m, n \in \mathbb{N}. \tag{A.5}$$

By integrating (A.4), it is easy to see that

$$f_{n,m}(\alpha) := n\alpha^{-1}\left[\frac{\alpha r_\alpha^n}{n} + \frac{r_\alpha^{n+2m}}{n+2m} - \frac{r_\alpha^{n+4m}}{n+4m}\right]. \tag{A.6}$$

Replacing the explicit α appearing inside the square brackets here using $\alpha = r_\alpha^{4m} - r_\alpha^{2m}$ and simplifying, one finds

$$\alpha\, f_{n,m}(\alpha) = \frac{4m\, r_\alpha^{n+4m}}{n+4m} - \frac{2m\, r_\alpha^{n+2m}}{n+2m}. \tag{A.7}$$

We shall have need of this expression shortly.

Next, some further properties of $f_{n,m}$ and its derivative will be developed. One has

$$\alpha\, f_{n,m}(\alpha) = n\int_0^{r_\alpha} [\alpha + r^{2m} - r^{4m}]\, r^{n-1}\, dr, \tag{A.8}$$

and therefore, by Leibniz's rule,

$$[\alpha\, f_{n,m}(\alpha)]' = n[\alpha + r_\alpha^{2m} - r_\alpha^{4m}]\, r_\alpha^{n-1}\, r'_\alpha + n\int_0^{r_\alpha} r^{n-1}\, dr = r_\alpha^n, \tag{A.9}$$

with the simplification in the last step occurring due to the implicit relation defining r_α. From (A.9) it follows that

$$\alpha\, f'_{n,m}(\alpha) = r_\alpha^n - f_{n,m}(\alpha), \tag{A.10}$$

and hence, using (A.7) and $\alpha = r_\alpha^{4m} - r_\alpha^{2m}$, that

$$\alpha^2 f'_{n,m}(\alpha) = \alpha\, r_\alpha^n - \left[\frac{4m\, r_\alpha^{n+4m}}{n+4m} - \frac{2m\, r_\alpha^{n+2m}}{n+2m}\right]$$

$$=n\left(r_\alpha^{2m} - \frac{n+4m}{n+2m}\right)\frac{r_\alpha^{n+2m}}{n+4m}. \tag{A.11}$$

It is now clear that $f_{n,m}(\alpha)$ has a minimum on $\alpha \in (0,\infty)$, and that it occurs at

$$r_\alpha^{2m} = \frac{n+4m}{n+2m} := \widetilde{r}_\alpha^{2m} \tag{A.12}$$

(one notes that this value is clearly larger than 1, and hence corresponds to an $\alpha > 0$). The corresponding value of α, denoted by $\widetilde{\alpha}$, may then be computed as

$$\widetilde{\alpha} = \widetilde{r}_\alpha^{2m}(\widetilde{r}_\alpha^{2m} - 1) = \frac{n+4m}{n+2m}\frac{2m}{n+2m} = \frac{2m(n+4m)}{(n+2m)^2}. \tag{A.13}$$

Finally one computes $\widetilde{f}_{n,m}$ using (A.7), (A.12), and (A.13), which leads to

$$\widetilde{f}_{n,m} = f_{n,m}(\widetilde{\alpha}) = \left(\frac{n+4m}{n+2m}\right)^{n/(2m)} = \left(1 + \frac{2m}{n+2m}\right)^{n/(2m)}, \quad m,n \in \mathbb{N}, \tag{A.14}$$

in accordance with our statement above. This completes the proof of Theorem 2.21.

We conclude with some remarks comparing the constant $\widetilde{f}_{n,m}$ found here with the corresponding constants $\widetilde{g}_{n,m}$ (our notation) found by Laptev in [114] (the comparison is most apt if we restrict our attention to the case of the Laplacian (i.e., $a = I_n$, $b = q = 0$), as that is the main case considered by Laptev [114]). Laptev's $\widetilde{g}_{n,m}$ are given by

$$\widetilde{g}_{n,m} = \left(1 + \frac{2m}{n}\right)^{n/(2m)}, \quad m,n \in \mathbb{N}. \tag{A.15}$$

It is clear from these expressions that

$$\widetilde{f}_{n,m} < \widetilde{g}_{n,m}, \quad m,n \in \mathbb{N}. \tag{A.16}$$

This shows that the bound given in Theorem 2.21 is always better than the bound (2.41) combined with the earlier work of Laptev [114]. Of course in the large n limit (for fixed m) both constants become arbitrarily close, since the limit of either $\widetilde{g}_{n,m}$

or $\widetilde{f}_{n,m}$ as $n \to \infty$ is $e \approx 2.71828$. On the other hand, in the large m limit (with n fixed) both constants go to 1 from above (with 1 being the best possible value of the constant that could be obtained in our upper bounds, at least in the case of the Laplacian).

In fact, it is generally true that

$$1 < \widetilde{f}_{n,m} < \widetilde{g}_{n,m} < e, \quad m, n \in \mathbb{N}, \tag{A.17}$$

that is, that

$$1 < (1 + 2m/(n + 2m))^{n/2m} < (1 + (2m/n))^{n/(2m)} < e, \quad m, n \in \mathbb{N}, \tag{A.18}$$

with 1 and e being the best possible lower and upper bounds for both $\widetilde{f}_{n,m}$ and $\widetilde{g}_{n,m}$ for all $m, n \in \mathbb{N}$. These claims can be proved using elementary calculus by focusing on the functions $G(x) := (\ln(1 + x))/x$ and $F(x) := (\ln(1 + x/(1 + x)))/x$ for $x > 0$ (note that with the identification $x = 2m/n$ these are the logarithms of $\widetilde{g}_{n,m}$ and $\widetilde{f}_{n,m}$, respectively, and that all $x > 0$ can be approximated arbitrarily closely by such ratios for $m, n \in \mathbb{N}$). In fact, one can show that the functions $G(x)$ and $F(x)$ are both strictly decreasing on $(0, \infty)$, with limiting value 1 as $x \to 0^+$, and with limiting value 0 as $x \to \infty$. This implies, in particular, that in all upper bound formulas for counting functions $N(\cdot)$ in this paper the bound would continue to hold (as a strict inequality) if the constant represented by $(1 + 2m/(n + 2m))^{n/2m}$ were replaced by the value e.

Bibliography

[1] A. Abbondandolo, *Morse Theory for Hamiltonian Systems.* Chapman & Hall/CRC Res. Notes Math. **425**, Chapman & Hall/CRC, Boca Raton, FL, 2001.

[2] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.

[3] A. Alonso and B. Simon, *The Birman-Krein-Vishik theory of selfadjoint extensions of semibounded operators*, J. Operator Th. **4**, 251–270 (1980); Addenda: **6**, 407 (1981).

[4] P. Alsholm and G. Schmidt, *Spectral and scattering theory of Schrödinger operators*, Arch. Rat. Mech. Anal. **40**, 281–311 (1971).

[5] V. I. Arnold, *Characteristic classes entering in quantization conditions,* Func. Anal. Appl. **1**, 1–14 (1967).

[6] V. I. Arnold, *Sturm theorems and symplectic geometry,* Func. Anal. Appl. **19**, 1–10 (1985).

[7] N. Arrizabalaga and M. Zubeldia, *Unique continuation for the magnetic Schrödinger operator with singular potentials*, Proc. Amer. Math. Soc. **143**, 3487–3503 (2015).

[8] M. S. Ashbaugh, F. Gesztesy, A. Laptev, M. Mitrea, and S. Sukhtaiev, *A Bound for the Eigenvalue Counting Function For Krein–von Neumann and Friedrichs Extensions,* Advances in Mathematics **304**, 1108-1155 (2017).

[9] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, and G. Teschl, *Spectral theory for perturbed Krein Laplacians in nonsmooth domains*, Adv. Math. **223**, 1372–1467 (2010).

[10] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, R. Shterenberg, and G. Teschl, *The Krein—von Neumann extension and its connection to an abstract buckling problem*, Math. Nachr. **283**, 165–179 (2010).

[11] M. S. Ashbaugh, F. Gesztesy, M. Mitrea, R. Shterenberg, and G. Teschl, *A survey on the Krein–von Neumann extension, the corresponding abstract buckling problem, and Weyl-type spectral asymptotics for perturbed Krein Laplacians in non smooth domains*, in *Mathematical Physics, Spectral Theory and Stochastic Analysis*, M. Demuth and W. Kirsch (eds.), Operator Theory: Advances and Applications, Vol. 232, Birkhäuser, Springer, Basel, 2013, pp. 1–106.

[12] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry III,* Math. Proc. Cambridge Philos. Soc. **79**, 71–99 (1976).

[13] M. F. Atiyah, I. M. Singer *Index theory for skew-adjoint Fredholm operators,* Inst. Hautes tudes Sci. Publ. Math. **37**, 5–26 (1969).

[14] A. A. Balinsky and W. D. Evans, *Spectral Analysis of Relativistic Operators*, Imperial College Press, London, 2011.

[15] J. Behrndt, F. Gesztesy, T. Micheler, and M. Mitrea, *The Krein-von Neumann realization of perturbed Laplacians on bounded Lipschitz domains*, arXiv1501.02896.

[16] M. Ben-Artzi, *Eigenfunction expansions and spacetime estimates for generators in divergence-form*, Reviews Math. Phys. **22**, 1209–1240 (2010); Erratum **26**, 1492001 (3 pages) (2014).

[17] M. Sh. Birman, *On the theory of self-adjoint extensions of positive definite operators*, Mat. Sbornik **38**, 431–450 (1956) (Russian).

[18] M. Sh. Birman and M. Z. Solomyak, *Leading term in the asymptotic spectral formula for "non-smooth" elliptic problems*, Funkcional. Anal. i Priložen **4**, no. 4, 1–13 (1970) (Russian); Engl. transl. in Funct. Anal. Appl. **4**, 265–275 (1970).

[19] M. Sh. Birman and M. Z. Solomyak, *On the asymptotic spectrum of "non-smooth" elliptic equations*, Funkcional. Anal. i Priložen **5**, no. 1, 69–70 (1971) (Russian); Engl. transl. in Funct. Anal. Appl. **5**, 56–57 (1971).

[20] M. Š. Birman and M. Z. Solomjak, *Spectral asymptotics of nonsmooth elliptic operators*, Sov. Math. Dokl. **13**, 906–910 (1972).

[21] M. Š. Birman and M. Z. Solomjak, *Spectral asymptotics of nonsmooth elliptic operators. I*, Trans. Moscow Math. Soc. **27**, 1–52 (1972).

[22] M. Š. Birman and M. Z. Solomjak, *Spectral asymptotics of nonsmooth elliptic operators. II*, Trans. Moscow Math. Soc. **28**, 1–32 (1973).

[23] M. Sh. Birman and M. Z. Solomyak, *Asymptotic behavior of the spectrum of differential equations*, Itogi Nauki i Tekhniki, Matematicheskii Analiz., **14**, 5–58 (1977) (Russian); Engl. transl. in J. Soviet Math. **12**, no. 3, 247–283 (1979).

[24] M. S. Birman and M. Z. Solomjak, *Quantitative Analysis in Sobolev Imbedding Theorems and Applications to Spectral Theory*, AMS Translations, Series 2, Vol. 114, Providence, RI, 1980, pp. 1–132.

[25] D. Bollé, F. Gesztesy, and C. Danneels, *Threshold scattering in two dimensions*, Ann. Inst. H. Poincaré **48**, 175–204 (1988).

[26] D. Bollé, F. Gesztesy, and M. Klaus, *Scattering theory for one-dimensional systems with* $\int dx\, V(x) = 0$, J. Math. Anal. Appl. **122**, 496–518 (1987); Erratum **130**, 590 (1988).

[27] D. Bollé, F. Gesztesy, and S. F. J. Wilk, *A complete treatment of low-energy scattering in one dimension*, J. Operator Theory **13**, 3–31 (1985).

[28] R. Bott, *On the iteration of closed geodesics and the Sturm intersection theory,* Comm. Pure Appl. Math. **9**, 171–206 (1956).

[29] B. Booss-Bavnbek, K. Furutani, *The Maslov Index: a functional analytical definition and the spectral flow formula*, Tokyo J. Math. **21**, 1–34 (1998).

[30] J. Behrndt, M. Langer, *Boundary value problems for elliptic partial differential operators on bounded domains*, J. Func. Anal. **243**, 536–565 (2007).

[31] J. Behrndt, T. Micheler, *Elliptic differential operators on Lipschitz domains and abstract boundary value problems*, J. Funct. Anal. **267**, 3657–3709 (2014).

[32] J. Behrndt, J. Rohleder, *An inverse problem of Calderon type with partial data*, Comm. Partial Diff. Eqns. **37**, 1141–1159 (2012).

[33] B. Booss-Bavnbek, K. Wojciechowski, *Elliptic boundary problems for Dirac operators,* Birkhäuser, Boston, MA, 1993.

[34] B. Booss-Bavnbek, C. Zhu, *General spectral flow formula for fixed maximal domain,* Cent. Eur. J. Math. **3**, 558-577 (2005).

[35] B. Booss-Bavnbek, C. Zhu, *The Maslov index in symplectic Banach spaces,* Preprint, http://arxiv.org/abs/1406.0569

[36] B. Booss-Bavnbek, C. Zhu, *The Maslov index in weak symplectic functional analysis,* Ann. Global Anal. Geom. **44**, 283-318 (2013).

[37] B. Booss-Bavnbek, C. Zhu, *Weak symplectic functional analysis and general spectral flow formula,* Preprint, http://arxiv.org/abs/math/0406139

[38] V. M. Bruk, *A certain class of boundary value problems with a spectral parameter in the boundary condition,* Mat. Sb. 100 (242), 210-216 (1976).

[39] A. Buffa, G. Geymonat, *On traces of functions in $W^{2,p}(\Omega)$ for Lipschitz domains in $\mathbb{R}^3$*, C. R. Acad. Sci. Paris Sr. I Math. **8**, 699–704 (2001).

[40] N. Burq, *Lower bounds for shape resonances widths of long range Schrödinger operators*, Amer. J. Math. **124**, 677–735 (2001).

[41] N. Burq, *Semi-classical estimates for the resolvent in nontrapping geometries*, Int. Math. Res. Notices **2002**, No. 5, 221–241.

[42] S. Cappell, R. Lee, E. Miller, *On the Maslov index*, Comm. Pure Appl. Math. **47**, 121–186 (1994).

[43] F. Chardard, T. J. Bridges, *Transversality of homoclinic orbits, the Maslov index, and the symplectic Evans function*, Nonlinearity **28** (2015), 77–102.

[44] F. Chardard, F. Dias, T. J. Bridges, *Fast computation of the Maslov index for hyperbolic linear systems with periodic coefficients,* J. Phys. A **39**, 14545–14557 (2006).

[45] F. Chardard, F. Dias, T. J. Bridges, *Computing the Maslov index of solitary waves. I. Hamiltonian systems on a four-dimensional phase space,* Phys. D **238**, 1841–1867 (2009).

[46] F. Chardard, F. Dias, T. J. Bridges, *Computing the Maslov index of solitary waves, Part 2: Phase space with dimension greater than four,* Phys. D **240**, 1334–1344 (2011).

[47] M. Cheney, *Two-dimensional scattering: The number of bound states from scattering data*, J. Math. Phys. **25**, 1449–1455 (1984).

[48] C. Conley, E. Zehnder, *Morse-type index theory for flows and periodic solutions for Hamiltonian equations,* Comm. Pure Appl. Math. **37**, 207–253 (1984).

[49] R. Courant, D. Hilbert, *Methods of mathematical physics. Vol. I,* Interscience Publishers, Inc., New York, N.Y., 1953.

[50] G. Cox, C. K. R. T. Jones, Y. Latushkin, and A. Sukhtayev, *The Morse and Maslov indices for multidimentional Schrödinger operators with matrix valued potential*, Trans. Amer. Math. Soc. **368**, 8145–8207 (2016).

[51] G. Cox, C. K. R. T. Jones, J. Marzuola, *A Morse index theorem for elliptic operators on bounded domains*, Comm. Partial Diff. Eqns. **40**, 1467–1497 (2015).

[52] G. Cox, C. K. R. T. Jones, and J. Marzuola, *Manifold decompositions and indices of Schrdinger operators*, to appear in Indiana Univ. Math. J., http://arxiv.org/abs/1506.07431.

[53] R. Cushman, J. J. Duistermaat, *The behavior of the index of a periodic linear Hamiltonian system under iteration,* Advances in Math. **23**, 1–21 (1977).

[54] Ju. Daleckii, M. Krein, *Stability of Solutions of Differential Equations in Banach Space,* Amer. Math. Soc., Providence, RI, 1974.

[55] K. Datchev, S. Dyatlov, and M. Zworski, *Resonances and lower resolvent bounds*, J. Spectr. Theory **5**, 599–615 (2015).

[56] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Math., Vol. 92, Cambridge Univ. Press, Cambridge, 1989.

[57] J. Deng, C. Jones, *Multi-dimensional Morse Index Theorems and a symplectic view of elliptic boundary value problems,* Trans. Amer. Math. Soc. **363**, 1487–1508 (2011).

[58] V. A. Derkach and M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95**, 1–95 (1991).

[59] J. J. Duistermaat, *On the Morse index in variational calculus,* Advances in Math. **21**, 173–195 (1976).

[60] R. G. Duran, M. A. Muschietti *On the traces of* $W^{2,p}(\Omega)$ *for a Lipschitz domain,* Revista Matem. Complutense **14**, 371–377 (2001).

[61] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1987.

[62] M. Erdogan, M. Goldberg, and W. Schlag, *Strichartz and smoothing estimates for Schrödinger operators with large magnetic potentials in* $\mathbb{R}^3$, J. Eur. Math. Soc. **10**, 507–531 (2008).

[63] M. Erdogan, M. Goldberg, and W. Schlag, *Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions*, Forum Math. **21**, 687–722 (2009).

[64] L. Fanelli, *Non-trapping magnetic fields and Morrey–Campanato estimates for Schrödinger operators*, J. Math. Anal. Appl. 357, 1–14 (2009).

[65] W. G. Faris, *Time decay and the Born series*, Rocky Mountain J. Math. **1**, 637–648 (1971).

[66] W. G. Faris, *Self-Adjoint Operators*, Lecture Notes in Mathematics, Vol. 433, Springer, Berlin, 1975.

[67] G. B. Folland *Introduction to partial differential equations, 2nd edition*, Princeton University Press, Princeton, New Jersey, 1995.

[68] R. Frank, A. Laptev, and T. Weidl, *Pólya's conjecture in the presence of a constant magnetic field*, J. Eur. Math. Soc. **11**, 1365–1383 (2009).

[69] L. Friedlander, *Some inequalities between Dirichlet and Neumann eigenvalues,* Arch. Rational Mech. Anal., **116**, 153–160 (1991).

[70] K. Furutani, *Fredholm-Lagrangian-Grassmannian and the Maslov index,* J. Geom. Phys. **51**, 269–331 (2004).

[71] A. Garcia, L^p–L^q estimates for electromagnetic Helmholtz equation. Singular potentials, J. Funct. Anal. **268**, 2787–2819 (2015).

[72] L. Geisinger, *A short proof of Weyl's law for fractional differential operators*, J. Math. Phys. **55**, 011504, 7 pp., (2014).

[73] L. Geisinger, A. Laptev, and T. Weidl, *Geometrical versions of improved Berezin–Li–Yau inequalities*, J. Spectral Theory **1**, 87–109 (2011).

[74] F. Gesztesy, A. Laptev, M. Mitrea, S. Sukhtaiev, *A bound for the eigenvalue counting function for higher-order Krein Laplacians on open sets.* In: *Mathematical Results in Quantum Mechanics, QMath12 Conference*, P. Exner, W. König, and H. Neidhardt (eds), World Scientific, Singapore, 2015, pp. 3–29.

[75] F. Gesztesy, Y. Latushkin, F. Sukochev, and Y. Tomilov, *Some operator bounds employing complex interpolation revisited*, in *Operator Semigroups Meet Complex Analysis, Harmonic Analysis and Mathematical Physics*, W. Arendt, R. Chill,

and Y. Tomilov (eds.), Operator Theory: Advances and Applications, Vol. 250, Birkhäuser, Springer, Basel, 2015, pp. 213–239.

[76] F. Gesztesy, M. Mitrea, *Generalized Robin Laplacians and some remarks on a paper by Filonov on eigenvalue inequalities*, J. Diff. Eq. **247**, 2871–2896 (2009).

[77] F. Gesztesy, M. Mitrea, *A description of all self-adjoint extensions of the Laplacian and Krein-type resolvent formulas on non-smooth domains*, J. Analyse Math. **113**, 53–172 (2011).

[78] F. Gesztesy, M. Mitrea, *Generalized Robin boundary conditions, Robin-to-Dirichlet maps, and Krein-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains*, In: Perspectives in partial differential equations, harmonic analysis and applications, 105 – 173, Proc. Sympos. Pure Math., **79**, Amer. Math. Soc., Providence, RI, 2008.

[79] F. Gesztesy, M. Mitrea, *Robin-to-Robin maps and Krein-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains,* Oper. Theory Adv. Appl. **191**, 81–113 (2009).

[80] G. Geymonat, *Trace theorems for Sobolev spaces on Lipschitz domains. Necessary conditions.*, Annales mathmatiques Blaise Pascal **14**, 187–197 (2007).

[81] I. Gohberg, S. Goldberg, and M. Kaashoek, *Classes of linear operators. Vol. I.*, Birkhäuser Verlag, Basel, 1990.

[82] V. I. Gorbachuk, M. L. Gorbachuk, *Boundary value problems for operator differential equations*, Kluwer Academic Publ., Dordrecht, 1991.

[83] M. A. de Gosson, *The principles of Newtonian and quantum mechanics*, Imperial College Press, London, 2001.

[84] G. Grubb, *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa (3), **22**, 425–513 (1968).

[85] G. Grubb, *Les problèmes aux limites généraux d'un opérateur elliptique, provenant de le théorie variationnelle*, Bull. Sci. Math. (2), **94**, 113–157 (1970).

[86] G. Grubb, *Spectral asymptotics for the "soft" selfadjoint extension of a symmetric elliptic differential operator*, J. Operator Th. **10**, 9–20 (1983).

[87] G. Grubb, *Distributions and Operators*, Graduate Texts in Mathematics, Vol. 252, Springer, New York, 2009.

[88] G. Grubb, *Krein-like extensions and the lower boundedness problem for elliptic operators*, J. Diff. Eq. **252**, 852–885 (2012).

[89] E. M. Harrell II and L. Hermi, *Differential inequalities for Riesz means and Weyl-type bounds for eigenvalues*, J. Funct. Anal. **254**, 3173–3191 (2008).

[90] E. M. Harrell II and L. Hermi, *On Riesz means of eigenvalues*, Commun. Partial Diff. Eq. **36**, 1521–1543 (2011).

[91] P. Howard, A. Sukhtayev, *The Maslov and Morse indices for Schrödinger operators on* $[0, 1]$, J. Diff. Eq., **260**, 4499-4549(2016).

[92] P. Howard, Y. Latushkin, and A. Sukhtayev, *The Maslov index for Lagrangian pairs on* $\mathbb{R}^{2n}$, Preprint, arXiv:1608.00632

[93] P. Howard, Y. Latushkin, and A. Sukhtayev, *The Maslov and Morse indices for Schrdinger operators on* $\mathbb{R}$, Preprint, arXiv:1608.05692

[94] T. Ikebe, *Eigenfunction expansions associated with the Schroedinger operators and their applications to scattering theory*, Arch. Rat. Mech. Anal. **5**, 1–34 (1960).

[95] T. Ikebe and Y. Saitō, *Limiting absorption method and absolute continuity for the Schrödinger operator*, J. Math. Kyoto Univ. **12**, 513–524 (1972).

[96] V. Isakov, *Inverse problems for partial differential equations, Second edition* Springer-Verlag, New York, 2006.

[97] C. K. R. T. Jones, Y. Latushkin, R. Marangel, The Morse and Maslov indices for matrix Hill's equations, Proceedings of Symposia in Pure Mathematics **87**, 205–233 (2013).

[98] C. K. R. T. Jones, Y. Latushkin, S. Sukhtaiev, Counting spectrum via the Maslov index for one dimensional $\theta-$periodic Schrödinger operators, Proc. AMS **145**, 363–377 (2017).

[99] Y. Karpeshina, *Perturbation theory for the Schrödinger operator with a periodic potential,* Lecture Notes Math. **1663**, Springer-Verlag, Berlin, 1997.

[100] T. Kato, *Notes on some inequalities for linear operators*, Math. Ann. **125**, 208–212 (1952).

[101] T. Kato, *Growth properties of solutions of the reduced wave equation with a variable coefficient*, Commun. Pure Appl. Math. **12**, 403–425 (1959).

[102] T. Kato, *Schrödinger operators with singular potentials*, Israel J. Math. **13**, 135–148 (1972).

[103] T. Kato, *Perturbation Theory for Linear Operators*, corr. printing of the 2nd ed., Springer, Berlin, 1980.

[104] T. Kapitula, K. Promislow, *Spectral and dynamical stability of nonlinear waves*, Springer, New York, 2003.

[105] J. B. Keller, *Corrected Bohr-Sommerfeld Quantum Conditions for Nonseparable Systems*, Ann. Physic **4**, 180-188 (1958).

[106] P. Kirk, M. Lesch *The $\theta-$invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary*, Forum Math. **16**, 553–629 (2004).

[107] H. Koch and D. Tataru, *Carleman estimates and unique continuation for second order parabolic equations with nonsmooth coefficients*, Commun. Part. Diff. Eqs. **34**, 305–366 (2009).

[108] A. N. Kočubeĭ, On extensions of symmetric operators and symmetric binary relations, Mat. Zametki **17**, 41-48 (1975).

[109] V. A. Kozlov, *Estimation of the remainder in a formula for the asymptotic behavior of the spectrum of nonsemibounded elliptic systems*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **1979**, no 4., 112–113, 125 (Russian).

[110] V. A. Kozlov, *Estimates of the remainder in formulas for the asymptotic behavior of the spectrum for linear operator bundles*, Funktsional. Anal. i Prilozhen

17, no. 2, 80–81 (1983). Engl. transl. in Funct. Anal. Appl. **17**, no. 2, 147–149 (1983).

[111] V. A. Kozlov, *Remainder estimates in spectral asymptotic formulas for linear operator pencils*, Linear and Nonlinear Partial Differential Equations. Spectral Asymptotic Behavior, pp. 34–56, Probl. Mat. Anal. **9**, Leningrad Univ., Leningrad, 1984; Engl. transl. in J. Sov. Math. **35**, 2180–2193 (1986).

[112] M. G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I*, Mat. Sbornik **20**, 431–495 (1947) (Russian).

[113] S. G. Krein, Ju. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators*, Transl. Math. Monographs, Vol. 54, Amer. Math. Soc., Providence, RI, 1982.

[114] A. Laptev, *Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces*, J. Funct. Anal. **151**, 531–545 (1997).

[115] Y. Latushkin, S. Sukhtaiev, *The Maslov index and the spectra of second order differential operators,* Preprint, https://arxiv.org/abs/1610.09765.

[116] Y. Latushkin, S. Sukhtaiev, A. Sukhtayev, *The Morse and Maslov indices for Schrödinger operators*, J. D'Analyse Math., to appear, http://arxiv.org/abs/1411.1656.

[117] Y. Latushkin, A. Sukhtayev, *Hadamard-type formulas via the Maslov form,* to appear in Journal of Evolution Equations, https://arxiv.org/abs/1601.07509.

[118] S. Levendorskiĭ, *Asymptotic Distribution of Eigenvalues of Differential Operators*, Kluwer, Dordrecht, 1990.

[119] P. Li and S.-T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Commun. Math. Phys. **88**, 309–318 (1983).

[120] E. H. Lieb, *The number of bound states of one-body Schroedinger operators and the Weyl problem*, in *Geometry of the Laplace Operator*, R. Osserman and A. Weinstein (eds.), Proc. Symposia Pure Math., Vol. 36, Amer. Math. Soc., Providence, RI, 1980, pp. 241–252.

[121] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Graduate Studies in Math., Vol. 14, Amer. Math. Soc., Providence, RI, 2001.

[122] M. M. Malamud, *Certain classes of extensions of a lacunary Hermitian operator*, Ukrainian Math. J. **44**, No. 2, 190–204 (1992).

[123] V. P. Maslov, *Theory of perturbations and asymptotic methods*, Izdat. Moskov. Gos. Univ., Moscow, 1965. French translation Dunod, Paris, 1972.

[124] D. McDuff and D. Salamon *Introduction to Symplectic Topology. Second Edition*, Clarendon Press, Oxford, 1998.

[125] W. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, 2000.

[126] D. McDuff, D. Salamon *Introduction to Symplectic Topology. Second Edition*, Clarendon Press, Oxford, 1998.

[127] M. Melgaard and G. V. Rozenblum, *Spectral estimates for magnetic operators*, Math. Scand. **79**, 237–254 (1996).

[128] G. Métivier, *Valeurs propres de problèmes aux limites elliptiques irégulières*, Mém. Soc. Math. France **51–52**, 125–219 (1977).

[129] V. A. Mikhaĭlets, *Distribution of the eigenvalues of finite multiplicity of Neumann extensions of an elliptic operator*, Differentsial'nye Uravneniya **30**, 178–179 (1994) (Russian); Engl. transl. in Diff. Eq. **30**, 167–168 (1994).

[130] V. A. Mikhailets, *Discrete spectrum of the extreme nonnegative extension of the positive elliptic differential operator*, in *Proceedings of the Ukrainian Mathematical Congress–2001, Section 7, Nonlinear Analysis*, Kyiv, 2006, pp. 80–94.

[131] J. Milnor, *Morse Theory*, Annals of Math. Stud. **51**, Princeton Univ. Press, Princeton, N.J., 1963.

[132] Yu. Netrusov, Yu. Safarov, *Weyl asymptotic formula for the Laplacian on domains with rough boundaries*, Comm. Math. Phys. **253**, no. 2, 481–509 (2005).

[133] L. I. Nicolaescu, *The Maslov index, the spectral flow, and decompositions of manifolds*, Duke Math. Journal **80**, 485–533. (1995)

[134] E. M. Ouhabaz, *Analysis of Heat Equations on Domains*, London Mathematical Society Monographs Series, Vol. 31, Princeton University Press, Princeton, NJ, 2005.

[135] K. Pankrashkin, *Resolvents of self-adjoint extensions with mixed boundary conditions*, Rep. Math. Phys. **58**, 207–221 (2006).

[136] A. Portaluri, N. Waterstraat, *A Morse-Smale index theorem for indefinite elliptic systems and bifurcation,* J. Diff. Eq. **258**, 1715–1748 (2015).

[137] J. Rauch and M. Taylor, *Regularity of functions smooth along foliations, and elliptic regularity*, J. Funct. Anal. **225**, 74–93 (2005).

[138] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. I: Functional Analysis*, Academic Press, New York, 1980.

[139] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II: Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[140] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. III: Scattering Theory*, Academic Press, New York, 1979.

[141] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV: Analysis of Operators*, Academic Press, New York, 1978.

[142] J. Robbin, D. Salamon, *The Maslov index for paths*, Topology **32**, 827–844 (1993).

[143] J. Robbin, D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27**, 1–33 (1995).

[144] I. Rodnianski and T. Tao, *Effective limiting absorption principles, and applications*, Commun. Math. Phys. **333**, 1–95 (2015).

[145] J. Rohleder, *Strict Inequality of Robin Eigenvalues for Elliptic Differential Operators on Lipschitz Domains*, Journal of Math. Anal. and App. **418**, 978–984 (2014).

[146] G. V. Rozenbljum, *On the distribution of eigenvalues of the first boundary value problem in unbounded regions*, Sov. Math. Dokl. **12**, 1539–1542 (1971).

[147] G. V. Rozenbljum, *On the eigenvalues of the first boundary value problem in unbounded domains*, Math. USSR Sb. **18**, 235–248 (1972).

[148] G. V. Rozenbljum, *The distribution of the discrete spectrum for singular differential operators*, Sov. Math. Dokl. **13**, 245–249 (1972).

[149] G. V. Rozenblyum, *Distribution of the discrete spectrum of singular differential operators*, Sov. Math. **20**, 63–71 (1976).

[150] G. V. Rozenblum, M. A. Shubin, and M. Z. Solomyak, *Spectral Theory of Differential Operators*, in *Partial Differential Equations VII*, M. A. Shubin (ed.), Encyclopaedia of Mathematical Sciences, Vol. 64, Springer, Berlin, 1994.

[151] Yu. Safarov, *Fourier Tauber theorems and applications*, J. Funct. Anal. **185**, 111–128 (2001).

[152] Yu. Safarov and D. Vassiliev, *The Asymptotic Distribution of Eigenvalues of Partial Differential Operators*, Transl. of Math. Monographs, Vol. 155, Amer. Math. Soc., Providence, RI, 1997.

[153] D. Salamon, K. Wehrheim, *Instanton Floer homology with Lagrangian boundary conditions*, Geom. Topol. **12**, 747–918 (2008).

[154] K. Schmüdgen, *Unbounded Self-Adjoint Operators on Hilbert Space*, Graduate Texts in Mathematics, Vol. 265, Springer, Dordrecht, 2012.

[155] S. Smale, *On the Morse index theorem*, J. Math. Mech. **14** (1965), 1049 – 1055; see also pp. 535–543 in: *The collected papers by Stephen Smale, V. 2*, F. Cucker and R. Wong, edts., City University of Hong Kong, 2000.

[156] B. Simon, *On positive eigenvalues of one-body Schrödinger operators*, Commun. Pure Appl. Math. **22**, 531–538 (1967).

[157] B. Simon, *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms*, Princeton University Press, Princeton, NJ, 1971.

[158] B. Simon, *Universal diamagnetism of spinless Bose systems*, Phys. Rev. Lett. **36**, 1083–1084 (1976).

[159] G. Stolz, *Expansions in generalized eigenfunctions of Schrödinger operators with singular potentials*, in *Topics in Operator Theory: Ernst D. Hellinger Memorial Volume*, L. De Branges, I. Gohberg, and J. Rovnyak (eds.), Operator Theory: Advances and Applications, Vol. 48, Birhäuser, Basel, 1990, pp. 353–372.

[160] R. C. Swanson, *Fredholm intersection theory and elliptic boundary deformation problems,* J. Differential Equations **28** (1978), *I*, 189–201, *II*, 202–219.

[161] M. E. Taylor, *Partial Differential Equations I. Basic Theory*, Appl. Math. Sci., **115**, Springer-Verlag, 2011.

[162] M. Taylor, *Regularity for a class of elliptic operators with Dini continuous coefficients*, J. Geom. Anal. **21**, 174–194 (2011).

[163] D. W. Thoe, *Eigenfunction expansions associated with Schroedinger operators in* $\mathbb{R}^n$*, $n \geq 4$*, Arch. Rat. Mech. Anal **26**, 335–356 (1967).

[164] K. Uhlenbeck, *The Morse index theorem in Hilbert spaces,* J. Diff. Geometry **8**, 555–564 (1973).

[165] M. L. Višik, *On general boundary problems for elliptic differential equations*, Trudy Moskov. Mat. Obsc. **1**, 187–246 (1952) (Russian); Engl. transl. in Amer. Math. Soc. Transl. (2), **24**, 107–172 (1963).

[166] G. Vodev, *On the exponential bound of the cutoff resolvent*, Serdica J. Math. **26**, 49–58 (2000).

[167] G. Vodev, *Exponential bounds of the resolvent for a class of noncompactly supported perturbations of the Laplacian*, Math. Res. Lett. **7**, 287–298 (2000).

[168] T. Weidl, *Improved Berezin–Li–Yau inequalities with a remainder term*, in *Spectral Theory of Differential Operators. M. Sh. Birman 80th Anniversary Collection*, Adv. Math. Sci., Vol. 62, Amer Math. Soc. Transl. (2) **225** (2008), pp. 253–263.

[169] J. Weidmann, *Linear Operators in Hilbert Spaces*, Graduate Texts in Mathematics, Vol. 68, Springer, New York, 1980.

[170] J. Wloka, *Partial differential equations*, Cambridge University Press, Cambridge, 1987.

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