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# Effect of relaxation parameters on the performance of a gas laser*

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This paper generalises the results of the well known Lamb theory of optical masers for a single mode operation by lifting the *Doppler Limit* restriction on $\gamma_{ab}$, the relaxation parameter and $ku$, the Doppler parameter. The equation for the intensity parameter $I$ is seen to be highly nonlinear in $\gamma_{ab}/ku$ and the detuning. The dependence of $I$ on $\gamma_{ab}/ku$ is more remarkable away from resonance. For comparable values of $\gamma_{ab}/ku$ the Lamb dip is sharper than the earlier predicted value. The effect of pressure and collisions have been incorporated in a simplified manner through $\gamma_{ab}/ku$. The results so derive show a qualitative similarly with the results of Stenholm for the same problem. A comparison with the results obtained from rate equation approximation is made.

## 1. Introduction

Lamb's theory of optical masers (Lamb 1964) is capable of explaining, at least qualitatively, most observed features of laser operations and also forms a basis of theories for Zeeman and ring lasers. However, departures from this theory occur even at moderate intensity. These departures are thought to be due to truncation of the solution at the third order or may be inherent in the formulation of the problem (Stenholm & Lamb 1969). Uehara & Shimoda (1965) and Shimoda & Uehara (1971) have extended the calculations to higher orders. Stenholm & Lamb (1969) have incorporated the effect of population change into the solution through continued fractions or rate equation approximation (REA). Except for the work of Stenholm (1970) the theory seems to be restricted to the Doppler limit. Lamb (1964) denotes the Doppler limit approximation as the *δ-function approximation* which restricts the usefulness of the solution because the exact effect of $\gamma_{ab}$, the relaxation parameter and $ku$, the Doppler parameter cannot be calculated. This restriction must be lifted to make Lamb's theory more universally usable. Lamb-Stenholm approach does not give an explicit expression for intensity parameter $I$. Indeed, even for moderate intensity of a laser, no explicit expression for $I$, showing the exact effect of $\gamma_{ab}$ and $ku$, seems to exist. Gyorffy, Borenstein & Lamb (1968) have calculated an expression for $I$ at a moderate intensity, but the Doppler limit restriction still

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remains. Further, their treatment is valid only for operations near resonance Lamb's results, when extended to higher orders (Uehara & Shimoda 1965) and with REA (Stenholm & Lamb 1969) seem to be fairly accurate, so that the knowledge of the explicit dependence of $I$ on $\gamma_{ab}/ku$ within the strict framework of the theory, seems desirable. This might also answer the question of the limit of validity of Lamb's theory.

Hence the derivation of explicit dependence of $I$ on $\gamma_{ab}$ and $ku$ is the prime aim of this paper. This value of $I$ will be shown to be quite accurate, subject to the limit of certain numerical computation, for moderate values of the relative excitation density $N$. As the relaxation terms are known to be influenced by pressure and collision effects in the material system, the results of this paper have been extended, albeit in an over simplified manner, to include these effects Though theories describing these effects are not lacking (Gyorffy *et al* 1968, Stenholm 1970), this step is desirable on two counts. First, as mentioned expressions for $I$, away from resonance and from Doppler limit restrictions, are lacking even for low and moderate excitation density. Second, considering the fact that experiments are unable to distinguish between the different descriptions of collisions, a simple theory displaying the essential features of laser operations at moderate intensities may not be out of place (Stenholm 1970).

In the next two sections the basis for calculation is established and expressions for frequency pulling and $I$ are derived. Section 4 introduces the idea of collision and pressure effects. The implications are discussed in section 5, incorporating the idea of saturation and comparing the results with the ones already existing.

## 2. First and Third Order Calculations

For the purpose of this paper the calculations are restricted to single frequency operations of an optical masor. Following Lamb the first order value of $C^{(1)}(t)$ and $S^{(1)}(t)$ are then the same as those given by Lamb (1964)

$$\begin{aligned} C^{(1)} &= (r^2/\hbar)E\bar{N}(\pi^{\frac{1}{2}}/ku)V(x, y) \\ S^{(1)} &= -(r^2/\hbar)E\bar{N}(\pi^{\frac{1}{2}}/ku)U(x, y) \end{aligned} \tag{1}$$

where $U(x, y)$ and $V(x, y)$ are the real and imaginary parts of the error function of the complex argument $z = (x+iy)$ with $x = (\nu-\omega)/ku$ and $y = \gamma_{ab}/ku$; $\gamma_{ab}$ being the relaxation constant for the non-diagonal elements of the density matrix. $E$ is the electric field and $r$ is the matrix element for the electric dipole moment between the lasing levels. $\bar{N}$ is the excitation density averaged over the cavity. The functions $U(x, y)$ and $V(x, y)$ are well tabulated (Feddeyeva & Terent'ev

1961) and are related to plasma dispersion function through a multiplying constant. $U(x, y)$ and $V(x, y)$ are defined by the equation

$$w(z) = U(x, y)+iV(x, y) = i\pi^{-1}\int_{-\infty}^{+\infty} \frac{\exp(-t^2)}{z-t}\,dt. \qquad \ldots \quad (2)$$

In the first order it is also seen

$$\frac{1}{Q} = \frac{\sqrt{\pi}r^2\bar{N}_T}{\epsilon_0\hbar ku}\,U(x, y) \qquad \ldots \quad (3)$$

and

$$(\nu-\Omega) = -\frac{r^2\sqrt{\pi}}{2\hbar\epsilon_0}\,\bar{N}\,\frac{\nu}{ku}\,V(x, y). \qquad \ldots \quad (4)$$

For a single frequency operation the third order values of $C$ and $S$, under Lamb's (1964) condition are given by

$$C^{(3)}(t) = -\frac{1}{8}\,r^4\hbar^{-3}E^3\,\frac{\bar{N}}{\gamma}\,\frac{\sqrt{\pi}}{(\omega-\nu)^2+\gamma_{ab}^2}\left[\frac{\nu-\omega}{ku}\,U(x, y)+\frac{\gamma_{ab}}{ku}\,V(x, y)\right] \qquad \ldots \quad (5)$$

$$S^{(3)}(t) = \frac{1}{4}\,r^4\hbar^{-3}E^3\,\frac{\bar{N}}{\gamma}\,\frac{\sqrt{\pi}}{(\omega-\nu)^2+\gamma_{ab}^2}\left[\frac{\gamma_{ab}}{ku}\,U(x, y)-\frac{\nu-\omega}{ku}\,V(x, y)\right.$$

$$\left.+\frac{1}{2}\,\frac{\nu-\omega}{\gamma_{ab}}\left\{\frac{\nu-\omega}{ku}\,U(x, y)+\frac{\gamma_{ab}}{ku}\,V(x, y)\right\}\right] \qquad \ldots \quad (6)$$

where $\dfrac{1}{\gamma} = \dfrac{1}{2}\left[\dfrac{1}{\gamma_a}+\dfrac{1}{\gamma_b}\right]$.

## 3. Intensity Parameter and Frequency Detuning

Up to third order, Lamb's (1964) amplitude determining equation becomes

$$\dot{E} = \alpha E-\beta E^3 \qquad \ldots \quad (7)$$

where

$$\alpha = -\tfrac{1}{2}\nu/Q\left[1-\bar{N}\frac{U(x, y)}{U(0, y)}\right] \qquad \ldots \quad (7)$$

and

$$\beta = \frac{1}{8}\,\frac{r^2}{\hbar^2 Q}\,\frac{\bar{N}}{\gamma}\,\frac{ku}{U(0, y)}\,\frac{\nu}{(\omega-\nu)^2+\gamma_{ab}^2}\left[\left\{\frac{\gamma_{ab}}{ku}\,U(x, y)-\frac{\nu-\omega}{ku}\,V(x, y)\right\}\right.$$

$$\left.+\frac{1}{2}\left(\frac{\nu-\omega}{\gamma_{ab}}\right)\left\{\frac{\nu-\omega}{ku}\,U(x, y)+\frac{\gamma_{ab}}{ku}\,V(x, y)\right\}\right] \qquad \ldots \quad (8)$$

where $\bar{N} = N/N_T$. We define the dimensionless intensity parameter $I$ by

$$\frac{1}{2}\left[\frac{(rE)^2}{\hbar^2\gamma\gamma_{ab}}\right].$$

$$I = \frac{1}{2}\left[\frac{(rE)^2}{\hbar^2\gamma\gamma_{ab}}\right] = 2\,\frac{\gamma_{ab}}{ku}\left[\left(\frac{\omega-\nu}{\gamma_{ab}}\right)^2+1\right]\times$$

$$\frac{[U(x,y)-\bar{N}^{-1}U(0,y)]}{\left[\frac{\gamma_{ab}}{ku}U(x,y)-\frac{\nu-\omega}{ku}V(x,y)\right]+\frac{1}{2}\left(\frac{\nu-\omega}{\gamma_{ab}}\right)\left[\frac{\nu-\omega}{ku}U(x,y)+\frac{\gamma_{ab}}{ku}V(x,y)\right]} \quad \ldots \quad (9)$$

The expression for $I$ at resonance ($\nu = \omega$) is independent of $\gamma_{ab}/ku$ and in this the present results are identical with Lamb's expression. In Fig. (1), we compare the intensity detuning curve of this paper with Lamb's (1964) results for $ku/\gamma_{ab}$

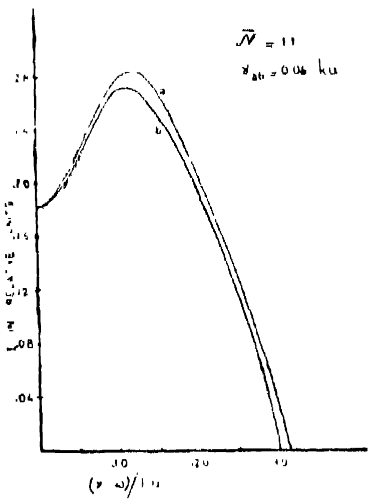



Fig. 1. Comparison of Lamb's result (b) with the present result (a).

= 20. Bennett (1962) has performed the experiment under conditions corresponding to $ku/\gamma_{ab} = 20$. It is seen in Fig. (1) that Lamb dip at resonance become more sharper than the Lamb's curve. The shallowness of Lamb's curve is due to the Doppler limit approximation.

The frequency determining equation of Lamb (1964) gives

$$\nu-\Omega = \frac{1}{2}\nu\frac{N}{Q}\frac{V(x,y)}{U(0,y)}+\frac{1}{4}\,\frac{\bar{N}U(x,y)-U(0,y)}{QU(0,y)}$$

$$\times\,\frac{\left[\frac{\nu-\omega}{ku}U(x,y)+\frac{\gamma_{ab}}{ku}V(x,y)\right]}{\left[\frac{\gamma_{a}}{ku}U(x,y)-\frac{\nu-\omega}{ku}V(x,y)\right]+\frac{1}{2}\,\frac{\nu-\omega}{\gamma_{ab}}\left[\frac{\nu-\omega}{ku}U(x,y)+\frac{\gamma_{ab}}{ku}V(x,y)\right]} \quad \ldots \quad (10)$$

From Fig. (2) points of stable operation can be computed from the points where the slope of the curve changes sign. When the condition

$$\left[\frac{\gamma_{ab}}{ku}U(x,y)-\frac{\nu-\omega}{ku}V(x,y)\right]+\frac{1}{2}\frac{\nu-\omega}{\gamma_{ab}}\left[\frac{\nu-\omega}{ku}U(x,y)+\frac{\gamma_{ab}}{ku}V(x,y)\right]$$

$$<\frac{1}{2}\frac{U(x,y)}{V(x,y)}\left[\frac{\nu-\omega}{ku}U(x,y)+\frac{\gamma_{ab}}{ku}V(x,y)\right] \qquad \ldots \quad (11)$$

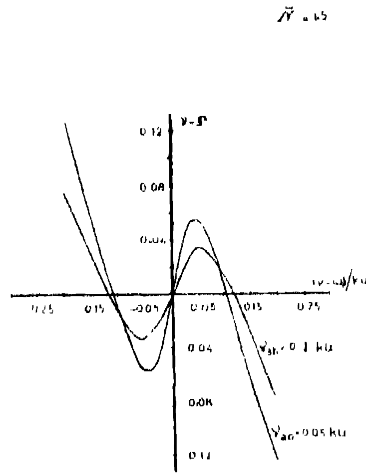



Fig. 2 Dependence of $(\nu-\Omega)$ on $(\nu-\omega)/ku$ for $ku/\gamma_{ab}$ = 10, 20

is satisfied, an increase of relative excitation $\bar{N}$ moves the operational frequency from $\omega$ to $\Omega$. For small detuning the condition (11) becomes

$$1 < \frac{1}{2}\sqrt{\pi}\,\frac{ku}{\gamma_{ab}}$$

4. Effect of Pressure and Collision on Intensity Parameter

For considering effects of pressure and collision on the operation of the gas laser, we define the intensity parameter

$$I = \frac{1}{2}\left[\frac{(rE)^2}{\hbar^2\gamma\gamma_{ab}}\right]\frac{\gamma\,\gamma_{ab}}{\Gamma\Gamma_{ab}} \qquad \ldots \quad (12)$$

where

$$\gamma_{ab} = \frac{1}{2}(\gamma_a+\gamma_b)+\frac{1}{T_2}, \qquad \ldots \quad (13)$$

$$\Gamma_{ab} = \gamma_{ab}+(1-\cos\phi)/T_1 \qquad \ldots \quad (14)$$

and

$$\Gamma \cong 2\left[\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}+\frac{1}{T_2}\right]. \qquad \ldots \quad (15)$$

Here $\phi$ is the phase change resulting due to the collisions occurring during the time $T_1$ and the average is over a set of possible collisions. The assumption and derivative leading to this is indicated an apprndix A. $T_2^{-1}$ is the collision frequency for the velocity changing collisions. We have followed the convention of Stenholm (1970). For purpose of convenience in later comparison with known results we adopt the value given by Foley (1969)

$$\overline{\sin\phi\,(1-\cos\phi)} = 0{\cdot}726. \qquad \ldots \quad (16)$$

The expression for $I$ is then

$$I = 2\,\frac{\gamma\gamma_{ab}}{\Gamma\Gamma_{ab}}\,\frac{\left[1+\left(\frac{\nu-\omega}{\Gamma_{ab}}\right)^2\right]\left[U(x,y_1)-\bar{N}^{-1}U(0,y_1)\right]}{\left[U(x,y_1)-\frac{\nu-\omega}{\Gamma_{ab}}V(x,y_1)\right]+\frac{1}{2}\frac{\nu-\omega}{\Gamma_{ab}}\times\left[\frac{\nu-\omega}{\Gamma_{ab}}U(x,y_1)+V(x,y_1)\right]} \qquad \ldots \quad (17)$$

where $y_1 = \Gamma_{ab}/ku$. From equation (16) we evaluate the value of $\Gamma_{ab}$ to be $\gamma_{ab}(1+1{\cdot}3n)$ where $n = 1/T_1\gamma_{ab}$ is the number of collisions in the transverse

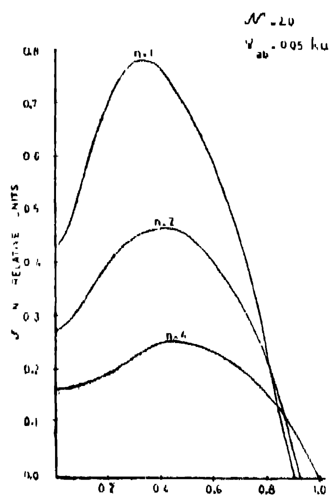



Fig. 3. Dependence of $I$ on $(\nu-\omega)/ku$ for various values of $n$.

decay time. The shift in the transition frequency due to phase changing collisions $\sin\phi/T_1$ is not taken into account explicitly (Stenholm 1970), as this is not important for calculation of laser operation. $I$ is plotted as a function of $(\nu-\omega)/ku$

for different values of $n$ in figure (3). The value of $I$ at resonance changes for different values of $n$, as $I$ at resonance is a function of $\Gamma_{ab}$. The Lamb dip vanishes at $n = 4$ for $ku/\gamma_{ab} = 10$. For $ku/\gamma_{ab} = 20$, $n$ should be greater than 4 for smearing the dip. In addition, the usual broadening due to collisions is also observed. A comparison of these results with that of Stenholm (1970) shows that the simple calculations of this paper brings out the essential characteristics of pressure and collisions effects on the performance of gas laser. Thus a simple treatment explaining these effects is not out of place because of the reason given in Section 1.

## 5. Discussions

It should be noted that our calculations in this paper are free from any approximation on $ku/\gamma_{ab}$. As a result, the calculations are valid for a wide range of values of $ku/\gamma_{ab}$. Thus the results of this paper are applicable to the gas lasers operating at low temperatures where $ku/\gamma_{ab}$ is low. In this case, the operational characteristics cannot be explained by the calculation of Lamb (1964) due to the Doppler limit approximation. Our calculations are also valid for laser operating at infrared and optical frequencies where $ku \gg \gamma_{ab}$. Bennett (1962) has shown that the value of $ku/\gamma_{ab}$ is around 20 when the wavelength of transition is 11522·76Å for He-Ne laser. For this value of $ku/\gamma_{ab}$, it is seen in figure 1 that Lamb's results (1964) show noticeable deviations from our calculations owing to the Doppler limit approximation. For a laser operating at the wavelength 6328Å the values of relaxation parameters used by Gyorffy *et al* (1968) are : $\gamma_a = 8{\cdot}3$ MHz, $\gamma_b = 18{\cdot}6$ MHz and $ku = 470$ MHz and thus the value of $ku/\gamma_{ab}$ comes around 40. This value of $ku/\gamma_{ab}$ has been used by Stenholm & Lamb (1969) for their theory calculated in the Doppler limit approximation. A comparison of the intensity curve even for $ku = 40\ \gamma_{ab}$ with that of Stenholm & Lamb (1969) shows that the shapes of the curves resulting from both the calculations are very similar for $(\nu-\omega) \geqslant 10\ \gamma_{ab}$ and $\bar{N} \leqslant 1{\cdot}5$. The peak of the intensity curves obtained from both the methods occur almost at the same value of detuning. But the peak value of $I$ in the present case differ from those of REA. However in both cases $I$ is arbitrary and as pointed out by Uehara & Shomoda (1965) the shapes not the exact value is important.

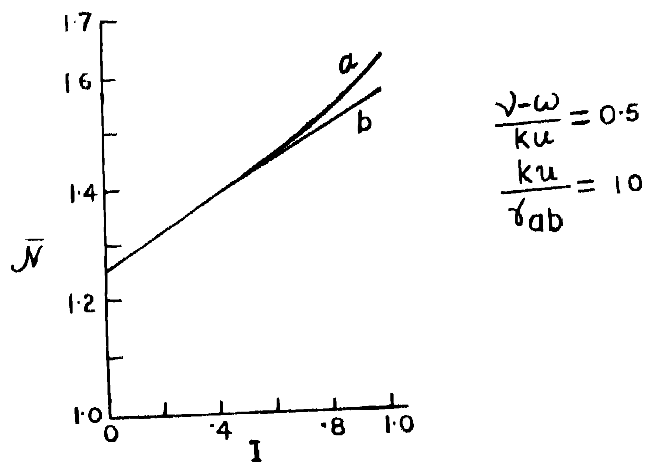
Now we compare the results of this paper with the results obtained from REA to get a range of validity of the parameters. For this purpose we take the eq. (95) in section (18) of Stenholm & Lamb (1969)

$$S = -r^2\bar{N}E(2\gamma_{ab}\hbar)^{-1}\int_{-\infty}^{+\infty}[\alpha(\omega-\nu-kv)+\alpha(\omega-\nu+kv)]W(v)$$

$$\times\{1+\tfrac{1}{2}I[\alpha(\omega-\nu+kv)+\alpha(\omega-\nu-kv)]\}^{-1}dv \qquad \ldots \quad (19)$$

The velocity integral in this equation is complicated and it is combersome to express the results in terms of the functions $U$ and $V$. At this stage we do not need a generalized expression as our results are equivalent to Lamb's results (1964) near resonance and comparison with REA can be found in Stenholm & Lamb (1969). So we are concerned only with the condition $|\nu-\omega| \gg \gamma_{ab}$. In this case, the velocity integral of eq (29) takes the form

$$\gamma_{ab}^2 \int_{-\infty}^{+\infty} [(\omega-\nu-kv)^2+\gamma_{ab}^2(1+\tfrac{1}{2}I)]^{-1} W(v)dv = \pi^{\frac{1}{2}} y(1+\tfrac{1}{2}I)^{-\frac{1}{2}} U(x, Y)$$



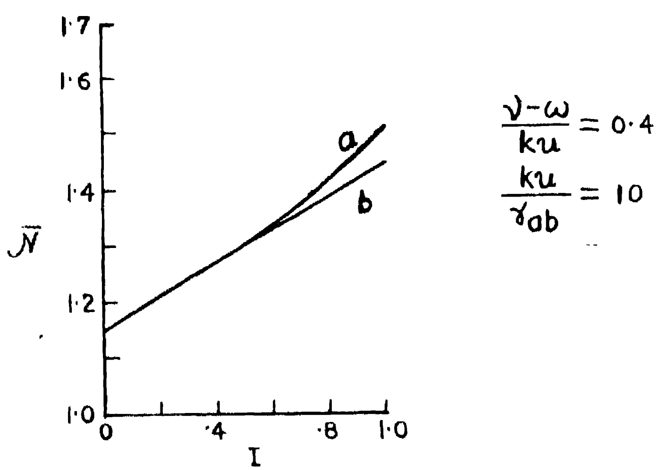




Fig 1 Comparison of intensity-excitation characteristics obtained from the present calculations (a) with that obtained from REA (b)

where $Y = y(1+\frac{1}{2}I)^{\frac{1}{2}}$. Utilizing this the amplitude equation in the steady state gives

$$\bar{N} = \frac{U(0\ ,y)}{U(x,\ Y)}(1+\tfrac{1}{2}I)^{\frac{1}{2}}. \tag{20}$$

The intensity-excitation characteristics are compared in figures 4. Is it seen that our results agree with the REA results within less than 2 per cent upto $\bar{N} \leqslant 1{\cdot}45$ when $\nu-\omega = 0{\cdot}5\ ku$ and $ku = 5\gamma_{ab}$. This agreement is adequate considering the computation error in the present theory and REA. Our results deviate only a little from this agreement for $ku = 10\ \gamma_{ab}$. The results of this paper appear more sensitive to a change in $ku/\gamma_{ab}$ than the REA results. Further a decrease of $(\nu-\omega)/ku$ reduces the range of similarity with REA, as under such a condition our results approach Lamb's (1964) results. A consideration of higher values of $\bar{N}$ is unnecessary since it is well known that $I \leqslant 1{\cdot}0$ for the present purpose. Further any similarity (or the lack of it) of theories at higher $I$ can only be of academic interest for a monomode operation due to the reasons given in section (7) of Stenholm & Lamb (1969).

It is evident from the above discussions that the similarity will increase when the value of $ku/\gamma_{ab}$ is decreased. Thus upto $\bar{N} \cong 1{\cdot}5$, our theory agrees quite well with REA for $(\nu-\omega) \geqslant 0{\cdot}5\ ku$ (figures 4). As expected, the theory is quite adequate at low $ku$, the best region of operation being $ku \lesssim 10\ \gamma_{ab}$. In addition, our method of calculation is less cumbersome and an explicit expression for $I$ is easily obtainable whereas the velocity integrals in REA make it difficult to obtain a generalised direct expression for $I$.

## Appendix A

For phase-changing collisions, $\rho_{ab}$ is given by $\rho_{ab}e^{-i\theta}$. The equation of motion for $\rho_{ab}$ in the field free case becomes

$$\dot{\rho}_{ab}e^{-i\theta}+\frac{d\ e^{-i\theta}}{dt}\rho_{ab} = -(i\omega+\gamma_{ab})\rho_{ab}e^{-i\theta}-(f+ig)\rho_{ab}e^{-i\theta} \qquad \ldots \quad \text{(A.1)}$$

The average is taken over a set of possible collisions. We have assumed that the collision changes $\omega$ to $\omega+g$ and $\gamma_{ab}$ to $\gamma_{ab}+f$. Separating the usual equation of motion from $\rho_{ab}$, we get

$$\frac{d\ \overline{e^{-i\theta}}}{dt}\rho_{ab} = -(f+ig)\rho_{ab}e^{-i\theta}$$

or

$$\int \frac{d(e^{-i\theta})}{e^{-i\theta}} = -\int_{c}^{T_1}(f+ig)dt$$

which gives $$\overline{\cos\phi}-i\,\overline{\sin\phi} = (1-fT_1-igT_1). \qquad \ldots \quad \text{(A.2)}$$

Since $1 \gg fT_1$ and $GT_1$. Separating the real and imaginary parts of eq. (A.2), we get

$$f = \frac{(1-\cos\phi)}{T_1} \qquad \ldots \quad (A.3)$$

and

$$g = \frac{\sin\phi}{T_1} \qquad \ldots \quad (A.4)$$

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