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ONE OPERATIONAL AMPLIFIER SIMULATES THIRD ORDER SYSTEMS WITH A LEADING-TIME CONSTANT

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ABSTRACT. The paper outlines a method for the simulation of third order linear systems with only one operational amplifier

A particular class of the general third order systems, that is, systems with a leading time constant is considered in this paper

Two basic circuits each consisting of one operational amplifier, four capacitors and five resistors are presented The circuits are analysed and the conditions of physical realisability discussed and obtained.

The design formulae and procedure are also given

INTRODUCTION

In previous communications (Wadhwa, 1961. 1962) on this subject three particular classes of the general third order linear systems were considered for simulation with only one operational amplifier. The purpose of this paper is to consider another particular class of systems, that is, third order systems with a leading time-constant, which are characterised by a transfer function of the form

$$F(S) = -\frac{b_0(b_1S+1)}{a_3S^3+a_2S^2+a_1S+1} \tag{1}$$

where a's and b's are positive and real constants, and S is the Laplace operator.

In principle, it should be possible to simulate the system of (1) with the aid of three capacitors and six resistors but the resulting network design formulae and the conditions of physical realisability become somewhat complicated. With the employment of four capacitors and five resistors, however, the design formulae and the conditions of physical realisability become simple and conveniently computable. It is primarily with a view to ensuring simplicity and convenience that in the networks presented in this paper four capacitors and five resistors have been used.

Of the various possible circuit each employing four capacitors and five resistors only two will be presented here; their design formulae obtained and conditions of physical realisability discussed.

Third order system simulation

A network for the simulation of third order systems is shown in Fig. 1 and its transfer function has been shown to be

$$\frac{E_0}{E_1} = -\frac{Y_1 Y_3 Y_5}{Y_6(Y_1+Y_2+Y_8)(Y_3+Y_4+Y_5+Y_7)+Y_3Y_6(Y_4+Y_5+Y_7)+Y_5Y_7(Y_1+Y_2+Y_3+Y_8)+Y_3Y_5Y_8} \quad \ldots \quad (2)$$

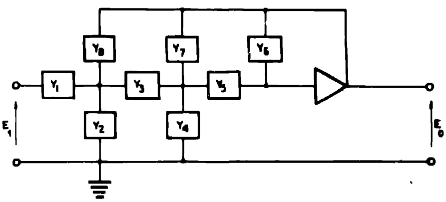

Fig 1 Network for the simulation of third systems.

Simulation of the system of (1) with the network of Fig. 1 is possible if the admittances (Y's) are properly chosen, and furthermore it should be obvious from (2) that at least three of the appropriate admittances will be required to be purely capacitative. As already mentioned the use of three capacitors gives inconveniently long design formulae and conditions of physical realisability while the use of four capacitors makes these simple and easily computable.

(a) Y_1, Y_2, Y_4 and Y_6 *capacitative*

A possible circuit for simulating the system of (1) is shown in Fig. (2*a*), in which

$$\left.\begin{aligned} Y_1 &= \left(SC_1 + \frac{1}{R}\right) \\ Y_2 &= SC_2 \\ Y_4 &= SC_4 \\ Y_6 &= SC_6 \\ Y_3 = Y_5 = Y_8 &= \frac{1}{R} \\ Y_7 &= \frac{1}{\alpha R} \end{aligned}\right\} \quad \ldots \quad (3)$$

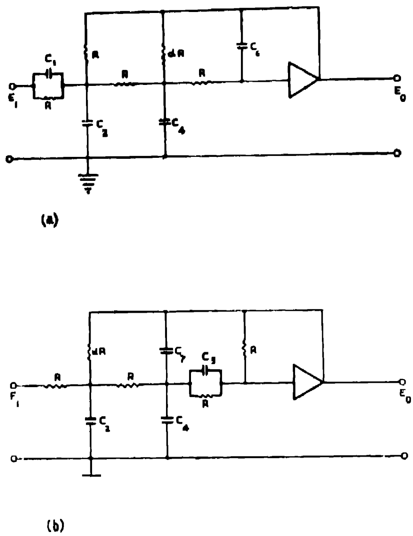

Fig 2. Network for the simulation of

$$\frac{E_0}{E_1} = -\frac{b_0(b_1S+1)}{a_3S^3+a_2S^2+a_1S+1}$$

Substituting (3) into (2) and simplifying

$$\frac{E_0}{E_1} = -\frac{\left(\frac{\alpha}{\alpha+3}\right)(RC_1S+1)}{\left(\frac{\alpha}{\alpha+3}\right)R^3(C_1+C_2)C_4C_6S^3+R^2C_6\left[\frac{2\alpha+1}{\alpha+3}(C_1+C_2)+\left(\frac{3\alpha}{\alpha+3}\right)C_4\right]S^2}$$

$$+\left[\left(\frac{\alpha}{\alpha+3}\right)R(C_1+C_2)+\left(\frac{5\alpha+3}{\alpha+3}\right)RC_6\right]S+1 \qquad \ldots (4)$$

Equations (1) and (4) will be identical if

$$b_0 = \left(\frac{\alpha}{\alpha+3}\right) \qquad \ldots (5)$$

$$b_1 = T_1 \qquad \ldots (6)$$

$$a_1 = \left(\frac{\alpha}{\alpha+3}\right)(T_1+T_2)+\left(\frac{5\alpha+3}{\alpha+3}\right)T_6 \qquad \ldots (7)$$

$$a_2 = T_6\left[\left(\frac{2\alpha+1}{\alpha+3}\right)(T_1+T_2)+\left(\frac{3\alpha}{\alpha+3}\right)\right]T_4 \qquad \ldots \quad (8)$$

$$a_3 = \left(\frac{\alpha}{\alpha+3}\right)(T_1+T_2)T_4T_6 \qquad \ldots \quad (9)$$

where

$$T_n = RC_n \qquad \ldots \quad (10)$$

Now, simulation of the system of (1) with the network of Fig. 2(a) is possible only if the values of α, T_1, T_2, T_4, T_6 obtained as the solution of (5) through (9) are real and positive. It is required to determine, therefore, in terms of the given real and positive a's and b's, the values of α, T_1, T_2, T_4, T_6 and find the conditions, if any, under which these can be real and positive.

Elimination of α, T_1, T_2, and T_6 from (5) through (9) gives a cubic

$$27b_0^3(a_1a_2-3a_3b_0)T_4^3-9b_0^2\{a_1a_3(5b_0+1)+a_2^2(4b_0+1)\}T_4^2+6a_2a_3b_0(4b_0+1)(5b_0+1)T_4-a_3^2(4b_0+1)(5b_0+1)^2=0 \qquad \ldots \quad (11)$$

which can have either one or three real roots depending on whether its discriminant is positive or negative.

Now, as shown in Appendix I, a set of real and positive α, T_1, T_2, T_4, T_6 exists, provided that

$$b_0 < 1 \qquad \ldots \quad (12)$$

and, either

$$\left.\begin{aligned} a_3 &> \frac{a_2b_1}{3} \\ a_1 &> \frac{3a_3b_0}{a_2} \end{aligned}\right\} \qquad \ldots \quad (13)$$

or

$$\left.\begin{aligned} a_3 &< \frac{a_2b_1}{3} \\ a_1 &> \frac{3(4b_0+1)(a_2b_1-3a_3)+b_0b_1}{b_1^2(5b_0+1)}+b_0b_1 \end{aligned}\right\} \qquad \ldots \quad (13a)$$

For the design of the network, circuit component values are required to be determined. The proper procedure for design would be to first check and see if the inequalities of (12) and either (13) or (13a) are satisfied. The satisfaction of these conditions signifies that the circuit of Fig. 2(a) for simulation of the system of (1) is physically realisable. The circuit component values may then be obtained by solving the cubic of (11) for T_4. Since α and T_1 are known directly from (5) and (6) respectively, then T_2 and T_6 may be obtained by solving (7) and (8). Having thus determined α, T_1, T_2, T_4, T_6, and choosing arbitrarily a convenient value for any one of the capacitors, the value of resistors and the remaining capacitors may be then determined with the aid of (10).

(b) *Y_2, Y_4, Y_5, and Y_7 capacitative*

Another possible circuit for the simulation of the system of (1) is shown in Fig. 2(b), where

$$\left.\begin{aligned} Y_2 &= SC_2 \\ Y_4 &= SC_4 \\ Y_5 &= (SC_5 + 1/R) \\ Y_7 &= SC_7 \\ Y_1 &= Y_3 = Y_6 = 1/R \\ Y_8 &= 1/\alpha R \end{aligned}\right\} \quad \ldots \quad (14)$$

Substituting (14) into (2) and simplifying

$$\frac{E_0}{E_1} = -\frac{\frac{\alpha}{3(\alpha+1)} \cdot (RC_5S+1)}{\frac{\alpha}{3(\alpha+1)} R^3C_2C_5C_7S^3 + \left[\frac{\alpha}{3(\alpha+1)} R^2(C_4+C_5)C_2 + \frac{2\alpha}{3(\alpha+1)} R^2C_2C_7 + \frac{(2\alpha+1)}{3(\alpha+1)} R^2C_5C_7\right] S^2 + \left[\frac{2\alpha}{3(\alpha+1)} RC_2 + \frac{(2\alpha+1)}{3(\alpha+1)} RC_4 + \frac{2}{3} RC_5 + \frac{2(2\alpha+1)}{3(\alpha+1)} RC_7\right] S + 1} \quad \ldots \quad (15)$$

Equations (1) and (15) will be identical if

$$b_0 = \frac{\alpha}{3(\alpha+1)} \quad \ldots \quad (16)$$

$$b_1 = T_5 \quad \ldots \quad (17)$$

$$a_1 = \frac{2\alpha}{3(\alpha+1)} T_2 + \frac{(2\alpha+1)}{3(\alpha+1)} T_4 + \frac{2}{3} T_5 + \frac{2(2\alpha+1)}{3(\alpha+1)} T_7 \quad \ldots \quad (18)$$

$$a_2 = \frac{\alpha}{3(\alpha+1)}(T_4+T_5)T_2 + \frac{2\alpha}{3(\alpha+1)} T_2T_7 + \frac{(2\alpha+1)}{3(\alpha+1)} T_5T_7 \quad \ldots \quad (19)$$

$$a_3 = \frac{\alpha}{3(\alpha+1)} T_2T_5T_7 \quad \ldots \quad (20)$$

where

$$T_n = RC_n \quad \ldots \quad (21)$$

Elimination of α, T_4, T_5 and T_7 from (16) through (20) gives a cubic

$$T_2^3 - \frac{1}{6b_0}[3a_1+b_1(3b_0-1)]T_2^2 + \frac{a_2(3b_0+1)}{6b_0^2} T_2 - \frac{a_3(3b_0+1)^2}{18b_0^3} = 0 \quad \ldots \quad (22)$$

which, as is obvious, can have no negative real roots and will have either one or three real positive roots depending on whether its discriminant Δ is positive or negative.

Now, as shown in Appendix II, if

$$\left.\begin{aligned} & b_0 < \text{Min}\left[\frac{1}{3}, \left\{\frac{b_1}{144a_3}(3a_1-2b_1)^2 - \frac{1}{3}\right\}, \left\{\frac{(a_2b_1-2a_3)^2}{4a_3b_1^3} - \frac{1}{3}\right. \right. \\ & (3a_1-2b_1[) > 0 \\ & (a_2b_1-2a_3) > 0 \end{aligned}\right\} \quad \ldots \quad (23)$$

then one set of positive real α, T_2, T_4, T_5 and T_7 exists, provided that either

$$\left.\begin{aligned} & \Delta = 4p^3+27q^2 > 0 \\ \text{and} \quad & OQ > OB > OP > OA \end{aligned}\right\} \quad \ldots \quad (24)$$

$$\left.\begin{aligned} \text{or} \quad & \Delta < 0 \\ \text{and} \quad & OB > OQ > OA > OP \end{aligned}\right\} \quad \ldots \quad (24a)$$

But, if (23) is satisfied and

$$\left.\begin{aligned} & \Delta < 0 \\ \text{and either} \quad & OB > OQ > OP > OA \\ \text{or} \quad & OQ > OB > OA > OP \end{aligned}\right\} \quad \ldots \quad (25)$$

then two sets of positive real values exist. And three sets of positive real values can exist if

$$\left.\begin{aligned} & \Delta < 0 \\ \text{and} \quad & OQ > OB > OP > OA \end{aligned}\right\} \quad \ldots \quad (26)$$

where

$$OA = \frac{b_1(3a_1-2b_1)-\sqrt{b_1^2(3a_1-2b_1)^2-48a_3b_1(3b_0+1)}}{12b_0b_1}$$

$$OB = \frac{b_1(3a_1-2b_1)+\sqrt{b_1^2(3a_1-2b_1)^2-48a_3b_1(3b_0+1)}}{12b_0b_1}$$

$$OP = \frac{(a_2b_1-2a_3)-\sqrt{(a_2b_1-2a_3)^2-4a_3b_1^3(b_0+\frac{1}{3})}}{2b_0b_1^2}$$

$$OQ = \frac{(a_2b_1-2a_3)+\sqrt{(a_2b_1-2a_3)^2-4a_3b_1^3(b_0+\frac{1}{3})}}{2b_0b_1^2} \qquad (27)$$

$$p = \frac{a_2(3b_0+1)}{6b_0^2}-\frac{1}{3}\left[\frac{3a_1+b_1(3b_0-1)}{6b_0}\right]^2$$

$$q = -\frac{a_3(3b_0+1)^2}{18b_0^3}+\frac{a_2(3b_0+1)[3a_1+b_1(3b_0-1)]}{108b_0^3}$$

$$-\frac{2}{27}\left[\frac{3a_1+b_1(3b_0-1)}{6b_0}\right]^3$$

To summarise, therefore, if (23) and either

$$OQ > OB > OP > OA \qquad (28)$$

or

$$\Delta' = 4p^3+27q^2 < 0 \qquad (29)$$

are satisfied then it is possible to simulate the system of (1) with the circuit of Fig. 2(*b*). The circuit component values may be obtained with the aid of (16), (17), (22), (20), (18) and (21).

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APPENDIX I

CONDITIONS UNDER WHICH THE CIRCUIT OF FIGURE 2(a) IS PHYSICALLY REALISABLE

Simulation of the system represented by (1) with the network of Fig. 2(a) is possible only if the values of α, T_1, T_2, T_4 and T_6 obtained as the solution of equations

$$b_0 = \frac{\alpha}{\alpha+3} \quad \dots (1.1)$$

$$b_1 = T_1 \quad \dots (1.2)$$

$$a_1 = \left(\frac{\alpha}{\alpha+3}\right)(T_1+T_2)+\left(\frac{5\alpha+3}{\alpha+3}\right)T_6 \quad \dots (1.3)$$

$$a_2 = T_6\left[\left(\frac{2\alpha+1}{\alpha+3}\right)(T_1+T_2)+\left(\frac{3\alpha}{\alpha+3}\right)T_4\right] \quad \dots (1.4)$$

$$a_3 = \left(\frac{\alpha}{\alpha+3}\right)(T_1+T_2)T_4T_6 \quad \dots (1.5)$$

are real and positive; where a's and b's are real and positive constants.

It is, therefore, required to determine the conditions under which α, T_1, T_2, T_4, T_6 can be real and positive; and graphical methods may be perhaps a convenient means of obtaining these.

Elimination of α, T_1 and T_4 from (1.1), (1.2), (1.3), (1.5) and (1.1), (1.2), (1.4), (1.5) give the following two equations

$$b_0T_2+(4b_0+1)T_6 = (a_1-b_0b_1) \quad \dots (1.6)$$

and

$$T_6 = \frac{3a_2}{(5b_0+1)(T_2+b_1)} - \frac{9a_3}{(5b_0+1)(T_2+b_1)^2} \quad \dots (1.7)$$

The intersection of the straight line of (1.6) and the curve of (1.7) in the first quadrant of the T_2-T_6 plane will give both T_2 and T_6 as real and positive. It is obvious from (1.1), (1.2) and (1.6) that the corresponding α, T_1 and T_4 are also real and positive, provided that

$$b_0 < 1 \quad \dots (1.8)$$

It should be clear, therefore, that only the portion of the curves lying on the right of the T_6-axis are of interest.

The intercepts that the straight line of (1.6) makes with the T_2 and T_6-axes respectively, are given by

$$OA = T'_2 = \left(\frac{a_1 - b_0 b_1}{b_0} \right) \qquad \ldots \quad (1.9)$$

$$OB = T'_6 = \left(\frac{a_1 - b_0 b_1}{4b_0 + 1} \right) \qquad \ldots \quad (1.10)$$

which are real and, also positive if

$$a_1 > b_0 b_1 \qquad \ldots \quad (1.11)$$

Similarly, the curve of (1.7) will cut the axes at points P and Q whose T_2 and T_6 coordinates are respectively given by

$$OP = T_2'' = \left(\frac{3a_3 - a_2 b_1}{a_2} \right) \qquad \ldots \quad (1.12)$$

$$OQ = T_6'' = \frac{3(a_2 b_1 - 3a_3)}{b_1^2 (5b_0 + 1)} \qquad \ldots \quad (1.13)$$

Now, if

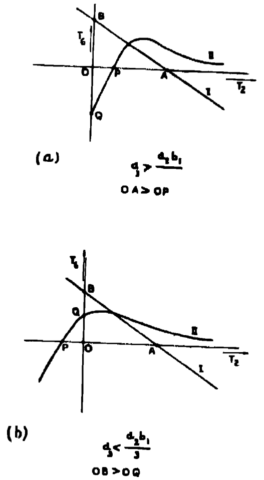
$$\left. \begin{aligned} (3a_3 - a_2 b_1) &> 0 \\ \text{i.e.} \qquad a_3 &> \frac{a_2 b_1}{3} \end{aligned} \right\} \qquad (1.14)$$

then the intercept OP is positive and OQ negative, but if

$$\left. \begin{aligned} (3a_3 - a_2 b_1) &< 0 \\ \textit{i.e.} \qquad a_3 &< \frac{a_2 b_1}{3} \end{aligned} \right\} \qquad \ldots \quad (1.15)$$

then the intercept OQ is positive and OP negative.

Therefore, if the conditions as expressed in (1.11) and either in (1.14) or (1.15) are satisfied then it is possible for a portion of the straight line and the curve to exist in the first quadrant and it may be possible, under certain conditions, for these to intersect each other at one or more points in that region. The sketches of the straight line of (1.6) and a portion of the curve of (1.7) lying on the right of the T_6-axis are shown in Fig. 1.1.



I $\quad b_0 T_2 + (4b_0+1)T_6 = (a_1 - b_0 b_1)$

II $\quad T_6 = \dfrac{3a_2}{(5b_0+1)(T_2+b_1)} - \dfrac{9a_3}{(5b_0+1)(T_2+b_1)^2}$

Fig. 1.1. Condition under which the straight line and the curve can intersect each other in the first quadrant.

It is evident from the sketch of Fig. 1.1(a) that if (1.11) and (1.14) are satisfied and

$$OA > OP$$

i.e.

$$a_1 > \frac{3a_3 b_0}{a_2} \qquad \cdots \quad (1.16)$$

or, as seen from figure 1.1(b), if (1.11) and (1.15) are satisfied, and

$$OB > OQ$$

i.e.

$$a_1 > \frac{3(4b_0+1)(a_2 b_1 - 3a_3)}{b_1^2(5b_0+1)} + b_0 b_1 \qquad \cdots \quad (1.17)$$

then it is possible for the straight line and the curve to intersect each other in the first quadrant giving T_2 and T_6 as real and positive.

To summarise, therefore, if

$$b_0 < 1 \qquad \dots (1.8)$$

and, either

$$\left.\begin{aligned} a_3 &> \frac{a_2 b_1}{3} \\ a_1 &> \frac{3a_3 b_0}{a_2} \end{aligned}\right\} \qquad \dots (1.16a)$$

or

$$\left.\begin{aligned} a_3 &< \frac{a_2 b_1}{3} \\ a_1 &> \frac{3(4b_0+1)(a_2 b_1 - 3a_3)}{b_1^2(5b_0+1)} + b_0 b_1 \end{aligned}\right\} \qquad \dots (1.17a)$$

then the circuit of Fig. 2(a) for simulating the system of (1) is physically realisable.

APPENDIX II

CONDITIONS UNDER WHICH THE CIRCUIT OF FIG. 2(b) IS PHYSICALLY REALISABLE

If the values of α, T_2, T_4, T_5 and T_7 obtained as the solution of equations

$$b_0 = \frac{\alpha}{3(\alpha+1)} \qquad \dots (2.1)$$

$$b_1 = T_5 \qquad \dots (2.2)$$

$$a_1 = \frac{2\alpha}{3(\alpha+1)} T_2 + \frac{(2\alpha+1)}{3(\alpha+1)} + \frac{2}{3} T_5 + \frac{2(2\alpha+1)}{3(\alpha+1)} T_7 \qquad \dots (2.3)$$

$$a_2 = \frac{\alpha}{3(\alpha+1)} T_2(T_4+T_5) + \frac{2\alpha}{3(\alpha+1)} T_2 T_7 + \frac{(2\alpha+1)}{3(\alpha+1)} \cdot T_5 T_7 \qquad \dots (2.4)$$

$$a_3 = \frac{\alpha}{3(\alpha+1)} T_2 T_5 T_7 \qquad \dots (2.5)$$

are real and positive, then it is possible to simulate the system of (1) with the circuit of Fig. 2(b).

Elimination of α, T_5 and T_7 from (2.1), (2.2), (2.3), (2.5) and (2.1), (2.2), (2.4), (2.5) give the following two equations:

$$T_4 = \frac{(3a_1-2b_1)}{(3b_0+1)} - \frac{2a_3}{b_0b_1T_2} - \frac{6b_0}{(3b_0+1)}T_2 \quad \ldots (2.6)$$

$$T_4 = \frac{(a_2b_1-2a_3)}{b_0b_1T_2} - \frac{a_3(3b_0+1)}{3b_0^2T_2^2} - b_1 \quad \ldots (2.7)$$

The intersection of the curves of (2.6) and (2.7) in the first quadrant of the T_2-T_4 plane will give both T_2 and T_4 as real and positive. It is evident from (2.1), (2.2) and (2.5) that the corresponding α, T_5 and T_7 will be also real and positive, provided that

$$b_0 < 1/3 \quad \ldots (2.8)$$

The curve of (2.6) will cut the T_2-axis (i.e. $T_4 = 0$) at two points A and B whose T_2 coordinates may be obtained by equating to zero the right hand side of (2.6) and solving the resulting quadratic

$$6b_0^2b_1T_2^2 - b_0b_1(3a_1-2b_1)T_2 + 2a_3(3b_0+1) = 0 \quad \ldots (2.9)$$

whose roots are given by

$$T_{2(A,B)} = \frac{b_1(3a_1-2b_1) \pm \sqrt{b_1^2(3a_1-2b_1)^2 - 48a_3b_1(3b_0+1)}}{12b_0b_1} \quad \ldots (2.10)$$

Now, A and B will be real, if

$$b_1(3a_1-2b_1)^2 > 48a_3(3b_0+1)$$

i.e.

$$b_0 < \frac{b_1}{144a_3}(3a_1-2b_1)^2 - \frac{1}{3} \quad \ldots (2.11)$$

and their coordinates will be positive, if

$$(3a_1-2b_1) > 0 \quad \ldots (2.12)$$

Similarly, (2.7) will cut the T_2-axis at two points P and Q whose T_2-coordinates are

$$T_{2(P,Q)} = \frac{(a_2b_1-2a_3) \pm \sqrt{(a_2b_1-2a_3)^2 - 4a_3b_1^3(b_0+1/3)}}{2b_0b_1^2} \quad \ldots (2.13)$$

and which will be real and positive, if

$$\left.\begin{aligned} &(a_2b_1-2a_3)^2 > 4a_3b_1^3(b_0+1/3) \\ \text{i.e.} \quad &b_0 < \frac{(a_2b_1-2a_3)^2}{4a_3b_1^3} - \frac{1}{3} \\ \text{and} \quad &(a_2b_1-2a_3) > 0 \end{aligned}\right\} \quad \ldots (2.14)$$

Therefore, if the conditions as expressed in (2.11), (2.12) and (2.14) are satisfied then it is possible for a portion of the curves of (2.6) and (2.7) to exist in the first quadrant, and it may be possible, under certain conditions, for these to intersect each other at one or more points in that region.

Elimination of T_4 from (2.6) and (2.7) gives a cubic

$$T_2^3 - \frac{1}{6b_0}\{3a_1+b_1(3b_0-1)\}T_2^2 + \frac{a_2(3b_0+1)}{6b_0^2}T_2 - \frac{a_3(3b_0+1)^2}{18b_0^3} = 0 \quad \ldots \quad (2.15)$$

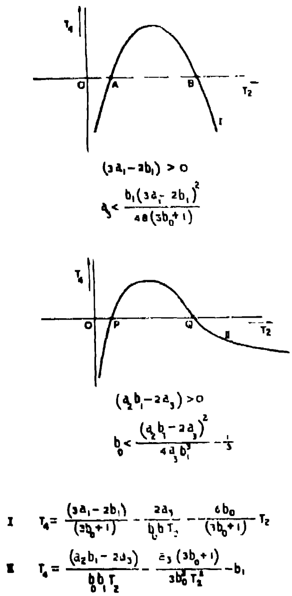


Fig. 2.1. Sketches of the curves for positive values of T_2.

The real roots of (2.15) give the real points of intersection of the curves of (2.6) and (2.7). It is obvious, in view of (2.12), that (2.15) can have no negative real roots, therefore, the curves do not intersect at real points on the left of the T_4-axis. If its discriminant Δ is positive then (2.15) will have one real root signifying that the curves intersect each other at one point on the right of T_4-axis; and if Δ is negative then the curves can intersect each other at three points on the right of T_4-axis. The sketches of portions of the curves lying on the right of the T_4-axis are shown in Fig. 2.1.

Now, as evident from figure 2.1, if the points A and B interlace with the points P and Q, such that

$$OQ > OB > OP > OA \quad \ldots \quad (2.16)$$

then the curves will intersect each other at one point in the first quadrant if

$$\Delta = 4p^3+27q^2 > 0$$

and at three points if

$$\Delta < 0$$

Hence, at least one set of positive real x, T_2, T_4, T_5, T_7 exist if (2.16) is satisfied, irrespective of whether Δ is positive or negative.

But if

$$\left.\begin{array}{ll} & \Delta < 0 \\ \text{and} & OB > OQ > OA > OP \end{array}\right\} \quad \ldots \quad (2.17)$$

then one set of real positive values exists, and two real positive sets of values exist if

$$\left.\begin{array}{ll} & \Delta < 0 \\ \text{and either} & OB > OQ > OP > OA \\ \text{or} & OQ > OB > OA > OP \end{array}\right\} \quad \ldots \quad (2.18)$$

where

$$\left.\begin{aligned} OA &= \frac{b_1(3a_1-2b_1)-\sqrt{b_1^2(3a_1-2b_1)^2-48a_3b_1(3b_0+1)}}{12b_0b_1} \\ OB &= \frac{b_1(3a_1-2b_1)+\sqrt{b_1^2(3a_1-2b_1)^2-48a_3b_1(3b_0+1)}}{12b_0b_1} \\ OP &= \frac{(a_2b_1-2a_3)-\sqrt{(a_2b_1-2a_3)^2-4a_3b_1^3(b_0+1/3)}}{2b_0b_1^2} \\ OQ &= \frac{(a_2b_1-2a_3)+\sqrt{(a_2b_1-2a_3)^2-4a_3b_1^3(b_0+1/3)}}{2b_0b_1^2} \\ p &= \frac{a_2(3b_0+1)}{6b_0^2} - \frac{1}{3}\left[\frac{3a_1+b_1(3b_0-1)}{6b_0}\right]^2 \\ q &= \frac{-a_3(3b_0+1)^2}{18b_0^3} + \frac{a_3(3b_0+1)[3a_1+b_1(3b_0-1)\}}{108b_0^3} \\ &\qquad - \frac{2}{27}\left[\frac{3a_1+b_1(3b_0-1)}{6b_0}\right]^3 \end{aligned}\right\} \quad \ldots \quad (2.19)$$

To summarise, therefore, if

$$b_0 < \text{Min} \left[\frac{1}{3}, \left\{ \frac{b_1}{144a_3} (3a_1 - 2b_1)^2 - \frac{1}{3} \right\}, \left\{ \frac{(a_2b_1 - 2a_3)^2}{4a_3b_1^3} - \frac{1}{3} \right\} \right] \qquad \text{... (2.20)}$$

$$(3a_1 - 2b_1) > 0$$

$$(a_2b_1 - 2a_3) > 0$$

and either $$OQ > OB > OP > OA \qquad \text{... (2.16)}'$$

or $$\Delta = 4p^3 + 27q^2 < 0 \qquad \text{... (2.21)}$$

then it is possible to simulate the system of (1) with the circuit of Fg. 2(*b*).