Treewidth and related graph parameters

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Abstract

For modeling some practical problems, graphs play very important roles. Since many modeled problems can be NP-hard in general, some restrictions for inputs are required. Bounding a graph parameter of the inputs is one of the successful approaches. We study this approach in this thesis. More precisely, we study two graph parameters, spanning tree congestion and security number, that are related to treewidth.

Let G be a connected graph and T be a spanning tree of G. For $e \in E(T)$, the *congestion* of e is the number of edges in G connecting two components of $T - e$. The *edge congestion of G in T* is the maximum congestion over all edges in T. The *spanning tree congestion of G* is the minimum congestion of G in its spanning trees. In this thesis, we show the spanning tree congestion for the complete k-partite graphs, the two-dimensional tori, and the two-dimensional Hamming graphs. We also address lower bounds of spanning tree congestion for the multi-dimensional hypercubes, the multi-dimensional grids, and the multi-dimensional Hamming graphs.

The security number of a graph is the cardinality of a smallest vertex subset of the graph such that any “attack” on the subset is “defendable.” In this thesis, we determine the security number of two-dimensional cylinders and tori. This result settles a conjecture of Brigham, Dutton and Hedetniemi [Discrete Appl. Math. 155 (2007) 1708–1714]. We also show that every outerplanar graph has security number at most three. Additionally, we present lower and upper bounds for some classes of graphs.

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Chapter 1

Introduction

Recently, graphs are used for modeling several practical problems such as VLSI design problems, network routing problems, and flight scheduling problems. Although the problems can be modeled without any lack of information by graphs, the modeled problems can be very hard, that is, NP-hard [24]. To cope with NP-hard problems, several approaches are introduced: approximation algorithms [54], randomized algorithms [40], exponential time exact algorithms [56], fixed parameter algorithms [20], and so on. On the other hand, it is known that some NP-hard problems can be solved in polynomial time if the inputs have some natural restrictions. For example, if the input graphs have bounded treewidth then many problems can be solved in polynomial time [7]. In this thesis, we concentrate on this approach, that is, the restrictions of the inputs. More precisely, we investigate the following question: “For which graphs, are useful graph parameters bounded?”

Graph parameters are properties of graphs representable by numbers such as: diameter, radius, maximum (or, minimum) degree, chromatic number. Among graph parameters, the treewidth has been studied intensively because of its usefulness. The notion of treewidth was introduced by Robertson and Seymour in their Graph Minor project. Roughly speaking, the treewidth is a graph parameter that indicates whether the graph has a tree-like structure of small width. It is known that if the treewidth of the graph is bounded by a constant then problems that can be expressible by Monadic Second Order Logic are solvable in linear time [17]. However, the problem to determine the treewidth of the input graph is NP-hard. Thus, to utilize treewidth, it is necessary to develop approximation algorithms for treewidth or to determine the treewidth of some natural graph classes.

Since treewidth and related graph parameters have been studied intensively, it is known that for some graph classes, such as outerplanar graphs, series parallel graphs, and chordal graphs, the treewidth and some related parameters can be determined in polynomial time. In this thesis, we study treewidth and related parameters for some important graph classes. We obtain lower and upper bounds, or exact bounds for those classes. We study treewidth related parameters, the spanning tree congestion and the security number, for graph classes complete k-partite graphs, outerplanar graphs, grids, cylinders, tori, hypercubes, Hamming graphs, and so on. These graph classes play important roles in the algorithmic graph theory or the graph minor theory.

In the following, we give an overview of the present thesis. For more precise definitions, see the corresponding chapters and sections.

Since a spanning tree of a graph has no cycle, a deletion of any edge in the tree derives a partition of the vertex set into two parts. The congestion of the deleted edge is the number of edges in the original graph between the two parts. The congestion of a spanning tree is the maximum congestion over all edges in the tree. The spanning tree congestion of a graph is the minimum congestion over all its spanning trees. In Chapter 2, we determine the spanning tree congestion of complete k-partite graphs, two-dimensional tori, and two-dimensional Hamming graphs. We also give lower and upper bounds on the spanning tree congestion of Hypercubes, Hamming graphs, and multi-dimensional grids. Additionally, we show that the treewidth of a graphs is at most the product of its spanning tree congestion and its maximum degree.

A secure set in a graph is a subset of the vertex set of the graph such that any “attack” on the subset from its outer boundaries is “defensible.” In other words, for any subset of a secure set, the number of its inner closed boundaries are at least the number of its outer boundaries. The security number of a graph is the cardinality of the smallest secure set in the graph. The notion of security number is introduced by Brigham, Dutton, and Hedetniemi [11] in 2007. They have shown lower and upper bounds on the security number of two-dimensional grids, cylinders, and tori. They conjectured that their upper bounds for cylinders and tori is the best possible. In Chapter 3, we settle this conjecture affirmatively. We also study the security number of outerplanar graphs, and show that any outerplanar graph has the security number at most three. We present lower and upper bounds on the security number of hypercubes as well.

1.1 Definitions

In this section, we give some definitions that will be used in this thesis.

1.1.1 Graph

A *graph* G is a pair of the vertex set $V(G)$ and the edge set $E(G)$. A *vertex* $v \in V(G)$ is an object, and an *edge* $e \in E(G)$ is an unordered pair of two distinct vertices. For $u, v \in V(G)$, if $\{u, v\} \in E(G)$ then we say that *u and v are adjacent*. In figures, we represent a vertex by a dot (or a circle) and an edge by a line. For example, if $V(G) = \{u, v, w\}$ and $E(G) = \{\{u, v\}, \{v, w\}\}$ then the graph G is represented by Fig. 1.1.

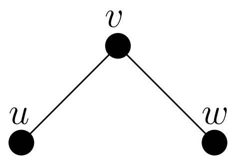


Fig. 1.1 An example of a graph.

In this thesis, all graphs are *simple* and *finite*, that is, there is at most one edge between a pair of vertices and the vertex set is a finite set.

Two graphs G and H are *isomorphic* if there is a bijection $\phi : V(G) \to V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{\phi(u), \phi(v)\} \in E(H)$. For example, it is easy to see that the graphs in Fig. 1.2 are isomorphic ($a \mapsto w$, $b \mapsto x$, $c \mapsto y$, and $d \mapsto z$).

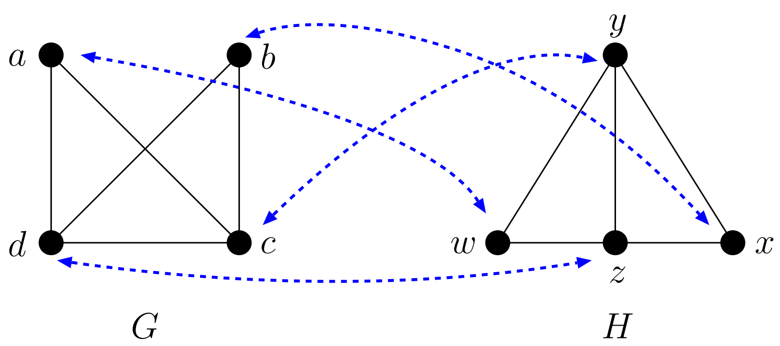


Fig. 1.2 Graphs G and H are isomorphic.

A *walk in a graph* G is a sequence of vertices $(p_1, \ldots, p_k)$ such that $\{p_i, p_{i+1}\} \in E(G)$ for each $1 \leq i < k$. For two vertices $u, v \in V(G)$, a *u-v path* in G is a walk $(p_1, \ldots, p_k)$ such that $p_1 = u$, $p_k = v$, and $p_i \neq p_j$ if $i \neq j$. We define the *distance between u and v*, denoted by $dist_G(u, v)$, as the number of edges in a shortest u-v path in G. Two paths P_1 and P_2 are *edge-disjoint* if they do not share any edge. A set of paths is *edge-disjoint* if the paths in the set are pairwise edge-disjoint. A *cycle in a graph* G is a walk $(p_1, \ldots, p_k)$ such that $p_i = p_j$ if and only if either $i = j$ or $\{i, j\} = \{1, k\}$. A graph G is *connected* if for every pair u, v of vertices, G has a u-v path. A graph F is a *forest* if F contains no cycle. A forest T is a *tree* if T is connected. A tree S is a *star* if S contains at most one vertex of degree greater than one.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is a *spanning subgraph* if $V(H) = V(G)$. If a spanning subgraph T of G is tree then T is a *spanning tree* of G. A subgraph H of a graph G is an *induced subgraph* if $u, v \in V(H)$ and $\{u, v\} \in E(G)$ imply $\{u, v\} \in E(H)$. For example, see Fig. 1.3. We denote by $G[S]$ the induced subgraph of G with the vertex set $S \subseteq V(G)$, that is, $V(G[S]) = S$. We call $G[S]$ a *subgraph of G induced by S*. If $S \subseteq V(G)$ induces a connected subgraph of G, we say that S is *connected*.

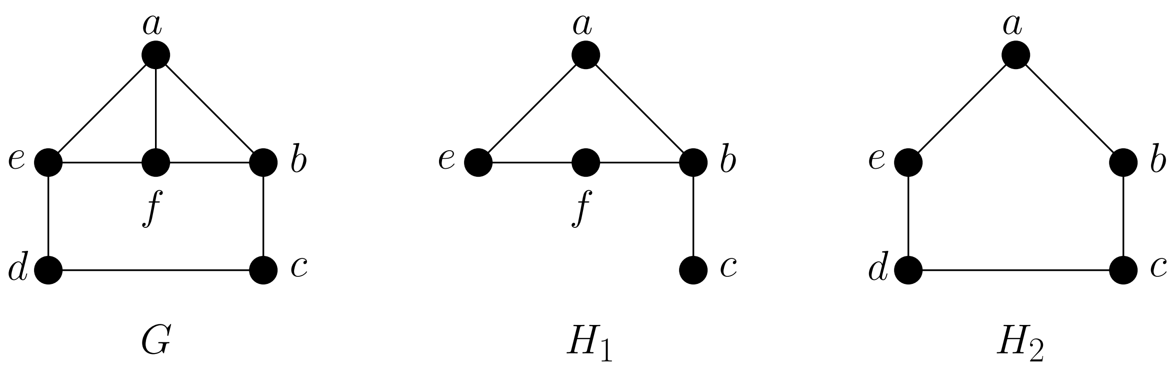


Fig. 1.3 A subgraph H_1 and an induced subgraph H_2 of G.

The *open neighborhood* of a vertex v in a graph G, denoted by $N_G(v)$, is the set of vertices such that for any $u \in N_G(v)$ there exists the edge $\{u, v\} \in E(G)$. We define the *closed neighborhood* of a vertex v in a graph G as $N_G[v] = \{v\} \cup N_G(v)$. The *degree* of a vertex v in a graph G, denoted by $deg_G(v)$, is the number of neighbors of v in G, that is, $deg_G(v) = |N_G(v)|$. We denote the *maximum degree* and the *minimum degree* of G by $\Delta(G)$ and $\delta(G)$, respectively, that is, $\Delta(G) = \max_{v \in V(G)} \deg_G(v)$ and $\delta(G) = \min_{v \in V(G)} \deg_G(v)$.

We can extend the notion of the neighborhood of a vertex to the neighborhood of a vertex set. For $S \subseteq V(G)$, let $N_G[S]$ denote the *closed neighborhood* of S, that is, $N_G[S] = S \cup \bigcup_{v \in S} N_G(v)$.

For $e \in E(G)$, we denote by $G - e$ the graph obtained by deleting e from G; that is, $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$. Similarly, for $F \subseteq E(G)$ let $G - F$ be the graph obtained by deletion of all edges in F from G.

1.1.2 Boundaries of a vertex set

We define the vertex boundary and edge boundary of a vertex set. These notions play very important roles in this thesis. For a vertex set $S \subseteq V(G)$, we define the *boundary edge set* $\theta_G(S)$ as

$$\theta_G(S) = \{\{u, v\} \in E(G) \mid \text{exactly one of } u, v \text{ is in } S\}.$$

We define the function θ also on positive integers $s \leq |V(G)|$ as $\theta_G(s) = \min_{S \subseteq V(G),\ |S|=s} |\theta_G(S)|$. For a vertex set $S \subseteq V(G)$, we denote the *vertex edge set* $\partial_G(S)$ as

$$\partial_G(S) = \{v \notin S \mid v \text{ is a neighbor of some } u \in S \text{ in } G\}.$$

Clearly, $\partial_G(S) = N_G[S] \setminus S$. We also define the function ∂ on positive integers $s \leq |V(G)|$ as $\partial_G(s) = \min_{S \subseteq V(G),\ |S|=s} |\partial_G(S)|$.

For example, see Fig. 1.4. In Fig. 1.4, $S = \{a, d, e\}$, $\partial(S) = \{b, c\}$, $\theta(S) = \{\{a, b\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}\}$.

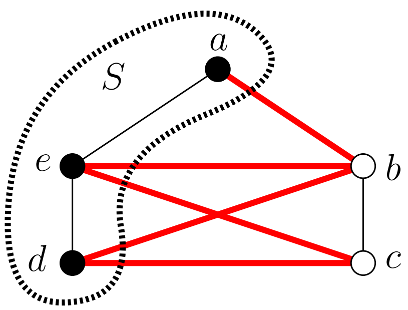


Fig. 1.4 A set $S = \{a, e, d\}$, its vertex boundary $\partial(S) = \{b, c\}$, and its edge boundary $\theta(S) = \{\{a, b\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}\}$.

1.1.3 Cartesian product

For graphs G and H, the *Cartesian product* of G and H, denoted by $G \square H$, is the graph whose vertex set is $V(G) \times V(H)$ and in which (g, h) is joined to (g', h') if and only if either $g = g'$ and $\{h, h'\} \in E(H)$ or $h = h'$ and $\{g, g'\} \in E(G)$ (see Fig. 1.5). Note that for any $h \in V(H)$, the induced subgraph of $G \square H$ induced by the set $\{(g, h) \mid g \in V(G)\}$ is isomorphic to G. For $d \geq 1$, the *dth Cartesian power* of a graph G, denoted by G^d, is defined as follows: $G^1 = G$ and $G^d = G \square G^{d-1}$ for $d \geq 2$.

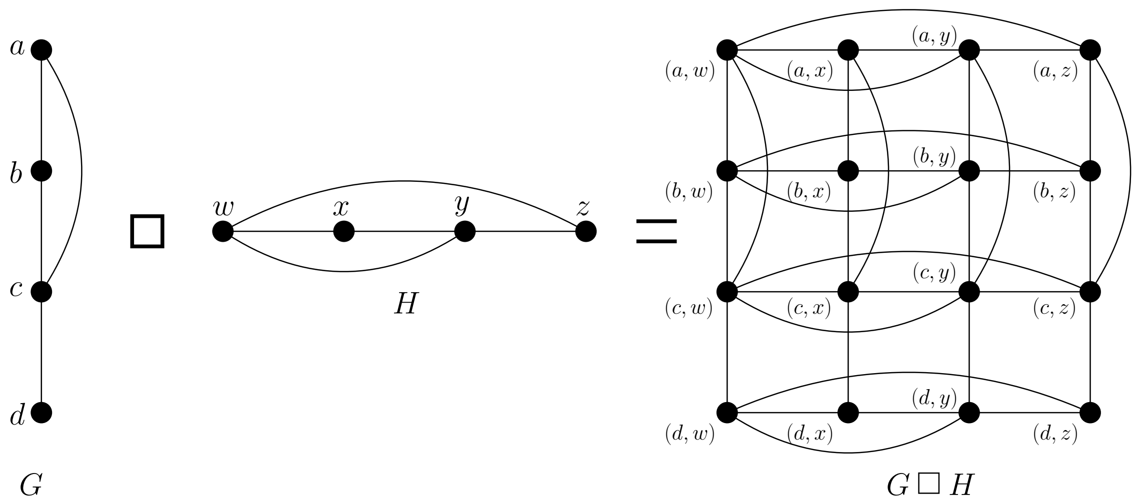


Fig. 1.5 The Cartesian product $G \square H$ of graphs G and H.

1.1.4 Graph classes

In this subsection, we define several important graph classes.

The *complete graph* K_n is a graph with the vertex set $\{0, \ldots, n-1\}$ and in which there is an edge between every pair of vertices. Let $V_1, V_2, \ldots, V_k$ be the disjoint vertex sets and $n_i = |V_i|$ for $1 \leq i \leq k$. The *complete k-partite graph* $K_{n_1,\ldots,n_k}$ is a graph such that the vertex set is $\bigcup_{1 \leq i \leq k} V_i$, and there exists an edge $\{u, v\}$ for $u \in V_i$ and $v \in V_j$ if and only if $i \neq j$. We call a complete 2-partite graph a *complete bipartite graph*. Note that if $n_i = 1$ for every i, $1 \leq i \leq k$, then the complete k-partite graph $K_{n_1,\ldots,n_k}$ is isomorphic to the complete graph K_k. See examples in Fig. 1.6.

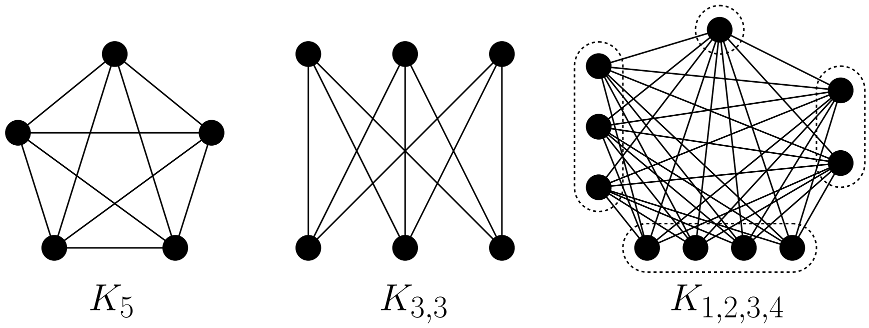


Fig. 1.6 A complete graph, a complete bipartite graph, and a complete 4-partite graph.

A graph is *planar* if it can be drawn in the plane with no pair of crossing edges. A *plane* graph is a planar graph with an embedding that causes no cross. A *face* of a plane graph is a topologically connected region surrounded by edges of the plane graph. A planar graph is *outerplanar* if there is a planar embedding in which all its vertices are in the outer-boundary. An outerplanar graph M is *maximal* if M is no longer outerplanar with the addition of a single edge. It is known that any maximal outerplanar graph M has $2|V(M)| - 3$ edges, and M has a unique Hamiltonian cycle (see [27, 18]).

Let $[n]$ denote the set $\{0, 1, \ldots, n - 1\}$. Recall that a complete graph K_n is a graph whose vertex set is $[n]$ and any two vertices are adjacent. A path P_n is a graph whose vertex set is $[n]$ and edge set is $\{\{i, i + 1\} \mid 0 \leq i \leq n - 2\}$. For $n \geq 3$, a cycle C_n is a graph whose vertex set is $[n]$ and edge set is $\{\{n - 1, 0\}\} \cup E(P_n)$. See examples in Fig. 1.7.

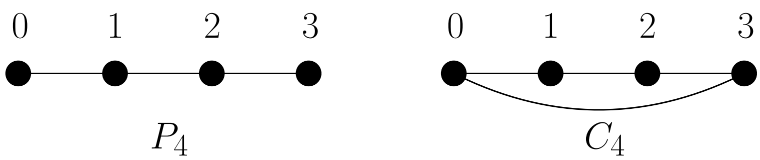


Fig. 1.7 A path and a cycle.

The graph $K_n^d = (K_n)^d$ is called a *d-dimensional Hamming graph*. The graph $P_n^d = (P_n)^d$ is called a *d-dimensional grid*. If n is even (odd) then we say that P_n^d is *even* (*odd*, respectively). The graph $C_n^d = (C_n)^d$ is called a *d-dimensional torus*. A *d-dimensional hypercube* Q^d is the dth Cartesian power of $P_2 = K_2$, that is, $Q^d = P_2^d = K_2^d$. Note that we sometimes call more general graphs $P_m \square P_n$ and $C_m \square C_n$ *two-dimensional grids* and *two-dimensional tori*, respectively.

1.1.5 Treewidth

The concept of treewidth was introduced by Robertson and Seymour in their project of Graph Minor Theory (see [46] for example). A *tree decomposition* of a graph G is a pair $(\mathcal{X}, T)$, where T is a tree and $\mathcal{X} = \{X_i \mid i \in V(T)\}$ is a collection of subsets of $V(G)$ such that

- $\bigcup_{i \in V(T)} X_i = V(G)$,
- for each edge $\{u, v\} \in E(G)$, there is a *node* $i \in V(T)$ such that $u, v \in X_i$, and
- for each $v \in V(G)$, the set of nodes $\{i \mid v \in X_i\}$ forms a subtree of T.

The elements in $\mathcal{X}$ are called *bags*. The *width* of a tree decomposition $(\mathcal{X}, T)$ equals $\max_{i \in V(T)} |X_i| - 1$. The *treewidth* of G, denoted by $tw(G)$, is the minimum width over all tree decompositions of G. A *path decomposition* of G is a tree decomposition $(\mathcal{X}, T)$ in which T is a path. The *pathwidth* of G, denoted by $pw(G)$, is the minimum width over all path decompositions of G.

For example, see Fig. 1.8. The graph depicted in Fig. 1.8 has treewidth at most two, since any bag has cardinality at most three. It is easy to see that the pathwidth of the graph in Fig. 1.8 is also at most two. To see this, remove the bag $\{f, g\}$ and insert a new bag $\{d, f, g\}$ between the bags $\{d, e, f\}$ and $\{d, f, h\}$; then marge the bags $\{i, j\}$ and $\{i, k\}$ into a new bag $\{i, j, k\}$. Clearly, the resultant structure is a path decomposition of the graph, and it has width three, as required. It is known that a graph has treewidth one if and only if the graph is a forest. Hence, we can conclude that the graph in Fig. 1.8 has treewidth two (and pathwidth two, also).

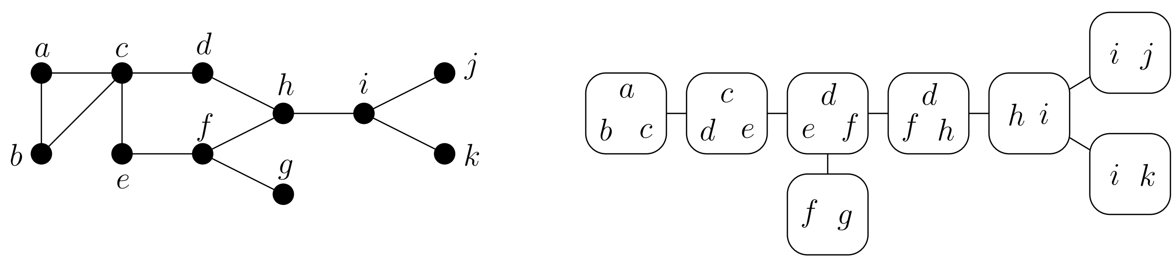


Fig. 1.8 A graph and its tree decomposition.

1.2 The vertex boundary-width of complete trees

In this section, we briefly review results on the vertex boundary-width of complete k-ary trees. The *vertex boundary-width problem* is to determine the value of

$$vbw(G) = \max_{1 \le i \le |V(G)|} \min_{S \subseteq V(G),\ |S|=i} |\partial(S)|$$

for a given graph G. The vertex boundary-width is also called the *vertex isoperimetric peak*. The complete k-ary tree of depth d, denoted by $T_{k,d}$, is defined recursively. The star $K_{1,k}$ is the complete k-ary tree of depth one. Let $d \ge 2$. For each vertex of degree one in $T_{k,d-1}$, we add k new vertices as neighbors of the vertex; The resultant tree is $T_{k,d}$.

The author and Yamazaki [43] proved the following lower and upper bounds on $vbw(T_{k,d})$.

Theorem 1.1 (Otachi and Yamazaki [43])**.**

$$\frac{\lg k}{k + 2\lg d + 6} \cdot d - 1 \le vbw(T_{k,d}) \le d.$$

The above theorem was improved by Bharadwaj and Chandran [5].

Theorem 1.2 (Bharadwaj and Chandran [5])**.** *Let $k \ge 2$ and $d \ge c_1 \log k$, where c_1 is a suitable chosen constant. Then, for some constant c_2,*

$$\frac{c_2}{\sqrt{k}} \cdot d \le vbw(T_{k,d}) \le d.$$

Finally, Vrt'o [55] has proved an asymptotically tight lower bound.

Theorem 1.3 (Vrt'o [55])**.** *For $k \ge 4$ and $d \ge 3$,*

$$\frac{3}{40} \cdot d - \frac{3}{20} \le vbw(T_{k,d}) \le d.$$

The above bound implies a somewhat unexpected fact $vbw(T_{k,d}) = \Theta(d)$, that is, the branching factor k does not effect the vertex boundary width of the complete trees. The exact value of $vbw(T_{k,d})$ is still open.

1.3 Related papers

The results in this thesis are based on the following two published papers.

1. Kyohei Kozawa, Yota Otachi, and Koichi Yamazaki, On spanning tree congestion of graphs, *Discrete Mathematics*, Volume 309, Issue 13, 6 July 2009, Pages 4215–4224. (doi:10.1016/j.disc.2008.12.021)
2. Kyohei Kozawa, Yota Otachi, and Koichi Yamazaki, Security number of grid-like graphs, *Discrete Applied Mathematics*, Volume 157, Issue 11, 6 June 2009, Pages 2555–2561. (doi:10.1016/j.dam.2009.03.020)
3. Yota Otachi and Koichi Yamazaki, A lower bound for the vertex boundary-width of complete k-ary trees, *Discrete Mathematics* Volume 308, Issue 12, 28 June 2008, Pages 2389–2395. (doi:10.1016/j.disc.2007.05.014)

The first paper is related to Chapter 2, and the second paper Chapter 3. The result of the last paper in the above list is mentioned in Section 1.2.

1.4 Other papers by the author

Here, we list the author's published papers that are not include in the list of the previous section.

1. Toshiki Saitoh, Yota Otachi, Katsuhisa Yamanaka, and Ryuhei Uehara, Random generation and enumeration of bipartite permutation graphs, ISAAC 2009, *Lecture Notes in Computer Science*, 5878 (2009) 1104–1113.
2. Katsuhisa Yamanaka, Yota Otachi, and Shin-ichi Nakano, Efficient enumeration of ordered trees with k leaves, WALCOM 2009, *Lecture Notes in Computer Science*, 5431 (2009) 141–150.
3. Tetsuya Ishizeki, Yota Otachi, and Koichi Yamazaki, An improved algorithm for longest induced path problem on k-chordal graphs, *Discrete Applied Mathematics*, Volume 156, Issue 15, 6 August 2008, Pages 3057–3059.
4. Yota Otachi, Yoshio Okamoto, and Koichi Yamazaki, Relationships between the class of unit grid intersection graphs and other classes of bipartite graphs, *Discrete Applied Mathematics*, Volume 155, Issue 17, 15 October 2007, Pages 2383–2390.

Chapter 2

Spanning tree congestion of graphs

2.1 Introduction

In this chapter, we study *the spanning tree congestion problem* for some classes of graphs. Let G be a graph and T a tree such that $V(G) \subseteq V(T)$. We say that T is a *host* and G is a *guest*. The *detour* for an edge $\{u, v\} \in E(G)$ is the unique u-v path in T. We define the *congestion* of $e \in E(T)$, denoted by $ec_G(e)$, as the number of detours that contain e. The *edge congestion of* G *in* T, denoted by $ec(G : T)$, is the maximum congestion over all edges in T. We define the *tree congestion* of G, denoted by $tc(G)$, and the *spanning tree congestion* of G, denoted by $stc(G)$, as

$$tc(G) = \min\{ec(G : T) \mid T \text{ is a tree and } V(T) = V(G)\},$$
$$stc(G) = \min\{ec(G : T) \mid T \text{ is a tree, } V(T) = V(G) \text{, and } E(T) \subseteq E(G)\}.$$

Several related problems have been studied. If the host graphs are paths, the problem is well-known *cutwidth* (or *minimum cut linear arrangement*) problem (see [53]). Liu and Yuan [37] have determined the cutwidth for several product graphs including two-dimensional grids and tori. When the host graphs are restricted to ternary trees, and all vertices of the guest graph are assigned to the leaves of the host trees, the problem is *carvingwidth* problem [49].

For some applications, host graphs are not restricted to acyclic graphs. For example, simple cycles [48], grids [4], and so on (see [44]). Note that if the host graph has a cycle, then the detour for an edge of the guest graph cannot be determined uniquely, and so, one should take the best one of the candidates.

Complexity results are known for several variants of tree congestion problem. Simonson [50] showed the problem is NP-hard if the host graphs are trees with bounded degree even when the guest graph is planar. Khuller, Raghavachari, and Young [32] have shown the NP-hardness for the following General Congestion Problem: *The input to the problem is two graphs $G = (V, E)$ and $F = (V, E')$. The problem is to find a minimum congestion tree T of G such that $E(T) \subseteq E'$.* They pointed out that if F is the complete graph, the problem can be solved in polynomial time [32], by using results of Gomory and Hu [25], and Gusfield [26]. It follows that the tree congestion problem is solvable in polynomial time. If $F = G$, the problem is exactly the spanning tree congestion problem. To the best of our knowledge, it is not known that whether the problem is NP-hard even when $F = G$. So the complexity of the spanning tree congestion problem is not known.[*1]

There are several results for the spanning tree congestion problem. Simonson [50] presented an algorithm for the spanning tree congestion problem on outerplanar graphs that outputs an embedding with the congestion at most one larger than the maximum degree of the input graph. Ostrovskii [41] showed some inequalities for the (spanning) tree congestion problem and studied the extremal graph problem of the spanning tree congestion. Hruska [31] studied the problem of the spanning tree congestion for the two-dimensional grids and the complete bipartite graphs. Castejón and Ostrovskii [12] gave asymptotic estimates for the spanning tree congestion of three-dimensional grids and tori. Löwenstein, Rautenbach, and Regen [38] have shown that the spanning tree congestion of a graph on n vertices is at most $n^{3/2}$.

In this chapter, we show the spanning tree congestion for some classes of graphs. We also show, with some applications, a technique to derive a lower bound of the spanning tree congestion. The rest of this chapter is organized as follows. In Section 2.2, we introduce some notations and state a general lower bound of the spanning tree congestion. In Section 2.3, we show the spanning tree congestion for the complete k-partite graphs. This properly extends the results of Ostrovskii [41] and Hruska [31] for the complete graphs

[*1] Very recently, Hans L. Bodlaender and the author have proved the NP-hardness of the problem [42]. See Subsection 2.9.1 for more details.

and the complete bipartite graphs, respectively. In Section 2.4, we show the spanning tree congestion for the two-dimensional tori. This problem is related to Hruska's result for the two-dimensional grids [31]. In Section 2.5, we show lower bounds of the spanning tree congestion for the hypercubes and the multi-dimensional grids by edge-isoperimetric inequalities. In Section 2.6, we show the spanning tree congestion of the two-dimensional Hamming graphs (a.k.a. rook's graphs). In Section 2.7, we give lower and upper bounds on the spanning tree congestion of multi-dimensional Hamming graphs. In Section 2.8, we show a relationship between the spanning tree congestion and the treewidth. In the last section, we state the concluding remarks.

2.2 Preliminaries

Let G be a connected graph. If $e \in E(G)$ has a vertex of degree one as one of its endpoints, e is called a *leaf edge*, otherwise e is called an *inner edge*. By using the function θ, the congestion $ec_G(e)$ of an edge $e \in E(T)$ can be defined in a different form as

$$ec_G(e) = |\theta_G(L_e)|$$

where L_e is the vertex set of one of the two components of $T - e$. Note that if e is a leaf edge of T, then $ec_G(e) = deg_G(v)$ where v is an endpoint of e such that $deg_T(v) = 1$. We omit the subscript of the function $ec_G(e)$ if the graph is clear from the context.

From a basic property of trees, we can derive a general lower bound for the spanning tree congestion.

Lemma 2.1 (Ostrovskii [41])**.** *For any tree T, there is an edge $e \in E(T)$ such that the number of vertices of the smaller component of $T - e$ is at least $(|V(T)| - 1)/\Delta(T)$.*

Corollary 2.2. *For a connected graph G, $stc(G) \geq \min_{s=\lceil(|V(G)|-1)/\Delta(G)\rceil}^{\lfloor |V(G)|/2 \rfloor} \theta(s)$.*

Proof. Let T be a spanning tree of G, $e \in E(T)$ be an edge in Lemma 2.1, and L_e and R_e be the vertex sets of the components of $T - e$. Without loss of generality, we may assume $|L_e| \leq |R_e|$. Since $V(T) = V(G)$, we have that

$$|L_e| \leq \lfloor |V(T)|/2 \rfloor = \lfloor |V(G)|/2 \rfloor .$$

Since $V(T) = V(G)$ and $\Delta(T) \leq \Delta(G)$, we have that

$$|L_e| \geq \lceil (|V(G)| - 1)/\Delta(G) \rceil .$$

Hence,

$$ec(G : T) \geq |\theta(L_e)| \geq \theta(|L_e|) \geq \min_{s=\lceil (|V(G)|-1)/\Delta(G)\rceil}^{\lfloor |V(G)|/2 \rfloor} \theta(s).$$

The lemma holds. □

2.3 Spanning tree congestion of complete k-partite graphs

In this section, we consider the spanning tree congestion of the complete k-partite graphs. Let n be the number of the vertices of $K_{n_1,\dots,n_k}$, that is, $n = \sum_{1 \leq i \leq k} n_i$. We assume $n_1 \leq \cdots \leq n_k$. We denote by $deg_i(K_{n_1,\dots,n_k})$ the degree of a vertex in V_i. Clearly, $deg_i(K_{n_1,\dots,n_k}) = n - n_i$. Note that $\delta(K_{n_1,\dots,n_k}) = deg_k(K_{n_1,\dots,n_k}) = n - n_k$ and $\Delta(K_{n_1,\dots,n_k}) = deg_1(K_{n_1,\dots,n_k}) = n - n_1$. In the following two subsections, we will show the following theorem.

Theorem 2.3. *For $k \geq 2$, $1 \leq n_1 \leq \cdots \leq n_k$, and $n = \sum_{1 \leq i \leq k} n_i$,*

$$stc(K_{n_1,\dots,n_k}) = \begin{cases} n - n_2 & \textit{if } n_1 = 1, \\ 2n - n_k - n_{k-1} - 2 & \textit{otherwise.} \end{cases}$$

2.3.1 Case $n_1 = 1$

First, we consider the case $n_1 = 1$. We use Ostrovskii's result [41]. For each two distinct vertices $u, v \in V(G)$, by $m(u, v)$ we denote the maximum number of edge-disjoint paths between u and v in G.

Lemma 2.4 (Ostrovskii [41]). *Let G be a graph and $u, v \in V(G)$ be distinct vertices. Then $tc(G) \geq m(u, v)$.*

Lemma 2.5. *Let $k \geq 2$ and $n_1 \leq \cdots \leq n_k$. If $n_1 = 1$ then*

$$stc(K_{n_1,\dots,n_k}) = n - n_2.$$

Proof. Let $V_1 = \{v_1\}$. We define a spanning tree T as a star $K_{1,n-1}$ with the center v_1. Since all edges of T are leaf edges,

$$ec(K_{n_1,\ldots,n_k} : T) = \max_{2 \le i \le k} deg_i(K_{n_1,\ldots,n_k}) = deg_2(K_{n_1,\ldots,n_k}) = n - n_2.$$

Therefore, $stc(K_{n_1,\ldots,n_k}) \le n - n_2$.

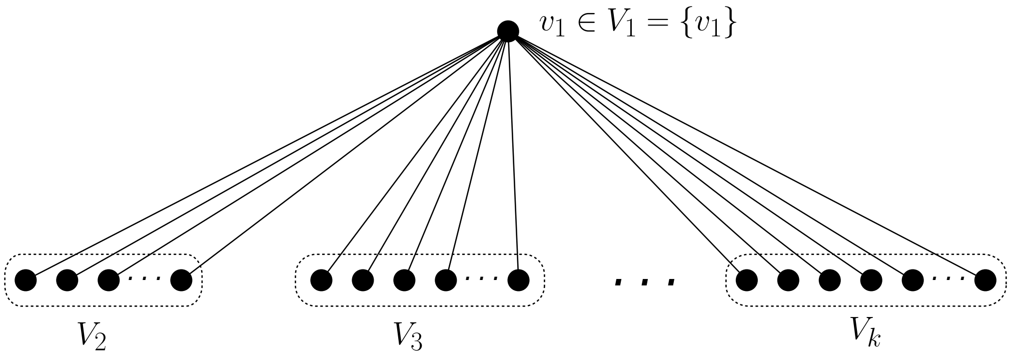


Fig. 2.1 An optimum spanning tree T for $K_{n_1,\ldots,n_k}$ in Lemma 2.5.

To show $stc(K_{n_1,\ldots,n_k}) \ge n - n_2$, we will demonstrate that $m(v_1, v_2) = n - n_2$ for any $v_2 \in V_2$. Clearly, there are $n - n_2 - 1$ disjoint paths of length two between v_1 and v_2, that is, the paths $\{(v_1, u, v_2) : u \in N(v_2) \setminus \{v_1\}\}$, and furthermore there is the edge $\{v_1, v_2\}$. Thus, $m(v_1, v_2) = deg(v_2) = n - n_2$. From Lemma 2.4, $stc(K_{n_1,\ldots,n_k}) \ge tc(K_{n_1,\ldots,n_k}) \ge n - n_2$. □

Note that Lemma 2.5 can be applied to the complete graphs as well. To see this, observe that $K_{n_1,\ldots,n_k}$ is the complete graph of k vertices if $n_i = 1$ for all $1 \le i \le k$.

2.3.2 Case $n_1 \ge 2$

Next, we consider the remaining case $n_1 \ge 2$. Recall that $n_1 \le \cdots \le n_k$ and $n = \sum_{1 \le i \le k} n_i$. The following two known lemmas can be integrated into Corollary 2.8.

Lemma 2.6 (Ostrovskii [41]). *If $k \ge 2$ and $n_i = 2$ for $1 \le i \le k$ then $stc(K_{n_1,\ldots,n_k}) = 2n - 6$.*

Lemma 2.7 (Hruska [31]). *For $2 \le n_1 \le n_2$, $stc(K_{n_1,n_2}) = n - 2$.*

Corollary 2.8. *Let $k \geq 2$ and $2 \leq n_1 \leq \cdots \leq n_k$. If either $n_k = 2$ or $k = 2$,*

$$stc(K_{n_1,\ldots,n_k}) = 2n - n_k - n_{k-1} - 2.$$

We will show that $stc(K_{n_1,\ldots,n_k}) = 2n - n_k - n_{k-1} - 2$ also holds for any $n_k \geq 3$ and $k \geq 3$. This properly extends the above lemmas.

First we show the upper bound.

Lemma 2.9. *If $2 \leq n_1 \leq \cdots \leq n_k$, $n_k \geq 3$, and $k \geq 3$ then*

$$stc(K_{n_1,\ldots,n_k}) \leq 2n - n_k - n_{k-1} - 2.$$

Proof. Let $v \in V_{k-1}$. We define a spanning tree T of $K_{n_1,\ldots,n_k}$ as follows (see Fig. 2.2):

$$V(T) = V(K_{n_1,\ldots,n_k}),$$
$$E(T) = E_v \cup E_{\mathrm{cm}},$$

where

$$E_v = \{\{u, v\} \mid u \in N_G(v)\},$$
$$E_{\mathrm{cm}} = \text{a complete matching from } V_{k-1} \setminus \{v\} \text{ to } V_k.$$

For any leaf edge $e_\ell \in E(T)$, $ec(e_\ell) \leq \Delta(K_{n_1,\ldots,n_k}) = n - n_1$. Let e_{in} be an inner edge of T. Then $ec(e_{\mathrm{in}}) = |\theta(\{x, y\})|$ for some $x \in V_{k-1} \setminus \{v\}$ and $y \in V_k$ such that the edge $\{x, y\} \in E_{\mathrm{cm}}$. It is easy to see that $|\theta(\{x, y\})| = deg(x) + deg(y) - 2 = (n - n_{k-1}) + (n - n_k) - 2 = 2n - n_k - n_{k-1} - 2$. Suppose $2n - n_k - n_{k-1} - 2 \leq n - n_1$. Then, we have $n \leq n_k + n_{k-1} + 2 - n_1 \leq n_k + n_{k-1}$, a contradiction. Thus, $2n - n_k - n_{k-1} - 2 > n - n_1$, and so,

$$ec(K_{n_1,\ldots,n_k} : T) = 2n - n_k - n_{k-1} - 2.$$

Hence, the lemma follows. □

Next we show the lower bound.

Lemma 2.10. *If $2 \leq n_1 \leq \cdots \leq n_k$, $n_k \geq 3$, and $k \geq 3$ then*

$$stc(K_{n_1,\ldots,n_k}) \geq 2n - n_k - n_{k-1} - 2.$$

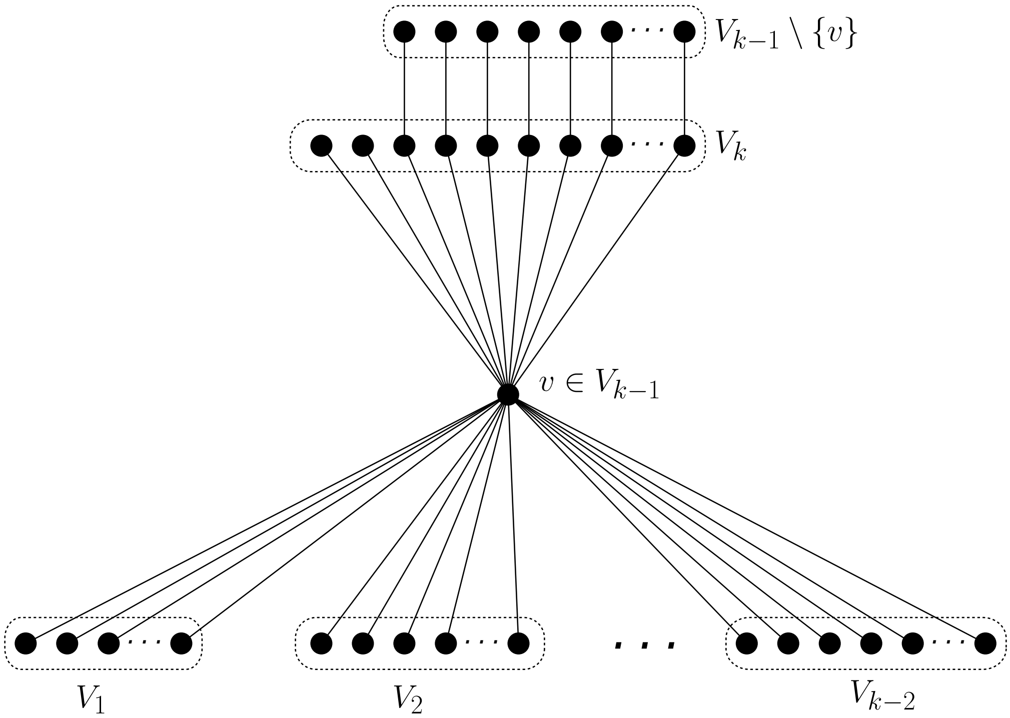


Fig. 2.2 An optimum spanning tree T for $K_{n_1,\ldots,n_k}$ in Lemma 2.9.

Proof. Let T be a spanning tree of $K_{n_1,\ldots,n_k}$. If T is a star, then the center of T has degree $n-1 > n-n_1 = \Delta(K_{n_1,\ldots,n_k})$, a contradiction. Thus, T has an inner edge. Let e be an inner edge of T. We shall show that the edge e has congestion at least $2n - n_k - n_{k-1} - 2$. We denote the vertex sets of the two components of $T - e$ by L_e and R_e. Since e is an inner edge, we have that $E(K_{n_1,\ldots,n_k}[L_e]) \neq \emptyset$ and $E(K_{n_1,\ldots,n_k}[R_e]) \neq \emptyset$. If a detour contains the edge e, we call it an *e-detour*. We divide the proof into following three cases:

1. $n_k < n/2$;
2. $n_k \geq n/2$ and either $V_k \cap L_e = \emptyset$ or $V_k \cap R_e = \emptyset$;
3. $n_k \geq n/2$, $V_k \cap L_e \neq \emptyset$, and $V_k \cap R_e \neq \emptyset$.

[Case 1] $n_k < n/2$: Without loss of generality, we may assume $|L_e| \leq n/2$. For each vertex $\ell \in L_e$, the number of e-detours connecting ℓ to its neighbors is at least $deg(\ell) - (|L_e| - 1)$, since ℓ has at most $|L_e| - 1$ neighbors in L_e. Therefore, we have

$$ec(e) \geq \sum_{\ell \in L_e} (deg(\ell) - (|L_e| - 1)) = \sum_{\ell \in L_e} deg(\ell) - |L_e|(|L_e| - 1).$$

Since $E(K_{n_1,\dots,n_k}[L_e]) \neq \emptyset$, it holds that $L_e \not\subseteq V_k$. Hence, there exists a vertex in L_e that has degree at least $deg_{k-1}(K_{n_1,\dots,n_k})$, and so,

$$\begin{aligned}\sum_{\ell \in L_e} deg(\ell) &\geq deg_{k-1}(K_{n_1,\dots,n_k}) + (|L_e| - 1)\delta(K_{n_1,\dots,n_k})\\ &= deg_{k-1}(K_{n_1,\dots,n_k}) + (|L_e| - 1)deg_k(K_{n_1,\dots,n_k}).\end{aligned}$$

Since $n_k < n/2$ and $|L_e| \leq n/2$, we can see that $|L_e| < n - n_k = deg_k(K_{n_1,\dots,n_k})$. This implies $|L_e| + 1 \leq deg_k(K_{n_1,\dots,n_k})$. Thus, we have

$$\begin{aligned}ec(e) &\geq deg_{k-1}(K_{n_1,\dots,n_k}) + (|L_e| - 1)deg_k(K_{n_1,\dots,n_k}) - |L_e|(|L_e| - 1)\\ &= deg_{k-1}(K_{n_1,\dots,n_k}) + deg_k(K_{n_1,\dots,n_k}) + (|L_e| - 2)deg_k(K_{n_1,\dots,n_k}) - |L_e|(|L_e| - 1)\\ &\geq deg_{k-1}(K_{n_1,\dots,n_k}) + deg_k(K_{n_1,\dots,n_k}) + (|L_e| - 2)(|L_e| + 1) - |L_e|(|L_e| - 1)\\ &= deg_{k-1}(K_{n_1,\dots,n_k}) + deg_k(K_{n_1,\dots,n_k}) - 2.\end{aligned}$$

Since $deg_i(K_{n_1,\dots,n_k}) = n - n_i$, the lemma holds in this case.

[Case 2] $n_k \geq n/2$ and either $V_k \cap L_e = \emptyset$ or $V_k \cap R_e = \emptyset$: Without loss of generality, we may assume $V_k \cap R_e = \emptyset$. This implies $V_k \subseteq L_e$, hence, we have that $ec(e) \geq |R_e|n_k$. Since $E(K_{n_1,\dots,n_k}[R_e]) \neq \emptyset$, $|R_e| \geq 2$. If $|R_e| \geq 3$ then $ec(e) \geq 3n_k = 4n_k - n_k \geq 2n - n_k$, since $n_k \geq n/2$. Otherwise $|R_e| = 2$. Let $R_e = \{r_1, r_2\}$. Then $\{r_1, r_2\} \in E(T)$, so r_1 and r_2 belong to different V_i's. Thus,

$$\begin{aligned}ec(e) &= deg(r_1) + deg(r_2) - 2\\ &\geq deg_k(K_{n_1,\dots,n_k}) + deg_{k-1}(K_{n_1,\dots,n_k}) - 2\\ &= 2n - n_k - n_{k-1} - 2.\end{aligned}$$

[Case 3] $n_k \geq n/2$, $V_k \cap L_e \neq \emptyset$, and $V_k \cap R_e \neq \emptyset$: First, note that we do not use the assumption $n_k \geq n/2$. This assumption is added here only for guaranteeing that the case analysis covers all cases exactly.

Without loss of generality, we may assume $|V_k \cap L_e| \geq \lceil n_k/2 \rceil$. Since $n_k \geq 3$, $|V_k \cap L_e| \geq 2$. Then there are three vertices $k_\ell^1, k_\ell^2, k_r \in V_k$ such that $k_\ell^1, k_\ell^2 \in L_e$ and $k_r \in R_e$. Since $E(K_{n_1,\dots,n_k}[R_e]) \neq \emptyset$, R_e contains a vertex $i_r \in V_i$ such that $i \neq k$. Similarly, L_e contains a vertex $j_\ell \in V_j$ such that $j \neq k$. We call the vertices $k_\ell^1, k_\ell^2, k_r, i_r$, and j_ℓ *initial vertices* and denote them by I (see Fig. 2.3). Observe that we can select i_r and j_ℓ so that $i \neq j$. Otherwise, every vertex except for vertices in V_k is in V_i. This contradicts $k \geq 3$. We will estimate the number of e-detours starting from one of the initial vertices. More precisely,

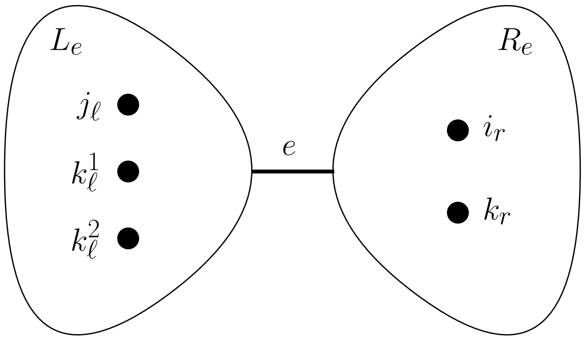


Fig. 2.3 Initial vertices $I = \left\{k_\ell^1, k_\ell^2, k_r, i_r, j_\ell\right\}$.

we estimate the number of e-detours from I to (1) I, (2) $V_k \setminus \{k_\ell^1, k_\ell^2, k_r\}$, (3) V_h ($h \notin \{i, j, k\}$), and (4) $V_i \cup V_j \setminus \{i_r, j_\ell\}$.

(1) From I to I: Since there are four edges $\{i_r, j_\ell\}$, $\{i_r, k_\ell^1\}$, $\{i_r, k_\ell^2\}$, and $\{j_\ell, k_r\}$ between L_e and R_e, there are four e-detours.

(2) From I to $V_k \setminus \{k_\ell^1, k_\ell^2, k_r\}$: We will show that there exist $n_k - 3$ e-detours. Recall that $|V_k| = n_k \geq 3$. If $n_k = 3$ there is no e-detour since $V_k \setminus \{k_\ell^1, k_\ell^2, k_r\} = \emptyset$. Otherwise, for each $v \in V_k \setminus \{k_\ell^1, k_\ell^2, k_r\}$, there is a detour, from i_r or j_ℓ to v. Thus, the number of e-detours is $|V_k \setminus \{k_\ell^1, k_\ell^2, k_r\}| = n_k - 3$.

(3) From I to V_h ($h \notin \{i, j, k\}$): For each $v \in V_h$, there exist at least two e-detours; from $\{i_r, k_r\}$ or $\{j_\ell, k_\ell^1, k_\ell^2\}$ to v. Hence, the number of e-detours from I to V_h is at least $2|V_h| = 2n_h$.

(4) From I to $V_i \cup V_j \setminus \{i_r, j_\ell\}$: For each $u \in V_i \setminus \{i_r\}$, there exists at least one e-detour; from k_r or $\{j_\ell, k_\ell^1, k_\ell^2\}$ to u. For each $v \in V_j \setminus \{j_\ell\}$, there are two e-detours; from $\{i_r, k_r\}$ or $\{k_\ell^1, k_\ell^2\}$ to v. So the number of e-detours from I to $V_i \cup V_j \setminus \{i_r, j_\ell\}$ is at least $|V_i \setminus \{i_r\}| + 2|V_j \setminus \{j_\ell\}| = n_i + 2n_j - 3$.

From the above observations (1–4),

$$\begin{aligned} ec(e) &\geq 4 + (n_k - 3) + \left(\sum_{\ell \in \{1,\ldots,k\} \setminus \{i,j,k\}} 2n_\ell \right) + (n_i + 2n_j - 3) \\ &= n_k + 2(n - n_i - n_j - n_k) + n_i + 2n_j - 2 \\ &= 2n - n_k - n_i - 2. \end{aligned}$$

Since $i \neq k$, $ec(e) \geq 2n - n_k - n_i - 2 \geq 2n - n_k - n_{k-1} - 2$. □

Corollary 2.8, Lemma 2.9, and Lemma 2.10 imply Theorem 2.3 for the case $n \geq 2$.

2.4 Spanning tree congestion of two-dimensional tori

Recently, Hruska [31] has determined the spanning tree congestion of the two-dimensional grids $P_m \square P_n$.

Theorem 2.11 (Hruska [31])**.** *For $m \leq n$,*

$$stc(P_m \square P_n) = \begin{cases} m & \text{if } m = n \text{ or } m \text{ odd}, \\ m + 1 & \text{otherwise}. \end{cases}$$

In this section, we consider a related problem. We will show the spanning tree congestion of the two-dimensional tori. A *two-dimensional torus* is the Cartesian product of two cycles, that is, $C_m \square C_n$ for some integers $m, n \geq 3$. The following result can be shown by Lemma 2.15 and Lemma 2.18 derived later.

Theorem 2.12. $stc(C_m \square C_n) = 2 \min\{m, n\}$.

Note that Castejón and Ostrovskii [12] showed the spanning tree congestion of square tori $C_n \square C_n$, independently. Clearly, our result is more general than theirs.

A vertex of $C_m \square C_n$ is represented as (i, j) for some integers $0 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$. $C_m \square C_n$ has an edge $\{(i, j), (i', j')\}$ if and only if either $i = i'$ and $j = ((j' + 1) \bmod n)$, or $j = j'$ and $i = ((i' + 1) \bmod m)$. We say that ith copy of C_n in $C_m \square C_n$ is the *ith column*, and jth copy of C_m in $C_m \square C_n$ is the *jth row*. We denote the ith column and the jth row by $Col(i)$ and $Row(j)$, respectively. Note that there are m columns and n rows in $C_m \square C_n$ (see Fig. 2.4).

The following lemma follows immediately from the definition of the function θ (see [3]).

Lemma 2.13. *For an r-regular graph G and a set $S \subseteq V(G)$,*

$$|\theta_G(S)| = r|S| - 2|E(G[S])|.$$

Since $C_m \square C_n$ is 4-regular, we have the following corollary from Lemma 2.13.

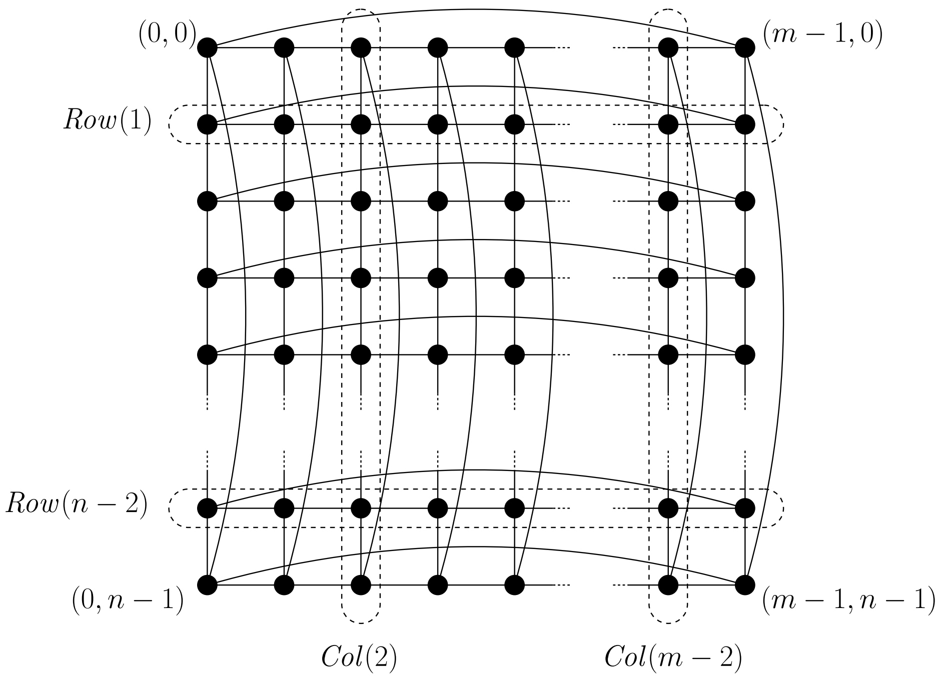


Fig. 2.4 A two-dimensional torus $C_m \,\square\, C_n$.

Corollary 2.14. *Let T be a spanning tree of $C_m \,\square\, C_n$, $e \in E(T)$, and L_e be the vertex set of a component of $T - e$. Then $ec(e) = 4|L_e| - 2|E((C_m \,\square\, C_n)[L_e])|$.*

Now, we show the upper bound.

Lemma 2.15. $stc(C_m \,\square\, C_n) \leq 2\min\{m, n\}$.

Proof. Without loss of generality, we may assume $m \geq n$. Our spanning tree T is defined as follows (see Fig. 2.5):

$$V(T) = V(C_m \,\square\, C_n),$$
$$E(T) = E_{\text{top}} \cup E_{\text{vert}},$$

where

$$E_{\text{top}} = \{\{(i, 0), (i+1, 0)\} \mid 0 \leq i \leq m-2\},$$
$$E_{\text{vert}} = \{\{(i, j), (i, j+1)\} \mid 0 \leq i \leq m-1, 0 \leq j \leq n-2\}.$$

Let $e_t \in E_{\text{top}}$ and $e_t = \{(i, 0), (i+1, 0)\}$ for some $0 \leq i \leq m-2$. Let L_{e_t} be a vertex set of the component of $T - e_t$ that contains $(i, 0)$. Then it is easy to see that $|L_{e_t}| = (i+1)n$ and $|E((C_m \square C_n)[L_{e_t}])| = (2i+1)n$ (see Fig. 2.5). So, from Corollary 2.14,

$$ec(e_t) = 4(i+1)n - 2(2i+1)n = 2n.$$

Let $e_v \in E_{\mathrm{vert}}$ and $e_v = \{(i, j), (i, j+1)\}$ for some $0 \le i \le m-1$ and $0 \le j \le n-2$. We denote by L_{e_v} the vertex set of a component of $T - e_v$ that contains $(i, j+1)$. Then clearly $|L_{e_v}| = n - j - 1$ and $|E((C_m \square C_n)[L_{e_v}])| = n - j - 2$ (see Fig. 2.5). So, from Corollary 2.14,

$$ec(e_v) = 4(n-j-1) - 2(n-j-2) = 2n - 2j \le 2n.$$

From the above observations, we have $ec(C_m \square C_n : T) = 2n$ as required. □

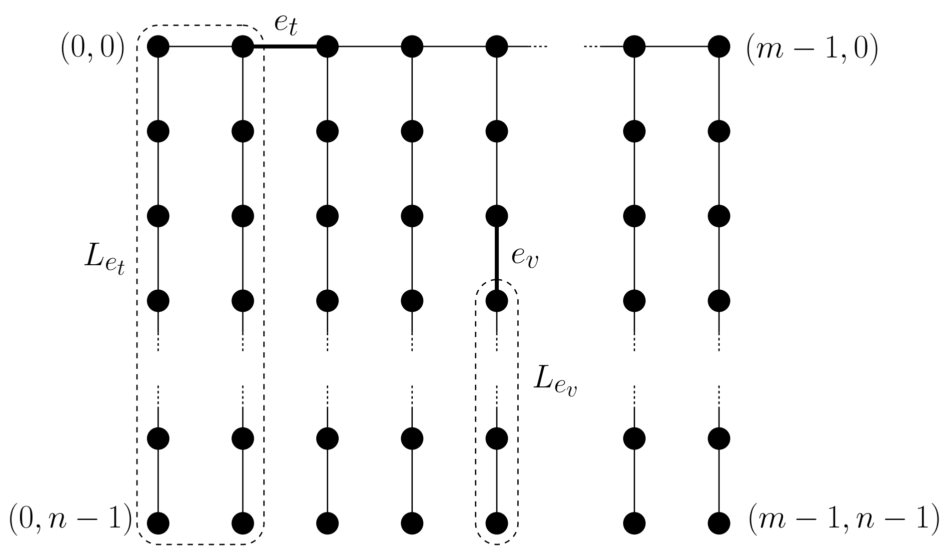


Fig. 2.5 An optimum spanning tree T for $C_m \square C_n$ in Lemma 2.15 ($m \ge n$).

Next we show the lower bound. To this end, we need some definitions and a corollary. Let S be a subset of $V(C_m \square C_n)$. We say that S *spans ith column* if S contains all vertices of $Col(i)$. Similarly, we say that S *spans jth row* if S contains all vertices of $Row(j)$. We say that S *touches ith column* if S contains some vertex of $Col(i)$ and S does not span $Col(i)$, and similarly, S *touches jth row* if S contains some vertex of $Row(j)$ and S does not span $Row(j)$. If an edge $e \in E(C_m \square C_n)$ is contained by some column then we say that e is *vertical*; otherwise e is *horizontal*.

Obviously, the following proposition holds.

Proposition 2.16. *If $S \subseteq V(C_m \square C_n)$ touches ith column (jth row) then the ith column (jth row) contains at least two vertical (horizontal, respectively) boundary edges.*

Since the set of vertical boundary edges and the set of horizontal boundary edges are disjoint for any $S \subseteq V(C_m \square C_n)$, the following corollary holds from Proposition 2.16.

Corollary 2.17. *Let $S \subseteq V(C_m \square C_n)$. If S touches c columns and r rows then $|\theta(S)| \geq 2(c + r)$.*

Now, we are ready to show the lower bound for $stc(C_m \square C_n)$.

Lemma 2.18. $stc(C_m \square C_n) \geq 2 \min\{m, n\}$.

Proof. Let T be an arbitrarily spanning tree of $C_m \square C_n$. Let $e \in E(C_m \square C_n)$ be an edge in Lemma 2.1, and L_e be the vertex set of the smaller component of $T - e$. Then $\lceil (mn - 1)/4 \rceil \leq |L_e| \leq \lfloor mn/2 \rfloor$ since $|V(C_m \square C_n)| = mn$ and $\Delta(T) \leq \Delta(C_m \square C_n) = 4$. By estimating $|\theta(L_e)|$, we will show that $ec(e)$ is large enough. Note that $|\theta(L_e)| = ec(e)$ here. We divide the proof into the following three cases:

1. L_e spans some columns and some rows;
2. L_e spans some columns but no row, or some rows but no column;
3. L_e spans neither columns nor rows.

[Case 1] L_e spans some columns and some rows: Without loss of generality, we may assume $m \geq n$. We denote by c and r the number of spanned columns and rows, respectively. Since each column is a copy of C_n and each row is a copy of C_m,

$$|L_e| \geq \max\{cn, rm\}.$$

Since L_e spans a column and a row, L_e intersects all columns and rows. So, L_e touches $m - c$ columns and $n - r$ rows. (Recall that $C_m \square C_n$ contains m columns and n rows.) Hence, from Corollary 2.17,

$$|\theta(L_e)| \geq 2(m - c + n - r).$$

Suppose $|\theta(L_e)| < 2n$. Then, we have that $2(m - c + n - r) < 2n$, which implies $m < c + r$. Therefore,

$$mn < (c + r)n \leq cn + rm \leq 2 \max\{cn, rm\} \leq 2|L_e|.$$

This implies $|L_e| > mn/2$ that contradicts $|L_e| \leq \lfloor mn/2 \rfloor$. Thus, $|\theta(L_e)| \geq 2n$.

[Case 2] L_e spans some columns but no row, or some rows but no column: If L_e spans a row then L_e touches all columns. So, $|\theta(L_e)| \geq 2m$ from Corollary 2.17. The opposite case can be proved by the symmetry argument.

[Case 3] L_e spans neither columns nor rows: Let r and c be the number of touched rows and touched columns, respectively. From Corollary 2.17, $|\theta(L_e)| \geq 2(r+c)$. Clearly, $rc \geq |L_e|$. It is well known that $(r+c)/2 \geq \sqrt{rc}$. Thus,

$$|\theta(L_e)| \geq 2(r+c) \geq 4\sqrt{rc} \geq 4\sqrt{|L_e|}.$$

Now we have the following three subcases:

[Case 3-a] $m \neq n$: If $m > n$, then $m \geq n+1$, and so,

$$|\theta(L_e)| \geq 4\sqrt{|L_e|} \geq 4\sqrt{(mn-1)/4} \geq 2\sqrt{n^2+n-1} \geq 2n.$$

Otherwise, that is, if $n > m$, we can derive $|\theta(L_e)| \geq 2m$ by the symmetry argument.

[Case 3-b] $m = n = 2\ell$ for some positive integer ℓ:

$$|\theta(L_e)| \geq 4\sqrt{|L_e|} \geq 4\sqrt{\lceil (mn-1)/4 \rceil} = 4\sqrt{\lceil \ell^2 - 1/4 \rceil} = 4\ell = 2n.$$

[Case 3-c] $m = n = 2\ell + 1$ for some positive integer ℓ:

$$|\theta(L_e)| \geq 4\sqrt{|L_e|} \geq 4\sqrt{(mn-1)/4} = 4\sqrt{\ell^2+\ell}.$$

Clearly, $4\sqrt{\ell^2+\ell} > 4\ell+1$ for $\ell \geq 1$. Thus, we have $|\theta(L_e)| > 4\ell+1 = 2n-1$, which implies $|\theta(L_e)| \geq 2n$. This completes the proof. □

The method used in the above proof is not essentially new. For example, Rolim, Sýkora, and Vrt'o used a similar method to show the cutwidth of cylinders $P_m \square C_n$ [47, Theorem 1].

2.5 Lower bounds for two classes of graphs

In this section, we show lower bounds of spanning tree congestion for two classes of graphs. We use Corollary 2.2 to derive the lower bounds.

2.5.1 Multi-dimensional grids

Recall that a *d-dimensional grid* P_n^d is the dth Cartesian power of a path P_n, that is, $P_n^1 = P_n$ and $P_n^d = P_n \,\square\, P_n^{d-1}$ for $d > 1$.

Lemma 2.19 (Bollobás and Leader [9])**.** *For* $1 \leq s \leq n^d$,

$$\theta_{P_n^d}(s) \geq \begin{cases} 4s/n & \textit{if } s < n^d/4, \\ n^{d-1} & \textit{if } n^d/4 \leq s \leq 3n^d/4, \\ 4(n^d - s)/n & \textit{if } s > 3n^d/4. \end{cases}$$

Theorem 2.20. $stc(P_n^d) \geq \left\lceil 2(n^d - 1)/(dn) \right\rceil$ *for* $d \geq 2$.

Proof. Obviously, $\Delta(P_n^d) = 2d$ and $|V(P_n^d)| = n^d$. So, from Corollary 2.2 and Lemma 2.19,

$$\begin{aligned} stc(P_n^d) &\geq \min_{s=\lceil (n^d-1)/(2d) \rceil}^{\lfloor n^d/2 \rfloor} \theta(s) \geq \min \left\{ n^{d-1}, \min_{s=\lceil (n^d-1)/(2d) \rceil}^{\lceil n^d/4 \rceil - 1} \frac{4s}{n} \right\} \\ &\geq \min \left\{ n^{d-1}, \frac{2(n^d - 1)}{dn} \right\}. \end{aligned}$$

Since $d \geq 2$, $n^{d-1} \geq 2(n^d - 1)/(dn)$. Thus, the theorem follows. $\square$

The above theorem has two applications. First, from Theorem 2.20,

$$stc(P_n \,\square\, P_n) \geq \left\lceil 2(n^2 - 1)/(2n) \right\rceil = \lceil n - 1/n \rceil = n.$$

This lower bound is the best possible (Hruska [31] has shown $stc(P_n \,\square\, P_n) = n$). Second, we can derive a lower bound for the *hypercube* $Q^d = P_2^d$. From Theorem 2.20,

$$stc(Q^d) = stc(P_2^d) \geq \left\lceil 2(2^d - 1)/(2d) \right\rceil = \left\lceil (2^d - 1)/d \right\rceil.$$

This bound, however, is not so good. In the following subsection, we will show a better lower bound for the hypercubes.

2.5.2 Hypercubes

Hruska [31] conjectured that $stc(Q^d) = 2^{d-1}$.[*2] In this subsection, we show that $stc(Q^d) = \Omega(2^d \log_2 d/d)$ and $stc(Q^d) \le 2^{d-1}$.

By the following lemma, we have an edge isoperimetric inequality for Q^d.

Lemma 2.21 (Chung, Füredi, Graham, and Seymour [16]). *Let G be a subgraph of a hypercube and $\overline{\delta}$ be the average degree of G. Then $|V(G)| \ge 2^{\overline{\delta}}$.*

Corollary 2.22 (See e.g. [3]). $\theta_{Q^d}(s) \ge s(d - \log_2 s)$ *for* $1 \le s \le 2^d$.

Proof. Let $S \subseteq V(Q^d)$ and $\overline{\delta}$ the average degree of $Q^d[S]$. Then $2|E(Q^d[S])| = \overline{\delta}|S|$. Since Q^d is d-regular, $|\theta(S)| = |S|\left(d - \overline{\delta}\right)$ from Lemma 2.13. By Lemma 2.21, we have $2^{\overline{\delta}} \le |S|$. It follows that $\overline{\delta} \le \log_2 |S|$. From the above observations, $|\theta(S)| \ge |S|(d - \log_2 |S|)$. Hence, the corollary follows. □

Chandran and Kavitha [13] have shown that the carvingwidth of Q^d is 2^{d-1}. To show this, they showed the following lemma.

Lemma 2.23 (Chandran and Kavitha [13]). $\theta_{Q^d}(s) \ge 2^{d-1}$ *for* $2^{d-2} \le s \le 2^{d-1}$.

We will show a lower bound for $stc(Q^d)$ by analyzing the function θ_{Q^d}.

Theorem 2.24. $stc(Q^d) \ge (2^d - 1)\log_2 d/d$.

Proof. Let $f(s) = s(d - \log_2 s)$ and $f'(s)$ be the derived function of $f(s)$. Then

$$f'(s) = d - \left(\log_2 s + \frac{1}{\ln 2}\right).$$

Thus, $f'(s) > 0$ for $1 \le s \le 2^{d-2}$. It follows that $f(s)$ is a monotonically increasing function on s for $1 \le s \le 2^{d-2}$. Hence, we have

$$\min_{s=\lceil(2^d-1)/d\rceil}^{2^{d-2}} f(s) \ge f\left(\frac{2^d-1}{d}\right) = \frac{2^d-1}{d}\left(d - \log_2 \frac{2^d-1}{d}\right) > \frac{2^d-1}{d}\log_2 d.$$

[*2] Recently, this conjecture has been disproved by Law [35]. See Subsection 2.9.1 for more detail.

Therefore, from Corollary 2.2, Corollary 2.22, and Lemma 2.23,

$$stc(Q^d) \geq \min\left\{2^{d-1}, \min_{s=\lceil (2^d-1)/d\rceil}^{2^{d-2}} f(s)\right\} \geq \min\left\{2^{d-1}, \frac{(2^d-1)\log_2 d}{d}\right\}.$$

It is easy to see that $(2^d - 1)\log_2 d/d \leq 2^{d-1}$ for $d \geq 1$. Hence, the theorem follows. □

The above bound for the hypercubes is not so strong to settle the conjecture. To show the upper bound 2^{d-1}, we use binomial trees. Binomial trees are introduced in the studies of the *minimum average distance spanning tree* of the hypercubes [19, 52]. A *d-level binomial tree* B_d is a spanning tree of Q^d: B_1 is an edge Q^1 rooted at 0; B_d consists of two $(d-1)$-level binomial trees and an edge between roots of the two trees; The root of B_d is one of the roots of two B_{d-1}'s. See Fig. 2.6 for example, and see references [19, 52] for formal definitions. From the construction of B_d, it is easy to see that for any edge $e \in B_d$, the smaller component C of $B_d - e$ induces a subcube Q^δ for some $\delta < d$. Since Q^d is d-regular and Q^δ is δ-regular, we have

$$|\theta_{Q^d}(C)| = |V(Q^\delta)|(d-\delta) = 2^\delta(d-\delta).$$

It is easy to verify that $2^\delta(d-\delta) \leq 2^{d-1}$ for $\delta < d$. Therefore, we have the upper bound.

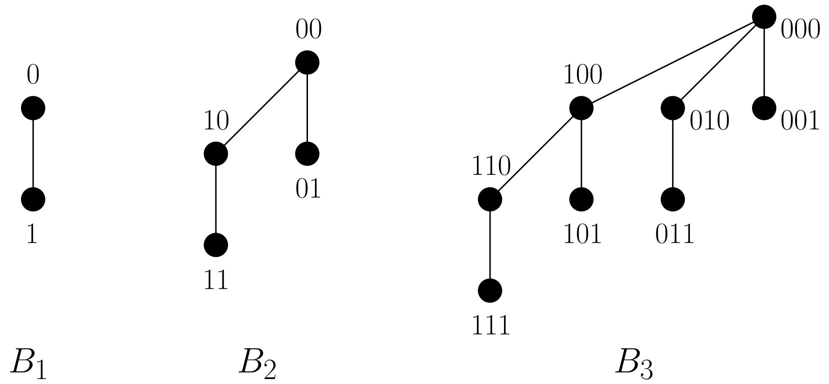


Fig. 2.6 Binomial trees.

2.6 Spanning tree congestion of rook's graphs

In this section, we exactly determine the spanning tree congestion of generalized two-dimensional Hamming graphs $K_m \square K_n$. These graphs have several natural characterizations. A *rook's graph* has the vertex set $\{(i, j) \mid i \in [m],\ j \in [n]\}$ which corresponds to the cells of the $m \times n$ chessboard; A vertex (i, j) in a rook's graph is adjacent to (i', j') if and only if a rook at the cell (i, j) can move to the cell (i', j') (see Fig. 2.7). In other words, (i, j) is adjacent to (i', j') if and only if either $i = i'$ and $j \neq j'$, or $i \neq i'$ and $j = j'$. Thus, the rook's graph on the $m \times n$ chessboard coincides with $K_m \square K_n$. It is also known that $K_m \square K_n$ is the line graph[*3] of the complete bipartite graph $K_{m,n}$. Line graphs of bipartite graphs are used in the proof of the Strong Perfect Graph Theorem [15]. Several properties of rook's graphs were studied [39, 30, 34, 1, 2].

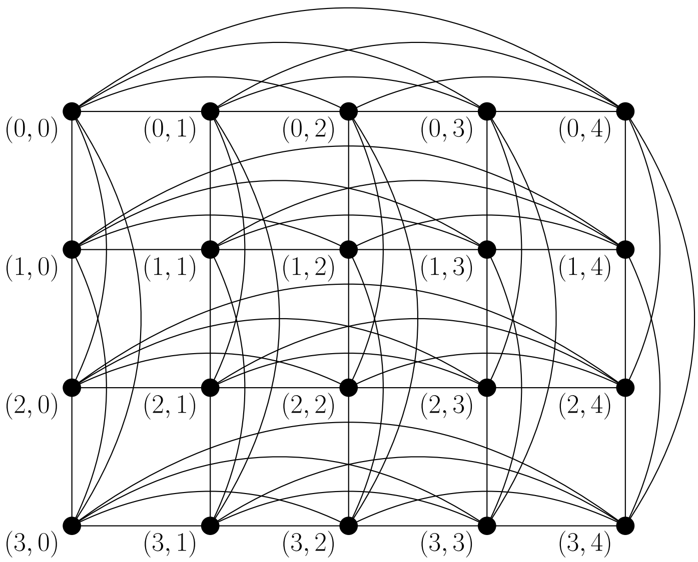


Fig. 2.7 A rook's graph $K_4 \square K_5$.

Lindsey [36] has solved the edge-isoperimetric problem for generalized d-dimensional Hamming graphs. In the *lexicographic order* $\prec_{lex}$, $(a_1, \ldots, a_d) \prec_{\mathrm{lex}} (b_1, \ldots, b_d)$ if and only if there exists i $(1 \leq i \leq d)$ such that

[*3] The *line graph* $L(G)$ of a graph G is a graph such that $V(L(G)) = E(G)$ and in which two vertices $e_1, e_2 \in V(L(G))$ are adjacent if and only if $e_1 \cap e_2 \neq \emptyset$.

$a_i < b_i$ and $a_{i'} = b_{i'}$ for each $i' < i$.

Lemma 2.25 ([36]). *Let $p_1 \le p_2 \le \cdots \le p_d$. Then for each s, $1 \le s \le \prod_{i=1}^{d} p_i$, the collection of the first s vertices of $K_{p_1} \square K_{p_2} \square \cdots \square K_{p_d}$ taken in the lexicographic order $\prec_{lex}$ provides minimum for the function θ.*

In the rest of this section, we assume without loss of generality that $2 \le m \le n$. In this section, $\theta = \theta_{K_m \square K_n}$. We call the vertices $\{(i, j) \mid j \in [n]\}$ the *row* i, and the vertices $\{(i, j) \mid i \in [m]\}$ the *column* j. The following lemma is our main tool.

Lemma 2.26. *Let $m \le n$, and $s = qn + r \le mn$ for nonnegative integers q and $r < n$. Then, $\theta(s) = (m - q)qn + (m + n - 2q - r - 1)r$.*

Proof. Let $S \subseteq V(K_m \square K_n)$ be the first s vertices taken in the order $\prec_{\mathrm{lex}}$. From Lemma 2.25, $|\theta(S)| = \theta(s)$. It is easy to see that S consists of q rows and r vertices contained by another row. Let R denote the r vertices (R may be empty). There are $\binom{n}{2}$ edges in each row, and n edges between each two rows. There are $\binom{r}{2}$ edges in R, and r edges between R and another row. So, we have that $|E((K_m \square K_n)[S])| = q\binom{n}{2} + \binom{q}{2}n + \binom{r}{2} + qr$. Since $K_m \square K_n$ is $(m + n - 2)$-regular, we have, from Lemma 2.13, that

$$\begin{aligned} |\theta(S)| &= (m + n - 2)(qn + r) - 2|E((K_m \square K_n)[S])| \\ &= (m - q)qn + (m + n - 2q - r - 1)r, \end{aligned}$$

as required. □

Using Lemma 2.26 and Corollary 2.2, we derive a lower bound for $stc(K_m \square K_n)$. We divide the range $\lceil (mn - 1)/(m + n - 2) \rceil \le s \le \lfloor mn/2 \rfloor$, in Corollary 2.2, into two ranges $\lceil (mn - 1)/(m + n - 2) \rceil \le s \le n$ and $n < s \le \lfloor mn/2 \rfloor$. This is possible since $n \ge \lceil (mn - 1)/(m + n - 2) \rceil$.

Lemma 2.27. $\theta(s) \ge \min\left\{\theta(n), \theta\left(\left\lceil \frac{mn-1}{m+n-2} \right\rceil\right)\right\}$ *for $m \le n$ and $\left\lceil \frac{mn-1}{m+n-2} \right\rceil \le s \le n$.*

Proof. From Lemma 2.26, $\theta(s) = -s(s - m - n + 1)$ for $s \le n$. Since $-s(s - m - n + 1)$ is a quadratic convex upward function on s, the lemma holds. □

Lemma 2.28. $\theta(s) \ge \theta(n)$ *for $m \le n$ and $n < s \le \lfloor mn/2 \rfloor$.*

Proof. Let q and r be two integers in Lemma 2.26. Clearly, $1 \le q \le m/2$.

From Lemma 2.26, we have $\theta(n) = (m-1)n$ and

$$\theta(s) = (m-q)qn + (m+n-2q-r-1)r.$$

Since $1 \leq q \leq m/2$, we have that $(m-q)q \geq m-1$. Thus,

$$(m-q)qn \geq (m-1)n.$$

Since $q \leq m/2$ and $r < n$, we have that $m+n-2q-r-1 \geq 0$, and hence,

$$(m+n-2q-r-1)r \geq 0.$$

Therefore, we have

$$\theta(s) = (m-q)qn + (m+n-2q-r-1)r \geq (m-1)n = \theta(n),$$

as required. □

Corollary 2.29. *For $m \leq n$, $stc(K_m \square K_n) \geq \min\left\{\theta(n), \theta\left(\left\lceil \frac{mn-1}{m+n-2} \right\rceil\right)\right\}$.*

Next, We show the upper bounds.

Lemma 2.30. $stc(K_m \square K_n) \leq \theta(n)$.

Proof. The spanning tree T is defined as follows (see Fig. 2.8):

1. For each row i, construct a star $K_{1,n-1}$ with the center $(i, 0)$;
2. For the column 0, construct a star $K_{1,m-1}$ with the center $(0, 0)$;
3. The union of the constructed stars is T.

Each edge e constructed in the first step is a leaf edge of T. Thus, $ec(e) = \theta(1)$. If an edge e is constructed in the second step, $ec(e) = \theta(n)$. Since $m, n \geq 2$, $\theta(1) = m+n-2 \leq (m-1)n = \theta(n)$. Hence, the lemma holds. □

Lemma 2.31. *For $m \leq n$, $stc(K_m \square K_n) \leq \theta\left(\left\lceil \frac{mn-1}{m+n-2} \right\rceil\right)$.*

Proof. For simplicity, let $x = \left\lceil \frac{mn-1}{m+n-2} \right\rceil$. The spanning tree T is constructed as follows (see Fig. 2.9):

1. Construct a star $K_{1,m+n-2}$ with the center $(0, 0)$;
2. For each column j, $1 \leq j \leq n-1$, construct a star $K_{1,x-1}$ with the center $(0, j)$ and the leaves $\{(h(i_j), j), (h(i_j+1), j), \ldots, (h(i_j+x-2), j)\}$, where $i_j = (j-1)(x-1)$ and $h(i) = (i \bmod m-1) + 1$ (see Fig. 2.9(a));

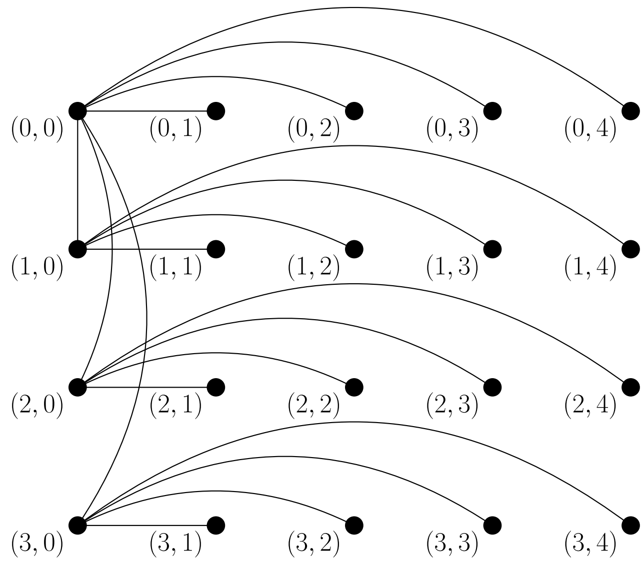


Fig. 2.8 The spanning tree of $K_4 \square K_5$ in Lemma 2.30.

3. For each row i, $1 \le i \le m-1$, construct a star with the center $(i, 0)$ whose leaves are the vertices of the row that are not contained any other star;
4. The union of the constructed stars is T (see Fig. 2.9(b)).

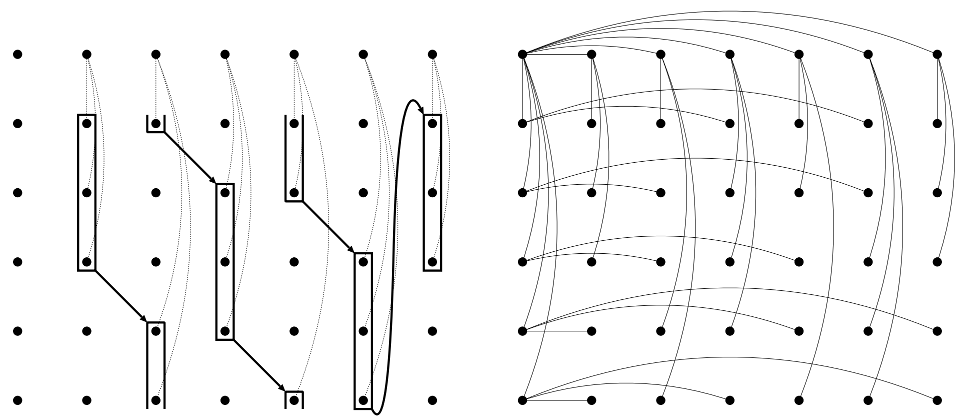
From the following claim, it suffices to show that for any edge e in T, the smaller component of $T - e$ has at most x vertices.

Claim 2.32. *$\theta(s) \le \theta(x)$ for $s \le x$.*

Proof. First, we show that $x \le \left\lceil \frac{m+n-1}{2} \right\rceil \le n$. Clearly, the second inequality is holds since $m \le n$. Suppose $x = \left\lceil \frac{mn-1}{m+n-2} \right\rceil > \left\lceil \frac{m+n-1}{2} \right\rceil$. This implies $\frac{mn-1}{m+n-2} > \frac{m+n-1}{2}$. Simplifying this inequation, we have that $(m-1)(m-2)+(n-1)(n-2) < 0$, which contradicts $n \ge m \ge 2$. Thus, we have $x \le \left\lceil \frac{m+n-1}{2} \right\rceil \le n$.

Lemma 2.26 implies $\theta(s) = -s(s-m-n+1)$ for $s \le n$. Clearly, $\theta\left(\left\lceil \frac{m+n-1}{2} \right\rceil\right) = \theta\left(\left\lfloor \frac{m+n-1}{2} \right\rfloor\right)$ is the peak of the function. Thus, the function is nondecreasing for $s \le x$. Hence, the claim holds. □

Without loss of generality, we assume that T is rooted at the vertex $(0, 0)$. If an edge e in T is not incident to the vertex $(0, 0)$, then e is a leaf edge, and e has congestion $\theta(1) \le \theta(x)$. Suppose that e is connected to the root $(0, 0)$. Then, either $e = \{(0, 0), (0, j)\}$ or $e = \{(0, 0), (i, 0)\}$ holds.



(a) Consecutive property of leaves of stars in the second step ($x = 4$).

(b) The union of the stars.

Fig. 2.9 The spanning tree of $K_6 \square K_7$ in Lemma 2.31

[Case 1] $e = \{(0,0),(0,j)\}$: Then $ec(e) = |\theta(V(T_{(0,j)}))|$, where $T_{(0,j)}$ is the subtree of T rooted at $(0, j)$. Clearly, $T_{(0,j)}$ is a star in the second step of the above construction. Thus, $|V(T_{(0,j)})| = x$ and $V(T_{(0,j)})$ is included in a clique. So, $ec(e) = \theta(x)$.

[Case 2] $e = \{(0,0),(i,0)\}$: Then $ec(e) = |\theta(V(T_{(i,0)}))|$, where $T_{(i,0)}$ is the subtree of T rooted at $(i, 0)$. Clearly, $T_{(i,0)}$ is a star in the third step, and thus, $|\theta(V(T_{(i,0)}))| = \theta(|V(T_{(i,0)})|)$. So, it suffices to show that $|V(T_{(i,0)})| \le x$. Since the vertices are consecutively taken in the second step, the numbers of the remaining vertices in any two rows can differ by at most one. For the root and the stars in the second step, $1 + x(n-1)$ vertices are used. So, the sum of the number of the remaining vertices is $mn - 1 - x(n-1)$, and so, each row contains at most $\lceil (mn-1-x(n-1))/(m-1) \rceil$ unused vertices. Suppose that $x < \lceil (mn-1-x(n-1))/(m-1) \rceil$. Then clearly $x < (mn-1-x(n-1))/(m-1)$ also holds. This implies that $x < (mn-1)/(m+n-2)$, which is a contradiction. □

Corollary 2.33. *For $m \le n$, $stc(K_m \square K_n) \le \min\left\{\theta(n), \theta\left(\left\lceil \frac{mn-1}{m+n-2} \right\rceil\right)\right\}$.*

Corollaries 2.29 and 2.33 together imply

$$stc(K_m \square K_n) = \min\left\{\theta(n), \theta\left(\left\lceil \frac{mn-1}{m+n-2} \right\rceil\right)\right\}$$

for $m \leq n$. We give the main theorem in a more transparent form.

Theorem 2.34. *For $m \leq n$,*

$$stc(K_m \square K_n) = \begin{cases} (m-1)n & \text{if } m^2 - 3m + 3 < n, \\ \left(m+n-1-\left\lceil \frac{mn-1}{m+n-2} \right\rceil\right)\left\lceil \frac{mn-1}{m+n-2} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. Let $x = \left\lceil \frac{mn-1}{m+n-2} \right\rceil$. From Lemma 2.26, $\theta(s) = (m+n-1-s)s$ for $x \leq s \leq n$. Let $f(s) = -s(s-m-n+1)$. Then $f(s)$ is a quadratic convex upward function, and its peak is taken at $s = \frac{m+n-1}{2}$. Thus, $f(n) = f(m-1) = \theta(n)$. Since $m \leq n$, it holds that $m-1 < \frac{m+n-1}{2} < n$. It is easy to see that $x \leq n$. Hence, $\theta(n) = f(m-1) < f(x) = \theta(x)$ if and only if $m-1 < x$ (see Fig. 2.10). Since $m-1$ is an integer, $m-1 < \left\lceil \frac{mn-1}{m+n-2} \right\rceil$ if and only if $m-1 < \frac{mn-1}{m+n-2}$. Simplifying this inequation, we have that $m^2 - 3m + 3 < n$. □

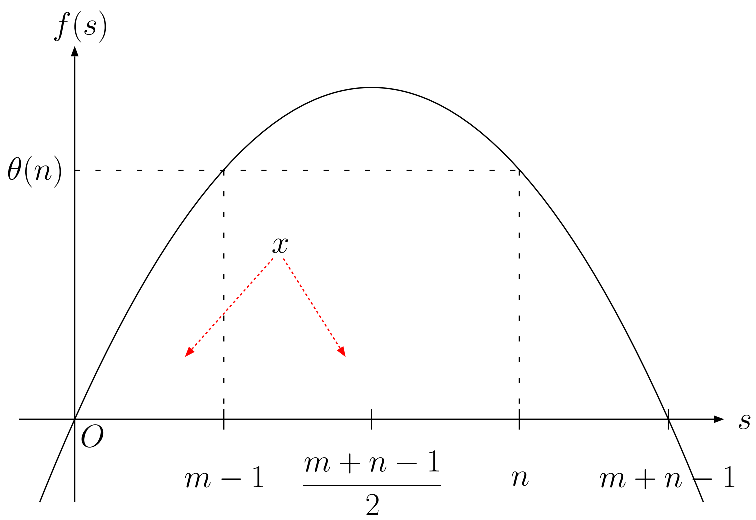


Fig. 2.10 The function $f(s)$ in Theorem 2.34.

For readers' convenience, we explicitly state the spanning tree congestion of the square rook's graph $K_n \square K_n = K_n^2$, which is a direct corollary of Theorem 2.34.

Corollary 2.35. *For $n \geq 2$,*

$$stc(K_n^2) = \begin{cases} (3n-4)(n+2)/4 & \text{if } n \text{ is even,} \\ 3(n-1)(n+1)/4 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. It is easy to see that $stc(K_2^2) = stc(C_4) = 2$, where C_4 is a simple cycle on four vertices. Obviously, $n^2 - 3n + 3 < n$ implies $n = 2$, and $\left\lceil \frac{mn-1}{m+n-2} \right\rceil = \lceil (n+1)/2 \rceil$ since $m = n$. Theorem 2.34 implies for $n \geq 3$ that

$$\begin{aligned} stc(K_n^2) &= (2n - 1 - \lceil (n+1)/2 \rceil) \lceil (n+1)/2 \rceil \\ &= \lfloor 3(n-1)/2 \rfloor \lceil (n+1)/2 \rceil . \end{aligned}$$

It is routine to verify that the corollary holds from the above equation. □

2.7 Multi-dimensional case

In this section, we study the spanning tree congestion of multi-dimensional Hamming graphs. More precisely, we show upper and lower bounds on $stc(K_n^d)$ for $n, d \geq 3$. For hypercubes Q^d, we have already shown that

$$(2^d - 1) \log_2 d/d \leq stc(K_2^d) \leq 2^{d-1}.$$

We extend the above bounds to the case $n \geq 3$.

First, we show a lower bound. In the previous section, Lemma 2.26 was the main tool. If we had such an exact closed formula for the multi-dimensional case, it would be easy to estimate bounds on $stc(K_n^d)$. However, since the graph in this section may have arbitrary high dimension, it is not easy to derive such a formula. So, we should use an asymptotic estimation. Fortunately, such an estimation is known.

Lemma 2.36 (Squier, Torrence, and Vogt [51]). *Let G be a graph with s vertices and t edges that is a subgraph of K_n^d, where $n \geq 2$. Then,*

$$2t \leq (n-1)s \log_n s.$$

Since K_n^d is $d(n-1)$-regular, Lemmas 2.13 and 2.36 imply the following corollary.

Corollary 2.37. $\theta_{K_n^d}(s) \geq (n-1)s(d - \log_n s)$.

For $n^{d-1} \le s \le n^d/2$, the following simple estimation is good enough.

Lemma 2.38. *$\theta_{K_n^d}(s) \ge (n-1)n^{d-1}$ for $n^{d-1} \le s \le n^d/2$.*

Proof. Let S be the first s vertices of K_n^d taken in the order $\prec_{\text{lex}}$. From Lemma 2.25, $\theta(s) = |\theta(S)|$. Let $s = n^{d-1}q + r$ for some integers q and r such that $1 \le q \le n/2$ and $0 \le r < n$. From the definition of $\prec_{\text{lex}}$, S consists of q copies of K_n^{d-1} and r vertices in another copy of K_n^{d-1}. We call the r vertices R and the remaining $n^{d-1} - r$ vertices T, in the copy of K_n^{d-1}. Note that R may be empty.

Each vertex in S has a neighbor in the ith copy of K_n^{d-1}, $q+2 \le i \le n$. Similarly, each vertex in T has a neighbor in any copy of K_n^{d-1} included by S. Thus,

$$\begin{aligned}\theta(S) &\ge (n^{d-1}q + r)(n-q-1) + (n^{d-1} - r)q \\ &= q(n-q)n^{d-1} + r(n-2q-1).\end{aligned}$$

If $q = n/2$ then $r = 0$ since $s = n^{d-1}q + r \le n^d/2$. If $q < n/2$ then $2q < n$, and so, $(n-2q-1) \ge 0$. Hence, $\theta(S) \ge q(n-q)n^{d-1}$. If $q(n-q) < n-1$ then $(q-1)(q-n+1) > 0$, and so, $q < 1$ or $q > n-1$. This contradicts the assumption. Thus, we have that $\theta(s) \ge q(n-q)n^{d-1} \ge (n-1)n^{d-1}$, as required. □

Lemma 2.39. *$stc(K_n^d) \ge (n^d - 1)\log_n d/d$ for $n, d \ge 3$.*

Proof. Let $f(s) = (n-1)s(d - \log_n s)$ and $f'(s)$ be the derived function of $f(s)$. Then $f'(s) = (n-1)(d - 1/\ln n - \log_n s) > (n-1)(d-1-\log_n s)$, and so $f'(s) > 0$ for $s \le n^{d-1}$. This implies that $f(s)$ is monotonically increasing for $1 \le s \le n^{d-1}$. Thus, we have that

$$\min_{s=\left\lceil \frac{n^d-1}{d(n-1)} \right\rceil}^{n^{d-1}} f(s) \ge f\left(\frac{n^d-1}{d(n-1)}\right) = \frac{n^d-1}{d}\left(d - \log_n \frac{n^d-1}{d(n-1)}\right) > \frac{n^d-1}{d}\log_n d.$$

Thus, with Corollary 2.2 and Lemma 2.38, we have that

$$stc(K_n^d) \ge \min\left\{(n-1)n^{d-1}, \frac{n^d-1}{d}\log_n d\right\}.$$

We claim that $(n^d - 1)\log_n d/d \le (n-1)n^{d-1}$ for $n, d \ge 3$, which implies the lemma. Suppose $(n^d - 1)\log_n d/d > (n-1)n^{d-1}$. Then we have

$$dn^{d-1} < \frac{n^d - 1}{n-1}\log_n d = \left(n^{d-1} + \frac{n^{d-1}-1}{n-1}\right)\log_n d,$$
$$(d - \log_n d)n^{d-1} < \frac{n^{d-1}-1}{n-1}\log_n d.$$

Clearly, $d - \log_n d \geq \log_n d$ since $n, d \geq 3$. Thus, we have that $n^{d-1} < (n^{d-1} - 1)/(n-1)$, which is a contradiction. □

Next, we show an upper bound.

Lemma 2.40. *$stc(K_n^d) \leq (n-1)n^{d-1}$ for $n, d \geq 3$.*

Proof. We recursively construct the required spanning tree T_d of K_n^d. For $d \geq 1$, T_d is rooted at the vertex $(0, \ldots, 0)$. If $d = 1$ then the spanning tree T_1 is the star $K_{1,n-1}$. If $d \geq 2$ then construct T_{d-1} for each copy of K_n^{d-1}, and construct the star $K_{1,n-1}$ with the center $(0, \ldots, 0)$ and the leaves $(i, 0, \ldots, 0)$, $1 \leq i \leq n-1$ (they are the root vertices of n copies of T_{d-1}). Note that the spanning tree in Lemma 2.30 coincides with T_2 if $m = n$.

It is easy to see that for any edge e in T_d, the smaller component C of $T_d - e$ induces a Hamming graph K_n^δ for some $\delta < d$. Since K_n^d and K_n^δ are $(n-1)d$-regular and $(n-1)\delta$-regular, respectively, we have $|\theta_{K_n^d}(C)| = |C|(n-1)(d-\delta) = n^\delta(n-1)(d-\delta)$ from Lemma 2.13. It is routine to verify that $n^\delta(n-1)(d-\delta) \leq (n-1)n^{d-1}$ for $\delta < d$ and $n \geq 3$. Therefore, the lemma holds. □

Lemmas 2.39 and 2.40 immediately imply the following theorem.

Theorem 2.41. $(n^d - 1)\log_n d/d \leq stc(K_n^d) \leq (n-1)n^{d-1}$ *for $n, d \geq 3$.*

2.8 Spanning tree congestion and treewidth

Bienstock [6] has shown some relationships between the carvingwidth and the treewidth. The *treewidth* of graphs has studied intensively. See Bodlaender's excellent survey [8]. We show that the treewidth of a graph is bounded by the product of its maximum degree and its spanning tree congestion.

Theorem 2.42. *For a connected graph G, $tw(G) < \Delta(G)(stc(G) + 1)$.*

Proof. Let T be a minimum congestion spanning tree of G. For each $v \in V(T)$, let E_v be the subset of $E(G)$ such that

$$E_v = \{e \in E(G) \mid \text{ the detour for } e \text{ in } T \text{ contains } v\}.$$

Then let B_v be the vertices contained by at least one edge in E_v, that is,

$$B_v = \bigcup_{\{u,w\}\in E_v} \{u, w\}.$$

Obviously, $|B_v| \leq 2|E_v|$. We define a tree $\mathcal{T}$ as

$$V(\mathcal{T}) = \{B_v \mid v \in V(G)\},$$
$$E(\mathcal{T}) = \{\{B_u, B_v\} \mid \{u, v\} \in E(T)\}.$$

It is not difficult to see that $\mathcal{T}$ is a tree decomposition of G, and so

$$tw(G) + 1 \leq \max_{v\in G} |B_v| \leq \max_{v\in G} 2|E_v|.$$

Let $e_1^v, e_2^v, \ldots, e_{deg_T(v)}^v$ be the edges in T that have $v \in V(G)$ as one of its ends. Then clearly,

$$|E_v| \leq \sum_{i=1}^{deg_T(v)} ec(e_i^v). \tag{2.1}$$

Observe that exactly $deg_G(v)$ edges in E_v have v as one of its ends. So, the remaining $|E_v| - deg_G(v)$ edges have v as an inner point of its detour. This means that $|E_v| - deg_G(v)$ edges are counted twice in the right hand side of the inequation (2.1). So, we have

$$2|E_v| \leq \sum_{i=1}^{deg_T(v)} ec(e_i) + deg_G(v) \leq \Delta(G) \cdot stc(G) + \Delta(G)$$

as required. □

Combining Theorem 2.42 and a result of Chandran and Kavitha [14] that determines the treewidth of Q^d, we have a lower bound of $stc(Q^d)$. Unfortunately, this bound is incomparably weaker than the bound in Theorem 2.24.

2.9 Concluding remarks

We have solved the spanning tree congestion problem for complete k-partite graphs, two-dimensional tori, and two-dimensional Hamming graphs. We also showed some bounds on the spanning tree congestion for multi-dimensional grids, hypercubes, and Hamming graphs.

As an analogue of the conjecture for hypercubes, one might conjecture that $stc(K_n^d) = n^{d-1}$ or $stc(K_n^d) = (n-1)n^{d-1}$. However, this straightforward analogue is not true in general. This is because that $stc(K_n^2)$ is approximately equal to $3n^2/4$ (see Corollary 2.35).

2.9.1 Additional remarks

Recently, Law [35] have disproved Hruska's conjecture "$stc(Q^d) = 2^{d-1}$" by showing that the lower bound in Theorem 2.24 is tight. That is, $stc(Q^d) = \Theta(2^d \log_2 d/d)$.

Very recently, the author and Hans L. Bodlaender have proved that the spanning tree congestion problem is NP-hard [42]. In their forthcoming paper, they will prove some negative complexity results as well as some positive ones.

Chapter 3

Security number of graphs

3.1 Introduction

The concept of *security in graphs* has been introduced by Brigham, Dutton and Hedetniemi [11] as a generalization of the concept of *alliances in graphs* [29]. Recently, Dutton, Lee, and Brigham [22] have shown some general lower and upper bounds on the security number.

For a graph G and a subset $S = \{s_1, s_2, \ldots, s_k\}$ of $V(G)$, let us imagine a situation in which each vertex s_i in S may be under attack from its neighbors other than S, and s_i can defend itself or one of its neighbors in S. And s_i fails to defend if the number of attackers of s_i is more than the number of defenders of s_i. Keeping the image in mind, let us see the following definition:

- An *attack* on S is any k mutually disjoint sets $\mathscr{A} = \{A_1, A_2, \ldots, A_k\}$ such that $A_i \subseteq N[s_i] \setminus S$ for $1 \leq i \leq k$.
- A *defense* of S is any k mutually disjoint sets $\mathscr{D} = \{D_1, D_2, \ldots, D_k\}$ such that $D_i \subseteq N[s_i] \cap S$ for $1 \leq i \leq k$.
- An attack $\mathscr{A}$ is said to be *defendable* if there exists a defense $\mathscr{D}$ such that $|D_i| \geq |A_i|$ for $1 \leq i \leq k$, and S is *secure* if every attack on S is defendable.

The *security number* $sn(G)$ of G is the cardinality of a smallest secure set of G. Clearly, a minimal secure set is connected. Brigham, Dutton and Hedetniemi [11] presented some characterizations of secure sets. We use the following characterization as the definition of secure sets.

Theorem 3.1 (Brigham, Dutton and Hedetniemi [11]). *Set $S \subseteq V(G)$ is a secure set of G if and only if $|N[X] \cap S| \geq |N[X] \setminus S|$ for all $X \subseteq S$.*

This work was motivated by a conjecture of Brigham, Dutton and Hedetniemi [11]. They showed upper bounds on the security number of two-dimensional cylinders (which will be defined later) and two-dimensional tori, and conjectured that the bound is the best possible. In Section 3.3, we show that their conjecture is true for tori. In Section 3.4, as a corollary of the result for tori, we show that the conjecture is also true for cylinders.

In Section 3.5, we show that any outerplanar graph has security number at most three. A *chord* of a maximal outerplanar graph M is an edge other than the edges on the outer-boundary. (In this thesis, it is enough to define chords only for maximal outerplanar graphs.) The arc distance of a chord $\{u, v\}$ in M is defined as the distance along the outer-boundary (that is, the unique Hamiltonian cycle) between vertices u and v.

3.2 Notation and related work

Recall that a two-dimensional grid is $P_m \square P_n$, and a two-dimensional torus is $C_m \square C_n$. We define similar graphs, cylinders. A *two-dimensional cylinder* $P_m \square C_n$ is the Cartesian product of a path P_m and a cycle C_n. We call these graphs *grid-like graphs*.

Some graph parameters of grid-like graphs are known: pathwidth [23], cutwidth and bisection width [47], spanning tree congestion [31, 33], powerful alliance number [10], and so on. Brigham, Dutton and Hedetniemi [11] have shown the following exact or upper bounds on the security number of two-dimensional grid-like graphs.

Proposition 3.2 (Brigham, Dutton and Hedetniemi [11])**.** *For two-dimensional grid-like graphs,*

1. $sn(P_m \square P_n) = \min\{m, n, 3\}$,
2. $sn(P_m \square C_n) \leq \min\{2m, n, 6\}$,
3. $sn(C_3 \square C_3) = 4$ *and* $sn(C_m \square C_n) \leq \min\{2m, 2n, 12\}$ *for* $\max\{m, n\} \geq 4$.

Brigham, Dutton and Hedetniemi [11] conjectured that the above upper bounds are tight. We will show that their conjecture is true.

3.3 Security number of two-dimensional tori

In this section, we show that $sn(C_m \square C_n) = \min\{2m, 2n, 12\}$ for $\max\{m, n\} \geq 4$. To this end, we need additional notation.

Recall the definitions of $Col(i)$ and $Row(j)$ in Section 2.4 (page 20). See also Fig. 2.4. Let $S \subseteq V(C_m \square C_n)$. We denote $\partial_i^c(S) = \partial(S) \cap Col(i)$ (the superscript c stands for "column"). Clearly, $\partial_{i_1}^c(S) \cap \partial_{i_2}^c(S) = \emptyset$ for $i_1 \neq i_2$, and $\partial(S) = \bigcup_{i \in \{0,\dots,m-1\}} \partial_i^c(S)$. We denote the indices of columns and rows that intersect with S by

$$\mathscr{C}(S) = \{i \mid Col(i) \cap S \neq \emptyset\} \quad \text{and} \quad \mathscr{R}(S) = \{j \mid Row(j) \cap S \neq \emptyset\},$$

respectively. For $k \geq 1$, we define partitions of $\mathscr{C}(S)$ and $\mathscr{R}(S)$, denoted by $\mathscr{C}_k(S)$ and $\mathscr{R}_k(S)$ respectively, as

$$\mathscr{C}_k(S) = \{i \mid |Col(i) \cap S| = k\} \quad \text{and} \quad \mathscr{R}_k(S) = \{j \mid |Row(j) \cap S| = k\}.$$

Obviously, $\mathscr{C}(S) \subseteq [m]$ and $\mathscr{R}(S) \subseteq [n]$. From the definitions, it is easy to see that $|\mathscr{C}(S)| = \sum_{k=1}^{n} |\mathscr{C}_k(S)|$ and $|S| = \sum_{k=1}^{n} k|\mathscr{C}_k(S)|$.

3.3.1 Some observations

In this subsection, we present some useful propositions. First, we can easily derive the following proposition.

Proposition 3.3. *If $i \in \mathscr{C}(S)$ then*

$$|\partial_i^c(S)| = \begin{cases} 0 & \text{if } i \in \mathscr{C}_n(S), \\ 1 & \text{if } i \in \mathscr{C}_{n-1}(S), \\ 2 \text{ or more} & \text{otherwise.} \end{cases}$$

We can directly derive the following corollary by the above proposition.

Corollary 3.4. *For $S \subseteq V(C_m \square C_n)$, $\left|\bigcup_{i \in \mathscr{C}(S)} \partial_i^c(S)\right| \geq 2|\mathscr{C}(S)| - 2|\mathscr{C}_n(S)| - |\mathscr{C}_{n-1}(S)|$.*

Since $C_m \square C_n$ is 4-regular, if a set $S \subseteq V(C_m \square C_n)$ contains a vertex v that has three neighbors not in S then S is not secure. (We call such a vertex v a *pendant vertex*.) From this property, we can estimate $|\partial_i^c(S)|$ for $i \notin \mathscr{C}(S)$.

Proposition 3.5. *Let S be a secure set of $C_m \square C_n$. If $i \notin \mathscr{C}(S)$ and $\{i-1, i+1\} \cap \mathscr{C}(S) \neq \emptyset$ then $|\partial_i^c(S)| \geq 2$.*

Proof. Suppose $|\partial_i^c(S)| = 1$. Then $|S \cap Col(i-1)| = 1$ or $|S \cap Col(i+1)| = 1$. Since $i \notin \mathscr{C}(S)$, there is a vertex in $S \cap Col(i-1)$ or $S \cap Col(i+1)$ that has at least three attackers. This contradicts that S is secure. $\square$

Corollary 3.6. *Let S be a secure set of $C_m \square C_n$. If $|\mathscr{C}(S)| \leq m-1$ then there exists $i_1 \notin \mathscr{C}(S)$ such that $|\partial^c_{i_1}(S)| \geq 2$. Moreover, if $|\mathscr{C}(S)| \leq m-2$ then there exists $i_2 \notin \mathscr{C}(S)$ such that $i_1 \neq i_2$ and $|\partial^c_{i_2}(S)| \geq 2$.*

Since any minimal secure set is connected, we can derive a lower bound of its size.

Proposition 3.7. *Let S be a connected subset of $V(C_m \square C_n)$. Then,*

$$|S| \geq |\mathscr{C}(S)| + |\mathscr{R}(S)| - 1.$$

Proof. We prove the proposition by induction on $|S|$. If $|S| = 1$, trivially the proposition holds. Let us assume $|S| \geq 2$ and for any connected set of size $|S|-1$, the proposition holds. Since S is connected and $|S|$, there is a vertex $(i, j) \in S$ such that $S \setminus \{(i, j)\}$ is also connected (for example, a leaf vertex of a spanning tree of $(C_m \square C_n)[S]$). Let S' denote $S \setminus \{(i, j)\}$. Clearly, $|S| = |S'|+1$. Then, from the inductive assumption, $|S'| \geq |\mathscr{C}(S')| + |\mathscr{R}(S')| - 1$. Hence,

$$|S| \geq |\mathscr{C}(S')| + |\mathscr{R}(S')|. \tag{3.1}$$

Since S is connected, there is a vertex $(i', j') \in S'$ such that $\{(i, j), (i', j')\} \in E(C_m \square C_n)$. From the definition of $C_m \square C_n$, either $i = i'$ or $j = j'$. This implies $i \in \mathscr{C}(S')$ or $j \in \mathscr{R}(S')$. Thus,

$$|\mathscr{C}(S)| + |\mathscr{R}(S)| \leq |\mathscr{C}(S')| + |\mathscr{R}(S')| + 1. \tag{3.2}$$

Combining the inequalities (3.1) and (3.2), we have

$$|S| \geq |\mathscr{C}(S)| + |\mathscr{R}(S)| - 1,$$

as required. □

Corollary 3.8. *Let S be a minimal secure set of $C_m \square C_n$. Then,*

$$|S| \geq |\mathscr{C}(S)| + |\mathscr{R}(S)| - 1.$$

The restriction on size of S bounds the size of $\mathscr{C}_n(S)$ and $\mathscr{C}_{n-1}(S)$.

Proposition 3.9. $|\mathscr{C}_n(S)| \leq \left\lfloor \frac{|S|}{n} \right\rfloor$ *and* $|\mathscr{C}_{n-1}(S)| \leq \left\lfloor \frac{|S|-|\mathscr{C}(S)|-(n-1)|\mathscr{C}_n(S)|}{n-2} \right\rfloor$.

Proof. Trivially, the first inequality holds. Since $|\mathscr{C}(S)| = \sum_{k=1}^{n} |\mathscr{C}_k(S)|$, $|S| = \sum_{k=1}^{n} k|\mathscr{C}_k(S)|$, and $n \geq 3$, we have

$$|S| - |\mathscr{C}(S)| = \sum_{k=1}^{n}(k-1)|\mathscr{C}_k(S)| \geq (n-1)|\mathscr{C}_n(S)| + (n-2)|\mathscr{C}_{n-1}(S)|.$$

Therefore, by simplifying the above inequality, we have

$$|\mathscr{C}_{n-1}(S)| \leq \frac{|S| - |\mathscr{C}(S)| - (n-1)|\mathscr{C}_n(S)|}{n-2}.$$

Since $|\mathscr{C}_{n-1}(S)|$ is integral, the second inequality in the proposition holds. □

As the last observation of this subsection, we present a property of adjacent columns.

Proposition 3.10. *Let $S \subseteq V(C_m \square C_n)$, $i \in \mathscr{C}_k(S)$ and $i' \in \mathscr{C}_{k'}(S)$ for some k, k'. If $|i - i'| = 1$ then $|\partial^c_{i'}(S)| \geq k - k'$.*

Proof. Each vertex $v \in Col(i) \cap S$ has a unique neighbor $u \in Col(i')$. The number of such neighbors is $|Col(i) \cap S| = k$, and at most k' of them can be in S. Thus, the lemma holds. □

3.3.2 Solution

We divide the problem into the following three cases.

1. $|\mathscr{C}(S)| \leq m-2$ or $|\mathscr{R}(S)| \leq n-2$ (Lemma 3.12),
2. $m \neq n$, $|\mathscr{C}(S)| \geq m-1$, and $|\mathscr{R}(S)| \geq n-1$ (Lemma 3.13),
3. $m = n$, $|\mathscr{C}(S)| \geq m-1$, and $|\mathscr{R}(S)| \geq n-1$ (Lemma 3.14).

From Proposition 3.2, and Lemmas 3.12, 3.13, and 3.14, we can conclude that the following theorem holds.

Theorem 3.11. *$sn(C_3 \square C_3) = 4$, and for $\max\{m, n\} \geq 4$,*

$$sn(C_m \square C_n) = \min\{2m, 2n, 12\}.$$

The 1st case: $|\mathscr{C}(S)| \leq m-2$ or $|\mathscr{R}(S)| \leq n-2$

This case is the easiest case.

Lemma 3.12. *Let S be a secure set of $C_m \square C_n$ such that $|\mathscr{C}(S)| \le m-2$ or $|\mathscr{R}(S)| \le n-2$. Then $|S| \ge \min\{2m, 2n, 12\}$.*

Proof. Observe that $|S| \le |\mathscr{C}(S)||\mathscr{R}(S)|$, since each row contains at most $|\mathscr{C}(S)|$ vertices of S. We claim that $\max\{|\mathscr{C}(S)|, |\mathscr{R}(S)|\} \ge \sqrt{|S|}$, which implies $\max\{|\mathscr{C}(S)|, |\mathscr{R}(S)|\} \ge \left\lceil \sqrt{|S|} \right\rceil$. Suppose $\max\{|\mathscr{C}(S)|, |\mathscr{R}(S)|\} < \sqrt{|S|}$. Then, we have $|\mathscr{C}(S)||\mathscr{R}(S)| < |S|$, which is a contradiction.

Without loss of generality, we assume that $|\mathscr{R}(S)| \le n-2$. Then $\mathscr{C}_n(S) = \mathscr{C}_{n-1}(S) = \emptyset$. It follows $\left|\bigcup_{i \in \mathscr{C}(S)} \partial_i^c(S)\right| \ge 2|\mathscr{C}(S)|$ from Corollary 3.4. So, if $|\mathscr{C}(S)| = m$, then $|\partial(S)| \ge 2m$. If $|\mathscr{C}(S)| = m-1$, then from Corollary 3.6, there is an index $i_1 \notin \mathscr{C}(S)$ such that $|\partial_{i_1}^c(S)| \ge 2$. So, $|\partial(S)| \ge 2|\mathscr{C}(S)| + 2 = 2m$.

If $|\mathscr{C}(S)| \le m-2$, then from Corollary 3.6, there are two distinct indices $i_1, i_2 \notin \mathscr{C}(S)$ such that $|\partial_{i_1}^c(S)| \ge 2$ and $|\partial_{i_2}^c(S)| \ge 2$. It follows that $|\partial(S)| \ge 2|\mathscr{C}(S)| + 4$. From the symmetry argument, we can also derive $|\partial(S)| \ge 2|\mathscr{R}(S)| + 4$. Thus,

$$|\partial(S)| \ge 2\max\{|\mathscr{C}(S)|, |\mathscr{R}(S)|\} + 4 \ge 2\left\lceil \sqrt{|S|} \right\rceil + 4.$$

It is routine to verify that for $|S| \le 11$, $|S| < 2\left\lceil \sqrt{|S|} \right\rceil + 4$. Thus, $|S| \ge 12$. □

The 2nd case: $m \ne n$, $|\mathscr{C}(S)| \ge m-1$, and $|\mathscr{R}(S)| \ge n-1$

Lemma 3.13. *Let S be a minimal secure set of $C_m \square C_n$ such that $|\mathscr{C}(S)| \ge m-1$ and $|\mathscr{R}(S)| \ge n-1$. If $m \ne n$ then $|S| \ge \min\{2m, 2n, 12\}$.*

Proof. Without loss of generality, we assume $m \ge n+1$. Suppose $|S| \le 2n-1$. We divide the proof into two cases.

[Case 1] $|\mathscr{C}(S)| = m$: If $|\mathscr{R}(S)| = n$, then $|S| \ge |\mathscr{C}(S)| + |\mathscr{R}(S)| - 1 = m+n-1 \ge 2n$ from Corollary 3.8. Thus, $|\mathscr{R}(S)| = n-1$, and so, $|\mathscr{C}_n(S)| = 0$. From Corollary 3.8 and $|S| \le 2n-1$, $m = n+1$. Hence, from Corollary 3.4 and Proposition 3.9, we have

$$|\partial(S)| \ge 2|\mathscr{C}(S)| - |\mathscr{C}_{n-1}(S)| \ge 2(n+1) - \left\lfloor \frac{2n-1-(n+1)}{n-2} \right\rfloor = 2n+1 > |S|,$$

which is a contradiction.

[Case 2] $|\mathscr{C}(S)| = m-1$: From Proposition 3.9 and the assumption $|S| \le 2n-1$, $|\mathscr{C}_n(S)| \le 1$. From Corollaries 3.4 and 3.6,

$$|\partial(S)| \ge 2(m-1) - 2|\mathscr{C}_n(S)| - |\mathscr{C}_{n-1}(S)| + 2 = 2m - 2|\mathscr{C}_n(S)| - |\mathscr{C}_{n-1}(S)|.$$

Then, from Proposition 3.9 and the assumption $|S| \leq 2n-1$,

$$\begin{aligned}|\partial(S)| &\geq 2m - 2|\mathscr{C}_n(S)| - \left\lfloor \frac{(2n-1)-(m-1)-(n-1)|\mathscr{C}_n(S)|}{n-2} \right\rfloor \\ &= 2m - \left\lfloor \frac{(n-3)|\mathscr{C}_n(S)| + 2n - m}{n-2} \right\rfloor \\ &\geq 2m - \left\lfloor \frac{3n-m-3}{n-2} \right\rfloor.\end{aligned}$$

From Corollary 3.8 and $|S| \leq 2n-1$, $m \in \{n+1, n+2\}$. So,

$$|\partial(S)| \geq \begin{cases} 2n+2-\left\lfloor \frac{2n-4}{n-2} \right\rfloor = 2n & \text{if } m = n+1, \\ 2n+4-\left\lfloor \frac{2n-5}{n-2} \right\rfloor = 2n + \left\lceil \frac{2n-3}{n-2} \right\rceil & \text{if } m = n+2. \end{cases}$$

Since $n \geq 3$, we have $|\partial(S)| \geq 2n > |S|$, a contradiction. □

The 3rd case: $m = n$, $|\mathscr{C}(S)| \geq m-1$, and $|\mathscr{R}(S)| \geq n-1$

Lemma 3.14. *Let S be a minimal secure set of $C_m \,\square\, C_n$ such that $|\mathscr{C}(S)| \geq m-1$ and $|\mathscr{R}(S)| \geq n-1$. If $m = n \geq 4$ then $|S| \geq \min\{2m, 2n, 12\}$.*

Proof. First we consider the smallest case $m = n = 4$. Riordan [45] has determined the ordering on the vertices of the multi-dimensional even torus such that the set S of the initial k vertices in the ordering has the minimum number of boundaries. By using the ordering, we can verify that $|S| < |\partial(S)|$ for any $S \subseteq V(C_4 \,\square\, C_4)$ such that $|S| \leq 6$. Thus, $sn(C_4 \,\square\, C_4) > 6$. So, it is sufficient to show that there is no secure set of $C_4 \,\square\, C_4$ with seven vertices, since $2m = 8$. It is routine to verify that there are only three non-isomorphic connected subsets of $V(C_4 \square C_4)$ that consist of seven vertices with no pendant vertex. The three subsets are depicted in Fig. 3.1. For each subset in Fig. 3.1, $|S| < |\partial(S)|$. So the lemma holds in this case.

In what follows, we assume $m = n \geq 5$, and by way of contradiction, assume $|S| \leq 2n-1$. Then from Proposition 3.9, $|\mathscr{C}_n(S)| + |\mathscr{C}_{n-1}(S)| \leq 1$. From Corollaries 3.4 and 3.6, and $|\mathscr{C}(S)| \in \{m-1, m\}$, if $|\mathscr{C}_n(S)| + |\mathscr{C}_{n-1}(S)| = 0$ then $|\partial(S)| \geq 2m$. Hence, $|\mathscr{C}_n(S)| + |\mathscr{C}_{n-1}(S)| = 1$. We have the following two cases.

[Case 1] $|\mathscr{C}(S)| = m$ and $|\mathscr{R}(S)| \geq n-1$: Without loss of generality, we assume $\mathscr{C}_n(S) \cup \mathscr{C}_{n-1}(S) = \{i_1\}$. From $|\mathscr{C}(S)| = m$, $|S| = \sum_{k=1}^{n} k|\mathscr{C}_k(S)|$, and

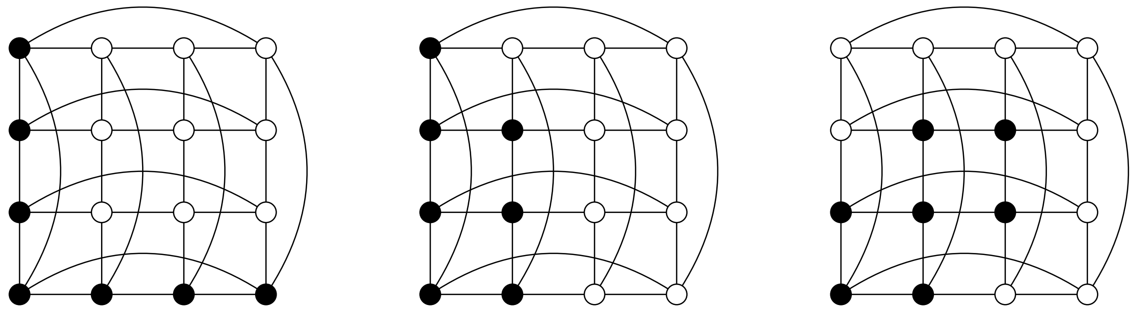


Fig. 3.1 Subsets of $V(C_4 \square C_4)$ that contain no pendant vertex ($\bullet \in S$).

$|S| \leq 2n-1$, we have $|\mathscr{C}_2(S)| \leq |\mathscr{C}_{n-1}(S)|$, $|\mathscr{C}_1(S)| = m-1-|\mathscr{C}_2(S)|$, and $|\mathscr{C}_k(S)| = 0$ for $3 \leq k \leq n-2$. Then, from Propositions 3.3 and 3.10,

$$\left|\partial^c_{i_1}(S)\right| + \left|\partial^c_{i_1-1}(S)\right| + \left|\partial^c_{i_1+1}(S)\right| \geq \begin{cases} (n-1)+(n-1) & \text{if } i_1 \in \mathscr{C}_n(S) \\ 1+(n-2)+(n-3) & \text{if } i_1 \in \mathscr{C}_{n-1}(S) \end{cases}$$
$$\geq 2n-4.$$

From Proposition 3.3, $|\partial^c_i(S)| \geq 2$ for $i \in \{0,\dots,m-1\} - \{i_1, i_1-1, i_1+1\}$. Thus, $|\partial(S)| \geq (2n-4)+2(m-3) = 4n-10$. Since $n \geq 5$, we have $|\partial(S)| \geq 4n-10 \geq 2n$, a contradiction.

[Case 2] $|\mathscr{C}(S)| = m-1$ and $|\mathscr{R}(S)| = n-1$: From $|\mathscr{R}(S)| = n-1$, $\mathscr{C}_n(S) = \emptyset$. Thus, $|\mathscr{C}_{n-1}(S)| = 1$. Let $\mathscr{C}_{n-1}(S) = \{i_1\}$. We have the following two subcases.

[Case 2-1] $i_1 - 1 \notin \mathscr{C}(S)$ or $i_1+1 \notin \mathscr{C}(S)$: Without loss of generality, we assume $i_1 - 1 \notin \mathscr{C}(S)$ (hence, $i_1+1 \in \mathscr{C}(S)$). Clearly, $|\partial^c_{i_1-1}(S)| \geq n-1$. Since $|\mathscr{C}(S)| = m-1$, $|S| \leq 2n-1$, and $|S| = \sum_{k=1}^{n} k|\mathscr{C}_k(S)|$, it follows that $i_1+1 \in \mathscr{C}_k(S)$ for some $k \leq 3$. From Proposition 3.10, $|\partial^c_{i_1+1}(S)| \geq n-4$. Then from Proposition 3.3 and Corollary 3.6,

$$|\partial(S)| = |\partial^c_{i_1}(S)| + |\partial^c_{i_1-1}(S)| + |\partial^c_{i_1+1}(S)| + \left|\bigcup_{i\in\{0,\dots,m-1\}-\{i_1,i_1-1,i_1+1\}} \partial^c_i(S)\right|$$
$$\geq 1+(n-1)+(n-4)+2(m-3) = 4n-10.$$

Since $n \geq 5$, we have $|\partial(S)| \geq 2n$, a contradiction.

[Case 2-2] $i_1-1, i_1+1 \in \mathscr{C}(S)$: By the symmetry argument, we can assume $\mathscr{R}_m(S) = \emptyset$, $\mathscr{R}_{m-1}(S) = \{j_1\}$, and $j_1-1, j_1+1 \in \mathscr{R}(S)$. Since

$|S| \leq 2n-1$, there are at most two vertices $u, v \in S$ such that $u, v \notin Col(i_1)$ and $u, v \notin Row(j_1)$ (not necessarily $u \neq v$). Since $|S|$ is connected, u and v must be in the masked area of Fig. 3.2. It is easy to see that S must have a pendant vertex since $m = n \geq 5$, a contradiction. □

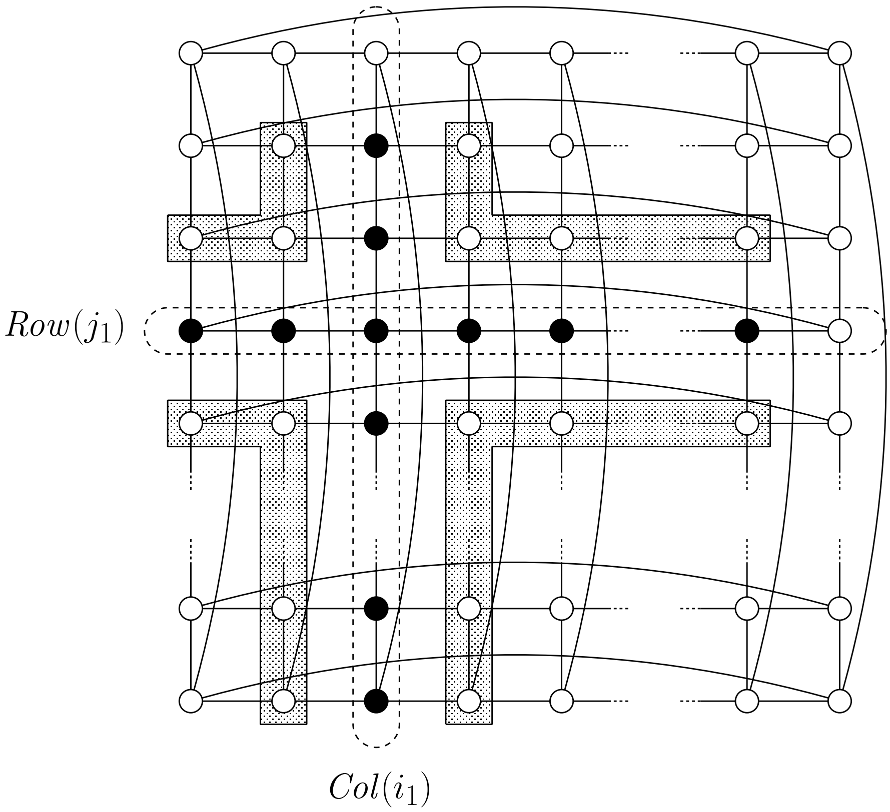


Fig. 3.2 Remaining vertices must be in the masked area ($\bullet \in S$).

3.4 Security number of two-dimensional cylinders

In this section, we show that the remaining part of the conjecture is also true, that is, $sn(P_m \square C_n) = \min\{2m, n, 6\}$. This result can be easily derived from the result of tori and the following lemma.

Lemma 3.15. $sn(C_{2m} \square C_n) \leq 2sn(P_m \square C_n)$.

Proof. Let S be an arbitrary secure set of $C_m \,\square\, P_n$. Let S' be the *reversed-shifted copy* of S, that is, $S' = \{(2m-1-u, v) \mid (u, v) \in S\}$ (see Fig. 3.3). We show that $S \cup S'$ is a secure set of $C_{2m} \,\square\, C_n$.

Let F denote the set of edges between the left half and the right half of $C_{2m} \,\square\, C_n$, that is,

$$F = \{\{(m-1, i), (m, i)\},\ \{(0, i), (2m-1, i)\} \mid 0 \le i \le n-1\}.$$

Clearly, $S \cup S'$ is a secure set of the graph obtained by deletion of F from $C_{2m} \,\square\, C_n$. Observe that $(m-1, i) \in S$ if and only if $(m, i) \in S'$. Similarly, $(0, i) \in S$ if and only if $(2m-1, i) \in S'$. Thus, any edge in F connects two vertices such that the both are in $S \cup S'$, or the both are not in $S \cup S'$. This means that F cannot contribute to any attack on $S \cup S'$. Therefore, $S \cup S'$ is also a secure set of $C_{2m} \,\square\, C_n$. $\square$

The above lemma implies that if $sn(P_m \square C_n) < \min\{2m, n, 6\}$ then $sn(C_{2m} \square C_n) < \min\{4m, 2n, 12\}$. However, this contradicts Theorem 3.11. So we have, with Proposition 3.2, the following theorem.

Theorem 3.16. $sn(P_m \,\square\, C_n) = \min\{2m, n, 6\}$.

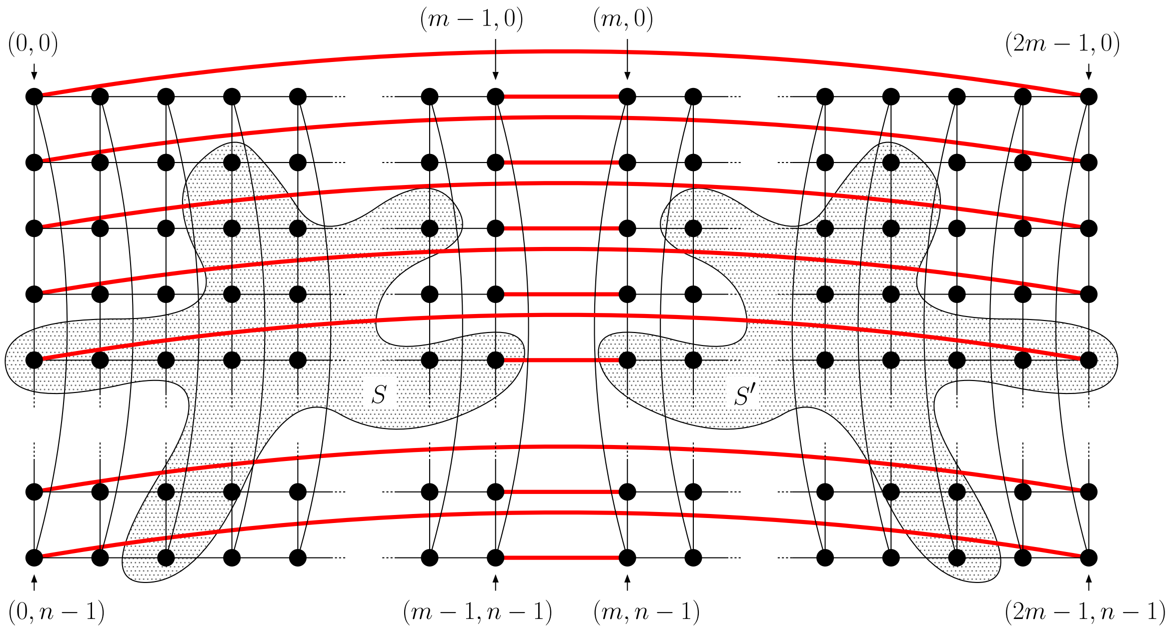


Fig. 3.3 The reversed-shifted copy S' of S.

3.5 Security number of outerplanar graphs

In this section, we show that any outerplanar graph has security number at most three.*1 To show the existence of such a small secure set, we use the following four lemmas.

Lemma 3.17. *Let $\{u, v\}$ be a chord of arc distance at least three in a maximal outerplanar graph M, and P_1 and P_2 be the set of vertices on two paths between u and v along the outer-boundary, except the endpoints u and v. Then, both P_1 and P_2 are secure sets of M.*

Proof. Clearly, the boundary of P_i, $\partial(P)$ is $\{u, v\}$, that is, only u and v are the attackers on P_i. Since $|P_i| \geq 2$ and P_i induces a connected subgraph of M, each vertex in P_i has two "candidates" of its defenders: itself and its neighbor in P_i. Hence, P_i is secure. □

Lemma 3.18. *Any maximal outerplanar graph has a secure set of size at most three.*

Proof. Let M be a maximal outerplanar graph. It is easy to verify that if $|V(M)| \leq 6$ then $sn(M) \leq 3$. Thus, we assume $|V(M)| \geq 7$.

From Lemma 3.17, it suffices to show that there is a chord of arc distance three or four. Let n denote $|V(M)|$ and c denote the number of chords with arc distance two in M. We first show that there is a chord $\{u, v\}$ of arc distance at least three. It is easy to check that $c \leq \lfloor n/2 \rfloor$. Since M has $(2n-3)-n = n-3$ chords and $n \geq 7$, we have $(n-3)-c \geq (n-3) - \lfloor n/2 \rfloor > 0$. This means that there is a chord $\{u, v\}$ of arc distance at least three.

Next, we demonstrate that the smallest arc distance among the chords with arc distance at least three is at most four. Hence, let $\{u, v\}$ denote a chord with the smallest arc distance among the chords with arc distance at least three, and $W = \{w_0, w_1, \ldots, w_k\}$ denote the vertices on the shortest path along the outer-boundary between $u = w_0$ and $v = w_k$, where k is the arc distance of the chord $\{u, v\}$. Consider the chords except $\{u, v\}$ in M whose endpoints are both in W. Let us denote such chords by C. From the choice of $\{u, v\}$, all chords in C have arc distance two in M. Therefore, C has at most $\lfloor k/2 \rfloor$ chords (not $\lfloor (k+1)/2 \rfloor$). On the other hand, the chords C are exactly the chords in

*1 The same result has been obtained independently by Dutton [21].

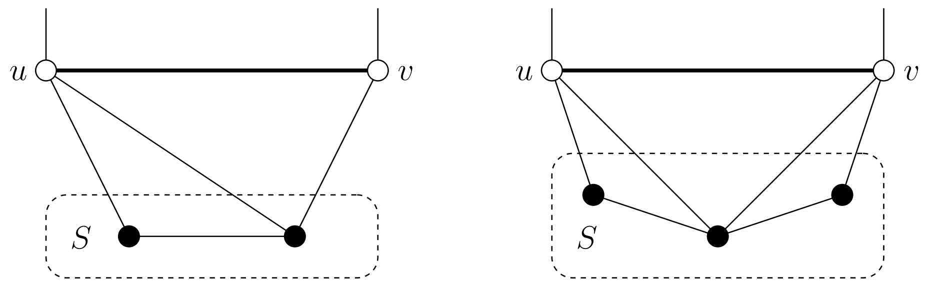
$M[W]$. Since $M[W]$ is a maximal outerplanar graph, the number of chords in $M[W]$ (that is, $|C|$) is $(2(k+1)-3)-(k+1)=k-2$. As a result, we have $k-2 \le \lfloor k/2 \rfloor$, which implies $k \le 4$. $\square$

The following lemma is immediate from the definition of secure sets.

Lemma 3.19. *Let S be a secure set of a graph G. For an edge set $F \subseteq E(G) \setminus E(G[S])$, S is also a secure set of the graph $G - F$.*

Lemma 3.20. *Let S be a secure set of a maximal outerplanar graph M obtained by Lemma 3.18. Then, for an edge subset F of $E(M[S])$, S includes a secure set of the graph $M - F$.*

Proof. It is easy to see that the secure set obtained by Lemma 3.18 can be divided into two types depicted in Fig. 3.4. In the both types, the deletion of any edge in $E(M[S])$ yields a vertex of degree one (see Fig. 3.4). Thus, S includes a secure set of the graph $M - F$. $\square$



(a) Arc distance three. (b) Arc distance four.

Fig. 3.4 Secure sets obtained by Lemma 3.18

Theorem 3.21. *For any outerplanar graph, its security number is at most three.*

Proof. Let G be an outerplanar graph, and M be a maximal outerplanar graph that has G as a spanning subgraph, that is, $V(M) = V(G)$ and $E(M) \supseteq E(G)$. Let $F = E(M) \setminus E(G)$ denote the additional edges, and let S be a secure set of M obtained by Lemma 3.18. Then let $F_{\text{in}} = F \cap E(M[S])$ and $F_{\text{out}} = F \setminus F_{\text{in}}$. Since $F_{\text{in}} \subseteq E(M[S])$ from Lemma 3.20, S includes a secure set of $M - F_{\text{in}}$

Since $F_{\text{out}} \subseteq (E(M - F_{\text{in}}) \setminus E((M - F_{\text{in}})[S]))$, from Lemma 3.19, S includes a secure set of $(M - F_{\text{in}}) - F_{\text{out}} = G$. □

The above bound is tight, that is, there are infinitely many outerplanar graphs of security number three. For $n \geq 3$, let H_n be a graph such that

$$V(H_n) = \{v_1, v_2, \ldots, v_{2n-1}, v_{2n} = v_0\},$$
$$E(H_n) = \{\{v_i, v_{i+1}\} \mid 0 \leq i \leq 2n - 1\} \cup \{\{v_{2i}, v_{2i+2}\} \mid 0 \leq i \leq n - 1\}.$$

See Fig. 3.5. It is easy to see that each vertex in H_n has at least two neighbors, and each pair of adjacent vertices has at least three boundary vertices. Thus, we can conclude that $sn(H_n) = 3$ for any $n \geq 3$. Note that $\{v_1, v_2, v_3\}$ is one of the minimum secure set of H_n.

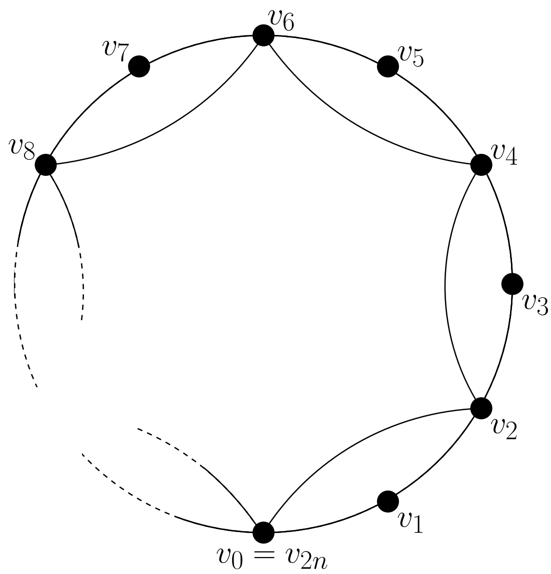


Fig. 3.5 An outerplanar graph of security number three.

3.6 Upper and lower bounds for hypercubes

In this section, we provide upper and lower bounds for hypercubes.

Lemma 3.22. *For any graphs G and H,*

$$sn(G \,\square\, H) \leq \min\{sn(G)|V(H)|, sn(H)|V(G)|\}.$$

Proof. Let $R \subseteq V(G)$ and $S = R \times V(H)$, that is, $S = \{(r, h) \mid r \in R,\ h \in V(H)\}$. Obviously, $|S| = |R||V(H)|$. Observe that edges between two copies of G cannot contribute any attack on S. Thus, S is secure in $G \square H$ if and only if R is secure in G. Choosing R as a minimum secure set, we can conclude that $sn(G \square H) \leq sn(G)|V(H)|$. The remaining relation can be shown by the symmetry argument. □

From the above lemma, $sn(G \square P_2) \leq 2sn(G)$. Thus, we have an upper bound on the security number of hypercubes.

Corollary 3.23. $s(Q^d) \leq 2^{d-1}$.

Note that $sn(G \square P_2)$ can be strictly less than $\min\{2sn(G), |V(G)|\}$ for some G (see Fig. 3.6).

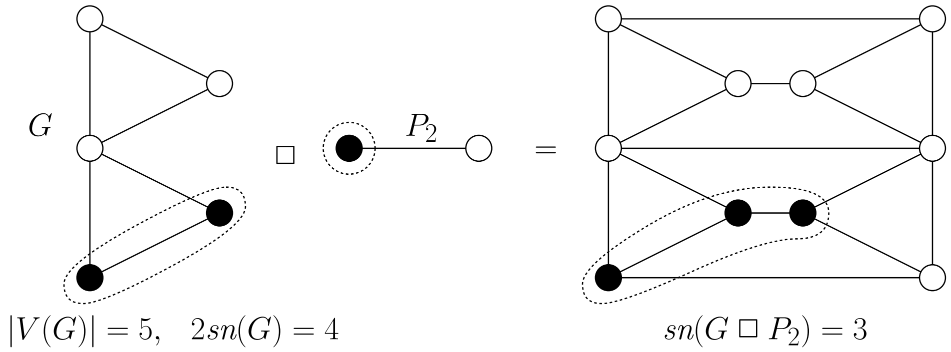


Fig. 3.6 $sn(G \square P_2) < \min\{2sn(G), |V(G)|\}$

From the definition, it is not difficult to see that if $|\partial_G(S)| > |S|$ then S is not secure. Thus, $\partial_G(k) > k$ implies there is no secure set of size k in G. Hence, we have the following lemma.

Lemma 3.24. *If $\partial_G(k) > k$ holds for all $1 \leq k \leq \ell$ then $sn(G) > \ell$.*

Using the above lemma, we present a lower bound for hypercubes. The vertex isoperimetric problem on hypercubes was settled by Harper [28]. Using his result, we will show that $\partial_{Q^d}(k) > k$ holds, for all $1 \leq k \leq \sum_{i=0}^{\lfloor (d-2)/3 \rfloor} \binom{d}{i}$. Namely, we show that $sn(Q^d) > \sum_{i=0}^{\lfloor (d-2)/3 \rfloor} \binom{d}{i}$.

First, we show a property of a partial sum over binomial coefficients.

Lemma 3.25. *For $d \geq 2$, $\sum_{i=0}^{r} \binom{d}{i} < \binom{d}{r+1}$ for $r \leq \lfloor (d-2)/3 \rfloor$.*

Proof. We will prove the lemma by induction on r. If $r = 0$, clearly the lemma holds. Let us assume $\sum_{i=0}^{r-1}\binom{d}{i} < \binom{d}{r}$ for some $1 \le r \le \lfloor (d-2)/3 \rfloor$. From $r \le \lfloor (d-2)/3 \rfloor$, we can derive $r+1 \le d-2r-1$. Therefore,

$$\sum_{i=0}^{r-1}\binom{d}{i} \Big/ \binom{d}{r} < 1 \le \frac{d-2r-1}{r+1} = \frac{d-r}{r+1} - 1,$$

$$\sum_{i=0}^{r-1}\binom{d}{i} < \binom{d}{r}\left(\frac{d-r}{r+1} - 1\right) = \binom{d}{r+1} - \binom{d}{r},$$

$$\sum_{i=0}^{r}\binom{d}{i} < \binom{d}{r+1}.$$

Thus, the lemma holds. □

Theorem 3.26 (Harper [28]). *For any integer k ($1 \le k \le |V(Q^d)|$), there exist a set $S \subseteq V(Q^d)$, a vertex $u_0 \in V(Q^d)$, and an integer r, such that $\{v \mid dist(u_0, v) \le r\} \subseteq S \subset \{v \mid dist(u_0, v) \le r+1\}$, $|S| = k$, and $|\partial(S)| = \min_{T \subseteq V(Q^d), |T|=k} |\partial(T)|$.*

By using Theorem 3.26, we can derive the next result.

Lemma 3.27. *If $k \le \sum_{i=0}^{\lfloor (d-2)/3 \rfloor}\binom{d}{i}$, then $\partial_{Q^d}(k) > k$.*

Proof. Let S, u_0, and r be the set, the vertex, and the integer in Theorem 3.26, respectively. Obviously $r \le \lfloor (d-2)/3 \rfloor$ since $k \le \sum_{i=0}^{\lfloor (d-2)/3 \rfloor}\binom{d}{i}$. Hence, from Lemma 3.25, we have $\sum_{i=0}^{r}\binom{d}{i} < \binom{d}{r+1}$. If $k = \sum_{i=0}^{r}\binom{d}{i}$, then $S = \{v \mid dist(u_0, v) \le r\}$ and $\partial(S) = \{v \mid dist(u_0, v) = r+1\}$. Thus, the lemma holds in this case. In the following, we will concentrate to the case $k > \sum_{i=0}^{r}\binom{d}{i}$. Note that in this case,

$$r \le \lfloor (d-2)/3 \rfloor - 1 \le (d-5)/3.$$

Let $S_\ell = \{v \mid v \in S,\ dist(u_0, v) = \ell\}$. Clearly,

$$|S| = |S_{r+1}| + \sum_{i=0}^{r}\binom{d}{i} < |S_{r+1}| + \binom{d}{r+1}.$$

It is easy to see that $\partial(S) = \partial(S_r) \cup \partial(S_{r+1})$. Thus, to estimate the size of $\partial(S)$, it is sufficient to show the sizes of $\partial(S_r)$ and $\partial(S_{r+1})$. Since S_r is exactly the set $\{v \mid dist(u_0, v) = r\}$, we have

$$|\partial(S_r)| = \binom{d}{r+1} - |S_{r+1}|.$$

We derive a lower bound for $|\partial(S_{r+1})|$. For any $v \in S_{r+1}$, $N(v) \cap \partial(S_{r+1}) = d - r - 1$. On the other hand, for any $v \in \partial(S_{r+1})$, $N(v) \cap S_{r+1} \leq r + 2$. See Fig. 3.7 to verify the above observations. It is easy to see that $|\partial(S_{r+1})|$ is minimized if for any $v \in \partial(S_{r+1})$, $N(v) \cap S_{r+1} = r + 2$. Therefore, we have

$$|\partial(S_{r+1})| \geq \frac{|S_{r+1}|(d - r - 1)}{r + 2}.$$

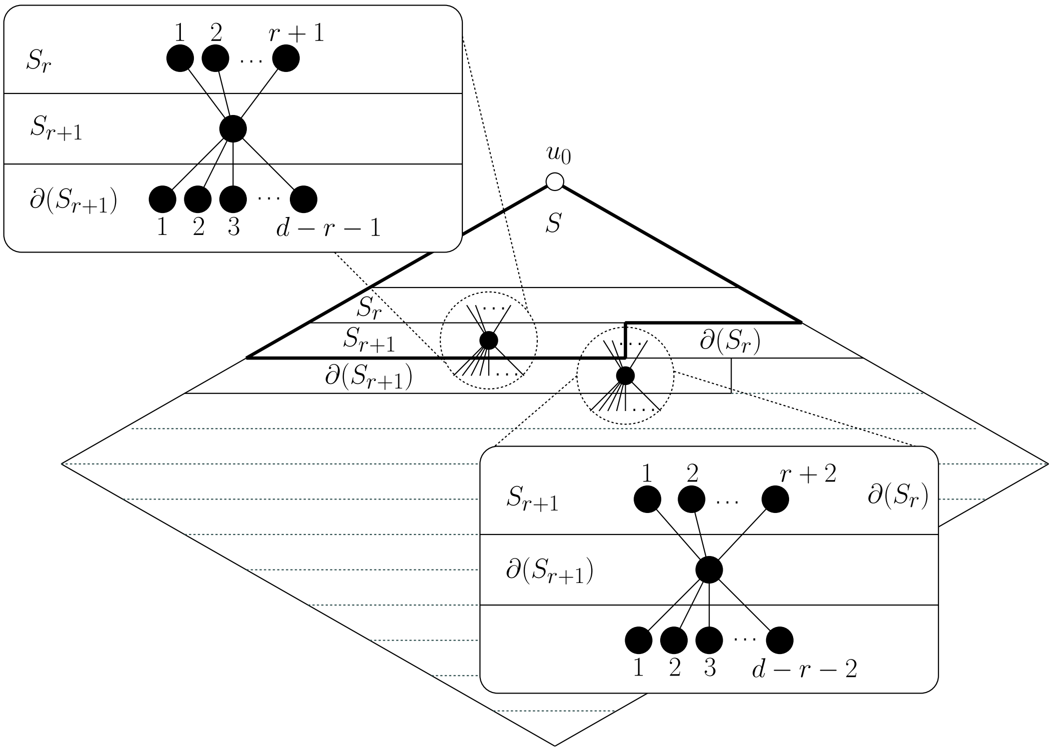


Fig. 3.7 Inner and outer degrees of vertices in S_{r+1} and $\partial(S_{r+1})$

From the above observations,

$$|\partial(S)| \geq \binom{d}{r+1} - |S_{r+1}| + \frac{|S_{r+1}|(d - r - 1)}{r + 2}.$$

Suppose $|S| \geq |\partial(S)|$. Then,

$$|S_{r+1}| + \binom{d}{r+1} > |S| \geq |\partial(S)| \geq \binom{d}{r+1} - |S_{r+1}| + \frac{|S_{r+1}|(d - r - 1)}{r + 2}.$$

Simplifying the above inequality, we have $r > (d - 5)/3$, a contradiction. □

From Lemmas 3.24 and 3.27, the following corollary holds.

Corollary 3.28. $sn(Q^d) > \sum_{i=0}^{\lfloor (d-2)/3 \rfloor} \binom{d}{i}$.

By combining Corollaries 3.23 and 3.28, we have the next result.

Theorem 3.29. $\sum_{i=0}^{\lfloor (d-2)/3 \rfloor} \binom{d}{i} < sn(Q^d) \leq 2^{d-1}$.

3.7 Concluding remarks

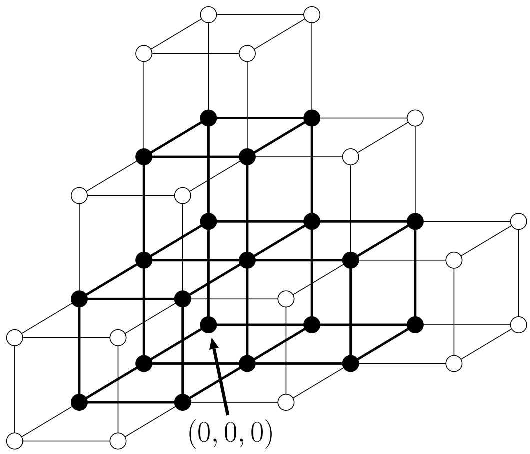
We have studied the security number of two-dimensional grid-like graphs and shown the best possible lower bounds for two-dimensional tori and two-dimensional cylinders. For future work, it is natural to study the security number of three-dimensional grid-like graphs. We believe that the upper bounds in the following proposition are the best possible except for small ℓ, m, n. (It is easy to see that $sn(C_3 \square C_3 \square C_3) \leq 12$, and $sn(P_2 \square C_3 \square C_3) \leq 8$.)

Proposition 3.30. *For three-dimensional grid-like graphs,*

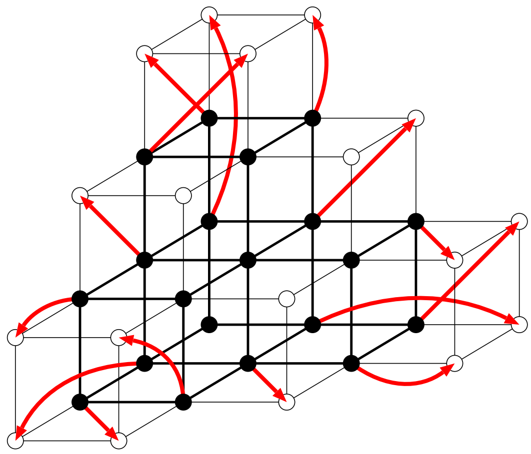
1. $sn(P_\ell \square P_m \square P_n) \leq \min\{\ell m, mn, n\ell, 20\}$,
2. $sn(P_\ell \square P_m \square C_n) \leq \min\{2\ell m, mn, n\ell, 40\}$,
3. $sn(P_\ell \square C_m \square C_n) \leq \min\{2\ell m, mn, 2n\ell, 80\}$,
4. $sn(C_\ell \square C_m \square C_n) \leq \min\{2\ell m, 2mn, 2n\ell, 160\}$.

Proof. (1) End vertices of the copies of P_n that lie in a single copy of $P_\ell \square P_m$ clearly form a secure set. Thus, $sn(P_\ell \square P_m \square P_n) \leq \ell m$. The upper bounds mn and $n\ell$ can be obtained by similar arguments. For the constant upper bound, let S be the set of corner vertices depicted in Fig. 3.8(a). Obviously, $|S| = 20$. For any attack on S, $u \in S$ can defend the vertex attacked by $v \in \partial(S)$ if $N(v) \cap S \subseteq N[u] \cap S$. Fig. 3.8(b) depicts such relations. White vertices marked with arcs are repelled by the corresponding black vertices. In Fig. 3.8(c), the remaining three white vertices can attack the three black vertices with a common unused defender. It is easy to see that the four black vertices can repel the three white vertices. Thus, S is secure.

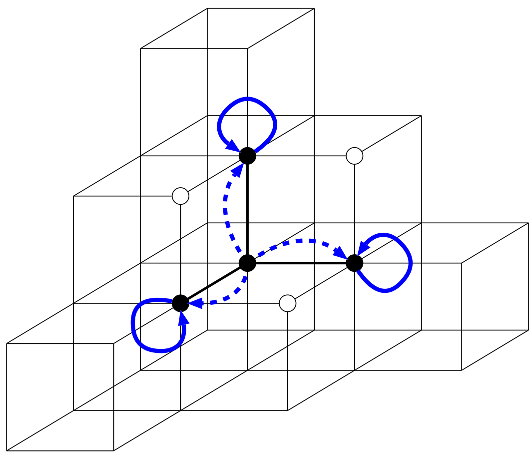
(2–4) For bounds like ab or $2ab$, corresponding secure set can be a single copy or two consecutive copies of $P_a \square P_b$, $P_a \square C_b$, or $C_a \square C_b$. For constant bounds, corresponding secure sets consist of two, four, or eight copies of the set S that are reversed and shifted. $\square$



(a) $\bullet \in S$, $\circ \in \partial(S)$.



(b) One-to-one marks.



(c) Self-defenses with help.

Fig. 3.8 A secure set S of $P_\ell \square P_m \square P_n$.

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