(t, k)-Diagnosable System: A Generalization of the PMC Models

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Abstract—In this paper, we introduce a new model for diagnosable systems called (t,k)-*diagnosable system* which guarantees that at least k faulty units (processors) in a system are detected provided that the number of faulty units does not exceed t. This system includes classical one-step diagnosable systems and sequentially diagnosable systems. We prove a necessary and sufficient condition for (t,k)-diagnosable system, and discuss a lower bound for diagnosability. Finally, we deal with a relation between (t,k)-diagnosability and diagnosability of classical basic models.

Index Terms—Fault diagnosis, PMC model, one-step t-diagnosis, sequential t-diagnosis, diagnosability, Cartesian product.

1 Introduction

The rapid development in digital technology has resulted in developing systems including a very large number of processors. As the number of processors in a system increases, it has become important to guarantee the reliability of such systems. This issue has prompted designers of such systems to study *fault-tolerant* systems that are capable of uninterrupted processing. Fault tolerance on multiprocessor systems is composed of two basic steps. The first step is called *fault diagnosis*. In this step, faulty processors are identified. The second step is called *system configuration*. In this step, the faulty units that have been previously identified are configured out of the system. Then, they are replaced by spare processors or the tasks that are assigned to the faulty processors are distributed to remaining fault-free processors in the system. In large network systems, it is impractical for each processor to be tested individually by another host. So, the concept of *system level diagnosis* is effective in this situation. Preparata et al. [1] first introduced a graph theoretical model (called *PMC model*) for system level diagnosis and the concept of two basic diagnosis of systems called one-step diagnosis and sequential diagnosis. After the PMC model, many theoretical or practical studies have been reported [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22].

In this paper, we propose a new model called the (t,k)-*diagnosable system*, which is a generalized sequentially diagnosable system. A diagnosis is said to be *complete* if all the faulty units can be identified. Otherwise, it is an *incomplete* diagnosis. Similarly, a diagnosis is *correct* if no fault-free units are identified as faulty. If not, the diagnosis is called *incorrect*. One-step diagnosis is a complete and correct diagnosis, that is, all faulty units in the system are correctly identified, while sequential diagnosis is a correct and incomplete diagnosis. It is known that the diagnoses proposed by Friedman [7] and by Somani and Peleg [20] are complete but incorrect diagnoses. Our (t,k)-diagnosis is a correct and incomplete diagnosis, hence, this is a generalization of sequential diagnosis.

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The rest of this paper is organized as follows: In Section 2, we describe backgrounds and definitions for diagnosable systems. In Section 3, we give the definition of a (t,k)-diagnosable system; the necessary and sufficient condition for this system is proven. In Section 4, we consider a lower bound of the (t,k)-diagnosability. Finally, in Section 5, we relate (t,k)-diagnosable systems to two basic systems using a graph product.

2 The PMC Model and Some Preliminaries

In the PMC model, a system S is decomposed into n independent units. Each unit v is assigned a subset of S to test and it is assumed that there is no unit tested by itself. The complete collection of tests in S, called *test assignment* of S, is represented by a directed graph $G(V,E)$, where each unit u is represented by a vertex $u \in V$ and there is a directed edge $(u,v) \in E$ if and only if u tests v in S. The following sets are associated with each unit u: $\Gamma u = \{v : (u,v) \in E\}$, $\Gamma^{-1}u = \{v : (v,u) \in E\}$. Similarly, the following sets are defined for a set of units $U \subset V$: $\Gamma U = \bigcup_{u \in U} \Gamma u - U$, $\Gamma^{-1}U = \bigcup_{u \in U} \Gamma^{-1}u - U$. The outcome of test (u,v) is represented by the weight $w(u,v)$ of the edge, where $w(u,v) = 0$ (resp. 1) if u evaluates v to be fault-free (resp. faulty). The set of all test outcomes of S is called the *syndrome* of S. The faults considered here are permanent, so the test outcome $w(u,v)$ is reliable if and only if u is fault-free. (Diagnosable systems with intermittent fault have been discussed in research such as in [12], [14].)

Preparata et al. [1] have introduced two basic notions of diagnosis of systems. One is the *one-step diagnosis*, which finds all faulty units at one time, and the other is the *sequential diagnosis*, which finds a subset of faulty units. A system is called *one-step t-diagnosable* (resp. *sequentially t-diagnosable*) if, given any complete collection of test outcomes, all (resp. at least one) faulty units in S can be identified, provided the number of faulty units does not exceed t.

The fault set and the consistent fault set have been defined as follows [9]:

Fault set. *A* fault set *is a set of the faulty units of a system S.*

Consistent fault set. *For a system S and a given syndrome, a subset $F \subset V$ is* a consistent fault set *(CFS) if and only if 1) $u \in V - F$ and $w(u,v) = 0$ imply $v \in V - F$ and 2) $u \in V - F$ and $w(u,v) = 1$ imply $v \in F$.*

Thus, F is a CFS for a given syndrome if and only if the assumption that the units in F are faulty and the units in $V - F$ are fault-free is consistent with the syndrome. Given a system S and a syndrome ω, let $\Omega_{\omega,t}$ be the set of possible CFSs for the syndrome in a t-fault situation (a fault situation means that t or fewer units in the system are faulty). Thus, $\Omega_{\omega,t} = \{F : F \text{ is a CFS for the syndrome } \omega \text{ and } |F| \leq t\}$. Clearly, we may regard all the units belonging to the intersection of CFSs as faulty. Thus, a system S is one-step t-diagnosable if and only if, for any given syndrome ω produced by a t-fault situation, $|\Omega_{\omega,t}| = 1$ and a system S is sequentially t-diagnosable if and only if, for any syndrome ω, $|\bigcap_{F \in \Omega_{\omega,t}} F| \geq 1$ or $\Omega_{\omega,t} = \{\emptyset\}$. In order to characterize sequentially diagnosable systems, the following terms are defined in [21]: For a set V, a set of subsets of V, $\pi = \{V_1, \ldots, V_r\}$, where $V_i \subset V$ $(i = 1, \ldots, r)$, is a *cover* of V if and only if $\bigcup_{1 \leq i \leq r} V_i = V$. Given a set V and a cover $\pi = \{V_1, \ldots, V_r\}$ of V, we associate the set $f(u) = \{V_i \in \pi : u \in V_i\}$ for each element $u \in V$. For each element $u \in V$ and a subset $U \subset V$, we define $N(u) = |f(u)|$ and $N(U) = \sum_{u \in U} N(u)$. That is, $f(u)$ is the set of the elements in π to which u belongs, and $N(u)$ is the number of subsets in π to which u belongs.

Hakimi and Amin [8] characterized one-step t-diagnosable systems as follows:

Theorem 2.1 (Hakimi and Amin [8]). *A system S is one-step t-diagnosable if and only if 1) $n \geq 2t+1$, 2) for any $v \in V$, $|\Gamma^{-1}v| \geq t$, and 3) for each integer p with $0 \leq p < t$ and each $V' \subset V$, $|V'| = n - 2t + p$, $|\Gamma^{-1}V'| > p$.*

Characterization for sequentially diagnosable systems was proved by Xu and Huang [21].

Theorem 2.2 (Xu and Huang [21]). *A system S is sequentially t-diagnosable if and only if, for any nonempty subset $F \subset V$ and for each cover π of F, $\pi = \{F_1, F_2, \ldots, F_r\}$ with $|F_i| \leq t$ $(i = 1, 2, \ldots, r)$, at least one of the following conditions holds: 1) $\bigcap_{1 \leq i \leq r} F_i \neq \emptyset$, 2) there exists a test (u, v) such that $u \in V - F$ and $v \in F$, 3) there exists a test (u, v) such that $f(v) \not\subset f(u)$ and $f(u) \cup f(v) \neq \pi$.*

For a diagnosable system, we call the maximum value of t the *diagnosability* of the system. We define the *one-step diagnosability* (resp. *sequential diagnosability*) of a system as the maximum value of t such that the system is one-step (resp. sequentially) t-diagnosable. That is, the diagnosability is the maximum number of faulty units that the system can guarantee to diagnose correctly. The result of Theorem 2.1 implies that the one-step diagnosability is bounded by the minimum vertex degree in the graph. However, most well-known interconnection networks, such as complete k-ary trees, hypercubes, grids, torus, cube-connected cycles, have small vertex degrees. Thus, the bound of the one-step diagnosability is very small in comparison with the total number of units.

It is known that there are two ways to increase the diagnosability under PMC models. One approach is sequential diagnosis. For example, a directed cycle with n units is known to be sequentially $(2\sqrt{n} - 2)$-diagnosable [1] (a directed cycle is referred as a *single-loop system* in [1]). Since most practical systems have Hamiltonian cycles, we can sequentially diagnose such systems for a large number of faulty units more than the smallest degree. Another way is to allow a certain number of units to be incorrectly diagnosed. Friedman's t/s-diagnosable system [7] and Somani and Peleg's t/k-diagnosable system [20] are the representative works in this area. These notions are extension of one-step diagnosis, and many theoretical results are reported. On the other hand, the studies of generalization of sequential diagnosis have not been reported.

The aim of sequential diagnosis is to identify iteratively subsets of faulty units until all faulty units are repaired. At the end of each iteration, the identified subset of faulty units is repaired and then next iteration is started. In each iteration, at least one faulty unit is guaranteed to be identified under sequential t-diagnosis. One approach to designing an algorithm for sequential diagnosis is that some fault-free units are identified in the first phase and, then, we iteratively search faulty units using the identified fault-free units and repair the identified faulty units. It should be noted that, in each iteration, there is a possibility that more than one faulty unit may be identified. Some algorithms [11], [10] adopted this approach in order to design efficient sequential t-diagnosis algorithms. As another approach, Somani et al. [19] treated the problem that the set of all faulty units with more than the one-step diagnosability is identified. Hence, it is an interesting problem to consider the number of faulty units a system can locate correctly. This problem may have the possibility of motivating the design of a new algorithm for diagnosis of systems.

For this reason, we propose a new framework for the sequential diagnosis, called the (t, k)-*diagnosis*, which is a generalized sequential diagnosis so that at least $k \geq 1$ faulty units are identified. The one-step t-diagnosable system and sequentially t-diagnosable system are basic diagnosis models and each of them treats the extreme case with respect to the number of the identifying faulty units. (One-step t-diagnosis identifies *all* faulty units, on the other hand, sequential t-diagnosis identifies *at least one* faulty unit.) The (t, k)-diagnosable system, with a new parameter k, gives a wide class of diagnosable systems which includes one-step diagnosable systems and sequentially diagnosable systems as extremal cases.

3 (t, k)-Diagnosable Systems and their Characterizations

A (t, k)-diagnosable system is defined as follows:

Definition 3.1. *For t and k, $t \geq k$, a system S is (t, k)-diagnosable if, given any syndrome produced by the system under the presence of a fault set F:*

1. *All faulty units can be identified for $|F| \leq k$ and*
2. *At least k faulty units can be identified for $k < |F| \leq t$.*

If $k = t$, then the system is one-step t-diagnosable and, if $k = 1$, then the system is sequentially t-diagnosable. Therefore, (t, k)-diagnosable systems are generalized fault diagnosable systems including the two basic diagnosable systems defined by Preparata et al.

Given an integer $k \geq 1$ and a system S, we define the (t, k)-*diagnosability* of the system S as the maximum value of t such that S is (t, k)-diagnosable. If there is no value t such that S is (t, k)-diagnosable for given k, we define the (t, k)-diagnosability as 0. By the definition, if a system S is (t, k)-diagnosable, then S is (t, k')-diagnosable for any $1 \leq k' \leq k$. Hence, the following proposition clearly holds:

Proposition 3.2. *For any system S, the (t, k')-diagnosability is greater than or equal to the (t, k)-diagnosability if $1 \leq k' < k$.*

Hence, it may be understood intuitively that the (t, k)-diagnosability becomes larger as k becomes smaller.

Let F be a fault set in a system S. We need $|F|$ times iterations of diagnosis in order to repair all faulty units in worst case under the sequential diagnosis. In (t, k)-diagnosable systems, it is guaranteed that the number of iterations of diagnosis is at most $\lceil |F|/k \rceil$. When the probability of processor to be faulty is low, (t, k)-diagnosable systems identify most faulty units (for appropriate value k) and, further, have higher reliability since the diagnosability is larger than the one-step diagnosability. (Somani [18] proposed a sequential diagnosis under the situation that faults occur sequentially. In this situation, identifying all faulty units is achieved by repeating one-step diagnosis.)

Definition 3.1 is equivalent to the following definition:

Definition 3.3. *A system S is (t, k)-diagnosable if and only if, given any syndrome ω for S in a t-fault situation, $|\bigcap_{F \in \Omega_{\omega,t}} F| \geq k$ or $|\Omega_{\omega,t}| = 1$.*

If $|\bigcap_{F \in \Omega_{\omega,t}} F| \geq k$, then at least k faulty units are identified and, if $|\Omega_{\omega,t}| = 1$, then all faulty units are identified.

Somani et al. [19] generalized the notion of one-step diagnosis. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be a family of fault sets. A fault set F is *uniquely diagnosable* with respect to a family of fault sets $\mathcal{F}$ if any syndrome corresponding to the fault set F is not producible by the system in the presence of any other fault set $F_i \in \mathcal{F}$. Clearly, a system is t-diagnosable if and only if any fault set F with $|F| \leq t$ is uniquely diagnosable with respect to the family of fault set $\mathcal{F}_t = \{F_i : |F_i| < t, F_i \subseteq V\}$. Somani et al. characterized a fault set to be uniquely diagnosable with respect to the family of fault set $\mathcal{F}_t$. If a system is (t, k)-diagnosable, then any fault set F with $|F| \leq k$ is uniquely diagnosable with respect to $\mathcal{F}_t$. However, Somani et al.'s work did not treat the case that two or more CFSs correspond to given syndrome. The (t, k)-diagnosis is a diagnosising scheme such that a fault set F with $|F| \leq k$ is uniquely

diagnosable and faulty units in F with $k < |F| \leq t$ are correctly and incompletely diagnosed.

A necessary and sufficient condition for a (t,k)-diagnosable system is stated as follows. This is a generalization of the characterization in Theorem 2.2 (Xu and Huang).

Theorem 3.4. *A system S represented by $G(V,E)$ is (t,k)-diagnosable if and only if, for every nonempty subset $F \subset V$ and for each cover π of F, $\pi = \{F_1, F_2, \ldots, F_r\}$ with $|F_i| \leq t$ $(i = 1,2,\ldots,r)$, at least one of the following conditions holds:*

1. $\left|\bigcap_{1\leq i\leq r} F_i\right| \geq k$,
2. $|\Gamma\overline{F}| \geq k$, *where* $\overline{F} = V(G) - F$,
3. *There exists an edge* $(u,v) \in E(G)$ *such that* $f(v) \not\subset f(u)$ *and* $f(u) \cup f(v) \neq \pi$, *and*
4. $r = 1$.

Proof. (Necessity) Suppose that there is a subset $U \subset V$ and a cover π of U, $\pi = \{F_1, F_2, \ldots, F_r\}$ with $|F_i| \leq t$ $(i = 1,2,\ldots,r)$, such that

1. $\left|\bigcap_{1\leq i\leq r} F_i\right| < k$,
2. $|\Gamma\overline{F}| < k$,
3. for all $(u,v) \in E(G)$, $f(v) \subset f(u)$ or $f(u) \cup f(v) = \pi$, and
4. $r \geq 2$.

Then, we can construct a syndrome ω such that $\pi \subset \Omega_{\omega,t}$ as follows: For any $(u,v) \in E$, let $w(u,v) = 0$ if $v \in \overline{F}$; let $w(u,v) = 1$ if $u \in \overline{F}$ and $v \in F$; let $w(u,v) = 0$ if $f(v) \subset f(u)$, otherwise $w(u,v) = 1$. Note that, for each $i = 1,2,\ldots,r$, if $u \notin F_i$ and $w(u,v) = 0$, then $v \notin F_i$ and, if $u \notin F_i$ and $w(u,v) = 1$, then $v \in F_i$. So, each V_i $(i = 1,2,\ldots,r)$ is a CFS for the syndrome ω. Since $|V_i| \leq t$ for any $i = 1,2,\ldots,r$, $V_i \in \Omega_{\omega,t}$. However, since $|\bigcap_{1\leq i\leq r} F_i| < k$, we have $|\bigcap_{F\in\Omega_{\omega,t}} F| < k$. Therefore, by Definition 3.3, S in not (t,k)-diagnosable.

(Sufficiency) Assume that one or more of conditions 1, 2, 3, and 4 hold, but S is not (t,k)-diagnosable. By Definition 3.3, there exists a syndrome ω such that $|\bigcap_{F\in\Omega_{\omega,t}} F| < k$ and $\Omega_{\omega,t} > 1$. Let $\pi = \Omega_{\omega,t} = \{F_1, F_2, \ldots, F_r\}$ and $F = \cup_{1\leq i\leq r} F_i = F$. So, π is a cover of $F \subset V$. Note that π does not satisfy the conditions 1 and 4. If there is an edge (u,v) such that $u \in V - F$ and $v \in F - \bigcap_{1\leq i\leq r} F_i$, then $v \notin F_x$ and $v \in F_y$ for some x. However, if $w(u,v) = 0$, then F_y is not a CFS. If $w(u,v) = 1$, then F_x is not a CFS. This is a contradiction. Thus, $v \in \bigcap_{1\leq i\leq r} F_i$. This implies that $\Gamma\overline{F} \subset \bigcap_{1\leq i\leq r} F_i$. Hence, $|\Gamma\overline{F}| \leq |\bigcap_{1\leq i\leq r} F_i| < k$. Therefore, π does not satisfy 2. Finally, assume π meets condition 3, that is, there exists an edge (u,v) such that $f(v) \not\subset f(u)$ and $f(u) \cup f(v) \neq \pi$. Then, there are two sets F_x, F_y such that $u \notin F_x$, $v \in F_x$ and $u, v \notin F_y$. However, $w(u,v) = 0$ implies F_x is not CFS, which is a contradiction; $w(u,v) = 1$ implies F_y is not CFS, which is also a contradiction. □

Setting $k = 1$ in the above theorem, we obtain the characterization for sequentially t-diagnosable systems (Theorem 2.2). By the proof of Theorem 3.4, $F \subset V$ and π satisfy condition 3 if and only if each $F_i \in \pi$ is CFS for some syndrome for S in a t-fault situation. When $k = t$, we find that the conditions in Theorem 3.4 are equivalent to 1) $r = 1$ or 2) there exits (u,v) such that $f(v) \not\subset f(u)$ and $f(u) \cup f(v) \neq \pi$, if $r \geq 2$. This condition can be put in other words: "A system is one-step t-diagnosable if and only if any syndrome does not have two or more CFSs." This is the definition of a one-step t-diagnosable system.

Lemma 3.5. *If a system S is (t,k)-diagnosable, then 1) $n \geq 2t+1$, 2) $|\Gamma^{-1}v| \geq k$ for any $v \in V$, and 3) $|\Gamma^{-1}U| > p$ for each $U \subset V$ with $|U| = 2(t-p)$ and $0 \leq p < k$.*

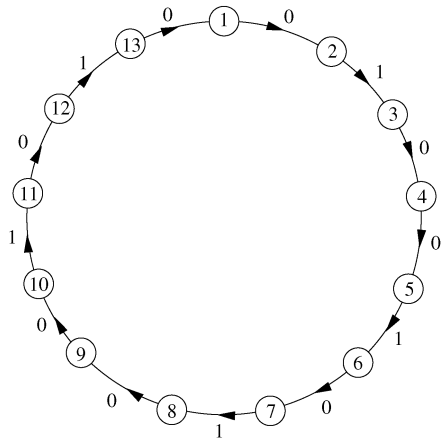

Fig. 1. An example of system that is not sequentially 6-diagnosable.

Proof. Similar to the proof of Theorem 2.1 [8]. □

The necessary condition in Lemma 3.5 is not sufficient when $k = t$. In fact, if $k = 1$, condition 3 means "for any U with $|U| = 2t$, $|\Gamma^{-1}U| > 0$." Hence, a single-loop system with 13 units illustrated in Fig. 1 satisfies conditions 1, 2, and 3 for $t = 6$. However, the system is not 6-diagnosable. The syndrome represented in Fig. 1 associates the following CFSs: $F_1 = \{3,6,8,9,10,13\}$, $F_2 = \{1,2,6,7,11,13\}$, $F_3 = \{3,4,5,8,11,12\}$. Since $F_1 \cap F_2 \cap F_3 = \emptyset$, no faulty units are identified. Hence, the system is not sequentially 6-diagnosable.

4 Diagnosability of (t, k)-Diagnosable Systems

It is known that the determination of the exact diagnosability of a given diagnosable system is difficult. Theorem 2.2 gives some conditions that must be satisfied by every subset of units. Since there are exponentially many subsets, checking each subset is impractical for large systems. Raghavan and Tripathi [16] have shown that determining the exact sequential diagnosability for any testing assignment is co-NP Complete. The characterization for (t,k)-diagnosable systems (Theorem 3.4) is essentially equivalent to sequentially t-diagnosable systems, hence there may be no efficient algorithm for determining the diagnosability of (t,k)-diagnosable systems. So, in this section, we consider a lower bound for the (t,k)-diagnosability using graph-theoretic properties.

Let $G(V,E)$ be a strongly connected digraph representing a system S with n units. Assume that S is not (t,k)-diagnosable for $n \geq 2t+1$ and that $\kappa(G) \geq k$, where $\kappa(G)$ denotes the *connectivity* of the digraph G defined by the minimum number of vertices whose removal from G results in a directed graph that is not strongly connected. By Theorem 3.4, there are a subset $F \subset V$ and a cover π of F, $\pi = \{F_1, F_2, \ldots, F_r\}$, $|F_i| \leq t$ $(i = 1,2,\ldots,r)$ such that all the following conditions are satisfied:

1. $\left|\bigcap_{1\leq i\leq r} F_i\right| < k$,
2. $|\Gamma\overline{F}| < k$,
3. For each edge $(u,v) \in E(G)$, $f(v) \subset f(u)$ or $f(u) \cup f(v) = \pi$, and
4. $r > 1$.

In general, there may be two or more subsets and covers. We adopt a subset F and a cover π that have the minimum value r. That is, F and π satisfy conditions 1-4 and, for values smaller than r, any pair of subsets and cover do not satisfy one or more conditions. Let Y and Y_i $(i = 1,2,\ldots,r)$ be $Y = \bigcap_{1\leq i\leq r} F_i$, and $Y_i = \bigcap_{1\leq j\leq r, j\neq i} F_j - Y$. The cardinality of the sets Y and Y_i is denoted by y and y_i, respectively.

Claim 4.1. $F = V$.

TABLE 1
Diagnosability of m-Dimensional Hypercubes

dimension	number of units	t	k
4	16	7	4
5	32	11	5
6	64	17	6
7	128	25	7
8	256	36	8

Proof. Assume that $F \neq V$. Let $U = \{u : u \notin F\}$ (of course, $U \neq \emptyset$). Since G is strongly connected, there is an edge $(u, v) \in E$ such that $u \in \overline{F}$ and $v \in F$. By condition 3, we have $f(v) = \pi$ (that is, $v \in Y$). Thus, removal of vertices in Y results in a subgraph of G that has no directed paths from U to $F - Y$. Hence, $|Y| \geq \kappa(G) \geq k$. This contradicts condition 1. □

Claim 4.2. *For each $i = 1, 2, \ldots, r$, $Y_i \neq \emptyset$.*

Proof. If $Y_i = \emptyset$, then $\pi' = \pi - \{F_i\}$ is a cover of $V' = \bigcup_{F_j \in \pi'} F_j$ that satisfies conditions 1-4. This is easily proven using the fact that $\bigcap_{F_j \in \pi'} F_j = Y \cup Y_i = Y$. □

Lemma 4.3 ([21]). *There is a set V_i such that $|F_i| \geq N(V)/r$.*

We show that $y + y_i \geq k$ for each $i = 1, 2, \ldots, r$. If $\pi - \{F_i\}$ is not a cover of V, then there exists a vertex u such that $f(u) = \{F_i\}$. Let $U = \{u : f(u) = \{F_i\}\}$, then $\Gamma U \subset \bigcap_{1 \leq j \leq r, j \neq i} F_i = Y \cup Y_i$. Thus, the graph obtained by removing the vertices in $Y \cup Y_i$ does not have directed path from a vertex $u \in U$ to a vertex $v \notin U \cup Y \cup Y_i$. This means $|Y \cup Y_i| = y + y_i \geq \kappa(G) \geq k$. If $\pi' = \pi - \{F_i\}$ is a cover of V, it is clear that π' satisfy conditions 2, 3, and 4. Hence, π' does not satisfy condition 1 by the way of selecting F and π. Thus, we obtain $y + y_i \geq k$. Therefore, we have

$$y + \sum_{1 \leq i \leq r} y_i = \sum_{1 \leq i \leq r} (y + y_i) - (r-1)y \geq kr - (r-1)(k-1) = k + r - 1.$$

Let $M = Y \cup Y_1 \cup \ldots \cup Y_r$, then

$$N(V) = N(M) + N(V - M) \geq ry + (r-1) \sum_{1 \leq i \leq r} y_i + (|V| - |M|)$$
$$= (r-1)y + (r-2) \sum_{1 \leq i \leq r} y_i + n$$
$$= y + (r-2)\left(y + \sum_{1 \leq i \leq r} y_i\right) + n \geq (r-2)(k + r - 1) + n.$$

From Lemma 4.3, $|F_i| \geq N(V)/r = r + (k-3) + (n - 2k + 2)/r$ for some i. Let $F(r) = r + (k-3) + (n - 2k + 2)/r$. It is obvious that $r \geq 3$ since $n \geq 2t + 1$. Thus, $F(r)$ has the minimum value $F_{min} = 2\sqrt{n - 2k + 2} + (k-3)$ for $3 \leq \sqrt{n - 2k + 2}$ and $F_{min} = (n + k + 2)/3$ for $3 > \sqrt{n - 2k + 2}$. Since $t \geq |F_i| \geq F_{min}$, we obtain the following theorem.

Theorem 4.4. *Let G be a strongly connected digraph representing a system S. For any positive integer k such that $\kappa(G) \geq k$, S is (t, k)-diagnosable if $n \geq 2t + 1$ and 1) $t < 2\sqrt{n - 2k + 2} + (k - 3)$ if $n \geq 2k + 7$, 2) $t < (n + k + 2)/3$ if $n < 2k + 7$.*

Example. m-dimensional hypercube Q_m has 2^m vertices, and connectivity $\kappa(Q_m) = m$. For $m \geq 4$, $2^m > 2m + 7$. Thus, for k with $k \leq m$ and t with $t < 2\sqrt{2^m - 2m + 2} + (m - 3)$, Q_m is (t, k)-diagnosable. Table 1 shows the (t, k)-diagnosability of hypercubes for small dimensions.

5 Construction of (t, k)-Diagnosable Systems by the Cartesian Product

The Cartesian product of graphs is defined as follows: For digraphs G and H, $G \times H$ is a digraph with a vertex set $V(G) \times V(H)$ and an edge from a vertex (u_1, v_1) to a vertex (u_2, v_2) exists if and only if $(u_1, u_2) \in E(G)$ and $v_1 = v_2$ or $u_1 = u_2$ and $(v_1, v_2) \in E(H)$. Many interconnection networks are defined by the Cartesian product, such as hypercubes, grids, torus, and meshes, and these networks constitute very important classes widely studied for interconnection networks.

It has been shown that one-step diagnosability has an intimate relation to Cartesian products [2].

Theorem 5.1 (Araki and Shibata [2]). *Let G and H be digraphs representing one-step t_G and t_H-diagnosable systems, respectively. Then, the system represented by $G \times H$ is one-step $(t_G + t_H)$-diagnosable.*

A relation between a (t, k)-diagnosable system and the two basic systems, one-step and sequentially diagnosable system, is stated using the Cartesian product.

Lemma 5.2. *Let G and H be digraphs representing a sequentially t_G-diagnosable and a one-step t_H-diagnosable system, respectively. Then, the system S represented by $G \times H$ is $(t_G + t_H, t_H + 1)$-diagnosable.*

Proof. Let ω be a syndrome for a system S provided the number of faulty units does not exceed $(t_G + t_H)$. The graph $G \times H$ has $|V(H)|$ copies of G as subgraphs. Let c be the number of copies of G containing test outcomes "1."

If $t_H < c \leq t_G + t_H$, then each copy of G contains at most t_G faulty units. Assume to the contrary that, if there exists a copy of G having more than t_G faulty units, then the total number of faulty units is at least $(t_G + 1) + (c - 1) = t_G + c > t_G + t_H$. This is a contradiction. Thus, in this case, each copy of G identifies at least one faulty units.

In the case of $1 \leq c \leq t_H$, we consider copies of H in $G \times H$. Obviously, the number of faulty units in the copies of H is at most t_H. Since the system represented by H is one-step diagnosable, all faulty units are identified.

By the above discussion, the system S identifies at least $t_H + 1$ faulty units or all faulty units. Thus, S is $(t_G + t_H, t_H + 1)$-diagnosable. □

Let K_2^+ be a digraph representing a system such that two units test each other.

Lemma 5.3. *Let G be a digraph representing a (t, k)-diagnosable system. Then, a system represented by $G \times K_2^+$ is $(t + 1, k + 1)$-diagnosable.*

Proof. Easily proven. □

The method of proof of Lemma 5.2 is very simple, so the supposable number of faulty units $(t_G + t_H)$ is small compared to the total number of units $(|V(G)||V(H)|)$. However, we can obtain a lower bound for diagnosability without considering the structure of graphs G and H using a property of Cartesian products.

6 Concluding Remarks

We introduce a notion of (t, k)-diagnosable systems and give a characterization of the system and discuss some relations to one-step and sequentially diagnosable systems. It is expected that there is a trade off relation between diagnosability t and a new parameter k. To solve the dependency of the diagnosability on the parameter k is a future subject. We think that the notion of (t, k)-diagnosis may motivate us to design new diagnosis algorithms.

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