DigitalCommons@WayneState

Journal of Modern Applied Statistical Methods

Volume 16 | Issue 2 Article 7

December 2017

On Variance Balanced Designs

Dilip Kumar Ghosh
Saurashtra University Rajkot, Gujarat, India, ghosh_dkg@yahoo.com

Sangeeta Ahuja
Indian Agricultural Statistics Research Institute, New Delhi, India

Recommended Citation

Ghosh, D. K., & Ahuja, S. (2017). On Variance Balanced Designs. Journal of Modern Applied Statistical Methods, 16(2), 124-137. doi: 10.22237/jmasm/1509494820

On Variance Balanced Designs

Cover Page Footnote
We are thankful to both the referees for their valuable suggestions in improving the paper. First author is thankful to UGC for providing him UGC BSR Faculty fellow.

Journal of Modern Applied Statistical Methods
November 2017, Vol. 16, No. 2, 124-137.
doi: 10.22237/jmasm/1509494820

ISSN 1538 − 9472

On Variance Balanced Designs

Dilip Kumar Ghosh
Saurashtra University Rajkot
Gujarat, India

Sangeeta Ahuja
Indian Agricultural Statistics Research Institute
New Delhi, India

Balanced incomplete block designs are not always possible to construct because of their parametric relations. In such a situation another balanced design, the variance balanced design, is required. This construction of binary, equal replicated variance balanced designs are discussed using the half fraction of the 2*n* factorial designs with smaller block sizes. This method was also extended to construct another variance balanced design by deleting the last block of the resulting variance balanced designs. Its efficiency factor compared with randomized block designs was compared and found to be highly efficient.

Keywords: Binary, efficiency factor, balanced incomplete block designs, information matrix and incidence matrix

Introduction

The balanced incomplete block design (BIBD) is binary, proper, connected, equi-replicated, balanced, and non-orthogonal. It is simple to construct and analyze. However, it is not available for all parameters because of its following parametric relations: (a) $vr = bk$; (b) $\lambda(v-1) = r(k-1)$; and (c) $b \geq v$. Thus, an incomplete block design is needed that should be connected and balanced. This type of incomplete block design is called a variance balanced (VB) design.

Rao (1958) noted that, if the information matrix **C** of a block design satisfies

$$\mathbf{C} = \theta\left[\mathbf{I}_v - \frac{1}{v}\mathbf{E}_{vv}\right] \quad (1)$$

where θ is the non-zero eigenvalue of the matrix **C**, $\mathbf{I}_v$ is an identity matrix of order v, and $\mathbf{E}_{vv}$ is a matrix of Unity with v rows and v columns, then such a design is a variance balanced design. BIBD satisfies this property and hence is a variance balanced design.

Dr. D. K. Ghosh is a UGC BSR Faculty Fellow in the Department of Statistics. Email him at: ghosh_dkg@yahoo.com. Dr. Sangeeta Ahuja is a Senior Scientist.

Chakrabarti (1963) gave the useful concept of a **C**-matrix of design. Pearce (1964) obtained VB design with varying block sizes. Mukerjee and Kageyama (1985) obtained a resolvable VB design with unequal replications. Das and Ghosh (1985) obtained unequal replicated, unequal/equal block size VB design from BIBD and partially balanced incomplete block design (PBIBD) using augmented blocks and treatments. Rao (1958), Hedayat and Federer (1974), Raghavarao (1962), and Puri and Nigam (1977) defined that a design is said to be variance balanced if every normalized estimable linear function of treatment effect can be estimated with same precision. Kageyama (1988) discussed the construction of VB design using BIBD, PBIBD, and some incidence matrices.

Khatri (1982) gave a formula to measure the overall A-efficiency of VB designs along with method of construction of VB designs. Gupta and Jones (1983) obtained VB designs using BIBD and PBIBD with two associate classes. Calvin and Sinha (1989) extended the technique of Calvin (1986) to construct VB designs with more than two distinct block sizes that permit fewer replications. Das and Ghosh (1985) defined generalized efficiency balanced (GEB) design which include both VB as well as efficiency balanced (EB) designs. Ghosh (1988), Ghosh and Karmoker (1988), Ghosh, Divecha, and Kageyama (1991), and Ghosh, Joshi, and Kageyama (1993) provided several methods for the construction of VB designs. Ghosh and Joshi (1995) constructed VB designs through a triangular design. Agarwal and Kumar (1985) constructed a VB design which is associated with group divisible design. Ghosh and Joshi (1991) constructed a VB design through a group divisible (GD) design.

Definition: A block design is said to be variance-balanced if it permits the estimation of all estimable normalized treatment contrasts with the same variance,

$$\mathrm{V}\left(\hat{t}_i - \hat{t}_j\right) = k\sigma^2, \quad \forall i \neq j = 1, 2, \ldots, v$$

Variance balance designs can also be further defined as in the following way: A connected block design is said to be variance balanced if and only if all the nonzero eigenvalues of the matrix **C** of the block design are equal.

Corollary: A connected block design is variance balanced if and only if its **C**-matrix has all its diagonal elements equal and all its off-diagonal element equal, i.e.

$$\mathbf{C} = \left(a - b\right)\mathbf{I} + bjj'$$

Remark: For an equi-replicated, proper, binary, variance balanced design, the concurrence matrix satisfies

$$\mathbf{NN}' = (r - \lambda)\mathbf{I}_v + \lambda jj' \tag{2}$$

Here, the variance balanced design is constructed using fractional factorial experiments, and by developing some incidence matrices.

Method of Construction of Variance Balanced Designs

Variance Balanced Design from Half Fraction of 2^n Factorial Experiment

Here, use the half fraction of 2^n factorial designs to construct binary, equi-replicated variance balanced designs with smaller block sizes.

Theorem 1: The half fraction of 2^n factorial designs always gives binary, equi-replicated variance balanced designs of unequal block sizes with smaller block sizes having parameters $v = n,\ b = 2^{n-1} - 1,\ r = 2^{n-2},\ k = \{2,\ldots,2;\ 4,\ldots,4;\ldots,p\}$, where $p = n$ if n is an even number; otherwise, $p = n - 1$.

Proof: Construct a half fraction of 2^n factorial design in blocks of sizes 2^{n-1}. That is, n factors or columns and 2^{n-1} rows. Consider the n factors as n treatments and the 2^{n-1} rows as blocks. Delete the first row having all the factors at zero levels. Consider this as an incidence matrix of an incomplete block design having n rows as treatments and $2^{n-1} - 1$ columns as blocks. Thus, $v = n, b = 2^{n-1} - 1$. Here, all the elements are either zero or one so the design is binary. Again, each row contains '1' 2^{n-2} times so $r = 2^{n-2}$. Similarly, each block contains '1' either 2, 4,…, n times or 2, 4,…, $n - 1$ times depending on n being even or odd. So $k = \{2,\ldots,2\ ;\ 4,\ldots,4;\ldots;\ n$ or $n - 1\}$.

The incidence matrix of the incomplete block design is

$$\begin{matrix} 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & 1 & \dots & 1 \end{matrix}$$

If it can be proven the incidence matrix satisfies (1) for

$$\mathbf{C} = \text{diag}(r_1, r_2, \ldots, r_v) - \mathbf{NK}^{-1}\mathbf{N}' \tag{3}$$

then the design will be variance balanced. Because each row of the incidence matrix has (half – 1) ‘0’ and half ‘1’, the diagonal and off-diagonal elements of $\mathbf{NK}^{-1}\mathbf{N}'$ are given as follows:

(a) Diagonal elements:

(i) $\dfrac{\binom{n-2}{0}}{2} + \dfrac{\binom{n-2}{1}}{2} + \dfrac{\binom{n-2}{2}}{4} + \dfrac{\binom{n-2}{3}}{4} + \ldots + \dfrac{1}{n}$ provided n is even,

(ii) $\dfrac{\binom{n-2}{0}}{2} + \dfrac{\binom{n-2}{1}}{2} + \dfrac{\binom{n-2}{2}}{4} + \dfrac{\binom{n-2}{3}}{4} + \ldots + \dfrac{1}{n-1}$ provided n is odd,

(b) Off-diagonal elements:

(i) $\dfrac{\binom{n-3}{0}}{2} + \dfrac{\binom{n-3}{1}}{4} + \dfrac{\binom{n-3}{2}}{4} + \dfrac{\binom{n-3}{3}}{6} + \ldots + \dfrac{1}{n}$ provided n is even,

(ii) $\dfrac{\binom{n-3}{0}}{2} + \dfrac{\binom{n-3}{1}}{4} + \dfrac{\binom{n-3}{2}}{4} + \dfrac{\binom{n-3}{3}}{6} + \ldots + \dfrac{1}{n-1}$ provided n is odd.

Using (3), we have the **C**-matrix of the incomplete block design as

$$\mathbf{C} = \begin{bmatrix} 2^{n-2} & 0 & \dots & 0 \\ 0 & 2^{n-2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 2^{n-2} \end{bmatrix} - \begin{bmatrix} A & B & \dots & B \\ B & A & \dots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \dots & A \end{bmatrix} \tag{4}$$

where

$$A = \frac{\binom{n-2}{0}}{2} + \frac{\binom{n-2}{1}}{2} + \frac{\binom{n-2}{2}}{4} + \frac{\binom{n-2}{3}}{4} + \dots + \frac{1}{p}$$

$$B = \frac{\binom{n-3}{0}}{2} + \frac{\binom{n-3}{1}}{4} + \frac{\binom{n-3}{2}}{4} + \frac{\binom{n-3}{3}}{6} + \dots + \frac{1}{p}$$

$p = n$ if n is even; otherwise, $p = n - 1$.

After solving (4),

$$\mathbf{C} = \begin{bmatrix} 2^{n-2} - A & -B & \dots & -B \\ -B & 2^{n-2} - A & \dots & -B \\ \vdots & \vdots & \ddots & \vdots \\ -B & -B & \dots & 2^{n-2} - A \end{bmatrix}$$

C can be rewritten in the form

$$\mathbf{C} = \theta\left[\mathbf{I}_v - \frac{1}{v}\mathbf{E}_{vv}\right] \tag{5}$$

where θ is the non-zero eigenvalue of the **C**-matrix of incomplete block design given by $\theta = (2^{n-2} - A + B)$. This satisfies the criteria of variance balanced design. Hence the design is variance balanced design with the required parameters.

Example 1: Let $n = 5$. Using method 1, the incidence matrix of the variance balanced design is given as

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Now, from the above incidence matrix and Theorem 1, it can be verified that the parameters of the VB design are $v = n = 5$, $b = 2^{n-1} - 1 = 15$, $r = 2^{n-2} = 8$, and $k = \{2,\ldots, 2; 4,\ldots, 4,\ldots, 4\}$, as n is an odd number.

Here,

$$A = \frac{\binom{3}{0}}{2} + \frac{\binom{3}{1}}{2} + \frac{\binom{3}{2}}{4} + \frac{1}{4} = 1 + \frac{3}{2} + \frac{3}{4} + \frac{1}{4} = \frac{12}{4}$$

$$B = \frac{1}{2} + \frac{2}{4} + \frac{1}{4} = \frac{5}{4}$$

so

$$\mathbf{NK}^{-1}\mathbf{N}' = \left(\frac{1}{4}\right) \begin{bmatrix} 12 & 5 & 5 & 5 & 5 \\ 5 & 12 & 5 & 5 & 5 \\ 5 & 5 & 12 & 5 & 5 \\ 5 & 5 & 5 & 12 & 5 \\ 5 & 5 & 5 & 5 & 12 \end{bmatrix}$$

Using (3), the **C**-matrix of the variance balanced design is computed as

$$\mathbf{C} = \left(\frac{1}{4}\right) \begin{bmatrix} 12 & -5 & -5 & -5 & -5 \\ -5 & 12 & -5 & -5 & -5 \\ -5 & -5 & 12 & -5 & -5 \\ -5 & -5 & -5 & 12 & -5 \\ -5 & -5 & -5 & -5 & 12 \end{bmatrix}$$

$$\mathbf{C} = \frac{25}{4}\left[\mathbf{I}_5 - \frac{1}{4}\mathbf{E}_{55}\right]$$

Therefore, the design illustrated in Example 1 is a variance balanced design with nonzero eigenvalue $\theta = 25/4$ with multiplicity four. The variance of the treatment effects t_i and t_j is computed as $\mathrm{V}\left(\hat{t}_i - \hat{t}_j\right) = \left(8/25\right)\sigma^2$.

Construction of Variance Balanced Designs by Deleting the Last Row of the Half Fraction of the 2^n Factorial Design

It was shown that the variance balanced design can be constructed by deleting the last row of the half fraction of the 2^n factorial designs discussed in the previous section, provided n is even and greater than 4. This is because, with $n = 4$, it gives a balanced incomplete block design with parameters $v = 4$, $b = 6$, $r = 3$, $k = 2$, and $\lambda = 1$. This design is currently known in the literature.

Theorem 2: Deleting the last row of the half fraction of the 2^n factorial designs, a variance balanced design can always be constructed provided that n is an even number and greater than four, with parameters $v = n$, $b = 2^{n-1} - 2$, $\mathrm{r} = 2^{n-2} - 1$, $\mathrm{k} = \{2,\ldots, 2; 4,\ldots, 4;\ldots, \mathrm{n} - 2\}$.

Proof: Construct a half fraction of the 2^n factorial designs when n is an even number and greater than four. This design contains n factors or columns and 2^{n-1} runs or rows. Delete the control treatment and last row which has '1' for all its elements. After deletion, we have $2^{n-1} - 2$ rows. Consider the n columns as treatments and $2^{n-1} - 2$ rows as blocks which form an incidence matrix. Call this an incidence matrix **N** of an incomplete block design d with n treatments arranged in $2^{n-1} - 2$ blocks. Using the previous section, we can easily see that, for this incomplete block design, $v = n$, $\mathrm{b} = 2^{n-1} - 2$, $r = 2^{n-2} - 1$, $k = \{2,\ldots, 2; 4,\ldots, 4;\ldots, n - 2\}$.

The design will be variance balanced if it can be proven the incidence matrix satisfies

$$\mathbf{C} = \theta\left[\mathbf{I}_v - \frac{1}{v}\mathbf{E}_{vv}\right] \qquad (6)$$

To prove this, derive the diagonal and off diagonal elements of $\mathbf{NK}^{-1}\mathbf{N}'$.

Diagonal elements:

$$\frac{\binom{n-2}{0}}{2}+\frac{\binom{n-2}{1}}{2}+\frac{\binom{n-2}{2}}{4}+\frac{\binom{n-2}{3}}{4}+\ldots+\frac{\binom{n-2}{n-3}}{n-2}$$

Off-diagonal elements:

$$\frac{\binom{n-3}{0}}{2}+\frac{\binom{n-3}{1}}{4}+\frac{\binom{n-3}{2}}{4}+\frac{\binom{n-3}{3}}{6}+\ldots+\frac{\binom{n-3}{n-4}}{n-2}$$

provided n is an even and greater than four. Using (3), we have that the **C**-matrix of the incomplete block design is

$$\mathbf{C}=\begin{bmatrix} 2^{n-2}-1 & 0 & \ldots & 0 \\ 0 & 2^{n-2}-1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 2^{n-2}-1 \end{bmatrix}-\begin{bmatrix} A & B & \ldots & B \\ B & A & \ldots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \ldots & A \end{bmatrix} \tag{7}$$

where

$$A=\frac{\binom{n-2}{0}}{2}+\frac{\binom{n-2}{1}}{2}+\frac{\binom{n-2}{2}}{4}+\frac{\binom{n-2}{3}}{4}+\ldots+\frac{\binom{n-2}{n-3}}{n-2}$$

$$B=\frac{\binom{n-3}{0}}{2}+\frac{\binom{n-3}{1}}{4}+\frac{\binom{n-3}{2}}{4}+\frac{\binom{n-3}{3}}{6}+\ldots+\frac{\binom{n-3}{n-4}}{n-2}$$

provided n is an even and greater than four. After solving (7),

$$\mathbf{C}=\begin{bmatrix} 2^{n-2}-1-A & -B & \ldots & -B \\ -B & 2^{n-2}-1-A & \ldots & -B \\ \vdots & \vdots & \ddots & \vdots \\ -B & -B & \ldots & 2^{n-2}-1-A \end{bmatrix}$$

and hence **C** can be rewritten as

$$\mathbf{C} = \theta\left[\mathbf{I}_v - \frac{1}{v}\mathbf{E}_{vv}\right] \tag{8}$$

where θ is the non-zero eigenvalue of the **C**-matrix of the incomplete block design given by $\theta = (2^{n-2} - 1 - A + B)$. This satisfies the criteria of variance balanced design. Hence, the design is variance balanced design with the required parameters.

Example 2: Let $n = 6$. Using Theorem 2, the incidence matrix of the variance balanced design is given as

$$\mathbf{N} = \begin{bmatrix} 1&0&0&0&0&1&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&1&1&1&1&1&1&1&0 \\ 0&1&0&0&0&1&0&0&0&1&1&1&0&0&0&1&1&1&1&1&1&0&0&0&0&1&1&1&0&1 \\ 0&0&1&0&0&0&1&0&0&1&0&0&1&1&0&1&0&0&1&1&0&1&1&1&0&1&1&0&1&1 \\ 0&0&0&1&0&0&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&1&1&0&1&1&0&1&1&1 \\ 0&0&0&0&1&0&0&0&1&0&0&1&0&1&1&0&0&1&0&1&1&1&0&1&1&0&1&1&1&1 \\ 1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0 \end{bmatrix}$$

From above incidence matrix and Theorem 2, it can be verified the parameters of the VB design are $v = n = 6$, $b = 2^{n-1} - 2 = 30$, $r = 2^{n-2} - 1 = 15$, $k = \{2,\ldots,2;\ 4,\ldots,4,\ \ldots,4\}$, as n is even and greater than 4.

$$A = \frac{\binom{4}{0}}{2} + \frac{\binom{4}{1}}{2} + \frac{\binom{4}{2}}{4} + \frac{\binom{4}{3}}{4} = \frac{20}{4}$$

$$B = \frac{\binom{3}{0}}{2} + \frac{\binom{3}{1}}{4} + \frac{\binom{3}{2}}{4} = \frac{8}{4}$$

so

$$\mathbf{NK}^{-1}\mathbf{N}' = \left(\frac{1}{4}\right)\begin{bmatrix} 20 & 8 & 8 & 8 & 8 & 8 \\ 8 & 20 & 8 & 8 & 8 & 8 \\ 8 & 8 & 20 & 8 & 8 & 8 \\ 8 & 8 & 8 & 20 & 8 & 8 \\ 8 & 8 & 8 & 8 & 20 & 8 \\ 8 & 8 & 8 & 8 & 8 & 20 \end{bmatrix}$$

Using (3), the **C**-matrix of the variance balanced design is computed as

$$\mathbf{C} = \left(\frac{1}{4}\right)\begin{bmatrix} 40 & -8 & -8 & -8 & -8 & -8 \\ -8 & 40 & -8 & -8 & -8 & -8 \\ -8 & -8 & 40 & -8 & -8 & -8 \\ -8 & -8 & -8 & 40 & -8 & -8 \\ -8 & -8 & -8 & -8 & 40 & -8 \\ -8 & -8 & -8 & -8 & -8 & 40 \end{bmatrix}$$

$$\mathbf{C} = \frac{48}{4}\left[\mathbf{I}_6 - \frac{1}{6}\mathbf{E}_{66}\right]$$

Thus, the design illustrated in Example 2 is a variance balanced design with non-zero eigenvalue $\theta = 48/4$ with multiplicity five. The variance of the treatment effects t_i and t_j is computed as $\mathrm{V}\left(\hat{t}_i - \hat{t}_j\right) = \left(8/48\right)\sigma^2$.

Efficiency Factor

The efficiency factor of variance-balanced designs compare to completely randomized design is given by

$$E = \frac{\mathrm{V}\left(\hat{t}_i - \hat{t}_j\right)_{\mathrm{RBD}}}{\mathrm{V}\left(\hat{t}_i - \hat{t}_j\right)_{\mathrm{VB}}}$$

The efficiency factor will be derived separately from equations (5) and (8):

From equation (5), $\theta = (2^{n-2} - A + B)$ so $\hat{t}_i = \left(1/\theta\right)Q_i$ and $\mathrm{V}\left(\hat{t}_i - \hat{t}_j\right) = \left(2/\theta\right)\sigma^2$. That is

$$\mathrm{V}\left(\hat{t}_i - \hat{t}_j\right)_{\mathrm{VB}} = \frac{2}{\left(2^{n-2} - A + B\right)}\sigma^2$$

$$\mathrm{V}\left(\hat{t}_i - \hat{t}_j\right)_{\mathrm{RBD}} = \frac{2}{r}\sigma^2$$

where r is the replication size of the randomized block designs. Hence the efficiency factor E is given by

$$E = \frac{2^{n-2} - A + B}{r}$$

The efficiency factor for Example 1 can be computed as

$$E = \frac{\frac{2}{8}\sigma^2}{\frac{8}{25}\sigma^2} = \frac{25}{32} = 0.78$$

The efficiency of the variance balanced design compared with the randomized block design is 78 percent.

From equation (8) we have $\theta = (2^{n-2} - 1 - A + B)$, so $\hat{t}_i = \left(1/\theta\right)Q_i$ and $\mathrm{V}\left(\hat{t}_i - \hat{t}_j\right) = \left(2/\theta\right)\sigma^2$. That is

$$\mathrm{V}\left(\hat{t}_i - \hat{t}_j\right)_{\mathrm{VB}} = \frac{2}{\left(2^{n-2} - 1 - A + B\right)}\sigma^2$$

$$\mathrm{V}\left(\hat{t}_i - \hat{t}_j\right)_{\mathrm{RBD}} = \frac{2}{r}\sigma^2$$

where r is the replication size of the randomized block designs. The efficiency factor E is given by

$$E = \frac{2^{n-2} - 1 - A + B}{r}$$

Similarly, the efficiency factor for Example 2 can be computed as

$$E = \frac{\frac{2}{15}\sigma^2}{\frac{8}{48}\sigma^2} = \frac{4}{5} = 0.8$$

The efficiency of the variance balanced design compared to the randomized block design is 80 percent.

Consider the parameters and efficiency factor of the variance balanced designs with the usual blocks and deletion of last block respectively in Table 1 and Table 2. The range of *n* from 3 to 10 was selected for convenience.

Remarks:

(i). From Table 1 and Table 2, it is obvious that efficiency factor increases as number of treatment increases.

(ii). If the efficiency factor of VB designs shown in Table 1 and Table 2 are compared, note the efficiency factor decreases when a block is deleted from a given VB design with the same number of treatment.

Table 1. The parameters and efficiency factor of the VB designs

SN	*n* = *v*	*b*	*r*	*k*	Efficiency Factor
1	4	7	4	{2,…, 2; 4}	0.7500
2	5	15	8	{2,…, 2; 4,…, 4; 4}	0.7800
3	6	31	16	{2,…, 2; 4,…, 4; 4,…, 4; 6}	0.8125
4	7	63	32	{2,…, 2; 4,…, 4; 4,…, 4; 6,…, 6; 6}	0.8300
5	8	127	64	{2,…, 2; 4,…, 4; 4,…, 4; 6,…, 6; 8}	0.8600
6	9	256	128	{2,…, 2; 4,…, 4; 4,…, 4; 6,…, 6; 8,…, 8, 8}	0.8814
7	10	512	256	{2,…, 2; 4,…, 4; 4,…, 4; 6,…, 6; 8,…, 8; 10}	0.9023

Table 2. The parameters and efficiency factor of the VB designs with deletion of last block

SN	*n* = *v*	*b*	*r*	*k*	Efficiency Factor
1	4	6	3	{2,…, 2}	0.6600
2	6	30	15	{2,…, 2; 4,…, 4}	0.8000
3	8	126	63	{2,…, 2; 4,…, 4; 4,…, 4; 6,…, 6}	0.8441
4	10	511	255	{2,…, 2; 4,…, 4; 4,…, 4; 6,…, 6; 8,…, 8}	0.9022

Acknowledgements

The first author is thankful to UGC for providing him UGC BSR Faculty Fellow. We are thankful to the referee for his valuable suggestions so as to bring this articles in better form.

References

Agarwal, G. G., & Kumar, S. (1985). A note on construction of variance balanced designs. *Journal of the Indian Society of Agricultural Statistics, 37*(2), 181-183.

Calvin, J. A. (1986). A new class of variance balanced designs. *Journal of Statistical Planning and Inference, 14*(2-3), 251-254. doi: 10.1016/0378-3758(86)90162-x

Calvin, J. A., & Sinha, K. (1989). A method of constructing variance balanced designs. *Journal of Statistical Planning and Inference*, *23*(1), 127-131.

Chakrabarti, M. C. (1963). On the C-matrix in design of experiments. *Journal of the Indian Statistical Association, 1*, 8-23.

Das, M. N., & Ghosh, D. K. (1985). Balancing incomplete block designs. *Sankhyā: The Indian Journal of Statistics, Series B, 47*(1), 67-77. Available from http://www.jstor.org/stable/25052375

Ghosh, D. K. (1988). A series of generalized efficiency balanced designs. *Gujarat Statistical Review, 15*(1), 33-38.

Ghosh, D. K., Divecha, J., & Kageyama, S. (1991). Equireplicate variance balanced designs from group divisible designs. *Journal of the Japan Statistical Society, 21*(1), 205-209.

Ghosh, D.K., & Joshi, K. (1991). Note on construction of variance balanced designs through group divisible designs. *Utilitas Mathematica, 39*, 249-253.

Ghosh, D. K., & Joshi, K. (1995). Construction of variance balanced designs through triangular PBIB designs. *Calcutta Statistical Association Bulletin*, *45*(1-2), 111-118. doi: 10.1177/0008068319950107

Ghosh, D. K., Joshi, K., & Kageyama, S. (1993). Ternary variance balanced designs through BIB and GD designs. *Journal of the Japan Statistical Society*, *23*(1), 75-81.

Ghosh, D. K., & Karmoker, P. K. (1988). Some series of efficiency balanced designs. *Australian & New Zealand Journal of Statistics, 30*(1), 47-51. doi: 10.1111/j.1467-842x.1988.tb00610.x

Gupta, S. C., & Jones, B. (1983). Equireplicate balanced block designs with unequal block sizes. *Biometrika*, *70*(2), 443-440. doi: 10.2307/2335557

Hedayat, A., & Federer, W. T. (1974). Pairwise and variance balanced incomplete block design. *Annals of the Institute of Statistical Mathematics*, *26*(1), 331-338. doi: 10.1007/bf02479828

Kageyama, S. (1988). Existence of variance-balanced binary designs with fewer experimental units. *Statistics & Probability Letters*, *7*(1), 27-28. doi: 10.1016/0167-7152(88)90083-1

Khatri, C. G. (1982). A note on variance balanced designs. *Journal of Statistical Planning and Inference*, *6*(2), 173-177. doi: 10.1016/0378-3758(82)90086-6

Mukerjee, R., & Kageyama, S. (1985). On resolvable and affine resolvable variance-balanced designs. *Biometrika*, *72*(1), 165-172. doi: 10.2307/2336346

Pearce, S. C. (1964). Experimenting with blocks of natural size. *Biometrics*, *20*(4), 699-706. doi: 10.2307/2528123

Puri, P. D., & Nigam, A. K. (1977). Balanced block designs. *Communications in Statistics – Theory and Methods*, *6*(12), 1171-1179. doi: 10.1080/03610927708827560

Raghavarao, D. (1962). On balanced unequal block designs. *Biometrika*, *49*(3/4), 561-562. doi: 10.2307/2333990

Rao, V. R. (1958). A note on balanced designs. *The Annals of Mathematical Statistics*, *29*(1), 290-294. doi: 10.1214/aoms/1177706729