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Periodicity In Iterated Algebraic K-Theory Of Finite Fields

Gabriel Angelini-Knoll
Wayne State University,

PERIODICITY IN ITERATED ALGEBRAIC K-THEORY OF FINITE FIELDS

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GABRIEL J. ANGELINI-KNOLL

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To my partner, Jess

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CHAPTER 1 INTRODUCTION

In the early 2000's, C. Ausoni and J. Rognes pioneered the study of the arithmetic of ring spectra by making the first computations of iterated algebraic K-theory. In [9] and later [8], C. Ausoni and J. Rognes computed $V(1)_*K(K(\cup\mathbb{F}_{q^{p^k}})_p)$ and $V(1)_*K(K(\mathbb{C})_p)$ and they showed that they are finitely generated free $P(v_2)$-modules. Observe that since $ku_p \simeq K(\mathbb{C})_p$ and $\ell_p \simeq K(\cup\mathbb{F}_{q^{p^k}})_p$, these spectra detect all the powers of v_1 and C. Ausoni and J. Rognes showed $V(1)_*K(K(\cup\mathbb{F}_{q^{p^k}})_p)$ and $V(1)_*K(K(\mathbb{C})_p)$ detect all the powers of v_2. This gives evidence for the red-shift conjecture that states, roughly, that applying algebraic K-theory increases chromatic complexity by one.

The goal of this thesis is to continue the study of arithmetic of ring spectra in the case of iterated algebraic K-theory of finite fields. Since due to J. F. Adams and D. Quillen, the spectrum $K(\mathbb{F}_q)_p$ detects the α family, one might hope that a Greek letter family one chromatic height higher is detected in $V(1)_*K(K(\mathbb{F}_q)_p)$. We will prove that, in fact this is the case.

Theorem 1.1. The v_2-periodic family generated by β_1 in $V(1)_*$ is detected in $V(1)_*K(K(\mathbb{F}_q)_p)$ where $p \geq 5$ and q is a prime power that topologically generates $\mathbb{Z}_p^\times$. Consequently, the elements β_{pk+1} in the homotopy groups of spheres are detected in $K(K(\mathbb{F}_q))$.

To compute algebraic K-theory of a commutative ring spectrum R, we take the approach of Bökstedt-Hsiang-Madsen [18] and approximate it using the highly non-trivial Bökstedt trace map to topological Hochschild homology

$$K(R) \longrightarrow THH(R)$$

where $THH(R) = S^1 \otimes R$ is the colimit in commutative ring spectra weighted by the simplicial circle [45]. Topological Hochschild homology is a linear approximation to the algebraic K-theory functor in the sense of Goodwillie calculus. In Chapter 3, we describe the results of joint work with Andrew Salch. we provide a tool for computing $X_\bullet \otimes R$ for general simplicial sets $X_\bullet$. In Chapter 4, we compute $V(1)_*THH(K(\mathbb{F}_q)_p)$ using this spectral sequence and give initial results towards $S/p_*THH(K(\mathbb{F}_q)_p)$:

Theorem 1.2. Let $p \geq 3$ and let q be a prime power that topologically generates $\mathbb{Z}_p^\times$. There is an isomorphism

$$V(1)_*THH(K(\mathbb{F}_q)_p) \cong P(\mu_2) \otimes \Gamma(\sigma b) \otimes \mathbb{F}_p\{1, \alpha_1, \lambda_1', \alpha_1\lambda_2, \lambda_1'\lambda_2, \alpha_1\lambda_1'\lambda_2\},$$

and there is an isomorphism

$$S/p_*THH(K(\mathbb{F}_q)_p; \ell) \cong P(v_1) \otimes \Gamma(\sigma b) \otimes \mathbb{F}_p\{1, y_{n,m}, y_{n,m}'\}/\sim$$

where the relations $\sim$ and the elements $y_{n,m}$ are explicitly defined in Chapter 4.

The next step in the approach of Bökstedt-Hsiang-Madsen, colloquially referred to as “trace methods," is to compute successive refinements of topological Hochschild homology using the extra structure that it has. In particular, it has an S^1-action by acting on the first coordinate and it is cyclotomic, which provides maps

$$THH(R)^{C_{p^n}} \underset{R}{\overset{F}{\rightrightarrows}} THH(R)^{C_{p^{n-1}}}$$

referred to as the Frobenius and Restriction maps. The homotopy limit of these maps is a model for topological cyclic homology after p completion

$$TC(R)_p \simeq \left(\mathrm{holim}_{F,R}\, THH(R)^{C_{p^n}}\right)_p .$$

As a consequence of work of B. Dundas [26], R. McCarthy [42], L. Hesselholt and I. Madsen [32], if R is connective and $\pi_0 R \cong \mathbb{Z}_p$, then there is an equivalence

$$K(R)_p \xrightarrow{\simeq} \tau_{\geq 0} TC(R)_p$$

where $\tau_{\geq 0}$ is the connective cover functor. (For more details on trace methods see Chapter 2.)

The ultimate goal is therefore to compute $V(1)_* TC(K(\mathbb{F}_q)_p)$, but this is beyond the scope of the present thesis. Instead, we compute enough of $V(1)_* THH(R)^{hS^1}$ to show that already in this approximation to $V(1)_* K(K(\mathbb{F}_q)_p)$ periodic classes of height 2 are visible. Specifically, in Chapter 5, we show that the v_2-periodic family generated by β_1 in $V(1)_*$ is detected in $V(1)_* THH(K(\mathbb{F}_q)_p)^{hS^1}$.

CHAPTER 2 K-THEORY AND CHROMATIC HOMOTOPY THEORY

This chapter provides the necessary definitions and framework for the thesis. It does not contain new results and the experienced reader may skip it, except for the last section, which contains conventions that will be used in the subsequent chapters.

2.1 Algebraic K-theory

There are many different models for algebraic K-theory of strictly associative ring spectra. Since the goal of this thesis to to make computations and not to prove a structural result about algebraic K-theory, we are not concerned with a specific model. We simply provide one here for completeness.

Definition 2.1. A Waldhausen category $\mathcal{C}$ is a category equipped with subcategories cof$\mathcal{C}$ and w$\mathcal{C}$ of cofibrations and weak equivalences satisfying some axioms described explicitly in [57]. In particular, we have a notion of cofiber sequence

$$A \hookrightarrow B \to B/A$$

where $A \hookrightarrow B$ denotes an arrow in cof$\mathcal{C}$.

Example 2.2. If $\mathcal{M}$ is a model category, then the subcategory $\mathcal{M}^{cof}$ of cofibrant objects in $\mathcal{M}$ forms a Waldhausen category by forgetting structure.

Definition 2.3. Let $\mathcal{C}$ be a small Waldhausen category, then form a simplicial Waldhausen category $S_\bullet\mathcal{C}$, whose n-th category $S_n\mathcal{C}$ consists of objects

$$A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_n$$

with choices of compatible quotients A_i/A_{i+1} where $A_i \hookrightarrow A_{i+1}$ is a morphism in cof$\mathcal{C}$ for all i. The morphisms of $S_n\mathcal{C}$ are commuting diagrams. The face maps d_i in $S_\bullet\mathcal{C}$ are given by omitting A_i and replacing the map with a composite and the degeneracy maps s_i are given by inserting the identity in the i-th position. We then define the zero-th space of the algebraic K-theory spectrum as

$$K(\mathcal{C})_0 := \Omega|wS_\bullet\mathcal{C}|$$

and the Ω-spectrum $K(\mathcal{C})$ is the sequence

$$\{\Omega|wS_\bullet\mathcal{C}|, |wS_\bullet\mathcal{C}|, |wS_\bullet^{(2)}\mathcal{C}|, |wS_\bullet^{(3)}\mathcal{C}|, \dots\}$$

where $S_\bullet^{(n)}\mathcal{C}$ is the n-th iterate of the $S_\bullet$ construction $S_\bullet(\dots(S_\bullet\mathcal{C}))$.

Example 2.4. If R is a commutative ring spectrum (or more generally a strictly associative ring spectrum) let $fc\mathrm{Mod}_R$ be the category of finite cell (left) R-modules and cellular maps. We define

$$K(R) := K(fc\mathrm{Mod}_R^{cof})$$

In particular, due to Gillet-Waldhausen [58, Chpt. V. Thm. 2.2] and Elmendorf-Kriz-Mandell-May [28, Thm. 4.3], we can identify $K(HA)$ where A is a ring and H is the Eilenberg-Maclane functor with Quillen's definition of $K(A)$.

2.2 Algebraic K-theory of finite fields and Waldhausen's program

Let p be a prime such that $p \geq 3$ and let q be a prime power that topologically generates $\mathbb{Z}_p^\times$, which denotes the units in the p-adic integers. Under these conditions, we claim that there are equivalences

$$j_p \simeq K(\mathbb{F}_q)_p \simeq \tau_{\geq 0} L_{K(1)} S$$

where $\tau_{\geq 0}$ indicates the connective cover functor; hence, these spectra are all different models for the same commutative ring spectrum. Due to Quillen [50], there is a fiber sequence

$$K(\mathbb{F}_q)_p \longrightarrow K(\overline{\mathbb{F}}_q)_p \xrightarrow{F-1} K(\overline{\mathbb{F}}_q)_p^{\geq 2}$$

where $R^{\geq 2}$ is the fiber of the map $R \to H\pi_0 R$ for a connective ring spectrum R and F is the map induced by the Frobenius map on $\overline{\mathbb{F}}_q$. Quillen [50] showed that this fiber sequence is homotopy equivalent to the fiber sequence

$$j_p \longrightarrow ku_p \xrightarrow{\psi^q - 1} ku_p^{\geq 2}$$

of Adams where ku is connective complex K-theory and j_p is the odd primary p-completion of the image of J spectrum. There is also a map of fiber sequences

$$\begin{tikzcd}
j_p \arrow[r] \arrow[d] & ku_p \arrow[r, "\psi^q - 1"] \arrow[d] & ku_p[2, \infty) \arrow[d] \\
L_{K(1)} S \arrow[r] & KU_p \arrow[r, "\psi^q - 1"] & KU_p
\end{tikzcd}$$

which exhibits j_p, and hence $K(\mathbb{F}_q)_p$, as the connective cover of $L_{K(1)}S$. The bottom fiber sequence is due to Devinatz-Hopkins [23]. It follows because when q topologically generates $\mathbb{Z}_p^\times \cong \mathbb{G}_1$, the fiber of $\psi^q - 1$ is a model for the homotopy fixed points $KU_p^{h\mathbb{G}_1}$, which Devinatz-Hopkins [23] showed is weakly equivalent to $L_{K(1)}S$.

This means that our computations of approximations to $K(K(\mathbb{F}_q)_p)$ are also approximations to $K(\tau_{\geq 0}L_{K(1)}S) \simeq K((\tau_{\geq 0}L_{E(1)}S)_p)$. This algebraic K-theory of the connective covers of the $E(n)$-localizations of the sphere fit into a tower

$$K(S_{(p)}) \to \ldots K(\tau_{\geq 0}L_{E(2)}S) \to K(\tau_{\geq 0}L_{E(1)}S) \to K(\mathbb{Z}_{(p)}),$$

which McClure-Staffledt [44] proved “converges" in the sense that

$$K(S_{(p)}) \simeq \mathrm{holim} K(\tau_{\geq 0}L_{E(n)}S)).$$

Waldhausen first suggested studying this tower and it’s non-connective version

$$K(S_{(p)}) \to \ldots K(L_{E(2)}S) \to K(L_{E(1)}S) \to K(\mathbb{Q}).$$

as an approach to computing $K(S_{(p)})$ [56]. Waldhausen’s idea was to study the localization sequences

$$K(\{\text{finite } E(n)\text{-acyclic } \tau_{\geq 0}L_{E(n)}S-\text{modules}\}) \to K(\tau_{\geq 0}L_{E(n)}S) \to K(L_{E(n)}S)$$

and

$$K(\{\text{finite } E(n)\text{-acyclic } L_{E(n+1)}S-\text{modules}\}) \to K(L_{E(n+1)}S) \to K(L_{E(n)}S)$$

to build up the tower, but there are two problems with this: 1) For his localization sequences, Waldhausen assumed the telescope conjecture $L_{E(n)} = L^f_{E(n)}$ where $L^f_{E(n)}$ is the finite localization, which is now widely believed to be false for $n > 1$, and 2) the fibers $K(\{\text{finite } E_n\text{-acyclic } \tau_{\geq 0}L_{E(n)}S - \text{modules}\})$ are not known to be $K(R)$ for some connective ring spectrum R, so they are not approachable using trace methods. In spite of these shortcomings, we believe that Waldhausen's program is interesting to study in its own right and it may still shed light on algebraic K-theory of the sphere spectrum.

2.3 Chromatic height in stable homotopy theory

Recall that a finite spectrum (finite cell S-module) V is said to have *type n* if $K(n)_*V \neq 0$, but $K(n-1)_*V = 0$, where $K(n)_*$ are cohomology theories, called Morava K-theory theories, with coefficients $K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}]$ where $|v_n| = 2p^n - 2$. By the influential work of Devinatz-Hopkins-Smith [24] and Hopkins-Smith [33] these Morava K-theories can be used to detect a vast amount of information about the category $\mathcal{F}$ of finite p-local spectra.

Theorem 2.5 (Devinatz-Hopkins-Smith [24], Hopkins-Smith [33])**.** We summarize three major theorems of the authors:

- The thick subcategories of $\mathcal{F}$ are classified by the filtration

  $$0 \subset \cdots \subset \mathcal{C}_n \subset \ldots \mathcal{C}_1 \subset \mathcal{C}_0 = \mathcal{F}$$

  where $\mathcal{C}_n$ is the full subcategory of $K(n-1)_*$ acyclic finite p-local spectra; i.e. every thick subcategory of $\mathcal{F}$ is of the form $\mathcal{C}_n$ for some n.

- If $V \in \mathcal{C}_n$, then V admits a periodic v_n-self map

  $$v_n^k : \Sigma^{(2p^n-2)k}V \to V;$$

i.e. no composite

$$(v_n^k)^{\circ n} : \Sigma^{(2p^n-2)kn} V \to \Sigma^{(2p^n-2)k(n-1)} V \to \cdots \to V$$

is null homotopic.

- A map $f : \Sigma^d W \to W$ is nilpotent, some composite of it with itself is null-homotopic, if and only if $K(n)_* f$ is nilpotent for all $0 \leq n < \infty$.

Since the notion of chromatic height referred to as *type* is used for *finite* spectra, we would like a notion of chromatic height that works well for spectra that are not finite spectra, and following Baas-Dundas-Rognes, we use the thick subcategory theorem of Devinatz-Hopkins-Smith to define this notion of height.

Definition 2.6 (Baas-Dundas-Rognes [11]). We say a spectrum X has *telescopic complexity* n if the thick subcategory $\mathcal{T}_X$ of $\mathcal{F}$, consisting of spectra V such that

$$V \wedge X \to v_n^{-1} V \wedge X$$

induces an isomorphism in homotopy groups π_k for k sufficiently large, is equal to $\mathcal{C}_n$.

2.4 Red-shift conjectures

Using the notion of telescopic complexity, we may describe two different versions of the red-shift conjecture.

Conjecture 2.7 (Ausoni-Rognes [1]). Suppose R is a (suitably finite) $K(n)$-local spectrum (for example $L_{K(n)} S \to R$ is a G-galois extension for some, possibly pro-finite, group G), then $K(R)$ has telescopic complexity $n + 1$.

We view this conjecture as an extension of the Lichtenbaum-Quillen conjecture to higher chromatic heights. In particular, the Lichtenbaum-Quillen conjecture may be phrased as the statement that for nice enough regular rings F with $\frac{1}{p} \in F$ the map

$$S/p_*K(F) \to v_1^{-1}S/p_*K(F)$$

induces an isomorphism in sufficiently high degrees (see Waldhausen [56]).

The version of the red-shift conjecture above, however, only takes non-connective spectra as input. Ausoni and Rognes were able to prove it in the case $R = KU_p$, p complete periodic complex K-theory, where $L_{K(1)}S \to KU_p$ is a $\mathbb{G}_1$-galois extension and $\mathbb{G}_1$ is the first Morava stabilizer group. They do this using the localization sequence of Blumberg-Mandell [16]

$$V(1)_*K(\mathbb{Z}_p) \to V(1)_*K(ku) \to V(1)_*K(KU).$$

Since they showed that $V(1)_*K(ku)$ is a finitely generated $P(v_2)$-module, and the, now proven, Lichtenbaum-Quillen conjecture implies that $K_*(\mathbb{Z}_p)$ has telescopic complexity 1, they can show that $K(KU)$ has the same telescopic complexity as $K(ku)$. This is account is a bit anachronistic since the Lichtenbaum-Quillen conjecture and the existence of such a localizations sequence were proven after the computation of Ausoni-Rognes. This approach does not work as well in the case of $K(K(\mathbb{F}_q)_p) \simeq K(\tau_{\geq 0}L_{K(1)}S)$, since the localization sequence associated to the map $K(\tau_{\geq 0}L_{K(1)}S) \to K(L_{K(1)}S)$ is not known to have algebraic K-theory of a connective spectrum as its fiber.

In the connective case, we may formulate the red-shift conjecture as follows.

Conjecture 2.8 (Barwick [12]). If R has telescopic complexity n, then $\overbrace{K(\dots K}^{\text{m}}(R))$ has telescopic complexity $n+m$.

Barwick described this form of the conjecture for $R = \mathbb{C}$ in a talk at MSRI in 2014 and it appears in [12]. The only cases where it is known for $R = \mathbb{C}$ are the cases where $n = 0$ and $m = 0, 1$, by work of Ausoni [8], since $K(\mathbb{C})_p \simeq ku_p$. The conjecture is known in the case $R = \mathbb{F}_q$ when $n = 0$, and $m = 1$ since $K(\mathbb{F}_q)_p \simeq j_p$, and the author is currently in studying the possible validity of the conjecture for $m = 2$ as well.

We propose a third version of this conjecture, which is in the same spirit of the red-shift conjecture.

Conjecture 2.9. If R detects a n-th Greek letter family, then $V_*K(R)$ detects a v_{n+1}-periodic family generated by the $n+1$-st Greek letter element $\alpha_1^{(n+1)}$ for some type $n+1$ spectrum V.

We will make this statement more precise in the next section. The reason we propose this version of the conjecture is two-fold:

1. The calculations in this thesis suggest that, in our main case of interest, $K(K(\mathbb{F}_q)_p)$ actually *does not* have telescopic complexity 2, even though $H\mathbb{F}_q$ certainly has telescopic complexity 0 and $K(\mathbb{F}_q)_p$ certainly has telescopic complexity 1. Unfortunately, this remains speculation at this time.

2. The spectrum $K(\mathbb{F}_q)$ detects the α-family, the first greek letter family, and the main theorem of this thesis is that $V(1)_*K(K(\mathbb{F}_q))$ detects the v_2-periodic family generated by β_1 in $V(1)_*$. To the author's knowledge, this is the first evidence for Conjecture 2.9 for $n = 1$.

2.5 Greek letter family elements

We will now discuss what is meant by the “n-th Greek letter family" and the “v_n-periodic family generated by the n-th Greek letter element $\alpha_1^{(n)}$" in the stable homotopy groups of spheres. At primes $p \geq 3$, the first Greek letter family is the α family, which Adams studied in [2]. To produce the α family, we note that at primes $p \geq 3$, the mod p Moore spectrum S/p has a periodic v_1-self map $\Sigma^{2p-2}S/p \to S/p$ and by mapping into the bottom cell, then composing this map with itself, and projecting onto the top cell, we may form α_k; i.e. the composite

$$\alpha_k : \Sigma^{(2p-2)k}S \xrightarrow{i_0} \Sigma^{(2p-2)k}S/p \xrightarrow{v_1^{\circ k}} S/p \xrightarrow{\delta_0} \Sigma S.$$

One may show that in fact this composite is not null-homotopic and therefore produces a family of elements in the stable homotopy groups of spheres $\alpha_k \in \pi_*S$. We say a ring spectrum R “detects the α family" if the classes α_k have non-trivial image in π_*R under the unit map $\pi_*S \to \pi_*R$.

Now let $p \geq 5$, then the cofiber of v_1, $V(1)$, admits a v_2-self map and we can construct the β-family

$$\beta_k : \Sigma^{(2p^2-2)k}S \xrightarrow{i_0} \Sigma^{(2p^2-2)k}S/p \xrightarrow{i_1} \Sigma^{(2p^2-2)k}V(1) \xrightarrow{v_2} \dots \xrightarrow{v_2} V(1) \xrightarrow{\delta_1} S/p \xrightarrow{\delta_0} S^0,$$

which was proven non-trivial by L. Smith [55]. By examining the long exact sequences in homotopy produced by the cofiber sequences

$$S_p \xrightarrow{p} S_p \xrightarrow{i_0} S/p$$

and

$$\Sigma^{2p-2}S/p \xrightarrow{v_1} S/p \xrightarrow{i_1} V(1)$$

we see that $\beta_1 \in \pi_{2p^2-2p-2}S$ maps non-trivially to $0 \neq i_0 i_1 \beta_1 \in \pi_{2p^2-2p-2}V(1)$. The class β_1 also maps non-trivially under the map $\pi_{2p^2-2p-2}S \to \pi_{2p^2-2p-2}L_{K(1)}V(1)$ to the class called $-g_1$ (See [48, Lem. 5.4] and [51]). Since $v_2 \in \pi_* V(1)$ maps to $v_2 \in L_{K(2)}V(1)$ and in $\pi_{2p^2-2p-2}L_{K(1)}V(1)$ the class $-g_1$ is v_2-periodic, the classes $\beta_1 v_2^k$ are non-trivial elements in $V(1)_*$. These are the classes that we refer to as a "v_2-periodic family generated by β_1."

One may want to know how the "v_2-periodic family generated by β_1" relates to the β-family itself. Due to Ravenel [51], we may define the Greek letter elements in the Adams-Novikov spectral sequence $\alpha_t^{(n)} \in E_2^{n,*}$ algebraically to be the elements $\delta_n \delta_{n-1} \ldots \delta_0(v_n^t)$ where

$$\delta_m : Ext_{BP_*BP}(BP_*, BP_*/(p, \ldots v_m)) \to Ext_{BP_*BP}(BP_*, BP_*/(p, \ldots v_{m-1})$$

One can show, by computing $\delta_0 \delta_1 v_2^t$ that β_t is represented by

$$\binom{t}{2} v_2^{t-2} k_0 + t v_2^{t-1} b_{1,0} \bmod (p, v_1)$$

in the Adams-Novikov spectral sequence where

$$b_{1,0} = \sum_{0<i<p} \frac{1}{p} \binom{p}{i} t_1^i \otimes t_1^{p-1}$$

and

$$k_0 = 2t_1^p \otimes t_2 - 2t_1^p \otimes t_1^{1+p} - t_1^{2p} \otimes t_1.$$

Notice that when $t \equiv 1 \bmod (p)$ then $\binom{t}{2} = \binom{1}{2}\binom{k_1}{0}\dots\binom{k_n}{0} = 0$ where $t = 1 + k_1 p + \dots k_n p^n$ by Lucas's theorem and therefore β_t is represented by $v_2^{t-1} b_{1,0}$ modulo p. Hence, when we detect the v_2-periodic family generated by β_1, we are detecting the spherical elements $\beta_t \in \pi_* S$ for $t \equiv 1 \bmod (p)$. The remaining classes project onto the top cell

$$\pi_* V(1) \to \pi_* \Sigma^{2p} S$$

to the classes $\beta_1 \beta_k$ where $k \not\equiv 0 \bmod (p)$.

2.6 Trace methods

The goal of this thesis is to give a close enough approximation of $K(K\mathbb{F}_q)_p)$ to detect periodic information of higher chromatic height. The approach we take was initiated by Bökstedt-Hsiang-Madsen [18] in the early 1990's. At that time, it was known that algebraic K-theory of rings was equipped with a non-trivial trace map

$$K_k(R) \to HH_k(R)$$

where $HH_k(R) = \pi_k |\mathrm{Bar}_{\otimes}^{\mathrm{cyc}}(R)|$ or equivalently the homology of the alternating sign chain complex of the simplicial ring $\mathrm{Bar}_{\otimes}^{\mathrm{cyc}}(R)$ by the Dold-Kan correspondence. The map is defined as a composite where the last map is the map induced by the trace map

$$HH_k(M_n(R)) \xrightarrow{\cong} HH_k(R)$$

which is an isomorphism since HH_* is Morita invariant. Bökstedt first constructed a version of this for ring spectra using functors with smash product (FSP's) to model spectra, since at the time there was not a good symmetric monoidal model category for spectra. There are now several models for symmetric monoidal model categories for spectra including S-modules [28], symmetric spectra [34], orthogonal spectra [39], and Γ-spaces [38] and these model categories have been unified in the sense that there are Quillen equivalences between all of them [40]. We can therefore define, for a strictly associative ring spectrum R

$$THH_k(R) = \pi_k|\mathrm{Bar}^{\mathrm{cyc}}_{\wedge}(R)|$$

When R is a commutative ring spectrum (a commutative monoid in a symmetric monoidal model category for spectra), we can describe this construction as a simplicial tensoring

$$THH_k(R) = \pi_k(S^1 \otimes R)$$

due to McClure-Schwanzl-Vogt [43] and this will be our approach in Chapter 3. From this description, it is clear S^1 acts on $S^1 \otimes R$ by acting on the first coordinate and hence C also acts on $S^1 \otimes R$ for any finite subgroup $C \subset S^1$. Using Bökstedt's model or the norm model for THH [5], we can also construct $THH(R)$ as a genuine C-equivariant spectrum for any finite subgroup C of S^1. The spectrum $THH(R)$ also has the structure of a cyclotomic spectrum, which means that there are compatible maps of S^1-spectra

$$\rho_C^{\#}\Phi^C(THH(R)) \to THH(R)$$

for each finite subgroup $C \subset S^1$, where $\phi^C(THH(R))$ has an S^1-action by pulling back along the isomorphism $\rho_C : S^1 \to S^1/C$. Together these properties allow us to construct the isotropy separation diagram, which is sometimes called the Norm-Restriction diagram for THH,

$$\begin{tikzcd} THH(R)_{hC_{p^n}} \arrow[r] \arrow[d] & THH(R)^{C_{p^n}} \arrow[r, "R"] \arrow[d] & THH(R)^{C_{p^{n-1}}} \arrow[d] \\ THH(R)_{hC_{p^n}} \arrow[r] & THH(R)^{hC_{p^n}} \arrow[r] & THH(R)^{tC_{p^n}} \end{tikzcd}$$

A key feature of this diagram is that the homotopy fixed points, homotopy orbits, and Tate fixed points can be computed using spectral sequences, so by computing $THH(R)$ you can, potentially, compute $THH(R)^{C_{p^n}}$ inductively.

Now if we write $F : THH(R)^{C_{p^n}} \to THH(R)^{C_{p^{n-1}}}$ for the inclusion of fixed points, then we can define

$$TF(R) = \mathrm{holim}_F\, THH(R)^{C_{p^n}}$$

and

$$TR(R) = \mathrm{holim}_R\, THH(R)^{C_{p^n}}.$$

we then define (p-typical) topological cyclic homology as

$$TC(R; p) = \mathrm{fib}\{\, TR(R) \xrightarrow{1-F} TR(R) \,\}$$

or equivalently

$$TC(R; p) = \mathrm{fib}\{\, TF(R) \xrightarrow{1-R} TF(R). \,\}$$

We can also define an integral version of topological cyclic homology TC and after p-

completion there is an equivalence

$$TC(R)_p \to TC(R;p)_p.$$

Bökstedt constructed a highly non-trivial trace map

$$K(R) \longrightarrow THH(R)$$

for strictly associative ring spectra and when $R = HA$ for a ring A, the Dennis trace map factors through Bökstedt's trace map. Bökstedt's trace map, in turn, factors through topological cyclic homology

$$\begin{tikzcd} K_k(R) \arrow[r] \arrow[dr] & TC_k(R) \arrow[d] \\ & THH_k(R). \end{tikzcd}$$

The benefit of the cyclic refinement of the trace map is that topological cyclic homology is a close approximation to algebraic K-theory.

Theorem 2.10 (Dundas-Goodwillie-McCarthy [27])**.** Let $f : R \to S$ be a map of connective strictly associative ring spectra such that $f : \pi_0 R \to \pi_0 S$ is surjective with nilpotent kernel, then the diagram

$$\begin{tikzcd} K(R) \arrow[r] \arrow[d] & K(S) \arrow[d] \\ TC(R) \arrow[r] & TC(S) \end{tikzcd}$$

is a homotopy pullback diagram.

Corollary 2.11. If $\mathbb{Z}_p$ surjects onto $\pi_0 R$ and R is a connective strictly associative ring spectrum, then

$$K(R)_p \simeq \tau_{\geq 0} TC(R)_p$$

This corollary depends on the Theorem 2.10 as well as computations of Hesselholt-Madsen [32].

We will need to use the multiplicativity of the cyclotomic trace map proven by Dundas.

Theorem 2.12 (Dundas [25])**.** The cyclotomic trace map

$$K(R) \to TC(R)$$

is a weak map of commutative ring spectra; i.e. it is a zigzag of commutative ring spectrum maps where each wrong way map is a weak equivalence of commutative ring spectra.

The approach we will take in this thesis is to compute the linear approximation to iterated algebraic K-theory of finite fields, topological Hochschild homology, and then compute enough of the homotopy fixed points of topological Hochschild homology to detect periodic information of higher chromatic height.

2.7 Organization

This thesis is organized into three main chapters of original research. Chapter 3 consists of the construction of a May-type spectral sequence in higher order topological Hochschild homology associated to filtered commutative ring spectrum. This chapter is based on joint work with Andrew Salch, and therefore some of the theorems are included without proof and the reader should read [3] for a thorough account. The goal is to highlight aspects of the joint project that were primarily the author's contribution as well as set up all the necessary machinery needed for Chapter 4. In Chapter 4, we compute mod (p, v_1)-homotopy of topological Hochschild homology of $K(\mathbb{F}_q)_p$ where $p \geq 3$ and q is a topological generator of $\mathbb{Z}_p^\times$. We also give initial calculations towards mod p homotopy of topological Hochschild homology of $K(\mathbb{F}_q)_p$. In Chapter 5, we prove the main theorem that the β-elements β_{pk+1} are detected in iterated algebraic K-theory of finite fields, giving evidence for red-shift phenomena.

2.8 Conventions

Throughout, we will work in the category of symmetric spectra of simplicial sets with the positive flat stable model structure, which we denote $\mathfrak{S}$. This particular model structure for spectra is chosen because the main theorem of the author's joint paper with Andrew Salch [4] then applies; i.e given a map of simplicial spectra $X_\bullet \to Y_\bullet$ where

1. the spectra X_n and Y_n are positive flat cofibrant for all n,
2. each degeneracy map $s_i : X_n \to X_{n+1}$ and $s_i' : Y_n \to Y_{n+1}$ is a levelwise cofibration, and
3. the map $X_n \to Y_n$ is a flat cofibration for each n

then the induced map on realizations $|X_\bullet| \to |Y_\bullet|$ is a cofibration. The category $\mathfrak{S}$ is also a combinatorial, cofibrantly generated, symmetric monoidal model category satisfying the Shipley-Schwede monoid axiom by [54] and it satisfies the strong commutative monoid axiom of White as he proved in [59].

We will write $\mathrm{Comm}\,\mathcal{C}$ for the category of commutative monoids in a symmetric monoidal category $\mathcal{C}$ and we will write $s\mathcal{C}$ for the category of simplicial objects in $\mathcal{C}$, in other words, functors $\Delta^{\mathrm{op}} \to \mathcal{C}$.

We will write $(-)_p$ for the p-completion of a spectrum or a group. We will write $P(x)$, $E(x)$, $P_h(x)$ and $\Gamma(x)$ for the polynomial algebra, exterior algebra, truncated polynomial, and divided power algebra on the generator x. Here $P_h(x) = P(x)/x^h$. Recall that $\Gamma(x)$ has generators $\gamma_i(x)$ for $i \geq 1$ satisfying $i!j!\gamma_i(x)\gamma_j(x) = (i+j)!\gamma_{i+j}(x)$. In particular, when $\Gamma(x)$ is an algebra over a field of characteristic p, there is an isomorphism

$$\Gamma(x) = P_p(x, \gamma_p(x), \gamma_{p^2}(x), \dots)$$

of $\mathbb{F}_p$-algebras. We will write $\doteq$ to indicate that the equality holds up to multiplication by a unit in $\mathbb{F}_p$.

We will write ℓ for $K(\bigcup_{i\geq 0} \mathbb{F}_{q^{p^i}})_p$, we will write j for $K(\mathbb{F}_q)_p$, and ku for $K(\overline{\mathbb{F}}_q)_p$ where $p \geq 3$ and q is a topological generator of $\mathbb{Z}_p^\times$. We will tacitly assume that ℓ, j and ku are cofibrant since we could cofibrantly replace them in Comm $\mathfrak{S}$ if they were not already. For chapter 5 we will assume $p \geq 5$ so that the finite spectrum $V(1) = \mathrm{cof}\{\Sigma^{2p-2}S/p \to S/p\}$ has a v_2-self map.

CHAPTER 3 A MAY-TYPE SPECTRAL SEQUENCE FOR HIGHER THH

The purpose of this chapter is to describe the construction of a spectral sequence in “higher order" topological Hochschild homology with coefficients associated to a decreasingly filtered commutative monoid in $\mathfrak{S}$ with filtered coefficients. The full details of the construction can be found in the author's joint paper with Andrew Salch [3].

3.1 Decreasingly filtered commutative monoids in symmetric spectra

We give a definition of decreasingly filtered commutative monoids in $\mathfrak{S}$ using categorical machinery that has the advantage of being clean and concise.

Definition 3.1. A *decreasingly indexed object in* $\mathfrak{S}$ is a functor $I : \mathbb{N}^{\text{op}} \to \mathfrak{S}$.

We write $\mathfrak{S}^{\mathbb{N}^{\text{op}}}$ for the category of such functors and, by convention, we write I_n for evaluation of a functor I in $\mathfrak{S}^{\mathbb{N}^{\text{op}}}$ on an object in $\mathbb{N}^{\text{op}}$.

Definition 3.2. We define the projective model structure on $\mathfrak{S}^{\mathbb{N}^{\text{op}}}$, by defining the fibrations to be those natural transformations $f : I \to J$ such that $f(n) : I_n \to J_n$ is a fibration for each $n \in \mathbb{N}$. The weak equivalences are the natural transformations $g : I \to J$ such that $g(n) : I_n \to J_n$ is a weak equivalence for each n. The cofibrations are natural transformations that have the left lifting property with respect to trivial fibrations.

Remark 3.3. The category $\mathbb{N}^{\text{op}}$ has the usual Reedy category structure as a partially ordered set. In particular, this gives $\mathbb{N}^{\text{op}}$ the structure of a direct category. The opposite Reedy structure makes $\mathbb{N}^{\text{op}}$ an indirect category and the projective model structure on the functor category $\mathfrak{S}^{\mathbb{N}^{\text{op}}}$ is the same as the Reedy model structure when $\mathbb{N}^{\text{op}}$ has the opposite Reedy model structure. The upshot of this description is that we may describe cofibrations in the projective model structure explicitly. The cofibrations in the projective model structure are those natural transformations $I \to J$ such that for each i the map

$$I_{i+1} \coprod_{I_i} J_{i+1} \to J_i$$

is a cofibration.

Note that in order for this model structure to exist we need the category $\mathfrak{S}$ to be cofibrantly generated, but this is the case due to Theorem 4.11 [53]. From now on the model structure on $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ will be understood to be the projective model structure specified above.

Definition 3.4. A *decreasingly filtered object in* $\mathfrak{S}$ is a cofibrant object in $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ in the projective model structure.

Remark 3.5. Note that this is the same data as a decreasingly indexed object in $\mathfrak{S}$ with the property that each map $f_i : I_i \to I_{i-1}$ is a cofibration and each object is cofibrant.

The category $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ has a symmetric monoidal product, denoted $\otimes_{\mathrm{Day}}$, called the Day convolution after B. Day who first constructed it in his thesis [22]. For this construction we enrich $\mathbb{N}^{\mathrm{op}}$ in $\mathfrak{S}$ by defining

$$\mathbb{N}^{\mathrm{op}}(n,m) = \begin{cases} S \text{ if } n \geq m \\ 0 \text{ otherwise} \end{cases}.$$

where S is a cofibrant replacement for the unit of the symmetric monoidal product in $\mathfrak{S}$.

Definition 3.6 (Day convolution)**.** The *Day convolution symmetric monoidal product* in $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ of I and J is the coend

$$(I \otimes_{Day} J)_n = \int^{(a,b)\in\mathbb{N}^{\mathrm{op}}\times\mathbb{N}^{\mathrm{op}}} \mathbb{N}^{\mathrm{op}}(a+b,n) \wedge I_a \wedge J_b.$$

By definition, this coend is the left Kan extension in the diagram

$$\begin{tikzcd}
\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}} \arrow[r, "I \bar{\otimes} J"] \arrow[d, "+"'] & \mathfrak{S} \\
\mathbb{N}^{\mathrm{op}} \arrow[ur, dashed, "I \otimes_{Day} J"'] &
\end{tikzcd},$$

which, in turn, is the colimit

$$\operatorname*{colim}_{a+b \in \mathbb{N}^{\mathrm{op}}/n} I_a \wedge I_b$$

where $\mathbb{N}^{\mathrm{op}}/n$ is the over category consisting of objects in $y \in \mathbb{N}^{\mathrm{op}}$ equipped with a map $y \to n$ and morphisms are commuting triangles.

Example 3.7. This construction can be visualized as follows. Consider the lattice,

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\ldots \arrow[r] & I_2 \wedge J_2 \arrow[r] \arrow[d] & I_1 \wedge J_2 \arrow[r] \arrow[d] & I_0 \wedge J_2 \arrow[d] \\
\ldots \arrow[r] & I_2 \wedge J_1 \arrow[r] \arrow[d] & I_1 \wedge J_1 \arrow[r] \arrow[d] & I_0 \wedge J_1 \\
\ldots \arrow[r] & I_2 \wedge J_0 \arrow[r] & I_1 \wedge J_0 & .
\end{tikzcd}$$

The colimit of this diagram is $(I \otimes_{\mathrm{Day}} J)_1$. To produce $(I \otimes_{\mathrm{Day}} J)_2$, we truncate the lattice further and take a colimit.

Theorem 3.8 (Day)**.** The category $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ forms a closed symmetric monoidal category. The category of commutative monoid objects in $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ is equivalent to the category of lax symmetric monoidal functors in $\mathbb{N}^{\mathrm{op}} \to \mathfrak{S}$.

Proof. This theorem is Example 3.2.2 in Day's thesis [22] and it follows form his work on promonoidal categories. This was also recently proven in the setting of quasi-categories by Glasman [31]. □

Definition 3.9. Recall that a lax symmetric monoidal functor between symmetric monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ is a functor

$$I : \mathcal{C} \to \mathcal{D}$$

together with natural transformations

$$\eta : \mathbb{1}_{\mathcal{D}} \to I(\mathbb{1}_{\mathcal{C}}) \text{ and}$$

$$\rho : I(-) \otimes_{\mathcal{D}} I(-) \to I(- \otimes_{\mathcal{C}} -)$$

satisfying the usual commutativity, associativity, and unitality axioms.

Definition 3.10. A *decreasingly indexed commutative monoid in* $\mathfrak{S}$ is a lax symmetric monoidal functor

$$I : (\mathbb{N}^{\mathrm{op}}, +, 0) \longrightarrow (\mathfrak{S}, \wedge, \mathbb{S})$$

or equivalently, due to Theorem 3.8, an object in $\mathrm{Comm}\, \mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$.

In order to have a good model structure on $\mathrm{Comm}\, \mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ we need $\mathrm{Comm}\, \mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ to satisfy a strong version of the commutative monoid axiom. Following White [59], a model category $\mathcal{C}$ satisfies the strong commutative monoid axiom if whenever $h : X \to Y$ is a (trivial) cofibration then $h^{\square n}/\Sigma_n$ is a (trivial) cofibration. The notation $h^{\square n}$ indicates that we are taking the pushout product of h with itself n times, for example if $f : A \to B$ and $g : X \to Y$, the pushout product of the two maps in $\mathcal{C}$ is the map

$$f\square g : A\otimes Y \coprod_{A\times X} B\otimes Y \to B\otimes Y$$

where $\otimes$ is a the symmetric monoidal product in $\mathcal{C}$.

Lemma 3.11. The category $\mathfrak{S}^{\mathbb{N}^{op}}$ with the projective model structure satisfies the strong commutative monoid axiom.

Proof. By White [59, Lem. A.1], it suffices to check the strong commutative monoid axiom on the generating cofibrations. The generating cofibrations in the projective model structure were determined in [14] and they are the natural transformations $f : I \to J$ of the form $\mathcal{I}\otimes\mathbb{N}^{op}$ where $\mathcal{I}$ is the set of generating cofibrations of $\mathfrak{S}$. This notation means that a map $f : I \to J$ is a cofibration if $f = g \wedge \mathbb{N}^{op}(i, -)$ for some $g \in \mathcal{I}$. Since

$$\mathbb{N}^{op}(i,-) \simeq \begin{cases} S \text{ if } j \leq i \\ 0 \text{ if} j > i \end{cases}$$

this is equivalent to saying that $I_0 \to J_0$ is a map in $\mathcal{I}$, $I_i = I_0$ for $i \leq j$ and $I_i = 0$ for $i > j$, and $J_i = J_0$ for $i \leq j$ and $J_i = 0$ for $i > j$. Let $h : I \to J$ be an (acyclic) map in $\mathcal{I}\otimes\mathbb{N}^{op}$, then we need to prove that $h^{\square n}/\Sigma_n$ is an (acyclic) cofibration in $\mathfrak{S}^{\mathbb{N}^{op}}$ in the projective model structure. In the case $n = 2$, we need to show that the map

$$h^{\square 2}/\Sigma_2 : (I\otimes_{Day} J \coprod_{I\otimes_{Day} I} J\otimes_{Day} I)/\Sigma_2 \to (J\otimes_{Day} J)/\Sigma_2$$

is an (acyclic) cofibration in the projective model structure. We therefore need to know if the map

$$\left((I \otimes_{Day} J \coprod_{I\otimes_{Day} I} J \otimes_{Day} I)_n/\Sigma_2 \coprod_{(I\otimes_{Day} J \coprod_{I\otimes_{Day} I} J\otimes_{Day} I)_{n+1}/\Sigma_2} (J \otimes_{Day} J)_{n+1}/\Sigma_2\right) \to (J \otimes_{Day} J)_n/\Sigma_2$$

is a cofibration in $\mathfrak{S}$. Note that for $X, Y \in \{I, J\}$, $(X \otimes_{Day} Y)_n = X_0 \wedge Y_0$ is $n \leq 2i$ and 0 if $n > 2i$, so the map is the same as the map

$$\left((I_0 \wedge J_0 \coprod_{I_0\wedge I_0} J_0 \wedge I_0)/\Sigma_2 \coprod_{(I_0\otimes_{Day} J_0 \coprod_{I_0\otimes_{Day} I_0} J_0\otimes_{Day} I_0/\Sigma_2} (J_0 \otimes_{Day} J_0)/\Sigma_2\right) = (J_0 \otimes_{Day} J_0)/\Sigma_2 \to (J_0 \wedge J_0)/\Sigma_2 \tag{3.1}$$

or in other words the identity map

$$(J_0 \otimes_{Day} J_0)/\Sigma_2 \to (J_0 \wedge J_0)/\Sigma_2$$

when $n < 2i$, it is the map

$$(I_0 \wedge J_0 \coprod_{I_0\wedge I_0} J_0 \wedge I_0)/\Sigma_2 \to (J_0 \otimes_{Day} J_0)/\Sigma_2 \tag{3.2}$$

when $n = 2i$ and it is the map $0 \to 0$ when $n > 2i$. Since the map 3.1 is the identity it is an (acyclic) cofibration in $\mathfrak{S}$. Since the map $I_0 \to J_0$ is an (acyclic) cofibration in $\mathfrak{S}$ by assumption and $\mathfrak{S}$ satisfies the strong commutative monoid axiom in $\mathfrak{S}$, the map 3.2 is also an (acyclic) cofibration in $\mathfrak{S}$. Hence, the map $h^{\square 2}/\Sigma_2$ is an (acyclic) cofibration as desired. The same type of argument works for $i > 2$ and it is therefore left to the reader. $\square$

Definition 3.12. The category $\mathrm{Comm}\,\mathfrak{S}^{\mathbb{N}^{op}}$ is equipped with the model structure created by the forgetful functor $U : \mathrm{Comm}\,\mathfrak{S}^{\mathbb{N}^{op}} \to \mathfrak{S}^{\mathbb{N}^{op}}$; i.e., fibrations in $\mathrm{Comm}\,\mathfrak{S}^{\mathbb{N}^{op}}$ are natural transformations $f : I \to J$ such that $U(f)$ is a projective fibration and weak equivalences are natural transformations $g : I \to J$ such that $U(g)$ is a weak equivalence in $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$. This model structure exists by Lemma 3.11 and White [59, Thm. 3.2]. We will call this model structure “the model structure inherited from the projective model structure on $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$."

Lemma 3.13. Let $\mathrm{Comm}\,\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ be equipped with the model structure inherited from the projective model structure on $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$, then cofibrations h between cofibrant objects in $\mathrm{Comm}\,\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ forget to cofibrations $U(h)$ in $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$.

Proof. This follows by White [59, Prop. 3.5] and Lemma 3.11. □

Definition 3.14. A decreasingly filtered commutative monoid in $\mathfrak{S}$ is a cofibrant object in $\mathrm{Comm}\,\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ with the model structure inherited from the projective model structure on $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$.

This definition also allows us to define a decreasingly filtered I-module in a simple way.

Definition 3.15. A *decreasingly filtered symmetric I-bimodule, M,* is a cofibrant object M in $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ that has the structure of a symmetric I-bimodule, where I is a cofibrant object in $\mathrm{Comm}\,\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$.

Definition 3.16. Let J be a decreasingly indexed object in $\mathfrak{S}$. We say J is *Hausdorff* if $\mathrm{holim}_n J_n \simeq 0$. We say that J is *finite* if there exists a non-negative integer n such that $f_m : J_m \to J_{m-1}$ is a weak equivalence whenever $m > n$.

The following appears as Definition 3.16 in the author’s joint paper with Andrew Salch [3].

Definition 3.17 (The associated graded commutative monoid)**.** Let I be a decreasingly filtered commutative monoid in $\mathfrak{S}$. We will write $E_0 I$ for the *associated graded commutative monoid of I*, which we define as follows:

- As an object of $\mathfrak{S}$,

$$E_0 I \cong \coprod_{n\in\mathbb{N}} I_n/I_{n+1}.$$

- As an object in $\mathrm{Comm}\,\mathfrak{S}$ we need to specify the unit map and multiplication map as well as show that it satisfies the axioms of a commutative monoid in $\mathfrak{S}$.

  – The unit map $\mathbb{S} \to E_0 I$ is the composite

$$\mathbb{S} \xrightarrow{\eta} I_0 \to I_0/I_1 \hookrightarrow E_0 I.$$

  – The multiplication on $E_0 I$ is given as follows. Since the smash product commutes with colimits, hence with coproducts, to specify a map $\mu_{E_0 I} : E_0 I \wedge E_0 I \to E_0 I$. it suffices to specify a component map

$$\nabla_{i,j} : I_i/I_{i+1} \wedge I_j/I_{j+1} \to E_0 I$$

  for every $i, j \in \mathbb{N}$. We define such a map $\nabla_{i,j}$ as follows: first, we have the commutative square

$$\begin{tikzcd} I_{i+1} \wedge I_j \arrow[r, "\rho_{i+1,j}"] \arrow[d, "f_{i+1}\wedge \mathrm{id}_{I_j}"] & I_{i+j+1} \arrow[d, "f_{i+j+1}"] \\ I_i \wedge I_j \arrow[r, "\rho_{i,j}"] & I_{i+j} \end{tikzcd}$$

  so, using the assumption that the maps f_i are cofibrations, we take vertical cofibers to get a map

$$\tilde{\nabla}_{i,j} : I_i/I_{i+1} \wedge I_j \to I_{i+j}/I_{i+j+1}.$$

Now we have the commutative diagram

$$
\begin{tikzcd}
I_{i+1} \wedge I_{j+1} \arrow[rr, "\rho_{i+1,j+1}"] \arrow[dr, "\mathrm{id}_{I_{i+1}} \wedge f_{j+1}"] \arrow[dd, "f_{i+1} \wedge \mathrm{id}_{I_{j+1}}"] & & I_{i+j+2} \arrow[dr, "f_{i+j+2}"] \arrow[dd, "f_{i+j+2}"] & \\
& I_{i+1} \wedge I_j \arrow[rr, "\rho_{i+1,j}"] \arrow[dd, "f_{i+1} \wedge \mathrm{id}_{I_j}"] & & I_{i+j+1} \arrow[dd, "f_{i+j+1}"] \\
I_i \wedge I_{j+1} \arrow[rr, "\rho_{i,j+1}"] \arrow[dr, "\mathrm{id}_{I_i} \wedge f_{j+1}"] \arrow[dd] & & I_{i+j+1} \arrow[dr, "f_{i+j+1}"] \arrow[dd] & \\
& I_i \wedge I_j \arrow[rr, "\rho_{i,j}"] \arrow[dd] & & I_{i+j} \arrow[dd] \\
I_i/I_{i+1} \wedge I_{j+1} \arrow[rr, "\tilde{\nabla}_{i,j+1}"] \arrow[dr, "\mathrm{id}_{I_i/I_{i+1}} \wedge f_{j+1}"'] & & I_{i+j+1}/I_{i+j+2} \arrow[dr, "0"] & \\
& I_i/I_{i+1} \wedge I_j \arrow[rr, "\tilde{\nabla}_{i,j}"] & & I_{i+j}/I_{i+j+1}
\end{tikzcd}
$$

in which the columns are cofiber sequences. So we have a choice of factorization of the composite map $\tilde{\nabla}_{i,j} \circ \left(\mathrm{id}_{I_i/I_{i+1}} \wedge f_{j+1}\right)$ through the zero object. So we have the commutative square

$$
\begin{tikzcd}
I_i/I_{i+1} \wedge I_{j+1} \arrow[r] \arrow[d, "\mathrm{id}_{I_i/I_{i+1}} \wedge f_{j+1}"] & 0 \arrow[d] \\
I_i/I_{i+1} \wedge I_j \arrow[r, "\tilde{\nabla}_{i,j}"] & I_{i+j}/I_{i+j+1}
\end{tikzcd}
$$

and, taking vertical cofibers, a map

$$I_i/I_{i+1} \wedge I_j/I_{j+1} \to I_{i+j}/I_{i+j+1},$$

which we compose with the inclusion map $I_{i+j}/I_{i+j+1} \hookrightarrow E_0 I$ to produce our desired map $\nabla_{i,j} : I_i/I_{i+1} \wedge I_j/I_{j+1} \to E_0 I$. We then define a map

$$\mu_{E_0 I} : E_0 I \wedge E_0 I \to E_0 I$$

using the universal property of the infinite wedge and the fact that the smash product distributes over the wedge.

3.2 The Loday construction

Let $\mathcal{D}$ be a closed symmetric monoidal model category. Let $fSet_+$ denote the category of finite pointed sets. We will write $*_S$ for the basepoint of a finite pointed set S and let Set_+ denote the category of pointed sets.

Definition 3.18 (Loday construction with coefficients)**.** Given a commutative monoid R in $\mathcal{D}$ and a symmetric R-bimodule, define a functor

$$- \otimes (R, M) : fSet_+ \to \mathrm{Mod}_R$$

in the following way:

- define the functor on objects by

$$S \otimes (R, M) = M\{*_S\} \wedge \bigwedge_{s \in S - \{*_S\}} R\{s\},$$

- define the functor on maps, by sending the map $f : S \to T$ to the map

$$f \otimes (R, M) : M\{*_S\} \wedge \bigwedge_{s \in S - \{*_S\}} R\{s\} \to M\{*_T\} \wedge \bigwedge_{t \in T - \{*_T\}} R\{t\}$$

defined by the composite

$$M\{*_S\} \wedge \bigwedge_{s\in S-\{*_S\}} R\{s\} \xrightarrow{\simeq} M\{*_S\} \wedge \bigwedge_{\{s\in f^{-1}(*_T)\}} R\{s\} \wedge \bigwedge_{t\in T} \bigwedge_{\{s\in f^{-1}(t)|t\in T-\{*_T\}\}} R\{s\} \to M\{*_T\} \wedge \bigwedge_{t\in T} R\{t\}$$

where the left map simply rearranges factors, and the right map is a smash product of two maps:

- the map

$$M\{*_S\} \wedge \bigwedge_{s\in f^{-1}(*_T)} R\{s\} \to M\{*_T\},$$

  given by the iterates of the module map, and

- the map

$$\bigwedge_{t\in T} \bigwedge_{\{s\in f^{-1}(t)|t\in T-\{*_T\}\}} R\{s\} \to \bigwedge_{t\in T} R\{t\}$$

  given by the iterating the multiplication map of R.

Note that the empty smash product is understood to be the unit of the symmetric monoidal product and a map from an empty smash product to R is given by the unit map of the commutative monoid R.

This extends to a functor

$$-\otimes(R, M) : sfSet_+ \to \mathrm{Mod}_R$$

using functoriality of $-\otimes(R, M)$ to define the face and degeneracy maps and then taking geometric realization of the resulting simplicial R-module.

Remark 3.19. This construction can again be extended to a functor

$$- \otimes (R, M) : sSet_+ \to s\mathrm{Mod}_R$$

by letting $- \otimes (R, M)(X_n) = \mathrm{colim}_{Y \subset X_n} Y \otimes (R, M)$ where Y ranges over all finite based subsets of X_n.

Remark 3.20. We use the notation $- \otimes (-; -)$ because of the relation to the tensoring of a simplicial set with a commutative ring spectrum. Recall, that McClure-Schwanzl-Vogt [43] proved that the category of commutative ring spectra have all weighted limits and colimits in simplicial sets and therefore, in particular, it is tensored and cotensored over simplicial sets. If we let $R = M$ and work in the category $\mathfrak{S}$, for example, then there is a commutative diagram

$$\begin{tikzcd} sSet_+ \arrow[r, "-\otimes(R;R)"] \arrow[d, "U"] & \mathrm{Comm}\,\mathfrak{S} \arrow[d, "id"] \\ sSet \arrow[r, "-\otimes R"] & \mathrm{Comm}\,\mathfrak{S} \end{tikzcd}$$

where the bottom functor is the weighted colimit in $\mathrm{Comm}\,\mathfrak{S}$ that defines the tensoring of a simplicial set with a commutative ring spectrum.

Remark 3.21. We will write $X_\bullet \tilde{\otimes} (R; M)$ for the simplicial R-module whose realization is $X_\bullet \otimes (R; M)$ and we will write $X_\bullet \otimes R$ for the simplicial commutative monoid in $\mathcal{D}$ when $R = M$ and $X_\bullet$ is unbased.

Example 3.22. In the case where $X_\bullet = S^1_\bullet := \Delta[1]/\delta\Delta[1]$, the minimal simplicial model for the circle, $S^1_\bullet \otimes R$ is the geometric realization of the simplicial object in $\mathfrak{S}$

$$S^1_\bullet\tilde{\otimes}R := \left\{ R \leftleftarrows R \wedge R \cdots R \wedge R \wedge R \cdots \ldots \right\} \tag{3.3}$$

with face and degeneracy maps given by the following formulas: the face maps are

$$d_i = \begin{cases} \mathrm{id}_R \wedge \ldots \mathrm{id}_R \wedge \mu \wedge \mathrm{id}_R \wedge \ldots \wedge \mathrm{id}_R & \text{if } i < n \\ (\mu \wedge \mathrm{id}_R \wedge \ldots \wedge \mathrm{id}_R) \circ t_n & \text{if } i = n \end{cases}$$

where the multiplication map $\mu : R \wedge R \to R$ is in the i-th position on the first line and $t_n : R^{\wedge n} \to R^{\wedge n}$ is the map that cyclicly permutes the factors to the right. The degeneracy maps are

$$s_i = \mathrm{id}_R \wedge \ldots \wedge \mathrm{id}_R \wedge \eta \wedge \mathrm{id}_R \wedge \ldots \wedge \mathrm{id}_R$$

where the unit map $\eta : S \to R$ from the sphere spectrum is in the i-th position.

Remark 3.23. The simplicial tensoring $S^1 \otimes R$ is the primary model for $THH(R)$ that we will work with even though it is not genuine S^1-equivariant. Since there is a model of THH, due to Bökstedt and developed by Hesselholt and Madsen which is genuine C-equivariant for all finite subgroups C of S^1, then we will tacitly use one of the genuine models for equivarant constructions. This will not cause an issue since the method of attack used here is by homotopy fixed point spectral sequences and the homotopy groups of each model for THH are the same. There is also a more recent model for THH as the norm $N_e^{S^1}(R) = I_{\mathbb{R}^\infty}^U(S^1 \otimes R)$ which is a genuine S^1-equivariant orthogonal spectrum [5]. In future joint work with C. Malkiewich, the author plans to construct a equivariant version of the THH-May spectral sequence using this construction, but that is beyond the scope of the present thesis.

3.3 The May filtration and the THH-May spectral sequence

Definition 3.24 (The May filtration)**.** Let S be a finite pointed set. We can equip the set of functions $x : S \to \mathbb{N}$, denoted $\mathbb{N}^S$, with the L_1 norm $|-|$ so that for $x \in \mathbb{N}^S$

$$|x| = \Sigma_{s \in S}\, x(s).$$

We then define a sub-poset of $\mathbb{N}^S$ by

$$\mathcal{D}_n^S := \{x \in \mathbb{N}^S; |x| \geq n\}.$$

Let I be a decreasingly filtered commutative monoid in $\mathfrak{S}$ and let M be a cofibrant symmetric I-bimodule. Define a functor

$$\mathcal{F}^S(I; M) : (\mathbb{N}^S)^{\mathrm{op}} \longrightarrow \mathfrak{S}$$

on objects by $\mathcal{F}^S(I; M)(x) = M_{x(*_S)} \bigwedge_{s \in S - \{*_S\}} I_{x(s)}$ for $x \in \mathbb{N}^S$ and on morphisms in the apparent way. We can precompose this functor with the inclusion functor to produce a functor

$$\mathcal{F}_n^S(I; M) : (\mathcal{D}_n^S)^{\mathrm{op}} \longrightarrow (\mathbb{N}^S)^{\mathrm{op}} \xrightarrow{\mathcal{F}^S(I)} \mathfrak{S}.$$

We then define the May filtration associated to the finite pointed set S to be the collection of objects

$$\mathcal{M}_n^S(I; M) := \mathrm{colim}\, \mathcal{F}_n^S(I; M)$$

in $\mathfrak{S}$ for $n \in \mathbb{N}$ along with maps

$$\mathcal{M}_n^S(I;M) \longrightarrow \mathcal{M}_{n-1}^S(I;M)$$

given by precomposing with the inclusion $(\mathcal{D}_n^S)^{\mathrm{op}} \longrightarrow (\mathcal{D}_{n-1}^S)^{\mathrm{op}}$ and taking colimits.

Remark 3.25. When $M = I$, we will simply write $\mathcal{F}^S(I)$ and $\mathcal{M}_n^S(I)$ for $\mathcal{F}^S(I;I)$ and $\mathcal{M}_n^S(I;I)$.

Using the May filtration in each simplicial degree, we produce a filtration of simplicial objects in $\mathfrak{S}$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\mathcal{M}_2^{X_0}(I;M) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{M}_2^{X_1}(I;M) \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{M}_2^{X_2}(I;M) \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \\
\mathcal{M}_1^{X_0}(I;M) \arrow[u, leftarrow] \arrow[r, shift left] \arrow[r, shift right] & \mathcal{M}_1^{X_1}(I;M) \arrow[u, leftarrow] \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{M}_1^{X_2}(I;M) \arrow[u, leftarrow] \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \\
\mathcal{M}_0^{X_0}(I;M) \arrow[u, leftarrow] \arrow[r, shift left] \arrow[r, shift right] & \mathcal{M}_0^{X_1}(I;M) \arrow[u, leftarrow] \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{M}_0^{X_2}(I;M) \arrow[u, leftarrow] \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

and when $X_\bullet = S^1_\bullet$ the bottom row is the simplicial object whose geometric realization is $THH(I_0, M_0)$.

Remark 3.26. Note that the definition of $\mathcal{M}_n^{X_\bullet}(I;M)$ can be extended to any simplicial pointed set $Y_\bullet$ by defining

$$\mathcal{M}_m^{Y_n}(I;M) = \operatorname*{colim}_{Y \subset Y_n} \mathcal{M}_m^Y(I;M)$$

in each simplicial degree where Y ranges over all finite subsets of Y_n.

When we write $\mathcal{M}_n^{X_\bullet}(I)$ we will mean that $X_\bullet$ is an un-based simplicial set and we have done the same construction otherwise.

Definition 3.27. If I is a decreasingly filtered commutative monoid in $\mathfrak{S}$ and M is a cofibrant symmetric I-bimodule, $X_\bullet$ is a simplicial pointed set, and G_* is a connective generalized homology theory then the *topological Hochschild-May spectral sequence* is the spectral sequence obtained by applying G_* to the tower of cofiber sequences

$$\begin{tikzcd}
\vdots \arrow[d] & \\
\left|\mathcal{M}_2^{X_\bullet}(I;M)\right| \arrow[r] \arrow[d] & \left|\mathcal{M}_2^{X_\bullet}(I;M)\right| / \left|\mathcal{M}_3^{S^\bullet}(I;M)\right| \\
\left|\mathcal{M}_1^{X_\bullet}(I;M)\right| \arrow[r] \arrow[d] & \left|\mathcal{M}_1^{X_\bullet}(I;M)\right| / \left|\mathcal{M}_2^{X_\bullet}(I;M)\right| \\
\left|\mathcal{M}_0^{X_\bullet}(I;M)\right| \arrow[r] & \left|\mathcal{M}_0^{X_\bullet}(I;M)\right| / \left|\mathcal{M}_1^{X_\bullet}(I;M)\right| .
\end{tikzcd} \tag{3.4}$$

That is, it is the spectral sequence of the exact couple

$$\begin{tikzcd}
D^1_{*,*} \cong \bigoplus_{i,j} G_i \left|\mathcal{M}_j^{X_\bullet}(I;M)\right| \arrow[rr] & & \bigoplus_{i,j} G_i \left|\mathcal{M}_j^{X_\bullet}(I;M)\right| \arrow[dl] \\
& E^1_{*,*} \cong \bigoplus_{i,j} G_i \left|\mathcal{M}_j^{X_\bullet}(I;M)\right| / \left|\mathcal{M}_{j+1}^{X_\bullet}(I;M)\right| \arrow[ul] &
\end{tikzcd} .$$

Remark 3.28. We need to know that the map

$$\left|\mathcal{M}_n^{X_\bullet}(I;M)\right| \longrightarrow \left|\mathcal{M}_{n-1}^{X_\bullet}(I;M)\right|$$

is a cofibration, and this relies on a theorem of the author and A. Salch, which states that a map between two “good" simplicial objects in $\mathfrak{S}$, which is a (positive) flat cofibration at each simplicial level, realizes to a (positive) flat cofibration [4]. This is the reason why we choose the category of symmetric spectra in simplicial sets with the positive flat model structure where this theorem holds and all necessary axioms hold. Otherwise, all the constructions are sufficiently general to work in any nice enough model category (see [3] for the exact conditions needed on a model category in order to construct the THH-May spectral sequence and identify the E_1-page). Consequently, the sequence

$$\ldots \longrightarrow \left|\mathcal{M}_2^{X_\bullet}(I)\right| \longrightarrow \left|\mathcal{M}_1^{X_\bullet}(I)\right| \longrightarrow \left|\mathcal{M}_0^{X_\bullet}(I)\right|$$

is again a decreasingly filtered commutative monoid in $\mathfrak{S}$, which we will call $\left|\mathcal{M}^{X_\bullet}(I)\right|$, and therefore, we can define $E_0\left|\mathcal{M}^{X_\bullet}(I)\right|$. The input of the THH-May spectral sequence as defined is $G_*\left(E_0\left|\mathcal{M}^{X_\bullet}(I)\right|\right)$.

The main theorem of [3] produces a more computable E_2-page. The idea is that the associated graded construction commutes with tensoring with a simplicial set.

Theorem 3.29 (Fundamental Theorem of the May filtration [3])**.** Let $X_\bullet$ be a simplicial pointed set, let I be a decreasingly filtered commutative monoid in $\mathfrak{S}$, and let M be a cofibrant symmetric I-bimodule, then there is a weak equivalence

$$E_0\left|\mathcal{M}^{X_\bullet}(I;M)\right| \simeq X_\bullet \otimes (E_0 I; E_0 M),$$

which is a weak equivalence in Comm $\mathfrak{S}$ when $M = I$ and $X_\bullet$ is an un-based simplicial set.

Proof. See [3] for a detailed proof. $\square$

We therefore produce a spectral sequence

$$E^2_{s,t} = G_{s,t}(X_\bullet \otimes (E_0 I; E_0 M)) \Rightarrow G_s(X_\bullet \otimes (I_0; M)) \tag{3.5}$$

with differential

$$d_r : E^r_{s,t} \longrightarrow E^r_{s-1,t+r}$$

for any connective generalized homology theory G_*. The spectral sequence strongly converges as long as I_i and M_i are Hausdorff and the differentials satisfy a Leibniz rule in the case $I = M$ and $X_\bullet$ is an un-based simplicial set [3].

Another construction of the THH-May spectral sequence

$$E^1_{*,*} \cong G_{*,*}(X_\bullet \otimes (E_0 I, E_0 M)) \Rightarrow G_*(X_\bullet \otimes (I_0, M)) \tag{3.6}$$

is possible using the Day convolution symmetric monoidal product. This construction is conceptually cleaner, but it does not simplify the process of proving that the resulting spectral sequence has the correct input term, output term and convergence properties.

The category $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ is a closed symmetric monoidal model category equipped with Day convolution and the projective model structure as proven in [22] and [35]. Now fix a based simplicial set $X_\bullet$, let I be a cofibrant commutative monoid object in $\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$, and let M be a cofibrant symmetric I-bimodule. We can form the Loday construction

$$- \otimes (I, M) : sSet_+ \to \mathrm{Mod}_I \subset \mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}.$$

For example, if $X_\bullet$ is the usual minimal simplicial model $S^1_\bullet = (\Delta[1]/\delta\Delta[1])_\bullet$ for the circle, then $S^1_\bullet\tilde{\otimes}(I,M)$ is the cyclic bar construction using the Day convolution as the tensor product:

$$S^1_\bullet\tilde{\otimes}(I,M) = \left(M \leftleftarrows M \otimes_{\mathrm{Day}} I \;\substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow}\; M \otimes_{\mathrm{Day}} I \otimes_{\mathrm{Day}} I \;\substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow}\; \dots \right)$$

Since I is a functor $\mathbb{N}^{\mathrm{op}} \to \mathfrak{S}$ and M is an I-module, we will write $I(n)$ and $M(n)$ for the evaluation of these functors at a nonnegative integer n just for the sake of this remark. We write $S^1_\bullet\tilde{\otimes}(I,M)(i)$ for the the simplicial object in $\mathfrak{S}$

$$S^1_\bullet\tilde{\otimes}(I,M)(i) = \left(M(i) \;\substack{\longleftarrow\\\longrightarrow\\\longleftarrow}\; (M \otimes_{\mathrm{Day}} I)(i) \;\substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow}\; (M \otimes_{\mathrm{Day}} I \otimes_{\mathrm{Day}} I)(i) \;\substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow}\; \dots \right)$$

Applying geometric realization to each $S^1\tilde{\otimes}(I,M)(i)$, we get a decreasingly filtered object in $\mathfrak{S}$

$$S^1_\bullet \otimes (I,M)(0) \leftarrow S^1_\bullet(I,M)(1) \leftarrow S^1_\bullet \otimes (I,M)(2) \leftarrow \dots$$

which is a decreasingly filtered object in $\mathfrak{S}$ by the main theorem of [4]. The spectral sequence obtained by applying a generalized homology theory G_* to the tower of cofiber sequences associated to this decreasingly filtered object in $\mathfrak{S}$ is precisely the THH-May spectral sequence.

Remark 3.30. The author expects that the construction of the THH-May spectral sequence may also work using the construction $\Lambda_{X_\bullet}(R)$ that appears in [52], which is a version of the Loday construction that does not require cofibrant input in order to be functorial and lends itself well to approximations of higher order topological cyclic homology, also known as covering homology [19].

Example 3.31. We conclude with an example. Suppose I is a trivially filtered commutative monoid in $\mathfrak{S}$; i.e., $I_n = 0$ for $n > 0$. Suppose M is a decreasingly filtered symmetric I-module object in $\mathfrak{S}$ with $M_n \simeq 0$ for $n > 1$. Then the sequence of simplicial commutative monoids becomes

$$\begin{tikzcd}
\mathcal{M}_1^{X_0}(I, M) \arrow[d] & \mathcal{M}_1^{X_1}(I, M) \arrow[l, shift left=2] \arrow[l, leftarrow] \arrow[l, shift right=2] \arrow[d] & \mathcal{M}_1^{X_2}(I, M) \arrow[l, shift left=4] \arrow[l, shift left=2, leftarrow] \arrow[l] \arrow[l, shift right=2, leftarrow] \arrow[l, shift right=4] \arrow[d] & \cdots \arrow[l, shift left=6] \arrow[l, shift left=4, leftarrow] \arrow[l, shift left=2] \arrow[l, leftarrow] \arrow[l, shift right=2] \arrow[l, shift right=4, leftarrow] \arrow[l, shift right=6] \\
\mathcal{M}_0^{X_0}(I, M) & \mathcal{M}_0^{X_1}(I, M) \arrow[l, shift left=2] \arrow[l, leftarrow] \arrow[l, shift right=2] & \mathcal{M}_0^{X_2}(I, M) \arrow[l, shift left=4] \arrow[l, shift left=2, leftarrow] \arrow[l] \arrow[l, shift right=2, leftarrow] \arrow[l, shift right=4] & \cdots \arrow[l, shift left=6] \arrow[l, shift left=4, leftarrow] \arrow[l, shift left=2] \arrow[l, leftarrow] \arrow[l, shift right=2] \arrow[l, shift right=4, leftarrow] \arrow[l, shift right=6]
\end{tikzcd}$$

where the realization of $\mathcal{M}_0^{X_\bullet}(I, M)$ is $X_\bullet \otimes (I_0; M_0)$, the realization of $\mathcal{M}_1^{X_\bullet}(I; M)$ is $X_\bullet \otimes (I_0, M_1)$ and the realization of the quotient $\mathcal{M}_0^{X_\bullet}(I, M)/\mathcal{M}_1^{X_\bullet}(I, M)$ is $\mathcal{M}^{X_\bullet}(I_0, M_0/M_1)$. The spectral sequence collapses to produce a long exact sequence coming from the cofiber sequence

$$X_\bullet \otimes (I_0, M_1) \to X_\bullet \otimes (I_0, M_0) \to X_\bullet \otimes (I_0, M_0/M_1).$$

When $X_\bullet = \Delta[1]/\delta\Delta[1]$ with the obvious basepoint, this specializes to a cofiber sequence,

$$THH(I_0, M_1) \to THH(I_0, M_0) \to THH(I_0, M_0/M_1)$$

which recovers a result of Pirashvili-Waldhausen [49, Prop. 2.13].

3.4 Filtered Commutative Ring Spectra

Let R be a cofibrant connective commutative monoid in $\mathfrak{S}$. In order to apply the THH-May spectral sequence, it is necessary to construct decreasingly filtered commutative monoids in spectra. The goal of this section is to produce a decreasingly filtered commutative monoid in $\mathfrak{S}$ as a specific multiplicative model for the Whitehead tower of a connective commutative monoid in $\mathfrak{S}$. This provides a large supply of decreasingly filtered commutative monoids. Part of the proof uses a Postnikov tower of a commutative ring spectrum constructed as a tower of square-zero extensions, so first we define square-zero extensions in this context.

3.4.1 Postnikov towers as towers of square-zero extensions

Definition 3.32. By a square-zero extension in $\mathfrak{S}$, we mean a fiber sequence

$$I \longrightarrow \tilde{A} \longrightarrow A$$

where $\tilde{A}$ is the pullback in $\mathrm{Comm}\,\mathfrak{S}$ of

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & A \arrow[d, "\epsilon"] \\ A \arrow[r, "d"] & A \ltimes \Sigma I, \end{tikzcd}$$

the map ϵ is defined to be the inclusion of A into $A \ltimes \Sigma I$ and d represents a class $[d] \in TAQ^0_S(A, \Sigma I)$. (For a definition of $TAQ^*_S(A, \Sigma I)$, see [13] or [41].) Note that, a priori, A must be a commutative monoid in $\mathfrak{S}$ and I must be a A-bimodule. By $A \ltimes \Sigma I$ we mean the trivial square-zero extension of A by ΣI; that is, additively $A \ltimes \Sigma I := A \vee \Sigma I$ and its multiplication is the map

$$\mu : A \wedge A \vee A \wedge I \vee I \wedge A \vee I \wedge I \longrightarrow A \vee I$$

determined, using the universal property of the coproduct, by the maps

$$\mu_A : A \wedge A \to A \hookrightarrow A \vee I$$
$$\psi^\ell : A \wedge I \to I \hookrightarrow A \vee I$$
$$\psi^r : I \wedge A \to I \hookrightarrow A \vee I$$
$$sq : I \wedge I \to 0 \hookrightarrow A \vee I$$

where μ_A is the multiplication on A, ψ^r and ψ^ℓ are the right and left action maps of I as an A-bimodule and sq is the usual map $I \wedge I \to I \hookrightarrow A \vee I$, which in this case factors through the zero object.

Definition 3.33. Let R be a connective commutative monoid in $\mathfrak{S}$. By a *Postnikov tower of square-zero extensions* associated to R, we mean a tower

$$\begin{tikzcd}
\ldots \arrow[r] & \tau_{\le 3}R \arrow[r] & \tau_{\le 2}R \arrow[r] & \tau_{\le 1}R \arrow[r] & \tau_{\le 0}R \\
& \Sigma^3 H\pi_3 R \arrow[u] & \Sigma^2 H\pi_2 R \arrow[u] & \Sigma^1 H\pi_1 R \arrow[u] & H\pi_0 R \arrow[u]
\end{tikzcd}$$

of fiber sequences where $\pi_k(\tau_{\le n}R) = \pi_k(R)$ for $k \le n$ and $\pi_k(\tau_{\le n}R) = 0$ for $k > n$, such that the fiber sequences

$$\Sigma^n H\pi_n R \longrightarrow \tau_{\le n}R \longrightarrow \tau_{\le n-1}R$$

are square-zero extensions.

As defined it is not clear that such Postnikov towers of square-zero extensions actually exist for a given commutative monoid in $\mathfrak{S}$, but it is a theorem that they do.

Theorem 3.34. Let R be a connective commutative monoid in $\mathfrak{S}$. Then there exists a model for the Postnikov tower associated to R which is a Postnikov tower of square-zero extensions.

Proof. See Theorem 4.3 and the comments after in [36] and Theorem 8.1 in [13]. Also, see Lurie's Corollary 3.19 from [37] for the result in the setting of quasi-categories. □

3.4.2 Constructing the Whitehead tower as a filtered commutative ring spectrum

Recall from Definition 3.14 that a cofibrant object in the category Comm $\mathfrak{S}^{\mathbb{N}^{op}}$ equipped with the projective model structure is a decreasingly filtered commutative monoid in $\mathfrak{S}$. We may define certain n-truncated decreasingly filtered commutative monoids in the following way.

Definition 3.35. Let $J_n \subset \mathbb{N}$ be the sub-poset of the natural numbers consisting of all $i \in \mathbb{N}$ such that $i \leq n$. We give this poset the structure of a symmetric monoidal category $(J_n, \dot{+}, 0)$ by letting

$$i \dot{+} j = \min\{i + j, n\}.$$

We may consider lax symmetric monoidal functors in $\mathfrak{S}^{J_n^{op}}$ for each n again as a consequence of [22, Ex. 3.2.2] these are equivalent to the commutative monoids in the functor category under the Day convolution symmetric monoidal product. We may also consider the model structure on $\mathrm{Comm}(\mathfrak{S}^{J_n^{op}})$ created by the forgetful functor to $\mathfrak{S}^{J_n^{op}}$, where $\mathfrak{S}^{J_n^{op}}$ has the projective model structure. Using the same considerations as the functor category $\mathfrak{S}^{\mathbb{N}^{op}}$, the model category structure created by the forgetful functor exists. In this model structure, it is an easy exercise to show that the cofibrant objects are functors $I^{\leq n}$ in $\mathfrak{S}^{J_n^{op}}$ such that each $I_i^{\leq n}$ is cofibrant in $\mathfrak{S}$ for $i \leq n$ and each map $f_i : I_i^{\leq n} \to I_{i-1}^{\leq n}$ is a cofibration in $\mathfrak{S}$ for each $i \leq n$.

Theorem 3.36. Let R be a cofibrant connective commutative monoid in $\mathfrak{S}$, then there exists a decreasingly filtered commutative monoid in $\mathfrak{S}$

$$R^{\geq \bullet} : \mathbb{N}^{\mathrm{op}} \to \mathfrak{S},$$

where we write $R^{\geq n}$ for the functor evaluated on an object in $\mathbb{N}^{\mathrm{op}}$, such that $\pi_k(R^{\geq n}) \cong \pi_k(R)$ for $k \geq n$ and $\pi_k(R^{\geq n}) \cong 0$ for $k < n$. In particular, there is a natural transformation

$$\rho_{i,j} : R^{\geq i} \wedge R^{\geq j} \longrightarrow R^{\geq i+j}$$

satisfying commutativity, associativity, and unitality.

Proof of Theorem 3.36. Let R be a cofibrant connective commutative monoid in $\mathfrak{S}$ and let

$$\begin{tikzcd}
\dots \arrow[r] & \tau_{\leq 2}R \arrow[r] & \tau_{\leq 1}R \arrow[r] & \tau_{\leq 0}R \\
& \Sigma^2 H\pi_2(R) \arrow[u] & \Sigma H\pi_1(R) \arrow[u] & H\pi_0(R) \arrow[u]
\end{tikzcd}$$

be a Postnikov tower of square-zero extensions of R in the sense of Definition 3.33. To prove the theorem we need to do the following:

1. Construct $R^{\geq n}$.

2. Construct natural transformations $\rho_{i,j} : R^{\geq i} \wedge R^{\geq j} \longrightarrow R^{\geq i+j}$.

3. Show that the maps $\rho_{i,j}$ satisfy commutativity, associativity and unitality.

The procedure will be inductive. First, define $R^{\geq 0} := R$ where R was assumed to be a cofibrant connective commutative monoid in $\mathfrak{S}$ and is therefore an object in $\mathrm{Comm}\,\mathfrak{S}^{J_0^{\mathrm{op}}}$.

To construct $R^{\geq 1}$, we consider the map of commutative ring spectra $R \longrightarrow H\pi_0 R$. We can assume this map is a fibration, since if it wasn't we could factor the map in commutative ring spectra into an acyclic cofibration and a fibration. We then define $R^{\geq 1}$ to be the fiber of this map. By design, we have constructed an object $I_\bullet^{\leq 1}$ in $\mathrm{Comm}\, \mathfrak{S}^{J_1^{\mathrm{op}}}$. Commutativity, associativity and unitality follow by the definition of a symmetric R-bimodule action of R on $R^{\geq 1}$. This completes the base step in the induction.

Suppose we have an object $I_\bullet^{\leq n-1} \in \mathrm{ob}\, \mathrm{Comm}\, \mathfrak{S}^{J_{n-1}{}^{\mathrm{op}}}$ for an arbitrary $n \geq 1$. As before, we define $R^{\geq i}$ to be $I_i^{\leq n-1}$ for all $i \leq n-1$. Define $P_n := \operatorname{colim}_{\mathcal{D}_n} \tau_{\geq i} R \wedge \tau_{\geq j} R$ where $\mathcal{D}_n$ is the full subcategory of $\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}$ with objects (i, j) such that $0 < i \leq j < n$ and $i + j \geq n$. Since $I_\bullet^{\leq n-1}$ is in $\mathrm{Comm}\, \mathfrak{S}^{J_{n-1}{}^{\mathrm{op}}}$, there is a unique map $P_n \longrightarrow R^{\geq n-1}$.

The fact that the fiber sequence $\Sigma^k H\pi_k R \longrightarrow \tau_{\leq k} R \longrightarrow \tau_{\leq k-1} R$ is a square-zero extension for each k implies that the natural maps

$$\Sigma^i H\pi_i R \wedge \Sigma^j H\pi_j R \longrightarrow \Sigma^{n-1} H\pi_{i+j} R,$$

factor through 0 for each $(i, j) \in \mathcal{D}_n$. We get an induced map on fibers by considering the diagrams

$$\begin{tikzcd}
R^{\geq k} \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & \tau_{\leq k-1} R \arrow[d] \\
\Sigma^k H\pi_k R \arrow[r] & \tau_{\leq k} R \arrow[r] & \tau_{\leq k-1} R
\end{tikzcd}$$

for $k < n$. There are therefore commutative diagrams

$$\begin{tikzcd} R^{\geq i} \wedge R^{\geq j} \arrow[r] \arrow[d] & \Sigma^i H\pi_i R \wedge \Sigma^j H\pi_j R \arrow[d, "0"] \\ R^{\geq n-1} \arrow[r] & \Sigma^{n-1} H\pi_n R \end{tikzcd}$$

for each $(i, j) \in \mathcal{D}_n$, hence, the map

$$R^{\geq i} \wedge R^{\geq j} \longrightarrow R^{\geq n-1} \longrightarrow \Sigma^{n-1} H\pi_n R$$

factors through zero for each $(i, j) \in \mathcal{D}_n$.

We need the map $R^{\geq n-1} \to \Sigma^{n-1} H\pi_n R$ to be a fibration, so we use the factorization

$$\begin{tikzcd} R^{\geq n-1} \arrow[rr] \arrow[dr] & & \Sigma^{n-1} H\pi_n R \\ & \overline{R}^{\geq n-1} \arrow[ur] & \end{tikzcd}$$

into a trivial cofibration followed by a fibration.

We can define $R^{\geq n}$ to be the pullback, in the category of R-modules in $\mathcal{C}$, of the diagram

$$\begin{tikzcd} R^{\geq n} \arrow[r] \arrow[d] & 0 \arrow[d] \\ \overline{R}^{\geq n-1} \arrow[r, "f"] & \Sigma^{n-1} H\pi_n R. \end{tikzcd}$$

We then also need to replace P_n by $\overline{P}_n$ where $\overline{P}_n$ is the same colimit as P_n except that each instance of $R^{\geq n-1}$ is replaced by $\overline{R}^{\geq n-1}$. There is therefore a map $P_n \to \overline{P}_n$ and there is a map $\overline{P}_n \to \Sigma^{n-1} H\pi_{n-1} R$ that factors through the zero map by the same considerations as above.

By the universal property of the pullback, there exists a unique map g

$$\begin{tikzcd}
\overline{P}_n \arrow[dr, "g"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& R^{\geq n} \arrow[r] \arrow[d] & 0 \arrow[d] \\
& \overline{R}^{\geq n-1} \arrow[r] & \Sigma^{n-1} H\pi_{n-1} R.
\end{tikzcd}$$

By composing the maps $R^{\geq i} \wedge R^{\geq j} \to \overline{P}_n$ and $\overline{R}^{\geq n-1} \wedge R^{\geq i} \to \overline{P}_n$ with the map g, we produce the necessary maps $\rho_{i,j} : R^{\geq i} \wedge R^{\geq j} \longrightarrow R^{\geq \min\{i+j,n\}}$ where $0 < i \leq j < n$. This also proves, by construction, that they satisfy the compatibility axiom (that is, naturality of the lax symmetric monoidal functor $J_n^{\mathrm{op}} \to \mathfrak{S}$). The factor swap map produces all the maps

$$\rho_{i,j} : R^{\geq i} \wedge R^{\geq j} \to R^{\geq \min\{i+j,n\}}$$

where $i > j$ and the commutativity and compatibility necessary for those maps as well. The maps $\rho_{0,n}$ and $\rho_{n,0}$ are the R-module action maps that we produced by working in the category of R-modules and again by construction these maps satisfy commutativity and compatibility with the other maps. Unitality is also easily satisfied for each $\rho_{i,j}$ with $i, j \in \{0, \ldots, n\}$, since all these maps are R-module maps.

We just need to check associativity. By assumption, we have associativity for all the maps $\rho_{i,j}$ where $i, j < n$, we therefore just need to show that the associativity diagrams involving the maps $\rho_{i,j}$ for i or j equal to n. Since the symmetric monoidal product on

R-modules is associative, we know that, for $i, j, k \in \{0, n\}$, the diagrams

$$\begin{tikzcd} R^{\geq i} \wedge R^{\geq j} \wedge R^{\geq k} \arrow[r] \arrow[d] & R^{\geq i} \wedge R^{\geq j+k} \arrow[d] \\ R^{\geq i+j} \wedge R^{\geq k} \arrow[r] & R^{\geq i+j+k} \end{tikzcd}$$

commute. We also know, by construction, that the diagram

$$\begin{tikzcd} R^{\geq i} \wedge R^{\geq j} \arrow[d] \arrow[dr] & \\ R^{\geq n-1} & R^{\geq n} \arrow[l] \end{tikzcd} \tag{3.7}$$

commutes for all $i + j \geq n$. The diagram

$$\begin{tikzcd} R^{\geq i} \wedge R^{\geq n} \arrow[r] \arrow[d] & R^{\geq n} \arrow[d] & R^{\geq n} \wedge R^{\geq i} \arrow[l] \arrow[d] \\ R^{\geq i} \wedge R^{\geq n-1} \arrow[r] & R^{\geq n-1} & R^{\geq n-1} \wedge R^{\geq i} \arrow[l] \end{tikzcd} \tag{3.8}$$

also commutes by construction.

We need to show that for $i, j, k \in \{0, 1, ..., n\}$ with either i, j, or k equal to n, then

$$\begin{tikzcd} R^{\geq i} \wedge R^{\geq j} \wedge R^{\geq k} \arrow[r] \arrow[d] & R^{\geq i} \wedge R^{\geq j+k} \arrow[d] \\ R^{\geq i+j} \wedge R^{\geq k} \arrow[r] & R^{\geq n} \end{tikzcd} \tag{3.9}$$

commutes. This follows by combining the commutativity of Diagram 3.7, Diagram 3.8, and the diagrams of the form of Diagram 3.9 when $i, j, k < n$, and using the fact that $R^{\geq n} \to R^{\geq n-1}$ is a monomorphism, since it is the pullback of a monomorphism in $\mathfrak{S}$ by construction, and hence it is retractile; i.e. when we say monomorphisms are retractile we mean that if $f \circ g$ is a monomorphism then f is also a monomorphism.

We have therefore produced an object in $\mathrm{Comm}\,\mathfrak{S}^{J_n^{\mathrm{op}}}$. By induction, we can therefore produce an object in $\mathrm{Comm}\,\mathfrak{S}^{\mathbb{N}^{\mathrm{op}}}$ and then cofibrantly replace it to produce a decreasingly filtered commutative monoid in $\mathfrak{S}$, denoted $R^{\geq\bullet}$, as desired. □

Remark 3.37. Since we have functorial factorizations of maps and functorial cofibrant replacement in our setting [29], the above theorem is entirely functorial, in other words, a map of connective commutative ring spectra $A \to B$ induces a map of Whitehead towers $A^{\geq\bullet} \to B^{\geq\bullet}$ compatible with the multiplication maps $\rho^A_{i,j}$ and $\rho^A_{i,j}$. This induces a map of associated graded commutative monoids in $\mathfrak{S}$

$$E_0A^{\geq\bullet} \longrightarrow E_0B^{\geq\bullet}$$

and a map of THH-May spectral sequences

$$\begin{tikzcd} E_{*,*}(X_\bullet \otimes E_0^*(A^{\geq\bullet})) \arrow[r, Rightarrow] \arrow[d] & E_*(X_\bullet \otimes A) \arrow[d] \\ E_{*,*}(X_\bullet \otimes E_0^*(B^{\geq\bullet})) \arrow[r, Rightarrow] & E_*(X_\bullet \otimes B). \end{tikzcd}$$

Example 3.38. Assume a prime $p \geq 3$ is fixed. Let j be a cofibrant replacement in $\mathrm{Comm}\,\mathfrak{S}$, for the commutative ring spectrum $K(\mathbb{F}_q)_p$ where q is a prime power that topologically generates $\mathbb{Z}_p^\times$. Then by Theorem 3.34, we produce a decreasingly filtered commutative monoid in $\mathfrak{S}$. We will let $j^{\geq\bullet}$ be the decreasingly filtered commutative monoid in $\mathfrak{S}$ that we produce. The associated graded $E_0^* j^{\geq \bullet}$ is additively equivalent to

$$H\pi_0 j \vee \Sigma^{2p-3} H\pi_{2p-3} j \vee \Sigma^{4p-5} H\pi_{4p-5} j \vee \ldots$$

or more succinctly $H\pi_*(j)$. Its homotopy groups $\pi_*(E_0^* j^{\geq \bullet}) \cong \pi_*(j)$, but it is a generalized Eilenberg-Maclane spectrum.

Corollary 3.39. By Theorem 3.36 and the construction of the THH-May spectral sequence, we produce a bound on topological Hochschild homology of any connective commutative ring spectrum R:

$$\#\pi_k THH(R) \leq \#\pi_k THH(H\pi_* R)$$

where $\#S$ for a set S indicates the cardinality of the set S.

CHAPTER 4 THH OF THE CONNECTIVE IMAGE OF J

We will assume that $p \geq 3$ and q is a prime power that topologically generates $\mathbb{Z}_p^\times$, which denotes the units in the p-adic integers. Recall that under these conditions, there are equivalences

$$j_p \simeq K(\mathbb{F}_q)_p \simeq \tau_{\geq 0} L_{K(1)} S$$

where $\tau_{\geq 0}$ indicates the connective cover functor. We will therefore simply write j in this chapter for $K(\mathbb{F}_q)_p$ and assume that it is cofibrant in Comm $\mathfrak{S}$, since we could cofibrantly replace it in Comm $\mathfrak{S}$ if it was not already cofibrant.

4.1 mod (p, v_1)-homotopy of THH of the connective image of J

Recall, from Chapter 2, the construction that takes a decreasingly filtered commutative monoid I in $\mathfrak{S}$ as input and produces a May-type spectral sequence

$$E^2_{s,t} = G_{s,t} THH(E_0 I) \Rightarrow G_s THH(I_0)$$

for any connective generalized homology theory G, which we we call the G-THH-May spectral sequence. Also, we produced a Whitehead-type decreasingly filtered commutative monoid in $\mathfrak{S}$, denoted $j^{\geq \bullet}$, associated to a cofibrant commutative ring spectrum model for p-complete connective image of J. We therefore have a spectral sequence

$$E^2_{s,t} = G_{s,t} THH(E_0 j^{\geq \bullet}) \Rightarrow G_s THH(j). \tag{4.1}$$

The purpose of this section is to compute this spectral sequence in the case $G = V(1)$.

In the case where $G = H\mathbb{F}_p$, the input of the spectral sequence is calculable, and the output is already known due the work of Angeltveit-Rognes [6]. This computation, therefore, will allow us to see the differentials in the $V(1)$-THH-May spectral sequence that are also detected in the $H\mathbb{F}_p$-THH-May spectral sequence. To begin, let us recall the computation of Angeltveit-Rognes.

Theorem 4.1 (Angeltveit-Rognes [6]). There is an isomorphism

$$H\mathbb{F}_{p_*}(j) \cong P(\tilde{\xi}_1^p, \tilde{\xi}_2, \bar{\xi}_3, \ldots) \otimes E(\tilde{\tau}_2, \bar{\tau}_3, \ldots) \otimes E(b) \cong (\mathcal{A}//A(1))_* \otimes E(b)$$

where all the elements in $(\mathcal{A}//A(1))_*$ besides $\tilde{\tau}_2$, $\tilde{\xi}_1^p$, and $\tilde{\xi}_2$, and b have the usual $\mathcal{A}_*$-coaction and the coaction on the remaining elements $\tilde{\tau}_2$, $\tilde{\xi}_1^p$, $\tilde{\xi}_2$, and b are

$$\begin{aligned}
\psi(b) &= 1 \otimes b \\
\psi(\tilde{\xi}_1^p) &= 1 \otimes \tilde{\xi}_1^p - \tau_0 \otimes b + \bar{\xi}_1^p \otimes 1 \\
\psi(\tilde{\xi}_2) &= 1 \otimes \tilde{\xi}_2 + \bar{\xi}_1 \otimes \tilde{\xi}_1^p + \tau_1 \otimes b + \bar{\xi}_2 \otimes 1 \\
\psi(\tilde{\tau}_2) &= 1 \otimes \tilde{\tau}_2 + \bar{\tau}_1 \otimes \tilde{\xi}_1^p + \bar{\tau}_0 \otimes \tilde{\xi}_2 - \tau_1\tau_0 \otimes b + \bar{\tau}_2 \otimes 1.
\end{aligned}$$

There is also an isomorphism

$$H\mathbb{F}_{p_*}(THH(j)) \cong H\mathbb{F}_{p_*}(j) \otimes E(\sigma\tilde{\xi}_1^p, \sigma\tilde{\xi}_2) \otimes P(\sigma\tilde{\tau}_2) \otimes \Gamma(\sigma b)$$

of $\mathcal{A}_*$-comodules and $H\mathbb{F}_{p_*}(j)$-algebras. The $\mathcal{A}_*$-coaction is given by using the formula

$$\psi(\sigma x) = (1 \otimes \sigma) \circ \psi(x)$$

and the previously stated coactions.

Note that Angeltveit and Rognes use a tilde over a symbol, for example $\tilde{x}$ to signify that the element has a different coaction then the coaction on x or $\bar{x}$. We now want to compute the input of the spectral sequence. First, we note that as described in Example 3.38, $S/p \wedge E_0 j^{\geq \bullet}$ is an $H\mathbb{F}_p$ algebra and hence $V(1) \wedge E_0 j^{\geq \bullet}$ is also an $H\mathbb{F}_p$ algebra. It is known more generally that $THH(R)$ is an R algebra, so $V(1) \wedge THH(E_0 j^{\geq \bullet})$ is a $V(1) \wedge E_0 j^{\geq \bullet}$-algebra and in particular an $H\mathbb{F}_p$-module. We can therefore apply the following lemma, which can be found in Ausoni-Rognes [10, Lem. 4.1], though certainly the lemma predates their work and they refer to Whitehead as the originator. We provide our own proof.

Lemma 4.2. Let M be an $H\mathbb{F}_p$-module. Then M is equivalent to a wedge of suspensions of $H\mathbb{F}_p$, and the Hurewicz map

$$\pi_*(M) \longrightarrow H\mathbb{F}_{p_*}(M)$$

induces an isomorphism between $\pi_*(M)$ and the subalgebra of $\mathcal{A}_*$-comodule primitives contained in $H\mathbb{F}_{p_*}(M)$.

Proof. We recall that in the language of Hopkins-Smith [33] the spectrum $H\mathbb{F}_p$ is a field spectrum, so any $H\mathbb{F}_p$-module is a wedge of suspensions of $H\mathbb{F}_p$. Observe that the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_p; H\mathbb{F}_{p_*}(M)) \Rightarrow \pi_{t-s}(M)$$

collapses to the $s = 0$ line, and therefore the input of the spectral sequence is

$$\mathrm{Hom}_{\mathcal{A}_*}(\mathbb{F}_p; H\mathbb{F}_{p_*}(M)).$$

Since $\mathcal{A}_*$-comodule maps from $\mathbb{F}_p$ to $H\mathbb{F}_{p_*}(M)$ are equivalent to $\mathcal{A}_*$-comodule primitives in $H\mathbb{F}_{p_*}(M)$ and $H\mathbb{F}_{p_*}(M)$ is a $\mathcal{A}_*$-comodule algebra, the result follows. □

Therefore, computing the algebra of comodule primitives in $H\mathbb{F}_{p_*}(V(1) \wedge THH(E_0 j^{\geq \bullet}))$ will suffice for computing the input of the $V(1)$-THH-May spectral sequence.

Lemma 4.3. There is an isomorphism

$$\pi_*(H\mathbb{F}_p \wedge E_0 j^{\geq \bullet}) \cong (A//E(0))_* \otimes P(v_1) \otimes E(\alpha_1).$$

Proof. As observed in Example 3.38

$$S/p \wedge E_0 j^{\geq \bullet} \simeq H\mathbb{F}_p \vee \bigvee_{i \geq 1} \Sigma^{(2p-2)i-1} H\mathbb{F}_p \vee \Sigma^{(2p-2)i} H\mathbb{F}_p$$

and $\pi_*(S/p \wedge E_0 j^{\geq \bullet}) \cong P(v_1) \otimes E(\alpha_1)$. By using the equivalence $H\mathbb{Z} \wedge S/p \simeq H\mathbb{F}_p$ we get

$$H\mathbb{F}_p \wedge E_0 j^{\geq \bullet} \simeq H\mathbb{Z} \wedge S/p \wedge E_0 j^{\geq \bullet}$$

so additively

$$H\mathbb{F}_p \wedge E_0 j^{\geq \bullet} \simeq H\mathbb{Z} \wedge (H\mathbb{F}_p \vee \bigvee_{i \geq 1} \Sigma^{(2p-2)i-1} H\mathbb{F}_p \vee \Sigma^{(2p-2)i} H\mathbb{F}_p).$$

We can write this as

$$(H\mathbb{Z} \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \left(H\mathbb{F}_p \vee \bigvee_{i \geq 1} \Sigma^{(2p-2)i-1} H\mathbb{F}_p \vee \Sigma^{(2p-2)i} H\mathbb{F}_p \right)$$

and use the collapse of the Künneth spectral sequence to produce the desired isomorphism.

□

Proposition 4.4. There is an isomorphism

$$H\mathbb{F}_{p_*}(THH(E_0 j^{\geq \bullet})) \cong (A//E(0))_* \otimes P(v_1) \otimes E(\alpha_1) \otimes E(\sigma\bar{\xi}_1, \sigma v_1) \otimes P(\sigma\bar{\tau}_2) \otimes \Gamma(\sigma\alpha_1)$$

where the $\mathcal{A}_*$-coaction is the usual one, that is the coproduct in $\mathcal{A}_*$, on elements in $(A//E(0))_*$. The coaction on α_1 and v_1 is given by the formulas

$$\psi(\alpha_1) = 1 \otimes \alpha_1$$

$$\psi(v_1) = \bar{\tau}_0 \otimes \alpha_1 + 1 \otimes v_1$$

and the coproduct on the rest of the classes, besides $\gamma_{p^k}(\sigma\alpha_1)$ uses the formula

$$\psi(\sigma x) = (1 \otimes \sigma) \circ \psi(x).$$

Proof. We already know that $\pi_*(H\mathbb{F}_p \wedge E_0 j^{\geq \bullet}) \cong (\mathcal{A}//E(0))_* \otimes P(v_1) \otimes E(\alpha_1)$. We can use the Bökstedt spectral sequence,

$$E_2^{s,t} = HH_*(H\mathbb{F}_{p_*} E_0 j^{\geq \bullet})) \Rightarrow H\mathbb{F}_{p_*}(THH(E_0 j^{\geq \bullet}))$$

to compute $H\mathbb{F}_{p_*}(THH(E_0 j^{\geq \bullet}))$. The input is

$$HH_*((\mathcal{A}//E(0))_* \otimes P(v_1) \otimes E(\alpha_1)) \cong (\mathcal{A}//E(0))_* \otimes P(v_1) \otimes E(\alpha_1) \otimes E(\sigma\bar{\xi}_i | i \geq 1) \otimes \Gamma(\sigma\bar{\tau}_i | i \geq 1) \otimes E(\sigma v_1) \otimes \Gamma(\sigma\alpha_1)$$

Using the well known facts that $\mathrm{Tor}^{E(x)}(k;k) \cong \Gamma(\sigma x)$ and $\mathrm{Tor}^{P(y)}(k,k) \cong E(\sigma y)$ for $|x| = 2i-1$ and $|y| = 2j$ for any $i, j \in \mathbb{N}$, one can show using classical methods that there are isomorphisms $HH_*(E(x)) \cong E(x) \otimes \Gamma(\sigma x)$ when $|x|$ is odd, and $HH_*(P(y)) \cong P(y) \otimes E(\sigma y)$ when $|y|$ is even (see [7], for example, for a detailed calculation). Using these calculations and the fact that the functor $HH_*(-)$ has the property that, when A and B are graded $\mathbb{F}_p$-algebras, there is an isomorphism

$$HH_*(A \otimes B) \cong HH_*(A) \otimes HH_*(B),$$

we can compute the input of the Bökstedt spectral sequence. We observe that by the definition of $E_0 j^{\geq \bullet}$ there is a map of commutative ring spectra $H\mathbb{Z} \longrightarrow E_0 j^{\geq \bullet}$, and therefore a map of Bökstedt spectral sequences,

$$\begin{tikzcd}
HH_*(H\mathbb{F}_{p*}H\mathbb{Z}) \arrow[r, Rightarrow] \arrow[d] & H\mathbb{F}_{p*}(THH(H\mathbb{Z})) \arrow[d] \\
HH_*(H\mathbb{F}_{p*}E_0 j^{\geq \bullet}) \arrow[r, Rightarrow] & H\mathbb{F}_{p*}(THH(E_0 j^{\geq \bullet}))
\end{tikzcd}$$

where the input of the top spectral sequence is

$$HH_*(H\mathbb{F}_{p*}H\mathbb{Z}) \cong (\mathcal{A}//E(0))_* \otimes E(\sigma\bar{\xi}_i | i \geq 1) \otimes \Gamma(\sigma\bar{\tau}_i | i \geq 1).$$

Recall that due to Bökstedt [17] (see also Ausoni [7]), there are differentials

$$d_{p-1}(\gamma_{p+k}(\sigma\bar{\tau}_i)) \doteq \sigma\bar{\xi}_{i+1}\gamma_k(\sigma\bar{\tau}_i)$$

for $k \geq 0$, $i \geq 1$. Since these classes map to classes of the same names in the Bökstedt spectral sequence for $E_0 j^{\geq \bullet}$, the same differentials occur in the Bökstedt spectral sequence for $E_0 j^{\geq \bullet}$.

The only remaining possible differentials, for bidegree reasons, are possible differentials on the classes $\gamma_{p^k}(\sigma\alpha_1)$. We claim that these differentials do not occur and we will prove this by contradiction. Suppose

$$d_r(\gamma_{p^k}(\sigma\alpha_1)) \neq 0 \tag{4.2}$$

for some r and some $k > 0$. Then we observe that in degree $2p^k(p^2 - p)$, the dimension of $(H\mathbb{F}_p)_*(THH(E_0 j^{\geq \bullet}))$ as an $\mathbb{F}_p$ vector space is strictly less than the dimension of $(H\mathbb{F}_p)_*(THH(j))$ in the same degree. Since the $H\mathbb{F}_p$-THH-May spectral sequence with input $(H\mathbb{F}_p)_*(THH(E_0 j^{\geq \bullet}))$ computes $(H\mathbb{F}_p)_*(THH(j))$, this leads to a contradiction. Therefore, the presence of any differential of the form (4.2) contradicts the known computation of $(H\mathbb{F}_p)_*(THH(j))$ due to Angeltveit-Rognes [6]. Thus, no differentials of the form (4.2) occur.

There is no further room for differentials for bidegree reasons so the E_∞-page for $E_0 j^{\geq \bullet}$ is

$$E_\infty^{*,*} \cong (\mathcal{A}//E(0))_* \otimes P(v_1) \otimes E(\alpha_1) \otimes E(\sigma\bar{\xi}_1) \otimes P_p(\sigma\bar{\tau}_i | i \geq 1) \otimes E(\sigma v_1) \otimes \Gamma(\sigma\alpha_1)$$

We therefore just need to resolve hidden extensions. Due to Bökstedt [17], we know that σ commutes with the Dyer-Lashof operations

$$Q^{p^i}(\bar{\tau}_i) = \bar{\tau}_{i+1}$$

computed by Steinberger [20] so we have relations

$$(\sigma\bar{\tau}_i)^p = Q^{p^i}(\sigma\bar{\tau}_i) = \sigma(Q^{p^i}(\bar{\tau}_i)) = \sigma\bar{\tau}_{i+1}$$

for all $k \geq 0$. These produce hidden multiplicative extensions

$$\sigma\bar{\tau}_{i+1} = (\sigma\bar{\tau}_i)^p$$

and therefore, the multiplicative structure on the output is

$$H\mathbb{F}_{p_*}(THH(E_0 j^{\geq\bullet})) \cong (\mathcal{A}//E(0))_* \otimes P(v_1) \otimes E(\alpha_1) \otimes E(\sigma\bar{\xi}_1) \otimes P(\sigma\bar{\tau}_1) \otimes E(\sigma v_1) \otimes \Gamma(\sigma\alpha_1).$$

For the coaction, recall that the class v_1 arose as the Bockstein on α_1 in $S/p \wedge E_0 j^{\geq\bullet}$, and therefore the coaction on v_1 is

$$\psi(v_1) = \bar{\tau}_0 \otimes \alpha_1 + 1 \otimes v_1.$$

The class dual to α_1 cannot be the Bockstein of a class because $\beta^2 = 0$. Since the class dual to α_1 is in degree $2p-3$ and the lowest class in $\mathcal{A}$ besides β is P^1 in degree $2p-2$, the class α_1 must be a comodule primitive with coaction

$$\psi(\alpha_1) = 1 \otimes \alpha_1.$$

The rest of the coactions are determined by the coproduct in $\mathcal{A}_*$ and the formula

$$\psi(\sigma x) = (1 \otimes \sigma) \circ \psi(x)$$

□

We now use the $H\mathbb{F}_p$-THH-May spectral sequence in a case where the output is known due to Angeltveit-Rognes [6] in order to detect differentials in the $V(1)$-THH-May spectral sequence.

Proposition 4.5. The only differentials in the $H\mathbb{F}_p$-THH-May spectral sequence

$$(H\mathbb{F}_p)_{s,t}(THH(E_0 j^{\geq \bullet})) \Rightarrow (H\mathbb{F}_p)_s(THH(j))$$

for j are as follows:

$$d_1(\bar{\xi}_1) \doteq \alpha_1 \quad d_1(\sigma\bar{\xi}) \doteq \sigma\alpha_1$$
$$d_1(\bar{\tau}_1) \doteq v_1 \quad d_1(\sigma\bar{\tau}_1) \doteq \sigma v_1.$$

The surviving classes $\bar{\xi}_1^{p-1}\alpha_1$, $\sigma\bar{\xi}\gamma_{p-1}\sigma\alpha_1$, $\gamma_p(\sigma\alpha_1)$, $(\sigma\bar{\tau}_1)^p$, and $(\sigma\bar{\tau}_1)^{p-1}\sigma v_1$ map to classes b, $\sigma\tilde{\xi}_1^p$, σb, $\sigma\tilde{\tau}_2$, and $\sigma\tilde{\xi}_2$ in $H\mathbb{F}_{p_*}THH(j)$ and all other surviving classes map to classes of the same name.

Proof. The output of the spectral sequence is trivial in the range $0 < s < 2p^2 - 2p - 1$, due to the computation of Angeltveit-Rognes [6], and each of the classes which are the source or target of one of the claimed nonzero differentials lie in this range. There are no other possible differentials besides d_1 differentials in this range, so this forces the specified d_1 differentials. The resulting $E_2 = E_\infty$-page is isomorphic to $H\mathbb{F}_{p_*}THH(j)$ with the specified correspondence in the proposition. $\square$

Remark 4.6. The behavior of the differentials above leads us to conjecture that d_1 commutes with the operation σ in the $H\mathbb{F}_p$-THH-May spectral sequence.

Proposition 4.7. There is an isomorphism

$$V(1)_*(THH(E_0 j^{\geq\bullet})) \cong E(\alpha_1, \lambda_1, \epsilon_1, \sigma\tilde{v_1}) \otimes P(\mu_1, \tilde{v}_1) \otimes \Gamma(\sigma\alpha_1)$$

where $|\epsilon_1| = |\lambda_1| = |\sigma\tilde{v}_1| = 2p-1$, $|\alpha_1| = 2p-3$, $|\mu_1| = 2p$, $|\tilde{v}_1| = 2p-2$, and $|\sigma\alpha_1| = 2p-2$.

Proof. We can compute $H\mathbb{F}_{p_*}(V(1) \wedge THH(j))$ where the input is $H\mathbb{F}_{p_*}(V(1) \wedge THH(E_0 j^{\geq\bullet}))$, using the $H\mathbb{F}_p \wedge V(1)$-THH-May spectral sequence. The differentials are the same and the classes $\bar{\tau}_0$ and $\bar{\tau}_1$ map to classes of the same name in the output. This is useful because there is a map of spectral sequences from the $V(1)$-THH-May spectral sequence to the $H\mathbb{F}_p \wedge V(1)$-THH-May spectral sequence induced by the map of S-algebras

$$S \wedge V(1) \xrightarrow{\eta \wedge \mathrm{id}_{V(1)}} H\mathbb{F}_p \wedge V(1)$$

where $\eta : S \to H\mathbb{F}_p$ is the unit map of $H\mathbb{F}_p$ as a ring spectrum. Due to Lemma 4.2, the map

$$V(1)_*(THH(E_0 j^{\geq\bullet})) \longrightarrow (H\mathbb{F}_p \wedge V(1))_*(THH(E_0 j^{\geq\bullet}))$$

includes $V(1)_*(THH(E_0 j^{\geq\bullet}))$ into $(H\mathbb{F}_p \wedge V(1))_*(THH(E_0 j^{\geq\bullet}))$ as the $\mathcal{A}_*$-comodule primitives. By Lemma 4.4, the elements

$$\{\alpha_1, v_1 - \bar{\tau}_0\alpha_1, \sigma\tilde{\tau}_1 - \bar{\tau}_0\bar{\xi}_1, \sigma\bar{\xi}_1, \sigma v_1 - \bar{\tau}_0\sigma\alpha_1, \tilde{\tau}_1 - \bar{\tau}_1\}$$

are comodule primitives where we write $\tilde{\tau}_1$ to distinguish the class in $H\mathbb{F}_{p_*}(THH(E_0 j^{\geq\bullet}))$ from the class $\bar{\tau}_1 \in H\mathbb{F}_{p_*}(V(1))$. We rename these classes respectively

$$\{\alpha_1, \tilde{v}_1, \mu_1, \lambda_1, \sigma\tilde{v}_1, \epsilon_1\}.$$

In particular,

$$(H\mathbb{F}_p \wedge V(1))_*(THH(E_0 j^{\geq\bullet})) \cong \mathcal{A}_* \otimes E(\epsilon_1) \otimes P(\sigma v_1) \otimes E(\alpha_1) \otimes E(\sigma\bar{\xi}_1) \otimes P(\sigma\bar{\tau}_1) \otimes E(\sigma v_1) \otimes \Gamma(\sigma\alpha_1)$$

so the classes $\gamma_{p^k}(\sigma\alpha_1)$, or possibly the difference of $\gamma_{p^k}(\sigma\alpha_1)$ and a correcting term, are also comodule primitives and by possible abuse of notation we still write $\gamma_{p^k}(\sigma\alpha_1)$ for these elements. Thus, the result follows from Lemma 4.4 and Lemma 4.2. □

We now consider the map of THH-May spectral sequences

$$\begin{tikzcd} V(1)_*(THH(E_0 j^{\geq\bullet})) \arrow[r, Rightarrow] \arrow[d, "f"] & V(1)_*(THH(j)) \arrow[d] \\ (H\mathbb{F}_p \wedge V(1))_*(THH(E_0 j^{\geq\bullet})) \arrow[r, Rightarrow] & (H\mathbb{F}_p \wedge V(1))_*(THH(j)) \end{tikzcd}$$

induced by the map

$$\eta \wedge \mathrm{id}_{V(1)} : S \wedge V(1) \longrightarrow H\mathbb{F}_p \wedge V(1)$$

where $\eta : S \to H\mathbb{F}_p$ is the unit map of the ring spectrum $H\mathbb{F}_p$.

Proposition 4.8. The only d_1 differentials are

$$d_1(\lambda_1) \doteq \sigma\alpha_1,$$
$$d_1(\epsilon_1) \doteq \tilde{v}_1, \text{ and}$$
$$d_1(\mu_1) \doteq \sigma\tilde{v}_1$$

in the $V(1)$-THH-May spectral sequence. The E_2-page of the $V(1)$-THH-May spectral sequence is therefore

$$E_2^{*,*} = E(\alpha_1, \lambda_1\gamma_{p-1}(\sigma\alpha_1), (\mu_1)^{p-1}\sigma\tilde{v}_1) \otimes P((\mu_1)^p) \otimes \Gamma(\sigma b)$$

Proof. The classes

$$\{\tilde{v}_1, \mu_1, \lambda_1, \sigma\alpha_1, \sigma\tilde{v}_1, \epsilon_1\}$$

in the $V(1)$-THH-May spectral sequence map to the classes

$$\{v_1 - \bar{\tau}_0\alpha_1, \sigma\tilde{\tau}_1 - \bar{\tau}_0\bar{\xi}_1, \sigma\bar{\xi}_1, \sigma\alpha_1, \sigma v_1 - \bar{\tau}_0\sigma\alpha_1, \tilde{\tau}_1 - \bar{\tau}_1\}$$

in the $H\mathbb{F}_p \wedge V(1)$-THH-May spectral sequence under the map of spectral sequences f. There are trivial differentials

$$d_1(\bar{\tau}_0) = d_1(\bar{\tau}_1) = 0$$

and nontrivial differentials

$$d_1(\bar{\xi}_1) \doteq \alpha_1 \quad d_1(\sigma\bar{\xi}) \doteq \sigma\alpha_1$$
$$d_1(\tilde{\tau}_1) \doteq v_1 \quad d_1(\sigma\bar{\tau}_1) \doteq \sigma v_1$$

in the $H\mathbb{F}_p \wedge V(1)$-THH-May spectral sequence by Propositions 4.5 and 4.7. We will use the formula $fd_1 = d_1 f$ to compute the differentials. Notice that the map f is injective on the E_2-page of the spectral sequences so it makes sense to use the formula $d_1(x) = f^{-1}d_1(f(x))$. We therefore produce differentials

$$d_1(\lambda_1) = f^{-1}(d_1(\sigma\bar{\xi}_1)) = f^{-1}(\sigma\alpha_1) = \sigma\alpha_1,$$
$$d_1(\epsilon_1) = f^{-1}d_1(\tilde{\tau}_1 - \bar{\tau}_1) = f^{-1}(v_1) = \tilde{v}_1,$$
$$d_1(\mu_1) = f^{-1}(d_1(\sigma\tilde{\tau}_1 - \bar{\tau}_0\bar{\xi}_1)) = f^{-1}(v_1 - \bar{\tau}_0\alpha_1) = \tilde{v_1}$$

in the $V(1)$-THH-May spectral sequence as desired. There are no other possible d_1 differentials for bidegree reasons. □

Lemma 4.9. There is an isomorphism

$$V(1)_*(THH(j;\ell)) \cong E(\lambda_1', \lambda_2) \otimes P(\mu_2) \otimes \Gamma(\sigma b)$$

Proof. Note that there are equivalences

$$V(1) \wedge THH(j;\ell) \simeq THH(j; H\mathbb{F}_p) \simeq H\mathbb{F}_p \wedge_j THH(j)$$

and that $H\mathbb{F}_p \wedge_j THH(j)$ is a $H\mathbb{F}_p \wedge_j j$-module, i.e. it is a $H\mathbb{F}_p$-module. We can therefore apply Lemma 4.2 and Theorem 4.1 to compute $V(1)_*THH(j;\ell) \cong \pi_*THH(j;H\mathbb{F}_p)$. The result is the algebra of comodule primitives in

$$H\mathbb{F}_{p_*}(THH(j;H\mathbb{F}_p)) \cong \mathcal{A}_* \otimes E(\sigma\tilde{\xi}_1^p, \sigma\tilde{\xi}_2) \otimes P(\sigma\tilde{\tau}_2) \otimes \Gamma(\sigma b),$$

which can be seen by the collapse of the Künneth spectral sequence. The algebra of comodule primitives is isomorphic to $E(\lambda_1', \lambda_2) \otimes P(\mu_2) \otimes \Gamma(\sigma b)$ where

$$\begin{aligned} \mu_2 &= \sigma\tau_2 - \bar{\tau}_0\sigma\tilde{\xi}_2 - \bar{\tau}_1\sigma\tilde{\xi}_1^p + \tau_0\tau_1\sigma b, & \lambda_1' &= \sigma\tilde{\xi}_1^p - \tau_0\sigma b, \\ \lambda_2 &= \sigma\bar{\xi}_2 - \bar{\xi}_1\sigma\tilde{\xi}_1^p - \tau_1\sigma b, \text{ and} & \sigma b &= \sigma b. \end{aligned}$$

□

We have another approach to computing $THH_*(j; j/(p, v_1)) = V(1)_*(THH(j))$, as a $V(1)_*$-module, but not as graded rings, by filtering the coefficients $j/(p, v_1)$ using the short filtration

$$0 \longrightarrow \Sigma^{2p-3} H\mathbb{F}_p \longrightarrow j/(p, v_1)$$

with associated graded j-module $H\mathbb{F}_p \ltimes \Sigma^{2p-3} H\mathbb{F}_p$, which multiplicatively has the structure of the trivial square-zero extension of $H\mathbb{F}_p$ by $\Sigma^{2p-3} H\mathbb{F}_p$. We use the THH-May spectral sequence with filtered coefficients as follows

$$THH_{s,t}(j; H\mathbb{F}_p \ltimes \Sigma^{2p-3} H\mathbb{F}_p) \to THH_s(j; j/(p, v_1)).$$

This spectral sequence reduces to the long exact sequence

$$\begin{tikzcd} \dots \arrow[r] & \pi_{k-2p+3}(THH(j; H\mathbb{F}_p) \arrow[r] & \pi_k(THH(j; j/(p, v_1)) \arrow[dll, rounded corners] \\ \pi_k(THH(j; H\mathbb{F}_p)) \arrow[r] & \pi_{k-2p+2}(THH(j; H\mathbb{F}_p)) \arrow[r] & \dots \end{tikzcd} \tag{4.3}$$

.

where two out of three terms are known. We claim that this exact sequence demonstrates that the $V(1)$-THH-May spectral sequence cannot collapse at E_2. The author owes

Eva Höning for giving some evidence that there must be a longer differential in personal communication, since the author originally had an argument that said that the differential on $\lambda_2 = (\mu_1)^{p-1}\sigma\tilde{v}_1$ was zero.

Proposition 4.10. There is a differential

$$d_{p-1}((\mu_1)^{p-1}\sigma\tilde{v}_1) \doteq \alpha_1\lambda_1\gamma_{p-1}(\sigma\alpha_1)$$

in the $V(1)$-THH-May spectral sequence and no remaining differentials.

Proof. There is only one remaining possible differential for bidegree reasons, which is the stated differential $d_{p-1}((\mu_1)^{p-1}\sigma\tilde{v}_1) \doteq \alpha_1\lambda_1\gamma_{p-1}(\sigma\alpha_1)$. Suppose the $V(1)$-THH-May spectral sequence computing $\pi_*(THH(j; j/(p, v_1))$ collapses at the E_2-page. Then, the long exact sequence (4.3) takes the form

$$\begin{tikzcd}
\Sigma^{2p-3}E(\lambda_1', \lambda_2) \otimes P(\mu_2) \otimes \Gamma(\sigma b) \arrow[r] & E(\alpha_1, \lambda_1\gamma_{p-1}(\sigma\alpha_1), (\mu_1)^{p-1}\sigma\tilde{v}_1) \otimes P((\mu_1)^p) \otimes \Gamma(\sigma b) \arrow[d] \\
 & E(\lambda_1', \lambda_2) \otimes P(\mu_2) \otimes \Gamma(\sigma b) \arrow[ul, dashed]
\end{tikzcd}$$

where the dotted arrow indicates a shift in degree by 1. In particular, in degree $2p^2 - 1$ and $2p^2 - 2$ we have the exact sequence

$$0 \longrightarrow \mathbb{F}_p\{(\mu_1)^{p-1}\sigma\tilde{v}_1\} \longrightarrow \mathbb{F}_p\{\lambda_2\} \longrightarrow \mathbb{F}_p\{\lambda_1'\} \longrightarrow \mathbb{F}_p\{\alpha_1\lambda_1\gamma_{p-1}(\sigma\alpha_1)\} \longrightarrow 0.$$

We can therefore determine if there should be a differential as stated by determining if the map

$$\mathbb{F}_p\{\lambda_2\} \longrightarrow \mathbb{F}_p\{\lambda_1'\}$$

is nontrivial. To determine this, we note that the boundary map is exactly the map

$$V(1)_*(THH(j;\ell)) \longrightarrow V(1)_*(THH(j;\Sigma^{2p-2}\ell))$$

induced by the map $\ell \longrightarrow \Sigma^{2p-2}\ell$ given by $1-\psi_q$ where q is the q-th Adams operation. This map induces multiplication by P^1 in cohomology

$$H\mathbb{F}_p^*(\Sigma^{2p-2}\ell) = \Sigma^{2p-2}\mathcal{A}//E(1) \xrightarrow{P^1} \mathcal{A}//E(1) = H\mathbb{F}_p^*(\ell).$$

In the dual, we therefore know that the map

$$(P^1)^* : H\mathbb{F}_{p_*}(\ell) = (\mathcal{A}//E(1))_* \longrightarrow \Sigma^{2p-2}(\mathcal{A}//E(1))_* = H\mathbb{F}_{p_*}(\Sigma^{2p-2}\ell)$$

sends classes of the form $\bar{\xi}_1 y$ to y and the map sends all other classes to zero. The same will therefore be true for the induced map

$$\begin{tikzcd}
H\mathbb{F}_{p_*}(V(1)\wedge THH(j;\ell)) \arrow[rr] & & H\mathbb{F}_{p_*}(V(1)\wedge THH(j;\Sigma^{2p-2}\ell)) \\
H\mathbb{F}_{p_*}(V(1)\wedge THH(j)) \otimes_{H\mathbb{F}_{p_*}(j)} H\mathbb{F}_{p_*}(\ell) \arrow[u, "\cong"] \arrow[rr, "\mathrm{id}\otimes_{H\mathbb{F}_{p_*}(j)}(P^1)^*"] & & H\mathbb{F}_{p_*}(V(1)\wedge THH(j)) \otimes_{H\mathbb{F}_{p_*}(j)} H\mathbb{F}_{p_*}(\Sigma^{2p-2}\ell) \arrow[u, "\cong"']
\end{tikzcd},$$

in particular $\bar{\xi}_1\sigma\tilde{\xi}_1^p$ maps to $\sigma\tilde{\xi}_1^p$. We therefore examine the square

$$\begin{tikzcd}
V(1)_{2p^2-1}(THH(j;\ell)) \arrow[r] \arrow[d, "g"] & V(1)_{2p^2-2}(THH(j;\Sigma^{2p-3}\ell)) \arrow[d, "h"] \\
(H\mathbb{F}_p \wedge V(1))_{2p^2-1}(THH(j;\ell)) \arrow[r] & (H\mathbb{F}_p \wedge V(1))_{2p^2-2}(THH(j;\Sigma^{2p-3}\ell)),
\end{tikzcd}$$

which is isomorphic to

$$\begin{tikzcd}
\mathbb{F}_p\{\lambda_2\} \arrow[r] \arrow[d, "g"] & \mathbb{F}_p\{\lambda_1'\} \arrow[d, "h"] \\
\mathbb{F}_p\{\sigma\tilde{\xi}_2, \bar{\xi}_1\sigma\tilde{\xi}_1^p, \bar{\tau}_1\sigma b, \bar{\xi}_1\bar{\tau}_0\sigma b\} \arrow[r] & \mathbb{F}_p\{\sigma\tilde{\xi}_1^p, \bar{\tau}_0\sigma b\}.
\end{tikzcd}$$

As stated in the proof of Proposition 4.8, the vertical maps send λ_2 and λ_1' to classes given by the formulas

$$g(\lambda_2) = \sigma\tilde{\xi}_2 - \bar{\xi}_1\sigma\tilde{\xi}_1^p - \bar{\tau}_1\sigma b$$

$$h(\lambda_1') = \sigma\tilde{\xi}_1^p - \bar{\tau}_0\sigma b.$$

The bottom horizontal map sends the class in the image of $\sigma\tilde{\xi}_2$ to the class $\sigma\tilde{\xi}_1^p$; i.e.,

$$\sigma\tilde{\xi}_2 - \bar{\xi}_1\sigma\tilde{\xi}_1^p - \bar{\tau}_1\sigma b \longmapsto \sigma\tilde{\xi}_1^p.$$

Since the inverse image of the Hurewicz map evaluated on this element is

$$h^{-1}(\sigma\tilde{\xi}_1^p) = h^{-1}(\sigma\tilde{\xi}_1^p - \bar{\tau}_0\sigma b) = \lambda_1'.$$

This proves that the top horizontal map is nontrivial and therefore, there must be a differ-

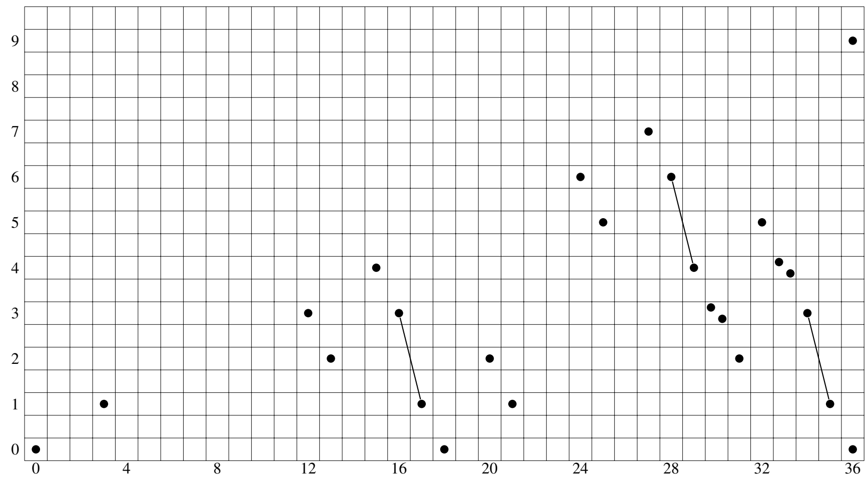

Figure 1: The E_{p-1}-page of the $V(1)$-THH-May spectral sequence at $p = 3$ for $s \leq 36$.

ential

$$d_{p-1}((\mu_1)^{p-1}\sigma\tilde{v}_1) \doteq \alpha_1\lambda_1\gamma_{p-1}(\sigma\alpha_1)$$

as stated. □

Remark 4.11. Due to Oka [47, Thm. 4.4], the obstruction to a ring structure on $V(1)$ at the prime 3 is a composite of maps including the composite map

$$\beta_1 : \Sigma^{11}S \longrightarrow \Sigma^{11}S/p \xrightarrow{\beta_{(1)}} S/p \longrightarrow \Sigma^1 S,$$

however we can easily compute that the induced map $\Sigma^{11} j \to \Sigma j$ is null homotopic and hence the obstruction vanishes after smashing with j. Thus, $V(1) \wedge j$ and hence $V(1) \wedge THH(j)$ are ring spectra, so the ring spectrum structure on $V(1)_*(THH(j))$ is also correct at the prime 3. This type of argument is also used by Ausoni in the case of $V(1) \wedge ku$ in [7].

Theorem 4.12. Let $p \geq 3$ be a prime number and let $V(1)$ be the cofiber of the map

$$v_1 : \Sigma^{2p-2} S/p \to S/p.$$

Then there is an isomorphism

$$V(1)_*(THH(j)) \cong P(\mu_2) \otimes \Gamma(\sigma b) \otimes \mathbb{F}_p\{\alpha_1, \lambda_1', \lambda_2\alpha_1, \lambda_2\lambda_1', \lambda_2\lambda_1'\alpha_1\}$$

where the products between the classes

$$\{\alpha_1, \lambda_1', \lambda_2\alpha_1, \lambda_2\lambda_1', \lambda_2\lambda_1'\alpha_1\}$$

are zero except for

$$\alpha_1 \cdot \lambda_2\lambda_1' = \lambda_1' \cdot \lambda_2\alpha_1 = \lambda_2\lambda_1'\alpha_1.$$

Proof. This proof follows from Proposition 4.5 and Proposition 4.10. There are no further possible differentials for bidegree reasons. This can be seen in Figure 1 since all the algebra generators are in the range specified. The only possible hidden multiplicative extension is easily ruled out by a filtration argument. □

4.2 THH of connective im J with coefficients in Morava K-theory

Note that $S/p_* THH(j;\ell) \cong \pi_* THH(j;k(1))$ where $k(1)$ is the connective cover of the first Morava K-theory spectrum. We can therefore compute $\pi_* THH(j;k(1))$ using the Bockstein spectral sequence

$$V(1)_* THH(j;\ell)[v_1] \Rightarrow \pi_* THH(j;k(1))$$

whose input we computed in the previous section.

Proposition 4.13. There is an isomorphism,

$$\pi_*(THH(j;k(1))) \Rightarrow (P(v_1)\otimes\Gamma(\sigma b)\otimes\mathbb{F}_p\{1,y_{n,m},y'_{n,m}\})/\sim$$

where $y_{n,m}=\lambda'_n\mu_2^{p^{n-1}m}$, $y'_{n,m}=\lambda'_n\lambda'_{n+1}\mu_2^{p^{n-1}m}$, $\lambda'_1=\sigma\tilde{\xi}_1^p$, $\lambda'_2=\lambda_2$,

$$\lambda'_n=\left\{\begin{array}{l} \lambda'_1\mu_2^{p^{n-3}(p-1)} \text{ if } n\geq 3 \text{ is odd} \\ \lambda_2\mu_2^{p^{n-3}(p-1)} \text{ if } n\geq 3 \text{ is even.}\end{array}\right\}$$

and the relations are given by $v_1^{l(n)}y_{n,m}=v_1^{l(n)}y'_{n,m}=0$ where $l(1)=1$, $l(2)=p^2$ and $l(n)=p^n+l(n-2)$.

Proof. The proof makes use of the map of spectral sequences,

$$\begin{tikzcd} \pi_*(V(1)\wedge THH(j;\ell))[v_1] \arrow[r, Rightarrow] \arrow[d, "f"] & \pi_*(S/p\wedge THH(j;\ell)) \arrow[d, "f'"] \\ \pi_*(V(1)\wedge THH(\ell))[v_1] \arrow[r, Rightarrow] & \pi_*(S/p\wedge THH(\ell)) \end{tikzcd}$$

induced by the map of ring spectra $\ell\wedge_j THH(j)\longrightarrow \ell\wedge_\ell THH(\ell)$ given by the map of ring spectra $j\longrightarrow\ell$.

In the bottom spectral sequence, we know due to McClure-Staffeldt [45] that the differentials satisfy the following formula.

$$d_r(\lambda_2)=d_r(\lambda_1)=0 \text{ for all } r\geq 0$$

$$d_{r(n)}(\mu^{p^{n-1}})=v_1^{r(n)}\lambda_n$$

where $r(1) = p$, $r(2) = p^2$ and $r(n) = p^n + r(n-2)$ for $n \geq 3$. The classes λ_n for $n \geq 3$ are defined to be

$$\lambda_n = \left\{ \begin{array}{l} \lambda_1 \mu^{p^{n-3}(p-1)} \text{ if } n \geq 3 \text{ is odd} \\ \lambda_2 \mu^{p^{n-3}(p-1)} \text{ if } n \geq 3 \text{ is even} \end{array} \right\}$$

We know, by using the Hurewicz map and the map of Bökstedt spectral sequences, that the map of spectral sequences, call it f, sends $\mu_2 = \sigma\bar{\tau}_2$ to μ. If n is odd, we see that if $d_1(\mu_2) = 0$ then there are no other possible differentials on μ for bidegree reasons. Therefore, $d_p(\mu_2) = 0$. But, this is a contradiction because it implies that $f(d_p(\mu_2)) = f(0) = 0$, when $f(d_p(\mu_2)) = d_p(f(\mu_2)) = d_p(\mu) = v_1^p\lambda_1 \neq 0$. Thus, $d_1(\mu_2) \neq 0$ so $d_1(\mu_2) \doteq v_1\sigma\bar{\xi}_1^p$. The map of spectral sequences implies that $d_{p^2}((\mu_2)^p) = v_1^{p^2}\sigma\bar{\xi}_2$ since there are no possible earlier differentials on $\sigma\bar{\tau}_2$ for bidegree reasons, and $f(\sigma\bar{\xi}_2) = \lambda_2$. If $\sigma\bar{\xi}_2$ died on an earlier page, there would be a contradiction because $f(d_{p^2}(\mu_2)) = f(0) = 0$ contradicts the known differential $d_{p^2}(f(\mu_2)) = d_{p^2}(\mu) = \lambda_2$. This implies that $d_1(\sigma\bar{\xi}_2) = 0$. The only other possible differential on $\sigma\bar{\xi}_2$ is $d_{p+1}(\sigma\bar{\xi}_2) = v_1^{p+1}$ but $f(v_1^p) = v_1^p$ so we have that $f(v_1) = f(d_{p+1}(\sigma\bar{\xi}_2)) = d_{p+1}(f(\sigma\bar{\xi}_2)) = d_{p+1}(\lambda_2) = 0$, which contradicts the fact that $f(v_1) = v_1$.

Letting $l(1) = 1$, $l(2) = p^2$ and $l(n) = p^n + l(n-2)$ for $n \geq 3$, we produce differentials $d_{l(n)}(\mu^{p^{n-1}}) = v_1^{l(n)}\lambda'_n$ by the same argument, where $\lambda'_1 = \sigma\bar{\xi}_1^p$, $\lambda'_2 = \lambda_2$ and for $n \geq 3$,

$$\lambda'_n = \left\{ \begin{array}{l} \lambda'_1 \mu^{p^{n-3}(p-1)} \text{ if } n \geq 3 \text{ is odd} \\ \lambda_2 \mu^{p^{n-3}(p-1)} \text{ if } n \geq 3 \text{ is even.} \end{array} \right\}$$

There is also a possible d_p differential on $\sigma\bar{\xi}_1^p$, but we claim that it is zero. We can prove this by contradiction. Suppose $d_p(\sigma\bar{\xi}_1^p) = v_1^p$ (Note that this is the only possible differential of this length), then $d_p(v_1^{p^2-p}\sigma\bar{\xi}_1^p\sigma\bar{\xi}_2) = v_1^{p^2}\sigma\bar{\xi}_2$. But, that would mean that $d_{p^2}(\mu_2) = 0$. This contradicts the known differential $d_{p^2}(\mu) = v_1^{p^2}\lambda_2$. Thus, the assumption that $d_p(\sigma\tilde{\xi}_1^p) \neq 0$ must be false.

There are no further possible differentials so

$$\pi_*(THH(j;k(1))) \cong P(v_1) \otimes \Gamma(\sigma b) \otimes \mathbb{F}_p\{1, y_{n,m}, y'_{n,m}\}/\sim$$

where $n \geq 1$, $m \geq 0$ and $m \not\equiv p-1 (\text{mod } p)$ and

$$y_{n,m} = \lambda'_n \mu_2^{p^{n-1}m}$$

$$y'_{n,m} = \lambda'_n \lambda'_{n+1} \mu_2^{p^{n-1}m}$$

□

4.3 Towards mod p-homotopy of THH of the connective image of J

We would really like to understand $S/p_*THH(j)$ and eventually $THH_*(j)$ if we want to compute $K(j)$, without smashing with a Smith-Toda complex, using trace methods. We conjecture the following description of $S/p_*THH(j)$

Conjecture 4.14. There is an isomorphism of graded $\mathbb{F}_p$-vector spaces

$$S/p_*THH(j) \cong H_*((P(v_1) \otimes \Gamma(\sigma b) \otimes \mathbb{F}_p\{1, y_{n,m}, y'_{n,m}\}) \otimes E(\alpha_1)/\sim, d) \tag{4.4}$$

where $H_*(-, d)$ indicates homology of the DGA with respect to the family of differentials

$$d(y_{2k,m}) = \alpha_1 y_{2k-1,m},$$

where α_1 has degree one and all other classes have degree zero, and $\sim$ indicates the same equivalence relation as in Proposition 4.13

The conjecture is really that the differentials stated are all the differentials. We know that the isomorphism of 4.4 is true for some family of differentials since there is a long exact sequence

$$\begin{tikzcd}
\dots \arrow[r] & S/p_kTHH(j;\ell) \arrow[r] & S/p_kTHH(j;\Sigma^{2p-2}\ell) \arrow[dll, rounded corners] \\
S/p_{k-1}THH(j) \arrow[r] & S/p_{k-1}THH(j;\ell) \arrow[r] & S/p_{k-1}THH(j;\Sigma^{2p-2}\ell) \arrow[dll, rounded corners] \\
S/p_{k-2}THH(j) \arrow[r] & S/p_{k-2}THH(j;\ell) \arrow[r] & \dots
\end{tikzcd}$$

We can compare this approach to the Bockstein spectral sequence

$$V(1)_*THH(j)[v_1] \Rightarrow S/p_*THH(j)$$

using the input computed in the previous section; i.e.

$$V(1)_*THH(j)[v_1] \cong P(\mu) \otimes \Gamma(\sigma b) \otimes P(v_1) \otimes \mathbb{F}_p\{1, \alpha_1, \lambda_1', \lambda_2\alpha_1, \lambda_2\lambda_1', \alpha_1\lambda_1'\lambda_2\}.$$

Comparing these two approaches forces the differentials $d(y_{2k,m}) = \alpha_1 y_{2k-1,m}$ above. It also forces differentials on μ^{p^k} for all $k \geq 0$, for example $d_1(\mu) = v_1\lambda_1'$. Some other possible differentials in both the long exact sequence and the Bockstein spectral sequence can be eliminated by comparing with the THH-May spectral sequence

$$S/p_*THH(E_0 j^{\geq \bullet}) \Rightarrow S/p_*THH(j).$$

The input of this spectral sequence is computable by similar methods to those in the previous section. The E_p-page of this spectral sequence is actually the same as the E_1-page of the Bockstein spectral sequence, but the advantage is that the S/p-THH-May spectral sequence has a slightly different grading convention than the Bockstein spectral sequence so some differentials that seem to be possible with the Bockstein grading are not possible using the THH-May grading. An example of this is the possible differential $d_4(\sigma b) = \alpha_1 v_1^4$ in the Bockstein spectral sequence and the long exact sequence, but this can be ruled out because in the S/p-THH-May spectral sequence $|\sigma b| = (p, 2p^2 - 2p)$ and $|\alpha_1 v_1^4| = (p, 2p^2 - 2p - 1)$ and therefore there is no possible differential. We can also eliminate possible differentials in the Bockstein spectral sequence by comparing to the long exact sequence, for example there are possible differentials $d_p(\lambda_2 \alpha_1) = \sigma b v_1 \alpha_1$ and $d_p(\lambda_1' \lambda_2) = v_1 \lambda_1' \sigma b$, but these can be ruled out since there is no boundary map in the long exact sequencethat would make this possible. The author plans to prove this conjecture in subsequent work.

CHAPTER 5 DETECTING V_2-PERIODICITY

The goal of this chapter is to prove a version of the red-shift conjecture in a specific case. We say that a spectrum R detects the n-th Greek letter family, if the family of elements maps nontrivially under the unit map

$$\pi_*S \to \pi_*R.$$

Let V be a finite cell p-local S-module of type $n+1$. We say that $V_*K(R)$ detects the v_{n+1}-periodic family generated by $\alpha_1^{(n+1)}$ if the classes $v_{n+1}^k\alpha_1^{(n+1)}$ map non-trivially to $V_*K(R)$ under the unit map

$$V_* \cong V_*S \xrightarrow{V_*\eta} V_*K(R).$$

Conjecture 5.1 (Greek letter family red-shift conjecture)**.** If R detects the Greek letter family $\alpha_k^{(n)}$, then $V_*K(R)$ detects the v_{n+1}-periodic family generated by $\alpha_1^{(n+1)}$ in V_* for some type $n+1$ spectrum V that detects $\alpha_1^{(n+1)}$.

The main example of interest is the spectrum $K(\mathbb{F}_q)_p$, which detects the alpha family $\{\alpha_k\}$. The main theorem of this chapter will be that $V(1)_*K(K(\mathbb{F}_q)_p)$ detects the v_2-periodic elements generated by β_1 verifying the conjecture for $R = K(\mathbb{F}_q)_p$ and $n = 1$. In particular, we can show that the classes β_{pk+1} are detected under the unit map $\pi_*S \to \pi_*K(K(\mathbb{F}_q)_p)$. Note that throughout this section we will assume $p \geq 5$ and q is a prime power that topologically generates $\mathbb{Z}_p^\times$.

5.1 Homotopy fixed point spectral sequences

Given a connective generalized homology theory E, we will refer to the spectral sequence

$$E^2_{*,*} \cong H^{-s}(S^1, E_t THH(R)) \Rightarrow E^c_{t+s} THH(R)^{hS^1}$$

as the "S^1-E homotopy fixed point spectral sequence associated to $THH(R)$" where $H^{-s}(S^1; M)$ indicates group cohomology with coefficients in M. We will call s the horizontal degree t the vertical degree and $t+s$ the topological degree or total degree, since we will use the Serre convention for grading the spectral sequence where the spectral sequence lies in the second quadrant if $THH(R)$ is connective. By definition, the "continuous" E-homology of $THH(R)^{hS^1}$ is

$$E^c_* THH(R)^{hS^1} \cong \lim_n \pi_*(E \wedge F((ES^1)^{(n)}_+, THH(R))^{S^1})$$

where the limit is taken with respect to the inclusion maps

$$(ES^1)^{(1)} \hookrightarrow (ES^1)^{(2)} \hookrightarrow \ldots \hookrightarrow ES^1.$$

Our choice of model for $(ES^1)^{(n)}$ is $S(\mathbb{C}^n)_+$ with the usual coordinate-wise action of S^1 on $\mathbb{C}^n$. If E is a finite cell complex, then since homotopy limits commute with homotopy (co)fiber sequences $E^c_* THH(R)^{hS^1} \cong E_* THH(R)^{hS^1}$, for example when $E = V(1) := \mathrm{cof}\{v_1 : \Sigma^{2p-2} S/p \to S/p\}$ for $p > 2$. We first give a general characterization of the d^2 differential in the S^1-E-homotopy fixed point spectral sequence. The following proof is adapted from Lemma 3.1 in Bruner-Rognes [21] to include a connective generalized cohomology theory E, though the argument is the same and Bruner-Rognes give a more general statement in the case $E = H\mathbb{F}_p$.

Proposition 5.2. Suppose $E_*THH(R)$ is a graded $\mathbb{F}_p$-vector space. All the d^2 differentials in the S^1-E homotopy fixed point spectral sequence associated to $THH(R)$ are of the form

$$d^2(x) = t\sigma x.$$

where t is a generator of $H^{-*}(S^1; \mathbb{F}_p)$ in degree -2.

Proof. An element $x \in E_*THH(R)$ is a non-equivariant map

$$S^t \longrightarrow E \wedge THH(R); . \tag{5.1}$$

that is, by adjunction a non-equivariant map

$$S^0 \longrightarrow F(S^t, E \wedge THH(R)).$$

By an adjunction in equivariant homotopy theory, this is equivalent to an S^1-equivariant map

$$S(\mathbb{C})_+ \wedge S^0 \longrightarrow F(S^t, E \wedge THH(R)).$$

We consider the diagram,

$$
\begin{tikzcd}
& S(\mathbb{C})_+ \arrow[d] \arrow[dr, "x"] & \\
(S^1 \times \delta D^2)_+ \arrow[r] \arrow[d] & S(\mathbb{C}^1) \arrow[r] \arrow[d] & F(S^t, E \wedge THH(R)) \\
(S^1 \times D^2)_+ \arrow[r] & S(\mathbb{C}^2). \arrow[ur, dashed] &
\end{tikzcd}
$$

The differential d^2 is the obstruction to lifting the composite map

$$(S^1 \times \delta D^2)_+ \to S(\mathbb{C}^1) \to F(S^t, E \wedge THH(R))$$

over the map $S(\mathbb{C}^2) \to F(S^t, E \wedge THH(R))$ where the attaching map is exactly the action of S^1. Using the equivariant adjunction again, this is the obstruction to lifting the map

$$\delta D^2_+ \to S(\mathbb{C}^1) \to F(S^t, E \wedge THH(R))$$

over the map $D^2_+ \to S(\mathbb{C}^2)$. Using the splitting $\delta D^2_+ \simeq S^1 \vee D^2_+$, we see that differential is the map

$$S^1 \to F(S^t, E \wedge THH(R)$$

which by adjunction is equivalent to the map

$$S^1 \wedge S^t \longrightarrow E \wedge THH(R),$$

which is exactly the element σx in $E_{t+1}(THH(R)$. $\square$

Corollary 5.3. In the S^1-$H\mathbb{F}_p$ homotopy fixed point spectral sequence associated to $THH(K(\mathbb{F}_q)_p)$ there are differentials

$$d^2(\tilde{\xi}_1^p) = t\sigma\tilde{\xi}_1^p, \quad d^2(\tilde{\xi}_2) = t\sigma\tilde{\xi}_2$$
$$d^2(\tilde{\tau}_2) = t\sigma\tilde{\tau}_2, \quad d^2(b) = t\sigma b$$

and all other d^2 differentials are zero.

Proof. The Corollary follows by direct application of 5.2 to the computation of Angeltveit-Rognes sumarized in Theorem 4.1. Note that the operator σ is a derivation, so in particular $d^2(\sigma x) = t\sigma^2 x = 0$ for all x. □

In the following section, we will write $T_k(R)$ for $F(S(\mathbb{C}^k)_+, THH(R))^{S^1}$. Note that there is a truncated homotopy fixed point spectral sequence with k columns converging to $E_*(T_k(R))$ and

$$\lim E_*T_k(R) = E_*^c(THH(R)^{hS^1}).$$

5.2 Detecting the classes β_1, β_1', and v_2

The following argument is inspired by the argument of Ausoni-Rognes [9, Prop. 4.8]. Recall from Chapter 2 that the class β_1 maps non-trivially under the map $S \xrightarrow{i_0} S/p \xrightarrow{i_1} V(1)$ where i_0 and i_1 are the maps that include in the bottom cell.

Proposition 5.4. The classes v_2, $i_0 i_1 \beta_1$, and $i_1\beta_1'$ in $V(1)_*$ map nontrivially to the classes $t\mu_2$, $t\sigma b$, and $t\sigma\tilde{\xi}_1^p$ in

$$V(1)_*THH(K(\mathbb{F}_q)_p)^{hS^1}$$

where β_1' is the mod p Bockstein on β_1.

Proof. First, v_2 is represented by $\bar{\tau}_2 \otimes 1$, β_1' is represented by $\bar{\xi}_1^p \otimes 1$ and β_1 is represented by $\Sigma_{i=1}^{p-1} \frac{1}{p}\binom{p}{i}\bar{\xi}_1^i \otimes \bar{\xi}_1^j \otimes 1$, in the cobar complex which is the E_1-page of the Adams spectral sequence that converges to $\pi_* V(1)$ [20]. We consider the map of Adams spectral sequences

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, H_* V(1)) \longrightarrow \mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, H_* V(1) \otimes T_2(K(\mathbb{F}_q)_p))$$

induced by the unit map

$$V(1) \wedge S \xrightarrow{1_{V(1)} \wedge \eta} V(1) \wedge T_2(K(\mathbb{F}_q)_p)) .$$

We see that $\bar{\tau}_2 \otimes 1$, $\bar{\xi}_1^p \otimes 1$, $\Sigma_{i=1}^{p-1} \frac{1}{p}\binom{p}{i}\xi_1^i \otimes \xi_1^j \otimes 1$ are permanent cycles in the source, which map to classes of the same name in the target. Since the elements in the source are infinite cycles, this implies that the elements that they map to are infinite cycles as well. We then have to check that these classes are not boundaries.

We can eliminate the possibility of a d_1 differential with $\bar{\tau}_2 \otimes 1$ as a co-boundary by computing the differential in the cobar complex for $H_* V(1) \otimes H_* T_2(K(\mathbb{F}_q)_p)$ on each class of the correct degree. Since the two column truncation of the homotopy fixed point spectral sequence converging to $H_* T_2(K(\mathbb{F}_q)_p)$ has a differential $d^2(\bar{\tau}_2) = t\mu_2$, by Corollary 5.3, the class $\bar{\tau}_2$ does not survive and can therefore not hit classes of the same name in the 1-co-chains.

The only other classes in the the right degree in $H_* V(1) \otimes H_* T_2(K(\mathbb{F}_q)_p)$ to be the source of a d_1 hitting $\bar{\tau}_2 \otimes 1$ are $\{\sigma\tilde{\xi}_2, \bar{\tau}_1 \sigma b\}$. However, σb is primitive so $d_1(\bar{\tau}_1 \sigma b) = d_1(\bar{\tau}_1)\sigma b \neq \bar{\tau}_2 \otimes 1$. Also, $d_1(\sigma\tilde{\xi}_2) = \bar{\xi}_1 \otimes \sigma\tilde{\xi}_1^p + \bar{\tau}_1 \otimes \sigma b \neq \bar{\tau}_2 \otimes 1$. Therefore, $\bar{\tau}_2 \otimes 1$ survives to the E_2-page. There are no possible longer differentials hitting $\bar{\tau}_2 \otimes 1$ because the class lies on the one-line of the Adams spectral sequence; hence, it is a permanent cycle.

We eliminate the possibility that the class $\bar{\xi}_1^p \otimes 1$ is a boundary of a d_1 by the same method. Consider the truncated homotopy fixed point spectral sequence converging to $H_*T_2(K(\mathbb{F}_q)_p)$. In that spectral sequence there is a d^2 on $\bar{\xi}_1^p$ hitting $t\sigma\bar{\xi}_1^p$ by Corollary 5.3. Therefore, the only classes that are in the right degree in $H_*V(1) \wedge T_2(K(\mathbb{F}_q)_p)$ to have $\bar{\xi}_1^p \otimes 1$ as their co-boundary are

$$\{\bar{\tau}_0\sigma\bar{\xi}_1^p, \sigma b.\}$$

However, $d_1(\sigma b) = 0$, since it is a comodule primitive, and

$$d_1(\bar{\tau}_0\sigma\tilde{\xi}_1^p) = 1 \otimes \bar{\tau}_0\sigma\bar{\xi}_1^p - \bar{\tau}_0 \otimes \sigma\tilde{\xi}_1^p - 1 \otimes \bar{\tau}_0\sigma\tilde{\xi}_1^p \neq \bar{\xi}_1^p \otimes 1.$$

The class $[\bar{\xi}_1^p \otimes 1]$ is in Adams filtration 1 so it can not be the target of a longer differential, therefore it is a permanent cycle.

For $\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i}\xi_1^i \otimes \xi_1^{p-i} \otimes 1$, we first need to check that it is not a boundary of an element in $\mathcal{A}_* \otimes H_*V(1) \wedge T_2(K(\mathbb{F}_q)_p)$. We check the differential in the cobar complex on all the elements here in the right degree. These classes are

$$\left\{ \begin{array}{l} 1 \otimes \sigma b,\ \bar{\tau}_0 \otimes \bar{\tau}_0 t\tilde{\xi}_1^p,\ \bar{\xi}_1^{p-1} \otimes \bar{\tau}_0\bar{\tau}_1,\ \bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1 \otimes \bar{\tau}_0\bar{\tau}_1, \\ \bar{\xi}_1^{p-1}\bar{\tau}_0 \otimes \bar{\tau}_1,\ \bar{\xi}_1^{p-1}\bar{\tau}_1 \otimes \bar{\tau}_0,\ \bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_0 \otimes 1,\ \bar{\xi}_1^p \otimes 1 \end{array} \right\}$$

where $\tilde{\xi}_1^p$ has a coproduct coming from $H_*K(\mathbb{F}_q)$ and $\bar{\xi}_1^p$ has the coproduct coming from the coaction on $\mathcal{A}_*$. Recall that Milnor computed the coaction of $\mathcal{A}_*$ on

$$H^*(\mathbb{C}P^\infty, \mathbb{F}_p) \cong H^*(BS^1; \mathbb{F}_p),$$

where the right side is the equivalent to the group cohomology $H^*(S^1;\mathbb{F}_p)$ [46]. The coaction on the class t is

$$\psi(t) = \Sigma_{i\geq 0}\bar{\xi}_1^i \otimes t^{p^i}.$$

Therefore, in the input of the truncated homotopy fixed point spectral sequence computing $V(1)_*T_2(K(\mathbb{F}_q)_p)$, the $\mathcal{A}_*$ coaction on t is primitive.

We compute the differential in the cobar complex on each of the elements that could possibly have the class representing β_1 as a target:

$$\begin{aligned}
d_1(1\otimes\sigma b) &= 1\otimes 1\otimes \sigma b\\
d_1(\bar{\tau}_0\otimes\bar{\tau}_0 t\tilde{\xi}_1^p) &= \bar{\tau}_0\otimes\bar{\tau}_0\otimes t\tilde{\xi}_1^p + \bar{\tau}_0\otimes\bar{\xi}_1^p\otimes t\bar{\tau}_0 + \bar{\tau}_0\otimes\bar{\tau}_0\bar{\xi}_1^p\otimes t + \bar{\tau}_0\otimes\bar{\tau}_0\otimes\bar{\tau}_0 tb\\
d_1(\bar{\xi}_1^{p-1}\otimes\bar{\tau}_0\bar{\tau}_1) &= 1\otimes\bar{\xi}_1^{p-1}\otimes\bar{\tau}_0\bar{\tau}_1 - \Delta(\bar{\xi}_1^{p-1})\otimes\bar{\tau}_0\bar{\tau}_1 + \bar{\xi}_1^{p-1}\otimes\psi(\bar{\tau}_0\bar{\tau}_1)\\
&= -\textstyle\sum_{i=1}^{p-2}\binom{p-1}{i}\bar{\xi}_1^{p-i-1}\otimes\bar{\xi}_1^i\otimes\bar{\tau}_0\bar{\tau}_1 + \bar{\xi}_1^{p-1}\otimes\bar{\tau}_0\otimes\bar{\tau}_1\\
&\quad +\bar{\xi}_1^{p-1}\otimes\bar{\tau}_0\bar{\tau}_1\otimes 1 + \bar{\xi}_1^{p-1}\otimes\bar{\tau}_1\otimes\bar{\tau}_0
\end{aligned}$$

$$\begin{aligned}
d_1(\bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1\otimes\bar{\tau}_0\bar{\tau}_1) &= 1\otimes\bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1 - \Delta(\bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1)\otimes\bar{\tau}_0\bar{\tau}_1 + \bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1\otimes\psi(\bar{\tau}_0\bar{\tau}_1)\\
&= 1\otimes\bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1\otimes\bar{\tau}_0\bar{\tau}_1 - \textstyle\sum_{i=0}^{p-2}\bar{\xi}_1^i\otimes\bar{\xi}_1^{p-i-2}\bar{\tau}_0\bar{\tau}_1\otimes\bar{\tau}_0\bar{\tau}_1\\
&\quad -\textstyle\sum_{i=0}^{p-2}\bar{\xi}_1^i\bar{\tau}_0\otimes\bar{\xi}_1^{p-i-2}\bar{\tau}_1\otimes\bar{\tau}_0\bar{\tau}_1 - \sum_{i=0}^{p-2}\bar{\xi}_1^i\bar{\tau}_0\bar{\tau}_1\otimes\bar{\xi}_1^{p-i-2}\otimes\bar{\tau}_0\bar{\tau}_1\\
&\quad -\textstyle\sum_{i=0}^{p-2}\bar{\xi}_1^i\bar{\tau}_1\otimes\bar{\xi}_1^{p-i-2}\bar{\tau}_0\otimes\bar{\tau}_0\bar{\tau}_1 - \sum_{i=0}^{p-2}\bar{\xi}_1^i\bar{\tau}_0\otimes\bar{\xi}_1^{p-i-2}\bar{\xi}_1\bar{\tau}_0\otimes\bar{\tau}_0\bar{\tau}_1\\
&\quad +\bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1\otimes 1\otimes\bar{\tau}_0\bar{\tau}_1 + \bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1\otimes\bar{\tau}_0\otimes\bar{\tau}_0\\
&\quad +\bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1\otimes\bar{\tau}_0\bar{\tau}_1\otimes 1 + \bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1\otimes\bar{\tau}_1\otimes\bar{\tau}_0
\end{aligned}$$

$$\begin{aligned}
d_1(\bar{\xi}_1^{p-1}\bar{\tau}_0\otimes\bar{\tau}_1) &= 1\otimes\bar{\xi}_1^{p-1}\bar{\tau}_0\otimes\bar{\tau}_1 - \Delta(\bar{\xi}_1^{p-1}\bar{\tau}_0)\otimes\bar{\tau}_1 + \bar{\xi}_1^{p-1}\bar{\tau}_0\otimes\psi(\bar{\tau}_1)\\
&= 1\otimes\bar{\xi}_1^{p-1}\bar{\tau}_0\otimes\bar{\tau}_1 - \textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^i\otimes\bar{\xi}_1^{p-i-1}\bar{\tau}_0\otimes\bar{\tau}_1\\
&\quad -\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^i\bar{\tau}_0\otimes\bar{\xi}_1^{p-i-1}\otimes\bar{\tau}_1 + \bar{\xi}_1^{p-1}\bar{\tau}_0\otimes 1\otimes\bar{\tau}_1\\
&\quad +\bar{\xi}_1^{p-1}\bar{\tau}_0\otimes\bar{\tau}_1\otimes 1
\end{aligned}$$

$$\begin{aligned}
d_1(\bar{\xi}_1^{p-1}\bar{\tau}_1 \otimes \bar{\tau}_0) &= 1 \otimes \bar{\xi}_1^{p-1}\bar{\tau}_1 \otimes \bar{\tau}_0 - \Delta(\bar{\xi}_1^{p-1}\bar{\tau}_1) \otimes \bar{\tau}_0 + \bar{\xi}_1^{p-1}\bar{\tau}_1 \otimes 1 \otimes \psi(\bar{\tau}_0) \\
&= 1 \otimes \bar{\xi}_1^{p-1}\bar{\tau}_1 \otimes \bar{\tau}_0 - \textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^i \otimes \bar{\xi}_1^{p-1-i}\bar{\tau}_1 \otimes \bar{\tau}_0 \\
&\quad - \textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^i\bar{\tau}_0 \otimes \bar{\xi}_1^{p-1-i}\bar{\xi}_1 \otimes \bar{\tau}_0 \\
&\quad - \textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^i\bar{\tau}_1 \otimes \bar{\xi}_1^{p-1-i}\bar{\tau}_1 \otimes \bar{\tau}_0 \\
&\quad + \bar{\xi}_1^{p-1}\bar{\tau}_1 \otimes 1 \otimes \bar{\tau}_0 + \bar{\xi}_1^{p-1}\bar{\tau}_1 \otimes \bar{\tau}_0 \otimes 1
\end{aligned}$$

$$\begin{aligned}
d_1(\bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1 \otimes 1) &= 1 \otimes \bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1 \otimes 1 - \Delta(\bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1) \otimes 1 + \bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1 \otimes 1 \otimes 1 \\
&= 1 \otimes \bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1 \otimes 1 - \textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^i \otimes \bar{\xi}_1^{p-1-i}\bar{\tau}_1\bar{\tau}_0 \otimes 1 \\
&\quad - \textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^i\bar{\tau}_0 \otimes \bar{\xi}_1^{p-1-i}\bar{\xi}_1\bar{\tau}_0 \otimes 1 \\
&\quad - \textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^i\bar{\tau}_1 \otimes \bar{\xi}_1^{p-1-i}\bar{\tau}_1\bar{\tau}_0 \otimes 1 \\
&\quad - \textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^i\bar{\tau}_0 \otimes \bar{\xi}_1^{p-1-i}\bar{\tau}_1 \otimes 1 \\
&\quad - \textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}\bar{\xi}_1^i\bar{\tau}_1\bar{\tau}_0 \otimes \bar{\xi}_1^{p-1-i}\bar{\tau}_1 \otimes 1 \\
&\quad + \bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1 \otimes 1 \otimes 1 \\
d_1(\bar{\xi}_1^p \otimes 1) &= 0.
\end{aligned}$$

Suppose that some combination of these elements could hit the class

$$\Sigma_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}\xi_1^i \otimes \xi_1^{p-i} \otimes 1.$$

Then we need to solve the equation

$$\begin{aligned}
\Sigma_{i=1}^{p-1}\tfrac{1}{p}\binom{p}{i}\xi_1^i \otimes \xi_1^{p-i} \otimes 1 &= a_1 d_1(1 \otimes \sigma b) + a_2 d_1(\bar{\tau}_0 \otimes \bar{\tau}_0 t\tilde{\xi}_1^p) + a_3 d_1(\bar{\xi}_1^{p-1} \otimes \bar{\tau}_0\bar{\tau}_1) \\
&\quad + a_4 d_1(\bar{\xi}_1^{p-2}\bar{\tau}_0\bar{\tau}_1 \otimes \bar{\tau}_0\bar{\tau}_1) + a_5 d_1(\bar{\xi}_1^{p-1}\bar{\tau}_0 \otimes \bar{\tau}_1) \\
&\quad + a_6 d_1(\bar{\xi}_1^{p-1}\bar{\tau}_1 \otimes \bar{\tau}_0) + a_7 d_1(\bar{\xi}_1^{p-1}\bar{\tau}_1\bar{\tau}_1 \otimes 1)
\end{aligned}$$

for some $a_i \in \mathbb{F}_p^\times$, but there are no solutions to this equation.

Since $\Sigma_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}\xi_1^i \otimes \xi_1^{p-i} \otimes 1$ is on the two-line, we still have to check that there is no d_2

differential hitting it in the Adams spectral sequence,

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, H_*(V(1) \wedge T_2(K(\mathbb{F}_q)_p)) \Rightarrow V(1)_* T_2(K(\mathbb{F}_q)_p).$$

Since a d_2 would have to have its source on the 0-line in degree $2p^2 - 2p - 1$, it would have to be a class in $H_{2p^2-2p-1} V(1) \wedge T_2(K(\mathbb{F}_q)_p)$.

We computed

$$H_{2p^2-2p-1} V(1) \wedge T_2(K(\mathbb{F}_q)_p) \cong \mathbb{F}_p\{\bar{\tau}_0 t \tilde{\xi}_1^p\},$$

since $d^2(b) = t\sigma b$ in the two column homotopy fixed point spectral sequence that computes $H_* T_2(K(\mathbb{F}_q)_p)$. The Leibniz rule implies

$$d_2(\bar{\tau}_0 t \tilde{\xi}_1^p) = d_2(\bar{\tau}_0) t \tilde{\xi}_1^p - \bar{\tau}_0 d_2(\tilde{\xi}_1^p) \neq \Sigma_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} \xi_1^i \otimes \xi_1^{p-i} \otimes 1,$$

so we can rule out this differential. Therefore, the class

$$\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} \xi_1^i \otimes \xi_1^{p-i} \otimes 1$$

is a permanent cycle.

We conclude that v_2, β_1' and β_1 map from $V(1)_* S$ to

$$V(1)_* T_2(K(\mathbb{F}_q)_p).$$

In $V(1)_* T_2(K(\mathbb{F}_q)_p)$, the only possible classes in the right degree to be v_2, β_1' and β_1 are $t\mu_2$,

$t\sigma b$ and $t\sigma\bar{\xi}_1^p$. The unit map factors through $V(1)_*THH(K(\mathbb{F}_q)_p)^{hS^1}$, so these classes pull back to classes in $V(1)_*THH(K(\mathbb{F}_q)_p)^{hS^1}$. This proves that these classes are permanent cycles in the S^1-$V(1)$ homotopy fixed point spectral sequence associated to $THH(K(\mathbb{F}_q)_p)$. □

Now, the classes $\beta_1 v_2^{k-1}$ have the property that in the Adams-Novikov spectral sequence for $V(1)$ they are represented by the classes

$$\Sigma_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}t_1^i \otimes t_1^{p-i} \otimes v_2^{k-1},$$

which are on the two-line. We can therefore give a similar argument to the previous one, except that we work in the Adams-Novikov spectral sequence, using the fact that the classes representing $\beta_1 v_2^{k-1}$ are in low Adams-Novikov filtration. To do this we must compute $BP \wedge V(1)_*T_k(K(\mathbb{F}_q)_p)$.

5.3 Computing $(BP \wedge V(1))_*THH(K(\mathbb{F}_q)_p)$

In this section, we compute

$$(BP \wedge V(1))_*THH(K(\mathbb{F}_q)_p$$

using the $BP \wedge V(1)$-THH-May spectral sequence. To accomplish this, we first need to compute the input of the $BP \wedge V(1)$-THH-May spectral sequence; i.e.,

$$(BP \wedge V(1))_*THH(E_0K(\mathbb{F}_q)_p^{\geq \bullet}).$$

Lemma 5.5. There is an isomorphism of $(BP \wedge V(1)_* E_0 K(\mathbb{F}_q)_p^{\geq \bullet}$-algebras

$$(BP \wedge V(1))_* THH(E_0 K(\mathbb{F}_q)_p^{\geq \bullet}) \cong P(t_1, t_2, \dots) \otimes E(\epsilon_1, \lambda_1, \sigma v_1, \alpha_1) \otimes P(v_1, \mu_1) \otimes \Gamma(\sigma \alpha_1)$$

and the Hurewicz map

$$(BP \wedge V(1))_* THH(E_0 K(\mathbb{F}_q)_p^{\geq \bullet}) \to (H\mathbb{F}_p \wedge BP \wedge V(1))_* THH(E_0 K(\mathbb{F}_q)_p^{\geq \bullet})$$

sends t_n to $\Phi(n)$ where $\Phi(n)$ is defined inductively with $\Phi(1) = \bar{\xi}_1 - \hat{\xi}_1$ and

$$\Phi(n) = \bar{\xi}_n - \hat{\xi}_n - \sum_{\substack{i+j=n; \\ i,j>0}} \bar{\xi}_i \cdot \Phi(j)^{p^i}.$$

Proof. Recall that $V(1) \wedge THH(E_0 K(\mathbb{F}_q)_p^{\geq \bullet})$ is a $V(1) \wedge E_0 K(\mathbb{F}_q)_p^{\geq \bullet}$-algebra, and hence an $H\mathbb{F}_p$ algebra, since $V(1) \wedge E_0 K(\mathbb{F}_q)_p^{\geq \bullet}$ is itself an $H\mathbb{F}_p$-algebra. Thus, there is an equivalence

$$BP \wedge V(1) \wedge THH(E_0 K(\mathbb{F}_q)_p^{\geq \bullet}) \simeq BP \wedge H\mathbb{F}_p \wedge_{H\mathbb{F}_p} V(1) \wedge THH(E_0 K(\mathbb{F}_q)_p^{\geq \bullet})$$

and by the collapse of the Künneth spectral sequence, an isomorphism

$$(BP \wedge V(1))_* THH(E_0 K(\mathbb{F}_q)_p^{\geq \bullet}) \cong P(t_1, t_2, \dots) \otimes E(\epsilon_1, \lambda_1, \sigma v_1, \alpha_1) \otimes P(v_1, \mu_1) \otimes \Gamma(\sigma \alpha_1) \quad (5.2)$$

as desired. Since $BP \wedge V(1) \wedge THH(E_0 K(\mathbb{F}_q)_p^{\geq \bullet})$ is an $H\mathbb{F}_p$-module we can use Lemma 4.2, which states that $(BP \wedge V(1))_* THH(E_0 K(\mathbb{F}_q)_p^{\geq \bullet})$ includes as the comodule primitives inside of

$$(H\mathbb{F}_p \wedge BP \wedge V(1))_* THH(E_0K(\mathbb{F}_q)_p^{\geq \bullet}).$$

We recall that by the Künneth isomorphism and Proposition 4.4 there is an isomorphism of graded rings

$$\begin{aligned}&(H\mathbb{F}_p \wedge BP \wedge V(1))_* THH(E_0K(\mathbb{F}_q)_p^{\geq \bullet}) \cong \\ &H_*(BP) \otimes E(\bar{\tau}_0, \bar{\tau}_1) \otimes (A//E(0))_* \otimes P(v_1) \otimes E(\alpha_1) \otimes E(\lambda_1) \otimes P(\mu_1) \otimes E(\sigma v_1) \otimes \Gamma(\sigma\alpha_1)\end{aligned}$$

where we write $(\mathcal{A}//E(0))_* \cong P(\hat{\xi}_1, \hat{\xi}_2, \dots) \otimes E(\hat{\tau}_1, \hat{\tau}_2, \dots)$ and $H_*(BP) \cong P(\bar{\xi}_1, \bar{\xi}_2, \dots)$ to distinguish the two sets of generators. The coaction on $\bar{\xi}_i$, $\bar{\tau}_i$, $\hat{\tau}_i$ and $\hat{\xi}_i$ are the same as the coproduct in the dual Steenrod algebra, and hence for example $\bar{\xi}_1 - \hat{\xi}_1$ is a comodule primitive, since

$$\psi(\bar{\xi}_1 - \hat{\xi}_1) = 1 \otimes \bar{\xi}_1 + \bar{\xi}_1 \otimes 1 - 1 \otimes \hat{\xi}_1 - \bar{\xi}_1 \otimes 1 = 1 \otimes \bar{\xi}_1 - 1 \otimes \hat{\xi}_1.$$

This is the base step in the induction. Suppose that $\Phi(j)$ is a comodule primitive for $j < n$, then

$$
\begin{aligned}
\psi(\bar{\xi}_n - \hat{\xi}_n - \textstyle\sum_{i+j=n;i,j>0} \bar{\xi}_i \Phi(j)^{p^i}) &= \psi(\bar{\xi}_n) - \psi(\hat{\xi}_n) - \textstyle\sum_{i+j=n;i,j>0} \psi(\bar{\xi}_i) \cdot \psi(\Phi(j))^p \\
&= \textstyle\sum_{i+j=n} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i} - \sum_{i+j=n} \bar{\xi}_i \otimes \hat{\xi}_j^{p^i} \\
&\quad - \textstyle\sum_{i+j=n;i,j>0} \psi(\bar{\xi}_i) \cdot (1 \otimes \Phi(j)^p) \\
&= 1 \otimes \bar{\xi}_{n-1} - 1 \otimes \hat{\xi}_n \\
&\quad + \textstyle\sum_{i+j=n;i,j>0} \bar{\xi}_i \otimes (\bar{\xi}_j^{p^i} - \hat{\xi}_j^{p^i}) \\
&\quad - \textstyle\sum_{i+j=n;i,j>0} \bar{\xi}_i \otimes (\bar{\xi}_j^{p^i} - \hat{\xi}_j^{p^i}) \\
&\quad - \textstyle\sum_{i+j=n;i,j>0} \sum_{\ell+k=i;\ell,k>0} \bar{\xi}_\ell \otimes \bar{\xi}_k^{p^\ell} \Phi(j)^{p^i} \\
&\quad + \textstyle\sum_{i'+j'=n;i',j'>0} \sum_{\ell'+k'=j;\ell',k'>0} \bar{\xi}_i \otimes \bar{\xi}_{\ell'}^{p^{i'}} \Phi(k)^{p^{\ell'+i'}} \\
&\quad -1 \otimes \textstyle\sum_{i'+j'=n;i',j'>0} \bar{\xi}_{i'} \Phi(j')^{p^{i'}} \\
&= 1 \otimes \bar{\xi}_{n-1} - 1 \otimes \hat{\xi}_n - 1 \otimes \textstyle\sum_{i'+j'=n;i',j'>0} \bar{\xi}_{i'} \Phi(j')^{p^{i'}}
\end{aligned}
$$

where the last equality can be seen by rearranging the indices in the sums

$$
-\sum_{i+j=n;i,j>0} \sum_{\ell+k=i;\ell,k>0} \bar{\xi}_\ell \otimes \bar{\xi}_k^{p^\ell} \Phi(j)^{p^i}
$$

and

$$
\sum_{i'+j'=n;i',j'>0} \sum_{\ell'+k'=j;\ell',k'>0} \bar{\xi}_i \otimes \bar{\xi}_{\ell'}^{p^{i'}} \Phi(k)^{p^{\ell'+i'}}
$$

so that $j = k'$, $i = \ell' + i'$, $k = \ell'$, $\ell = i'$, $j + k = j'$, and $n = k' + \ell' + i'$. Therefore,

$$
\bar{\xi}_n - \hat{\xi}_n + \Sigma_{i+j=n;i,j>0} \bar{\xi}_i \Phi(j)^{p^i}
$$

is a comodule primitive for each integer $n \geq 2$.

The coaction on the remaining elements is

$$\begin{array}{ll} \psi(\alpha_1) = 1 \otimes \alpha_1 & \psi(\sigma v_1) = 1 \otimes \sigma v_1 + \bar{\tau}_0 \otimes \sigma\alpha_1 \\ \psi(\sigma\alpha_1) = 1 \otimes \sigma\alpha_1 & \psi(\lambda_1) = 1 \otimes \lambda_1 \\ \psi(\gamma_{p^k}(\sigma\alpha_1)) = 1 \otimes \gamma_{p^k}(\sigma\alpha_1) & \psi(\mu_1) = 1 \otimes \mu_1 + \bar{\tau}_0 \otimes \lambda_1. \\ \psi(v_1) = 1 \otimes v_1 + \bar{\tau}_0 \otimes \alpha_1 & \end{array}$$

We therefore know that the elements

$$\{\lambda_1, \sigma v_1 - \bar{\tau}_0\sigma\alpha_1, \alpha_1, \gamma_{p^k}(\sigma\alpha_1), v_1 - \bar{\tau}_0\alpha_1, \mu_1 - \bar{\tau}_0\lambda_1, \bar{\xi}_n - \hat{\xi}_n + \Sigma_{i+j=n; i,j>0}\bar{\xi}_i\Phi(j)^{p^i} \mid n \geq 1\}$$

as well as products and sums of these classes are comodule primitives. By comparing the dimension as an $\mathbb{F}_p$-vector space in each degree to the isomorphism 5.2, we know this must be all the comodule primitives. Note that we are using the fact that a product or sum of comodule primitives is primitive since the coaction map $M \to \mathcal{A}_* \otimes M$ is also a ring map when M is a comodule algebra. □

Proposition 5.6. As a (BP_*, BP_*BP)-comodule

$$(BP \wedge V(1))_*THH(K(\mathbb{F}_q)_p) \cong P(t_1^p, t_2, \ldots) \otimes E(b) \otimes E(\sigma\bar{\xi}_1^p, \sigma\bar{\xi}_2) \otimes P(\mu_2) \otimes \Gamma(\sigma b)$$

where the coaction is given by

$$\begin{array}{ll} \psi(t_1^p) = 1 \otimes t_1^p + t_1^p \otimes 1 & \psi(\mu_2) = 1 \otimes \mu_2 \\ \psi(t_n) = \Delta(t_n) \text{ for } n \geq 2 & \psi(\gamma_{p^k}(\sigma b)) = 1 \otimes \gamma_{p^k}(\sigma b) \\ \psi(b) = 1 \otimes b & \psi(\sigma x) = (1 \otimes \sigma) * \psi(x) \end{array}$$

Proof. We need to compute differentials in the $BP \wedge V(1)$-THH-May spectral sequence

$$E^1_{*,*} = (BP \wedge V(1))_{*,*}THH(E_0K(\mathbb{F}_q)_p^{\geq \bullet}) \Rightarrow BP \wedge V(1)_*THH(K(\mathbb{F}_q)_p)$$

so we examine the map of spectral sequences

$$\begin{tikzcd}
(BP \wedge V(1))_{*,*}THH(E_0K(\mathbb{F}_q)_p^{\geq \bullet}) \arrow[r, Rightarrow] \arrow[d, "h"] & BP \wedge V(1)_*THH(K(\mathbb{F}_q)_p) \arrow[d] \\
(H\mathbb{F}_p \wedge BP \wedge V(1))_{*,*}THH(E_0K(\mathbb{F}_q)_p^{\geq \bullet}) \arrow[r, Rightarrow] & (H\mathbb{F}_p \wedge BP \wedge V(1))_*THH(K(\mathbb{F}_q)_p).
\end{tikzcd}$$

induced by the Hurewicz map $BP \to H\mathbb{F}_p$. Recall from Lemma 5.5 that

$$(BP \wedge V(1))_*THH(E_0K(\mathbb{F}_q)_p^{\geq \bullet}) \cong P(\xi_1, \xi_2, \dots) \otimes E(\epsilon_1, \lambda_1, \sigma v_1, \alpha_1) \otimes P(v_1, \mu_1) \otimes \Gamma(\sigma\alpha_1).$$

We know that in the $H\mathbb{F}_p \wedge BP \wedge V(1)$-THH-May spectral sequence the classes $\bar{\xi}_i$ for $i \geq 1$ and $\bar{\tau}_j$ for $j = 0, 1$ survive to E^∞, since the output of the spectral sequence is known to be

$$\begin{aligned}
&(H\mathbb{F}_p \wedge BP \wedge V(1))_*THH(K(\mathbb{F}_q)_p) \cong \\
&P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_0, \bar{\tau}_1) \otimes H_*(K(\mathbb{F}_q)_p) \otimes E(\sigma\bar{\xi}_1^p, \sigma\bar{\xi}_2) \otimes P(\sigma\bar{\tau}_2) \otimes \Gamma(\sigma b)
\end{aligned}$$

by Theorem 4.1 and the Künneth isomorphism. This forces differentials

$$\begin{aligned}
d^1(\hat{\xi}_1) &= \alpha_1 & d^1(\hat{\tau}_i) &= 0 \text{ for } i > 0 \\
d^1(\bar{\xi}_i) &= 0 & d^1(\lambda_1) &= \sigma\alpha_1 \\
d^1(\hat{\tau}_1) &= v_1 & d^1(\mu_1) &= \sigma v_1 \\
d^1(\bar{\tau}_i) &= 0
\end{aligned}$$

and there are no further nontrivial differentials. Since the Hurewicz map h is injective and it sends t_1 to $\bar{\xi}_1 - \hat{\xi}_1$, the differential $d^1(t_1)$ in the top spectral sequence can be computed using the formula

$$d^1(t_1) = d^1(h^{-1}(\bar{\xi}_1 - \hat{\xi}_1)) = h^{-1}d^1(\bar{\xi}_1 - \hat{\xi}_1) = h^{-1}(\alpha_1) = \alpha_1.$$

Similarly, ϵ_1 maps to $\bar{\tau}_1 - \hat{\tau}_1$ implying $d^1(\epsilon_1) = v_1$. Hence, in the $BP \wedge V(1)$-THH-May spectral sequence there are differentials

$$\begin{aligned} d^1(t_1) &= \alpha_1 & d^1(\lambda_1) &= \sigma\alpha_1 \\ d^1(\epsilon_1) &= v_1 & d^1(\mu_1) &= \sigma v_1. \end{aligned}$$

On E^2-pages the map of spectral sequences induced by the Hurewicz map is again injective. Since $E^2 \cong E^\infty$ in the target spectral sequence, the same is true in the source. This implies that the $BP \wedge V(1)$-THH-May spectral sequence collapses at the E^2-page.

By examining the long exact sequence

$$BP_*(V(1) \wedge j) \to BP_*(V(1) \wedge \ell) \to BP_*(V(1) \wedge \Sigma^{2p-2}\ell)$$

we can determine that the coaction on t_1^p and t_i for $i \geq 2$ is the same as the coaction on these elements in $BP_*(V(1) \wedge \ell) \cong P(t_1, t_2, \dots)$. Note that there is no hidden comultiplication on t_1^p since there are no classes in degrees $2p^2 - 2p - (2p - 2)$ or lower and the lowest degree element in BP_*BP is in degree $2p - 2$. The class b in lowest degree and therefore it is primitive, so this gives the coaction on b, t_1^p, t_i for $i \geq 2$ by using the splitting of BP_*BP-

comodules

$$BP_*(V(1) \wedge j) \rightleftarrows BP_*(V(1) \wedge THH(j)).$$

The coaction on μ_2 is primitive because $|\mu_2| = 2p^2$ and there are no classes in degrees $2p^2 - 2p + 2$ or $2p^2 - 4p + 4$ or lower and the classes in BP_*BP are in degrees congruent to zero mod $2p^n - 2$ for some n. Similarly, the coaction on λ_1' is primitive because there are no classes in degree $2p^2 - 2p + 1 - (2p - 2)$ or lower.

To determine the coaction on λ_2, we use the map of BP_*BP-comodules

$$(BP \wedge V(1))_*THH(K(\mathbb{F}_q)_p) \to (BP \wedge V(1))_*THH(\ell_p).$$

We claim that in $(BP \wedge V(1))_*THH(\ell_p)$, the coaction on λ_2 is

$$\psi(\lambda_2) = 1 \otimes \lambda_2 + t_1 \otimes \lambda_1'.$$

Note that there is an isomorphism

$$\begin{aligned} BP_*(V(1) \wedge \ell) \cong BP_*H\mathbb{F}_p &\cong P(\bar{\xi}_1, \bar{\xi}_2, \dots) \\ &\cong BP_*BP \otimes_{BP_*} \mathbb{F}_p \\ &\cong P(t_1, t_2, \dots) \end{aligned}$$

so $\bar{\xi}_2$ and t_2 are two names for the same basis element up to multiplication by a unit. The operation σ gives

$$\lambda_2 = \sigma\bar{\xi}_2 \doteq \sigma t_2, \quad \lambda_1' = \sigma\bar{\xi}_1^p \doteq \sigma t_1^p$$

and we can therefore compute the coaction on λ_2 as

$$\psi(\lambda_2) = (1 \otimes \sigma)\Delta(t_2)$$

which produces the desired coaction modulo (p, v_1). We then just check that there are no other terms that could be added on to $\psi(\lambda_2)$ in $(BP \wedge V(1))_* THH(K(\mathbb{F}_q)_p)$ that would map to zero, but there aren't any because the only class in degree $2p^2 - 1 - (2p - 2)$ is λ_1' and there are no classes in degree $2p^2 - 1 - (4p - 4)$ or lower.

We know $\psi(\sigma b) = (1 \otimes \sigma)(1 \otimes b) = 1 \otimes \sigma b$. This just leaves the classes $\gamma_{p^k}(\sigma b)$ for $k > 0$. Note that we already showed that in the input of $BP \wedge V(1)$-THH-May spectral sequence the classes $\gamma_{p^{k+1}}(\sigma\alpha_1) = \gamma_{p^k}(\sigma b)$ are primitive. Therefore, it suffices to check that there is not a hidden coaction in the THH-May spectral sequence. If the coaction contains terms of the form $x \otimes m$ where $|m| < |\gamma_{p^k}(\sigma\alpha_1)|$, then the May filtration of m must be greater or equal to the May filtration of $\gamma_{p^k}(\sigma b)$.

Suppose the May filtration of m is greater or equal to p^{k+1}, the May filtration of $\gamma_{p^k}(\sigma b)$. Then, since the only classes with positive May filtration are $\gamma_{p^j}(\sigma b)$, b, λ_1', and λ_2, the class m must be of the form

$$(\gamma_{p^j}(\sigma b))^\ell b^{\epsilon_1} \lambda_1'^{\epsilon_2} \lambda_2^{\epsilon_3} z,$$

for some possibly zero element z, where $0 \geq \ell < p$, ϵ_1, ϵ_2, ϵ_3. Write mfilt(x) for the May filtration of an element, then

$$\begin{aligned} \mathrm{mfilt}(\gamma_{p^j}(\sigma b)) &= p^{j+1} & \mathrm{mfilt}(\lambda_1') &= p - 1 \\ \mathrm{mfilt}(b) &= 1 & \mathrm{mfilt}(\lambda_2) &= 1. \end{aligned}$$

so j, ℓ, ϵ_1, ϵ_2, and ϵ_3 must satisfy

$$\ell p^{j+1} + \epsilon_1 + \epsilon_2(p-1) + \epsilon_3 \geq p^{k+1}. \tag{5.3}$$

We split into cases. If $k = 1$, then $j \geq k-1$, and if $j = k-1$, then the inequality (5.3) only holds if $\ell = p-1$. In that case, ϵ_2 must be 1 and either ϵ_1 or ϵ_3 must be 1. Thus,

$$|(\gamma_{p^j}(\sigma b))^\ell b^{\epsilon_1} \lambda_1'^{\epsilon_2} \lambda_2^{\epsilon_3}| \geq (2p^2-2p)(p-1) + 2p^2 - 2p + 1 + 2p^2 - 2p - 1 = 2p^3 - 2p$$

But, $2p^3 - 2p > 2p^3 - 2p^2 = |\gamma_p(\sigma b)|$ contradicting the assumption that $|m| < |\gamma_p(\sigma b)|$. In the case $k > 1$, then the inequality (5.3) only holds if $j \geq k$, but if $j \geq k$, then

$$|(\gamma_{p^j}(\sigma b))^\ell| \geq 2p^{k+2} - 2p^{k+1} = |\gamma_{p^k}(\sigma b)|$$

so again m does not satisfy $|m| < |\gamma_{p^k}(\sigma b)|$. Thus, no such m such that $|m| < |\gamma_{p^k}(\sigma b)|$ and $\mathrm{mfilt}(m) \geq \mathrm{mfilt}(\gamma_p^k(\sigma b))$ exists. This implies that there are no hidden coactions and $\gamma_{p^k}(\sigma b)$ remains a co-module primitive. □

Corollary 5.7. In the S^1–BP-homotopy fixed point spectral sequence associated to $THH(K(\mathbb{F}_q)_p)$ there are differentials

$$d^2(\tilde{\xi}_1^p) = t\lambda_1'$$
$$d^2(\tilde{\xi}_2) = t\lambda_2$$
$$d^2(b) = t\sigma b$$

and no further d^2 differentials.

Proof. This follows from Proposition 5.2 and the fact that $\lambda_2 = \sigma\tilde{\xi}_2$ and $\lambda_1' = \sigma\tilde{\xi}_1^p$. □

Remark 5.8. We will also need to know the coaction of BP_*BP on

$$BP_*(V(1) \wedge T_{k+1}(K(\mathbb{F}_q)_p)),$$

which is isomorphic to

$$P(\bar{\xi}_1^p, \bar{\xi}_2, ...) \otimes E(b) \otimes E(\sigma\bar{\xi}_1^p, \sigma\bar{\xi}_2) \otimes P(\mu_2) \otimes \Gamma(\sigma b) \otimes P(t)/t^k$$

modulo differentials. This just amounts to describing the coaction on the class t. On the homotopy fixed point spectral sequence

$$H^*(S^1, BP_*(V(1) \wedge THH(K(\mathbb{F}_q)_p)))$$

BP_*BP only has a natural coaction on the coefficients, so the class $t \in H^2(S^1, \mathbb{F}_p)$ is a comodule primitive.

5.4 Detecting the periodic families of height two in iterated K-theory

We first recall a theorem of Ausoni-Rognes that aids in producing the v_2-periodic family generated by β_1 in $V(1)_*THH(K(\mathbb{F}_q)_p)^{hS^1}$.

Theorem 5.9 (Ausoni-Rognes [9])**.** The classes v_2^k map to nonzero classes $(t\mu)^k$ under the unit map

$$V(1)_*S \to V(1)_*THH(\ell)^{hS^1}$$

and hence pullback to nontrivial classes v_2^k in $V(1)_*K(\ell)$.

Remark 5.10. Since we showed v_2 maps to $t\mu_2$ under the unit map $V(1)_*S \to V(1)_*THH(j)^{hS^1}$

and the maps

$$V(1)_*S \to V(1)_*THH(K(\mathbb{F}_q)_p)^{hS^1} \to V(1)_*THH(\ell)^{hS^1}$$

are ring maps, the classes v_2^k also map to $(t\mu_2)^k$ under the unit map

$$V(1)_*S \to V(1)_*THH(K(\mathbb{F}_q)_p)^{hS^1}.$$

We therefore know that $(t\mu)^k$ are permanent cycles in the Adams, Adams-Novikov, and homotopy fixed point spectral sequences computing $V(1)_*THH(K(\mathbb{F}_q)_p)^{hS^1}$. This fact will be used in the following proof.

Theorem 5.11. The elements $\beta_1 v_2^i$ in $V(1)_*S$ map to a non-trivial element $(t\mu_2)^i t\sigma b$ in $V(1)_*THH(K(\mathbb{F}_q)_p)^{hS^1}$ under the unit map.

Proof. We claim that $v_2 \in BP_*(V(1))$ also maps to $t\mu_2$ in

$$BP_*(V(1) \wedge F(S(\mathbb{C}^2)_+, THH(K(\mathbb{F}_q)_p))^{S^1})$$

in the cobar complex, which is the E_1-page of the Adams-Novikov spectral sequence for $BP_*(V(1) \wedge F(S(\mathbb{C}^2)_+, THH(K(\mathbb{F}_q)_p))^{S^1})$. The class v_2 in the Adams-Novikov E_∞ page for $V(1)$ must map nontrivially to a class in the Adams-Novikov E_∞ page in the target, since v_2 in the associated graded represents a class in $V(1)_*S$ which maps nontrivially to $V(1)_*T_2(K(\mathbb{F}_q)_p)$

by Proposition 5.4. Since we have a commutative diagram of spectral sequences with input

$$\begin{tikzcd}
Ext_{BP_*BP}(BP_*, BP_*(V(1))) \arrow[r, "g"] \arrow[d] & Ext_{BP_*BP}(BP_*, BP_*(V(1) \wedge T_2(K(\mathbb{F}_q)_p))) \arrow[d] \\
Ext_{\mathcal{A}}(\mathbb{F}_p, H_*V(1)) \arrow[r] & Ext_{\mathcal{A}}(\mathbb{F}_p, H_*V(1) \wedge T_2(K(\mathbb{F}_q)_p))
\end{tikzcd}$$

and the map of spectral sequences $Ext_{BP_*BP}(BP_*, BP_* \otimes X) \to Ext_{\mathcal{A}_*}(\mathbb{F}_p, H_*X)$ always raises filtration, the class v_2 must map to a class in filtration 0 or 1. Since we know v_2 maps to $u \cdot t\mu$, where $u \in \mathbb{F}_p^\times$, modulo classes in higher filtration, we just need to check classes in topological degree $2p^2 - 2$ in filtration 1. The classes in topological degree $2p^2 - 2$ and filtration 1 are $1 \otimes \lambda_2$, $t_1 \otimes \lambda_1'$, and $v_1 \otimes \lambda_1'$, so

$$g(v_2) = t\mu_2 + a_0 \cdot 1 \otimes \lambda_2 + a_1 \cdot t_1 \otimes \lambda_1' + a_2 \cdot v_1 \otimes \lambda_1'$$

for some $a_0, a_1, a_2 \in \mathbb{F}_p$. However, we know that the composite map of spectral sequences

$$Ext_{BP_*BP}(BP_*, BP_*(V(1))) \to Ext_{\mathcal{A}}(\mathbb{F}_p, H_*(V(1) \wedge T_2(K(\mathbb{F}_q)_p)))$$

sends v_2 to $\bar{\tau}_2 \otimes 1$ by Proposition 5.4. We also know that the map

$$Ext_{BP_*BP}(BP_*, BP_*(V(1) \wedge T_2(K(\mathbb{F}_q)_p))) \to Ext_{\mathcal{A}}(\mathbb{F}_p, H_*V(1) \wedge T_2(K(\mathbb{F}_q)_p))$$

sends $1 \otimes \lambda_2$, $t_1 \otimes \lambda_1'$, and $v_1 \otimes \lambda_1'$ to classes of the same name by Proposition 5.6. Thus, a_0, a_1, and a_2 must be zero.

This argument along with Remark 5.10 also implies that v_2^k maps to $(u \cdot t\mu_2)^k$ in

$$BP_*(V(1) \wedge T_{k+1}(K(\mathbb{F}_q)_p)).$$

and, hence, the class $(u \cdot t\mu_2)^k$ is a comodule primitive for each k.

The element β_1 is represented by the class

$$b_{1,0} = \Sigma_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} t_1^i \otimes t_1^{p-i} \otimes 1$$

in the E_1-page of the Adams-Novikov spectral sequence for $V(1)$ [51]. It maps to a class of the same name in the cobar complex for the BP_*BP-comodule

$$BP_*(V(1) \wedge T_2(K(\mathbb{F}_q)_p)),$$

i.e. the E_1 page of the Adams-Novikov spectral sequence for $V(1) \wedge T_2(K(\mathbb{F}_q)_p)$ We know that the class $b_{1,0}$ survives to E_∞ in the Adams-Novikov spectral sequence for $V(1) \wedge T_2(K(\mathbb{F}_q)_p)$ and becomes the class $t\sigma b$ by Proposition 5.4.

The class $b_{1,0}v_2^{k-1}$ represents $\beta_1 v_2^{k-1}$ in the Adams-Novikov spectral sequence for $V(1)$. It maps to $b_{1,0}(u \cdot t\mu_2)^{k-1}$ in the Adams-Novikov spectral sequence for $V(1) \wedge T_k(K(\mathbb{F}_q)_p)$ by the argument above. Since the class representing $\beta_1 v_2^{k-1}$ is a permanent cycle in the Adams-Novikov spectral sequence for $V(1)$, the class $b_{1,0}(u \cdot t\mu_2)^{k-1}$ is an infinite cycle in the Adams-Novikov spectral sequence for $V(1) \wedge T_k(K(\mathbb{F}_q)_p)$, but it could still be a co-boundary. It is on the two-line of the Adams-Novikov spectral sequence, so we just need to check that it is not the co-boundary of a d^1 or d^2 differential.

1. If the class $b_{1,0}(u \cdot t\mu_2)^{k-1}$ is the co-boundary of a d^1, then there is a sum of classes

$$\sum_i a_i \otimes m_i \in BP_*BP \otimes_{BP_*} BP_*(V(1) \wedge T_{k+1}(K(\mathbb{F}_q)_p)).$$

such that

$$d_1(\sum_i a_i \otimes m_i) = \Sigma_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} t_1^i \otimes t_1^{p-i} \otimes (u \cdot t\mu_2)^{k-1}.$$

Recall that the coaction on m is of the form $\psi(m) = 1 \otimes m + \sum_j a_j \otimes m_j$ where $|m_j| < |m|$. Observe that the only elements in $(BP \wedge V(1))_* T_k(K(\mathbb{F}_q)_p)$ whose coaction contains $(u \cdot t\mu)^{k-1}$ as either m or m_j for some j are classes of the form $(u \cdot \mu)^{k-1} y$ for some $y \in (BP \wedge V(1))_* T_k(K(\mathbb{F}_q)_p)$ and unit $v \in \mathbb{F}_p^{\times}$. The coaction of such a class is

$$\psi((u \cdot t\mu)^{k-1} y) = (1 \otimes (u \cdot t\mu)^{k-1})\psi(y),$$

and $\psi(y)$ must be of the form

$$\psi(y) = 1 \otimes y + z \otimes 1 + \sum b_i \otimes y_i$$

since $\psi((u \cdot t\mu)^{k-1} y)$ must have $1 \otimes (u \cdot t\mu)^{k-1}$ as a term. Since the only classes in $(BP \wedge V(1))_* T_k(K(\mathbb{F}_q)_p)$ that have a term $z \otimes 1$ in their coaction are the classes t_1^p, t_i for $i \geq 2$ the class y must be a product of these. Since $|(u \cdot t\mu)^{k-1} y| = (2p^2 - 2)(k-1) + |y|$ and the degree must equal $(2p^2 - 2)k - 2p$, the degree of y must be $2p^2 - 2p - 2$. However, the class t_1^p is the element of lowest degree in the set $\{t_1^p, t_2, \dots\}$ and $|t_1^p| = 2p^2 - 2p$, so no product of classes in this set can be in degree $2p^2 - 2p - 2$. Thus, $m_i = (u \cdot t\mu)^{k-1}$

for at least one i.

Now, if $m_i = (u \cdot t\mu)^{k-1}$ for only one i, then the element a_i corresponding to m_i must have reduced co-product $e \cdot b_{1,0} + z$ for some unit $e \in \mathbb{F}_p^{\times}$ and some class z; i.e.

$$\bar{\Delta}(a_i) = \Delta(a_i) - a_i \otimes 1 - 1 \otimes a_i = e \cdot b_{1,0} + z$$

The degree of a_i must be $2p^2 - 2p$, so $a_i = f \cdot t_1^j v_1^{p-j}$ where $f \in \mathbb{F}_p^{\times}$.

However,

$$\begin{aligned} \bar{\Delta}(f \cdot t_1^j v_1^{p-j}) &= f \cdot v_1^{p-j} \bar{\Delta}(t_1^j) \\ &= f \cdot v_1^{p-j}(t_1 \otimes 1 + 1 \otimes t_1)^j - 1 \otimes f \cdot t_1^j v_1^{p-j} - f_i t_1^j v_1^{p-j} \otimes 1 \end{aligned}$$

and this does not equal

$$e \Sigma_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} t_1^i \otimes t_1^{p-i} + z$$

for any $j, e, f \in \mathbb{F}_p^{\times}$, and any class z.

Suppose that $m_i = (u \cdot t\mu)^{k-1}$ for $i \in I$ where I contains more than one natural number. Then

$$\psi(\sum_{i \in I} a_i) = g \cdot b_{1,0} + z'$$

for some unit $g \in \mathbb{F}_p^{\times}$ and some possibly trivial class z'. However, we checked in the proof of Proposition 5.4 that no class of the form $\sum_{i \in I} a_i \otimes 1$ has coaction

$$g \cdot \Sigma_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} t_1^i \otimes t_1^{p-i} \otimes 1 + z'$$

as described above and the same proof applies here.

Thus, there is no sum of classes $\sum_i a_i \otimes m_i$ such that

$$d_1(\sum_i a_i \otimes m_i) = b_{1,0} \otimes (u \cdot t\mu_2)^{k-1}$$

and therefore the class $b_{1,0}(u \cdot t\mu_2)^{k-1}$ survives to the E_2-page.

2. Now suppose there is a class in bidegree $(2p^2k-2k+2p^2-2p+1, 0)$ that is the source of a d^2 differential hitting $b_{1,0}(u \cdot t\mu_2)^{k-1}$. This class is therefore in $BP_{2p^2k-2k+2p^2-2p+1}V(1) \wedge T_{k+1}(K(\mathbb{F}_q)_p)$. Since this class is in an odd degree, we can classify all the classes that could possibly be in this degree as a linear combination of elements in the three families $\{\lambda_1' z_1, \lambda_2 z_2, t^k b z_3\}$ where z_1 and z_2 are some nontrivial product of even dimensional classes and z_3 is some nontrivial product of even dimensional classes that does not include $t\sigma b$ or $(t\mu_2)^j$ for any $j \geq 1$ as a factor since $t^{k+1} = 0$. We therefore compute the differential d_2 on each of these classes.

   Since λ_1' is a co-module primitive, $d_1(\lambda_1') = 0$, so we need to check what $d_2(\lambda_1')$. However, $|d_2(\lambda_1')| = (2p^2 - 2p, 2)$ using the convention $(t - s, s)$. So $d_2(\lambda_1')$ is represented in the cobar complex by an element of the form $a \otimes b \otimes c$ where $|a \otimes b \otimes c| = 2p^2 - 2p + 2$, but by checking the degrees of every class, we see that no products of classes are in this degree and hence no such class exists. Hence, $d_2(\lambda_1') = 0$ and consequenctly $d_2(\lambda_1' z_1) = \lambda_1' d_2(z_1)$.

   The class λ_2 has coaction $\psi(\lambda_2) = 1 \otimes \lambda_2 + t_1 \otimes \lambda_1'$ so there is a differential $d_1(\lambda_2) = t_1 \otimes \lambda_1'$. Hence, $d_1(\lambda_2 z_2) = (t_1 \otimes \lambda_1')z_2 + \lambda_2 d_1(z_3) \neq 0$ so $\lambda_2 z_3$ does not survive to the E_2-page.

We therefore just need to check that a class of the form $t^k b z_3$ where z_3 does not contain $t\mu$ or $t\sigma b$ as a factor. Note that the Leibniz rule implies

$$d_2(t^k b z_3) = d_2(t^k b) z_3 + t^k b d_2(z_3).$$

We now need to check if

$$d_2(a_1\lambda_1' z_1 + a_2 t^k b z_3) = m\Sigma_{i=1}^{p-1}\frac{1}{p}\binom{p}{i} t_1^i \otimes t_1^{p-i} \otimes (t\mu_2)^{k-1}$$

for some unit $m \in \mathbb{F}_p^{\times}$, but

$$d_2(a_1\lambda_1' z_1 + a_2 t^k b z_3) = a_1\lambda_1' d_2(z_1) \pm a_2(d_2(t^k b) z_3 + t^k b d_2(z_3))$$

and there are no values of $d_2(z_1)$, $d_2(t^k b)$, and $d_2(z_3)$ that make this hold since z_3 cannot contain $t\mu$ or $t\sigma b$ as a factor and the classes λ_1' and $t^k b$ are not factors of $b_{1,0}(u \cdot t\mu)^k$.

□

To produce the following corollary to Theorem 5.11 we will need to use the property that the trace map $K(R) \to THH(R)^{hS^1}$ is a map of commutative ring spectra when R is a commutative ring spectrum. The proof that the trace maps $K(R) \to TC(R)$ and $K(R) \to THH(R)$ are maps of ring spectra when R is a commutative ring spectrum may be attributed to Hesselholt-Geisser [30], Blumberg-Gepner-Tabuada [15], and Dundas [25]. We recall the theorem of Dundas.

Theorem 5.12 (Dundas [25])**.** Let R be a commutative ring spectrum. The cyclotomic trace map $K(R) \to TC(R)$ is a weak map of commutative ring spectra; i.e. there is a zizag of commutative ring spectrum maps where all wrong way maps are equivalences of commutative ring spectra.

We also need to know that the map $\pi_* TC(R) \to \pi_* THH(R)^{hS^1}$ is a map of rings.

Lemma 5.13. The map $TC(R;p) \to THH(R)^{hS^1}$ is a map in $\mathrm{Ho}(\mathrm{Comm}\,\mathfrak{S})$; hence, the map $\pi_* TC(R)_p \to \pi_* THH(R)^{hS^1}$ is a map of rings.

Proof. First, recall that $TF(R) \simeq \operatorname{holim}_F THH(R)^{C_{p^n}}$ and, since $THH(R)^{C_{p^n}}$ are commutative ring spectra when R is a commutative ring spectrum, we can take the homotopy limit in the category of commutative ring spectra and $TF(R)$ will be a commutative ring spectrum as well. Define $TC(R;p)$ as the equalizer

$$TC(R;p) \simeq \mathrm{eq}\,\{TF(R) \underset{R}{\overset{Id}{\rightrightarrows}} TF(R)\}$$

in the category of commutative ring spectra, since Id and R are commutative ring spectrum maps. Thus, the map $TC(R;p) \to TF(R)$ is a map of commutative ring spectra. Now, the maps $THH(R)^{C_{p^n}} \to THH(R)^{hC_{p^n}}$ are maps of commutative ring spectra since there is a commuting diagram

$$\begin{tikzcd}
F(S^0, THH(R))^{C_{p^n}} \arrow[r] & F((EC_{p^n})_+, THH(R))^{C_{p^n}} \\
F(S^0 \wedge S^0, THH(R) \wedge THH(R))^{C_{p^n}} \arrow[r] \arrow[u] & F((EC_{p^n})_+ \wedge (EC_{p^n})_+, THH(R) \wedge THH(R))^{C_{p^n}} \arrow[u] \\
F(S^0, THH(R))^{C_{p^n}} \wedge F(S^0, THH(R))^{C_{p^n}} \arrow[r] \arrow[u] & F((EC_{p^n})_+, THH(R))^{C_{p^n}} \wedge F((EC_{p^n})_+, THH(R))^{C_{p^n}} \arrow[u]
\end{tikzcd}$$

where the horizontal maps are induced by the map $(EC_{p^n})_+ \to S^0$ and the vertical maps are induced by the S^1-equivariant commutative multiplication map $THH(R) \wedge THH(R) \to THH(R)$ and the diagonal maps $S^0 \to S^0 \wedge S^0$ and $(EC_{p^n})_+ \to (EC_{p^n})_+ \wedge (EC_{p^n})_+$, which are co-commutative maps such that the diagram

$$\begin{tikzcd}
(EC_{p^n})_+ \arrow[r] \arrow[d] & (EC_{p^n})_+ \wedge (EC_{p^n})_+ \arrow[d] \\
S^0 \arrow[r] & S^0 \wedge S^0
\end{tikzcd}$$

commutes. Since the maps $THH(R)^{hC_{p^n}} \to THH(R)^{hS^1}$ is also a map of commutative ring spectra such that the diagram

$$\begin{tikzcd}
THH(R)^{C_{p^{n+1}}} \arrow[r] & THH(R)^{hC_{p^{n+1}}} \arrow[dr] & \\
THH(R)^{C_{p^n}} \arrow[r] \arrow[u] & THH(R)^{hC_{p^n}} \arrow[r] \arrow[u] & THH(R)^{hS^1}
\end{tikzcd}$$

commutes in the category of commutative ring spectra, we can take homotopy limits in the category of commutative ring spectra to produce a composite map

$$TC(R;p) \to TF(R) \to \operatorname{holim}_F THH(R)^{hC_{p^n}} \to THH(R)^{hS^1}$$

in the category of commutative ring spectra. Since the derived functor of the forgetful functor $U : \mathrm{Comm}\,\mathfrak{S} \to \mathfrak{S}$ is a Quillen right adjoint, it preserves homotopy limits, so the homotopy limits that we compute are actually the same as the homotopy limits in the category $\mathrm{Ho}\,\mathfrak{S}$ and hence, they are equivalent to the way that TF and TC are usually defined. Thus, the map

$$TC(R;p) \to THH(R)^{hS^1}$$

is a map in $\operatorname{Ho}\operatorname{Comm}\mathfrak{S}$, which suffices to induce a ring map in homotopy groups. $\square$

Since $TC(R)_p \simeq TC(R;p)_p$ as commutative ring spectra, we therefore produce a commutative ring spectrum map

$$K(R) \to THH(R)^{hS^1},$$

which will be used in the proof below.

Corollary 5.14. Let $p \geq 5$ be a prime and q be a prime power that topologically generates $\mathbb{Z}_p^\times$. The classes $\beta_1 v_2^k$ map from $\pi_* V(1)$ to nonzero elements in $V(1)_* K(K(\mathbb{F}_q))$ under the unit map. Consequently, the classes β_{pk+1} for $k \geq 0$ map to $K(K(\mathbb{F}_q))$ under the unit map $\pi_* S \to K(K(\mathbb{F}_q))$.

Proof. The classes $\beta_1 v_2^k$ in $V(1)_*$ map to $V(1)_* K(K(\mathbb{F}_q))$ under the unit map since the cyclotomic trace is multiplicative and therefore the maps

$$V(1)_* S \to V(1)_* THH(K(\mathbb{F}_q)_p)^{hS^1} \to V(1)_* THH(K(\mathbb{F}_q)_p)$$

factor through $V(1)_* K(K(\mathbb{F}_q))$; i.e, there is a commutative diagram

$$\begin{tikzcd}
V(1)_* S \arrow[r, "V(1)_*\eta"] \arrow[dr, "V(1)_*\eta"'] & V(1)_* K(K(\mathbb{F}_q)) \arrow[r, "V(1)_* K(f_p)"] \arrow[d, "V(1)_*(K(f_p)\circ tr)"] \arrow[dr, "V(1)_*(K(f_p)\circ tr)"] & V(1)_* K(K(\mathbb{F}_q)_p) \arrow[d, "V(1)_* tr"] \\
 & V(1)_* THH(K(\mathbb{F}_q)_p) & V(1)_* THH(K(\mathbb{F}_q)_p)^{hS^1} \arrow[l, "V(1)_* F"]
\end{tikzcd}$$

where $f_p : K(\mathbb{F}_q) \to K(\mathbb{F}_q)_p$ denotes the p-completion map, $F : X^{hS^1} \to X$ indicates inclusion of homotopy fixed points, and we abuse notation and write tr for the trace maps from algebraic K-theory to THH and its S^1-homotopy fixed points.

There is also a commuting diagram of ring spectra

$$\begin{tikzcd}
S \simeq S \wedge S \arrow[r, "1_S \wedge \eta"] \arrow[d, "i_0 i_1 \wedge 1_S"'] & S \wedge K(K(\mathbb{F}_q)) \arrow[r] \arrow[d, "i_0 i_1 \wedge 1_{K(K(\mathbb{F}_q))}"'] \arrow[dr, "i_0 i_1 \wedge K(f_p)"] & V(1) \wedge K(K(\mathbb{F}_q)_p)_p \\
V(1) \wedge S \arrow[r, "1_{V(1)} \wedge \eta"] & V(1) \wedge K(K(\mathbb{F}_q)) \arrow[r, "1_{V(1)} \wedge K(f_p)"] & V(1) \wedge K(K(\mathbb{F}_q)_p) \arrow[u, "1_{V(1)} \wedge g_p"']
\end{tikzcd}$$

where $f_p : K(\mathbb{F}_q) \to K(\mathbb{F}_q)_p$, and $g_p : K(K(\mathbb{F}_q)_p) \to K(K(\mathbb{F}_q)_p)_p$ are p-completion maps and η is the unit map. Since the classes $v_2^{pk}\beta_1 \in V(1)_*$ pullback to classes β_{pk-1} in $\pi_* S$ along the unit map and since they map nontrivially to classes in $\pi_* V(1) \wedge K(K(\mathbb{F}_q)_p)$, they must map to nontrivial classes in $\pi_* K(K(\mathbb{F}_q))$ under the unit map

$$\pi_* S \to \pi_* K(K(\mathbb{F}_q)).$$

$\square$

Remark 5.15. We expect that more of the divided β-family in the homotopy groups of spheres is detected in $\pi_* K(K(\mathbb{F}_q))$ and we are currently in the process of studying how much more we can detect. To detect all of the divided β-family would require more knowledge of $\pi_* K(K(\mathbb{F}_q))$, which is beyond the scope of the present thesis.

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ABSTRACT

PERIODICITY IN ITERATED ALGEBRAIC K-THEORY OF FINITE FIELDS

by

GABRIEL J. ANGELINI-KNOLL

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Advisor: Dr. Andrew Salch

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In this dissertation, we study the interactions between periodic phenomena in the homotopy groups of spheres and algebraic K-theory of ring spectra. C. Ausoni and J. Rognes initiated a program to study the arithmetic of ring spectra using algebraic K-theory and gave a higher chromatic version of the Lichtenbaum-Quillen conjecture, called the red-shift conjecture, that is expected to govern this arithmetic. This dissertation provides a proof of a special case of a variation on the red-shift conjecture. Specifically, we show that, under conditions on the order of the fields, iterated algebraic K-theory of finite fields detects a periodic family of chromatic height two.

To prove that iterated algebraic K-theory of finite fields detects a periodic family of chromatic height two, we compute approximations to iterated algebraic K-theory using the theory of trace methods. We develop a tool for computing higher order topological Hochschild homology (THH) using a filtration of a commutative ring spectrum. We then compute THH of algebraic K-theory of finite fields after smashing with a finite complex. We then detect height two periodic elements in the circle homotopy fixed points of THH and show that periodic families of height two are detected in iterated algebraic K-theory of finite fields.

AUTOBIOGRAPHICAL STATEMENT

Gabriel James Angelini-Knoll was born in the Cass Corridor neighborhood of Detroit, Michigan in 1988. He grew up on the east side of Detroit and attended University of Detroit Jesuit High School. He went on to attend Kalamazoo College and earn a B.A. in Mathematics and a B.A. in Psychology in 2011. In 2013, he earned a M.A. in mathematics from Wayne State University with a Master's thesis "Galois cohomology and algebraic K-theory of finite fields" under the direction of Dr. Andrew Salch. In 2017, he earned a Ph.D. from Wayne State University receiving a Rumble fellowship for his final year of research as well as other honors. He accepted a post-doctoral position at Michigan State University in 2017.