Weighted Linear Matroid Parity*

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Abstract

The matroid parity (or matroid matching) problem, introduced as a common generalization of matching and matroid intersection problems, is so general that it requires an exponential number of oracle calls. Nevertheless, Lovasz (1978) showed that this problem admits a min-max formula and a polynomial algorithm for linearly represented matroids. Since then efficient algorithms have been developed for the linear matroid parity problem.

This talk presents a recently developed polynomial-time algorithm for the weighted linear matroid parity problem. The algorithm builds on a polynomial matrix formulation using Pfaffian and adopts a primal-dual approach based on the augmenting path algorithm of Gabow and Stallmann (1986) for the unweighted problem.

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1 Introduction

The concept of matroids was introduced by Whitney [33] as a combinatorial abstraction of linear dependence. A matroid is a pair $(S, \mathcal{I})$ of a finite set E and its subset family $\mathcal{I}$ that satisfy the following axioms.

(I0) $\emptyset \in \mathcal{I}$.
(I1) $I \subseteq J \in \mathcal{I} \Rightarrow I \in \mathcal{I}$.
(I2) $I, J \in \mathcal{I}$, $|I| < |J| \Rightarrow \exists e \in J \setminus I, I \cup \{e\} \in \mathcal{I}$.

A primary example is a set S of vectors in a certain linear space, where $\mathcal{I}$ is the collection of vector subsets that are linearly independent. Such a matroid representable in this way is called a linear matroid.

The importance of matroids in the context of combinatorial otimization was established by Edmonds [6, 8]. In particular, the framework of matroid intersection generalizes bipartite matching and captures various combinatorial optimizations problems solvable in polynomial time.

As a common generalization of matroid intersection and nonbipartite matching, Lawler [19] introduced matroid parity. Suppose that the ground set S of a matroid $(S, \mathcal{I})$ is partitioned into pairs, called lines. The matroid parity problem asks for finding a maximum cardinality independent set that is a disjoint union of lines. It turned out, however, that this framework is too general to be solvable. In fact, it includes NP-hard problems and requires exponential number of independence oracle calls [17, 21]. A PTAS for this general framework has been developed only recently [20].

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For linear matroids, however, Lovász [21, 23, 24] showed a min-max formula and presented a polynomial algorithm that is applicable if the linear representation is available. Since then, efficient combinatorial algorithms have been developed for this linear matroid parity problem [11, 29, 30]. Gabow and Stallmann [11] developed an augmenting path algorithm with the aid of a linear algebraic trick, which was later extended to the linear delta-matroid parity problem [13]. Orlin and Vande Vate [30] provided an algorithm that solves this problem by repeatedly solving matroid intersection problems coming from the min-max theorem. Later, Orlin [29] improved the running time bound of this algorithm. The current best deterministic running time bound due to [11, 29] is $O(nm^{\omega})$, where n is the cardinality of the ground set, m is the rank of the linear matroid, and ω is the matrix multiplication exponent, which is at most 2.38.

Since matching and matroid intersection algorithms [4, 7] have been successfully extended to their weighted version [5, 9, 15, 18], it is natural to expect polynomial algorithms for the weighted linear matroid parity problem. In fact, a recent work [16] has presented a combinatorial, deterministic, polynomial-time algorithm for the weighted linear matroid parity problem. The algorithm builds on a polynomial matrix formulation, which naturally extends the one discussed in [12] for the unweighted problem.

2 The Linear Matroid Parity Problem

Let A be a matrix of row-full rank over an arbitrary field $\mathbf{K}$ with row set U and column set V. Assume that $n = |V|$ are even. The column set V is partitioned into pairs, called *lines*. Each $v \in V$ has its *mate* $\bar{v}$ such that $\{v, \bar{v}\}$ is a line. We denote by L the set of lines.

The linear dependence of the column vectors naturally defines a matroid $\mathbf{M}(A)$ on V. The independent set family $\mathcal{I}$ is given by $\mathcal{I} = \{J \mid \operatorname{rank} A[U, J] = |J|\}$ A subset $X \subseteq V$ is called a *parity set* if it consists of lines. The linear matroid parity problem asks for finding an independent parity set of maximum cardinality. We denote the optimal value by $\nu(A, L)$ This problem generalizes finding a maximum matching in graphs and a maximum common independent set of a pair of linear matroids on the same ground set.

For a skew-symmetric matrix Φ whose rows and columns are indexed by W, the *support graph* of Φ is the graph $G = (W, E)$ with edge set $E = \{(u, v) \mid \Phi_{uv} \neq 0\}$. We denote by $\operatorname{Pf} \Phi$ the *Pfaffian* of Φ, which is defined as follows:

$$\operatorname{Pf} \Phi = \sum_{M} \sigma_M \prod_{(u,v)\in M} \Phi_{uv},$$

where the sum is taken over all perfect matchings M in G and σ_M takes ± 1 in a suitable manner, see [25]. It is well-known that $\det \Phi = (\operatorname{Pf} \Phi)^2$ and $\operatorname{Pf} (S \Phi S^{\top}) = \operatorname{Pf} \Phi \cdot \det S$ for any square matrix S.

Associated with the linear matroid parity problem, we consider a skew-symmetric matrix Φ_A defined by

$$\Phi_A = \begin{pmatrix} O & A \\ -A^{\top} & D \end{pmatrix},$$

where D is a block-diagonal matrix in which each block is a 2×2 skew-symmetric matrix $D_\ell = \begin{pmatrix} 0 & -\tau_\ell \\ \tau_\ell & 0 \end{pmatrix}$ corresponding to a line $\ell \in L$. Assume that the coefficients τ_ℓ are independent parameters (or indeterminates).

▶ **Lemma 1** ([12]). *The optimal value $\nu(A, L)$ of the linear matroid parity problem is given by*

$$\nu(A, L) = \operatorname{rank} \Phi_A - n.$$

This characterization leads to an efficient randomized algorithm for solving the linear matroid parity problem in high probability by substituting randomly generated numbers to the indeterminates. In fact, Lovász [22] introduced such an approach using another skew-symmetric matrix, and Cheung, Lau, and Leung [3] improved it to run in $O(nm^{\omega-1})$ time, extending the techniques of Harvey [14] developed for matching and matroid intersection.

3 The Minimum-Weight Parity Base Problem

In the same setting as the linear matroid parity problem, suppose that each line $\ell \in L$ has a weight $w_\ell \in \mathbb{R}$. Let $\mathcal{B}$ be the base family of $\mathbf{M}(A)$, i.e., $\mathcal{B} = \{B \mid \operatorname{rank} A[U, B] = |B| = |U|\}$. A base $B \in \mathcal{B}$ is called a *parity base* if it consists of lines. As a weighted version of the linear matroid parity problem, we will consider the problem of finding a parity base of minimum weight, where the weight of a parity base is the sum of the weights of lines in it. We denote the optimal value by $\zeta(A, L, w)$. This problem generalizes finding a minimum-weight perfect matching in graphs and a minimum-weight common base of a pair of linear matroids on the same ground set.

As another weighted version of the matroid parity problem, one can think of finding an independent parity set of maximum weight. This problem can be easily reduced to the minimum-weight parity base problem.

Associated with the minimum-weight parity base problem, we consider a skew-symmetric polynomial matrix $\Phi_A(\theta)$ in variable θ defined by

$$\Phi_A(\theta) = \begin{pmatrix} O & A \\ -A^\top & D(\theta) \end{pmatrix},$$

where $D(\theta)$ is a block-diagonal matrix in which each block is a 2×2 skew-symmetric polynomial matrix $D_\ell(\theta) = \begin{pmatrix} 0 & -\tau_\ell \theta^{w_\ell} \\ \tau_\ell \theta^{w_\ell} & 0 \end{pmatrix}$ corresponding to a line $\ell \in L$. Assume that the coefficients τ_ℓ are independent parameters (or indeterminates).

We have the following lemma that associates the optimal value of the minimum-weight parity base problem with $\operatorname{Pf} \Phi_A(\theta)$.

▶ **Lemma 2** ([16]). *The optimal value of the minimum-weight parity base problem is given by*

$$\zeta(A, L, w) = \sum_{\ell \in L} w_\ell - \deg_\theta \operatorname{Pf} \Phi_A(\theta).$$

In particular, if $\operatorname{Pf} \Phi_A(\theta) = 0$*, then there is no parity base.*

Note that Lemma 2 does not immediately lead to a polynomial-time algorithm for the minimum weight parity base problem. This is because computing the degree of the Pfaffian of a skew-symmetric polynomial matrix is not so easy. Indeed, randomized algorithms in [2, 3] for the weighted linear matroid parity problem compute the degree of the Pfaffian of another skew-symmetric polynomial matrix, which results in pseudopolynomial complexity.

Starting with the characterization in Lemma 2, we have developed a combinatorial, deterministic polynomial-time algorithm for the minimum-weight parity base problem [16].

The algorithm employs a modification of the augmenting path search procedure for the unweighted problem by Gabow and Stallmann [11]. The correctness proof for the optimality is based on the idea of combinatorial relaxation for polynomial matrices due to Murota [28].

▶ **Theorem 3** ([16]). *The minimum-weight parity base problem can be solved with $O(mn^3)$ arithmetic operations over $\mathbf{K}$, where $m = |U|$ and $n = |V|$.*

This leads to a strongly polynomial algorithm for linear matroids represented over a finite field. For linear matroids represented over the rational field, one can exploit that algorithm to solve the problem in polynomial time.

4 Applications

The linear matroid parity problem finds various applications: structural solvability analysis of passive electric networks [27], pinning down planar skeleton structures [25], and maximum genus cellular embedding of graphs [10]. We describe two interesting applications of the weighted matroid parity problem in combinatorial optimization.

A T-path in a graph is a path between two distinct vertices in the terminal set T. Mader [26] showed a min-max characterization of the maximum number of openly disjoint T-paths. The problem can be equivalently formulated in terms of $\mathcal{S}$-paths, where $\mathcal{S}$ is a partition of T and an $\mathcal{S}$-path is a T-path between two different components of $\mathcal{S}$. Lovász [24] formulated the problem as a matroid matching problem and showed that one can find a maximum number of disjoint $\mathcal{S}$-paths in polynomial time. Schrijver [32] has described a more direct reduction to the linear matroid parity problem.

As a weighted version of the disjoint $\mathcal{S}$-paths problem, it is quite natural to think of finding disjoint $\mathcal{S}$-paths of minimum total length. It is not immediately clear that this problem reduces to the weighted linear matroid parity problem. A recent paper of Yamaguchi [34] clarifies that this is indeed the case.

The weighted linear matroid parity has also been used in the design of approximation algorithms. Prömel and Steger [31] provided a 5/3-approximation algorithm for the Steiner tree problem with the aid of the weighted parity problem for graphic matroids. Even though the performance ratio is larger than the current best one for the Steiner tree problem [1], this suggests that there may be other combinatorial optimization problems that admit new approximation algorithms using weighted linear matroid parity.

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