Embedding Graphs into Embedded Graphs*†

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Abstract

A (possibly degenerate) drawing of a graph G in the plane is approximable by an embedding if it can be turned into an embedding by an arbitrarily small perturbation. We show that testing, whether a drawing of a planar graph G in the plane is approximable by an embedding, can be carried out in polynomial time, if a desired embedding of G belongs to a fixed isotopy class, i.e., the rotation system (or equivalently the faces) of the embedding of G and the choice of outer face are fixed. In other words, we show that c-planarity with embedded pipes is tractable for graphs with fixed embeddings.

To the best of our knowledge an analogous result was previously known essentially only when G is a cycle.

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1 Introduction

In the theory of graph visualization a drawing of a graph $G = (V, E)$ in the plane is usually assumed to be free of degeneracies, i.e., edge overlaps and edges passing through a vertex. However, in practice degenerate drawings often arise and need to be dealt with.

Recent papers [1, 7] address a certain aspect of this problem for simple polygons which can be thought of as straight-line (rectilinear) embeddings of graph cycles. Chang et al. [7] gave an $O(n^2 \log n)$-time algorithm to detect if a given polygon with n vertices can be turned into a simple (non self-intersecting) one by small perturbations of its vertices, or in other words if the polygon is **weakly simple**. We mention that there exists an earlier closely related definition of weakly simple polygons by Toussaint [6, 26], however, as pointed out in [7] this notion is not well-defined for general polygons with "spurs", see [7] for an overview of attempts at combinatorial definitions of a polygon not crossing itself.

An $O(n \log n)$ improvement on the running time of the algorithm by Chang et al. was announced very recently by Akitaya et al. [1]. The combinatorial formulation of this problem corresponds to the setting of **c-planarity with embedded pipes** introduced by Cortese et al. [10] well before the two aforementioned papers. Therein only an $O(n^3)$-time algorithm for the problem was given. Nevertheless, the algorithms in [1, 7] were built upon the ideas from [10]. Moreover, to the best of our knowledge the complexity status of the c-planarity with embedded pipes is essentially known only for cycles. Recently the problem was studied for general planar graphs by Angelini and Da Lozzo [3], but they gave only an FPT algorithm.

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The introduction of this problem was motivated by a more general and well known problem of **c-planarity** by Feng et al. [13, 14], whose tractability status was open since 1995 even in much more restricted cases than the one that we consider. Biedl [4] gave a polynomial-time algorithm for c-planarity with two clusters. Beyond two clusters a polynomial time algorithm for c-planarity was obtained only in special cases, e.g., [9, 18, 19, 20, 21], and most recently in [5, 8, 15].

There is, however, another tightly related line of research on approximability or realizations of maps pioneered by Siekluckі [24], Minc [22] and M. Skopenkov [25] that is completely independent from the aforementioned developments, and that is also a major source of inspiration for our work. It can be easily seen that the result [25, Theorem 1.5] implies that c-planarity is tractable for flat instances with three clusters or cyclic clustered graphs [17, Section 6] with a fixed isotopy class of a desired embedding. An algorithm with a better running time was given by the author in [15].

The aim of the present work is to show that c-planarity with embedded pipes is tractable for planar graphs with a fixed isotopy class of embeddings, which extends results of [2, 3, 15]. Our work also implies the tractability of deciding whether a drawing is approximable by an embedding in a fixed isotopy class, which extends results of [1, 7]. This also answers in the affirmative a question posed in [7, Section 8.2] if the isotopy class of an embedding of G is fixed.

Roughly, we are to decide if in the given isotopy of G an embedding approximating a given (possibly degenerate) drawing of G class exists. The degenerate drawing of G is viewed as a plane graph H and the degeneracies (if any) are captured by a simplicial map between G and H. Let G and H be a pair of graphs such that H contains neither loops nor multiple edges, i.e., H is **simple**. A map $\gamma : V(G) \to V(H)$ is **simplicial** if for every edge $uv \in E(G)$ either $\gamma(u) = \gamma(v)$ or $\gamma(u)\gamma(v)$ is an edge of H. We partition $V(G)$ into **clusters** V_ν so that $\gamma(v) = \nu$ if and only if $v \in V_\nu$. If it leads to no confusion, we do not distinguish between a vertex or an edge and its representation in the drawing and we use the words "vertex" and "edge" in both contexts. We are in the position to state our problem formally.

We are given an ordered triple (G, H, γ), where G is a planar graph (possibly with loops and multiple edges) given by the isotopy class of an embedding of G in the plane, H is a plane simple graph[1], and $\gamma : V(G) \to V(H)$ is a simplicial map. We assume that the drawing given by H is piece-wise linear. The **feature size** of H is the minimum of the set consisting of the Euclidean non-zero distances between the endpoints of the line segments defining the drawing of H, and the Euclidean distances between the line segments defining the drawing of H. By treating a graph as a 1-dimensional topological space we extend the definition of γ linearly to the edges of G. We want to decide if the given isotopy class of G contains an embedding $\mathcal{E}$ such that $\|\mathcal{E}(x) - \gamma(x)\|_2 \ll \varepsilon$, for all $x \in G$, where $\varepsilon := \varepsilon(H) > 0$ is smaller than half of the feature size of H. Thus, by the choice of ε a desired embedding of G lies in a small neighborhood of H preserving the facial structure of the embedding of H.

However, to view the problem from a perspective that is more combinatorial, we put further restrictions on a desired embedding of G, which lead to the equivalent problem of **c-planarity with embedded pipes**, see Figure 1. To this end we need to introduce a couple of notions. Let $\mathrm{dist}(\mathbf{p}, \mathbf{q})$ denote the Euclidean distance between $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$. Let $\mathrm{dist}(\mathbf{p}, S) = \min_{\mathbf{q} \in S} \mathrm{dist}(\mathbf{p}, \mathbf{q})$, where $S \subset \mathbb{R}^2$. Let $N_\varepsilon(S)$ for $S \subset \mathbb{R}^2$ denote the ε-neighborhood of S, i.e., $N_\varepsilon(S) = \{\mathbf{p} \in \mathbb{R}^2 | \ \mathrm{dist}(\mathbf{p}, S) \leq \varepsilon\}$. Let $\varepsilon' > 0$ be a small value as described later. The

[1] In other words, a (planar) graph drawn in the plane without edge crossings.

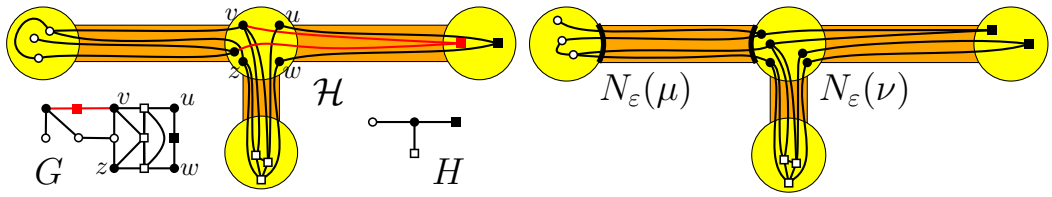

Figure 1 Instance of c-planarity with embedded pipes. The partition of the vertex set of G into clusters is encoded by the shape of vertices. An H-compatible embedding of a subgraph of G that cannot be extended to the whole G (left). An H-compatible embedding of G (right) inside $\mathcal{H}$. The valves of $\rho = \nu\mu$ at $N_\varepsilon(\nu)$ and $N_\varepsilon(\mu)$ are highlighted by bold arcs.

thickening $\mathcal{H}$ of H is the union of $N_\varepsilon(\nu)$, for all $\nu \in V(H)$ and $N_{\varepsilon'}(\rho)$, for all $\rho \in E(H)$[2]. Let the **pipe** of $\rho \in E(H)$ be the closure of $N_{\varepsilon'}(\rho) \setminus (N_\varepsilon(\nu) \cup N_\varepsilon(\mu))$, where $\rho = \nu\mu$. Let the **valve** of ρ at ν be the curve obtained as the intersection of $N_\varepsilon(\nu)$ and the pipe of ρ. We put $\varepsilon' < \varepsilon = \varepsilon(H)$, where $\varepsilon(H)$ is the same as in the previous paragraph, so that the valves are pairwise disjoint in $\mathcal{H}$.

In the combinatorial formulation of the problem, we are to decide if the given isotopy class of G contains an embedding contained in $\mathcal{H}$, where the vertices in V_ν, for every ν, are drawn in the interior of $N_\varepsilon(\nu)$ and every edge crosses the boundary of $N_\varepsilon(\nu)$, for every $\nu \in V(H)$, at most once. This does not change the problem as observed in [7]. Such an embedding of G is **H-compatible**. Let $E_{\nu\mu} = \{uv \in E(G) |\ v \in V_\nu, u \in V_\mu\}$. An H-compatible embedding of G is encoded by G, H, and a set of total orders $(E_{\nu\mu}, <_\omega)$, for every $\nu\mu \in E(H)$ and a valve ω of $\nu\mu$, where $(E_{\nu\mu}, <_\omega)$ encodes the order of crossings of ω with edges along ω. The isotopy class of G is encoded by a choice of the outer face, a set of rotations at its vertices and a containment relation of its connected components as described in Section 2. Since we are interested only in combinatorial aspects of the problem, H is also given by the isotopy class of its embedding. Throughout the paper we assume that G and H are given as above.

▶ **Theorem 1.** *There exists an $O(n^2)$-time algorithm that decides if the given isotopy class of G contains an H-compatible embedding. An H-compatible embedding of G can be also constructed in $O(n^2)$ time if it exists. In other words, c-planarity with embedded pipes is tractable, when an isotopy class of a desired embedding of G is fixed.*

As a corollary of our result we obtain that we can test in polynomial time if a piecewise linear drawing of a graph in the plane is approximable by an embedding and construct such an embedding if it exists. As we previously discussed, this extends results in [1, 7] and also [25].

▶ **Corollary 2.** *There exists an $O(n^4)$-time algorithm that decides if a piecewise linear (possibly degenerate) drawing of a graph in the plane is approximable by an embedding, and constructs such an embedding if it exists, where n is the size of the representation of the drawing.*

Extensions of our results. By [23, Theorem 3.1] and Fáry–Wagner theorem [12], our result holds also in the setting of rectilinear, i.e., straight-line, drawings of graphs. To extend it further in this setting by allowing "forks" seems to be just a little bit technical.

[2] Throughout the paper we denote vertices and edges of H by Greek letters.

In a recent manuscript [16], we verified a conjecture of M. Skopenkov [25, Conjecture 1.6] implying that our problem is tractable, when we lift the restriction on the isotopy class G. This does not imply that the problem with the restriction on the isotopy class G is tractable except when G is connected. The running time of the algorithm, that is implied by [16], is $O(|V|^{2\omega})$, where $O(n^\omega)$ is the running time of the fastest algorithm for multiplying a pair of n by n matrices. Since $\omega > 2$ due to the matrix size, this is much worse that the running time claimed by Theorem 1. Furthermore, the algorithm is not constructive.

As noted by Chang et al. [7], the technique of Cortese et al. [10] extends directly from the plane to any closed two-dimensional surface. The same holds for our method, but since considering general two-dimensional surfaces does not bring anything substantially new to our treatment of the problem, for the sake of simplicity we consider only the planar case.

Strategy of the proof of Theorem 1. Recall that the **input** of our algorithm is a triple (G, H, γ), where the partition of the vertex set of G corresponds to the map γ from the set of vertices of G to the set of vertices of H. Hence, for $v \in V_\nu$, where $\nu \in V(H)$, we have $\gamma(v) = \nu$. The input (G, H, γ) is **positive** if there exists an H-compatible embedding of G in the given isotopy class of G, and **negative** otherwise.

Main troubles in constructing a polynomial time algorithm for our problem are caused by so called "spurs" such as the red vertex in Figure 1 (left), i.e., connected components in subgraphs of G induced by clusters, whose all adjacent vertices belong to the same cluster. Due to the presence of spurs it is hard to see that our problem is tractable even in the case, when G is a path.

The centerpiece of our method is an extension of the definition of the derivative of maps of intervals/loops (corresponding to the case, when G is a path/cycle, in our terminology) in the plane introduced by Minc [22]. We adapt this notion to the setting of c-planarity with embedded pipes. The derivative is an operator that takes (G, H, γ), and either detects that there exists no H-compatible embedding of G in the given isotopy class of G, or outputs (G', H', γ'), that is also a valid input for our algorithm, such that (G, H, γ) is positive if and only if (G', H', γ') is positive (Lemma 5). Intuitively, H' is reminiscent of the line graph of H and the subgraphs of G, that are mapped by γ to the edges of H, are turned into subgraphs of G' mapped by γ' into vertices of H'. This results in a shortening of problematic spurs, and zooming into the structure of the map γ. We show that by iterating the derivative $|E(G)|$ times we either detect that there exists no H-compatible embedding of G in the given isotopy class of G, or we arrive at an input without problematic spurs (Lemma 6). Since it is fairly easy to solve the problem for the latter inputs; the derivative at every iteration can be computed in linear time in $|V(G)|$; and by derivating the size of the input is increased only by a little, the tractability follows.

The operation of node expansion and base contraction introduced by Cortese et al. [10] resemble the derivative. The main difference is that these two operations affect only a single cluster or a pair of clusters in (G, H, γ), and therefore they are local, whereas the derivative changes the whole input. We are very positive that our method is applicable to other graph drawing problems related to c-planarity whose tractability is open. This is documented by our recent manuscript [16] in which a similar technique was applied.

The derivative is applied to an input (G, H, γ), in which every cluster V_ν induces in G an independent set. Such an input is in the **normal form**. The detailed description of the algorithm proving Theorem 1 is in Section 3. We show in Section 3.1 that an input can be assumed to be in the normal form. The definition of the derivative is given in Section 3.2, and sufficiently simplified inputs are dealt with in Section 3.3.

2 Preliminaries

Throughout the paper we tacitly use Jordan-Schönflies theorem for polygons.

Let $G = (V, E)$ denote a planar graph possibly with multiple edges and loops. For $V' \subseteq V$ we denote by $G[V']$ the sub-graph of G induced by V'. A **star** $St(v)$ of a vertex v in a graph G is the subgraph of G consisting of all the edges incident to v. Throughout the paper we use standard graph theoretical notions such as path, cycle, walk, vertex degree $deg(v)$ etc., see [11].

A **drawing** $\mathcal{D}(G)$ is a representation of G in the plane, where every vertex in V is represented by a point and every edge $e = uv$ in E is represented by a simple piecewise linear curve joining the points that represent u and v. Thus, a drawing can be thought of as a map from G understood as a topological space into the plane. In a drawing, we additionally require every pair of distinct curves representing edges to meet only in finitely many points each of which is a proper crossing or a common endpoint. In a **degenerate** drawing, we allow a pair of distinct vertices to be represented by the same point and a pair of edges to be represented by the same curve. Note that we do not allow an edge to pass through a vertex by the definition of the drawing, or in other words, we do not allow a drawing to contain **forks** [7]. A drawing in which every vertex is represented by a unique point and every edge by a unique curve is **non-degenerate**. In a non-degenerate drawing, multiple edges are mapped to distinct arcs meeting at their endpoints. In the paper we consider non-degenerate drawings, except for Corollary 2. An **embedding** is a non-degenerate drawing with no edge crossings. A graph given by an embedding in the plane is a **plane graph**. If it leads to no confusion, we do not distinguish between a vertex or an edge and its representation in the drawing and we use the words "vertex" and "edge" in both contexts.

The following lemma is well known.

▶ **Lemma 3.** *Let G be a plane graph with n vertices such that G does not contain a pair of multiple edges joining the same pair of vertices that form a face of size two, i.e., a lens, except for the outer face. The graph G has $O(n)$ edges.*

The **rotation** at a vertex in an embedding of G is the counterclockwise cyclic order of the edges, that are incident to the vertex, which is defined by the order of their end pieces at the vertex in the embedding. The rotation at a vertex is stored as a doubly linked list of edges. Furthermore, we assume that for every edge of G we store a pointer to its preceding and succeeding edge in the rotation at both of its end vertices. The **interior** and **exterior** of a cycle in an embedded graph is the bounded and unbounded, respectively, connected component of its complement in the plane. Similarly, the **interior** of an inner face and outer face in an embedded connected graph is the bounded and unbounded, respectively, connected component of the complement of its facial walk in the plane bounded by the walk. An embedding of a connected graph G is up to an isotopy described by the rotations at its vertices and the choice of its outer (unbounded) face. If G is not connected the isotopy class of its embedding is described by isotopy classes of its connected components $G_1, \ldots, G_l$ and the containment relation $G_i \subset f$, for every G_i, where f is a face of G_j, $j \neq i$, such that G_i is embedded in the interior of f.

3 Proof of Theorem 1

Let (G, H, γ) be the input of our algorithm. We naturally extend γ to edges: $\gamma(vu) = \rho = \nu\mu$, for $v \in V_\nu$ and $u \in V_\mu$, and to subgraphs G_1 of G: $\gamma(G_1) = H_1 = (V(H_1), E(H_1))$ such that

$V(H_1) = \{\nu \in V(H) | \gamma(v) = \nu,\ v \in V(G_1)\}$ and $E(H_1) = \{\rho \in E(H) |\ \gamma(e) = \rho, e \in E(G_1)\}$.

A vertex $\nu \in V(H)$ of degree two is **redundant** if $V_\nu \subseteq V(G)$ is an independent set consisting of vertices of degree two such that for every $v \in V_\nu$ we have $\gamma(vu) \neq \gamma(vw)$, where u and w are the two neighbors of v. We assume that every edge of H is used by at least one edge of G, i.e., for every $\rho \in E(H)$ there exists $e \in E(G)$ such that $\gamma(e) = \rho$.

3.1 The normal form

Similarly as in [15], the input (G, H, γ) is in the **normal form** if

1. every cluster $V_\nu \subseteq V(G)$, for $\nu \in V(H)$, is an independent set without isolated vertices; and
2. H does not contain a pair of redundant vertices joined by an edge.

We remark that (2) is required only due to the running time analysis. We do not forbid redundant vertices completely, since we do not allow H to contain multiple edges. Indeed, suppressing all vertices of degree two in a graph can lead to multiple edges. In what follows we show how to either detect that no H-compatible embedding in the given isotopy class of G exists just by considering the subgraph of G induced by a single cluster V_ν, or construct an input (G^N, H^N, γ^N) in the normal form, which is positive if and only if the input (G, H, γ) is positive. Clearly, (2) can be assumed without loss of generality. Before establishing the other condition we introduce a couple of definitions.

A **contraction** of an edge $e = uv$ in an embedding of a graph is an operation that turns e into a vertex by moving v along e towards u while dragging all the other edges incident to v along e. By a contraction we can introduce multiple edges or loops at the vertices. We will also use the following operation which can be thought of as the inverse operation of the edge contraction in an embedding of a graph. A **vertex split**, see Figure 2 Left, in an embedding of a graph G is an operation that replaces a vertex v by two vertices u and w joined by a crossing-free edge so that the neighbors of v are partitioned into two parts according to whether they are joined with u or w in the resulting drawing. The rotations at u and w are inherited from the rotation at v. When applied to G, the operations are meant to return a graph given by an isotopy class of its embedding; the same applies to vertex multisplit defined later. Note that a contraction can be carried out in $O(1)$ time, since it amounts to merging a pair of doubly linked lists, and redirecting at most four pointers. The same applies to the vertex split.

In order to satisfy (1), by a series of successive edge contractions we contract each connected component of $G[V_\nu]$, for all $\nu \in V(H)$, to a vertex. Since rotations are stored as doubly linked lists, contracting all such connected components can be carried out in linear time. We delete any created loop and isolated vertices. If a loop at a vertex from V_ν contains a vertex from a different cluster V_μ, $\nu \neq \mu$, in its interior we know that the input is negative, since for every μ all the vertices in V_μ must be contained in the outer face of $G[V_\nu]$ if the input is positive. All this can be easily checked in time linear in $|V(G)|$ by the breadth-first or depth-first search algorithm. If a loop at a vertex from V_ν does not contain a vertex from a different cluster, deleting the loop preserves the existence of an H-compatible embedding in the given isotopy class of G. Indeed, isolated vertices and deleted empty loops can be reintroduced in an H-compatible embedding of the resulting graph, and contracted edges recovered via vertex splits. Let (G^N, H^N, γ^N) denote the resulting input in the normal form. We proved the following.

▶ **Lemma 4.** *If a loop at a vertex v of G obtained during the previously described procedure contains in its interior a vertex u of G satisfying $\gamma(v) \neq \gamma(u)$, then the input (G, H, γ) is negative. Otherwise, the input (G, H, γ) is positive if and only if (G^N, H^N, γ^N) is positive.*

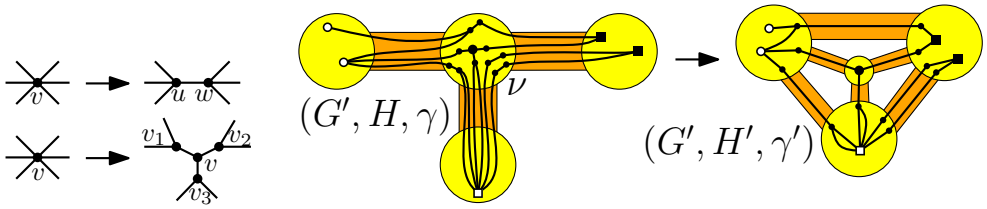

Figure 2 Left: Operation of vertex split (top) and multisplit (bottom). Right: The derivative of (G, H, γ) in the normal form. On the left the input after splitting vertices, and on the right the obtained derivative; in the example we have $H' = H'_\nu$, since every other H_μ, for $\nu \neq \mu$, is a trivial graph with one vertex.

3.2 Derivative

We present the operation of the derivative that simplifies the input, and whose iterating results in an input that is easy to deal with. Such inputs are treated in Section 3.3. Before we describe the derivative we give a couple of definitions.

A **vertex multisplit**, see Figure 2 Left, in an embedding of a graph G is an operation producing an embedding of a graph obtained from G by replacing a vertex v and its adjacent edges with a star $(\{v, v_1, \ldots, v_l\}, \{vv_1, \ldots, vv_l\})$, where $l \leq deg(v)$, so that the resulting underlying graph has vertex set $V(G) \cup \{v_1, \ldots, v_l\}$ and edge set $(E(G) \setminus \{vu_1, \ldots, vu_{deg(v)}\}) \cup \{v_{i_j}u_j |\ j = 1, \ldots, deg(v)\} \cup \{vv_1, \ldots, vv_l\}$, where $u_1, \ldots, u_{deg(v)}$ are neighbors of v in G and $1 \leq i_j \leq l$, for all j. The rotations at $v_1, \ldots, v_l$ are inherited from the rotation at v so that by contracting all the edges of $St(v)$ in the resulting graph we obtain the original embedding of G. Note that a vertex multisplit can be carried out in $O(deg(v))$ time.

The rotation of $\nu \in V(H)$ is **consistent** with the rotation of $v \in V_\nu$ if the rotation given by $(\gamma(vv_1), \ldots, \gamma(vv_{deg(v)}))$, where $(vv_1, \ldots, vv_{deg(v)})$ is the rotation at v in an embedding of G in the given isotopy class, is the rotation at $\nu \in V(H)$ in the embedding of H.

The **derivative** of (G, H, γ) is the input (G', H', γ') obtained as follows, see Figure 2 Right.

First, we construct the graph G' from G by applying the following procedure to every vertex $v \in V(G)$ such that the star $\gamma(St(v))$ has at least two edges, and thus, v is not a "spur". In fact, we construct an auxiliary input (G', H, γ), where by slightly abusing the notation we will extend γ to take values on the vertices of G'. (In the second step we use (G', H, γ) to construct (G', H', γ').) The input (G, H, γ) is clearly negative, if there exists a vertex v in G with four incident edges $vv_1, \ldots, vv_4 \in E(G)$ such that vv_1, vv_2, vv_3 and vv_4 appear in the rotation at v in the given order and $\gamma(vv_1) = \gamma(vv_3) \neq \gamma(vv_2), \gamma(vv_4)$. Otherwise, the following operations of vertex split and multisplit are applicable to G. Let the **valency** of a vertex $v \in V(G)$ be $val(v) := |E(\gamma(St(v)))|$. Thus, the valency count the size of the set of edges of H that the edges incident to v are mapped to.

If $val(v) = 2$, we apply the operation of vertex split to v thereby turning it into an edge uw as follows. Let $E(\gamma(St(v))) = \{\rho_1, \rho_2\}$. Let $v_1, \ldots, v_{deg(v)}$ be the neighbors of v. Let $\{v_1 \ldots v_l\} \cup \{v_{l+1} \ldots v_{deg(v)}\}$ be the partition of the neighbors of v such that $\gamma(vv_1) = \ldots = \gamma(vv_l) = \rho_1$ and $\gamma(vv_{l+1}) = \ldots = \gamma(vv_{deg(v)}) = \rho_2$. We put $\gamma(u), \gamma(w) := \gamma(v)$, and join u by an edge with the vertices in $\{v_1 \ldots v_l\}$ and w with the vertices in $\{v_{l+1} \ldots v_{deg(v)}\}$.

If $val(v) \geq 3$, we analogously apply the operation of vertex multisplit to v so that we replace v with a star $(\{v, v_1, \ldots, v_l\}, \{vv_1, \ldots, vv_l\})$ with $l := val(v)$ edges, in which the set of incident edges of every leaf vertex v_i is $\{vv_i\} \cup \{v_iu |\ vu \in \gamma^{-1}[\rho_i]\}$, where $E(\gamma(St(v))) = \{\rho_1, \ldots, \rho_l\}$, and for every such leaf $\gamma(v_i) := \gamma(v)$.

Let $V_{\geq 3} \subset V(G)$ denote the set of vertices in G consisting of the vertices $v \in V(G)$ with $val(v) \geq 3$. Note that $V_{\geq 3}$ can be treated also as a subset of $V(G')$. Let $E_2 \subset E(G')$ denote the set of edges in G' consisting of every edge uw obtained by splitting $v \in V(G)$ such that $val(v) = 2$. Let $\mathcal{C}$ denote the set of connected components of $G' \setminus E_2 \setminus V_{\geq 3}$. Note that every connected component of $\mathcal{C}$ is mapped to an edge of H by γ.

Second, we construct H': $V(H') := \{\rho^* | \rho \in E(H)\} \cup \{\nu_v |\ v \in V_{\geq 3}\}$, and $E(H') := \{\nu_v \rho^* |\ \rho \in E(\gamma(St(v)))\} \cup \{\gamma(C)^* \gamma(D)^* |\ C, D \in \mathcal{C}$ *s.t.* there exists $e \in E_2$ joining C with $D\}$. We put $\gamma'(v) := \gamma(C)^*$, for $v \in V(C)$ where $C \in \mathcal{C}$; and $\gamma(v) := \nu_v$, for $v \in V_{\geq 3}$.

Finally, the embedding of H', if it exists, is constructed as follows. For $\nu \in V(H)$, let C_ν be the cycle with the vertex set $\{\rho^* |\ \rho = \nu\mu \in E(H)\}$ that captures the rotation at ν, i.e., a pair of vertices ρ_0^* and ρ_1^* is joined by an edge in C_ν if ρ_0 and ρ_1 are consecutive in the rotation at ν. Let H'_ν, for $\nu \in V(H)$, denote the subgraph of H' induced by $\{\rho^* |\ \rho = \nu\mu \in E(H)\} \cup \{\nu_v | \gamma(v) = \nu\}$. Let $\hat{H}'_\nu$ be obtained from H'_ν by adding to H'_ν (1) the missing edges of the cycle C_ν; and (2) a new vertex joined by the edges exactly with all the vertices of C_ν. Note that $\hat{H}'_\nu$ is vertex three-connected, and hence, if $\hat{H}'_\nu$ is planar, then the rotations at vertices in its embedding are determined up to the choice of orientation.

Suppose that every $\hat{H}'_\nu$, for $\nu \in V(H)$, is a planar graph. Let us fix for every $\nu \in V(H)$ an embedding of H'_ν, in which the cycle C_ν bounds the outer face and its orientation corresponds to the rotation of ν. Such an embedding is obtained as a restriction of an embedding of $\hat{H}'_\nu$. Note that for every ν the graph H'_ν does not have multiple edges. Since H also does not have multiple edges, H'_ν and H'_μ, for $\nu \neq \mu$, are either disjoint (if $\nu\mu \notin E(H)$) or intersect in a single vertex $(\nu\mu)^*$ (if $\nu\mu \in E(H)$). It follows that H' does not have multiple edges. The desired embedding of H' is obtained by combining embeddings of H'_ν, for $\nu \in V(H)$, in the same isotopy class as the embeddings of H'_ν, that we fixed above, by identifying the corresponding vertices so that the restriction of the obtained embedding of H' to every H'_ν has the rest of H' in the interior of the outer face (of this restriction).

Note that the construction of (G', H', γ') can be carried out in $O\left(\sum_{v \in V(G')} deg(v)\right) = O(|V(G)|)$ thanks to the doubly-linked lists that we use to store the rotations of the vertices of G and H.

▶ **Lemma 5.** *The input (G, H, γ) is negative if one of the following three conditions is satisfied. (1) There exists a vertex v in G with four incident edges $vv_1, \ldots, vv_4 \in E(G)$ such that vv_1, vv_2, vv_3 and vv_4 appear in the rotation at v in the given order and $\gamma(vv_1) = \gamma(vv_3) \neq \gamma(vv_2), \gamma(vv_4)$. (2) The graph $\hat{H}'_\nu$, for some $\nu \in V(H)$, is not planar. (3) The rotation of a vertex $\nu_v \in V(H'_\nu)$, for some $\nu \in V(H)$ and $v \in V(G')$, in the obtained embedding of H'_ν is not consistent with the rotation of v in G'.*

Otherwise, the input (G, H, γ) is positive if and only if the input (G', H', γ') is positive.

3.3 Locally injective inputs

Let the **potential** $p(G, H, \gamma) = |E(G)| - |E(H)|$. Obviously, $p(G, H, \gamma) \geq 0$, and $p(G, H, \gamma) = 0$ if G is isomorphic to H. The input in the normal form (G, H, γ) is **locally injective** if

- **(i)** the restriction of γ to $V(St(v))$ is injective, for all $v \in V(G)$; and
- **(ii)** for every degree one vertex v in G its unique incident edge e satisfies the following. If $\gamma(e) = \gamma(f)$ then $e = f$ for all $f \in E(G)$.

Given an input (G, H, γ), the vertex $v \in V(G)$ is **fixed** if the condition of property (i) holds for v, and v is alone in its cluster, i.e., $\gamma(u) = \gamma(v)$ implies $u = v$. If v is fixed then

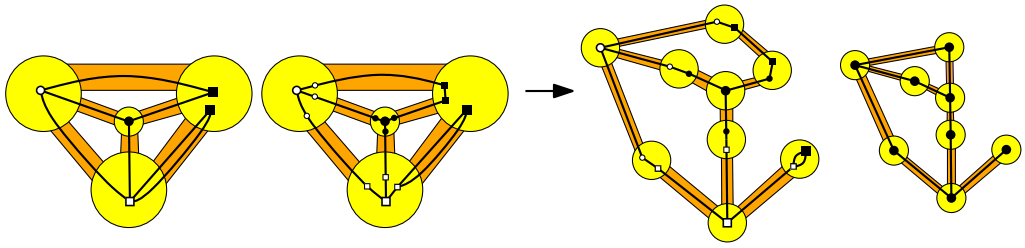

Figure 3 Constructing the normal form and derivating one more time the derivative from Figure 2 on the left, we obtain an input that is strongly locally injective in the normal form on the right.

we call $\gamma(v) = \nu \in V(H)$ also **fixed**. Note that the edges incident to fixed vertices do not contribute to the potential.

For a non-locally injective (G, H, γ) in the normal form, by Lemma 5 we either easily detect that there does not exist an H-compatible embedding of G in the given isotopy class, or we construct the input (G', H', γ') having a smaller potential after being brought to the normal form, such that (G', H', γ') is positive if and only if (G, H, γ) is positive. The following lemma implies that by iterating the derivative at most $|E(G)| = O(|V(G)|)$ many times we obtain an input that is locally injective.

▶ **Lemma 6.** *If (G, H, γ) is in the normal form then $p((G')^N, (H')^N, (\gamma')^N) \leq p(G, H, \gamma)$. If additionally (G, H, γ) is not locally injective then the inequality is strict.*

Given an input (G, H, γ) in the normal form. As in Section 3.2, let $V_{\geq 3} \subseteq V(G)$ denote the set of vertices in G consisting of the vertices $v \in V(G)$ with $val(v) \geq 3$. The input is **strongly locally injective** if it is locally injective and

(iii) every vertex in $V_{\geq 3}$ is fixed.

For convenience, we would like to work with strongly locally injective inputs, see Figure 3. The following lemma shows that if the input (G, H, γ) is locally injective, but not strongly, we just derivate it one more time in order to arrive at a strongly locally injective input.

▶ **Lemma 7.** *Suppose that (G, H, γ) in the normal form is locally injective. Then in $((G')^N, (H')^N, (\gamma')^N)$, every vertex $v \in V((G')^N)$, such that $val(v) \geq 3$, is fixed. Moreover, $((G')^N, (H')^N, (\gamma')^N)$ is still locally injective.*

Proof. The lemma follows directly from the definition of the derivative. ◀

Deciding in, roughly, quadratic time in $p(G, H, \gamma)$, which is sufficient for us, whether the strongly locally injective input (G, H, γ) is positive, is quite straightforward. The reason is that in this case the order of crossings of a valve with edges, that are incident to the same vertex v of G, along the valve in an H-compatible embedding of G is determined by the rotation at v. In order to decide if a desired H-compatible embedding of G exists, we just detect if for every valve ω such an order of all the edges crossing ω exists, such that together the orders are compatible. To this end we consider relations between unordered pairs of edges of G such that the edges in a pair are mapped by γ to the same edge of H, and two pairs are related if they intersect in a pair of vertices. In the following we assume that (G, H, γ) is strongly locally injective.

Let $\Xi = \{\{e, f\} |\ e, f \in E(G)\ s.t.\ e \neq f \text{ and } \gamma(e) = \gamma(f)\}$. Two elements $\{e_1, f_1\} \in \Xi$ and $\{e_2, f_2\} \in \Xi$ are **neighboring** if $|e_1 \cap e_2| = 1$, $|f_1 \cap f_2| = 1$ and $\gamma(e_1 \cap e_2) = \gamma(f_1 \cap f_2)$; we write $\{e_1, f_1\} \sim \{e_2, f_2\}$. An element $\{e_1, f_1\} \in \Xi$ is a **boundary pair** if there exists

at most one $\{e_2, f_2\} \in \Xi$ such that $\{e_1, f_1\}$ and $\{e_2, f_2\}$ are neighboring. Let $\Xi_1, \dots \Xi_l$ be equivalence classes of the transitive closure of the relation $\sim$. A boundary pair $\{e_1, f_1\} \in \Xi$ is **determined** if there exists a pair of edges e_2 and f_2 such that $|e_1 \cap e_2| = 1$, $|f_1 \cap f_2| = 1$, $\gamma(e_1 \cap e_2) = \gamma(f_1 \cap f_2)$ and $\gamma(e_2) \neq \gamma(f_2)$. By properties (i) and (iii) of strong local injectivity, the subgraph G_Ξ of G induced by $\bigcup_{\{e,f\}\in\Xi}\{e, f\}$ has maximum degree two. First, we consider the case when a connected component of G_Ξ does not contain a vertex of degree one.

▶ **Lemma 8.** *If there exists an equivalence class Ξ_c, such that the subgraph G_{Ξ_c} of G induced by $\bigcup_{\{e,f\}\in\Xi_c}\{e, f\}$ is a cycle, then (G, H, γ) is a negative input.*

Note that Lemma 8 does not cover the case when G_{Ξ_c} is a union of two cycles. By (ii), it must be that if Ξ_c contains a boundary pair, then it, in fact, contains exactly two boundary pairs, both of which are determined. Hence, in the following we assume that every Ξ_c either gives rise to a pair of cycles, or contains exactly two determined boundary pairs. We construct for every valve ω of $\rho \in E(H)$ the relation $(E_\rho, <_\omega)$, where $E_\rho = \{e \in E(H) |\ \gamma(e) = \rho\}$. We define relations $(E_\rho, <_\omega)$ by propagating relations enforced by the determined boundary pairs, for every determined pair contained in Ξ. We assume that $(E_\rho, <_\omega)$ encodes the increasing order of the crossing points of edges with ω as encountered when traversing $\omega \subset N_\varepsilon(\nu)$ in the direction inherited from the counterclockwise orientation of the boundary of $N_\varepsilon(\nu)$.

Let $\{e_1, f_1\} \in \Xi_c \subseteq \Xi$ be determined. Let $\Xi_c = \{\{e_1, f_1\}, \dots \{e_m, f_m\}\}$ such that $\{e_p, f_p\} \sim \{e_{p+1}, f_{p+1}\}$. Let $\gamma(e_1) = \gamma(f_1) = \nu\mu, \gamma(e_0) = \nu\mu', \gamma(f_0) = \nu\mu''$, where $\mu' \neq \mu''$ and $|e_0 \cap e_1| = 1$ and $|f_0 \cap f_1| = 1$. W.l.o.g. we suppose that $\nu\mu, \nu\mu'$ and $\nu\mu''$ appear in the rotation of ν in this order counterclockwise. Let ω_1 be the valve of $\nu\mu$ at ν. Let ω_2 be the valve of $\nu\mu$ at μ. We put the relation $f_1 <_{\omega_1} e_1$ into $(E_{\nu\mu}, <_{\omega_1})$ and $e_1 <_{\omega_2} f_1$ into $(E_{\nu\mu}, <_{\omega_2})$. Recursively, we put $f_{p+1} <_{\omega_{2p+1}} e_{p+1}$ into $(E_{\nu\mu}, <_{\omega_{2p+1}})$ and $e_{p+1} <_{\omega_{2(p+1)}} f_{p+1}$ into $(E_{\nu\mu}, <_{\omega_{2(p+1)}})$, if $f_p <_{\omega_{2p-1}} e_p$ and $e_p <_{\omega_{2p}} f_p$, and vice-versa, where ω_{2p} and ω_{2p+1} are valves contained in the boundary of the same disc.

If G_{Ξ_c} is a union of two disjoint cycles we add $f_p <_{\omega_{2p}} e_p$ and $e_p <_{\omega_{2p-1}} f_p$, or $f_p >_{\omega_{2p}} e_p$ and $e_p >_{\omega_{2p-1}} f_p$ for every p, in correspondence with the isotopy class of G.

▶ **Lemma 9.** *Suppose that every equivalence class Ξ_c contains exactly two determined boundary pairs, or G_{Ξ_c} is a union of two disjoint cycles. We can test in $O((p(G, H, \gamma))^2 + |V(G)|)$ time if (G, H, γ) is positive or negative.*

3.4 Algorithm

We give a description of the decision algorithm proving the first part of the theorem. The running time analysis using Lemmas 6, 7 and 9, and the constructive algorithm is omitted in this extended abstract.

Decision Algorithm. Let $(G, H, \gamma) = (G_0, H_0, \gamma_0)$ be the input. We work with inputs in which G contains multiple edges and loops. However, w.l.o.g. we assume that G does not contain a pair of multiple edges joining the same pair of vertices that form a face of size two, i.e., a lens, except for the outer face. Moreover, we assume that whenever a lens is created during the execution of the algorithm, the lens is eliminated by deleting one of its edges.

An execution of the algorithm is divided into steps. During the s-th step we process (G_s, H_s, γ_s) and output $(G_{s+1}, H_{s+1}, \gamma_{s+1})$ as follows.

First, by following the procedure described in Section 3.1 we either construct an instance $((G_s)^N, (H_s)^N, (\gamma_s)^N)$ in the normal form that is positive if and only if (G_s, H_s, γ_s) is positive, or output that (G, H, γ) is negative, if the hypothesis of the first part of Lemma 4 is satisfied.

Second, if $((G_s)^N, (H_s)^N, (\gamma_s)^N)$ is not strongly locally injective we proceed as follows. If (G_s, H_s, γ_s) satisfies the hypothesis of the first part of Lemma 5 with $((G_s)^N, (H_s)^N, (\gamma_s)^N)$ playing the role of (G, H, γ) we output that (G, H, γ) is negative; otherwise we construct the derivative $((G_s^N)', (H_s^N)', (\gamma_s^N)') = (G_{s+1}, H_{s+1}, \gamma_{s+1})$ defined in Section 3.2 and proceed to the $(s+1)$-st step. Otherwise, $((G_s)^N, (H_s)^N, (\gamma_s)^N)$ is strongly locally injective and we construct equivalence classes $\Xi_1, \dots \Xi_l$ from Section 3.3 defined by $((G_s)^N, (H_s)^N, (\gamma_s)^N)$ and proceed as follows.

We check if there exists a class Ξ_c satisfying the hypothesis of Lemma 8. If this is the case, then we output that (G, H, γ) is negative. Otherwise, we construct relations $(E_\rho, <_\omega)$, for every $\rho \in (H_s)^N$ and its valve ω. If there exists $(E_\rho, <_\omega)$ that is not a total order we output that (G, H, γ) is negative; otherwise we check if the isotopy class of an H-compatible embedding of G_s enforced by relations $(E_\rho, <_\omega)$ is the same as the given one and output that (G, H, γ) is positive if and only if this is the case.

The correctness of the algorithm follows directly from Lemma 4,5,8, and 9.

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