Envy-free Matchings with Lower Quotas*†

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Abstract

While every instance of the Hospitals/Residents problem admits a stable matching, the problem with lower quotas (HR-LQ) has instances with no stable matching. For such an instance, we expect the existence of an envy-free matching, which is a relaxation of a stable matching preserving a kind of fairness property.

In this paper, we investigate the existence of an envy-free matching in several settings, in which hospitals have lower quotas. We first provide an algorithm that decides whether a given HR-LQ instance has an envy-free matching or not. Then, we consider envy-freeness in the Classified Stable Matching model due to Huang (2010), i.e., each hospital has lower and upper quotas on subsets of doctors. We show that, for this model, deciding the existence of an envy-free matching is NP-hard in general, but solvable in polynomial time if quotas are paramodular.

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1 Introduction

Since the seminal work of Gale and Shapley [11], the *Hospitals/Residents problem* (HR, for short), or the *College Admission problem*, has been studied extensively [14, 20, 27]. They proposed an algorithm that finds a stable matching in linear time for every instance. In this problem, each hospital has an upper quota for the number of doctors assigned to it. In some applications, each hospital also has a lower quota for the number of doctors it receives. That is, we want to consider the Hospitals/Residents problem with lower quotas (HR-LQ, for short). Unfortunately, for HR-LQ, we cannot ensure the existence of a stable matching. However, it is easy to decide whether there is a stable matching or not for a given HR-LQ instance, because the number of doctors assigned to each hospital is identical for any stable matching (according to the well-known Rural Hospitals Theorem [12, 24, 25, 26]).

When a given HR-LQ instance has no stable matching, one natural approach is to weaken stability concept while preserving some kind of fairness. *Envy-freeness* [30] (also called *fairness* in the school choice literature [8, 13]) of matchings is a relaxation of stability obtained by giving up efficiency. Similarly to stability, envy-freeness forbids the existence of a doctor who has justified envy toward some other doctor, but it tolerates the existence of a doctor who claims a hospital's vacant seat. The importance of envy-freeness and its variants has recently been recognized in the context of constrained matching [8, 13, 18, 19, 4], and structural properties of envy-free matchings were investigated in [30].

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Envy-free matchings naturally arise when we find a matching in the following ad hoc manner. For an HR-LQ instance, suppose that we find a stable matching while disregarding the lower quotas, and that the obtained matching does not meet the lower quotas. Let us reduce the upper quotas of hospitals that receive many doctors, and again find a stable matching while disregarding the lower quotas, and repeat. If we find a stable matching that meets the lower quotas after repeating such adjustments, then the obtained matching is an envy-free matching of the original instance (see Proposition 4).

Because an envy-free matching is a relaxation of a stable matching, it is more likely to exist. Indeed, if all doctor-hospital pairs are acceptable and the sum of lower quotas of all hospitals does not exceed the number of doctors, then we can ensure the existence of an envy-free matching. (This follows from the results of Fragiadakis et al. [8]). However, if not all pairs are acceptable, then even an envy-free matching may fail to exist. Moreover, deciding the existence of an envy-free matching is not so simple because envy-free matchings have different sizes unlike stable matchings.

Our Contribution

In this paper, we study envy-free matchings for the HR-LQ model and its generalizations. In our models, not all doctor-hospital pairs are acceptable (i.e., preference lists are incomplete).

We first investigate envy-free matchings in the setting of HR-LQ. We provide the following characterization of the existence of an envy-free matching. Let I be a given HR-LQ instance and let I' be an HR instance obtained from I by removing lower quotas and replacing upper quotas with the original lower quotas. We prove that I has an envy-free matching if and only if every hospital is full in a stable matching of I' (Theorem 6). Combined with the rural hospitals theorem, this characterization yields an efficient algorithm to decide the existence of an envy-free matching for an HR-LQ instance. That is, we can decide it by finding a stable matching for the HR instance whose upper quotas are the original lower quotas, and checking whether all hospitals are full or not.

Next, we move to a generalized model, in which each hospital imposes an upper and a lower quota on each subset of doctors. That is, we consider an envy-free matching version of Huang's *Classified Stable Matching* [17] (CSM, for short). (See "Related Works" below for results on stable matchings of CSM and its generalizations.) In Huang's original model, each hospital has a family of sets of doctors, called *classes*, and each class has an upper and a lower quota. We formulate this setting by letting each hospital have a pair of set functions defined on the set of acceptable doctors. These two functions respectively represent upper quotas and lower quotas. For this model, we show that it is NP-hard to decide the existence of an envy-free matching, even if the number of non-trivial quotas is linear (Theorem 6). The proof is by a reduction from the NP-complete problem (3,B2)-SAT [2].

Then, we provide a tractable special case of CSM. We show that if the pair of lower and upper quota functions of each hospital is *paramodular* [9] (see Section 4 for the definition), then we can decide the existence of an envy-free matching in polynomial time. This means that the problem is tractable if the family of acceptable doctor sets forms a generalized matroid for each hospital. A *generalized matroid* [28] (also called an $M^\natural$*-convex family* [22]) is a family of subsets satisfying a certain axiom called the exchange axiom. It is known that a paramodular function pair defines a generalized matroid and vice versa. Because constraints defined on a laminar (or hierarchical) family yield a generalized matroid, our tractable special case includes a case in which each hospital defines quotas on a laminar family of doctors.

Related Works

Recently, the study of matching models with lower quotas has developed substantially [1, 7, 13, 15, 16, 17, 20, 21]. The Hospitals/Residents problem with lower quotas (HR-LQ) was first studied by Hamada et al. [15, 16], who considered the minimization of the number of blocking pairs subject to upper and lower quotas. They showed the NP-hardness of the problem, gave an inapproximability result, and provided an exponential-time exact algorithm. Motivated by the matching scheme used in Hungary's higher education sector, Biró et al. [3] considered a version of HR-LQ in which hospitals (i.e., colleges) are allowed to be closed, i.e., each hospital is assigned enough doctors or no doctor. They showed the NP-completeness to decide the existence of a stable matching.

The Classified Stable Matching problem (CSM), proposed by Huang [17], is a generalization of HR-LQ without hospital closures. In this model, each hospital (or institute, in Huang's terminology) has a classification of doctors (i.e., applicants) based on their expertise and gives an upper and lower quota for each class. Huang showed that it is NP-complete in general to decide the existence of a stable matching, and proved that it is solvable in polynomial time if classes form a laminar family. For this tractable special case, Fleiner and Kamiyama [7] gave a concise explanation in terms of matroids, and their framework is generalized by Yokoi [31] to a framework with generalized matroids.

To cope with the nonexistence of a stable matching in constrained matching models (not only models with lower quotas but also with other types of constraints such as regional constraints), several relaxations of stability have been proposed. See, e.g., Kamada and Kojima [18, 19], Fragiadakis et al. [8], and Goto et al. [13]. Envy-freeness is one of them that places emphasis on fairness rather than efficiency. Fragiadakis et al. [8] provided a strategy-proof algorithm that always finds an envy-free matching (or fair matching, in their terminology) of HR-LQ under the assumption that all doctor-hospital pairs are acceptable. The outcome of their mechanism also fulfills a second-best efficiency (i.e., nonwastefulness) property. Their framework is generalized in Goto et al. [13] so that regional quotas can be handled.

Here we compare our models with the above models. Unlike the models of Goto et al. [13] and Kamada and Kojima [18, 19], our models cannot handle regional quotas. Instead, our CSM model (in Sections 3 and 4) allows each hospital to have quotas on classes of doctors, which are not dealt with in their models. The setting of a tractable special case of CSM described in Section 4 is equivalent to a many-to-one case of Yokoi's model [31], which studied stable matchings. Neither [31] nor the study in this paper relies on the results of the other, while both of them utilize the matroid framework of Fleiner [5, 6].

The remainder of this paper is organized as follows. Section 2 investigates envy-free matchings in the Hospitals/Residents problem with lower quotas (HR-LQ). In Section 3, we define an envy-free matching in the classified stable matching (CSM) model, and show the NP-hardness of its existence test. As its tractable special case, Section 4 presents results on CSM with paramodular quota functions. Due to space constraints, we defer the proofs for the theorems and corollary in Section 4 to the full version.

2 Envy-freeness in HR with lower quotas

In this section, we investigate envy-free matchings in the Hospitals/Residents problem with lower quotas (HR-LQ).

There are two disjoint sets D and H, which represent doctors and hospitals, respectively. A set of acceptable doctor-hospital pairs is denoted by $E \subseteq D \times H$.

For each doctor $d \in D$, its acceptable hospital set is denoted by

$$A(d) := \{\, h \in H \mid (d,h) \in E \,\} \subseteq H,$$

and d has a preference list (strict order) $\succ_d$ on $A(d)$. Similarly, for each hospital $h \in H$,

$$A(h) := \{\, d \in D \mid (d,h) \in E \,\} \subseteq D,$$

and h has a preference $\succ_h$ on $A(h)$. Each hospital h has a lower quota $l_h \in \mathbf{Z}$ and an upper quota $u_h \in \mathbf{Z}$ with $0 \le l_h \le u_h \le |A(h)|$.

We call a tuple $I = (D, H, E, \succ_{DH}, \{(l_h, u_h)\}_{h \in H})$ an **HR-LQ instance**, where $\succ_{DH}$ is an abbreviated notation for the union of $\{\succ_d\}_{d \in D}$ and $\{\succ_h\}_{h \in H}$. In particular, if $l_h = 0$ for all $h \in H$, we call it an **HR instance**. An arbitrary subset M of E is called an **assignment**. For any assignment M, we denote $M(d) = \{\, h \in A(d) \mid (d,h) \in M \,\}$ for each $d \in D$ and $M(h) = \{\, d \in A(h) \mid (d,h) \in M \,\}$ for each $h \in H$. If $|M(d)| = 1$, the notation $M(d)$ is also used to refer its single element.

An assignment M is called a **matching** (or, said to be **feasible**) if $|M(d)| \le 1$ for each $d \in D$ and $l_h \le |M(h)| \le u_h$ for each $h \in H$. In a matching M, a doctor d is **unassigned** (resp., **assigned**) if $M(d) = \emptyset$ (resp., $|M(d)| = 1$), and h is **undersubscribed** (resp., **full**) if $|M(h)| < u_h$ (resp., $|M(h)| = u_h$).

▶ **Definition 1.** For a matching M, an unassigned pair $(d,h) \in E \setminus M$ is a **blocking pair** if (i) d is unassigned or $h \succ_d M(d)$, and (ii) h is undersubscribed or there is $d' \in M(h)$ with $d \succ_h d'$. A matching M is **stable** if there is no blocking pair.

For an HR instance, it is known that the algorithm of Gale and Shapley [11] always finds a stable matching. The set of stable matchings has the following property.

▶ **Proposition 2** ("Rural Hospitals" Theorem [12, 24, 26])**.** *For a given HR instance, any two stable matchings M, M' satisfy $|M(h)| = |M'(h)|$ for every $h \in H$. Moreover $M(h) = M'(h)$ if h is undersubscribed in M or M'.*

As mentioned in the Introduction, if hospitals have lower quotas, then we cannot guarantee the existence of a stable matching anymore. By Proposition 2, however, we can easily check the existence by finding a stable matching while disregarding lower quotas, and checking whether the obtained matching meets lower quotas.

For an instance that has no stable matching, we want to obtain some matching that still has a kind of fairness. As a relaxation of the concept of stability, envy-freeness (also called fairness) of matchings has been proposed [8, 30].

▶ **Definition 3.** For a matching M, a doctor d has **justified envy** toward d' with $M(d') = h$ if (i) d is unassigned or $h \succ_d M(d)$ and (ii) $d \succ_h d'$. A matching M is **envy-free** if no doctor has justified envy.

Note that, if d has justified envy toward d' with $M(d) = h$, then it means that (d,h) is a blocking pair. Thus, stability implies envy-freeness. The envy-freeness of a matching is also regarded as the stability with reduced upper quotas, as follows.

▶ **Proposition 4.** *For $I = (D, H, E, \succ_{DH}, \{(l_h, u_h)\}_{h \in H})$, an assignment M is an envy-free matching if and only if M is a stable matching of $I' = (D, H, E, \succ_{DH}, \{(l_h, u'_h)\}_{h \in H})$ for some $\{u'_h\}_{h \in H}$ with $u'_h \le u_h$ $(h \in H)$.*

Doctor's preferences	Hospitals' preferences	
$d_1 : h_1$	$h_1 : d_2 \ \ d_1$	$(l_{h_1} = 1,\ u_{h_1} = 2)$
$d_2 : h_1 \ \ h_2$	$h_2 : d_2$	$(l_{h_2} = 1,\ u_{h_2} = 2)$

Figure 1 An instance of HR-LQ with no envy-free matching.

Proof. The "if" part is clear because feasibility in I' implies that in I, and stability implies envy-freeness. For the "only if" part, suppose that M is envy-free in I and set $u'_h := |M(h)|$ for each $h \in H$. Then, M is feasible for I' and all hospitals are full, and hence there is no doctor who claims a hospital's vacant seat. Because M is envy-free, it is stable in I'. ◀

By Proposition 4, to check whether we can obtain a stable matching by reducing upper quotas, it suffices to check for the existence of an envy-free matching.

Under the assumption that all doctor-hospital pairs are acceptable and the sum of lower quotas does not exceed the number of doctors, Fragiadakis et al. [8] provided a strategy-proof mechanism that always finds an envy-free matching. As a corollary, we have the following.

▶ **Proposition 5.** *For an instance $I = (D, H, E, \succ_{DH}, \{(l_h, u_h)\}_{h\in H})$ such that $E = D \times H$ and $|D| \geq \sum_{h\in H} l_h$, there exists an envy-free matching.*

However, if not all pairs are acceptable, then even an envy-free matching may not exist. Figure 1 shows an instance with $D = \{d_1, d_2\}$, $H = \{h_1, h_2\}$, $E = \{(d_1, h_1), (d_2, h_1), (d_2, h_2)\}$, $l_{h_1} = l_{h_2} = 1$, and $u_{h_1} = u_{h_2} = 2$. For this instance, $M = \{(d_1, h_1), (d_2, h_2)\}$ is the unique feasible matching, but it is not envy-free because d_2 has justified envy toward d_1. Hence, there is no envy-free matching.

Note that an envy-free matching does exist if there is no lower quota, because empty matching is clearly envy-free. Therefore, the existence test of an envy-free matching is non-trivial when incomplete lists and lower quotas are introduced simultaneously. Here we provide a characterization.

▶ **Theorem 6.** *$I = (D, H, E, \succ_{DH}, \{(l_h, u_h)\}_{h\in H})$ has an envy-free matching if and only if some stable matching M' of the HR instance $I' = (D, H, E, \succ_{DH}, \{(0, l_h)\}_{h\in H})$ satisfies $|M'(h)| = l_h$ for all $h \in H$.*

Proof. For the "if" part, let M' be a stable matching of I' satisfying $|M'(h)| = l_h$ for all $h \in H$. Then, M' is feasible for I' and no doctor has justified envy because M' has no blocking pair. Thus, M' is an envy-free matching of I.

For the "only if" part, assume that I has an envy-free matching M. Suppose, to the contrary, a stable matching M' of I' satisfies $|M'(h^*)| < l_{h^*}$ for some $h^* \in H$. Let us denote $N = M \setminus M'$ and $N' = M' \setminus M$. For every $h \in H$, because $|M'(h)| \leq l_h \leq |M(h)|$, we have $|N'(h)| \leq |N(h)|$. In particular, $|N'(h^*)| < |N(h^*)|$ follows from $|M'(h^*)| < l_{h^*}$.

Consider a bipartite graph $G = (D, H; N \cup N')$, i.e., a graph between doctors and hospitals with edge set $N \cup N' = M \triangle M'$. Let G^* be a connected component of G including h^*, and denote by D^* and H^* the sets of doctors and hospitals in G^*, respectively. Because there is no edge connecting G^* and the outside, $\sum_{d\in D^*} |N(h)| = \sum_{h\in H^*} |N(h)|$ and $\sum_{d\in D^*} |N'(h)| = \sum_{h\in H^*} |N'(h)|$. As $|N'(h^*)| < |N(h^*)|$ and $|N'(h)| \leq |N(h)|$ for any $h \in H^*$, we obtain

$$\textstyle\sum_{d\in D^*} |N'(h)| = \sum_{h\in H^*} |N'(h)| < \sum_{h\in H^*} |N(h)| = \sum_{d\in D^*} |N(h)|.$$

Then, there exists $d^* \in D^*$ with $|N'(d^*)| < |N(d^*)|$, which implies $N'(d^*) = \emptyset$ and $|N(d^*)| = 1$ because $N' = M' \setminus M$ and $N = M \setminus M'$ are subsets of matchings. As G^* is a connected bipartite graph, there is a path $d_0\, h_0\, d_1\, h_1 \dots d_k\, h_k$ with $d_0 = d^*$ and $h_k = h^*$. Also, as $|N(d_i)| \le 1$ and $|N'(d_i)| \le 1$ for $i = 0, 1, \dots k$, this path alternately uses edges in $N = M \setminus M'$ and $N' = M' \setminus M$. Because $N'(d^*) = \emptyset$ and $|N(d^*)| = 1$, we have

$$\begin{aligned} M'(d_0) &= \emptyset, \\ (d_i, h_i) &\in M \setminus M' \qquad (i = 0, 1, \dots, k), \\ (d_{i+1}, h_i) &\in M' \setminus M \qquad (i = 0, 1, \dots, k-1). \end{aligned}$$

The doctor d_0 is unassigned in M' and finds h_0 acceptable because $(d_0, h_0) \in M$. Hence, the stability of M' implies that h_0 prefers $d_1 \in M'(h_0)$ to d_0. Then, the envy-freeness of M implies that d_1 prefers $h_1 = M(d_1)$ to h_0. In this way, we obtain

$$\begin{aligned} &d_{i+1} \succ_{h_i} d_i \qquad (i = 0, 1, \dots, k-1), \\ &h_{i+1} \succ_{d_{i+1}} h_i \qquad (i = 0, 1, \dots, k-1). \end{aligned}$$

Thus, $M(d_k) = h_k \succ_{d_k} h_{k-1} = M'(d_k)$. Because $h_k = h^*$ satisfies $|M'(h_k)| < l_{h_k}$, then (d_k, h_k) is a blocking pair in I', which contradicts the stability of M'. ◀

Theorem 6 ensures that the following algorithm decides the existence of an envy-free matching of an HR-LQ instance $I = (D, H, E, \succ_{DH}, \{(l_h, u_h)\}_{h \in H})$.

Algorithm EF-HR-LQ
Step1. Find a stable matching M' of $I' = (D, H, E, \succ_{DH}, \{(0, l_h)\}_{h \in H})$.
Step2. return M' if $|M'(h)| = l_h$ for all $h \in H$, and otherwise "there is no envy-free matching."

Since the Gale-Shapley algorithm finds a stable matching of an HR instance in $O(|E|)$ time, we obtain the following theorem.

▶ **Theorem 7.** *For any HR-LQ instance $I = (D, H, E, \succ_{DH}, \{(l_h, u_h)\}_{h \in H})$, the algorithm* EF-HR-LQ *decides whether I has an envy-free matching or not in $O(|E|)$ time.*

3 Envy-freeness in Classified Stable Matching

In this section, we consider the envy-freeness in a model in which each hospital has lower and upper quotas on subsets of doctors. This can be regarded as an envy-free matching version of the Classified Stable Matching, proposed by Huang [17]. Similarly to Section 2, we have doctors D, hospitals H, acceptable pairs $E \subseteq D \times H$, and preferences $\succ_{DH}$.

The only difference from HR-LQ is that, in the current model, each hospital $h \in H$ has a pair of functions $p_h, q_h : 2^{A(h)} \to \mathbf{Z}$, instead of a pair of numbers l_h, u_h. These functions define a lower and an upper quota for each subset of acceptable doctors. Throughout this paper, we assume that for any hospital h, the functions p_h and q_h satisfy

$$0 \le p_h(B) \le q_h(B) \le |B| \qquad (B \subseteq A(h)).$$

We call such a tuple $(D, H, E, \succ_{DH}, \{(p_h, q_h)\}_{h \in H})$ a **CSM instance**. For each $h \in H$, the family of **acceptable** subsets of doctors is denoted by

$$\mathcal{F}(p_h, q_h) := \{\, X \subseteq A(h) \mid \forall B \subseteq A(h) : p_h(B) \le |X \cap B| \le q_h(B) \,\}.$$

For any $h \in H$, we say that $B \subseteq A(h)$ has a **non-trivial lower** (resp., **upper**) **constraint** if $p_h(B) > 0$ (resp., $q_h(B) < |B|$). We denote the family of constrained subsets by

$$\mathcal{C}(p_h, q_h) := \{\, B \subseteq A(h) \mid p_h(B) > 0 \;\text{ or }\; q_h(B) < |B| \,\}.$$

Then, we see that $\mathcal{F}(p_h, q_h)$ is represented as

$$\mathcal{F}(p_h, q_h) = \{\, X \subseteq A(h) \mid \forall B \subseteq \mathcal{C}(p_h, q_h) : p_h(B) \leq |X \cap B| \leq q_h(B) \,\}.$$

For a CSM instance $I = (D, H, E, \succ_{DH}, \{(p_h, q_h)\}_{h \in H})$, $M \subseteq E$ is called a **matching** (or, said to be **feasible**) if $|M(d)| \leq 1$ for each $d \in D$ and $M(h) \in \mathcal{F}(p_h, q_h)$ for each $h \in H$.

▶ **Definition 8.** For a matching M, an unassigned pair $(d, h) \in E \setminus M$ is a **blocking pair** if (i) d is unassigned or $h \succ_d M(d)$, and (ii) $M(h) + d \in \mathcal{F}(p_h, q_h)$ or $M(h) + d - d' \in \mathcal{F}(p_h, q_h)$ for some $d' \in M(h)$ with $d \succ_h d'$. A matching M is **stable** if there is no blocking pair.

In Definition 8, the condition $M(h) + d \in \mathcal{F}(p_h, q_h)$ means that h can add d to the current assignment without violating any upper quota, and $M(h) + d - d' \in \mathcal{F}(p_h, q_h)$ means that h can replace d' with d without violating any upper or lower quota. The Classified Stable Matching, introduced by Huang [17], is the problem to decide the existence of a stable matching for a given CSM instance[1]. Because this is a generalization of HR-LQ, there are instances that have no stable matching. Let us consider envy-freeness for a CSM instance.

▶ **Definition 9.** For a matching M, a doctor d has **justified envy** toward d' with $M(d') = h$ if (i) d is unassigned or $h \succ_d M(d)$ and (ii) $M(h) + d - d' \in \mathcal{F}(p_h, q_h)$ and $d \succ_h d'$. A matching M is **envy-free** if no doctor has justified envy.

As in the case of HR-LQ, an envy-free matching can be regarded as a stable matching with reduced upper quotas as follows. For any $h \in H$ and $k \in \mathbf{Z}$ with $0 \leq k \leq q(A(h))$, a function $q'_h : 2^{A(h)} \to \mathbf{Z}$ is called a k**-truncation** of q_h if $q'(A(h)) = k$ and $q'(B) = q(B)$ for every $B \subsetneq A(h)$. Also, we simply say that q'_h is a **truncation** of q_h if there is such $k \in \mathbf{Z}$.

▶ **Proposition 10.** *For $I = (D, H, E, \succ_{DH}, \{(p_h, q_h)\}_{h \in H})$, an assignment M is an envy-free matching if and only if M is a stable matching of $I' = (D, H, E, \succ_{DH}, \{(p_h, q'_h)\}_{h \in H})$ such that each q'_h is some truncation of q_h.*

Proof. To show the "only if" part, let M be an envy-free matching of I. For each $h \in H$, let q'_h be the $|M(h)|$-truncation of q_h. Then $M(h) \in \mathcal{F}(p_h, q'_h)$ and $M(h) + d \notin \mathcal{F}(p_h, q'_h)$ for every $d \in A(h) \setminus M(h)$. That is, M is feasible for I' and there is no doctor who claims a hospital's vacant seat. Therefore, if there is a blocking pair $(d, h) \in E \setminus M$ for I', it follows that d has a justified envy toward some d' with $M(d') = h$, which contradicts the envy-freeness of M. Thus, M is a stable matching of I'.

For the "if" part, let M be a stable matching of I'. Clearly, M is feasible for I. Suppose, to the contrary, some doctor d has justified envy toward d' with $M(d') = h$ with respect to I. Then d is unassigned or $h \succ_d M(d)$. Also, we have $d \succ_h d'$ and $M(h) + d - d' \in \mathcal{F}(p_h, q_h)$. Then, $M(h) + d - d' \in \mathcal{F}(p_h, q'_h)$ follows because $|M(h) + d - d'| = |M(h)|$. Hence, (d, h) is a blocking pair in I', a contradiction. ◀

We provide a hardness result for deciding the existence of an envy-free matching. Here, we assume that evaluation oracles of set functions p_h and q_h are available for each hospital h.

▶ **Theorem 11.** *It is NP-hard to decide whether a CSM instance $I = (D, H, E, \succ_{DH}, \{(p_h, q_h)\}_{h \in H})$ has an envy-free matching or not. The problem is NP-complete even if the size of $\mathcal{C}(p_h, q_h)$ is at most 4 for each $h \in H$.*

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[1] In his original model, each hospital h has a classification $\mathcal{C}_h \subseteq 2^{A(h)}$ and sets a lower and an upper quota for each member of $\mathcal{C}_h$. That is, we are provided $\mathcal{C}(p_h, q_h)$ and the values of p_h, q_h on it, rather than set functions p_h, q_h. Our formulation uses set functions to simplify the arguments in the next section.

Proof. We use reduction from the NP-complete problem (3, B2)-SAT [2], which is a restriction of SAT such that each clause contains exactly three literals and each variable occurs exactly twice as a positive literal and exactly twice as a negative literal. Let $\varphi = c_1 \wedge c_2 \wedge \cdots \wedge c_m$ be an instance of (3, B2)-SAT with Boolean variables $v_1, v_2, \ldots, v_n$. Then, each clause c_j is a disjunction of three literals, (e.g., $c_j = v_1 \vee \neg v_2 \vee \neg v_3$) and each of literals v_i and $\neg v_i$ appears in exactly two clauses. For each variable v_i, denote by $j^*(i,1)$, $j^*(i,2)$ the indices of two clauses that contain v_i. Similarly, denote by $j^*(i,-1)$, $j^*(i,-2)$ the indices of two clauses that contain $\neg v_i$.

We now define a CSM instance corresponding to φ. We have a variable-hospital h_i for each variable v_i, and a clause-hospital h_j for each clause c_j. For each variable v_i, we have four doctors $\{\, d_{i,t} \mid t \in \{1, 2, -1, -2\} \,\}$. For each doctor $d_{i,t}$, we have

$$A(d_{i,t}) = \{h_i, h_{j^*(i,t)}\}, \quad h_i \succ_{d_{i,t}} h_{j^*(i,t)}.$$

The set E is defined as the set of all pairs $(d_{i,t}, h)$ such that $h \in A(d_{i,t})$. Then, for each variable-hospital h_i and clause-hospital h_j, we have

$$A(h_i) = \{\, d_{i,t} \mid t \in \{1, 2, -1, -2\} \,\},$$
$$A(h_j) = \{\, d_{i,t} \mid j^*(i,t) = j \,\}.$$

Note that $d_{i,t} \in A(h_j)$ implies $v_i \in c_j$ or $\neg v_i \in c_j$. Also, each of $v_i \in c_j$ and $\neg v_i \in c_j$ implies $d_{i,t} \in A(h_j)$ for some unique $t \in \{1, 2, -1, -2\}$. Therefore, $|A(h_j)| = 3$ for each clause-hospital h_j. For each variable-hospital h_i, define p_{h_i} and q_{h_i} so that

$$\mathcal{C}(p_{h_i}, q_{h_i}) = \bigcup \{\, \{d_{i,t}, d_{i,t'}\} \mid t \in \{1,2\},\ t' \in \{-1,-2\} \,\},$$
$$p_{h_i}(\{d_{i,t}, d_{i,t'}\}) = q_{h_i}(\{d_{i,t}, d_{i,t'}\}) = 1 \qquad (t \in \{1,2\},\ t' \in \{-1,-2\}).$$

Then, we see that $\mathcal{F}(p_{h_i}, q_{h_i}) = \{D_i^+, D_i^-\}$, where $D_i^+ := \{d_{i,1}, d_{i,2}\}$ and $D_i^- := \{d_{i,-1}, d_{i,-2}\}$. For each clause-hospital h_j, define p_{h_i} and q_{h_i} so that

$$\mathcal{C}(p_{h_j}, q_{h_j}) = \{A(h_j)\}, \quad p_{h_j}(A(h_j)) = 1, \quad q_{h_j}(A(h_j)) = |A(h_j)| = 3.$$

We define preference lists of hospitals arbitrarily. Note that $|\mathcal{C}(p_h, q_h)| \leq 4$ for every hospital. We show that this CSM instance has an envy-free matching if and only if $\varphi = c_1 \wedge c_2 \wedge \cdots \wedge c_m$ is satisfiable.

The "only if" part: Suppose that there is an envy-free matching M. Then, for every variable-hospital h_i, $M(h_i)$ is D_i^+ or D_i^-. For each h_i, set variable v_i to FALSE if $M(h_i) = D_i^+$, and to TRUE if $M(h_i) = D_i^-$. This Boolean assignment satisfies every clause c_j of φ as follows. Because $M(h_j) \in \mathcal{F}(p_{h_j}, q_{h_j})$, we have $|M(h_j)| \geq 1$. Hence, some $d_{i,t}$ with $j^*(i,t) = j$ is assigned to h_j. Then, $d_{i,t} \notin M(h_i)$. There are two cases: (i) $t \in \{1,2\}$, (ii) $t \in \{-1,-2\}$. In the case (i), $d_{i,t} \notin M(h_i)$ implies $M(h_i) \neq D_i^+$, and hence v_i is set to TRUE. Also, $t \in \{1,2\}$ and $j^*(i,t) = j$ imply $v_i \in c_j$. Hence, clause c_j is satisfied. Similarly, in the case (ii), we see that v_i is set to FALSE and we have $\neg v_j \in c_j$. Hence, clause c_j is satisfied.

The "if" part: Suppose that there is a Boolean assignment satisfying φ. Define an assignment M so that

- $M(h_i) = D_i^-$ if v_i is TRUE, and $M(h_i) = D_i^+$ if v_i is FALSE, and
- $M(h_j) = \{\, d_{i,t} \in A(h_j) \mid d_{i,t} \in D_i^+,\ v_i \text{ is TRUE} \,\} \cup \{\, d_{i,t} \in A(h_j) \mid d_{i,t} \in D_i^-,\ v_i \text{ is FALSE} \,\}$.

We can observe that $|M(d)| = 1$ for every doctor d, and $M(h_i) \in \mathcal{F}(p_{h_i}, q_{h_i})$ for every variable-hospital h_i. Also, because all clauses are satisfied, the above definition implies $M(h_j) \in \mathcal{F}(p_{h_j}, q_{h_j})$ for every clause-hospital h_j. Then, M is feasible. We now show the envy-freeness of M. Suppose, to the contrary, $d_{i,t}$ has justified envy toward d'. Because we have $|M(d_{i,t})| = 1$, $A(d_{i,t}) = \{h_i, h_{j^*(i,t)}\}$, and $h_i \succ_{d_{i,t}} h_{j^*(i,t)}$, this justified envy implies conditions $d' \in M(h_i)$, $d_{i,t} \notin M(h_i)$ and $M(h_i) + d_{i,t} - d' \in \mathcal{F}(p_{h_i}, q_{h_i})$. As $M(h_i) \in \mathcal{F}(p_{h_i}, q_{h_i}) = \{D_i^+, D_i^-\}$, then we have $\{M(h_i) + d_{i,t} - d', M(h_i)\} = \{D_i^+, D_i^-\}$, which contradicts $|D_i^+ \setminus D_i^-| = |D_i^- \setminus D_i^+| = 2$. ◀

4 Envy-freeness in CSM with Paramodular Quotas

In Section 3, we showed that it is NP-hard in general to decide whether a CSM instance has an envy-free matching or not. This section shows that the problem is solvable in polynomial time if the pair of quota functions is paramodular for each hospital. The proofs of the theorems and corollary in this section can be found in the full version. We first introduce the notion of paramodularity [9].

Let A be a finite set and let $p, q : 2^A \to \mathbf{Z}$. The pair (p, q) is **paramodular** (or, called a **strong pair** [10]) if

- p is **supermodular**, i.e., $p(B) + p(B') \le p(B \cup B') + p(B \cap B')$ for every $B, B' \subseteq A$,
- q is **submodular**, i.e., $q(B) + q(B') \ge q(B \cup B') + q(B \cap B')$ for every $B, B' \subseteq A$, and
- the **cross-inequality** $q(B) - p(B') \ge q(B \setminus B') - p(B' \setminus B)$ holds for every $B, B' \subseteq A$.

Here we provide examples of constraints that can be represented by paramodular pairs. (See Yokoi [31, Appendices A and B].)

▶ **Example 12** (Laminar Constraints)**.** Let $\mathcal{L} \subseteq 2^A$ be a laminar (or hierarchical) classification (i.e., any $X, Y \subseteq \mathcal{L}$ satisfy $X \subseteq Y$ or $X \supseteq Y$ or $X \cap Y \neq \emptyset$). Let $\hat{p}, \hat{q} : \mathcal{L} \to \mathbf{Z}$ be functions that define a lower and an upper quota for each class. Denote the acceptable set family by $\mathcal{J}(\mathcal{L}, \hat{p}, \hat{q}) := \{\, B \subseteq A \mid \forall X \in \mathcal{L} : \hat{p}(X) \le |B \cap X| \le \hat{q}(X) \,\}$. If $\mathcal{J}(\mathcal{L}, \hat{p}, \hat{q})$ is nonempty, then $\mathcal{J}(\mathcal{L}, \hat{p}, \hat{q}) = \mathcal{F}(p, q)$ for some paramodular pair (p, q).

▶ **Example 13** (Staffing Constraints)**.** For a finite set S (e.g., a set of sections of a hospital), let $\Gamma : S \to 2^A$ and $\hat{l}, \hat{u} : S \to \mathbf{Z}$ be functions such that $\Gamma(s) \subseteq A$ represents members acceptable to $s \in S$ and $\hat{l}(s), \hat{u}(s) \in \mathbf{Z}$ represent a lower and an upper quota of each $s \in S$. Let $\mathcal{J}(S, \Gamma, \hat{l}, \hat{u}) \subseteq 2^A$ be a family of subsets $B \subseteq A$ such that there exists a function $\pi : B \to S$ satisfying $\forall d \in B : d \in \Gamma(\pi(d))$ and $\forall s \in S : \hat{l}(s) \le |\,\{\, d \in B \mid \pi(d) = s \,\}\,| \le \hat{u}(s)$. If $\mathcal{J}(S, \Gamma, \hat{l}, \hat{u})$ is nonempty, then $\mathcal{J}(S, \Gamma, \hat{l}, \hat{u}) = \mathcal{F}(p, q)$ for some paramodular pair (p, q).

For a set function $p : 2^A \to \mathbf{Z}$, its **complement** $\overline{p} : 2^A \to \mathbf{Z}$ is defined by

$$\overline{p}(B) = p(A) - p(A \setminus B) \quad (B \subseteq A).$$

Recall that a CSM instance is represented as a tuple $(D, H, E, \succ_{DH}, \{(p_h, q_h)\}_{h \in H})$, where it is assumed that $0 \le p_h(B) \le q_h(B) \le |B|$ for every $h \in H$ and $B \subseteq A(h)$. Here is the main theorem of this section. We denote by $\mathbf{0}$ a set function that is identically zero.

▶ **Theorem 14.** *For a CSM instance $I = (D, H, E, \succ_{DH}, \{(p_h, q_h)\}_{h \in H})$, suppose that (p_h, q_h) is paramodular for each $h \in H$. Then, an instance $I' = (D, H, E, \succ_{DH}, \{(\mathbf{0}, \overline{p_h})\}_{h \in H})$ has at least one stable matching and the following three conditions are equivalent.*

(a) *I has an envy-free matching.*
(b) *Some stable matching M' of I' satisfies $|M'(h)| = p_h(A(h))$ for all $h \in H$.*
(c) *Every stable matching M' of I' satisfies $|M'(h)| = p_h(A(h))$ for all $h \in H$.*

Also, if (b) *holds, then M' is an envy-free matching of I.*

Algorithm 1: EF-Paramodular-CSM

Input: $I = (D, H, E, \succ_{DH}, \{(p_h, q_h)\}_{h \in H})$ such that each (p_h, q_h) is paramodular
Output: return an envy-free matching M', or "there is no envy-free matching."

Set $N_D \leftarrow E$, $N_H \leftarrow \emptyset$, and let M' be undefined;
while *M' is undefined* **do**
  $R_D \leftarrow \bigcup_{d \in D} \{ (d,h) \mid h \in N_D(d),\ h \neq \max_{\succ_d} N_D(d) \}$;
  $R_H \leftarrow \bigcup_{h \in H} \{ (d,h) \mid d \in N_H(h),\ p(A(h) \setminus N_H(h)_{\succeq_h d}) = p(A(h) \setminus N_H(h)_{\succ_h d}) \}$;
  if $(N_D, N_H) = (E \setminus R_H, E \setminus R_D)$ **then**
    let $M' \leftarrow N_D \cap N_H$ and **break**;
  else
    update $(N_D, N_H) \leftarrow (E \setminus R_H, E \setminus R_D)$;
  end
end
if $|M'(h)| = p_h(A(h))$ *for all* $h \in H$ **then**
  return M';
else
  return "there is no envy-free matching";
end

Here we sketch the proof of Theorem 14. See the full version for the detailed proof. The existence of a stable matching of I' and the equivalence between (b) and (c) can be shown by using Fleiner's results on the matroid framework [5, 6]. The most difficult part is showing the equivalence between conditions (a) and (b). To show that (a) implies (b), we construct a stable matching M' of I' from an envy-free matching M of I. This construction is achieved by using the fixed-point method of Fleiner [6]. The paramodularity of each (p_h, q_h) (or a generalized matroid structure of each $\mathcal{F}(p_h, q_h)$) is essential to show the existence of a fixed-point satisfying a required condition.

Theorem 14 implies that, when quota function pairs are paramodular, we can decide the existence of an envy-free matching of $I = (D, H, E, \succ_{DH}, \{(p_h, q_h)\}_{h \in H})$ by the following algorithm.

Step1. Find a stable matching M' of $I' = (D, H, E, \succ_{DH}, \{(\mathbf{0}, \overline{p_h})\}_{h \in H})$.

Step2. If $|M'(h)| = p_h(A(h))$ for every $h \in H$, then return M'. Otherwise, return "there is no envy-free matching."

Step 1 (i.e., finding a stable matching of I') can be done by the generalized Gale-Shapley algorithm studied in [5, 6] (for the details see the full version). Then, the detailed description of the algorithm is given as follows. Here, for each $h \in H$, $N \subseteq E$, and $d \in N(h)$, we use notations $N(h)_{\succ_h d} := \{ d' \in N(h) \mid d' \succ_h d \}$ and $N(h)_{\succeq_h d} := \{ d' \in N(h) \mid d' \succ_h d \text{ or } d' = d \}$.

In the full version, we show that the assignment M' obtained in the above algorithm is indeed a stable matching of I'. Also, it is shown that N_D is monotone decreasing and N_H is monotone increasing in the algorithm, and hence the "while loop" is iterated at most $2|E|$ times. Thus, we obtain the following theorem.

▶ **Theorem 15.** *For a CSM instance $I = (D, H, E, \succ_{DH}, \{(p_h, q_h)\}_{h \in H})$ such that each (p_h, q_h) is paramodular, the algorithm* EF-Paramodular-CSM *decides whether I has an envy-free matching or not in $O(|E|^2)$ time, provided that evaluation oracles of $\{p_h\}_{h \in H}$ are available.*

As is shown in Examples 12 and 13, if the acceptable family of each hospital h is defined by a laminar constraint $\mathcal{J}_h := \mathcal{J}(\mathcal{L}_h, \hat{p}_h, \hat{q}_h)$ or by a staffing constraint $\mathcal{J}_h := \mathcal{J}(S_h, \Gamma_h, \hat{l}_h, \hat{u}_h)$, there is a paramodular pair (p_h, q_h) such that $\mathcal{J}_h = \mathcal{F}(p_h, q_h)$. The following corollary states that, in such a case, we can decide the existence of an envy-free matching of $I = (D, H, E, \succ_{DH}, \{(p_h, q_h)\}_{h \in H})$ even if evaluation oracles of $\{p_h\}_{h \in H}$ are not provided.

▶ **Corollary 16.** *Suppose that, for each $h \in H$, the family of acceptable doctor sets is defined in the form $\mathcal{J}_h := \mathcal{J}(\mathcal{L}_h, \hat{p}_h, \hat{q}_h) \neq \emptyset$ (resp., $\mathcal{J}_h := \mathcal{J}(S_h, \Gamma_h, \hat{l}_h, \hat{u}_h) \neq \emptyset$). Let (p_h, q_h) be a paramodular pair such that $\mathcal{J}_h = \mathcal{F}(p_h, q_h)$. Then, given $\mathcal{L}_h, \hat{p}_h, \hat{q}_h$ (resp., $S_h, \Gamma_h, \hat{l}_h, \hat{u}_h$) for each $h \in H$, one can decide whether $I = (D, H, E, \succ_{DH}, \{(p_h, q_h)\}_{h \in H})$ has an envy-free matching or not in time polynomial in $|E|$ (resp., in $|E|$ and $\max_{h \in H} |S_h|$).*

Proof. As we have Theorem 15, it completes the proof to show that we can simulate an evaluation oracle of each p_h in time polynomial in $|E|$ (resp., in $|E|$ and $|S_h|$). For a paramodular pair (p_h, q_h) with $\mathcal{J}_h = \mathcal{F}(p_h, q_h)$, it is known that, for any $B \subseteq A(h)$, we have $p_h(B) = \min\{\, |X \cap B| \mid X \in \mathcal{J}_h\}$ (see, e.g., [9, Theorem 14.2.8]). Consider a weight function w_B on $A(h)$ such that $w_B(d) = 1$ for every $d \in B$ and $w_B(d) = 0$ for every $d \in A(h) \setminus B$. Then, $p_h(B) = \min\{\, w_B(X) \mid X \in \mathcal{J}_h \,\}$, which is a weight minimization problem on $\mathcal{J}_h$. As shown in [31, Appendix C], if $\mathcal{J}_h$ is defined in the form in the statement, this problem can be reduced to the minimum cost circulation problem, which can be solved in strongly polynomial time [29, 23]. (See [31] for the details of the reduction.) Thus, the proof is completed. ◀

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