Computational Complexity of Graph Partition under Vertex-Compaction to an Irreflexive Hexagon

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Abstract

In this paper, we solve a long-standing graph partition problem under vertex-compaction that has been of interest since about 1999. The graph partition problem that we consider in this paper is to decide whether or not it is possible to partition the vertices of a graph into six distinct non-empty sets A, B, C, D, E, and F, such that the vertices in each set are independent, i.e., there is no edge within any set, and an edge is possible but not necessary only between the pairs of sets A and B, B and C, C and D, D and E, E and F, and F and A, and there is no edge between any other pair of sets. We study the problem as the vertex-compaction problem for an irreflexive hexagon (6-cycle). Determining the computational complexity of this problem has been a long-standing problem of interest since about 1999, especially after the results of open problems obtained by the author on a related compaction problem appeared in 1999. We show in this paper that the vertex-compaction problem for an irreflexive hexagon is NP-complete. Our proof can be extended for larger even irreflexive cycles, showing that the vertex-compaction problem for an irreflexive even k-cycle is NP-complete, for all even $k \geq 6$.

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1 Introduction

The vertex-compaction problem and the compaction problem are special graph colouring problems, and can also be viewed as graph partition problems. The colouring problem is a classic problem in graph theory. The graph homomorphism problem, also called the H-colouring problem, is a generalization of the colouring problem. The vertex-compaction problem is the graph homomorphism problem with additional constraints. The compaction problem is the vertex-compaction problem with additional constraints. We describe our motivation and results after introducing the following definitions and problems.

1.1 Definitions

The pair of vertices of an edge in a graph are called the *endpoints* of the edge. An edge with the same endpoints in a graph is called a *loop*. A vertex v of a graph is said to have a loop if vv is an edge of the graph. A *reflexive graph* is a graph in which every vertex has a loop. An *irreflexive graph* is a graph in which no vertex has a loop. Any graph, in general, is a *partially reflexive graph*, in which its vertices may or may not have loops. Thus reflexive and irreflexive graphs are special partially reflexive graphs. A *bipartite graph* G is a graph whose

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vertex set can be partitioned into two distinct subsets G_A and G_B, such that each edge of G has one endpoint in G_A and the other endpoint in G_B; we say that (G_A, G_B) is a bipartition of G. Thus a bipartite graph is irreflexive by definition. If uv is an edge of a graph then vu is also an edge of the graph, i.e., we assume graphs to be undirected graphs. A cycle of length k is called a k-cycle, $k \geq 3$. A *hexagon* will be used as a synonym for a 6-cycle. We shall denote an irreflexive k-cycle by C_k.

Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G respectively. Given an induced subgraph H of G, we denote by $G - H$, the subgraph obtained by deleting from G the vertices of H together with the edges incident with them; thus $G - H$ is a subgraph of G induced by $V(G) - V(H)$. The vertices in a set $I \subseteq V(G)$ are said to be *independent* if there is no edge in the subgraph of G induced by I. When a set S is an argument of a mapping f, we define $f(S) = \{f(s)|s \in S\}$. The *distance* between a pair of vertices u and v in G, denoted as $d_G(u, v)$ or $d_G(v, u)$, is the length of a shortest path from u to v in G, if u and v are connected in G; we define $d_G(u, v)$ (and $d_G(v, u)$) to be infinite, if u and v are disconnected in G. The *diameter* of G is the maximum distance between any two vertices in G. The distance between two sets X and Y of vertices in G, denoted as $d_G(X, Y)$ or $d_G(Y, X)$, is the minimum distance between any vertex of X and any vertex of Y in G, i.e., $d_G(X, Y) = min\{d_G(x, y)|x \in X, y \in Y\}$, where $min\ A$ gives the minimum element in a set A. If a set has only one vertex, we may just write the vertex instead of the set. In the following, let G and H be graphs.

A *homomorphism* $f : G \to H$, of G to H, is a mapping f of the vertices of G to the vertices of H, such that if g and g' are adjacent vertices of G then $f(g)$ and $f(g')$ are adjacent vertices of H. If there exists a homomorphism of G to H then G is said to be *homomorphic* to H. Note that if G is irreflexive then G is k-colourable if and only if G is homomorphic to the irreflexive complete graph K_k having k vertices. Thus the concept of a homomorphism generalises the concept of a k-colourability, and the *H-colouring* problem is to decide whether or not G is homomorphic to H. The *H-colouring* problem is trivial and easily seen to be polynomial time solvable if H is bipartite or H has a loop. For any fixed non-bipartite irreflexive graph H, it is shown in [Hell and Nesetril, 1990] that the *H-colouring* problem is NP-complete.

A *compaction* $c : G \to H$, of G to H, is a homomorphism of G to H, such that for every vertex x of H, there exists a vertex v of G with $c(v) = x$, and for every edge hh' of H, $h \neq h'$, there exists an edge gg' of G with $c(g) = h$ and $c(g') = h'$. Note that the first part of the definition for a compaction (the requirement for every vertex x of H) follows from the second part unless H has isolated vertices. If there exists a compaction of G to H then G is said to *compact* to H. Given a compaction $c : G \to H$, if for a vertex v of G, we have $c(v) = x$, where x is a vertex of H, then we say that the vertex v of G *covers the vertex* x of H under c; and if for an edge gg' of G, we have $c(\{g, g'\}) = \{h, h'\}$, where hh' is an edge of H, then we say that the edge gg' of G *covers the edge* hh' of H under c (note in the definition of compaction, it is not necessary that a loop of H be covered by any edge of G under c).

We notice that the notion of a *homomorphic image* described in [Harary, 1969] (also cf. [Hell & Miller, 1979]) coincides with the notion of a compaction in case of irreflexive graphs (i.e., when G and H are irreflexive in the above definition for compaction).

A *vertex-compaction* $c : G \to H$, of G to H, is a homomorphism of G to H, such that for every vertex x of H there exists a vertex v of G with $c(v) = x$. If there exists a vertex-compaction of G to H then G is said to *vertex-compact* to H. We define vertex and edge covering under a vertex-compaction c similarly as for a compaction. Note that every compaction is also a vertex-compaction.

A *retraction* $r : G \to H$, of G to H, with H as an induced subgraph of G, is a homomorphism of G to H, such that $r(h) = h$, for every vertex h of H. If there exists a retraction of G to H then G is said to *retract* to H. Note that every retraction $r : G \to H$ is necessarily also a compaction, and hence a vertex-compaction.

1.2 Vertex-Compaction, Compaction, and Retraction Problems

The problem of deciding the existence of a vertex-compaction to a fixed graph H, called the *vertex-compaction problem for H*, and denoted as *VCOMP-H*, asks whether or not an input graph G vertex-compacts to H.

Our graph partition problem is to decide whether or not it is possible to partition the vertices of a graph into six distinct non-empty sets A, B, C, D, E, and F, such that the vertices in each of these sets are independent, and an edge is possible but not necessary only between the pairs of sets A and B, B and C, C and D, D and E, E and F, and F and A, and there is no edge between any other pair of sets. We note that our graph partition problem is the problem *VCOMP-C_6*.

The problem of deciding the existence of a compaction to a fixed graph H, called the *compaction problem for H*, and denoted as *COMP-H*, asks whether or not an input graph G compacts to H. The compaction problem is a well studied problem over last several years, and includes some popular problems. Results on the compaction problem can be found in [Vikas, 1999, 2002, 2003, 2004a, 2004b, 2004c, 2005, 2011, 2013].

Note that unlike the *H-colouring* problem, the problems *VCOMP-H* and *COMP-H* are still interesting if H is bipartite or H has a loop. Some work on graph partition problems have also been studied in [Feder, Hell, Klein, and Motwani, 1999, 2003] and [Hell, 2014].

The problem of deciding the existence of a retraction to a fixed graph H, called the *retraction problem for H*, and denoted as *RET-H*, asks whether or not an input graph G, containing H as an induced subgraph, retracts to H. Retraction problems have been of continuing interest in graph theory for a long time and have been studied in various literature including [Hell, 1972], [Hell, 1974], [Nowakowski and Rival, 1979], [Pesch and Poguntke, 1985], [Bandelt, Dahlmann, and Schutte, 1987], [Hell and Rival, 1987], [Pesch, 1988], [Bandelt, Farber, and Hell, 1993], [Feder and Hell, 1998], [Feder and Vardi, 1993, 1998], [Feder, Hell, and Huang, 1999], [Vikas, 2004b, 2004c, 2005], etc.

1.3 Motivation and Results

It can be shown that for every fixed graph H, if the problem *COMP-H* is solvable in polynomial time then the problem *VCOMP-H* is also solvable in polynomial time (similarly as in [Vikas, 2004b]). Whether the converse is true is not known. The problem *COMP-C_6* is shown to be NP-complete in [Vikas, 1999, 2004a]. It turns out that the unique smallest bipartite graph H for which *COMP-H* is NP-complete is C_6 [Vikas, 2004a]. Therefore, with respect to the preceding question on converse, we are motivated to specifically determine the computational complexity of our partition problem *VCOMP-C_6*, to see whether like *COMP-C_6*, it is also NP-complete in support of the converse. We show in this paper that *VCOMP-C_6* is NP-complete. Determining the computational complexity of *VCOMP-C_6* has been a long-standing problem of interest since about 1999, especially after results on the computational complexity of *COMP-C_6* obtained by the author appeared in 1999 [Vikas, 1999]. Determining the computational complexity of *COMP-C_6* was also a long-standing problem of interest since about 1988, solved by the author in [Vikas, 1999, 2004a]. Although the problem *VCOMP-C_6* is only a little variation of the problem *COMP-C_6*, it turns out to be another difficult problem to determine its computational complexity.

Similarly, our motivation for the partition problem *COMP-C_6* was with respect to the retraction problem. It can be shown that for every fixed graph H, if the problem *RET-H* is polynomial time solvable then the problem *COMP-H* is also polynomial time solvable [Vikas, 2004b]. However, whether the converse is true is again not known. As discussed in [Vikas, 1999, 2003], the question on converse was also asked by Peter Winkler in 1988 in the context of reflexive graphs, and this was the general problem that motivated Winkler for asking the computational complexity of *COMP-H* when H is a reflexive square, as the unique smallest reflexive graph H for which *RET-H* is NP-complete is a reflexive square. It has been shown in [Vikas, 1999, 2003] that when H is a reflexive square, *COMP-H* is NP-complete. As discussed in [Vikas, 2004a], since the unique smallest bipartite graph H for which *RET-H* is NP-complete is C_6, we are therefore motivated, with respect to the above question on converse, to know whether the problem *COMP-C_6* is also NP-complete like the problem *RET-C_6* supporting the converse. As mentioned above, it is shown in [Vikas, 1999, 2004a] that *COMP-C_6* is NP-complete.

The problem *RET-C_6* is shown to be NP-complete in [Feder, Hell, and Huang, 1999], and also independently by G. MacGillivray in 1988. Since C_4 is a complete bipartite graph, it is easy to see that *RET-C_4*, and hence *COMP-C_4* and also *VCOMP-C_4*, are all polynomial time solvable. In fact, when H is a chordal bipartite graph (which includes C_4), the problem *RET-H* is polynomial time solvable [Bandelt, Dahlmann, and Schutte, 1987], and hence *COMP-H* and *VCOMP-H* are also polynomial time solvable. Thus it follows that the unique smallest bipartite graph H for which *RET-H*, *COMP-H*, and *VCOMP-H* are NP-complete is C_6.

It has been shown in [Hell and Nesetril, 1990] that the *H-colouring* problem is NP-complete for any fixed irreflexive non-bipartite graph H. It follows that *RET-H*, *COMP-H*, and *VCOMP-H* are also NP-complete for any non-bipartite irreflexive graph H, which includes an irreflexive odd k-cycle, for all $k \geq 3$.

As we mentioned earlier, the *H-colouring* problem is trivial and easily seen to be polynomial time solvable when H is a bipartite graph. The natural question for bipartite graphs H, which motivated Pavol Hell and Jaroslav Nesetril (personal communications) around 1988, was to ask for the computational complexity of the *H-colouring* problem with added constraints, namely the problem *COMP-H*, and in particular for the problem *COMP-C_6*.

It can also be shown that for every fixed graph H, if the problem *RET-H* is polynomial time solvable then the problem *VCOMP-H* is also polynomial time solvable (similarly as in [Vikas, 2004b]), but whether the converse is true is not known. Hence, once again, in relation to the converse and the problem *RET-C_6*, we are motivated to know whether the problem *VCOMP-C_6* is NP-complete.

The algorithms given in [Vikas, 2011, 2013] yield a polynomial time algorithm for *VCOMP-C_6* for any input graph of diameter more than four, and it is suggested in [Vikas, 2011, 2013] as a guidance that an input graph of diameter four could be a candidate for *VCOMP-C_6* to be possibly NP-complete. We are thus motivated to see whether *VCOMP-C_6* is indeed NP-complete for an input graph of diameter four, guided by the algorithmic aspects of the vertex-compaction problem studied in [Vikas, 2011, 2013]. The instance of the input graph for which we show *VCOMP-C_6* to be NP-complete in this paper is indeed of diameter four.

Our proof and technique of construction for C_6 can be extended for larger irreflexive even cycles to show that *VCOMP-C_k* is NP-complete, for all even $k \geq 6$. Our proof showing NP-completeness of *VCOMP-C_6* directly uses graphs that we construct just by adding vertices

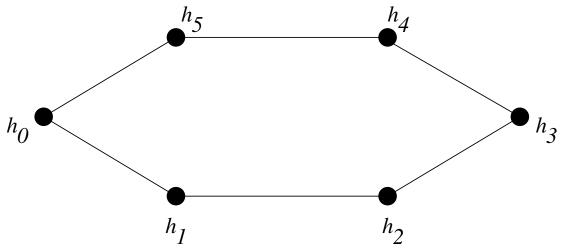

Figure 1 Irreflexive Hexagon H

and edges. Our graphs therefore lay down the foundation for construction of graphs for the case of a general irreflexive even k-cycle, by extending the paths constructed and adding edges appropriately, showing NP-completeness of *VCOMP-C_k*, for all even $k \geq 6$.

In Section 2, we present the proof showing NP-completeness of deciding the existence of a vertex-compaction to an irreflexive hexagon, i.e., the problem *VCOMP-C_6*. In Section 3, we address how our NP-completeness proof of *VCOMP-C_6* can be extended for an irreflexive even k-cycle, showing NP-completeness of *VCOMP-C_k*, for all even $k \geq 6$.

2 Vertex-Compaction to an Irreflexive Hexagon

▶ **Theorem 2.1.** *The problem of deciding the existence of a vertex-compaction to an irreflexive hexagon is NP-complete.*

Proof. Let H be the irreflexive hexagon $h_0h_1h_2h_3h_4h_5h_0$ shown in Figure 1.

We shall prove that the problem of deciding the existence of a vertex-compaction to H, i.e., the problem *VCOMP-H*, is NP-complete. Clearly, the problem *VCOMP-H* is in NP. We give a polynomial transformation from the problem *RET-H* to *VCOMP-H*. As mentioned earlier, it is known that the problem *RET-H* is NP-complete. Since only a bipartite graph can be homomorphic to H, the problem *RET-H* remains to be NP-complete if the instance of *RET-H* is restricted to be only a bipartite graph.

Let a bipartite graph G, containing H as an induced subgraph, be an instance of *RET-H*. We construct in time polynomial in the size of G, a graph G', containing G as an induced subgraph, such that the following statements (i), (ii), and (iii) are equivalent:

(i) G retracts to H.
(ii) G' retracts to H.
(iii) G' vertex-compacts to H.

Since *RET-H*, with the instance restricted to be a bipartite graph, is NP-complete, this shows that *VCOMP-H* is also NP-complete. We prove that (i) is equivalent to (ii), and (ii) is equivalent to (iii), in two separate lemmas, Lemma 2.2 and Lemma 2.3, respectively.

One of the main challenges is to construct such a graph of diameter four. Let (G_A, G_B) be a bipartition of G, and (H_A, H_B) be a bipartition of H, with $H_A \subseteq G_A$, and $H_B \subseteq G_B$. We shall assume for convenience that $h_0 \in H_B$.

The construction of G' is as follows. For each vertex $a \in G_A - H_A$, we add a new vertex z_a adjacent to a and h_1. For every pair of vertices a and b, with $a \in G_A - H_A$, $b \in G_B - H_B$, we add a new vertex z_{ab} adjacent to z_a and b. Thus for each $a \in G_A - H_A$, we have paths $az_az_{ab}b$, $az_az_{ab'}b'$, for all $b, b' \in G_B - H_B$. See Figure 2. In the figure, we have taken three distinct vertices a, a', and a'' of $G_A - H_A$, and three distinct vertices b, b', and b'' of $G_B - H_B$. Also, in the figures in this section, we are not depicting any edge that may be present between a vertex of $G_A - H_A$ and a vertex of $G_B - H_B$.

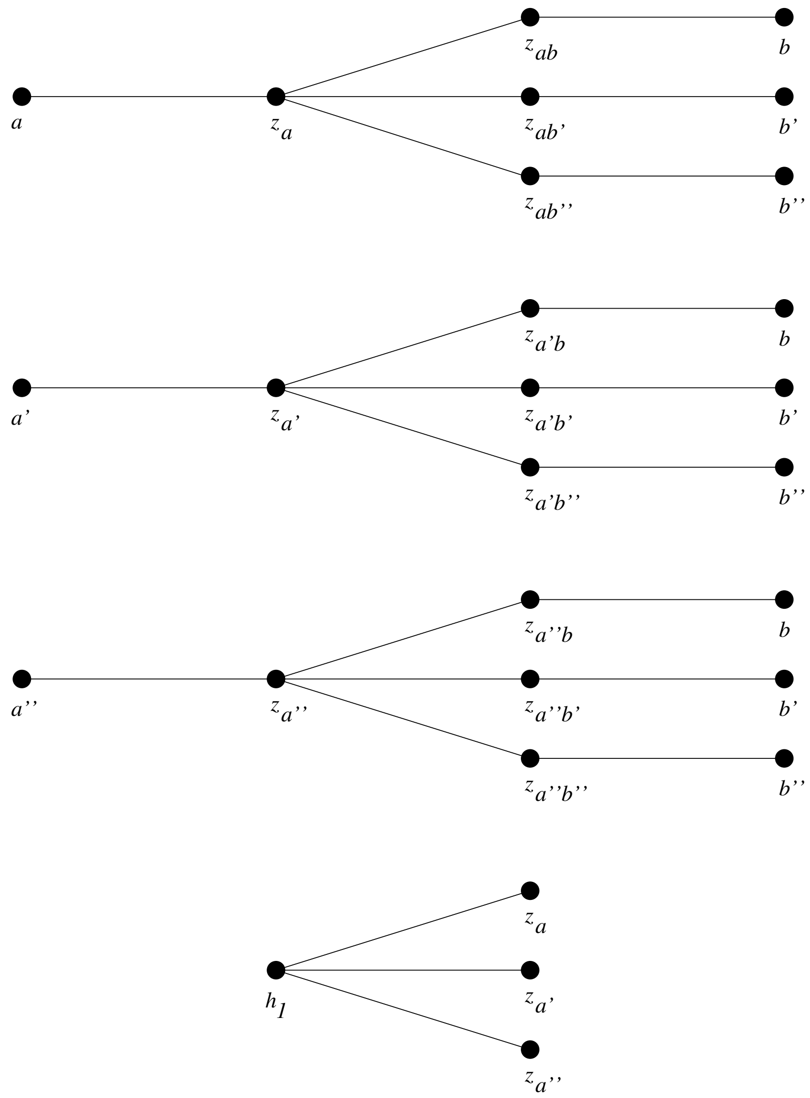

Figure 2 Construction of G', with z_a and z_{ab}, for every pair of vertices a and b, with $a \in G_A - H_A$, $b \in G_B - H_B$

In the graph G' constructed so far, the maximum distance between any pair of vertices in $V(G') - V(H)$ is already four, i.e., $d_{G'}(v, v') \leq 4$, with $v, v' \in V(G') - V(H)$. This can be observed due to the following paths and subpaths within those paths of length at most four : $az_az_{ab}bz_{a'b}$, $z_{ab}z_az_{ab'}b'z_{a'b'}$, $az_ah_1z_{a'}a'$, $bz_{ab}z_az_{ab'}b'$.

If we constructed the graph G' including the vertices of H also, i.e., if we constructed G' for every pair of vertices a and b, with $a \in G_A$, $b \in G_B$ then the diameter of G' would be four.

We continue further with the construction of G'. For each vertex $b \in G_B - H_B$, we add a new vertex x_b adjacent to z_{ab} and h_5, for all $a \in G_A - H_A$. For every pair of vertices a and b, with $a \in G_A - H_A$, $b \in G_B - H_B$, we add a new vertex x_{ba} adjacent to x_b and z_a. Thus for each $b \in G_B - H_B$, we have paths $z_{ab}x_bx_{ba}z_a$, $z_{a'b}x_bx_{ba'}z'_a$, for all $a, a' \in G_A - H_A$. See Figure 3.

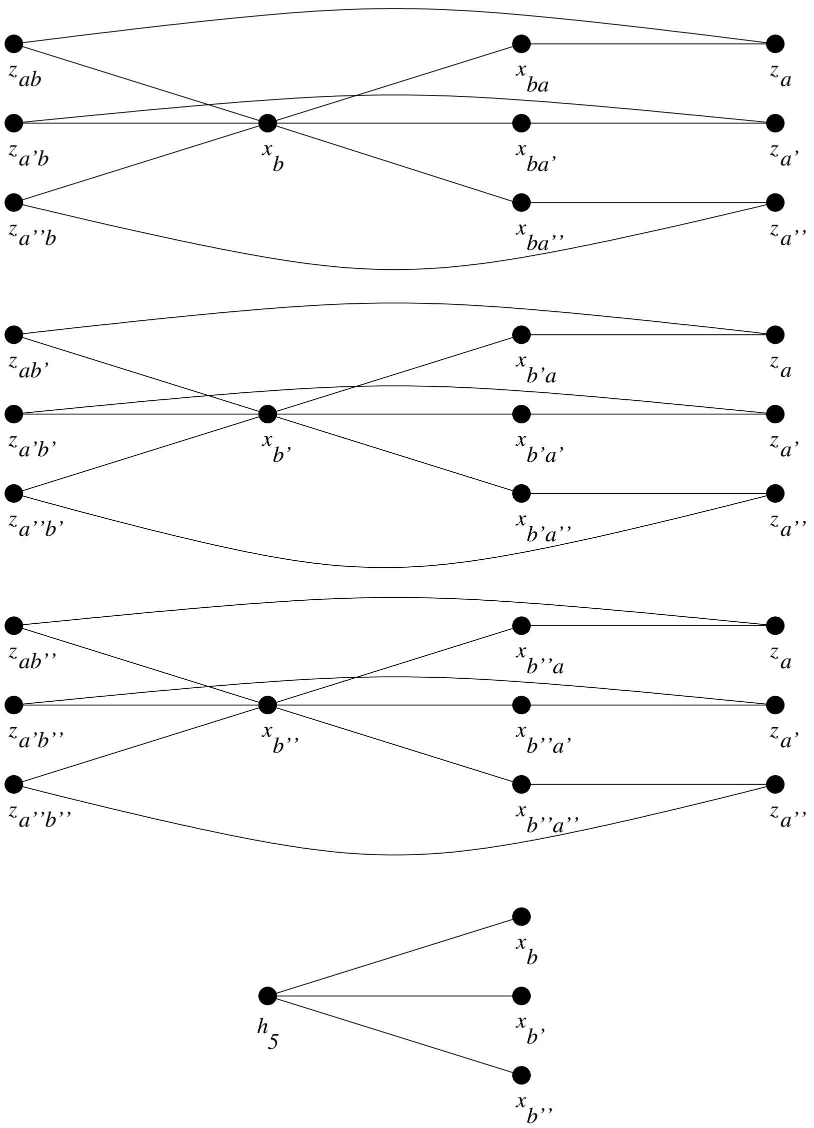

Figure 3 Construction of G', with x_b and x_{ba}, for every pair of vertices a and b, with $a \in G_A - H_A$, $b \in G_B - H_B$

The maximum distance between any pair of vertices in $V(G') - V(H)$ is still four. This can be observed due to the following paths and subpaths within those paths of length at most four : $x_{ba}x_bx_{ba'}z_{a'}x_{b'a'}$, $x_{ba}x_bx_{ba'}z_{a'}z_{a'b'}$, $x_{ba}x_bx_{ba'}z_{a'}a'$, $x_bx_{ba}z_az_{ab'}b'$.

For each vertex $a \in G_A - H_A$ (hence $d_G(h_0, a)$ is odd as we are assuming that $h_0 \in H_B \subseteq G_B$), we add to G new vertices u_1^a adjacent to h_0; u_2^a adjacent to u_1^a, a, and h_1; w_1^a adjacent to h_3, u_1^a, and a; y_1^a adjacent to h_1, u_1^a, and a; y_2^a adjacent to y_1^a, h_4, w_1^a, and u_2^a. See Figure 4. Note that there could be edges in G from a to some vertices of H but in Figure 4, we are not depicting these edges.

For each vertex $b \in G_B - H_B$ (hence $d_G(h_0, b)$ is even), we add to G new vertices u_1^b adjacent to h_0 and b; w_1^b adjacent to h_3 and u_1^b; w_2^b adjacent to w_1^b, b, and h_2; y_1^b adjacent to h_5, u_1^b, and w_2^b; y_2^b adjacent to y_1^b, h_2, w_1^b, and b. See Figure 5. There could be edges in G from b to some vertices of H but in Figure 5, we are not depicting these edges.

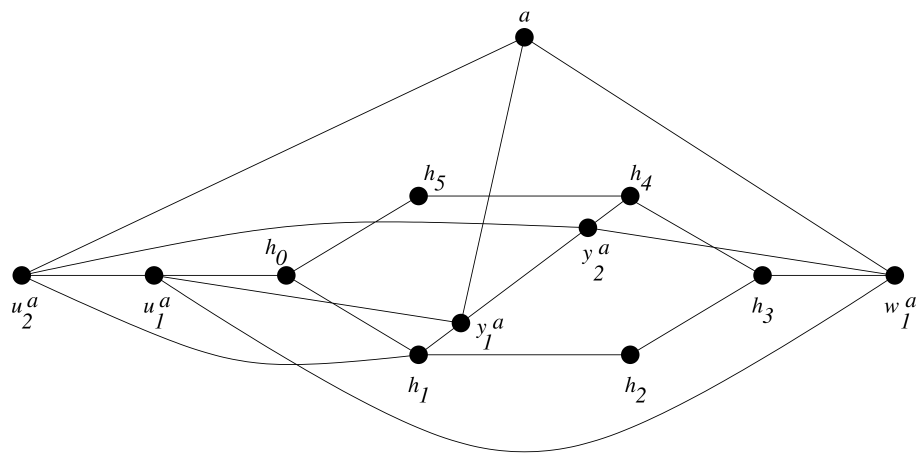

Figure 4 Construction of G' for a vertex a in $G_A - H_A$

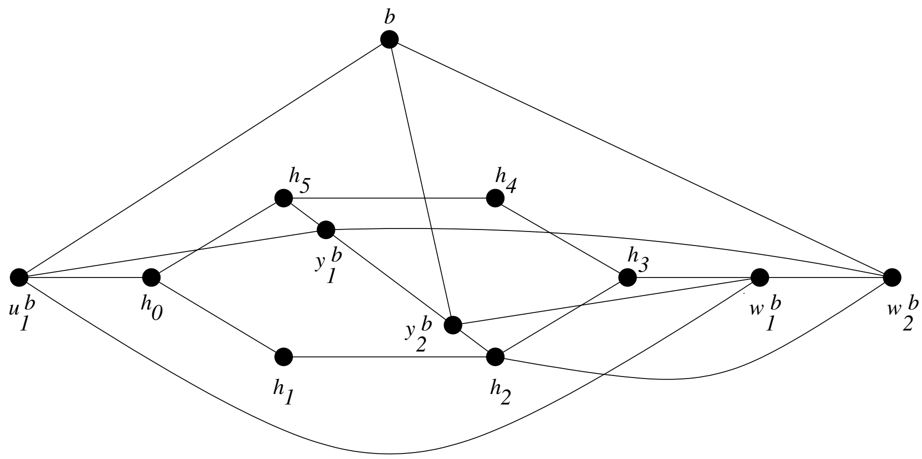

Figure 5 Construction of G' for a vertex b in $G_B - H_B$

For every vertex $a \in G_A - H_A$, we further make z_a adjacent to u_1^a and y_2^a. For every pair of vertices a and b, with $a \in G_A - H_A$, $b \in G_B - H_B$, we further also make z_{ab} adjacent to w_1^b and y_1^b.

For every vertex $b \in G_B - H_B$, we also further make x_b adjacent to u_1^b, y_2^b, and w_2^b. For every pair of vertices a and b, with $a \in G_A - H_A$, $b \in G_B - H_B$, we further make x_{ba} adjacent to w_1^a, y_1^a, and u_2^a. See Figure 6.

This completes the construction of G'. The diameter of the graph G' is four. We now prove the following two lemmas in order to prove the theorem.

▶ **Lemma 2.2.** *G retracts to H if and only if G' retracts to H.*

Proof. If G' retracts to H then it is clear that G also retracts to H, as G is a subgraph of G'. Now suppose that G retracts to H, and let $r : G \to H$ be a retraction. We define a retraction $r' : G' \to H$ as follows.

We define r' for the vertices v of G (that are also vertices of G') as

$r'(v) = r(v)$.

We define r' for the newly added vertices of G', with $a \in G_A - H_A$, as follows.

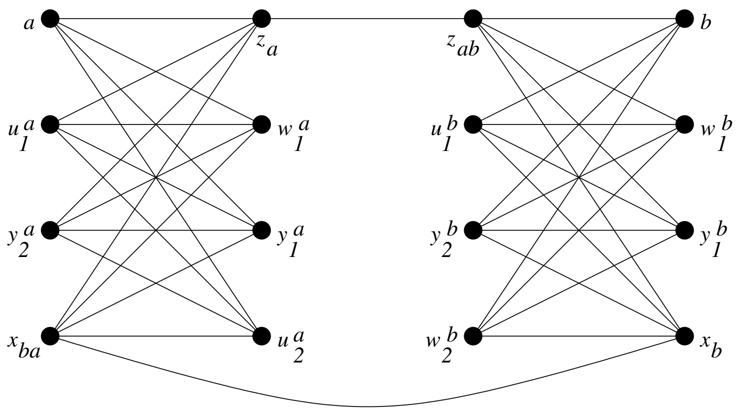

Figure 6 Construction of G' for a pair of vertices a and b, with $a \in G_A - H_A$, $b \in G_B - H_B$

If $r(a) = h_1$ or h_3, then we define
$r'(u_1^a) = h_1$, $r'(u_2^a) = h_2$,
$r'(w_1^a) = h_2$,
$r'(y_1^a) = h_2$, $r'(y_2^a) = h_3$,
$r'(z_a) = h_2$.
If $r(a) = h_5$, then we define
$r'(u_1^a) = h_5$, $r'(u_2^a) = h_0$,
$r'(w_1^a) = h_4$,
$r'(y_1^a) = h_0$, $r'(y_2^a) = h_5$,
$r'(z_a) = h_0$.
We define r' for the newly added vertices of G', with $b \in G_B - H_B$, as follows.
If $r(b) = h_0$ or h_2, then we define
$r'(u_1^b) = h_1$,
$r'(w_1^b) = h_2$, $r'(w_2^b) = h_1$,
$r'(y_1^b) = h_0$, $r'(y_2^b) = h_1$,
$r'(x_b) = h_0$.
If $r(b) = h_4$, then we define
$r'(u_1^b) = h_5$,
$r'(w_1^b) = h_4$, $r'(w_2^b) = h_3$,
$r'(y_1^b) = h_4$, $r'(y_2^b) = h_3$,
$r'(x_b) = h_4$.
We define r' for the vertices z_{ab} and x_{ba} of G', with $a \in G_A - H_A$, $b \in G_B - H_B$, as follows.
If $r(a) = h_1$ or h_3, and $r(b) = h_0$ or h_2, then we define
$r'(z_{ab}) = h_1$, $r'(x_{ba}) = h_1$.
If $r(a) = h_1$ or h_3, and $r(b) = h_4$, then we define
$r'(z_{ab}) = h_3$, $r'(x_{ba}) = h_3$.
If $r(a) = h_5$, and $r(b) = h_0$ or h_2, then we define
$r'(z_{ab}) = h_1$, $r'(x_{ba}) = h_5$.
If $r(a) = h_5$, and $r(b) = h_4$, then we define
$r'(z_{ab}) = h_5$, $r'(x_{ba}) = h_5$.

We now verify that $r' : G' \to H$ is indeed a homomorphism (and hence a retraction). We do this by considering all the edges ab of G', and proving that $r'(a)r'(b)$ is an edge of H. Before verifying, we point out that, as far as C_6 is concerned, we could use the vertices y_1^a

and y_1^b instead of the vertices z_a and x_b, respectively, but we have continued to keep z_a and x_b, as construction similar to them would be needed in the construction for larger cycles, with $a \in G_A - H_A$, $b \in G_B - H_B$.

Consider first an edge gg', with $gg' \in E(G)$. We have from our definition of r' that $r'(g) = r(g)$ and $r'(g') = r(g')$. Since $r : G \to H$ is a homomorphism, $r(g)r(g')$ must be an edge of H. Hence $r'(g)r'(g') = r(g)r(g')$ is an edge of H.

Now consider the edges $u_1^a h_0$, $u_2^a h_1$, $u_1^b h_0$, $w_1^a h_3$, $w_1^b h_3$, $w_2^b h_2$, $y_1^a h_1$, $y_2^a h_4$, $y_1^b h_5$, $y_2^b h_2$, $z_a h_1$, and $x_b h_5$, with $a \in G_A - H_A$, $b \in G_B - H_B$. From the definition of r', we have $r'(h_i) = r(h_i) = h_i$, for all $i = 0, 1, 2, 3, 4, 5$. Depending on the value of $r(a)$, we note from our definition of r' that $r'(u_1^a) = h_1$ or h_5, $r'(u_2^a) = h_2$ or h_0, $r'(w_1^a) = h_2$ or h_4, $r'(y_1^a) = h_2$ or h_0, $r'(y_2^a) = h_3$ or h_5, and $r'(z_a) = h_2$ or h_0. Hence $r'(u_1^a)r'(h_0)$, $r'(u_2^a)r'(h_1)$, $r'(w_1^a)r'(h_3)$, $r'(y_1^a)r'(h_1)$, $r'(y_2^a)r'(h_4)$, and $r'(z_a)r'(h_1)$ are always edges of H. Similarly, depending on the value of $r(b)$, from our definition of r', we have $r'(u_1^b) = h_1$ or h_5, $r'(w_1^b) = h_2$ or h_4, $r'(w_2^b) = h_1$ or h_3, $r'(y_1^b) = h_0$ or h_4, $r'(y_2^b) = h_1$ or h_3, and $r'(x_b) = h_0$ or h_4. Thus $r'(u_1^b)r'(h_0)$, $r'(w_1^b)r'(h_3)$, $r'(w_2^b)r'(h_2)$, $r'(y_1^b)r'(h_5)$, $r'(y_2^b)r'(h_2)$, and $r'(x_b)r'(h_5)$ are always edges of H.

The remaining edges of G' can also be verified. Since $r'(h) = r(h) = h$, for all $h \in V(H)$, the homomorphism $r' : G' \to H$ is a retraction. We have thus proved the lemma. ◀

▶ **Lemma 2.3.** *G' retracts to H if and only if G' vertex-compacts to H.*

Proof. If G' retracts to H then by definition G' vertex-compacts to H. Now suppose that G' vertex-compacts to H. We shall prove that G' also retracts to H. Let $c : G' \to H$ be a vertex-compaction. We let $U = \{u_1^v | v \in V(G - H)\} \cup \{h_1, h_0, h_5\}$ and $W = \{w_1^v | v \in V(G - H)\} \cup \{h_2, h_3, h_4\}$.

Since h_0 is adjacent to every other vertex in U, and G' is bipartite, the subgraph of G' induced by the vertices in U is of diameter two. Hence, the vertices of $c(U)$ induce a path of length one or two in H, as H is irreflexive. Thus $c(U)$ has either two or three vertices. Similarly, $c(W)$ has either two or three vertices. We shall prove that $c(U)$ and $c(W)$ both have three vertices.

Suppose that $c(U)$ has only two vertices. Then we know that the vertices in $c(U)$ are adjacent in H. Without loss of generality, let $c(U) = \{h_0, h_1\}$ and $c(h_0) = h_0$ (due to symmetry of vertices in H). Hence $c(U - \{h_0\}) = \{h_1\}$. We note that $d_{G'}(U - \{h_0\}, g) < 3$, for all $g \in V(G')$. Hence $d_{G'}(U - \{h_0\}, g) < d_H(c(U - \{h_0\}) = h_1, h_4) = 3$, for all $g \in V(G')$. This implies that $c(g) \neq h_4$, for all $g \in V(G')$, which is impossible, as $c : G' \to H$ is a vertex-compaction. Hence it must be that $c(U)$ has three vertices. We also note that $d_{G'}(W - \{h_3\}, g) < 3$, for all $g \in V(G')$, and hence, similarly, it must be that $c(W)$ also has three vertices.

Thus $c(U)$ and $c(W)$ both induce paths having three vertices in H. Without loss of generality, let $c(U) = \{h_1, h_0, h_5\}$ (due to symmetry). This implies that $c(h_0) = h_0$. We first prove that $c(h_3) = h_3$. We note that the diameter of G' is 4, and hence our vertex-compaction $c : G' \to H$ must also be a compaction, as otherwise the diameter of G' will be greater than 4. Let some edge gg' of G' cover the edge h_3h_4 or h_3h_2 of H under c, with $c(g) = h_3$ and $c(g') = h_4$ or h_2 (indeed there exists such an edge in G', as the vertex-compaction c is also a compaction). We note that h_3 is at distance 2 from $c(U)$ in H, as $d_H(c(U), h_3) = d_H(h_1, h_3) = 2$. Further, both h_4 and h_2 are at distance 1 from $c(U)$ in H, as $d_H(c(U), h_4) = d_H(h_5, h_4) = 1$, and $d_H(c(U), h_2) = d_H(h_1, h_2) = 1$. Thus it must be that $d_{G'}(U, g) \geq 2$ and $d_{G'}(U, g') \geq 1$. Since there is no vertex at distance more than 2 from U in G', we have $d_{G'}(U, g) = 2$ and $d_{G'}(U, g') = 1$ or 2. Further, since G' is bipartite, it must be that $d_{G'}(U, g) = 2$ and $d_{G'}(U, g') = 1$.

We note that the only vertices that could possibly be at distance 1 from U in G', as possible candidates for g', are : h_2, h_4, w_1^a, w_1^b, y_1^a, y_1^b, u_2^a, b, z_a, and x_b, with $a \in G_A - H_A$, $b \in G_B - H_B$. The only vertices that could possibly be at distance 2 from U in G', as possible candidates for g, are : h_3, y_2^a, y_2^b, w_2^b, a, z_{ab}, and x_{ba}, with $a \in G_A - H_A$, $b \in G_B - H_B$. Thus g and g' are among these vertices.

Since $c(h_0) = h_0$ and H is bipartite, $c(h_3) \neq h_4$ or h_2. Suppose that $c(h_3) \neq h_3$. Then no edge of G' with h_3 as an endpoint covers the edge h_3h_4 or h_3h_2 of H under c. Hence gg' must be an edge among $y_2^a w_1^a$, $y_2^b w_1^b$, $y_2^a y_1^a$, $y_2^b y_1^b$, $y_2^a z_a$, $y_2^a u_2^a$, $y_2^a h_4$, $y_2^b h_2$, $y_2^b b$, $y_2^b x_b$, $a y_1^a$, $a u_2^a$, $a w_1^a$, $a z_a$, $w_2^b w_1^b$, $w_2^b b$, $w_2^b x_b$, $w_2^b y_1^b$, $z_{ab} w_1^b$, $z_{ab} y_1^b$, $z_{ab} b$, $z_{ab} z_a$, $z_{ab} x_b$, $x_{ba} z_a$, $x_{ba} w_1^a$, $x_{ba} y_1^a$, $x_{ba} u_2^a$, $x_{ba} x_b$, and possibly ab, with $a \in G_A - H_A$, $b \in G_B - H_B$, where the first vertex in each of these edges stand for g and the second for g', and in order to meet the rquirements of the edge gg', the first vertex in each of these edges is assumed to achieve distance 2 from U in G' and hence may map to h_3 under c, and the second vertex in each of these edges is assumed to achieve distance 1 from U in G' and hence may map to h_4 or h_2 under c. We shall be always mentioning these edges in this order. Further, if ah_2 or ah_4 is an edge of G, for some vertex $a \in G_A - H_A$, then we need to include such an edge also for gg'. These edges for gg' are all the possible edges of G' that may cover the edge h_3h_4 or h_3h_2 of H under c assuming that $c(h_3) \neq h_3$. Since $c(h_3) \in H_A$ (as $c(h_0) = h_0 \in H_B$) and $c(h_3) \neq h_3$, we have $c(h_3) = h_1$ or h_5. The outline for proving that $c(h_3) \neq h_3$ is impossible is as follows. We suppose that $c(h_3) = h_1$, and consider each of the possible edges for gg' mentioned above, and show that they do not cover the edge h_3h_4 under c (i.e., $c(\{g, g'\}) \neq \{h_3, h_4\}$). Symmetrically, if $c(h_3) = h_5$ then it can be shown that none of the possible edges for gg' mentioned above can cover the edge h_3h_2 under c (i.e., $c(\{g, g'\}) \neq \{h_3, h_2\}$). Thus let $c(h_3) = h_1$.

Consider first the edges $y_2^a w_1^a$ and $y_2^b w_1^b$, with $a \in G_A - H_A$, $b \in G_B - H_B$. We consider them together as an edge $y_2^v w_1^v$, with $v \in V(G - H)$. Suppose that $y_2^v w_1^v$ covers the edge h_3h_4 under c. Then $c(y_2^v) = h_3$ and $c(w_1^v) = h_4$. By assumption, we have $c(h_3) = h_1$. Since w_1^v is adjacent to h_3, this implies that $c(w_1^v) = h_0$ or h_2. Thus $c(w_1^v) \neq h_4$, and we have a contradiction.

Next consider the edges $a w_1^a$, $w_2^b w_1^b$, and $z_{ab} w_1^b$, with $a \in G_A - H_A$, $b \in G_B - H_B$. Similar to the above, since $c(w_1^v)$ must be adjacent to $c(h_3) = h_1$, it is impossible that $c(w_1^v) = h_4$, with $v \in V(G - H)$, and hence the above edges cannot cover the edge h_3h_4 under c.

Now consider the edges $y_2^a y_1^a$ and $y_2^b y_1^b$, with $a \in G_A - H_A$, $b \in G_B - H_B$. We consider them together as an edge $y_2^v y_1^v$, with $v \in V(G - H)$. Suppose that $y_2^v y_1^v$ covers the edge h_3h_4 under c. Then $c(y_2^v) = h_3$ and $c(y_1^v) = h_4$. Since $c(u_1^v)$ must be adjacent to both $c(h_0) = h_0$ and $c(y_1^v) = h_4$, this implies that $c(u_1^v) = h_5$. Since $c(w_1^v)$ must be adjacent to both $c(u_1^v) = h_5$ and $c(y_2^v) = h_3$, it must be that $c(w_1^v) = h_4$. This implies that $y_2^v w_1^v$ covers the edge h_3h_4 under c, which we have already proved does not hold.

The remaining edges for gg' can be verified also. Symmetrically, if $c(h_3) = h_5$ then no possible edge for gg' can cover the edge h_3h_2 under c. We thus establish that $c(h_3) = h_3$, and hence $c(W) = \{h_2, h_3, h_4\}$.

We now prove that $c(h_1) \neq c(h_5)$. Suppose to the contrary that $c(h_1) = c(h_5)$. Since $c(h_0) = h_0$, we have $c(h_1), c(h_5) \in \{h_1, h_5\}$. Without loss of generality, let $c(h_1) = c(h_5) = h_1$ (due to symmetry). Since $c(U) = \{h_1, h_0, h_5\}$, it must be that $c(u_1^v) = h_5$ for some vertex v of $G - H$. Since $c(w_1^v)$, $c(h_2)$, and $c(h_4)$ must all be adjacent to $c(h_3) = h_3$, we have $c(w_1^v)$, $c(h_2)$, $c(h_4) \in \{h_2, h_4\}$. Since $c(w_1^v)$ must be adjacent to $c(u_1^v) = h_5$, it must be that $c(w_1^v) \neq h_2$, and hence $c(w_1^v) = h_4$. Since $c(h_2)$ must be adjacent to $c(h_1) = h_1$, it must be that $c(h_2) \neq h_4$, and hence $c(h_2) = h_2$. Since $c(h_4)$ must be adjacent to $c(h_5) = h_1$, it must be that $c(h_4) \neq h_4$, and hence $c(h_4) = h_2$. Now $c(y_2^a)$ must be adjacent to $c(h_4) = h_2$ and

$c(w_1^a) = h_4$, implying that $c(y_2^a) = h_3$, with $a \in G_A - H_A$. Also, $c(y_2^b)$ must be adjacent to $c(h_2) = h_2$ and $c(w_1^b) = h_4$, implying that $c(y_2^b) = h_3$ also, with $b \in G_B - H_B$. Thus we have, in general, $c(y_2^v) = h_3$. We also have that $c(y_1^a)$ must be adjacent to $c(h_1) = h_1$ and $c(u_1^a) = h_5$, implying that $c(y_1^a) = h_0$, with $a \in G_A - H_A$. Also, we have that $c(y_1^b)$ must be adjacent to $c(h_5) = h_1$ and $c(u_1^b) = h_5$, implying that $c(y_1^b) = h_0$ also, with $b \in G_B - H_B$. Thus we have, in general, $c(y_1^v) = h_0$. This is impossible as $c(y_1^v)$ must be adjacent to $c(y_2^v) = h_3$.

Thus $c(h_1) \neq c(h_5)$, i.e., $c(\{h_1, h_5\}) = \{h_1, h_5\}$. Without loss of generality, suppose that $c(h_1) = h_1$ and $c(h_5) = h_5$ (due to symmetry). Since $c(h_3) = h_3$, we have $c(h_2)$, $c(h_4)$ $\in \{h_2, h_4\}$. Since $c(h_2)$ must be adjacent to $c(h_1) = h_1$, it must be that $c(h_2) \neq h_4$, and hence $c(h_2) = h_2$. Since $c(h_4)$ must be adjacent to $c(h_5) = h_5$, it must be that $c(h_4) \neq h_2$, and hence $c(h_4) = h_4$. We already have $c(h_0) = h_0$ and $c(h_3) = h_3$. Thus we have $c(h_i) = h_i$, for all $i = 0, 1, 2, 3, 4, 5$. Hence $c : G' \to H$ is a retraction, proving the lemma. ◀

We have thus proved Theorem 2.1. ◀

3 Vertex-Compaction to an Irreflexive k-Cycle

Our proof of Theorem 2.1 showing NP-completeness of *VCOMP-*C_6 directly uses graphs that we construct simply by adding vertices and edges. Our technique of construction of graphs therefore lays down the foundation for construction of graphs for the case of a general irreflexive even k-cycle, by extending the paths constructed and adding edges appropriately, showing NP-completeness of *VCOMP-*C_k, for all even $k \geq 6$.

In [Vikas, 1999, 2004a], it is shown that the problem *COMP-*C_k is NP-complete, for all even $k \geq 6$. The problem *RET-*C_k is shown to be NP-complete, for all even $k \geq 6$, in [Feder, Hell, and Huang, 1999], and independently by G. Macgillivray in 1988. To prove NP-completeness of *VCOMP-*C_k, we give a transformation from *RET-*C_k to *VCOMP-*C_k, for all even $k \geq 6$. In our construction to prove NP-completeness of *VCOMP-*C_k, with even $k \geq 6$, we now have for example paths Z_{ab}, X_{ba}, U_a, U_b, W_a, W_b, Y_a, and Y_b of appropriate lengths instead of the vertices z_{ab}, x_{ba}, u_1^a, u_2^a, u_1^b, w_1^a, w_1^b, and w_2^b that we used in the construction in the proof of Theorem 2.1 for proving NP-completeness of *VCOMP-*C_6, and we add edges chosen appropriately.

Considering $k = 4$, it is easy to see that the problems *VCOMP-*C_4, *COMP-*C_4, and *RET-*C_4 are polynomial time solvable, as C_4 is a complete bipartite graph. We now consider odd $k \geq 3$. Note that a graph G is homomorphic to a graph H if and only if the disjoint union $G \cup H$ vertex-compacts, compacts, and retracts to H. Thus we have a polynomial transformation from the *H-colouring* problem to the problems *VCOMP-H*, *COMP-H*, and *RET-H*. The *H-colouring* problem is shown to be NP-complete for any fixed non-bipartite irreflexive graph H in [Hell & Nesetril 1990]. Hence, it follows that the problems *VCOMP-H*, *COMP-H*, and *RET-H* are also NP-complete when H is any non-bipartite irreflexive graph. Thus, in particular, the problems *VCOMP-*C_k, *COMP-*C_k, and *RET-*C_k are NP-complete, for all odd $k \geq 3$.

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