Induced Embeddings into Hamming Graphs*

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Abstract

Let d be a positive integer. Can a given graph G be realized in $\mathbb{R}^d$ so that vertices are mapped to distinct points, two vertices being adjacent if and only if the corresponding points lie on a common line that is parallel to some axis? Graphs admitting such realizations have been studied in the literature for decades under different names. Peterson asked in [Discrete Appl. Math., 2003] about the complexity of the recognition problem. While the two-dimensional case corresponds to the class of line graphs of bipartite graphs and is well-understood, the complexity question has remained open for all higher dimensions.

In this paper, we answer this question. We establish the NP-completeness of the recognition problem for any fixed dimension, even in the class of bipartite graphs. To do this, we strengthen a characterization of induced subgraphs of 3-dimensional Hamming graphs due to Klavžar and Peterin. We complement the hardness result by showing that for some important classes of perfect graphs – including chordal graphs and distance-hereditary graphs – the minimum dimension of the Euclidean space in which the graph can be realized, or the impossibility of doing so, can be determined in linear time.

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1 Introduction

The main question addressed in this paper is the following: How difficult is it to determine if a given graph G can be realized in $\mathbb{R}^d$ so that vertices are mapped to distinct points and two vertices are adjacent if and only if the corresponding points are on a common line that is parallel to some axis? Let us refer to any such mapping as a *d-realization* of G and say that a graph is *d-realizable* if it has a d-realization. The class of d-realizable graphs was studied in the literature for decades, under diverse names such as arrow graphs (Cook, 1974 [13]), $(d-1)$-plane graphs and $(d-1)$-line graphs of d-partite d-uniform hypergraphs (Bermond et al., 1977 [3]; see also [29]), d-dimensional cellular graphs (Gurvich and Temkin, 1992 [25]), d-dimensional chessboard graphs (Staton and Wingard, 1998 [50]), and d-dimensional gridline

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graphs (Peterson, 2003 [46]). Recently, Sangha and Zito studied d-realizable graphs in the more general context of the so-called Line-of-Sight (LoS) networks [49] and showed that the independent set problem, known to be polynomially solvable in the class of 2-realizable graphs, is NP-complete in the class of 3-realizable graphs. For the small-dimensional cases, $d \in \{2, 3\}$, Peterson suggested an application of d-realizable graphs to robotics [46]: if the movement of a robot is restricted to be along axis-parallel directions only and turns are allowable only at certain points, then a shortest path in a d-realized graph gives the number of turns required. Further possible applications of d-realizable graphs belong to the area of wireless networks, via their connection with Line-of-Sight networks [23].

Despite many studies on d-realizable graphs in the literature, determining the computational complexity of recognizing d-realizable graphs has been elusive except for $d \in \{1, 2\}$, when d-realizable graphs coincide with complete graphs and with line graphs of bipartite graphs, respectively (and can be recognized in polynomial time). The main aim of this paper is to settle the question about recognition complexity of d-realizable graphs for $d \geq 3$, asked explicitly by Peterson in 2003 [46]. We show that for all $d \geq 3$, determining if a given graph is d-realizable is NP-complete, even for bipartite graphs. We also identify some tractable cases. We characterize d-realizable graphs (for any positive integer d) in the class of HHD-free graphs, a large class of perfect graphs containing chordal graphs and distance-hereditary graphs. The characterization leads to a linear time recognition algorithm.

Our approach is based on the fact that a graph G is d-realizable if and only if G is an induced subgraph of a Cartesian product of d complete graphs. Given two graphs G and H, their *Cartesian product* is the graph $G \square H$ with vertex set $V(G) \times V(H)$ in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1v_1 \in E(G)$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2v_2 \in E(H)$. The Cartesian product is associative and commutative (in the sense that $G \square H \cong H \square G$ where $\cong$ denotes the graph isomorphism relation). Another name for Cartesian products of complete graphs is *Hamming graphs*; a Hamming graph is *d-dimensional* if it is the Cartesian product of d nontrivial complete graphs. The 3-dimensional Hamming graphs having all factors of the same size were studied in the literature under the name *cubic lattice graphs* [37, 12, 11, 1, 16], hence, 3-realizable graphs are exactly the induced subgraphs of cubic lattice graphs. Our results are based on a characterization of induced subgraphs of d-dimensional Hamming graphs due to Klavžar and Peterin [33], expressed in terms of the existence of a particular edge labeling. For the 3-dimensional case, we develop a more specific characterization based on induced cycles of the graph and use it to prove hardness of recognizing 3-realizable graphs via a reduction from the 3-edge-coloring problem in cubic graphs. The hardness of the 3-dimensional case forms the basis for establishing hardness for all higher dimensions.

Since a d-realizable graph is also $(d+1)$-realizable, the notion of d-realizability suggests a natural graph parameter. The *Cartesian dimension* of a graph $G = (V, E)$, denoted $\mathrm{Cdim}(G)$, is defined as the minimum non-negative integer d such that G is *d-realizable*, if such an integer exists, and ∞, otherwise. The infinite case can indeed occur, even some small graphs – the diamond, the 5-cycle, and the complete bipartite graph $K_{2,3}$, for example – cannot be realized in any dimension. Note that $\mathrm{Cdim}(G)$, when finite and strictly positive, is the minimum positive integer d such that G is an induced subgraph of the Cartesian product of d complete graphs. This point of view adds the Cartesian dimension of a graph to the list of graph dimensions studied in the literature related to various embeddings of graphs into Cartesian product graphs [24, 20, 27, 34]. Other dimensions were studied related to the strong product [22, 15, 32, 47] and the direct product of graphs [41, 48].

Related work. As already mentioned, concepts equivalent to d-realizable graphs were studied in the literature in various contexts [13, 3, 29, 25, 50, 46, 49]. Much further work in the literature deals exclusively with the two-dimensional case [26, 28, 14, 46, 2], which (as we will discuss in Section 2) corresponds to the class of line graphs of bipartite graphs, one of the basic building blocks in the structural decomposition of perfect graphs [10].

Among the many dimension parameters of graphs defined via product graphs, let us mention two that seem to be most closely related to the Cartesian dimension. A d-realization is said to be *irredundant* [34] (or: *d-dimensionally spanning* [49]) if for each $i \in \{1, \ldots, d\}$ some pair of adjacent vertices of G is mapped to a pair of points spanning a line that is parallel to the i-th coordinate axis. Based on this notion, Klavžar et al. [34] defined the Hamming dimension of a graph G, denoted by $\mathrm{Hdim}(G)$, as the largest integer d such that G has an irredundant d-realization, if such an integer exists, and ∞, otherwise. Note that the Cartesian dimension can be defined analogously, with "smallest" instead of "largest"; in particular, $\mathrm{Cdim}(G) \leq \mathrm{Hdim}(G)$. Strict inequality is possible (for example, if P_4 denotes the 4-vertex path, then $\mathrm{Cdim}(P_4) = 2$ and $\mathrm{Hdim}(P_4) = 3$) and the two dimensions are finite on the same set of graphs.

The second relevant dimension is a Dushnik-Miller type dimension of a graph, the so-called *product dimension*. This parameter, denoted simply by $\dim(G)$, is defined analogously to the Cartesian dimension but with respect to the direct product. Given two graphs G and H, their direct product is the graph $G \times H$ with vertex set $V(G) \times V(H)$ in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)$. The product dimension was introduced by Nešetřil and Rödl in [41] and studied by Lovász et al. in [39] and more recently by Chandran et al. [8]; see also [21]. Unlike the Cartesian dimension, the product dimension is finite for all graphs. The problem of computing the product dimension of a given graph was shown to be NP-hard [40], even in the special case of recognizing three-dimensional instances [36]. The Cartesian and the product dimensions of graphs are closely related in the two-dimensional case: since the Cartesian product of two complete graphs is isomorphic to the complement of their direct product, we have $\mathrm{Cdim}(G) \leq 2$ if and only if $\mathrm{Hdim}(\overline{G}) \leq 2$, where $\overline{G}$ denotes the complement of G.

The Cartesian dimension of graphs introduced in this paper should not be confused with any of the "Cartesian dimensions" of a graph studied by Burosch and Ceccherini [7]. They are defined similarly to the Hamming dimension $\mathrm{Hdim}(G)$ from [34], but with respect to various inclusion relations and with the relaxation that the factors are not restricted to be complete.

Structure of the paper. In Section 2 we collect the necessary definitions, summarize some characterizations of the two-dimensional case and a necessary condition for the general, d-dimensional case. In Section 3 we review a characterization of induced subgraphs of d-dimensional Hamming graphs due to Klavžar and Peterin and introduce two related results regarding the three-dimensional case. We build on these results in Section 4, where the NP-completeness of recognizing d-realizable graphs is established for all $d \geq 3$. A linear time algorithm for computing the Cartesian dimension of a given HHD-free graph is developed in Section 5, after the general problem is reduced to the biconnected case. We conclude the paper in Section 6. Due to space limitations, several proofs are omitted.

2 Preliminaries

All graphs considered in this paper will be finite, simple and undirected. By K_n, P_n, and C_n we denote the complete graph, the path, and the cycle with n vertices. By $K_{m,n}$ we denote the complete bipartite graph with parts of sizes m and n; the *claw* is the graph $K_{1,3}$. A *clique* (resp., *independent set*) in a graph G is a set of pairwise adjacent (resp., pairwise non-adjacent) vertices. By $\alpha(G)$ we denote the independence number of G, that is, the maximum size of an independent set in G. A *triangle* in G is a clique of size 3 in G. The *diamond* is the graph obtained by removing an edge from a K_4. For a vertex v in G, the *neighborhood* of v is the set of vertices in G adjacent to v. It is denoted by $N_G(v)$ (or simply by $N(v)$ if the graph will be clear from the context). The *degree* of v (in G) is the size of its neighborhood. A graph is *cubic* if all its vertices have degree 3. The *girth* of a graph G is the length of the shortest cycle in G (and ∞ if G is acyclic). Given a graph G and a set $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of G induced by U. Given a set of graphs $\mathcal{F}$, a graph G is said to be $\mathcal{F}$*-free* if no induced subgraph of G is isomorphic to a graph from $\mathcal{F}$. A *cut vertex* in a connected graph G is a vertex whose removal disconnects the graph. Given a graph G, a *block* of G is a maximal connected subgraph of G without cut vertices. A graph G is *biconnected* if G itself is its only block. The disjoint union of two graphs G and H is denoted by $G + H$. For graph theoretic terms not defined here, see, e.g., [51].

Given a positive integer d, a d*-realization* of a graph $G = (V, E)$ is an injective mapping $\varphi_G : V \to \mathbb{R}^d$ such that two vertices $u, v \in V$ are adjacent if and only if $\varphi_G(u)$ and $\varphi_G(v)$ differ in exactly one coordinate. A graph G is said to be d*-realizable* if it has a d-realization. Note that G is d-realizable if and only if G has a d-realization $\varphi_G : V \to \mathbb{N}^d$. The *Cartesian dimension* of a graph $G = (V, E)$, denoted $\mathrm{Cdim}(G)$, is defined as the minimum non-negative integer d such that G is d*-realizable*, if such an integer exists, and ∞, otherwise. (Note that K_1 is the only graph of Cartesian dimension 0.)

Clearly, the only graphs of Cartesian dimension 1 are complete graphs of order at least two. Graphs of Cartesian dimension at most 2 coincide with line graphs of bipartite graphs, for which various characterizations and linear time recognition algorithms are known. Recall that a graph G is said to be *bipartite* it has a *bipartition*, that is, a pair (X, Y) of disjoint independent sets such that $X \cup Y = V(G)$. The *line graph* of a graph G is the graph denoted by $L(G)$ with vertex set $E(G)$, in which two distinct vertices are adjacent if and only if they have a common endpoint as edges in G. Line graphs of bipartite graphs were studied in the literature under various names such as graphs of $(0,1)$-matrices [28], matrix graphs [14], two-dimensional chessboard graphs [50], (two-dimensional) gridline graphs [46], cellular graphs [25], and rooks graphs [2]. The characterization of line graphs of bipartite graphs in terms of forbidden induced subgraph was discovered and rediscovered many times: by Chartrand in 1964 [9], by Hedetniemi in 1971 [28], by Harary and Holzman in 1974[26], by Staton and Wingard in 1998 [50], and by Peterson in 2003 [46]. Furthermore, Staton and Wingard [50] and Peterson [46] established the connection with the Cartesian dimension. These characterizations are summarized in the following theorem.

▶ **Theorem 1.** *For every graph G, the following conditions are equivalent:*

1. $\mathrm{Cdim}(G) \leq 2$.
2. *G is the line graph of a bipartite graph.*
3. *G is $\{$claw, diamond, C_5, $C_7, \ldots\}$-free.*

For any positive integer d, Staton and Wingard proved the following necessary condition for a graph to be d-realizable.

▶ **Theorem 2** (Staton and Wingard [50]). *Every d-realizable graph is $\{K_{1,d+1}$, diamond, $K_{2,3}, C_5\}$-free.*

Staton and Wingard asked whether for $d \geq 3$, the list of forbidden induced subgraphs for the class of d-realizable graphs given by Theorem 2 is complete. This is not the case: Peterson constructed an infinite family of graphs that are minimally forbidden for d-realizability for all $d \geq 3$ [46, Figure 4] (see also [45]). However, the complete list of forbidden induced subgraphs is not known for any $d \geq 3$.

3 The Klavžar-Peterin characterization

In this section, we recall the characterization of induced subgraphs of d-dimensional Hamming graphs due to Klavžar and Peterin [33] and strengthen it in the 3-dimensional case. The characterization is expressed in terms of the existence of a particular edge labeling. Given a graph G, a *d-edge-labeling of G* is a mapping from $E(G)$ to some set L of *labels*, where $|L| = d$ (we often have $L = \{1, \ldots, d\}$). Given a d-edge-labeling ℓ of G and a set $F \subseteq E(G)$, we say that F is *ℓ-monochromatic* (or simply *monochromatic* if the labeling is clear from the context) if the labeling is constant on F, that is, if $e, e' \in F$ implies $\ell(e) = \ell(e')$. We extend the definition of monochromaticity to subgraphs of G in the obvious way. A *(d-)edge-coloring* is a (d-)edge-labeling such that no two incident edges share the same label. In the case of edge-colorings, labels may also be referred to as *colors*.

We say that a d-edge-labeling of G is a *(d-)KP-labeling* if it satisfies the following two conditions:

- **Condition 1**: every triangle is monochromatic.
- **Condition 2**: for every pair of distinct non-adjacent vertices u, v, there exist different labels i and j which both appear on every induced u,v-path.

Note that in a KP-labeling, every induced P_3 will be 2-edge-colored due to Condition 2; in particular, this implies that for triangle-free graphs, KP-labelings coincide with edge-colorings.

Since induced subgraphs of Hamming graphs are exactly the graphs of finite Cartesian dimension, the result of Klavžar and Peterin given by [33, Theorem 3.3] can be equivalently stated as follows.

▶ **Theorem 3** (Klavžar and Peterin [33]). *Let G be a connected graph. Then* $\mathrm{Cdim}(G) < \infty$ *if and only if G has a KP-labeling.*

The proof of Theorem 3 given in [33] actually shows the following more specific equivalence:

▶ **Theorem 4.** *For every connected graph G and a positive integer d, we have* $\mathrm{Cdim}(G) \leq d$ *if and only if G has a d-KP-labeling.*

We can find d-realizations of two graphs G and H such that $\mathrm{Cdim}(G) \leq \mathrm{Cdim}(H) = d$ when $d > 1$, using d-tuples over disjoint sets for the two graphs. The case $d = 1$ is exceptional: by definition, two different 1-tuples result in a pair of adjacent vertices. Thus, as all graphs of Cartesian dimension 1 are complete, the Cartesian dimension of any disconnected graph is at least 2. We record these observations for later use.

▶ **Observation 5.** *For every two graphs G and H, we have* $\mathrm{Cdim}(G + H) = \max\{\mathrm{Cdim}(G), \mathrm{Cdim}(H), 2\}$.

We now present two results for the 3-dimensional case. Both are related to the Klavžar-Peterin characterization and will be needed in our hardness proof for recognizing 3-realizable graphs developed in Section 5. First, we show that the defining properties of a 3-KP-labeling are satisfied for a graph as soon as they are satisfied for the family of all its induced subgraphs isomorphic to a cycle or to a P_3.

▶ **Theorem 6.** *Let G be a graph. A 3-edge-labeling of G is a KP-labeling if and only if it satisfies the following two conditions:*

- **Condition 3***: for every induced cycle C of G, the restriction of the labeling to $E(C)$ is a KP-labeling of C.*
- **Condition 4***: no induced P_3 is monochromatic.*

Proof. The necessity of the two conditions is easy to see. If G is 3-KP-labeled and H is an induced subgraph of G, then the restriction of the labeling to $E(H)$ is a 3-KP-labeling of H, hence Condition 3 is necessary. Condition 4 follows from Condition 2.

In order to prove sufficiency, note that Condition 3 immediately implies Condition 1. Now, by way of contradiction suppose that there is a 3-edge-labeling $\ell : E(G) \to \{1, 2, 3\}$ satisfying Conditions 3 and 4, but not Condition 2. Since G violates Condition 2, it contains two different induced paths of length at least two, say P and Q, intersecting at their endpoints – call these vertices u and v – such that no pair of different labels appears on both P and Q. Due to Condition 4, on each of the paths P and Q at least two different labels appear. Since no pair of different labels appears on both P and Q, we may assume that P and Q take – alternatingly – labels $1, 2$ and $1, 3$, respectively. Moreover, assume that

(∗) P and Q were chosen so as to minimize $|V(P)| + |V(Q)|$.

Given a path R and two of its vertices x and y, denote by R_{xy} the subpath of R between x and y, and by V_R^{-xy} the set $V(R) \setminus \{x, y\}$. We say that a path is *k-labeled* if exactly k different labels appear on its edges.

We claim that $V_P^{-uv} \cap V_Q^{-uv} = \emptyset$. Indeed, suppose for a contradiction that $w \in V_P^{-uv} \cap V_Q^{-uv}$. Then, P_{uw} and Q_{uw} would be both 2-labeled (u and w cannot be adjacent due to (∗)), only agreeing on label 1; thus, $P_{uw} \cup Q_{uw}$ would be 3-labeled, contradicting (∗).

For $t \in \{u, v\}$ and $xy \in E(G)$ with $(x, y) \in V_P^{-uv} \times V_Q^{-uv}$, a cycle $C = P_{tx}\text{-}xy\text{-}Q_{yt}$ such that either $P_{tx}\text{-}xy$ or $xy\text{-}Q_{yt}$ is an induced path will be called a *PQ-cycle.* Note that a PQ-cycle cannot be 3-labeled: if – say – $P' = P_{tx}\text{-}xy$ was an induced path, then P' and Q_{yt} would make evident a violation to (∗).

Condition 3 implies that the cycle $C_0 = P \cup Q$ cannot be induced. Let xy be a chord in C_0 ($\{x, y\} \cap \{u, v\} = \emptyset$) such that $x \in V(P)$ is closest to u (where the distance is measured within P), and y is the neighbor of x in Q closest to v (where the distance is measured within Q). Observe that each of $C_1 = P_{ux}\text{-}xy\text{-}Q_{yu}$ and $C_2 = P_{vx}\text{-}xy\text{-}Q_{yv}$ is either a PQ-cycle or a triangle, implying that neither of them is 3-labeled. Neither of them can be monochromatic either: if – say – C_1 was monochromatic then, as $E(C_0) \subset E(C_1) \cup E(C_2)$ while C_1 and C_2 share the label of xy, it would follow that C_2 was 3-labeled. Thus, C_1 and C_2 are 2-labeled.

As C_1 and C_2 are 2-labeled, they share exactly one label. By definition, any PQ-cycle contains a P_3 from either P or Q, hence (recalling that P and Q alternate labels $1, 2$ and $1, 3$, respectively), C_1 and C_2 share label 1. Such is then the label of xy. However, one of the two edges incident to x in P is also labeled with 1, forming with xy a monochromatic induced P_3 (as part of either C_1 or C_2), which contradicts Condition 4. ◀

Next, we characterize 3-KP-labelings of cycles. By Condition 1 in the definition of a KP-labeling, every 3-KP-labeling of a 3-cycle is constant. The next lemma analyzes longer cycles.

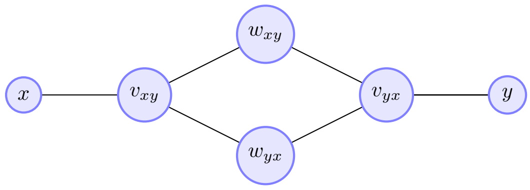

Figure 1 A gadget replacing each edge xy.

▶ **Lemma 7.** *Let C be a cycle of length at least 4. A 3-edge-coloring of C with colors $1, 2, 3$ is a KP-labeling if and only if*

- *either it is a 2-edge-coloring of C, or*
- *possibly after permuting the labels $1, 2, 3$, cycle C contains a cyclically ordered sequence of 6 distinct (not necessarily consecutive) edges labeled $1, 2, 3, 1, 2, 3$, respectively. We call this the* ***123123-condition****.*

4 NP-completeness of testing realizability in $d \geq 3$ dimensions

In this section, we show that for every $d \geq 3$, determining whether $\mathrm{Cdim}(G) \leq d$ is NP-complete. First we establish the result for $d = 3$ and then derive from it the general case.

▶ **Theorem 8.** *Given a graph G, determining whether $\mathrm{Cdim}(G) \leq 3$ is NP-complete, even for connected bipartite graphs of maximum degree at most 3.*

Proof. A polynomially checkable certificate of the fact that $\mathrm{Cdim}(G) \leq 3$ is any 3-realization of G of the form $\varphi_G : V \to \mathbb{N}^3$. Therefore, the problem is in NP (on any class of input graphs).

To show hardness, we make a reduction from the 3-edge-coloring problem in cubic graphs, proved to be NP-complete by Holyer [30]. Let G be a cubic graph that is the input for the 3-edge-coloring problem. We may assume that G is connected. Construct a graph G' from G by replacing each edge xy of G with the structure shown in Fig. 1. Formally,

$$\begin{aligned} V(G') &= V(G) \cup \bigcup_{xy \in E(G)} \{v_{xy}, w_{xy}, w_{yx}, v_{yx}\}\,, \\ E(G') &= \bigcup_{xy \in E(G)} \{xv_{xy}, v_{xy}w_{xy}, v_{xy}w_{yx}, v_{yx}w_{xy}, v_{yx}w_{yx}, yv_{yx}\}\,. \end{aligned}$$

Letting $V_1 = V(G) \cup \bigcup_{xy \in E(G)} \{w_{xy}, w_{yx}\}$ and $V_2 = \bigcup_{xy \in E(G)} \{v_{xy}, v_{yx}\}$, we see that (V_1, V_2) is a bipartition of G'. Thus, G' is a bipartite graph with vertices of degrees 2 and 3 only. We will show that G is 3-edge-colorable if and only if $\mathrm{Cdim}(G') \leq 3$.

We first prove the (simpler) backward direction. Let $\mathrm{Cdim}(G') \leq 3$. By Theorem 4, G' has a 3-KP-labeling. Then for each $xy \in E(G)$ the 4-cycle $C = v_{xy}\text{-}w_{xy}\text{-}v_{yx}\text{-}w_{yx}\text{-}v_{xy}$ in G' must be 2-KP-labeled. This implies that the edges xv_{xy} and yv_{yx} must have the same label ℓ_{xy} – the one not used in C. Since G' is triangle-free, any KP-labeling of G' is an edge-coloring (otherwise, Condition 4 would be violated). Therefore, by labeling each edge $xy \in E(G)$ with ℓ_{xy}, we get a 3-edge-coloring of G.

Now suppose that G has a 3-edge-coloring using colors $1, 2, 3$. For each edge xy of G labeled $i \in \{1, 2, 3\}$, let $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$ and label the associated edges of G' as follows: edges xv_{xy} and yv_{yx} with i, edges $v_{xy}w_{xy}$ and $v_{yx}w_{yx}$ with j, and edges $v_{xy}w_{yx}$ and $v_{yx}w_{xy}$ with k.

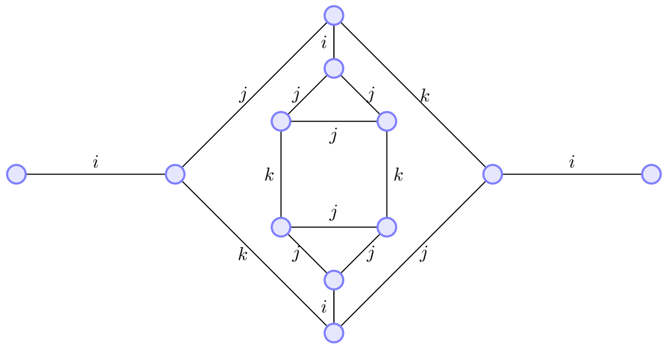

Figure 2 A gadget replacing each edge for proving hardness in cubic graphs.

We claim that the so obtained labeling of G' is a KP-labeling. By Theorem 6, it suffices to check that Conditions 3 and 4 are satisfied. The latter condition is obviously satisfied.

In order to verify that Condition 3 holds, note that G' has two types of induced cycles:

- 4-cycles. They only appear in the gadget of Fig. 1; they are properly 2-edge-colored and hence KP-labeled by Lemma 7.
- Cycles of length greater than 4. Each such cycle C has length $4p$ for some $p \geq 3$, and arises from a (not necessarily induced) p-cycle C' in G. We will show that such cycles satisfy the 123123-condition and apply Lemma 7. Let $x_1, x_2, \ldots, x_p$ be a cyclic order of vertices in C'. Without loss of generality, let $1, 2, 3, 1$ be the labels (in this order) on some shortest path from x_1 to x_2 in C. Then, the sequence of labels on the edges of any shortest path from x_2 to x_3 in C is one of the following: $(2, 1, 3, 2)$, $(2, 3, 1, 2)$, $(3, 1, 2, 3)$, or $(3, 2, 1, 3)$. Thus, along cycle C we find 6 distinct edges labeled $1, 2, 3, 1, 2, 3$ in order. This shows that C satisfies the 123123-condition.

It follows that Condition 3 is satisfied, hence by Theorem 6 G' has a 3-KP-labeling. By Theorem 4, we conclude that $\mathrm{Cdim}(G') \leq 3$. ◀

▶ Remark. A simple modification of the above construction, using a somewhat more involved gadget, can be used to show NP-completeness of testing whether $\mathrm{Cdim}(G) \leq 3$ for cubic (non-bipartite) graphs. We omit the details but show the gadget in Fig. 2 together with edge labels indicating how to extend a 3-edge-coloring of G to a 3-KP-labeling of G'.

▶ Remark. Recall that Peterson constructed an infinite family of graphs that are minimally forbidden for 3-realizability [46]. All those graphs are of girth 3. The above proof implies that the landscape of forbidden induced subgraphs for 3-realizability is much more complicated, consisting of graphs of arbitrarily large girth. To see this, note that for every positive integer g, there exists a graph F_g of maximum degree at most 3 and of girth at least g with $\mathrm{Cdim}(F_g) > 3$. This follows from the proof of Theorem 8 and the fact that there exist cubic graphs of arbitrarily large girth that are not 3-edge-colorable [35]. Since $\mathrm{Cdim}(F_g) > 3$, graph F_g contains a forbidden induced subgraph for 3-realizability, say F'_g. Since every acyclic graph of maximum degree at most 3 is 3-realizable (this follows, e.g., from Corollary 15 in Section 5.2), graph F'_g has a cycle and is therefore of (finite) girth at least g.

From Theorem 8 we derive hardness of recognizing graphs of any constant Cartesian dimension.

▶ **Theorem 9.** *For every $d \geq 3$, determining whether a given graph G satisfies $\mathrm{Cdim}(G) \leq d$ is NP-complete, even for connected bipartite graphs.*

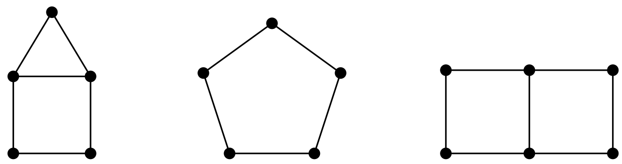

Figure 3 The house (left), the smallest hole (middle), and the domino (right).

Proof idea. The base case, $d = 3$, is given by Theorem 8. The inductive step can be established using the observation that for every connected bipartite graph G, the Cartesian product $G\square K_2$ is also connected and bipartite, and satisfies $\mathrm{Cdim}(G\square K_2) = \mathrm{Cdim}(G)+1$. ◀

5 Tractable cases: chordal graphs and distance-hereditary graphs

Since bipartite graphs are perfect, Theorem 8 implies that the problem of recognizing graphs of Cartesian dimension 3 is NP-complete in the class of perfect graphs. In this section, we show that the problem can be solved in linear time in two well-studied classes of perfect graphs: chordal graphs and distance-hereditary graphs. A graph G is *chordal* if it has no induced cycle of length at least four and *distance-hereditary* if in every connected induced subgraph of G, the distance between any pair of vertices is the same as in G. We characterize chordal graphs and distance-hereditary graphs of given Cartesian dimension. The characterizations will imply linear time algorithms for computing the Cartesian dimension of a given chordal or distance-hereditary graph.

We develop a unified approach that will imply both results, by considering the class of HHD-free graphs. We define a *hole* to be a cycle of length at least five.[1] A graph G is said to be *HHD-free* if it does not contain an induced subgraph isomorphic to the house, a hole, or the domino (see Fig. 3).

HHD-free graphs were introduced by Olariu [44] as a class of perfect graph generalizing both chordal and distance-hereditary graphs. They can be equivalently defined as the *(5,2)-chordal graphs*, that is, graphs in which every cycle of length at least five has at least two chords (see, e.g., [4]). Jamison and Olariu [31] characterized HHD-free graphs in terms of properties of the Lexicographic Breadth First Search algorithm, and Nikolopoulos and Palios gave an $O(|V(G)||E(G)|)$ time recognition algorithm [43]. Many other studies looked into metric, structural, and algorithmic properties of HHD-free graphs (see, e.g., [18, 6, 19, 42, 19, 17, 5]).

We characterize HHD-free graphs of a given Cartesian dimension and derive the corresponding results for chordal and distance-hereditary graphs as corollaries. We do this by first showing that the problem of computing the Cartesian dimension of an arbitrary graph can be reduced to its blocks (Lemma 10), and by identifying two particularly nice cases of this reduction (Lemmas 11 and 12). Next, we characterize biconnected HHD-free graphs of a given Cartesian dimension. To this end, we apply the necessary conditions for graphs of finite Cartesian dimension given by Theorem 2 to reduce the problem to the biconnected $\{$diamond, $K_{2,3}\}$-free HHD-free graphs, which we characterize in Lemma 13. Finally, the simple structure of the biconnected $\{$diamond, $K_{2,3}\}$-free HHD-free graphs (they can only be complete or 4-cycles) is used to prove the desired characterization (Theorem 14) and a linear time algorithm for computing the Cartesian dimension of an HHD-free graphs (Theorem 16).

[1] We remark that the terminology on holes is not completely uniform in the graph theory literature. In many papers, holes are defined as cycles of length at least four.

5.1 Reduction to blocks

For a graph G and a vertex $v \in V(G)$, we set $\alpha_G(v) = \alpha(G[N(v)])$ and $\alpha_1(G) = \max\{\alpha_G(v) : v \in V(G)\}$. Note that $\alpha_1(G)$ is the maximum value of n such that $K_{1,n}$ is an induced subgraph of G. Hence, by Theorem 2, every graph G has $\mathrm{Cdim}(G) \geq \alpha_1(G)$. The following lemma specifies the reduction for the problem of computing the Cartesian dimension of a given graph to the biconnected case.

▶ **Lemma 10.** *Let G be a connected graph with a cut vertex v, let $V(G) = \{v\} \cup V_1 \cup V_2$ where V_1 and V_2 are disjoint non-empty subsets of $V(G) \setminus \{v\}$ such that no vertex from V_1 is adjacent to a vertex in V_2, and let $G_i = G[\{v\} \cup V_i]$ for $i \in \{1, 2\}$. Then,* $\mathrm{Cdim}(G) = \max\{\mathrm{Cdim}(G_1), \mathrm{Cdim}(G_2), \alpha_G(v)\}$.

Lemma 10 has two useful consequences. For a connected graph G, we denote by C_G the set of cut vertices of G and by $\mathcal{B}_G$ the set of blocks of G. The *block-cutpoint tree* of a connected graph G is the bipartite graph T with vertex set $\mathcal{B}_G \cup C_G$ in which $B \in \mathcal{B}_G$ is adjacent to $v \in C_G$ if and only if $v \in V(B)$. It is well known that T is a tree such that all leaves of T are blocks of G (see, e.g., [51]). A class of graphs is *hereditary* if it is closed under vertex deletion. We say that a graph G is *maxstar-dimensional* if $\mathrm{Cdim}(G) = \alpha_1(G)$.

▶ **Lemma 11.** *Let $\mathcal{G}$ be a hereditary class of graphs such that every biconnected graph in $\mathcal{G}$ is maxstar-dimensional. Then every connected graph in $\mathcal{G}$ is maxstar-dimensional.*

Let us call a graph *star-dimensional* if $\mathrm{Cdim}(G) = \alpha_G(v)$ for every $v \in V(G)$.

▶ **Lemma 12.** *Let $\mathcal{G}$ be a hereditary class of graphs such that every biconnected graph in $\mathcal{G}$ is star-dimensional. Then, every connected graph $G \in \mathcal{G}$ with a cut vertex satisfies*

$$\mathrm{Cdim}(G) = \max_{v \in C_G} \sum_{B \in \mathcal{B}_G : v \in B} \mathrm{Cdim}(B)\,.$$

5.2 Cartesian dimension of HHD-free graphs

The following lemma characterizes biconnected {diamond, $K_{2,3}$}-free HHD-free graphs. In the proof we will need the notion of a *block graph*, that is, a connected graph every block of which is complete.

▶ **Lemma 13.** *Let G be a biconnected {diamond, $K_{2,3}$}-free HHD-free graph. Then, G is either complete or a C_4.*

Proof. Let G be a biconnected {diamond, $K_{2,3}$}-free HHD-free graph. Consider first the case when G is chordal. Since connected diamond-free chordal graphs are exactly the block graphs (see, e.g., [38]), G is a block graph. Thus, since G is biconnected, it is complete.

Suppose now that G is not chordal. Since G has no induced cycles of length 5 or more but is not chordal, G has an induced C_4, say C. We want to show that $G = C$. First, note that every vertex of G not in C has at most one neighbor in C. Indeed, the neighborhood on C of a vertex $v \in V(G) \setminus V(C)$ consisting of at least three neighbors, exactly two neighbors that are adjacent, or exactly two neighbors that are non-adjacent, would lead to an induced subgraph of G isomorphic to an diamond, a house, or a $K_{2,3}$, respectively.

Let (v_1, v_2, v_3, v_4) be a cyclic order of vertices along C and let U denote the set of vertices in $V(G) \setminus V(C)$ adjacent to a vertex of C. Since every vertex in U has exactly one neighbor in C, the set U can be partitioned into pairwise disjoint sets, $U = U_1 \cup U_2 \cup U_3 \cup U_4$, where U_i is the set of vertices in U adjacent to v_i. Note that since G is domino-free, no vertex in

U_i is adjacent to a vertex in U_{i+1} (indices modulo 4). Moreover, since G is C_5-free, no vertex in U_i is adjacent to a vertex in U_{i+2} (indices modulo 4). Thus, if $i \neq j$, then no vertex in U_i is adjacent to a vertex in U_j.

Suppose for a contradiction that $G \neq C$. Since G is connected, the fact that $G \neq C$ implies that one of the sets U_i is non-empty, say (w.l.o.g.) $U_1 \neq \emptyset$. Let $X = V(C) \setminus \{v_1\}$. Since G is biconnected, it contains a U_1, X-path avoiding v_1. Let P be a shortest such path. Let us enumerate the vertices of P along the path as $w_1, \ldots, w_k$ where $w_1 \in U_1$ and $w_k \in X$, more specifically, $w_k = v_i$ for some (unique) $i \in \{2, 3, 4\}$. By minimality, P is an induced path in $G - v_1$; moreover, since there are no edges connecting a vertex in U_1 with a vertex in U_j for $j \neq 1$, we infer that $k \geq 4$. By the minimality of P, no internal vertex of P is in $U_1 \cup X$, moreover, $w_{k-1} \in U_i$ and $V(P) \cap (U_2 \cup U_3 \cup U_4) = \{w_{k-1}\}$. It follows that w_1 and possibly w_k are the only neighbors of v_1 on P. Now, if $i \in \{2, 4\}$, then $V(P) \cup \{v_1\}$ induces a cycle of length at least five in G, which is not possible. Similarly, if $i = 3$, then $V(P) \cup \{v_1, v_2\}$ induces a cycle of length at least six in G, again a contradiction. This shows that $G = C$, as claimed, and completes the proof. ◀

It is not difficult to verify that every graph $G \in \{C_4\} \cup \{K_n : n \geq 1\}$ is star-dimensional, with

$$\mathrm{Cdim}(G) = \alpha_1(G) = \begin{cases} 0, & \text{if } G \text{ is a } K_1; \\ 1, & \text{if } G \text{ is a } K_n \text{ with } n \geq 2; \\ 2, & \text{if } G \text{ is a } C_4. \end{cases}$$

By Lemma 13, every biconnected {diamond, $K_{2,3}$}-free HHD-free graph is star-dimensional.

Recall that the inequality $\mathrm{Cdim}(G) \geq \alpha_1(G)$ holds for every graph G, where $\alpha_1(G)$ is the maximum value of n such that $K_{1,n}$ is an induced subgraph of G. Lemmas 11 and 13 imply that equality holds in the case of HHD-free graphs of finite Cartesian dimension.

▶ **Theorem 14.** *For every connected HHD-free graph G,*

$$\mathrm{Cdim}(G) = \begin{cases} \alpha_1(G), & \text{if } G \text{ is } \{\text{diamond}, K_{2,3}\}\text{-free;} \\ \infty, & \text{otherwise.} \end{cases}$$

Since the house, the domino, and each hole contain an induced cycle of length at least four, every chordal graph is HHD-free. Every distance-hereditary graph is also HHD-free; in fact, distance-hereditary graphs are known to be exactly the gem-free HHD-free graphs (see, e.g. [4]), where the *gem* is the graph obtained from the four-vertex path by adding to it a universal vertex. Theorem 14 therefore implies the following result.

▶ **Corollary 15.** *If a connected graph G is chordal or distance-hereditary, then*

$$\mathrm{Cdim}(G) = \begin{cases} \alpha_1(G), & \text{if } G \text{ is } \{\text{diamond}, K_{2,3}\}\text{-free;} \\ \infty, & \text{otherwise.} \end{cases}$$

Observation 5 and Lemma 12 imply a linear time algorithm for computing the Cartesian dimension of a given HHD-free graph. We summarize its pseudocode in Algorithm 1 below and prove its correctness in Theorem 16.

▶ **Theorem 16.** *Algorithm 1 runs in time $O(|V(G)| + |E(G)|)$ and correctly computes the Cartesian dimension of a given HHD-free graph G (in particular, G may be a chordal graph or a distance-hereditary graph).*

Algorithm 1: Computing the Cartesian dimension of an HHD-free graph

Input: An HHD-free graph $G = (V, E)$.
Output: The value of $\mathrm{Cdim}(G)$.

compute the connected components $G_1, \ldots, G_r$ of G;
if $r > 1$ **then**
  run the algorithm recursively on each component of G;
  return $\max\{\max_{1 \le i \le r} \mathrm{Cdim}(G_i), 2\}$;
`// from now on,` G `is connected`
compute T, the block-cutpoint tree of G, C_G, the set of cut vertices of G, and $\mathcal{B}_G$, the set of its blocks;
if *G has a block that is not complete or a C_4* **then**
  return ∞;
`// from now on, each block of` G `is either complete or a` C_4
foreach $B \in \mathcal{B}_G$ **do**
  $$\mathrm{Cdim}(B) \leftarrow \begin{cases} 0, & \text{if } |V(B)| = 1 \\ 1, & \text{if } |V(B)| \ge 2 \text{ and } B_i \text{ is complete} \\ 2, & \text{if } B \text{ is a } C_4; \end{cases}$$
if $|\mathcal{B}_G| = 1$ **then**
  let $B \in \mathcal{B}_G$ and **return** $\mathrm{Cdim}(B)$;
foreach $v \in \mathcal{C}_G$ **do**
  $\alpha_G(v) \leftarrow \sum_{B \in \mathcal{B}_G : v \in B} \mathrm{Cdim}(B)$;
return $\max_{v \in C_G} \alpha_G(v)$;

▶ Remark. Lemma 10 determines how to efficiently combine KP-labelings of the blocks of a given graph G into a KP-labeling of G. Moreover, the proof of [33, Theorem 3.3] shows that a d-realization of a given d-KP-labeled graph can be computed in polynomial time. Hence, there exists a polynomial time algorithm that takes as input an HHD-free graph G of finite Cartesian dimension and outputs a d-realization of G where $d = \mathrm{Cdim}(G)$.

6 Conclusion

The main contribution of the present work is settling the computational complexity status of recognizing d-realizable graphs for any $d \ge 3$, answering thereby a question by Peterson from 2003. While the hardness result is valid already for the class of bipartite graphs, we identified an important class of perfect graphs for which the problem is solvable in linear time – the class of HHD-free graphs. Besides the question of identifying further graph classes where the problem of d-realizability is (in)tractable, the main question left open by this work is to determine the complexity status of the problem of deciding if a given graph G is d-realizable for some d (or, equivalently, whether its Cartesian dimension is finite). It would also be interesting to study in more detail the relation between the Cartesian and the Hamming dimensions of a graph, as both parameters can be defined in terms of the set of integers d such that the graph has an irredundant d-realization.

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