Eilenberg Theorems for Free

Henning Urbat[*,1], Jiří Adámek[*,2], Liang-Ting Chen[3], and Stefan Milius[†,4]

1 Institut für Theoretische Informatik, Technische Universität Braunschweig, Germany
2 Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University in Prague, Czech Republic
3 Department of Computer Science, Swansea University, United Kingdom
4 Lehrstuhl für Theoretische Informatik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany

──── **Abstract** ────

Eilenberg-type correspondences, relating varieties of languages (e.g., of finite words, infinite words, or trees) to pseudovarieties of finite algebras, form the backbone of algebraic language theory. We show that they all arise from the same recipe: one models languages and the algebras recognizing them by monads on an algebraic category, and applies a Stone-type duality. Our main contribution is a variety theorem that covers e.g. Wilke's and Pin's work on ∞-languages, the variety theorem for cost functions of Daviaud, Kuperberg, and Pin, and unifies the two categorical approaches of Bojańczyk and of Adámek et al. In addition we derive new results, such as an extension of the local variety theorem of Gehrke, Grigorieff, and Pin from finite to infinite words.
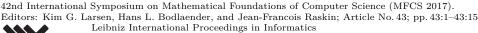
1 Introduction

Algebraic language theory studies the behaviors of machines by relating them to algebraic structures. This has proved extremely fruitful. For example, regular languages can be described as the languages recognized by finite monoids, and the decidability of star-freeness rests on Schützenberger's theorem [30]: a regular language is star-free iff it is recognized by a finite aperiodic monoid. At the heart of algebraic language theory are results establishing generic correspondences of this kind. The prototype is Eilenberg's variety theorem [12], which states that *varieties of languages* (classes of regular languages closed under boolean operations, derivatives, and homomorphic preimages) correspond to *pseudovarieties of monoids* (classes of finite monoids closed under quotients, submonoids, and finite products). This together with Reiterman's theorem [25], stating that pseudovarieties of monoids can be specified by profinite equations, establishes a firm connection between automata, languages, and algebras.

Inspired by Eilenberg's work, over the past four decades numerous further variety theorems were discovered for regular languages [14,20,24,31], treating notions of varieties with modified closure properties, but also for machine behaviors beyond finite words, including weighted languages over a field [26], infinite words [21,35], words on linear orderings [5,6], ranked

trees [4], binary trees [29], and cost functions [11]. This plethora of structurally similar results has raised interest in category-theoretic approaches which allow to derive all the above results as special instances of only *one* general variety theorem. The first step in this direction was achieved in our previous work [1–3, 10]: there we replaced monoids by monoid objects in a category $\mathcal{D}$ of (ordered) algebras such as sets, posets, semilattices or vector spaces, and proved a variety theorem for $\mathcal{D}$-monoids that subsumes five different Eilenberg theorems for regular languages [12, 20, 24, 26, 31]. Subsequently, Bojańczyk [8] took an orthogonal approach: he keeps the category of sets but considers, in lieu of monoids, algebras for a *monad* on sorted sets as recognizing structures, improving earlier generic results from [4, 28].

In order to obtain the desired general variety theorem, a unification of the two approaches is required. On the one hand, one needs to take the step from sets to more general categories $\mathcal{D}$ to capture the proper notion of language recognition; for example, for the treatment of weighted languages [26] one needs to work over the category of vector spaces. On the other hand, to deal with machine behaviors beyond finite words, one has to replace monoids by other algebraic structures. The main contribution of this paper is a variety theorem that achieves the desired unification, and in addition directly encompasses many Eilenberg-type results captured by neither of the previous generic results, including the work [5, 6, 11, 21, 29, 35].

Traditionally, Eilenberg-like correspondences are proved in essentially the same four steps:

1. Identify an algebraic theory T such that the languages in mind are the ones recognized by finite T-algebras. For example, for regular languages one takes the theory of monoids.
2. Describe the syntactic T-algebras, i.e. the minimal algebraic recognizers of languages.
3. Infer the form of the derivatives under which varieties of languages are closed.
4. Establish the Eilenberg correspondence between varieties of languages and pseudovarieties of algebras by relating the languages of a variety to their corresponding syntactic algebras.

The key insight provided by our paper is that all steps can be simplified or even automated.

For Step 1, putting a common roof over Bojańczyk's and our own previous work, we consider an algebraic category $\mathcal{D}$ and algebras for a *monad* $\mathbf{T}$ on $\mathcal{D}^S$, the category of S-sorted $\mathcal{D}$-algebras for some finite set S of sorts. For example, to capture regular languages one takes the free-monoid monad $\mathbf{T}\Sigma = \Sigma^*$ on **Set**. For regular ∞-languages one takes the monad $\mathbf{T}(\Sigma, \Gamma) = (\Sigma^+, \Sigma^\omega + \Sigma^* \times \Gamma)$ on **Set**^{2} representing ω-semigroups. For weighted languages over a finite field K, ones takes the free K-algebra monad $\mathbf{T}$ on the category of vector spaces.

For Steps 2 and 3, we develop our main technical tool, the concept of a *unary presentation* of a monad. A unary presentation expresses in categorical terms how to present $\mathbf{T}$-algebras by unary operations; for example, a monoid M is presented by the translations $x \mapsto yx$ and $x \mapsto xy$ for $y \in M$. It turns out that unary presentations are, in a precise sense, necessary and sufficient for constructing syntactic algebras (Theorem 4.7). This clarifies the role of syntactic algebras in earlier work on Eilenberg-type theorems, and the nature of derivatives appearing in varieties of languages. In our paper, syntactic algebras are *not* used for proving the variety theorem: we rely solely on the more elementary concept of a unary presentation.

We emphasize that, in general, nontrivial work lies in finding a good unary presentation for a monad $\mathbf{T}$. However, our work here shows that then Steps 3 and 4 are entirely generic: after choosing a unary presentation, we obtain a notion of *variety of $\mathbf{T}$-recognizable languages* (involving a notion of derivatives directly inferred from on the presentation) and the following

Variety Theorem. *Varieties of $\mathbf{T}$-recognizable languages are in bijective correspondence with pseudovarieties of $\mathbf{T}$-algebras.*

The proof has two main ingredients. The first one is duality: besides $\mathcal{D}$ we also consider a variety $\mathcal{C}$ that is dual to $\mathcal{D}$ on the level of finite algebras. Varieties of languages live in $\mathcal{C}$, while over $\mathcal{D}^S$ we form pseudovarieties of **T**-algebras. This is much inspired by the work of Gehrke, Grigorieff, and Pin [14] who interpreted the original Eilenberg theorem [12] in terms of Stone duality ($\mathcal{C}$ = boolean algebras, $\mathcal{D}$ = sets). The second ingredient is the *profinite monad* of **T**, introduced in [9]. It extends the classical construction of the free profinite monoid, and allows for the introduction of topological methods to our setting. The key to our approach is a categorical Reiterman theorem (Theorem 5.10) asserting that pseudovarieties of **T**-algebras correspond to profinite equational theories. The Variety Theorem then boils down to the fact that (i) varieties of **T**-recognizable languages dualize to profinite theories, and (ii) by the Reiterman theorem, profinite theories correspond to pseudovarieties of **T**-algebras.

Our results establish a conceptual and highly parametric, yet easily applicable framework for algebraic language theory. In fact, to derive an Eilenberg correspondence in our framework, the traditional four steps indicated above are replaced by a simple three-step procedure:

1. Find a monad **T** whose finite algebras recognize the desired languages.
2. Choose a unary presentation for **T**.
3. Spell out what a variety of **T**-recognizable languages and a pseudovariety of **T**-algebras is (by instantiating our general definitions), and invoke the Variety Theorem.

To illustrate the strength of this approach, we will show that roughly a dozen Eilenberg theorems in the literature emerge as special instances. In addition, we get several new results for free, e.g. an extension of the local variety theorem of [14] from finite to infinite words.

2   The Profinite Monad

We start by setting up our categorical framework for algebraic language theory. Recall that for a finitary one-sorted signature Γ, a *variety of algebras* is a class of Γ-algebras presented by equations between Γ-terms. A *variety of ordered algebras* is a class of ordered Γ-algebras (i.e. Γ-algebras on a poset with monotone Γ-operations) presented by inequations between Γ-terms. Morphisms of (ordered) Γ-algebras are (monotone) Γ-homomorphisms.

▶ **Assumptions 2.1.** Fix a variety $\mathcal{C}$ of algebras and a variety $\mathcal{D}$ of algebras/ordered algebras such that (i) $\mathcal{C}$ and $\mathcal{D}$ are *locally finite*, i.e. all finitely generated algebras are finite; (ii) the full subcategories $\mathcal{C}_f$ and $\mathcal{D}_f$ on finite algebras are dually equivalent; (iii) the signature of $\mathcal{C}$ contains a constant; (iv) epimorphisms in $\mathcal{D}$ are surjective. Further, fix a finite set S of sorts and a monad $\mathbf{T} = (T, \eta, \mu)$ on the product category $\mathcal{D}^S$ with T preserving epimorphisms.

▶ **Example 2.2.** The following categories $\mathcal{C}$ and $\mathcal{D}$ satisfy our assumptions:

1. $\mathcal{C} = \mathbf{BA}$ (boolean algebras) and $\mathcal{D} = \mathbf{Set}$: Stone duality [15] yields a dual equivalence $\mathbf{BA}_f^{op} \simeq \mathbf{Set}_f$, mapping a finite boolean algebra to the set of its atoms.
2. $\mathcal{C} = \mathbf{DL}_{01}$ (distributive lattices with $0, 1$) and $\mathcal{D} = \mathbf{Pos}$ (posets): Birkhoff duality [7] gives a dual equivalence $(\mathbf{DL}_{01})_f^{op} \simeq \mathbf{Pos}_f$, mapping a finite distributive lattice to the poset of its join-irreducible elements.
3. $\mathcal{C} = \mathcal{D} = \mathbf{JSL}_0$ (join-semilattices with 0): the self-duality $(\mathbf{JSL}_0)_f^{op} \simeq (\mathbf{JSL}_0)_f$ maps a finite semilattice $(X, \vee, 0)$ to its opposite semilattice $(X, \wedge, 1)$.
4. $\mathcal{C} = \mathcal{D} = \mathbf{Vec}_K$ (vector spaces over a finite field K): the familiar self-duality of $(\mathbf{Vec}_K)_f$ maps a finite (= finite-dimensional) vector space X to its dual space $X^* = \mathbf{Vec}_K(X, K)$.

▶ **Example 2.3.** Our focus is on monads **T** whose algebras represent formal languages.

1. Let $\mathbf{T}_*$ be the free-monoid monad on **Set**. Languages of finite words correspond to subsets of $T_*\Sigma = \Sigma^*$. The category of $\mathbf{T}_*$-algebras is isomorphic to the category of monoids.

2. Languages of finite and infinite words (∞-*languages* for short) are represented by the monad $\mathbf{T}_\infty$ on $\mathbf{Set}^2$ associated to the algebraic theory of ω-semigroups. Recall that an ω-*semigroup* is a two-sorted set $A = (A_+, A_\omega)$ with a binary product $A_+ \times A_+ \xrightarrow{\cdot} A_+$, a mixed binary product $A_+ \times A_\omega \xrightarrow{\cdot} A_\omega$ and an ω-ary product $A_+^\omega \xrightarrow{\pi} A_\omega$ satisfying (mixed) associative laws [19]. The free ω-semigroup on a two-sorted set (Σ, Γ) is $(\Sigma^+, \Sigma^\omega + \Sigma^* \times \Gamma)$ with products given by concatenation of words. Thus $T_\infty(\Sigma, \Gamma) = (\Sigma^+, \Sigma^\omega + \Sigma^* \times \Gamma)$, and ∞-languages over the alphabet Σ are two-sorted subsets of $T_\infty(\Sigma, \emptyset) = (\Sigma^+, \Sigma^\omega)$.

3. Weighted languages $L \colon \Sigma^* \to K$ over a finite field K are represented by the free-K-algebra monad $\mathbf{T}_K$ on $\mathbf{Vec}_K$. Thus for the space K^Σ with finite basis Σ we have $T_K(K^\Sigma) = K\langle\Sigma\rangle$, the space of all polynomials $\sum_{i<n} k_i w_i$ ($k_i \in K$, $w_i \in \Sigma^*$) in non-commuting variables. Since $K\langle\Sigma\rangle$ has the basis Σ^*, weighted languages correspond to linear maps $T_K(K^\Sigma) \to K$.

▶ **Remark 2.4.** We denote by $\mathbf{Alg}_f\, \mathbf{T}$ and $\mathbf{Alg}\, \mathbf{T}$ the categories of (finite) **T**-algebras. The category $\mathcal{D}^S$ has the factorization system of sortwise surjective morphisms and sortwise injective/order-reflecting morphisms. This lifts to $\mathbf{Alg}\, \mathbf{T}$ since T preserves surjections: every **T**-homomorphism factors into a surjective **T**-homomorphism followed by an injective/order-reflecting one. *Quotients* and *subalgebras* in $\mathbf{Alg}\, \mathbf{T}$ are taken in this factorization system.

In the theory of regular languages, topology enters the stage via the Stone space $\widehat{\Sigma^*}$ of *profinite words*, formed as the inverse (a.k.a. cofiltered) limit of all finite quotient monoids of Σ^*. In [9] we generalized this construction from monoids to algebras for a monad as follows:

▶ **Notation 2.5.** For a variety $\mathcal{D}$ of algebras, let $\mathbf{Stone}(\mathcal{D})$ denote the category of *Stone-topological $\mathcal{D}$-algebras*; its objects are $\mathcal{D}$-algebras endowed with a Stone topology and continuous $\mathcal{D}$-operations, and its morphisms are continuous $\mathcal{D}$-morphisms. For a variety $\mathcal{D}$ of ordered algebras, let $\mathbf{Priest}(\mathcal{D})$ be the category of ordered topological $\mathcal{D}$-algebras with a Priestley topology, and monotone continuous $\mathcal{D}$-morphisms. Denote by $\widehat{\mathcal{D}}$ the full subcategory of $\mathbf{Stone}(\mathcal{D})/\mathbf{Priest}(\mathcal{D})$ on *profinite $\mathcal{D}$-algebras*, i.e. inverse limits of algebras in $\mathcal{D}_f$. Here we view $\mathcal{D}_f$ as a full subcategory of $\widehat{\mathcal{D}}$ by equipping objects of $\mathcal{D}_f$ with the discrete topology.

▶ **Example 2.6.** For the varieties $\mathcal{D}$ of Example 2.2, every algebra in $\mathbf{Stone}(\mathcal{D})/\mathbf{Priest}(\mathcal{D})$ is profinite: we have $\widehat{\mathbf{Set}} = \mathbf{Stone}$ (Stone spaces), $\widehat{\mathbf{Pos}} = \mathbf{Priest}$ (Priestley spaces), $\widehat{\mathbf{JSL}_0} = \mathbf{Stone}(\mathbf{JSL}_0)$ (Stone semilattices) and $\widehat{\mathbf{Vec}_K} = \mathbf{Stone}(\mathbf{Vec}_K)$ (Stone vector spaces); see [15].

▶ **Construction 2.7.** For any object $D \in \mathcal{D}_f^S$ form the poset $\mathsf{Quo}_f(\mathbf{T}D)$ of all finite quotient algebras $e \colon \mathbf{T}D \twoheadrightarrow (A, \alpha)$ of the free **T**-algebra $\mathbf{T}D = (TD, \mu_D)$, ordered by $e \leq e'$ iff e factorizes through e'. Define the object $\hat{T}D$ in $\widehat{\mathcal{D}}^S$ to be the inverse limit of the diagram $\mathsf{Quo}_f(\mathbf{T}D) \to \widehat{\mathcal{D}}^S$ mapping $(e \colon \mathbf{T}D \twoheadrightarrow (A, \alpha))$ to A. We denote the limit projection associated to e by $e^+ \colon \hat{T}D \twoheadrightarrow A$. In particular, for any finite **T**-algebra (A, α) we have the projection $\alpha^+ \colon \hat{T}A \to A$, since $\alpha \colon \mathbf{T}A \twoheadrightarrow (A, \alpha)$ is a surjective **T**-homomorphism by the **T**-algebra laws.

▶ **Theorem 2.8** (see [9]). *The object map $D \mapsto \hat{T}D$ from $\mathcal{D}_f^S$ to $\widehat{\mathcal{D}}^S$ extends (by taking inverse limits) to a functor $\hat{T} \colon \widehat{\mathcal{D}}^S \to \widehat{\mathcal{D}}^S$. Further, $\hat{T}$ can be equipped with the structure of a monad $\widehat{\mathbf{T}} = (\hat{T}, \hat{\eta}, \hat{\mu})$ called the* profinite monad *of* **T**. *Its unit $\hat{\eta}_D$ and multiplication $\hat{\mu}_D$ for $D \in \mathcal{D}_f^S$ are uniquely determined by the commutativity of the diagrams*

$$
\begin{array}{ccc}
D \xrightarrow{\hat{\eta}_D} \hat{T}D \xleftarrow{\hat{\mu}_D} \hat{T}\hat{T}D \\
\,_{e\eta_D}\searrow \quad \downarrow{e^+} \qquad \downarrow{\hat{T}e^+} \\
A \xleftarrow{\ \ \alpha^+\ \ } \hat{T}A
\end{array}
\qquad \text{for all } e \colon \mathbf{T}D \twoheadrightarrow (A, \alpha) \text{ in } \mathsf{Quo}_f(\mathbf{T}D). \tag{1}
$$

In categorical terms, $\widehat{\mathbf{T}}$ is the codensity monad of the functor $\mathbf{Alg}_f\,\mathbf{T} \to \widehat{\mathcal{D}}$, $(A,\alpha) \mapsto A$.

▶ **Example 2.9.** The monad $\widehat{\mathbf{T}_*}$ on **Stone** assigns to each finite set Σ the Stone space $\widehat{\Sigma^*}$ of profinite words. The monad $\widehat{\mathbf{T}_K}$ on $\mathbf{Stone}(\mathbf{Vec}_K)$ assigns to each finite vector space K^Σ the Stone vector space arising as the limit of all finite quotient spaces of $K[\Sigma]$.

▶ **Remark 2.10.**

1. If (A,α) is a finite $\mathbf{T}$-algebra, then (A,α^+) is a $\widehat{\mathbf{T}}$-algebra: putting $e = \alpha$ in (1) gives the unit and associative law. By [9, Prop. 3.10] this yields an isomorphism $\mathbf{Alg}_f\,\mathbf{T} \cong \mathbf{Alg}_f\,\widehat{\mathbf{T}}$ given by $(A,\alpha) \mapsto (A,\alpha^+)$ and $h \mapsto h$.
2. Let $V \colon \widehat{\mathcal{D}}^S \to \mathcal{D}^S$ denote the forgetful functor. If $D \in \widehat{\mathcal{D}}_f^S \ (= \mathcal{D}_f^S)$ we usually write D for VD. By [9, Rem. B.6] there is a natural transformation $\iota \colon TV \to V\hat{T}$ whose component $\iota_D \colon TVD \to V\hat{T}D$ for $D \in \mathcal{D}_f^S$ is determined by $Ve^+ \cdot \iota_D = e$ for all $e \in \mathsf{Quo}_f(\mathbf{T}D)$.

▶ **Remark 2.11.**

1. $\widehat{\mathcal{D}}$ is the *pro-completion* (the free completion under inverse limits) of $\mathcal{D}_f$, see [15, Rem. VI.2.4]. Further, since the variety $\mathcal{C}$ is locally finite, $\mathcal{C}$ is the *ind-completion* (the free completion under filtered colimits) of $\mathcal{C}_f$. Therefore the dual equivalence between $\mathcal{C}_f$ and $\mathcal{D}_f$ extends to a dual equivalence between $\mathcal{C}$ and $\widehat{\mathcal{D}}$. We denote the equivalence functors by $P \colon \widehat{\mathcal{D}} \xrightarrow{\cong} \mathcal{C}^{op}$ and $P^{-1} \colon \mathcal{C}^{op} \xrightarrow{\cong} \widehat{\mathcal{D}}$. For example, for $\mathcal{C}/\mathcal{D} = \mathbf{BA}/\mathbf{Set}$ with $\widehat{\mathcal{D}} = \mathbf{Stone}$, this is the classical Stone duality [15]: P maps a Stone space to the boolean algebra of clopens, and P^{-1} maps a boolean algebra to the Stone space of all ultrafilters.
2. Denote by $|-|$ the forgetful functors of $\mathcal{C}$ and $\widehat{\mathcal{D}}$ into **Set** and by $\mathbb{1}_{\mathcal{C}}/\mathbb{1}_{\mathcal{D}}$ the free one-generated objects in $\mathcal{C}/\widehat{\mathcal{D}}$. The two finite objects $O_{\mathcal{C}} := P\mathbb{1}_{\mathcal{D}}$ and $O_{\mathcal{D}} := P^{-1}\mathbb{1}_{\mathcal{C}}$ play the role of a *dualizing object* [15] of $\mathcal{C}$ and $\widehat{\mathcal{D}}$. This means that there is a natural isomorphism between the functors $|-| \cdot P^{op}$ and $\widehat{\mathcal{D}}(-, O_{\mathcal{D}}) \colon \widehat{\mathcal{D}}^{op} \to \mathbf{Set}$, given for any $D \in \widehat{\mathcal{D}}$ by

$$|PD| \cong \mathcal{C}(\mathbb{1}_{\mathcal{C}}, PD) \cong \widehat{\mathcal{D}}(P^{-1}PD, P^{-1}\mathbb{1}_{\mathcal{C}}) = \widehat{\mathcal{D}}(P^{-1}PD, O_{\mathcal{D}}) \cong \widehat{\mathcal{D}}(D, O_{\mathcal{D}}).$$

   Similarly, $|-| \cdot P^{-1} \cong \mathcal{C}(-, O_{\mathcal{C}})$. Thus $O_{\mathcal{C}}$ and $O_{\mathcal{D}}$ have essentially the same underlying set, as $|O_{\mathcal{D}}| \cong \widehat{\mathcal{D}}(\mathbb{1}_{\mathcal{D}}, O_{\mathcal{D}}) \cong |P\mathbb{1}_{\mathcal{D}}| = |O_{\mathcal{C}}|$. For our pairs $\mathcal{C}/\mathcal{D}$ of Ex. 2.2, we get $O_{\mathbf{BA}} = \{0 < 1\}/O_{\mathbf{Set}} = \{0,1\}$, $O_{\mathbf{DL}_{01}} = \{0 < 1\} = O_{\mathbf{Pos}}$, $O_{\mathbf{JSL}_0} = \{0 < 1\}$, $O_{\mathbf{Vec}_K} = K$.
3. *Subobjects* in $\mathcal{C}$ are represented by monomorphisms, i.e. injective morphisms. Dually, *quotients* in $\widehat{\mathcal{D}}$ are represented by epimorphisms, which can be shown to be the surjective morphisms. Thus *quotients* of $\widehat{\mathbf{T}}$-algebras are represented by surjective $\widehat{\mathbf{T}}$-homomorphisms.

3 Recognizable Languages

A language $L \subseteq \Sigma^*$ of finite words may be identified with its characteristic function $L \colon \Sigma^* \to \{0,1\}$. To model languages in our categorical setting, we replace the one-sorted alphabet Σ by an S-sorted alphabet Σ in $\mathbf{Set}_f^S$, and represent it in $\mathcal{D}^S$ via the free object $\not\prec \in \mathcal{D}_f^S$ generated by Σ (w.r.t. the forgetful functor $|-| \colon \mathcal{D}^S \to \mathbf{Set}^S$). Note that $\not\prec$ is finite because $\mathcal{D}$ is locally finite. The output set $\{0,1\}$ is replaced by a finite "object of outputs" in $\mathcal{D}_f^S$, viz. the object with $O_{\mathcal{D}} \in \mathcal{D}_f$ in each sort. By abuse of notation, we denote this object of $\mathcal{D}_f^S$ also by $O_{\mathcal{D}}$. This leads to the following definition, unifying concepts from [2] and [8].

▶ **Definition 3.1.** A *language* over the alphabet $\Sigma \in \mathbf{Set}_f^S$ is a morphism $L \colon T\not\prec \to O_{\mathcal{D}}$ in $\mathcal{D}^S$. It is called $\mathbf{T}$-*recognizable* if there exists a $\mathbf{T}$-homomorphism $h \colon \mathbf{T}\not\prec \to (A,\alpha)$ with finite codomain and a morphism $p \colon A \to O_{\mathcal{D}}$ in $\mathcal{D}^S$ with $L = p \cdot h$. In this case, we say that L is *recognized by h (via p)*. We denote by $\mathsf{Rec}(\Sigma)$ the set of all $\mathbf{T}$-recognizable languages over Σ.

▶ **Example 3.2.**

1. $\mathbf{T} = \mathbf{T}_*$ on **Set** with $O_{\mathbf{Set}} = \{0,1\}$: a language $L\colon T_*\Sigma \to O_{\mathbf{Set}}$ corresponds to a language $L \subseteq \Sigma^*$ of finite words. It is recognized by a monoid morphism $h\colon \Sigma^* \to A$ iff $L = h^{-1}[Y]$ for some subset $Y \subseteq A$. Recognizable languages coincide with regular languages, i.e. languages accepted by finite automata; see e.g. [22].

2. $\mathbf{T} = \mathbf{T}_\infty$ on $\mathbf{Set}^2$ with $O_{\mathbf{Set}} = \{0,1\}$: since $T_\infty(\Sigma, \emptyset) = (\Sigma^+, \Sigma^\omega)$, a language $L\colon T_\infty(\Sigma, \emptyset) \to (\{0,1\}, \{0,1\})$ corresponds to an ∞-language $L \subseteq (\Sigma^+, \Sigma^\omega)$. It is recognized by an ω-semigroup morphism $h\colon (\Sigma^+, \Sigma^\omega) \to A$ iff $L = h^{-1}[Y]$ for some two-sorted subset $Y \subseteq A$. Recognizable ∞-languages are the ones accepted by finite Büchi automata [19].

The topological perspective on regular languages rests on the important observation that regular languages over Σ correspond to the clopen subsets of the Stone space $\widehat{\Sigma^*}$ of profinite words, or equivalently to continuous maps from $\widehat{\Sigma^*}$ into the discrete space $\{0,1\}$; see e.g. [22, Prop. VI.3.12]. This generalizes from the monad $\mathbf{T}_*$ on **Set** to arbitrary monads $\mathbf{T}$:

▶ **Theorem 3.3.** *$\mathbf{T}$-recognizable languages over $\Sigma \in \mathbf{Set}_f^S$ correspond bijectively to morphisms from $\hat{T}\!\not\Sigma$ to $O_{\mathcal{D}}$ in $\widehat{\mathcal{D}}^S$. The bijection is given by $(\hat{T}\!\not\Sigma \xrightarrow{\hat{L}} O_{\mathcal{D}}) \mapsto (T\!\not\Sigma \xrightarrow{\iota\!\not\Sigma} V\hat{T}\!\not\Sigma \xrightarrow{V\hat{L}} O_{\mathcal{D}})$.*

▶ **Remark 3.4** (𝒞-algebraic structure on $\mathsf{Rec}(\Sigma)$)**.** By the above and Remark 2.11.2, we deduce

$$\mathsf{Rec}(\Sigma) \cong \widehat{\mathcal{D}}^S(\hat{T}\!\not\Sigma, O_{\mathcal{D}}) = \prod_s \widehat{\mathcal{D}}((\hat{T}\!\not\Sigma)_s, O_{\mathcal{D}}) \cong \prod_s |P(\hat{T}\!\not\Sigma)_s|. \tag{2}$$

Thus we can consider $\mathsf{Rec}(\Sigma)$ as an object of 𝒞 isomorphic to $\prod_s P(\hat{T}\!\not\Sigma)_s$. One can show that $\mathsf{Rec}(\Sigma)$ is a subobject of the product $\prod_s O_{\mathcal{C}}^{|T\!\not\Sigma|_s}$ in 𝒞: the embedding $\mathsf{Rec}(\Sigma) \rightarrowtail \prod_s O_{\mathcal{C}}^{|T\!\not\Sigma|_s}$ maps a language $L\colon T\!\not\Sigma \to O_{\mathcal{D}}$ to the S-tuple $(|T\!\not\Sigma|_s \xrightarrow{|L|_s} |O_{\mathcal{D}}| \xrightarrow{\cong} |O_{\mathcal{C}}|)_{s\in S}$, using the bijection $|O_{\mathcal{D}}| \cong |O_{\mathcal{C}}|$ of Remark 2.11.2. Consequently the 𝒞-algebraic structure of $\mathsf{Rec}(\Sigma)$ is determined by $O_{\mathcal{C}}$. For example, for $\mathcal{C} = \mathbf{BA}$ with $O_{\mathbf{BA}} = \{0,1\}$, the boolean structure on $\mathsf{Rec}(\Sigma)$ is given by union, intersection and complement. Taking $\mathbf{T} = \mathbf{T}_*$ on **Set**, we recover an important result of Pippenger [23]: the boolean algebra of regular languages over Σ is dual to the Stone space $\widehat{\Sigma^*}$ of profinite words. In fact, in this case (2) yields $\mathsf{Rec}(\Sigma) \cong P(\widehat{\Sigma^*})$.

4   Unary Presentations

In this section we introduce *unary presentations* of $\mathbf{T}$-algebras that later, in Section 6, will serve as our key tool for defining the derivatives of a language. For motivation, let A be an algebra over a finitary single-sorted signature Γ. By a standard result in universal algebra, an equivalence relation $\equiv$ on A is a Γ-congruence (i.e. stable under Γ-operations) iff it is stable under *elementary translations*, i.e. $a \equiv a'$ implies $u(a) \equiv u(a')$ for all maps $u\colon A \to A$ of the form $a \mapsto \gamma^A(a_0, \ldots, a_i, a, a_{i+1}, \ldots a_n)$, where $\gamma \in \Gamma$ and $a_0, \ldots a_n \in A$. (For the sorted case, see [17,18].) If Γ contains infinitary operations, this statement generally fails, but remains valid if $\equiv$ is refinable to a Γ-congruence of finite index; see Examples 4.3.3/5. Identifying equivalence relations with quotients, we first state the concept of refinement categorically:

▶ **Definition 4.1.** Let $(A, \alpha) \in \mathbf{Alg\,T}$. A quotient $e\colon A \twoheadrightarrow B$ in $\mathcal{D}^S$ is $\mathbf{T}$-*refinable* if there is a finite quotient $\overline{e}\colon (A, \alpha) \twoheadrightarrow (C, \gamma)$ in $\mathbf{Alg\,T}$ and a morphism $p\colon C \to B$ in $\mathcal{D}^S$ with $e = p \cdot \overline{e}$.

Then the description of congruences via translations has the following categorical formulation:

▶ **Definition 4.2.** A *unary operation* on $A \in \mathcal{D}^S$ is a morphism $u: A_s \to A_t$ in $\mathcal{D}$, where $s, t \in S$. A *unary presentation* of a **T**-algebra (A, α) is a set $\mathbb{U}$ of unary operations on A such that, for any **T**-refinable quotient $e: A \twoheadrightarrow B$ in $\mathcal{D}^S$, the following are equivalent:

(U1) e carries a quotient of (A, α) in **Alg T**, i.e. there exists a **T**-algebra structure (B, β) on B for which $e: (A, \alpha) \twoheadrightarrow (B, \beta)$ is a **T**-homomorphism.

(U2) Each unary operation $u: A_s \to A_t$ in $\mathbb{U}$ admits a lifting along e, i.e. a morphism $u_B: B_s \to B_t$ in $\mathcal{D}$ with $e_t \cdot u = u_B \cdot e_s$.

▶ **Example 4.3.**

1. $\mathbf{T} = \mathbf{T}_*$ on **Set**: every monoid M has a unary presentation given by the unary operations $x \mapsto yx$ and $x \mapsto xy$ on M, where y ranges over all elements of M.

2. $\mathbf{T} = \mathbf{T}_\infty$ on $\mathbf{Set}^2$: every ω-semigroup $A = (A_+, A_\omega)$ has a unary presentation given by the operations (i) $x \mapsto yx$ and $x \mapsto xy$ on A_+, (ii) $x \mapsto xz$ and $x \mapsto x^\omega = \pi(x, x, x, \ldots)$ from A_+ to A_ω, and (iii) $z \mapsto yz$ on A_ω, where $y \in A_+ \cup \{1\}$ and $z \in A_\omega$. The proof uses Ramsey's theorem and appears implicitly in the work of Wilke [35]; see also [19].

3. Let **T** be a monad on $\mathbf{Set}^S$. Every **T**-algebra (A, α) has a generic unary presentation given as follows. Let $1_s \in \mathbf{Set}^S$ be the S-sorted set with one element in sort s and otherwise empty; thus a morphism $1_s \to A$ in $\mathbf{Set}^S$ chooses an element of A_s. A *polynomial* over A is a morphism $p: 1_t \to T(A + 1_s)$ with $s, t \in S$, i.e. a "term" of output sort t in a variable of sort s. Denote by $A_s \xrightarrow{[p]} A_t$ the evaluation map that substitutes elements of A_s for the variable. The maps $[p]$ (where p ranges over polynomials over A) form a unary presentation of (A, α). The proof rests on the fact that **T**-algebras can be viewed as algebras over a (possibly large and infinitary) signature [16]. Note that for monoids and ω-semigroups, the polynomial presentation is much larger than the one in Example 1/2; e.g., for a monoid M it contains all operations $x \mapsto y_0 x y_1 x \ldots x y_n$ with $y_0, \ldots, y_n \in M$. Polynomials appeared in [8] in the context of syntactic congruences; see Example 4.8.

4. In contrast to Example 4.3.3, in general not every **T**-algebra admits a unary presentation if $\mathcal{D} \neq \mathbf{Set}$. Indeed, let $\mathcal{D} = \mathbf{Set}_{c,d}$ be the variety of sets with two constants c, d, and $\mathbf{Set}_{c \neq d}$ its full reflective subcategory on the terminal object 1 and all $X \in \mathcal{D}$ with $c^X \neq d^X$. The right adjoint $\mathbf{Set}_{c \neq d} \rightarrowtail \mathcal{D}$ induces a monad **T** on $\mathcal{D}$ with $\mathbf{Alg\,T} \cong \mathbf{Set}_{c \neq d}$. One can show that the **T**-algebra corresponding to $\{x, c, d\} \in \mathbf{Set}_{c \neq d}$ has no unary presentation.

5. If **T** represents algebras with finitary operations, the equivalence (U1)⇔(U2) often holds for arbitrary quotients e. However, the restriction to **T**-*refinable* quotients is crucial in the presence of infinitary operations. For example, let **T** be the free Γ-algebra monad on **Set** for the signature Γ with one ω-ary operation. Thus TX is the set of well-founded Γ-trees (= ω-branching trees without infinite paths) with X-labeled leaves. Let $X \neq \emptyset$ and $e: TX \twoheadrightarrow \{0, 1\}$ be the map sending a tree t to 0 iff t has finite height. Then for the polynomial presentation of **T**, see Ex. 4.3.3, the direction (U2)⇒(U1) does not hold for e.

Digression: Syntactic **T**-algebras

Languages are often analyzed by means of *syntactic algebras*, i.e. their minimal recognizing algebras. This language-theoretic concept is closely related to our algebraic notion of a unary presentation, as we now explain. The results of this subsection serve to put our concepts into the context of classical algebraic language theory; they are, however, not used in the sequel and may be skipped by readers interested only in the variety theorem and its applications.

▶ **Definition 4.4.** Let $L: T\not\!\kappa \to O_\mathcal{D}$ be recognizable. A *syntactic* **T**-*algebra for* L is a finite **T**-algebra A_L together with a surjective **T**-homomorphism $e_L: T\not\!\kappa \twoheadrightarrow A_L$ (called a *syntactic morphism* for L) such that (i) e_L recognizes L, and (ii) e_L factors through any surjective **T**-homomorphism $e: T\not\!\kappa \twoheadrightarrow A$ recognizing L, i.e. $e_L = h \cdot e$ for some $h: A \twoheadrightarrow A_L$ in **Alg T**.

▶ **Example 4.5.** Let $\mathbf{T} = \mathbf{T}_*$ on **Set**. The *syntactic monoid* [22] of a recognizable language $L \colon \Sigma^* \to \{0, 1\}$ is the quotient monoid $e_L \colon \Sigma^* \twoheadrightarrow \Sigma^*/{\equiv_L}$, where $\equiv_L$ is the monoid congruence on Σ^* defined by $x \equiv_L x'$ iff $L(yxz) = L(yx'z)$ for all $y, z \in \Sigma^*$.

The above definition of $\equiv_L$ involves for each $y, z \in \Sigma^*$ the unary operation $x \mapsto yxz$ on Σ^*, which can be expressed as the composite of the operations $x \mapsto yx$ and $x \mapsto xz$ appearing in the unary presentation of Σ^* in Example 4.3.1. This is no coincidence: one can always derive a syntactic congruence from a unary presentation, and vice versa. For brevity, we discuss only the case where $\mathcal{D}$ is a variety of algebras; see the full paper [34] for the ordered case.

▶ **Notation 4.6.** Let $\mathbb{U}$ be a set of unary operations on $T{\not\curvearrowright}$, and denote by $\overline{\mathbb{U}}$ its closure under composition and identity morphisms $id \colon (T{\not\curvearrowright})_s \to (T{\not\curvearrowright})_s$. Given a language $L \colon T{\not\curvearrowright} \to O_{\mathcal{D}}$, the S-sorted equivalence relation $\equiv_{\mathbb{U},L}$ on $|T{\not\curvearrowright}|$ is defined as follows: for $x, x' \in |T{\not\curvearrowright}|_s$, put

$$x \equiv_{\mathbb{U},L} x' \quad \Leftrightarrow \quad L_t \cdot u(x) = L_t \cdot u(x') \text{ for all sorts } t \text{ and all } u \colon (T{\not\curvearrowright})_s \to (T{\not\curvearrowright})_t \text{ in } \overline{\mathbb{U}}.$$

One readily verifies that $\equiv_{\mathbb{U},L}$ is a congruence on $T{\not\curvearrowright}$ in $\mathcal{D}^S$, i.e. sortwise stable under all $\mathcal{D}$-operations. We denote the induced quotient in $\mathcal{D}^S$ by $e_L \colon T{\not\curvearrowright} \twoheadrightarrow T{\not\curvearrowright}/{\equiv_{\mathbb{U},L}}$.

▶ **Theorem 4.7.** *For any set $\mathbb{U}$ of unary operations on $T{\not\curvearrowright}$, the following are equivalent:*
 (i) $\mathbb{U}$ *is a unary presentation of $\mathbf{T}{\not\curvearrowright}$.*
 (ii) *Every recognizable language $L \colon T{\not\curvearrowright} \to O_{\mathcal{D}}$ has a syntactic $\mathbf{T}$-algebra, and $e_L \colon T{\not\curvearrowright} \twoheadrightarrow T{\not\curvearrowright}/{\equiv_{\mathbb{U},L}}$ carries a quotient of $\mathbf{T}{\not\curvearrowright}$ in $\mathbf{Alg}\,\mathbf{T}$ that forms a syntactic morphism for L.*

▶ **Example 4.8.** Let $\mathbb{U}$ be the unary presentation of Σ^* in Example 4.3.1. Then $\equiv_{\mathbb{U},L}$ is precisely the congruence $\equiv_L$ of Example 4.5, and Theorem 4.7 shows that $\equiv_L$ is a syntactic congruence for L. Similarly, Example 4.3.2/3 and Theorem 4.7 give a description of the syntactic ω-semigroup for any recognizable ∞-language [19], and of the syntactic $\mathbf{T}$-algebra for any monad $\mathbf{T}$ on $\mathbf{Set}^S$ and any $\mathbf{T}$-recognizable language [8]. We omit the details.

Theorem 4.7 explains why syntactic algebras are presented as a key technique in earlier work on Eilenberg theorems: they implicitly contain unary presentations. However, the latter are the "heart of the matter", and it is easier to directly work with presentations in lieu of syntactic algebras to derive Eilenberg-type theorems. We will demonstrate this in Section 7.

5 **Pseudovarieties of $\mathbf{T}$-algebras and Profinite Theories**

In this section we investigate pseudovarieties of $\mathbf{T}$-algebras and establish a categorical Reiterman theorem: pseudovarieties correspond to profinite equational theories. The results of the present section are largely independent of our Assumptions 2.1: they hold for any locally finite variety $\mathcal{D}$ of (ordered) algebras and any monad $\mathbf{T}$ on $\mathcal{D}^S$ that preserves surjections.

▶ **Definition 5.1.** A Σ-*generated $\mathbf{T}$-algebra* is a quotient $e \colon \mathbf{T}{\not\curvearrowright} \twoheadrightarrow A$ of $\mathbf{T}{\not\curvearrowright}$ in $\mathbf{Alg}\,\mathbf{T}$. The *subdirect product* of two quotients $e_i \colon \mathbf{T}{\not\curvearrowright} \twoheadrightarrow A_i$ $(i = 0, 1)$ is the image $e \colon \mathbf{T}{\not\curvearrowright} \twoheadrightarrow A$ of the $\mathbf{T}$-homomorphism $\langle e_0, e_1 \rangle \colon \mathbf{T}{\not\curvearrowright} \to A_0 \times A_1$. We say that e_1 is a *quotient* of e_0 if e_1 factors through e_0, i.e. $e_1 = q \cdot e_0$ for some q. A *local pseudovariety of Σ-generated $\mathbf{T}$-algebras* is a class of Σ-generated finite $\mathbf{T}$-algebras closed under subdirect products and quotients.

In order-theoretic terms, local pseudovarieties are precisely the ideals of the poset $\mathsf{Quo}_f(\mathbf{T}{\not\curvearrowright})$.

▶ **Definition 5.2.** A $\widehat{\mathbf{T}}$-algebra is *profinite* if it is an inverse limit of finite $\widehat{\mathbf{T}}$-algebras. By a Σ-*generated profinite $\widehat{\mathbf{T}}$-algebra* is meant a quotient $\varphi \colon \widehat{\mathbf{T}}{\not\curvearrowright} \twoheadrightarrow P$ of $\widehat{\mathbf{T}}{\not\curvearrowright}$ in $\mathbf{Alg}\,\widehat{\mathbf{T}}$ with P profinite. Σ-generated profinite $\widehat{\mathbf{T}}$-algebras are ordered by $\varphi \leq \varphi'$ iff φ factors through φ'.

▶ **Theorem 5.3** (Local Reiterman Theorem). *For each $\Sigma \in \mathbf{Set}_f^S$, the lattice of local pseudovarieties of Σ-generated $\mathbf{T}$-algebras (ordered by inclusion) is isomorphic to the lattice of Σ-generated profinite $\widehat{\mathbf{T}}$-algebras. The isomorphism maps a Σ-generated profinite $\widehat{\mathbf{T}}$-algebra $\varphi \colon \widehat{\mathbf{T}}\kern-0.6em\not\kern0.2em\kern-0.4em\twoheadrightarrow P$ to the local pseudovariety of all finite quotients $e \colon \mathbf{T}\kern-0.6em\not\kern0.2em\kern-0.4em\twoheadrightarrow A$ with $e^+ \leq \varphi$.*

▶ **Remark 5.4.** Theorem 5.3 can be interpreted in terms of profinite (in-)equations. If $\mathcal{D}$ is a variety of ordered algebras, a *profinite inequation $u \leq v$ over Σ* is a pair of elements $u, v \in |\hat{T}\kern-0.6em\not\kern0.2em\kern-0.4em|_s$ in some sort s. We say that a Σ-generated finite $\mathbf{T}$-algebra $e \colon \mathbf{T}\kern-0.6em\not\kern0.2em\kern-0.4em\twoheadrightarrow A$ *satisfies* $u \leq v$ if $e^+(u) \leq e^+(v)$. Using 5.3 one can show that local pseudovarieties are precisely the classes of Σ-generated finite $\mathbf{T}$-algebras presentable by profinite inequations over Σ. Similarly, if $\mathcal{D}$ is a variety of algebras, local pseudovarieties are presentable by *profinite equations*.

Eilenberg's theorem considers regular languages over arbitrary alphabets. In contrast, in a sorted setting one may need to make a *choice* of alphabets to capture the proper languages (e.g. alphabets of the form $(\Sigma, \emptyset)$ in 2.3.2). On the algebraic side, this requires us to restrict to $\mathbf{T}$-algebras with certain generators. From now on, let $\mathbb{A} \subseteq \mathbf{Set}_f^S$ be a fixed class of alphabets.

▶ **Definition 5.5.** A $\mathbf{T}$-algebra (A, α) is $\mathbb{A}$-*generated* if there is a surjective $\mathbf{T}$-homomorphism $e \colon \mathbf{T}\kern-0.6em\not\kern0.2em\kern-0.4em\twoheadrightarrow (A, \alpha)$ with $\Sigma \in \mathbb{A}$. A *pseudovariety of $\mathbf{T}$-algebras* is a class $\mathcal{V}$ of $\mathbb{A}$-generated finite $\mathbf{T}$-algebras closed under $\mathbb{A}$-generated subalgebras of finite products (i.e. for $A_1, \ldots, A_n \in \mathcal{V}$ and any $\mathbb{A}$-generated subalgebra $A \rightarrowtail \prod_{i=1}^n A_i$, one has $A \in \mathcal{V}$) and quotients.

N.B. We emphasize that, in contrast to Definition 5.1, an $\mathbb{A}$-generated $\mathbf{T}$-algebra (A, α) is not equipped with a *fixed* quotient $e \colon \mathbf{T}\kern-0.6em\not\kern0.2em\kern-0.4em\twoheadrightarrow (A, \alpha)$. Only the existence of e is required.

▶ **Example 5.6.**

1. Every finite $\mathbf{T}$-algebra (A, α) is $\mathbf{Set}_f^S$-generated: since $\mathcal{D}$ is locally finite, there is a surjective morphism $e \colon \kern-0.6em\not\kern0.2em\kern-0.4em \twoheadrightarrow A$ with $\Sigma \in \mathbf{Set}_f^S$, so (A, α) is a quotient of $\mathbf{T}\kern-0.6em\not\kern0.2em\kern-0.4em$ via $(\mathbf{T}\kern-0.6em\not\kern0.2em\kern-0.4em \xrightarrow{Te} TA \xrightarrow{\alpha} (A, \alpha))$. Consequently, for $\mathbb{A} = \mathbf{Set}_f^S$, a pseudovariety is a class of finite $\mathbf{T}$-algebras closed under quotients, subalgebras, and finite products. This concept was studied in [9]. For the monad $\mathbf{T}_*$ on $\mathbf{Set}$ we recover the original concept of Eilenberg [12]: a class of finite monoids closed under quotients, submonoids, and finite products.

2. Let $\mathbf{T} = \mathbf{T}_\infty$ on $\mathbf{Set}^2$. As suggested by Example 2.3.2, we choose $\mathbb{A} = \{ (\Sigma, \emptyset) : \Sigma \in \mathbf{Set}_f \}$. A finite $\mathbf{T}_\infty$-algebra (= finite ω-semigroup) A is $\mathbb{A}$-generated iff it is *complete*, i.e. every element $a \in A_\omega$ can be expressed as an infinite product $a = \pi(a_0, a_1, \ldots)$ for some $a_i \in A_+$. Clearly complete ω-semigroups are closed under finite products. Thus a pseudovariety of $\mathbf{T}_\infty$-algebras is a class of finite complete ω-semigroups closed under quotients, complete ω-subsemigroups, and finite products. This concept is due to Wilke [35]; see also [19].

▶ **Remark 5.7.** Every $\mathbf{T}$-homomorphism $g \colon \mathbf{T}D' \to \mathbf{T}D$ with $D, D' \in \mathcal{D}_f^S$ extends uniquely to a $\widehat{\mathbf{T}}$-homomorphism $\hat{g} \colon \widehat{\mathbf{T}}D' \to \widehat{\mathbf{T}}D$ with $\iota_D \cdot g = V\hat{g} \cdot \iota_{D'}$ (for ι_D recall Remark 2.10.2).

▶ **Definition 5.8.** A *profinite theory* is a family $\varrho = (\varrho_\Sigma \colon \widehat{\mathbf{T}}\kern-0.6em\not\kern0.2em\kern-0.4em \twoheadrightarrow P_\Sigma)_{\Sigma \in \mathbb{A}}$ such that (i) ϱ_Σ is a Σ-generated profinite $\widehat{\mathbf{T}}$-algebra for each $\Sigma \in \mathbb{A}$, and (ii) for every $\mathbf{T}$-homomorphism $g \colon \mathbf{T}\kern-0.6em\underline{\not}\kern0.2em\kern-0.4em \to \mathbf{T}\kern-0.6em\not\kern0.2em\kern-0.4em$ with $\Sigma, \Delta \in \mathbb{A}$, there exists a $\widehat{\mathbf{T}}$-homomorphism $g_P \colon P_\Delta \to P_\Sigma$ with $\varrho_\Sigma \cdot \hat{g} = g_P \cdot \varrho_\Delta$. Profinite theories are ordered by $\varrho \leq \varrho'$ iff ϱ_Σ factors through ϱ'_Σ for each $\Sigma \in \mathbb{A}$.

▶ **Remark 5.9.** Profinite theories generalize the *varieties of filters of congruences* introduced by Almeida [4] for algebras over a finitary signature, and earlier by Thérien [32] for monoids.

▶ **Theorem 5.10** (Reiterman Theorem)**.** *The lattice of pseudovarieties of* **T**-*algebras (ordered by inclusion) is isomorphic to the lattice of profinite theories. The isomorphism maps a theory* $(\varrho_\Sigma \colon \widehat{\mathbf{T}}\!\diagup\!\!\!\!\!\diagdown \twoheadrightarrow P_\Sigma)_{\Sigma \in \mathbb{A}}$ *to the class of all finite* **T**-*algebras arising as a quotient of some* P_Σ.

▶ **Remark 5.11.** Theorem 5.10 has again an interpretation in terms of profinite (in-)equations. If $\mathcal{D}$ is a variety of ordered algebras, we say that a finite **T**-algebra (A, α) *satisfies* a profinite inequation $u \leq v$ over $\Sigma \in \mathbb{A}$ if $h(u) \leq h(v)$ for all $\widehat{\mathbf{T}}$-homomorphisms $h \colon \widehat{\mathbf{T}}\!\diagup\!\!\!\!\!\diagdown \to (A, \alpha^+)$. Using 5.10 one can show that pseudovarieties are precisely the classes of $\mathbb{A}$-generated finite **T**-algebras presentable by profinite inequations over $\mathbb{A}$. In the unordered case, one takes profinite equations $u = v$. For $\mathbb{A} = \mathbf{Set}_f^S$, this was proved in [9, Thm. 4.12, Rem. 5.7].

6   The Variety Theorem

To investigate varieties of languages in our categorical setting, we need a notion of *language derivatives* extending the classical concept. To this end, we make use of our Assumption 2.1(iii) that the variety $\mathcal{C}$ has a constant in the signature. Choosing a constant gives a natural transformation from the constant functor $C_{\mathbb{1}_\mathcal{C}}$ on $\mathbb{1}_\mathcal{C}$ to the identity functor $\mathsf{Id}_\mathcal{C}$. It dualizes to a natural transformation $\perp \colon \mathsf{Id}_{\widehat{\mathcal{D}}} \to C_{O_\mathcal{D}}$. The purpose of $\perp$ is to model the empty set:

▶ **Example 6.1.** For our categories $\mathcal{D}$ of Example 2.2 and the corresponding objects $O_\mathcal{D}$ (see Remark 2.11.2) we choose $\perp \colon D \to O_\mathcal{D}$ for $D \in \widehat{\mathcal{D}}$ to be the constant morphism with value 0.

▶ **Definition 6.2.** Let $L \colon T\!\diagup\!\!\!\!\!\diagdown \to O_\mathcal{D}$ be a language. Then we define the following languages:

1. the *preimage* $g^{-1}L$ of L under a **T**-homomorphism $g \colon \mathbf{T}\!\gneqq \to \mathbf{T}\!\diagup\!\!\!\!\!\diagdown$ by $T\!\gneqq \xrightarrow{g} T\!\diagup\!\!\!\!\!\diagdown \xrightarrow{L} O_\mathcal{D}$;
2. the *derivative* $u^{-1}L \colon T\!\diagup\!\!\!\!\!\diagdown \to O_\mathcal{D}$ of L w.r.t. a unary operation $u \colon (T\!\diagup\!\!\!\!\!\diagdown)_s \to (T\!\diagup\!\!\!\!\!\diagdown)_t$ by

$$(u^{-1}L)_s = (T\!\diagup\!\!\!\!\!\diagdown)_s \xrightarrow{u} (T\!\diagup\!\!\!\!\!\diagdown)_t \xrightarrow{L_t} O_\mathcal{D} \quad ; \quad (u^{-1}L)_r = (T\!\diagup\!\!\!\!\!\diagdown)_r \xrightarrow{\iota_{\diagup\!\!\!\!\!\diagdown}} (V\hat{T}\!\diagup\!\!\!\!\!\diagdown)_r \xrightarrow{V\perp} O_\mathcal{D} \ (r \neq s)$$

N.B.   In the single-sorted case $S = 1$ the derivative $u^{-1}L$ is equal to $L \cdot u$ and the natural transformation $\perp$ is not used. Therefore Assumption 2.1(iii) can be dropped.

▶ **Example 6.3.**

1. $\mathbf{T} = \mathbf{T}_*$ on **Set**: consider the unary operations of Example 4.3.1 presenting Σ^*. The induced derivatives of a language $L \subseteq \Sigma^*$ are exactly the classical ones, i.e. the languages $y^{-1}L = \{\, x \in \Sigma^* : yx \in L \,\}$ and $Ly^{-1} = \{\, x \in \Sigma^* : xy \in L \,\}$ for $y \in \Sigma^*$.
2. $\mathbf{T} = \mathbf{T}_\infty$ on $\mathbf{Set}^2$: consider the unary operations of Example 4.3.2 presenting $\mathbf{T}_\infty(\Sigma, \emptyset) = (\Sigma^+, \Sigma^\omega)$. The induced derivatives of $L \subseteq \Sigma^+ \cup \Sigma^\omega$ are $\{\, x \in \Sigma^+ : yx \in L \,\}$, $\{\, x \in \Sigma^+ : xy \in L \,\}$, $\{\, x \in \Sigma^+ : xz \in L \,\}$, $\{\, x \in \Sigma^+ : x^\omega \in L \,\}$, and $\{\, z \in \Sigma^\omega : yz \in L \,\}$, where $y \in \Sigma^*$ and $z \in \Sigma^\omega$. These are the derivatives for ∞-languages studied by Wilke [35].
3. Let $\mathbf{T}$ be any monad on $\mathbf{Set}^S$, and consider the unary operations $[p]$ of Example 4.3.3 presenting $\mathbf{T}\Sigma$. The induced derivatives of $L \subseteq T\Sigma$ are the languages $p^{-1}L \subseteq T\Sigma$ with $(p^{-1}L)_s = \{\, x \in (T\Sigma)_s : [p](x) \in L_t \,\}$ and $(p^{-1}L)_r = \emptyset$ for $r \neq s$, where $p \colon 1_t \to T(T\Sigma + 1_s)$ is a polynomial. These *polynomial derivatives* were studied by Bojańczyk [8].

▶ **Definition 6.4.** Given an S-indexed family $\mathcal{L} = (\, L^s \colon T\!\diagup\!\!\!\!\!\diagdown \to O_\mathcal{D} \,)_{s \in S}$ of languages over Σ, the *diagonal* of $\mathcal{L}$ is the language $\Delta\mathcal{L}$ over Σ with $(\Delta\mathcal{L})_s = L^s_s \colon (T\!\diagup\!\!\!\!\!\diagdown)_s \to O_\mathcal{D}$ for all $s \in S$.

▶ **Notation 6.5.** Recall that we work with a fixed class $\mathbb{A} \subseteq \mathbf{Set}_f^S$ of alphabets. Fix for each $\Sigma \in \mathbb{A}$ a unary presentation $\mathbb{U}_\Sigma$ of the free **T**-algebra $\mathbf{T}\!\diagup\!\!\!\!\!\diagdown$.

▶ **Definition 6.6.**

1. A *local variety of languages* over $\Sigma \in \mathbb{A}$ is a subobject $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$ in $\mathcal{C}$ closed under $\mathbb{U}_\Sigma$-derivatives ($L \in W_\Sigma$ implies $u^{-1}L \in W_\Sigma$ for $u \in \mathbb{U}_\Sigma$) and diagonals.
2. A *variety of languages* is a family of local varieties $(W_\Sigma \subseteq \mathsf{Rec}(\Sigma))_{\Sigma \in \mathbb{A}}$ closed under preimages, i.e. $L \in W_\Sigma$ implies $g^{-1}L \in W_\Delta$ for all $\Sigma, \Delta \in \mathbb{A}$ and $g \colon \mathbf{T}\underline{\underline{\Sigma}} \to \mathbf{T}\not\sphericalangle$ in $\mathbf{Alg\,T}$.

▶ **Remark 6.7.**

1. Using the isomorphism $\mathsf{Rec}(\Sigma) \cong \prod_s P(\hat{T}\not\sphericalangle)_s$ of Remark 3.4, one can show that a subobject $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$ is closed under diagonals iff it has the form $\prod_s m_s \colon \prod_s (W'_\Sigma)_s \rightarrowtail \prod_s P(\hat{T}\not\sphericalangle)_s$ where $m_s \colon (W'_\Sigma)_s \rightarrowtail P(\hat{T}\not\sphericalangle)_s$ is a monomorphism in $\mathcal{C}$.
2. There are two important cases where the closure under diagonals in Definition 6.6.1 is trivially satisfied and thus can be dropped. First, if $S = 1$, clearly every subobject of $\mathsf{Rec}(\Sigma)$ is closed under diagonals. Secondly, if $\mathcal{C}$ is one of the categories of Example 2.2 and $\mathbb{U}_\Sigma$ contains all identity morphisms, one can show that every subobject of $\mathsf{Rec}(\Sigma)$ closed under $\mathbb{U}_\Sigma$-derivatives is closed under diagonals. This will hold in all our applications.

We are ready to state the main result of our paper, which holds under the Assumptions 2.1.

▶ **Theorem 6.8** (Variety Theorem).
1. *For each $\Sigma \in \mathbb{A}$, local varieties of languages over Σ and local pseudovarieties of Σ-generated $\mathbf{T}$-algebras form isomorphic lattices.*
2. *Varieties of languages and pseudovarieties of $\mathbf{T}$-algebras form isomorphic lattices. The isomorphism maps a pseudovariety $\mathcal{V}$ to the variety of all languages recognized by some algebra in $\mathcal{V}$.*

Proof sketch. Duality + Reiterman! For 1. one shows that a diagonal-closed subobject $W_\Sigma \subseteq \mathsf{Rec}(\Sigma)$, given by a monomorphism $(m_s \colon (W'_\Sigma)_s \rightarrowtail P(\hat{T}\not\sphericalangle)_s)_{s\in S}$ in $\mathcal{C}^S$ by Rem. 6.7.1, is closed under derivatives iff the dual epimorphism $(P^{-1}m_s \colon (\hat{T}\not\sphericalangle)_s \twoheadrightarrow P^{-1}(W'_\Sigma)_s)_{s\in S}$ in $\widehat{\mathcal{D}}^S$ carries a Σ-generated profinite $\widehat{\mathbf{T}}$-algebra. Then the Local Reiterman Theorem 5.3 gives the isomorphism. For 2. one shows that a family $(W_\Sigma)_{\Sigma \in \mathbb{A}}$ of local varieties is closed under preimages iff its dual family in $\widehat{\mathcal{D}}^S$ is a profinite theory, and uses the Reiterman Theorem. ◀

7 Applications

In this section, we derive some concrete variety theorems, including *new* results, as special instances of Theorem 6.8. In each case, we follow the three-step plan from the introduction.

Languages of finite words. Eilenberg's theorem [12] relates varieties of regular languages to pseudovarieties of monoids. It was later extended to ordered monoids [20], idempotent semirings [24] and K-algebras [26]. In [1, 3, 10] we unified all these results to an Eilenberg theorem for $\mathcal{D}$-monoids in a commutative variety $\mathcal{D}$, based on the dual view of automata as algebras and coalgebras. Recall that a variety $\mathcal{D}$ is *commutative* if for any $A, B \in \mathcal{D}$ the hom-set $\mathcal{D}(A, B)$ carries a subalgebra of $B^{|A|}$ in $\mathcal{D}$. A $\mathcal{D}$-*monoid* is an object $D \in \mathcal{D}$ with a monoid structure $(|D|, \bullet, 1)$ whose multiplication is a $\mathcal{D}$-*bimorphism*, i.e. for every $y \in |D|$ the maps $y \bullet - \colon |D| \to |D|$ and $- \bullet y \colon |D| \to |D|$ carry endomorphisms on D. Monoids in $\mathcal{D} = \mathbf{Set}$, $\mathbf{Pos}$, $\mathbf{JSL}_0$, $\mathbf{Vec}_K$ are classical monoids, ordered monoids, idempotent semirings and K-algebras, respectively. For any set Σ, the free $\mathcal{D}$-monoid $(\not\sphericalangle^*, \bullet, \varepsilon)$ on $\not\sphericalangle$ consists of the free $\mathcal{D}$-object $\not\sphericalangle^*$ on Σ^*, the multiplication $\bullet$ extending the concatenation of words, and the empty word ε. The variety theorem for $\mathcal{D}$-monoids emerges from our three steps as follows:

1. Let $\mathbf{T}_M$ be the free-monoid monad on $\mathcal{D}$. Thus $T_M \measuredangle = \measuredangle^*$ and $\mathbf{Alg}\,\mathbf{T}_M$ is isomorphic to the category of $\mathcal{D}$-monoids. A language $L\colon \measuredangle^* \to O_\mathcal{D}$ is $\mathbf{T}_M$-recognizable iff its restriction $L^@\colon \Sigma^* \to |O_\mathcal{D}|$ is a regular behavior, i.e. a function computed by some finite automaton with output set $|O_\mathcal{D}|$. If $|O_\mathcal{D}| \cong \{0,1\}$ (e.g., for $\mathcal{D} = \mathbf{Set}$, $\mathbf{Pos}$, $\mathbf{JSL}_0$), regular behaviors are exactly the characteristic functions of regular languages over Σ.

2. Generalizing Example 4.3.1, the free $\mathcal{D}$-monoid $\mathbf{T}_M \measuredangle = \measuredangle^*$ has the unary presentation $\mathbb{U}_\Sigma = \{\,\measuredangle^* \xrightarrow{y\bullet-} \measuredangle^*,\ \measuredangle^* \xrightarrow{-\bullet y} \measuredangle^* : y \in \Sigma^*\,\}$. Thus the $\mathbb{U}_\Sigma$-derivatives of a language $L\colon \measuredangle^* \to O_\mathcal{D}$ are (after identifying L with $L^@$) the classical derivatives of Example 6.3.1.

3. Let $\mathbb{A} = \mathbf{Set}_f$. Instantiating Definition 6.6 gives the notion of a *variety of regular behaviors in* $\mathcal{C}$: a family $(W_\Sigma \subseteq \mathsf{Rec}(\Sigma))_{\Sigma \in \mathbb{A}}$ of regular behaviors closed under $\mathcal{C}$-algebraic operations (see Rem. 3.4), derivatives and preimages of $\mathcal{D}$-monoid morphisms, i.e. $g^{-1}L \in W_\Delta$ for any $L\colon \measuredangle^* \to O_\mathcal{D}$ in W_Σ and any $\mathcal{D}$-monoid morphism $g\colon \gneqq^* \to \measuredangle^*$. Theorem 6.8 gives

▶ **Theorem 7.1** ([1,3,10])**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be varieties satisfying the Assumptions 2.1(i),(ii),(iv), and suppose that the variety* $\mathcal{D}$ *is commutative. Then the lattice of (local) varieties of regular behaviors in* $\mathcal{C}$ *is isomorphic to the lattice of (local) pseudovarieties of* $\mathcal{D}$-*monoids.*

Four special instances are listed below. The third column describes the $\mathcal{C}$-algebraic operations under which (local) varieties are closed, and the fourth one states what $\mathcal{D}$-monoids are. All correspondences are known in the literature, and are uniformly covered by Theorem 7.1.

$\mathcal{C}$	$\mathcal{D}$	(local) var. of behav. closed under	$\cong$	(local) pseudovarieties of	proved in
BA	**Set**	boolean operations		monoids	[12,14]
$\mathbf{DL}_{01}$	**Pos**	finite union and finite intersection		ordered monoids	[14,20]
$\mathbf{JSL}_0$	$\mathbf{JSL}_0$	finite union		idempotent semirings	[24]
$\mathbf{Vec}_K$	$\mathbf{Vec}_K$	addition of weighted languages		K-algebras	[26]

Polynomial varieties. Next, we derive Bojańczyk's polynomial variety theorem [8]. **1.** Let $\mathbf{T}$ be a monad on $\mathbf{Set}^S$. **2.** Choose the polynomial presentation of $\mathbf{T}\Sigma$. **3.** Let $\mathbb{A} = \mathbf{Set}_f^S$. Applying Def. 6.6, a *polynomial variety of languages* is a family of $\mathbf{T}$-recognizable languages closed under boolean operations, polynomial derivatives (see Ex. 6.2.3), and preimages of $\mathbf{T}$-homomorphisms. Thm. 6.8 gives Bojańczyk's variety theorem [8] and a new local version:

▶ **Theorem 7.2.** *The lattice of (local) polynomial varieties of* $\mathbf{T}$-*recognizable languages is isomorphic to the lattice of (local) pseudovarieties of* $\mathbf{T}$-*algebras.*

∞**-languages.** Finally, we derive two variety theorems for ∞-languages. For the first one, **1.** let $\mathbf{T} = \mathbf{T}_\infty$ on $\mathbf{Set}^2$. **2.** Choose $\mathbb{A} = \{\,(\Sigma, \emptyset) : \Sigma \in \mathbf{Set}_f\,\}$, and for each $\Sigma \in \mathbf{Set}_f$ the unary presentation of $\mathbf{T}_\infty(\Sigma, \emptyset) = (\Sigma^+, \Sigma^\omega)$ as in Example 4.3.2. **3.** Def. 6.6 yields the notion of a *variety of* ∞-*languages*: a family of ∞-regular languages closed under boolean operations, derivatives (see Example 6.3.2) and preimages of ω-semigroup morphisms. Theorem 6.8 gives

▶ **Theorem 7.3.** *The lattice of (local) varieties of* ∞-*languages is isomorphic to the lattice of (local) pseudovarieties of* ω-*semigroups.*

The non-local part is due to Wilke [19,35] and the local part is a new result, extending the local variety theorem of [14] to infinite words. Similarly, we can obtain an ordered version of Theorem 7.3: **1.** take the monad $\mathbf{T}_{\infty,\leq}$ on $\mathbf{Pos}^2$ representing *ordered* ω-*semigroups* (i.e. ω-semigroups on a poset with monotone products). **2.** Choose $\mathbb{A}$ and the unary presentation

of $\mathbf{T}_{\infty,\leq}(\nleq, \emptyset)$ as above. **3.** Since $\mathcal{C} = \mathbf{DL}_{01}$, Def. 6.6 gives *positive varieties of ∞-languages*, emerging from varieties by dropping closure under complement. Then Theorem 6.8 yields the theorem below. Its non-local part is due to Pin [21], and the local part is again a new result.

▶ **Theorem 7.4.** *The lattice of (local) positive varieties of ∞-languages is isomorphic to the lattice of (local) pseudovarieties of ordered ω-semigroups.*

The two above theorems do not follow from Theorem 7.1/7.2, which shows that our framework has a wider scope than the earlier work in [1, 3, 8, 10]. For many more applications, including variety theorems for tree languages [29] and cost functions [11], see the full paper [34].

8 Conclusions and Future Work

We presented a categorical framework for algebraic language theory that captures, as special instances, the bulk of the Eilenberg theorems in the literature for pseudovarieties of finite algebras and varieties of recognizable languages. Let us mention directions for future work.

First, we aim to investigate if it is possible to obtain a variety theorem for data languages based on nominal Stone duality [13]. On a similar note, it would also be interesting to see whether dualities modeling probabilistic phenomena (e.g. Gelfand or Kadison duality) lead to a meaningful algebraic language theory for probabilistic automata and languages.

Secondly, although *finite* structures are of most relevance from the automata-theoretic perspective, there has been some work on variety theorems with relaxed finiteness restrictions. One example is Reutenauer's theorem [26] for weighted languages over arbitrary fields K. To cover this in our setting the results of Section 5 should be presented for $(\mathcal{E}, \mathcal{M})$-structured categories $\mathcal{D}$ in lieu of varieties. This has been worked out in [33] and, independently, in the recent preprint [27]. In the latter, also a formal "Eilenberg correspondence" is stated for dual $(\mathcal{E}, \mathcal{M})$-categories. An important conceptual difference to our present work is that in loc. cit. one uses discrete dualities (e.g. complete atomic boolean algebras/sets instead of boolean algebras/Stone spaces) and that unary presentations do not appear. This makes the concept of a variety of languages (called a *coequational theory*) and the Eilenberg correspondence in [27] easy to state, but much harder to apply in practice. The results of [27] and of our paper do not entail each other, and we leave it for future work to find a common roof.

References

1   J. Adámek, S. Milius, R. Myers, and H. Urbat. Generalized Eilenberg Theorem I: Local Varieties of Languages. In A. Muscholl, editor, *Proc. FoSSaCS'14*, volume 8412 of *LNCS*, pages 366–380. Springer, 2014. Full version: `http://arxiv.org/pdf/1501.02834v1.pdf`.

2   J. Adámek, S. Milius, and H. Urbat. Syntactic monoids in a category. In *Proc. CALCO'15*, LIPIcs. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2015. Full version: `http://arxiv.org/abs/1504.02694`.

3   J. Adámek, R. Myers, S. Milius, and H. Urbat. Varieties of languages in a category. In *30th Annual ACM/IEEE Symposium on Logic in Computer Science*. IEEE, 2015.

4   J. Almeida. On pseudovarieties, varieties of languages, filters of congruences, pseudoidentities and related topics. *Algebra Universalis*, 27(3):333–350, 1990.

5   N. Bedon and O. Carton. An Eilenberg theorem for words on countable ordinals. In *Proc. LATIN'98*, volume 1380 of *LNCS*, pages 53–64. Springer, 1998.

6   N. Bedon and C. Rispal. Schützenberger and Eilenberg theorems for words on linear orderings. In *Proc. DLT'05*, volume 3572 of *LNCS*, pages 134–145. Springer, 2005.

7   G. Birkhoff. Rings of sets. *Duke Mathematical Journal*, 3(3):443–454, 1937.

8 M. Bojańczyk. Recognisable languages over monads. In I. Potapov, editor, *Proc. DLT'15*, volume 9168 of *LNCS*, pages 1–13. Springer, 2015. `http://arxiv.org/abs/1502.04898`.

9 L.-T. Chen, J. Adámek, S. Milius, and H. Urbat. Profinite monads, profinite equations and Reiterman's theorem. In B. Jacobs and C. Löding, editors, *Proc. FoSSaCS'16*, volume 9634 of *LNCS*. Springer, 2016. `http://arxiv.org/abs/1511.02147`.

10 L.-T. Chen and H. Urbat. A fibrational approach to automata theory. In L. S. Moss and P. Sobocinski, editors, *Proc. CALCO'15*, LIPIcs. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2015.

11 L. Daviaud, D. Kuperberg, and J.-É. Pin. Varieties of cost functions. In N. Ollinger and H. Vollmer, editors, *Proc. STACS 2016*, volume 47 of *LIPIcs*, pages 30:1–30:14. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2016.

12 S. Eilenberg. *Automata, Languages, and Machines Vol. B.* Academic Press, 1976.

13 M. J. Gabbay, T. Litak, and D. Petrişan. Stone duality for nominal boolean algebras with new. In A. Corradini, B. Klin, and C. Cîrstea, editors, *Proc. CALCO'11*, volume 6859 of *LNCS*, pages 192–207. Springer, 2011.

14 M. Gehrke, S. Grigorieff, and J.-É. Pin. Duality and equational theory of regular languages. In L. Aceto and al., editors, *Proc. ICALP'08, Part II*, volume 5126 of *LNCS*, pages 246–257. Springer, 2008.

15 P. T. Johnstone. *Stone spaces.* Cambridge University Press, 1982.

16 E. G. Manes. *Algebraic Theories*, volume 26 of *Graduate Texts in Mathematics*. Springer, 1976.

17 G. Matthiessen. Theorie der heterogenen Algebren. Technical report, Universität Bremen, 1976.

18 G. Matthiessen. A heterogeneous algebraic approach to some problems in automata theory, many-valued logic and other topics. In *Proc. Klagenfurt Conf.*, 1979.

19 D. Perrin and J.-É. Pin. *Infinite Words.* Elsevier, 2004.

20 J.-É. Pin. A variety theorem without complementation. *Russ. Math.*, 39:80–90, 1995.

21 J.-É. Pin. Positive varieties and infinite words. In *LATIN 98*, volume 1380 of *LNCS*, pages 76–87. Springer, 1998.

22 J.-É. Pin. Mathematical foundations of automata theory. Available at `http://www.liafa.jussieu.fr/~jep/PDF/MPRI/MPRI.pdf`, November 2016.

23 N. Pippenger. Regular languages and Stone duality. *Th. Comp. Sys.*, 30(2):121–134, 1997. URL: `http://link.springer.com/10.1007/BF02679444`, `doi:10.1007/BF02679444`.

24 L. Polák. Syntactic semiring of a language. In J. Sgall, A. Pultr, and P. Kolman, editors, *Proc. MFCS'01*, volume 2136 of *LNCS*, pages 611–620. Springer, 2001.

25 J. Reiterman. The Birkhoff theorem for finite algebras. *Algebra Universalis*, 14(1):1–10, 1982.

26 C. Reutenauer. Séries formelles et algèbres syntactiques. *J. Algebra*, 66:448–483, 1980.

27 J. Salamánca. Unveiling Eilenberg-type Correspondences: Birkhoff's Theorem for (finite) Algebras + Duality. `https://arxiv.org/abs/1702.02822`, February 2017.

28 S. Salehi and M. Steinby. Varieties of many-sorted recognizable sets. TUCS Technical Report 626, Turku Center for Computer Science, 2004.

29 S. Salehi and M. Steinby. Tree algebras and varieties of tree languages. *Theor. Comput. Sci.*, 377(1-3):1–24, 2007.

30 M. P. Schützenberger. On finite monoids having only trivial subgroups. *Inform. and Control*, 8:190–194, 1965.

31 H. Straubing. On logical descriptions of regular languages. In S. Rajsbaum, editor, *LATIN 2002 Theor. Informatics*, volume 2286 of *LNCS*, pages 528–538. Springer, 2002.

32 D. Thérien. *Classification of Regular Languages by Congruences*. PhD thesis, University of Waterloo, 1980.

33 H. Urbat. A note on HSP theorems. `https://www.tu-braunschweig.de/Medien-DB/iti/hspnote.pdf`.

34 H. Urbat, J. Adámek, L.-T. Chen, and S. Milius. Eilenberg Theorems for Free. `https://arxiv.org/abs/1602.05831`. February 2017.

35 T. Wilke. An Eilenberg theorem for ∞-languages. In *Proc. ICALP'91*, volume 510 of *LNCS*, pages 588–599. Springer, 1991.