Another Characterization of the Higher K-Trivials*

Paul-Elliot Angles d'Auriac[1] and Benoit Monin[2]

1 **UPEC LACL, Creteil, France**
panglesd@lacl.fr
2 **UPEC LACL, Creteil, France**
benoit.monin@computability.fr

Abstract

In algorithmic randomness, the class of K-trivial sets has proved itself to be remarkable, due to its numerous different characterizations. We pursue in this paper some work already initiated on K-trivials in the context of higher randomness. In particular we give here another characterization of the non hyperarithmetic higher K-trivial sets.

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1 Introduction

Algorithmic randomness defines what it is for an infinite 0-1 valued sequence to be random. It takes its roots deeply in computability : lots of definition and techniques from pure computability are used in algorithmic randomness, as hierarchies, reducibilites and forcing constructions. The research in this field led to the identification of many different randomness notions, the most known being perhaps Martin-Löf randomness: a sequence is Martin-Löf random if it is in no Π^0_2 set $\bigcap_n \mathcal{U}_n$ where the Lebesgue measure of each $\mathcal{U}_n$ is smaller than 2^{-n}. The reader can refer to [21] and [8] for more details on algorithmic randomness. One of the main research area is to study how the different classes of random sequences relate. For a given such class of randoms, another important research area is to study the sets relative to which this class does not change. This is called *lowness for randomness*. For example, the class of K-trivials are exactly the low for Martin-Löf randomness. This class is defined as the set of infinite sequences having minimal (up to a constant) Kolmogorov complexity on their prefixes, that is the Kolmogorov complexity of a prefix should not be bigger than its length [1]. The class of K-trivials proved itself to be remarkable, due to its numerous very different characterizations [20], [12], [6], [1], [9].

Another field has a lot of interactions with computability theory : descriptive set theory. This field can be studied completely independently from recursion theory as in [14]. However, the study of descriptive set theory in close relation with computability appeared to be a fruitful approach. The mix of these two fields is called effective descriptive set theory and can be used to prove lots of results from the classical version of descriptive set theory. This is done mainly in [19]. Effective descriptive set theory also gave rise to higher computability.

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1 Here it is important that we use the so called *prefix-free Kolmogorov complexity*, as it is the case in general with algorithmic randomness.

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The notion of computation for this field comes from a very logical point of view, far from any implementation. Nonetheless it is possible to give an intuition of higher computation which is closer to what computer scientists are used to: one can view a higher computation as a regular computation (by a Turing machine, say) where the steps of computation are carried through the computable ordinals. This new way of computing has several things in common with the classical one, and of course some differences (as the lack of continuity in a computation) that may cause trouble when trying to lift some computability theorems from the classical setting to the higher setting. The reader can refer to [22], [5] and [18] for more details on higher recursion theory. The reader can also see [11] for more information about what could be a Turing machine which keeps running over ordinal times of computation.

Algorithmic randomness naturally arises from mixing probability theory and computability. Following the same ideas, researchers defined notions of higher randomness, obtained analogously, but by considering higher computability instead of computability. After the founder paper of the field [13], a lots of advances were made by several researchers ([4], [3], [2], [10]). The reader can also refer to [5] and [18] for more details on higher randomness. One of the notion which has previously been studied and which is the core subject of the paper, is the notion of higher K-triviality, the direct higher analogue of K-triviality. In particular, we give in this paper a characterization of the non-Δ^1_1 higher K-trivials, by proving that they are exactly the sets that shrink the class $\mathrm{W}\Pi^1_1\mathrm{R}$ to the class Π^1_1-$\mathrm{ML}\langle\mathcal{O}\rangle$ when relativizing continuously. This characterization is specific to the higher setting: the randomness notions that are equivalent to $\mathrm{W}\Pi^1_1\mathrm{R}$ and Π^1_1-$\mathrm{ML}\langle\mathcal{O}\rangle$ in the lower setting, coincide.

2 Preliminaries

2.1 Notations

In this paper, we work in the space of infinite sequences of 0's and 1's, called the Cantor space, denoted by 2^ω. We call *strings* finite sequences of 0's and 1's and *sequences* or *sets* elements of the Cantor space. For a sequence A we write $A\upharpoonright_n$ to denote the string equal to the n first bits of A. The space of strings is denoted by $2^{<\omega}$ and the space of strings of length smaller than n is denoted by $2^{<n}$. For a string σ, we denote the set of sequences extending σ by $[\sigma]$.

The topology on Cantor space is generated by the basic intervals $[\sigma] = \{X \in 2^\omega \mid X \succ \sigma\}$ for any string σ. For a set of strings $W \subseteq 2^{<\omega}$, we let $[W] = \bigcup_{\sigma\in W}[\sigma]$. A set of string W is said to be prefix-free if no string in W is a prefix of another string in W.

For $\mathcal{A} \subseteq 2^\omega$ Lebesgue-measurable, $\lambda(\mathcal{A})$ denotes the Lebesgue measure of $\mathcal{A}$, which is the unique Borel measure such that $\lambda([\sigma]) = 2^{-|\sigma|}$ for all strings σ.

We assume that the reader is familiar with basic notions of computability. For $A, B \in 2^\omega$ we write $A \leq_T B$ if A is Turing reducible to B. We denote by $\emptyset'$ the halting problem. We also assume that the reader is familiar with the basics of effective descriptive set theory, in particular with the notations $\Sigma^0_1, \Pi^0_1, \Sigma^0_2, \Pi^0_2$, *etc...*

We finally also assume that the reader is familiar with the notion of Kolmogorov complexity. In this paper, we will only consider a prefix-free version of the Kolmogorov complexity (used in the definition of K-triviality): using compressors $M : 2^{<\omega} \to 2^{<\omega}$ such that the domain of M is prefix-free. It is an easy exercise to show that an optimal prefix-free compressor exists (optimal up to a constant of course).

2.2 Background on algorithmic randomness

In 1966, Martin-Löf gave in [16] a definition capturing elements of the Cantor space that can be considered 'random'. Many nice properties of the Martin-Löf random sequences make this notion of randomness one of the most interesting and one of the most studied.

Intuitively a random sequence should not have any atypical property. A property is here considered atypical if the set of sequences sharing this property is of measure 0. It also makes sense to consider only properties which can be described in some effective way (because any X has the property of being in the set $\{X\}$ and thus nothing would be random).

▶ **Definition 1.** An intersection of measurable sets $\bigcap_n \mathcal{A}_n$ is said to be *effectively of measure* 0 if the function which to n associates the measure of $\mathcal{A}_n$ is bounded by 2^{-n}. A *Martin-Löf test*, or an ML-test is a Π^0_2 set $\bigcap_n \mathcal{U}_n$ effectively of measure 0. We say that $X \in 2^\omega$ is *Martin-Löf random* if it is in no Martin-Löf test. The class of Martin-Löf randoms is also referred to as the class MLR.

The requirement for a Martin-Löf test to be effectively of measure 0 is important and leads to very nice properties. In particular there exists a universal Martin-Löf test, i.e. a test containing all the others (see [16]). This is not the case anymore if we drop the 'effectively of measure 0' condition. Instead we get a notion known as weak-2-randomness.

▶ **Definition 2.** A Π^0_2 nullset is called a *weak-2 test* or a W2 test. We say that $X \in 2^\omega$ is *weakly-2-random* if it is in no weak-2 test. The class of weakly-2-randoms is also referred to as the class W2R.

As a randomness notion, weak-2-randomness is a strictly stronger than 1-randomness: tests can capture more elements and thus there are fewer randoms. For any given randomness notion, it makes sense to relativize it to any oracle:

▶ **Definition 3.** Let $A \in 2^\omega$. An ML^A test is a $\Pi^0_2(A)$ set $\bigcap_n O_n$ effectively of measure 0. We say that $X \in 2^\omega$ is MLR^A if it is in no ML^A test. Similarly a $\mathrm{W2}^A$ test is a $\Pi^0_2(A)$ nullset, and we say that X is $\mathrm{W2R}^A$ if it is in no $\mathrm{W2}^A$ test.

A nice characterization of W2R has been given from restricting the relativization $\mathrm{MLR}^{\emptyset'}$: we can only use $\emptyset'$ to find the indices of the open sets in a test.

▶ **Definition 4.** Let $(W_e)_{e\in\omega}$ be an effective enumeration of the c.e. sets of strings. A $\mathrm{ML}\langle\emptyset'\rangle$ test is a set $\bigcap_n [W_{f(n)}]$ with $\lambda([W_{f(n)}]) \leq 2^{-n}$ were $f : \omega \to \omega$ is computable from $\emptyset'$. A set is $\mathrm{MLR}\langle\emptyset'\rangle$ if it is in no $\mathrm{ML}\langle\emptyset'\rangle$ test.

Note that with the full relativization of an ML test to A, the oracle A itself is not needed to find the index of the n-th $\Sigma^0_1(A)$ open set of the test: the use of A for that can be swallowed in the process of enumerating each $\Sigma^0_1(A)$ component of the test.

Going back to the previous definition, we have the following easy theorem:

▶ **Theorem 5** ([2], section 7)**.** $\mathrm{W2R} = \mathrm{ML}\langle\emptyset'\rangle$.

Proof. Let's start with $\mathrm{W2R} \subseteq \mathrm{ML}\langle\emptyset'\rangle$, given a $\mathrm{ML}\langle\emptyset'\rangle$ test $\bigcap_n \mathcal{U}_{f(n)}$, we will show that it is included in a W2R test. We define $V_{\langle m,t\rangle} = \bigcup_{s\geq t} \mathcal{U}_{f_s(m)}$. As $\bigcap_n V_n = \bigcap_n \mathcal{U}_{f(n)}$, we have that $\lambda(\bigcap_n \mathcal{V}_n) = 0$, so this is a W2R test.

Now, let $\bigcap_n \mathcal{U}_n$ be a Π^0_2 nullset, one can use $\emptyset'$ to find uniformly in n the first $m = f(n)$ such that $\lambda(\mathcal{U}_m) \leq 2^{-n}$. ◀

The sets relative to which $\mathrm{MLR}^A = \mathrm{MLR}$ have been extensively studied, and have been identified as the class of K-trivial sets.

▶ **Definition 6.** A set $A \in 2^\omega$ is K-trivial if for any n, the prefix-free Kolmogorov complexity of $A\restriction_n$ is smaller than the prefix-free Kolmogorov complexity of n (up to a constant).

The reader can refer to [21] for more details on the K-trivials. They are also the sets relative to which $\mathrm{W2R}^A = \mathrm{W2R}$:

▶ **Theorem 7** ([20], [15], [7])**.** *The following are equivalent for a set A:*
1. *A is K-trivial*
2. $\mathrm{W2R}^A = \mathrm{W2R}$
3. $\mathrm{MLR}^A = \mathrm{MLR}$

As we will see, this characterization fails in the higher setting, but it fails in a way that will help us provide another characterization of the higher K-trivials.

2.3 Background on higher computability

We assume that the reader is familiar with the concepts of Δ^1_1, Π^1_1 and Σ^1_1 subsets of ω and of 2^ω. A known result is that an open set $\mathcal{U}$ is Π^1_1 if and only if there exists a Π^1_1 set of strings W such that $\mathcal{U} = [W]^\prec$.

There is a strong analogy between classical concepts in computability (referred to as the *lower setting*) and their analogue in higher computability (referred to as the *higher setting*). For instance, Δ^1_1 can be seen as a higher analogue of computable, and Π^1_1 can be seen as a higher analogue of computably enumerable, with the difference that the times at which elements are enumerated are now computable ordinals.

We refer to the set of codes for computable ordinals (using whichever equivalent coding) as Kleene's $\mathcal{O}$. As usual, the smallest non-computable ordinal is denoted by ω_1^{CK}.

We recall here a few definitions about continuous higher Turing reductions. In [2] (Definition 1.1) higher Turing reductions are defined to compute elements of 2^ω. In [10] (section 3.2) this definition is extended in a straightforward way, to compute elements of $(\omega_1^{CK})^\omega$. We also extend this definition here in a straightforward way, to compute elements of $(\omega_1^{CK})^{\omega_1^{CK}}$.

An absolutely formal definition of computations of functions from ω_1^{CK} to ω_1^{CK} should either use the language of set theory and deals with actual ordinals, or use a unique notation system for computable ordinals. There exists such a Π^1_1 notation system $\mathcal{O}_1 \subseteq \omega$ (see [22] or [18], 3.6.1) and up to this notation system, one can view a function from ω_1^{CK} to ω_1^{CK} as a function from $\mathcal{O}_1$ to $\mathcal{O}_1$, and thus simply defined on integers.

▶ **Definition 8** ([10] [2])**.** We say that A higher Turing computes (or higher computes) $f : \omega_1^{CK} \mapsto \omega_1^{CK}$ (respectively $g : \omega \mapsto \omega_1^{CK}$) if there exists a Π^1_1 set $C \subseteq 2^{<\omega} \times \omega_1^{CK} \times \omega_1^{CK}$ (respectively $C \subseteq 2^{<\omega} \times \omega \times \omega_1^{CK}$) such that $f(o_1) = o_2$ iff $\exists \sigma \prec A\ (\sigma, o_1, o_2) \in C$ (respectively $g(n) = o$ iff $\exists \sigma \prec A\ (\sigma, n, o) \in C$). We say that A higher Turing computes $B \in 2^\omega$ if A higher Turing computes the characteristic function of B.

In [2] it is shown that Kleene's $\mathcal{O}$ higher Turing computes a set $A \in 2^\omega$ iff $\mathcal{O}$ Turing computes A.

2.4 Background on higher randomness

Higher randomness goes back to Martin-Löf who promoted the notion of Δ^1_1-randomness (already defined by Sacks [22]), defending the idea that it was the appropriate mathematical concept of randomness [17]. Even if his first definition undoubtedly became the most successful over the years, this other definition recently got a second wind on the initiative of Hjorth and Nies who started to study the analogy between the usual notions of randomness and their higher counterparts. In order to do so they created in [13] a higher analogue of Martin-Löf randomness.

▶ **Definition 9** (Hjorth, Nies)**.** A Π^1_1-Martin-Löf test, or a Π^1_1-ML test, is given by an effectively null intersection of open sets $\bigcap_n \mathcal{U}_n$, each $\mathcal{U}_n$ being Π^1_1 uniformly in n. A sequence X is Π^1_1-Martin-Löf random if it is in no Π^1_1-Martin-Löf test. The class of Π^1_1-Martin-Löf randoms is also referred to as the class Π^1_1-MLR.

The higher analogue of weak-2-randomness has also been studied (see [4] [2]):

▶ **Definition 10.** We say that X is *weakly-*Π^1_1*-random* if it belongs to no $\bigcap_n \mathcal{U}_n$ with each $\mathcal{U}_n$ open set Π^1_1 uniformly in n and with $\lambda(\bigcap_n \mathcal{U}_n) = 0$. The class of weakly-$\Pi^1_1$-randoms is also referred to as the class $\mathrm{W}\Pi^1_1\mathrm{R}$.

It is also possible to define an analogue of $\mathrm{MLR}\langle\emptyset'\rangle$ in the higher setting, using Kleene's $\mathcal{O}$ in place of $\emptyset'$.

▶ **Definition 11.** Let $(W_e)_{e\in\omega}$ be an enumeration of the Π^1_1 sets of strings. A Π^1_1-$\mathrm{ML}\langle\mathcal{O}\rangle$ test is a set $\bigcap_n [W_{f(n)}]$ with $\lambda([W_{f(n)}]) \leq 2^{-n}$ were $f : \omega \to \omega$ is Turing computable from Kleene's $\mathcal{O}$. A set is Π^1_1-$\mathrm{MLR}\langle\mathcal{O}\rangle$ if it is in no Π^1_1-$\mathrm{ML}\langle\mathcal{O}\rangle$ test.

Theorem 5 does not lift to the higher setting. The proof in the lower setting uses what has been defined in [2] to be a 'time trick': we use the fact that time and space are the same objects: the natural numbers. In the higher setting, this is not anymore true as the time goes along the ordinals. It is in fact possible to show that the class Π^1_1-$\mathrm{MLR}\langle\mathcal{O}\rangle$ is strictly contained in the class $\mathrm{W}\Pi^1_1\mathrm{R}$. To be more specific, let us introduce maybe the most important notion of higher randomness, first given by Sacks, and made possible by a theorem of Lusin saying that even though Π^1_1 sets are not necessarily Borel, they remain all measurable.

▶ **Definition 12** (Sacks)**.** We say that $X \in 2^\omega$ is Π^1_1-Random if it is in no Π^1_1 nullset.

We have the following:

▶ **Theorem 13** ([2])**.** Π^1_1-$\mathrm{MLR}\langle\mathcal{O}\rangle \subsetneq \Pi^1_1$-Randoms $\subsetneq \mathrm{W}\Pi^1_1\mathrm{R} \subsetneq \Pi^1_1$-MLR

We finally give another characterization of Π^1_1-$\mathrm{ML}\langle\mathcal{O}\rangle$, that has no counterpart in the lower setting (with $\mathrm{ML}\langle\emptyset'\rangle$ in place of Π^1_1-$\mathrm{ML}\langle\mathcal{O}\rangle$), and which will be useful in the paper.

▶ **Property 14** ([2])**.** *The following are equivalent for a sequence $X \in 2^\omega$:*
1. *X is Π^1_1-$\mathrm{ML}\langle\mathcal{O}\rangle$ random*
2. *X does not belong to any test $(\mathcal{U}_s)_{s<\omega_1^{CK}}$ not necessarily nested where each $\mathcal{U}_s$ is a Π^1_1 open set uniformly in s, and such that $\lambda(\bigcap_{s<\omega_1^{CK}} \mathcal{U}_s) = 0$*

2.5 Continuous relativization of higher randomness

It is also possible to define an analogue of K-triviality in the higher setting. The higher K-trivials are defined analogously, but using a version of Kolmogorov complexity with Π^1_1-prefix-free compression machines.

▶ **Definition 15** ([13]). We define:
- The *higher prefix-free Kolmogorov complexity* is given by $K(y) = \min\{|\sigma| : U(\sigma) = y\}$ for U the universal prefix-free Π^1_1-machine given by $U(0^e 1\tau) = M_e(\tau)$ and $(M_e)_{e\in\omega}$ a uniform enumeration of the Π^1_1 prefix-free machines.
- A is *higher K-trivial* if $\exists b\ \forall n\ K(A \upharpoonright n) \leq K(n) + b$.

However, for the higher K-trivials to also be low for Π^1_1-MLR, one has to be careful about the way things are relativized to oracles. In higher computability we don't have anymore the continuity aspect of the lower setting : if B is $\Delta^1_1(A)$, it does not mean that a finite quantity of A suffices to know a finite quantity of B. However, we can force this state of things, as done previously with the notion of higher Turing computations. We next define what it means to relativize the notion of Π^1_1 set, continuously to an oracle.

▶ **Definition 16** ([2]). An oracle-continuous Π^1_1 set of integers is given by a set $W \subseteq 2^{<\mathbb{N}} \times \mathbb{N}$. For a string σ we write W^σ to denote the set $\{n : \exists\tau \prec \sigma, (\tau, n) \in W\}$. For a sequence X we write W^X to denote the set $\{n : \exists\tau \prec X, (\tau, n) \in W\}$. The set W^X is then called an X-continuous Π^1_1 set of integers.

An open set $\mathcal{U}$ is X-continuously Π^1_1 if there is an X-continuously Π^1_1 set of strings W such that $\mathcal{U} = [W^X]$.

We are now ready to define continuous relativization of randomness notions :

▶ **Definition 17.** If A is a set, we say that X is $\mathrm{W}\Pi^1_1\mathrm{R}^A$ if it is in no $\mathcal{U} = \bigcap_n \mathcal{U}_n$ where $(\mathcal{U}_n)_{n\in\omega}$ is a uniform family of A-continuous Π^1_1 open sets, such that $\lambda(\mathcal{U}) = 0$. We say that X is Π^1_1-MLR^A if it is in no $\bigcap_n \mathcal{U}_n$ where $(\mathcal{U}_n)_{n\in\omega}$ is a uniform family of A-continuous Π^1_1 open sets, such that $\lambda(\mathcal{U}_n) \leq 2^{-n}$.

We now have the following:

▶ **Theorem 18** ([2]). *The higher K-trivials are exactly the low for* Π^1_1-MLR*, using continuous relativization.*

Unlike in the lower setting, the higher K-trivials are not anymore the low for $\mathrm{W}\Pi^1_1\mathrm{R}$. For A Δ^1_1 (a special case of being higher K-trivial), it is still obviously the case that $\mathrm{W}\Pi^1_1\mathrm{R}^A = \mathrm{W}\Pi^1_1\mathrm{R}$. But if A is K-trivial and not Δ^1_1, we will actually see that $\mathrm{W}\Pi^1_1\mathrm{R}^A = \Pi^1_1\text{-ML}\langle\mathcal{O}\rangle$.

3 Another characterization of the higher K-trivials

3.1 Collapsing approximations

When trying to lift the Δ^0_2 definitions from the lower to the higher setting, some new possibilities appear. In the lower setting, for an approximation of A the set $\{A_t : t < s\}$ is always finite as s ranges over the natural numbers. So in particular it is closed. At the contrary, when s is an ordinal, the set $\{A_t : t < s\}$ may not have this property, which leads us to define a different type of approximation, which depends on the topological properties of $\{A_t : t < s\}$.

▶ **Property/Definition 19** ([2]).
1. *A sequence A is higher Δ^0_2 if it satisfies the following equivalent properties :*
   (a) $A \leq_T \mathcal{O}$
   (b) *There is a higher computable sequence $(A_s)_{s<\omega_1^{CK}}$ with $\lim_{s\to\omega_1^{CK}} A_s = A$*
2. *A computable approximation $(A_s)_{s<\omega_1^{CK}}$ converging to A is said to be collapsing if for every stage s, the set A is not in the closure of $\{A_t : t < s\}$.*

Such approximations are called collapsing, because they can be used to "collapse" ω_1^{CK} to a computable ordinal in a strong way: such approximations can be used to compute an ω-sequence of computable ordinals, with ω_1^{CK} as a supremum:

▶ **Property 20** ([2]). *Every sequence A with a collapsing approximation, higher Turing computes a function $f : \omega \to \omega_1^{CK}$ which is cofinal in ω_1^{CK}.*

In classical computability, given an effectively open set $\mathcal{U}$, it is uniformly possible to obtain a c.e. and prefix-free set W such that $[W] = \mathcal{U}$. However, in the setting of higher computability, it can be proved that this is no more possible: there is a Π^1_1 open set $\mathcal{U}$ such that for every prefix-free Π^1_1 set of strings W, $\mathcal{U} \neq [W]$. But working relative to some sets that have a collapsing approximation allows us to use time tricks, and in a way brings us "closer" to classical computability.

▶ **Property 21** ([2],end of page 20). *If A has a collapsing approximation $(A_s)_{s<\omega_1^{CK}}$, and $\mathcal{U}$ is an oracle-continuous Π^1_1 open set, then there exists an oracle-continuous Π^1_1 set $W \subseteq 2^{<\omega} \times 2^{<\omega}$ such that $\mathcal{U}^A = [W^A]$ and for all B, the set W^B is prefix-free.*

Proof. We define an enumeration of a Π^1_1 oracle-continuous set W. The enumeration of W will compute throughout its stages a collapsing approximation $(A_s)_{s<\omega_1^{CK}}$ of A. At stage s, if A_s is not in the closure of $\{A_t\}_{t<s}$, then let $\tau \prec A_s$ be the smallest such that τ has never been a prefix of some A_t for $t < s$. Then enumerate into W^{τ}_{s+1} all strings σ of length smaller than or equal to $|\tau|$ such that $[\sigma] \subseteq \mathcal{U}^{\tau}_s$ but $[\sigma]$ is disjoint from $[W^{\tau}_s]$.

It is clear that $[W^A] \subseteq \mathcal{U}^A$. Let us argue that $\mathcal{U}^A \subseteq [W^A]$. Suppose $\sigma \in \mathcal{U}^A$. There are sequences $\{\tau_n\}_{n\in\omega}$ and $\{s_n\}_{n\in\omega}$ such that for every n, the ordinal s_n is the first for which we have $A_{s_n}\upharpoonright_{|\tau_n|} = A\upharpoonright_{|\tau_n|} = \tau_n$, and such that $\sup_n s_n = \omega_1^{CK}$.

Let n be the smallest such that $|\tau_n| > |\tau_{n-1}| \geq |\sigma|$ and such that $\sigma \in \mathcal{U}^{\tau_n}_{s_n}$. Then we have by construction that $\sigma \subseteq [W^{\tau_n}_{s_n+1}]$. Therefore $[W^A] = \mathcal{U}^A$. Also by construction W^B is prefix-free for every B. ◀

3.2 Properties of higher K-Trivials

One key property of the higher K-trivial sequences is that they have a collapsing approximation as long as they are not Δ^1_1.

▶ **Property 22** ([2]). *Every higher K-trivial, but not Δ^1_1, sequence has a collapsing approximation.*

▶ **Corollary 23.** *If A is higher K-trivial but not Δ^1_1, then:*
- *A higher Turing computes a function $f : \omega \to \omega_1^{CK}$ whose range is unbounded in ω_1^{CK} ;*
- *if $\mathcal{U}$ is an oracle-continuous Π^1_1 open set, one can uniformly find a Π^1_1 oracle-continuous set of strings W such that $\mathcal{U}^A = [W^A]$ and $\forall B \in 2^\omega$, W^B is prefix-free.*

Proof of the corollary. By property 22, together with property 20 and 21. ◀

The proof that the low for Martin-Löf randoms are exactly the K-trivials requires a big machinery. Using the fact that higher K-trivials have a collapsing approximation, it is possible to transpose this proof and to show that the continuously low for Π^1_1-MLR are the higher K-trivials. The machinery developed in this proof can also be used to show a slightly more general statement, known in the lower setting as the "Main Lemma". One can find a detailed proof and explanation of this result for the higher setting in [2]. We give here a version of the Main Lemma which is closer to our need than the one in [2] (using oracle-continuous open sets in place of oracle-continuous discrete semi-measures) :

▶ **Theorem 24** (Main Lemma). *If A is higher K-trivial, $(A_s)_{s<\omega_1^{CK}}$ is any collapsing approximation of A, and W is an oracle-continuous Π^1_1 set of strings such that there exists $c \in \omega$ such that for all X we have $\sum_{\sigma \in W^X} 2^{-|\sigma|} \leq c$, then there exists a higher computable function $q : \omega_1^{CK} \to \omega_1^{CK}$ such that:*

$$S = \sum_{r<\omega_1^{CK}} \sum_{\sigma \in E_r} 2^{-|\sigma|} \text{ is finite}$$

where

$$E_r = \left\{ \sigma : \begin{array}{l} \sigma \in W^A[q(r)] \text{ with use } u, \text{ and} \\ A[q(r)] \upharpoonright u \neq A[q(r+1)] \upharpoonright u \end{array} \right\}.$$

Intuitively, if A is higher K-trivial, we can slow down its approximations in such a way that not too much measure is added in the open set, with pieces of oracle that were believed at some point to be prefixes of A but in fact are not: the total sum of 'wrong' measure added this way over the times of computation can be made finite.

3.3 A higher K-trivial and not Δ^1_1 implies $\Pi^1_1\text{-ML}\langle\mathcal{O}\rangle = \text{W}\Pi^1_1\text{R}^A$

▶ **Theorem 25.** *If A is higher K-Trivial and not Δ^1_1, then $\text{W}\Pi^1_1\text{R}^A \subseteq \Pi^1_1\text{-MLR}\langle\mathcal{O}\rangle$.*

Proof. Fix an A. By contrapositive, we prove that if X is captured by a $\Pi^1_1\text{-ML}\langle\mathcal{O}\rangle$ test, then it is also captured by a $\text{W}\Pi^1_1\text{R}^A$ test. We use the characterization 14 of $\Pi^1_1\text{-ML}\langle\mathcal{O}\rangle$ tests, so let $\mathcal{U} = \bigcap_{s<\omega_1^{CK}} \mathcal{U}_s$ be such a test.

We make use of the corollary 23 that A higher computes a function f with cofinality ω_1^{CK}. Let $g(\langle m,n\rangle)$ be the m-th element of $\mathcal{O}_{\leq f(n)} \subseteq \mathbb{N}$ (where $\mathcal{O}_{\leq\alpha}$ is the set of codes for computable ordinals smaller than α). Then g is also higher computable from A, and its range is all the computable ordinals. Now, we consider $\bigcap_n \mathcal{U}_{g(n)}$. As the range of g is ω_1^{CK}, the intersection is equal to $\mathcal{U}$, so its measure is 0 and as g is higher computable from A, this set is a $\text{W}\Pi^1_1\text{R}^A$ test. ◀

The other inclusion will be a corollary of a more general theorem, whose proof follows the same spirit than the proof in the lower setting that K-trivials are low for W2R.

▶ **Theorem 26.** *Let A be higher K-trivial. Let $G = \bigcap_n \mathcal{U}_n$ where $(\mathcal{U}_n)_{n\in\omega}$ is a uniform family of Π^1_1 open sets, continuously in A. Then there exists a set $S = \bigcap_{s<\omega_1^{CK}} \mathcal{V}_s$ where $(\mathcal{V}_s)_{s<\omega_1^{CK}}$ is a uniformly Π^1_1 family of open sets, such that $\lambda(S) = \lambda(G)$ and $S \supseteq G$.*

We will first prove the result for the simplest G, that is when the family is reduced to a single open set $\mathcal{U}$, and then extend this result to a uniform countable intersection of such open sets.

▶ **Lemma 27.** *Let A be higher K-trivial. Let G be a A-continuously Π^1_1 open set. Then there exists a set $S = \bigcap_{s<\omega_1^{CK}} \mathcal{V}_s$ where $(\mathcal{V}_s)_{s<\omega_1^{CK}}$ is a uniformly Π^1_1 family of open sets, such that $\lambda(S) = \lambda(G)$ and $S \supseteq G$.*

Proof. Using the property 21, there exists an oracle-continuous Π^1_1 set of strings W such that $G = [W^A]$, and such that W^B is prefix-free for all B.

If A is Δ^1_1 we are done. Otherwise, as it is higher K-trivial, it has a collapsing approximation, so we can try to use it to approximate G with Π^1_1 open sets $\mathcal{V}_s$. A first candidate for $\mathcal{V}_s$ could be $\bigcup_{s\leq r<\omega_1^{CK}} W^A[r]$, because every such $\mathcal{V}_s$ would contain G, but this approximation is "too large", because W and approximations of A can be such that $W^A[s]$ enumerates the empty word for a family of s cofinal in ω_1^{CK}.

The trick to prevent the measure to increase is to restrain the computation of $W^A[s]$ only to some special stages and parts of the oracle, so that the weight of all errors is finite. These stages are given by the Main Lemma : let $q : \omega_1^{CK} \to \omega_1^{CK}$ be the function given by the Main Lemma, applied to W. We then have:

$$\sum_{r<\omega_1^{CK}} \sum_{\sigma \in E_r} 2^{-|\sigma|} \text{ is finite}$$

where

$$E_r = \left\{ \sigma : \begin{array}{l} \sigma \in W^A[q(r)] \text{ with use } u, \text{ and} \\ A[q(r)] \upharpoonright u \neq A[q(r+1)] \upharpoonright u \end{array} \right\}.$$

Now we define $\mathcal{V}_s$ by computing only over the special stages and prefixes, that is

$$\sigma \in \mathcal{V}_s \Leftrightarrow \exists r \geq s \text{ such that } \sigma \in W^A[q(r)].$$

Every $\mathcal{V}_s$ contains G as any string σ enumerated in W^A, with use u, will be in every $W^A[q(r)]$ for $r \geq t$ such that $A[q(t+1)]$ has settled on $A \upharpoonright u$.

Now consider the errors of the $\mathcal{V}_s$, that is the strings σ enumerated in $\mathcal{V}_s$ but such that $[\sigma] \not\subseteq G$. There must exists an $r \geq s$ such that $\sigma \in W^A[q(r)]$ with use u, and such that $A[q(r)] \upharpoonright u \neq A[q(r+1)] \upharpoonright u$. Then $\sigma \in E_r$ for some $r \geq s$. It follows that:

$$\lambda(\mathcal{V}_s \setminus G) \leq \sum_{s \leq r < \omega_1^{CK}} \sum_{\sigma \in E_r} 2^{-|\sigma|}.$$

But as the total sum is finite, the partial sum goes to zero as s increases:

$$\lim_{s \to \omega_1^{CK}} \lambda(\mathcal{V}_s \setminus G) = 0.$$

Finally with $S = \bigcap_{s<\omega_1^{CK}} \mathcal{V}_s$, we have $\lambda(S \setminus G) = 0$ and $S \supseteq G$, which concludes the proof of the lemma. ◀

proof of Theorem 26. It remains to prove using this lemma the more general case when $G = \bigcap_{n\in\omega} \mathcal{U}_n$. We can apply what we just proved to $R = \bigcup_{e\in\omega} 0^e 1[W_e]$ where $(W_e)_{e\in\omega}$ is an effective listing of the A-continuous Π_1^1 sets. We then find $T = \bigcap_{s<\omega_1^{CK}} T_s$ with $T \supseteq R$ and $\lambda(T) = \lambda(R)$.

Let f be a computable function such that $\mathcal{U}_n = [W_{f(n)}]$. Writing $A|w = \{X : wX \in A\}$, we let $S = \bigcap_{n\in\omega}(T \mid 0^{f(n)}1)$. Let us show that S works for our purpose. First S is a Π_1^1-ML$\langle\mathcal{O}\rangle$ test, by the characterization 14 of these tests, as

$$S = \bigcap_{n\in\omega} \left(\left(\bigcap_{s<\omega_1^{CK}} T_s \right) \mid 0^{f(n)}1 \right) = \bigcap_{\omega s+n<\omega_1^{CK}} T_s \mid 0^{f(n)}1.$$

Then $S \supseteq G$ as for every n, we have $T \mid 0^{f(n)}1 \supseteq R \mid 0^{f(n)}1 = [W_{f(n)}] = \mathcal{U}_n \supseteq G$. Finally, we show that $\lambda(S \setminus G) = 0$. We have:

$$S - G = \bigcap_{n\in\omega} S - [W_{f(n)}] \subseteq \bigcap_{n\in\omega} T \mid 0^{f(n)}1 - [W_{f(n)}].$$

But $\lambda(T \mid 0^{f(n)}1 - [W_{f(n)}]) \leq 2^{f(n)+1}\lambda(T - R) = 0$, so finally $\lambda(S - G) = 0$. ◀

▶ **Corollary 28.** *If A is higher K-Trivial and not Δ^1_1, then* $\mathrm{W}\Pi^1_1\mathrm{R}^A \supseteq \Pi^1_1\text{-ML}\langle\mathcal{O}\rangle$.

Proof. We proceed by contrapositive, and show that every $\mathrm{W}\Pi^1_1\mathrm{R}^A$ test G is included in a $\Pi^1_1\text{-ML}\langle\mathcal{O}\rangle$ test. Given a $\mathrm{W}\Pi^1_1\mathrm{R}^A$ test, we just apply the theorem to this test and get $S = \bigcap_{s<\omega_1^{CK}} \mathcal{V}_s$ such that $S \supseteq G$ and $\lambda(S) = \lambda(G) = 0$, that is S is a $\Pi^1_1\text{-ML}\langle\mathcal{O}\rangle$ test containing G. ◀

3.4 $\Pi^1_1\text{-ML}\langle\mathcal{O}\rangle = \mathrm{W}\Pi^1_1\mathrm{R}^A$ implies A higher K-trivial and not Δ^1_1

In this section, we will suppose that A is not higher K-trivial, and we will prove that under this assumption there exists a $\mathrm{W}\Pi^1_1\mathrm{R}^A$ sequence that is not $\Pi^1_1\text{-ML}\langle\mathcal{O}\rangle$ random. To do this we need the existence of a particular set, that will allow us to build a specific sequence by forcing.

This proof follows the lines of the proof of lowness for Π^1_1-randomness [10]: if A is not Δ^1_1 and not higher K-trivial, then there exists a Π^1_1-ML test relative to A, which captures a Π^1_1-random. In [10] the proof has been done using full relativization and not continuous relativization. Full relativization helps in particular to work with tests whose captured sequences are closed under suppression of prefixes. It is not necessarily obvious using continuous relativization that we can work with such tests. In particular, for some oracles A it might be the case that there is no universal Π^1_1-ML test continuously relativized to A. Thus we first need to show the following lemma:

▶ **Lemma 29.** *Let A be any set, and $\mathcal{U}$ a $\Pi^1_1\text{-ML}^A$ test. Then there exists an $\Pi^1_1\text{-ML}^A$ test $\mathcal{V}$ such that if $\sigma X \in \mathcal{U}$ then $X \in \mathcal{V}$.*

Proof. First we establish some notation. For $\mathcal{A} \subseteq 2^\omega$, we write $\mathcal{A}^{-n}$ for $\{X : \exists \sigma \in 2^n, \sigma X \in \mathcal{A}\}$ that is the set of strings of $\mathcal{A}$, for which we remove the first n bits. We remark that $\lambda(\mathcal{A}^{-n}) \leq 2^n \lambda(\mathcal{A})$. Now say $\mathcal{U} = \bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{U}_m) \leq 2^{-m}$ and $(\mathcal{U}_m)$ is uniformly Π^1_1-open, continuously in A. We define $\mathcal{V} = \bigcap_n \mathcal{V}_n$ by :

$$\mathcal{V}_n = \{X : \exists m > n, \exists \sigma \in 2^{<m}, \sigma X \in \mathcal{U}_{2m}\} = \bigcup_{m>n} \bigcup_{i<m} (\mathcal{U}_{2m})^{-i}.$$

We now only need to verify that this proves the theorem. We need this set to be a $\Pi^1_1\text{-ML}^A$ test. It is easily a uniform intersection of Π^1_1 open sets continuously in A, but we need to check that it is effectively of measure 0. We have

$$\lambda(\mathcal{V}_n) \leq \sum_{m>n} \sum_{i<m} \lambda(\mathcal{U}_{2m}^{-i}) \leq \sum_{m>n} \sum_{i<m} 2^i \lambda(\mathcal{U}_{2m})$$

by the remark after the definition of $\mathcal{A}^{-n}$, and then

$$\lambda(\mathcal{V}_n) \leq \sum_{m>n} \sum_{i<m} 2^i 2^{-2m} \leq \sum_{m>n} 2^m 2^{-2m} \leq \sum_{m>n} 2^{-m} \leq 2^{-n}.$$

So $\mathcal{V}$ is a test. ◀

▶ Remark. We did not proved that the test $\bigcap \mathcal{V}_n$ is closed under deletion of prefixes. It's own closure under deletion of prefixes may need to be bigger, but this state of things will be enough for our needs.

Recall we will suppose in this part that A is not higher K-trivial. The next lemma makes use of this fact to define a set that will be useful in our next construction.

▶ **Lemma 30.** *If A is not higher K-trivial, then there exists a Π^1_1-ML^A test $\bigcap_n \mathcal{U}_n$ such that for every n and every Π^1_1 open set $\mathcal{V}$ with $\lambda(\mathcal{V}) < 1$, we have $\mathcal{U}_n \cap \mathcal{V}^c \neq \emptyset$ (that is $\mathcal{U}_n \not\subseteq \mathcal{V}$).*

Proof. By contrapositive, we will show that if the conclusion of the theorem does not hold, then every Π^1_1-ML^A test is contained in a Π^1_1-ML test. As the sequences which are continuously low for Π^1_1-MLR are exactly the higher K-trivials (Theorem 18), we can conclude that A is higher K-trivial. Following this plan, our hypothesis becomes: "For every Π^1_1-ML^A test $\cap_n \mathcal{U}_n$ there exists n and a Π^1_1 open set $\mathcal{V}$ with $\lambda(\mathcal{V}) < 1$ and such that $\mathcal{U}_n \subseteq \mathcal{V}$."

Our goal is to show that A is low for Π^1_1-MLR. Let $\mathcal{U} = \bigcap \mathcal{U}_n$ be a Π^1_1-ML^A test. By the previous lemma, we find a test $\widetilde{\mathcal{U}} = \bigcap \widetilde{\mathcal{U}_n}$ containing all the suffixes of elements in $\mathcal{U}$. Then by the hypothesis, we find $\mathcal{V} \supseteq \bigcap \widetilde{\mathcal{U}_n} (\supseteq \bigcap \mathcal{U}_n)$ where $\mathcal{V}$ is Π^1_1 open and $\lambda(\mathcal{V}) < 1$. Let W be such that $\mathcal{V} = [W]$ and $\mathrm{wg}(W) = \sum_{\sigma \in W} 2^{-|\sigma|} < 1 - \varepsilon$ for some ε (we make W almost prefix-free , that is $\mathrm{wg}(W) \leq \lambda(\mathcal{V}) + \varepsilon'$ for ε' sufficiently small, as allowed by [18], Lemma 3.7.1). We define:

$$\mathcal{V}^n = [W^n] = [\{\sigma_1 \sigma_2 \cdots \sigma_n : \sigma_i \in W\}].$$

We show that $\bigcap \mathcal{V}^n \supseteq \bigcap \mathcal{U}_n$ and that it is a valid test. Let $X \in \bigcap \mathcal{U}_n - \bigcap \mathcal{V}^n$ toward a contradiction. There exists a n such that $X \in \mathcal{V}^n$ and $X \notin \mathcal{V}^{n+1}$ (we must have $X \in \mathcal{V}^1$ by definition of $\mathcal{V}^1$). As $X \in \mathcal{V}^n$, there exists $\sigma \in W^n$ such that $X = \sigma Y$. But as $X \in \bigcap \mathcal{U}_n$, $Y \in \bigcap \widetilde{\mathcal{U}_n} \subseteq \mathcal{V}$, and there exists $\tau \in W$ such that $Y = \tau Z$. But then, $\sigma\tau \in W^{n+1}$ and $X \in \mathcal{V}^{n+1}$, a contradiction.

It remains to prove that $\bigcap \mathcal{V}^n$ is a test which is the case if it is effectively of measure 0. To do so we can easily prove by induction that $\lambda(\mathcal{V}^n) \leq \mathrm{wg}(W)^n$. Indeed, $\lambda(\mathcal{V}^{n+1}) \leq \sum_{\sigma \in W^n} \sum_{\tau \in W} 2^{-|\sigma\tau|} \leq (\sum_{\sigma \in W^n} 2^{-|\sigma|})(\sum_{\tau \in W} 2^{-|\tau|}) = \mathrm{wg}(W)^n$. Then $\lambda(\mathcal{V}^{n+1}) \leq (1 - \varepsilon)^{n+1}$.

We covered every Π^1_1-ML^A test with a test without oracle, so A is low for Π^1_1-MLR, that is, higher K-trivial. ◀

▶ **Theorem 31.** *Suppose A is not higher K-trivial. Then, there is a Π^1_1-ML$\langle\mathcal{O}\rangle$-random which is not $\mathrm{W}\Pi^1_1\mathrm{R}^A$.*

Proof. Let us denote by $\mathcal{R}_\mathcal{O}$ (respectively $\mathcal{R}_\mathrm{W}$) the set of Π^1_1-ML$\langle\mathcal{O}\rangle$ (respectively $\mathrm{W}\Pi^1_1\mathrm{R}^A$) randoms. We are trying to prove that the set $\mathcal{R}_\mathcal{O} \cap \overline{\mathcal{R}_\mathrm{W}}$ is not empty. We will build an element inside this intersection by forcing. The main thing needed for the construction is to clarify how we will layer these two sets.

First we have $\mathcal{R}_\mathcal{O} = \bigcap_m \bigcup_n \mathcal{F}_{m,n}$ where the $\mathcal{F}_{m,n}$ are Σ^1_1 closed sets, increasing over n. Neither the intersection or the union need to be effective. Each union is in fact effective in Kleene's $\mathcal{O}$ (by definition of a Π^1_1-ML$\langle\mathcal{O}\rangle$ test) and each intersection is effective in the double jump of Kleene's $\mathcal{O}$ (to select the functions Turing computable from Kleene's $\mathcal{O}$ which are totals and which pick the right indices for a Π^1_1-ML$\langle\mathcal{O}\rangle$ test). Note that we can also require without loss of generality that each $\mathcal{F}_{m,n}$ contains only Π^1_1-ML randoms: to do so we simply replace each $\mathcal{F}_{m,n}$ by the uniform union of its intersection with each Σ^1_1 closed component in the complement of a universal Π^1_1-ML test.

Then $\overline{\mathcal{R}_\mathrm{W}}$ is the union of all the $\mathrm{W}\Pi^1_1\mathrm{R}^A$ tests. In particular, it contains the test $\bigcap_n \mathcal{U}_n$ given by the lemma 30: as A is not higher K-trivial, every $\mathcal{U}_n$ intersects every Σ^1_1 closed set of positive measure. Furthermore if this closed set contains only Π^1_1-ML randoms, this intersection must be of positive measure (it is a fact that no Π^1_1-ML random can be in a Σ^1_1 closed set of measure 0).

In conclusion, it is sufficient to construct a Z such that $Z \in \mathcal{U}_n$ for every n and $Z \in \bigcup_n \mathcal{F}_{m,n}$ for every m. It is now clear how to do so by forcing with a decreasing sequence of Σ^1_1 closed sets of positive measure: We start with $\mathcal{F}_{0,0}$ which intersects with positive measure some $[\sigma_0] \subseteq \mathcal{U}_0$.

Suppose now by induction, that for some m we have closed sets $\mathcal{F}_{i,n_i}$ for $i \leq m$ and strings $\sigma_1 \prec \cdots \prec \sigma_m$, such that $\lambda(\bigcap_{i\leq m} \mathcal{F}_{i,n_i} \cap [\sigma_m]) > 0$ and such that $[\sigma_m] \subseteq \bigcap_{i\leq m} \mathcal{U}_i$. Let us find n_{m+1} and $\sigma_{m+1} \succ \sigma_m$ with $[\sigma_{m+1}] \subseteq \bigcap_{i\leq m+1} \mathcal{U}_i$ such that $\lambda(\bigcap_{i\leq m+1} \mathcal{F}_{i,n_i} \cap [\sigma_m]) > 0$.

As $\bigcap_{i\leq m} \mathcal{F}_{i,n_i} \cap [\sigma_m]$ is a Σ^1_1 closed set of positive measure, it intersects with positive measure the set $\mathcal{U}_{m+1}$. Thus there exists $\sigma_{m+1} \succ \sigma_m$ with $\sigma_{m+1} \subseteq \bigcap_{i\leq m+1} \mathcal{U}_i$ such that $\lambda(\bigcap_{i\leq m} \mathcal{F}_{i,n_i} \cap [\sigma_{m+1}]) > 0$. Now as $\lambda(\bigcup_n \mathcal{F}_{m+1,n}) = 1$, there is some n_{m+1} such that $\lambda(\bigcap_{i\leq m+1} \mathcal{F}_{i,n_i} \cap [\sigma_{m+1}]) > 0$.

By construction, the unique sequence $Z \in \bigcap_i [\sigma_i]$ is such that $Z \in \bigcap_m \bigcup_n \mathcal{F}_{m,n}$ and $Z \in \bigcap_n \mathcal{U}_n$ which concludes the proof. ◀

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