The Transitive Minimum Manhattan Subnetwork Problem in 3 Dimensions

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Abstract

We consider the Minimum Manhattan Subnetwork (MMSN) Problem which generalizes the already known Minimum Manhattan Network (MMN) Problem: Given a set P of n points in the plane, find shortest rectilinear paths between all pairs of points. These paths form a network, the total length of which has to be minimized. From a graph theoretical point of view, a MMN is a 1-spanner with respect to the L_1 metric. In contrast to the MMN problem, a solution to the MMSN problem does not demand L_1-shortest paths for all point pairs, but only for a given set $R \subseteq P \times P$ of pairs. The complexity status of the MMN problem is still unsolved in ≥ 2 dimensions, whereas the MMSN was shown to be NP-complete considering general relations R in the plane. We restrict the MMSN problem to transitive relations R_T (*Transitive* Minimum Manhattan Subnetwork (TMMSN) Problem) and show that the TMMSN problem in 3 dimensions is NP-complete.

Key words: Manhattan network, 1-spanner, grid graph, 3 dimensions

1 Introduction

The Minimum Manhattan Network Problem was first introduced in 2001 by Gudmundson et al. [5] and can be briefly described as follows: Given a set P of points in the plane and two orthogonal directions (X- and Y-axis), connect all pairs (p, q) of points of P. For *every* pair (p, q) a path from p to q must be a shortest path with respect to the Manhattan or L_1 metric and only consist of axis parallel line segments. The set of line segments, containing all shortest paths is called *Manhattan Network (MN)*. We measure the length of such a network by summing up the lengths of all line segments and call a solution to

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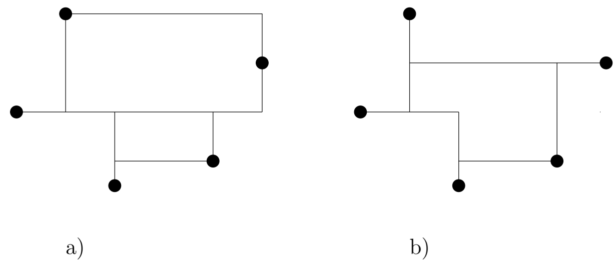

Fig. 1. a) A Manhattan Network; b) A Minimum Manhattan Network.

the MMN problem, i.e. a MN with minimum length, a *Minimum Manhattan Network (MMN)* of P (Fig. 1).

The MMN problem, as well as the closely related rectilinear Steiner Tree and Steiner Arborescence problem, has application in VLSI design, where connecting the chip components with minimum total wire length is desirable. Especially a MMN solution is useful, as it minimizes the total wire length while the connections between pairs of components are as short (and thus as fast) as possible. Further, Lam et. al. [8] use an MMN approach as a preprocessing step to accelerate the Viterbi algorithm for an alignment problem.

Thus, efficient algorithms for these problems would be highly appreciable, but both of the above Steiner problems already turned out to be NPO-complete (see [4], [12]). Up to now, the complexity status of the MMN problem is still unknown, but mostly suspected to be NP-hard as well. Hence, the previous work on the MMN problem solely features approximation algorithms: Gudmundson et al. [5] presented an 8-approximation in time $O(n \log n)$ and a 4-approximation in time $O(n^3)$ (which is used in [8]). Benkert et al. [1] introduced a 3-approximation in time $O(n \log n)$. Kato et al. [6] proposed a 2-approximation in time $O(n^3)$. However, the proof of the correctness seems to be incomplete [1]. Chepoi et al. [2] gave a 2-approximation based on an LP with $O(n^3)$ variables and constraints dominating the running time. The LP-formulation was given by K.Nouioua [9], who also developed a 3-approximation which runs in $O(n \log n)$ time [10]. The best approximation factor so far was achieved by Seibert and Unger [13] who proposed a 1.5-approximation in time $O(n^3)$, although the correctness of their algorithm and the completeness of its analysis is discussed critically in [3].

The problem considered in this paper differs from the original MMN problem in that the point pairs to be connected by shortest L_1-paths are explicitly given due to a relation $R \subseteq P \times P$. This problem is mentioned in [7] as the F-restricted Minimum Manhattan Network problem. A special case of the FMMN problem arises for $R_{onetoall} = \{\{p, q_i\}\}$ and the point set $P = \{p, q_1, \ldots, q_{n-1}\}$, e.g. every point q_i has to be connected to the point p by a shortest L_1-path. The resulting problem is known as the Rectilinear Steiner Minimum Arborescence (RSMA) problem mentioned above. It was shown to be NP-complete in $\mathbb{R}^2$ by W. Shi and C. Su [12].

But this result features a rather special relation, whereas we consider all transitive relations R_T, i.e. R such that $\forall\{p,q\},\{p,r\} \in R : \{q,r\} \in R$. The transitivity of the considered relations result in a kind of double layer MMN problem: We solve the MMN problem for subsets of points of P under the restriction that not only the sum of the shortest path lengths has to be minimized as in the original MMN problem, but also the sum of the lengths of the subset-MMNs is to be minimized (see Section 2 and Figure 2). The set of all transitive relations R_T includes $R_{alltoall} = \{\{p,q\}|p \neq q \in P\}$. This means that the original MMN problem is a special case of TMMSN. On the other hand the set R_T obviously does not contain $R_{onetoall}$ and RSMA is no special case of TMMSN. Thus the question of the complexity of the TMMSN problem is open.

The remaining part of the article is organized as follows. In Section 2, we give some formal definitions regarding the different problem formulations. Then we proceed to an NP-completeness proof for the decision version of TMMSN in Section 3. We conclude with a short summary in Section 4.

2 Definitions

In this section, we briefly introduce some definitions and notations used throughout the paper. Let P be a set of points $p = (x_p, y_p, z_p)$ in $\mathbb{R}^3$ and consider three pairwise orthogonal directions (x-, y- and z-axis).

Definition 1 *Let s be a line segment and S a set of such segments. Then $|s|$ denotes the length of s, and we define the* total length *of S as $|S| = \sum_{s\in S} |s|$.*

Definition 2 *π_{pq} denotes a set of axisparallel line segments which form an arbitrary shortest path between two points $p, q \in P$ with respect to the L_1-metric.*

Definition 3 *Let $B(P)$ be the bounding box of P, i.e. the smallest axis aligned cuboid that contains all points of P, and let $\partial B(P)$ be its boundary.*

In the following section we refer to π_{pq} as L_1-shortest path or simply (shortest) path. The length of a L_1-shortest path is always given by $|\pi_{pq}| = |p_x - q_x| + |p_y - q_y| + |p_z - q_z|$. Points lying on $\partial B(P)$ are considered as "contained in $B(P)$". $B(P)$ can degenerate to a rectangle or even a line segment. Further, all $\pi_{pq}, p, q \in P$, are contained in $B(P)$. Recall the Minimum Manhattan Network (MMN) Problem in $\mathbb{R}^2$ which transfers most easily into three dimensions as follows :

Given a set P of points in $\mathbb{R}^3$ and three pairwise orthogonal directions (x-, y- and z-axis), a *Minimum Manhattan Network* of P, MMN(P), is a set of axis parallel line segments with the following properties:

- $\forall p, q \in P : \exists \pi_{pq} \subseteq MMN(P)$.
- $|MMN(P)|$ is minimal amongst the lengths of all sets of axis parallel line segments containing at least one shortest L_1 path between each pair of points $p, q \in P$.

We call a given set P an *instance* of the MMN problem.

Analogously define the Minimum Manhattan Subnetwork (MMSN) Problem:

Given a set P of points in $\mathbb{R}^3$, 3 pairwise orthogonal directions (x-, y- and z-axis) and a relation $R \subseteq P \times P$, a *Minimum Manhattan Subnetwork* of P with regard to R, MMSN($[P, R]$), is a set of axis parallel line segments with the following properties:

- $\forall \{p, q\} \in R : \exists \pi_{pq} \subseteq MMSN([P, R])$.
- $|MMSN([P, R])|$ is minimal amongst the lengths of all of all sets of axis parallel line segments containing at least one shortest L_1 path between each pair of points $\{p, q\} \in R$.

We call a given pair $[P, R]$ an *instance* of the MMSN problem.

The Transitive MMSN is then defined as the Minimum Manhattan Subnetwork Problem for instances $[P, R_T]$. Due to the transitivity of R, the set P can be partitioned uniquely into subsets $P_1, \ldots, P_k$ of points such that:

- $\forall p, q \in P_i : \{p, q\} \in R$
- $\forall p \in P_i, q \in P_j, i \neq j : \{p, q\} \notin R$

We call $\mathcal{P} = P_1, \ldots, P_k$ the R-induced partition of P. Points forming a set $P_i, 1 \leq i \leq k$ have to be *fully interconnected*.

To solve the TMMSN problem it may seem to be sufficient to solve independent MMN problems for all P_i, but this impression is misleading: Consider the points $P = \{p, q, r, s\}$ and the (transitive) relation $R = \{\{p, q\}, \{r, s\}\}$ which induces the partition $\mathcal{P} = P_1 \cup P_2, P_1 = \{p, q\}, P_2 = \{r, s\}$. Solving the two MMN problems on P_1 and P_2 may yield a "solution" for the TMMSN([P,R]) like the one in Figure 2 a), whereas the solution in Figure 2 b) is minimal.

This problem only arises for instances $[P, R]$, where shortest paths between point pairs from different sets P_i can share line segments (like the paths π_{pq} and π_{rs} in Figure 2). Thus combining solutions for the MMN problem on the point sets P_i yields a valid solution TMMSN([P,R]) if paths $\pi_{pq}, p, q \in P_i$ and $\pi_{rs}, r, s \in P_{j \neq i}$ cannot share any line segments. This holds if:

$$\forall P_i, P_{j \neq i} : B(P_i) \cap B(P_j) = \emptyset.$$

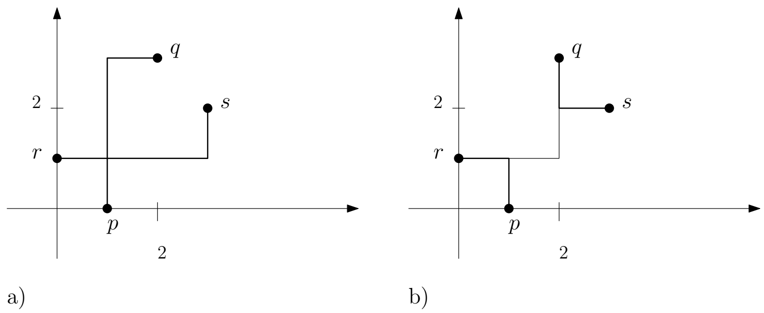

Fig. 2. a) Combining MMN-solutions for P_i; b) TMMSN solution for $[P, R]$.

We call point sets P_i and P_j with the above property *geometrically independent.*

Finally, we formulate the decision version of the TMMSN problem:

Given an instance $[P, R]$ and a constant c, is $|TMMSN([P, R])| \leq c$?
We call a triple $[P, R, c]$ an *instance* of the TMMSN([P,R]) decision problem.

3 TMMSN is NP-complete

We show that the TMMSN decision problem in three dimensions is NP-hard. For this a polynomial time reduction from E3SAT is presented. Since the decision version of the TMMSN is obviously in NP,[1] we obtain the NP-completeness result. The reduction works as follows. At first we construct an instance $[P_\alpha, R_\alpha]$ of the TMMSN problem from a given E3SAT instance α in polynomial time. We then determine a lower bound c_α to the length of the minimal transitive Manhattan subnetwork of this instance. Finally we prove that α is satisfiable if and only if the length of the solution attends the lower bound, i.e. $|TMMSN([P_\alpha, R_\alpha])| = c_\alpha$.

Construction of $[P_\alpha, R_\alpha]$. Consider the structure of an instance of the E3SAT problem: Given a Boolean formula α with n clauses, $c_1, \ldots, c_n$, over m variables, $x_1, \ldots, x_m$, where each clause consists of three literals; that is:

$$\alpha = c_1 \wedge \ldots \wedge c_n$$

$$\text{with } c_i = (l_i^1 \vee l_i^2 \vee l_i^3) \text{ and } l_i^1, l_i^2, l_i^3 \in \{\, x_j, \neg x_j;\ 1 \leq j \leq m \,\}\,.$$

[1] Ask an oracle for the TMMSN([P,R]), which is a set of $O(n^2)$ line segments, sum up their length to l and check whether $l < c$, which only needs polynomial time due to the size of TMMSN([P,R]).

Let $occ(x_j)$ denote the number of occurrences of variable x_j (literal x_j or $\neg x_j$) in α. Then P_α consists of the following point sets:

- For each variable x_j, $1 \leq j \leq m$, we add a set $P_j = E_j \dot\cup A_j$ of points with $|E_j| = 4$ and $|A_j| = 2(m+1)occ(x_j)$ to P_α.
- For each clause c_i, $1 \leq i \leq n$, we add $m+1$ point sets $P_i^k = \{p_i^k, q_i^k, r_i^k, s_i^k\}$, $1 \leq k \leq m+1$ to P_α.

Further, we define R_α such that:

- For each variable x_j , $1 \leq j \leq m$: $P_j \times P_j \in R_\alpha$.
- For each clause c_i, $1 \leq i \leq n$ and $1 \leq k \leq m+1$: $\{\{p_i^k, q_i^k\}, \{r_i^k, s_i^k\}\} \in R_\alpha$.

Next we define the placement of the points P_j and P_i^k in $\mathbb{R}^3$. We start with $E_j = \{e_j^1, e_j^2, e_j^3, e_j^4\}$, where:

$$e_j^1 = (0,0,j),\ e_j^2 = (5(m+1)n, 0, j),$$
$$e_j^3 = (5(m+1)n, 2, j),\ e_j^4 = (0, 2, j).$$

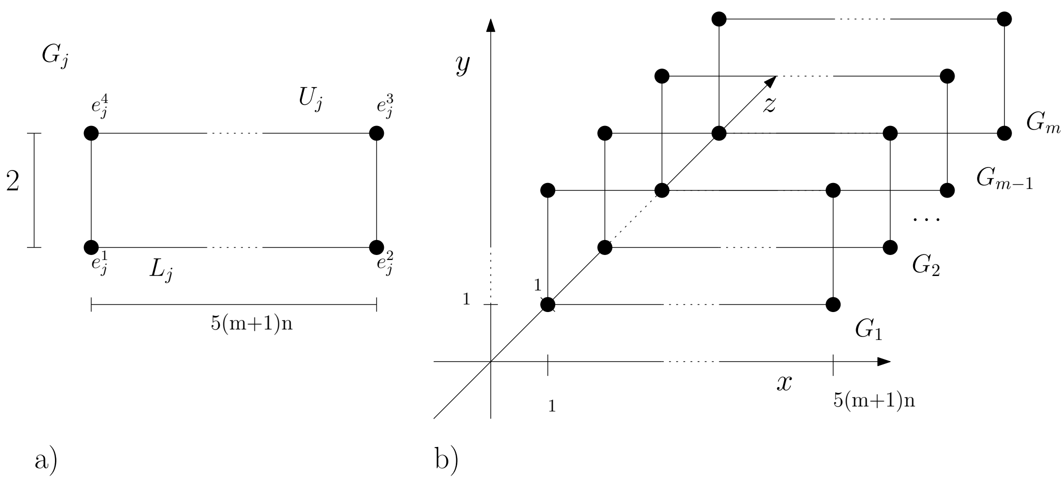

Fig. 3. a) Single basic variable gadget G_j. b) Arrangement of basic gadgets in $\mathbb{R}^3$.

Thus E_j determines the corner points of a rectangle of height 2 and length depending on the number of variables and clauses of α and which lies in the plane $z = j$ of $\mathbb{R}^3$ (Fig. 3). The shortest L_1-paths between pairs of neighboring corner points (in counter-clockwise order) are reduced to the straight line segments between the points, e.g. $\pi_{e_j^1 e_j^2} = \{\overline{e_j^1 e_j^2}\}$. Thus the rectangle $e_j^1, e_j^2, e_j^3, e_j^4$ identifies with $\partial B(E_j)$ when we consider the two-dimensional boundary. We call the points E_j together with $\partial B(E_j)$ the *basic variable gadget*. The *basic variable gadget* together with the points A_j forms the *variable gadget* G_j. We abbreviate: $L_j := \pi_{e_j^1 e_j^2}$ and $U_j := \pi_{e_j^3 e_j^4}$.

Before we proceed with the coordinates of the points from A_j (which will all lie on L_j or U_j), we place the four points of each $P_i^k, 1 \leq i \leq n, 1 \leq k \leq m+1$ as follows:

$$p_i^k = (5(k-1)n + 5(i-1) + 2, -1, 0),$$
$$q_i^k = (5(k-1)n + 5(i-1) + 4, 3, m+1),$$
$$r_i^k = (5(k-1)n + 5(i-1) + 4, 0.5, m+1),$$
$$s_i^k = (5(k-1)n + 5(i-1) + 4, 1.5, m+1).$$

$B(P_i^k)$ is a cube of width 2 height 4 and depth $m+1$ with p_i^k as front lower left and q_i^k as back upper right point. Together with the points P_i^k, $B(P_i^k)$ forms the *clause gadget* G_i^k. All L_j and U_j pass through each G_i^k in x-direction (see Fig. 4), i.e. $B(P_i^k) \cap L_j = l$, where l is a line segment parallel to the x-axis with $|l| = 2$ and $B(P_i^k) \cap U_j = u$ for a line segment u with the same properties as l.

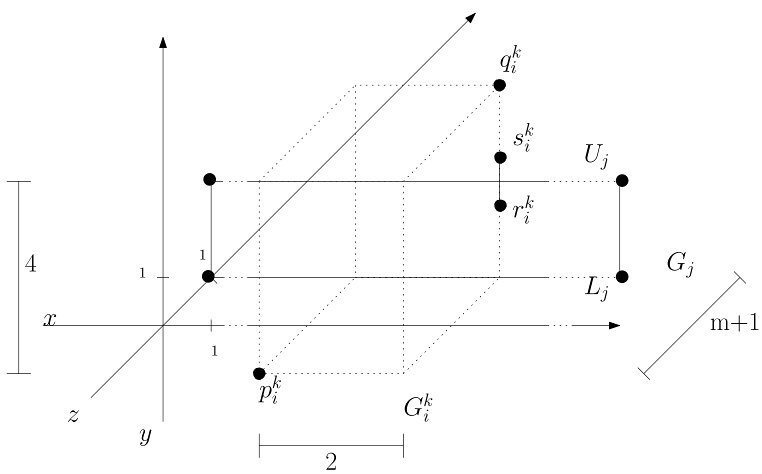

Fig. 4. A clause gadget G_i^k (cube defined by p_i^k and q_i^k) with L_j, U_j passing through. (Note that $\pi_{s_i^k r_i^k}$ is not crossed by any L_j or U_j because with z-coordinate $m+1$ the points s_i^k and r_i^k are located behind the variable gadgets G_j.)

From the placement of the points it is evident that all $m+1$ clause gadgets G_i^k are copies of G_i^1 shifted in x-direction which we need for technical reasons. The G_i^k, $1 \leq i \leq n, 1 \leq k \leq m+1$, follow each other with a horizontal spacing of 3 along the x-axis in the following order:

[Clause Gadget Order] $G_1^1, \ldots, G_n^1, G_1^2, \ldots, G_n^2, \ldots, G_1^{m+1}, \ldots G_n^{m+1}$.

Now we determine the placement of the $2(m+1)occ(x_j)$ remaining points from A_j on each variable gadget. These points are essential as they represent the occurrence of a variable x_j as literal x_j or $\neg x_j$ respectively in a clause c_i. Their positions influence the structure of the TMMSN$[P_\alpha, R_\alpha]$ and the lower bound on its length. The lower bound to $|TMMSN[P_\alpha, R_\alpha]|$ is then attained by the solution if we can save a fixed length for every clause. In this case, every clause is satisfiable by a consistent truth assignment. This again is true if and only if α is satisfiable. We achieve this by a placement of the points of A_j,$1 \leq j \leq m$, with respect to the following rule.

Placement Rule:

- If literal x_j occurs in c_i, we place
  $p^k_{i,j} = (5n(k-1) + 5(i-1) + 3, 0, j)$ (on L_j) *and*
  $\overline{p^k_{i,j}} = (5n(k-1) + 5i, 2, j)$ (on U_j) for $1 \leq k \leq m+1$.
- If literal $\neg x_j$ occurs in c_i, we place
  $p^k_{i,j} = (5n(k-1) + 5(i-1) + 1, 0, j)$ (on L_j) *and*
  $\overline{p^k_{i,j}} = (5n(k-1) + 5(i-1) + 3, 2, j)$ (on U_j) for $1 \leq k \leq m+1$.

This means that for each variable x_j that occurs in clause c_i there is one point of A_j (on L_j or U_j) *inside* each G^k_i *and* one point to the left *or* to the right of G^k_i. Whether the point inside G^k_i is located on L_j or U_j depends on which literal of x_j occurs in c_i. As an example consider formula α with only one variable and two clauses $\alpha = (a) \wedge (\neg a)$ for the sake of simplicity, although this is no exact E3SAT formula (see Fig. 5).

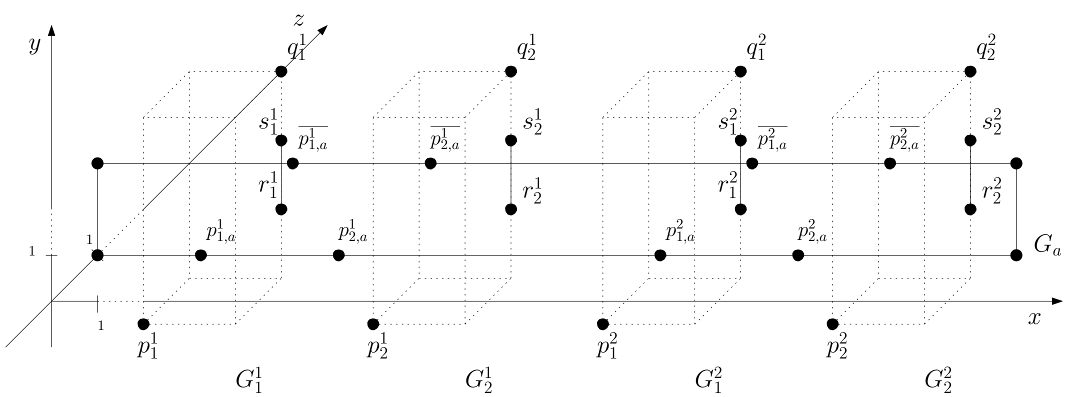

Fig. 5. Example: P_α for $\alpha = (a) \wedge (\neg a)$.

We have determined the coordinates for all points of P_α which together with the relation R_α completes the construction of the instance $[P_\alpha, R_\alpha]$. The size of our construction is in $O(m^2 n)$ and thus with $m \leq 3n$ polynomial in $O(n^3)$.

Lower bound on $|$TMMSN$([P_\alpha, R_\alpha])|$ and structural properties. Proceeding with the reduction, we give a lower bound c_α to the length of the solution to the TMMSN problem on $[P_\alpha, R_\alpha]$. Let M be any Manhattan network on $[P_\alpha, R_\alpha]$, i.e. a set of axis parallel line segments containing at least one shortest path for each point pair from R_α (which need not be of minimal length amongst such sets). Further, denote with π^*_{uv} a shortest path between a point pair $\{u, v\} \in R_\alpha$, that is contained in the solution M.[2]

Each network M can be partitioned into the following three sets of segments U_M, V_M and C_M:

The set $U_M \subseteq M$ contains all fixed line segments. These are all segments that have to be contained in the solution, because they constitute unique L_1-shortest paths i.e. the paths between the corner points of the variable gadgets and the paths between each two points r_i^k and s_i^k:

$$U_M = \bigcup_{1 \le j \le m} \partial B(E_j) \cup \bigcup_{1 \le i \le n, 1 \le k \le m+1} \pi_{r_i^k s_i^k}. \tag{1}$$

The set $V_M \subseteq M$ contains all segments from M that contribute to L_1-shortest paths between point pairs $\{u, v\} \in \bigcup_{1 \le j \le m} A_j \times A_j$ without segments that already occur in U_M :

$$V_M = \bigcup_{u \ne v \in A_j, 1 \le j \le m} \pi^*_{uv} \setminus U_M. \tag{2}$$

The set $C_M \subseteq M$ contains all segments from M that contribute to L_1-shortest paths between the points pairs (p_i^k, q_i^k) without segments that already occur in U_M or V_M:

$$C_M = \bigcup_{1 \le i \le n, 1 \le k \le m+1} \pi^*_{p_i^k q_i^k} \setminus U_M \cup V_M. \tag{3}$$

Let min_X, $X \in \{U, V, C\}$ be defined as: $min_X = min_M |X_M|$.Then $|M| \ge c_\alpha$ with $c_\alpha := min_U + min_V + min_C$ is true for all M and especially:

Property 1 $|$*TMMSN(*$[P_\alpha, R_\alpha]$*)*$| \ge c_\alpha$

Thus we can determine the desired lower bound c_α as the sum of min_U, min_V, min_C by minimizing the lengths of U_M, V_M and C_M independently over all M. The set U_M contains the fixed segments of $\partial B(E(x_j))$ with length $10(m+1)n+4$ for each of the m variable gadgets and the line segments $\overline{r_i^k s_i^k}$ of length 1 for the $(m+1)n$ clause gadgets. Thus the length of U_M is always $|U_M| = min_U$ and min_U only depends on m and n:

$$min_U = (10(m+1)n + 4)m + (m+1)n. \tag{4}$$

[2] In general there are different π^*_{uv}, but in cases where we discuss the geometric structure of one such π^*_{uv} in detail, it will be unique in M.

As the structure of U_M is also the same for all M, we set $U = U_M$. To determine min_V, we further partition V_M: Let V_j be the subset of V_M containing all segments from the paths $\pi^*_{uv} \setminus U$, $\{u, v\} \in A_j \times A_j$. Then $V_M = \bigcup_{1 \leq j \leq m} V_j$ and $|V_M| = \sum_{1 \leq j \leq m} |V_j|$. The latter holds, because A_j and $A_{l \neq j}$ are geometrically independent by the placement of the parallel planes on the z-Axis.Therefore π^*_{uv}, $u, v \in A_j$, and π^*_{wt}, $w, t \in A_{l \neq j}$ can not share any segments.

Thus it is sufficient to determine the minimum length of V_j for all variables. For this, we consider a basic variable gadget which is the (2-D) rectangle $\partial B(E_j)$ including the corner points. All points A_j also lie on $\partial B(E_j)$, especially on L_j and U_j. Moreover all pairs $\{p, q\}$ of *(x-)neighbouring* points, i.e. $p = (x_p, y_p, z_p)$ and $q = (x_q, y_q, z_q)$ such that $p, q \in A_j$ and $\nexists r = (x_r, y_r, z_r) \in A_j : x_p < x_r < x_q$, are located on different of the two segments L_j and U_j. We call such a $\{p, q\}$ an *alternating point pair* or state that points (of a set) *alternate*, if all neighboring point pairs are alternating point pairs.

Property 2 *The points of A_j alternate on L_j and U_j.*

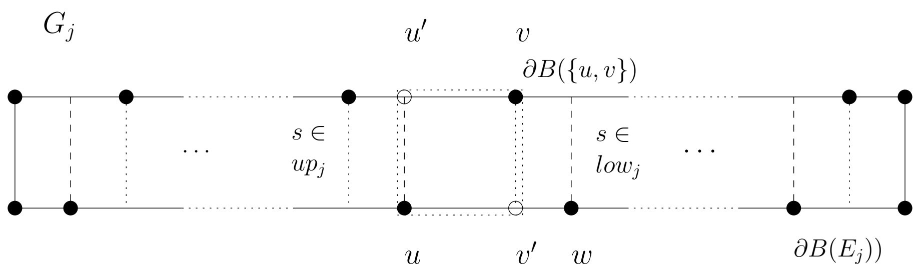

Fig. 6. Alternating points of A_j: All shortest paths π^*_{uv} between (x)-neighboured points contain induced segments.

This can be seen by the following inductive argument: Let c_i be the first clause of α containing x_j as a literal, which results in a point $p^k_{i,j}$ located on L_j and $\overline{p^k_{i,j}}$ on U_j to the right of $p^k_{i,j}$ as determined by the placement rule. Let c_l be the next clause of α where variable x_j occurs. If literal x_j occurs in c_l, a point $p^k_{l,j}$ is again located on L_j (and to the right of $\overline{p^k_{i,j}}$) followed by $\overline{p^k_{l,j}}$ on U_j: The alternation is kept. The same is true, if $\neg x_j$ occurs in c_l, because $\overline{p^k_{l,j}}$ is located on U_j inside G^k_l, but $p^k_{l,j}$ is located to the left of G^k_l, between $\overline{p^k_{l,j}}$ and $\overline{p^k_{i,j}}$ The same is true for all other possible sequences of occurrence of a variable x_j.

With Property 2, it suffices to consider the interconnection of alternating points on the lower and upper horizontal edge of a given 2D rectangle to determine the minimal length of each V_j. This situation corresponds to the *original* MMN problem on the point set P_j. All points on L_j (U_j resp.) are already interconnected by L_1-shortest paths via L_j (U_j resp.). L_j and U_j are contained in U and thus do not contribute to $|V_j|$. Further, we only need to connect a point u on L_j to both its neighbours on U_j directly. All other points on U_j are then connected to u via its neighbours automatically. Thus V_j only contains vertical segments s of length $|s| = 2$ which start on L_j and end on U_j (or vice versa). The minimal length of V_j is then determined by the minimal number $min_s(V_j)$ of such segments used.

Lemma 1 $min_s(V_j) = (m + 1)occ(x_j)$

PROOF. Consider the neighbors u on L_j and v on U_j to the right of u and their vertical projections u' on U_j and v' on L_j (Fig. 6). Let s be a vertical segment which starts on L_j and ends on U_j (or vice versa). Any such segment contributing to π_{uv} has to be situated in $B(\{u, v\})$ which identifies with the rectangle (u, v', v, u') in this case. Thus s can contribute to at least one π_{uv}. Let w be the right neighbour of v on L_j. Then a segment s contributing to π_{vw} has again to lie within $B(\{v, w\})$. Further $B(\{u, v\}) \cap B(\{v, w\}) = \overline{vv'}$ and $\overline{vv'}$ can contribute to both π_{uv} and π_{vw}. We call such a segment, starting from a point q on either L_j or U_j respectively and ending in its vertical projection q' on U_j resp. L_j, an *induced* segment (by q). Thus any induced segment can contribute to two shortest paths π_{pq} and π_{qr}, p and r being q's left and right neighbor. For any three rectangles $B(\{u, v\})$, $B(\{w, t\})$, $B(\{x, y\})$, $u, v, w, t, x, y \in A_j$ the intersection is empty (even if $v = w$ and $t = x$). Thus no segment s can contribute to more than two shortest paths between neighboring points in A_j. As $|A_j| = 2(m+1)occ(x_j)$, we have to connect $2(m+1)occ(x_j)-1$ neighbouring point pairs. Using induced segments leads to the following minimal number of segments needed:

$$\left\lceil \frac{2(m+1)occ(x_j) - 1}{2} \right\rceil = \left\lceil \frac{2(m+1)occ(x_j)}{2} \right\rceil - \left\lfloor \frac{1}{2} \right\rfloor = (m+1)occ_x$$

□

We can now determine the minimal length of V_M as the sum of all $min_s(V_j)$ multiplied by the length of the segments $|s| = 2$. With Lemma 1, $min_s(V_j)$ depends on the number of occurrences of variable x_j in α which is not known for the general α. On the other hand the total number of variable occurrences in α is $3n$ as the definition of E3SAT claims three literals in each of n clauses. Thus we obtain $|V_M| = min_V$, which again only depends on n and m:

$$min_V = 2 \sum_{1 \le j \le m} min_s(V_j) = 2 \sum_{1 \le j \le m} (m+1)occ(x_j) = 2(m+1)3n. \qquad (5)$$

Before we proceed, we make some remarks on the structure of the sets $V_j \subset V_M$ of minimal length. The proof of Lemma 1 suggests only to use induced segments, but does not specify which segments should be chosen. Let $S =$

$s_1, \ldots, s_{2occ(x_j)}$ be the x-ordered sequence of segments induced by the points of A_j. Of course, V_j can not only contain the left half of S, as they would only complete half the shortest paths $\pi^*_{uv}, u, v \in A_j$. But we obtain all necessary shortest paths, if we take every second next induced segment. Starting from the left, these are all segments induced by a point on L_j and starting from the right we obtain all segments induced by a point on U_j. We will refer to the set of segments induced by all points of A_j on L_j as the set low_j (dashed in Fig. 6) and to the set of segments induced by all points of A_j on U_j as the set up_j (dotted in Fig. 6) respectively. We refer to both sets as the *parities* of the variable gadget G_j. The parities of G_j identify with the two possible *nice minimum vertical covers* of A_j as defined in [1] and [6] and obviously together with the rectangle edges of G_j each of the sets contains all desired $\pi_{pq}, \{pq\} \in A_j$.

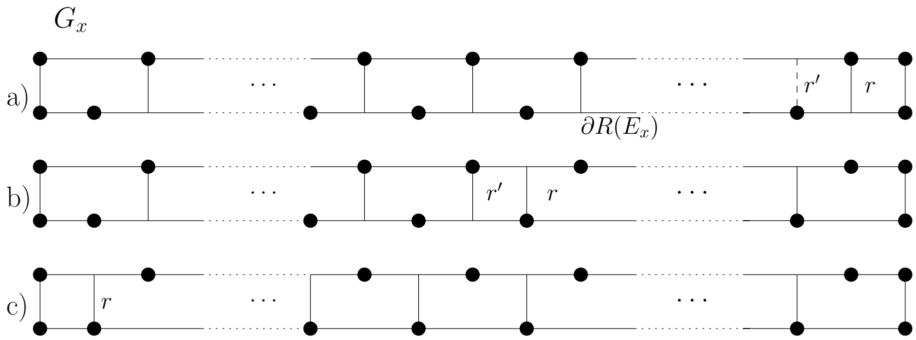

Fig. 7. a) $V_j = up_j$; b) Parity change between r and r'; c) $V_j = low_j$.

Later on, the parities of V_j are interpreted as the possible truth values of x_j. Unfortunately, low_j and up_j are not the only possible sets V_j with minimum length. As there are $2(m+1)occ(x_j) - 1$ point pairs $\{u, v\} \in A_j \times A_j$ to be fully interconnected, only $(m+1)occ(x_j) - 1$ induced segments are necessary and the last π^*_{uv} can be completed by a non-induced segment. Further, not all induced segments in V_j have to belong to the same parity of G_j (see Fig. 7). This would lead to an inconsistency of a derived truth assignment. We define:

Definition 4 *Assume the segments of V_j are sorted from left to right (by x-order of the inducing points). Then any two subsequent segments are considered as a* parity change *if they are of different parities, i.e. one segment belongs to low_j and the other to up_j or vice versa (see r and r' in Fig. 7b).*

Lemma 2 *Any set V_j of minimal length contains at most one parity change* or *one non-induced segment.*

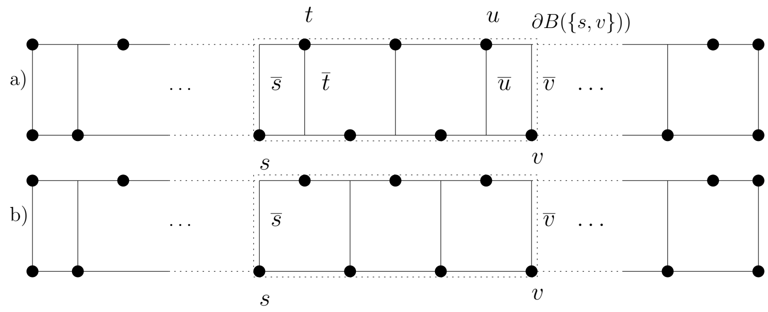

Fig. 8. a) Two parity changes in V_j; b) No parity change in V_j with $|V_j| = |V'_j| - 2$: V'_j was not minimal.

PROOF. First, we see that V_j cannot contain two non-induced segments. If we assume otherwise, we know that each of them contributes to only one of the $2(m+1)occ(x_j) - 1$ shortest paths. Even if we use only induced segments for the remaining $2(m+1)occ(x_j) - 3$ paths we need the following total number of segments:

$$2 + \left\lceil \frac{2(m+1)occ(x_j) - 3}{2} \right\rceil = 2 + \left\lceil \frac{2(m+1)occ(x_j)}{2} \right\rceil - \left\lfloor \frac{3}{2} \right\rfloor =$$
$$2 + (m+1)occ_x - 1 = (m+1)occ_x + 1 = m_s(V_j) + 1$$

This contradicts the minimality of V_j by Lemma 1 and we just have to consider two cases left: First, assume V_j contains two parity changes. Let $\overline{s}, \overline{t}$ and $\overline{u}, \overline{v}$ be the two pairs of subsequent segments forming the parity changes and s, t, u, v the points from which the segments extend (Fig. 8 a). W.l.o.g. the segment $\overline{s}$ belongs to low_j, $\overline{t}$ and $\overline{u}$ belong to up_j and $\overline{v}$ again to low_j: We can divide V_j into three parts. Segments from low_j situated at the left and right end of G_j and segments from up_j in between. We look at the part of G_j containing the segments of up_j which is specified by $B(\{s, v\})$ (Fig. 8 a): The left and right sides of $B(\{s, v\})$, $\overline{s}$ and $\overline{v}$ are both from low_j, which means s and v are both on L_j. As the points on L_j and U_j are alternating (Property 2), $B(\{s, v\})$ contains one more point on U_j than on L_j inside. Moreover V_j contains *all* segments extending from those points on U_j, as there are no segments from low_j inside $B(\{s, v\})$ and otherwise there would be a pair of neighboring points (x, y) inside $B(\{s, v\})$ such that $\pi_{xy} \nsubseteq U \cup V_j$. Thus, we can exchange all segments from up_j inside $B(\{s, v\})$ for all segments from low_j inside $B(\{s, v\})$, which is one segment less (Fig. 8 b). This contradicts the minimality of V_j.

Second, assume V_j contains a parity change and a non-induced segment. Then the non-induced segment induces a parity change in the following sense: Let s be the non-induced segment, u the next point of A_j to the left and v the next point of A_j to the right of s in x-order (see Fig. 9). W.l.o.g. u lies on L_j and v on U_j. Then s only contributes to $\pi^*_{uv} \in U \cup V_j$. Thus neither $\overline{u} \in V_j$ nor $\overline{v} \in V_j$. Otherwise s would be superfluous and contradict the minimality of V_j.

Now let $t \in A_j$ be the next point to the left of u and $w \in A_j$ the next point to the right of v. As we have seen above, there cannot be any more non-induced segment in V_j ($\overline{s'}$ or $\overline{s''}$ in Fig.9), $\overline{t} \in V_j$ and $\overline{w} \in V_j$. Otherwise there would be no π_{tu} and π_{vw} in $U \cup V_j$ (see Fig. 9). Now neither the structure nor the number of locally needed segments changes, if s is shifted to the left *or* to the right until $s = \overline{u}$ or $s = \overline{v}$. In both cases, we either have a second parity change and thus again case 1 or u or v coincide with corner points from E_j and thus s has been superfluous from the beginning. This again contradicts the minimality of V_j.

□

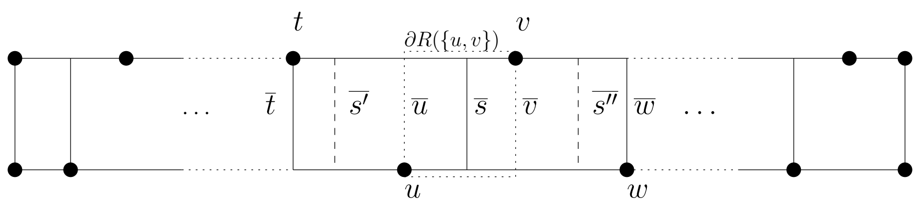

Fig. 9. Non-induced segment $\overline{s}$ causes a parity change.

Each parity change causes the later derived truth assignment to be inconsistent. Thus we must avoid any parity change, but with Lemma 2 the derived truth value from the whole construction for each variable can still be inconsistent. Yet we show that there is always at least one part of our construction that has the desired property and neither contains a non-induced segment nor a parity change.

Definition 5 *First we define the eight corner points $b_1^k, \ldots, b_8^k$ of a cuboid $cub^k, 0 \leq k \leq m+1$ as follows:*

$$
\begin{aligned}
cub^k = \{ & b_1^k = (5kn, -1, 0),\ b_2^k = (5(k+1)n, -1, 0) \\
& b_3^k = (5(k+1)n, 3, 0),\ b_4^k = (5kn, 3, 0) \\
& b_5^k = (5kn, -1, m+1),\ b_6^k = (5(k+1)n, -1, m+1) \\
& b_7^k = (5(k+1)n, 3, m+1),\ b_8^k = (5kn, 3, m+1)\}
\end{aligned}
$$

Then let a block $b^k, 1 \leq k \leq m+1$ be defined as: $b^k = B(P_\alpha) \cap B(cub^k)$.

Lemma 3 *There exists a block $b^* \in \{b^1, \ldots, b^k\}$ such that* no *parity change and* no *non-induced segments occurs on the parts of all variable gadgets within b^*.*

PROOF. Remember that $[P_\alpha, R_\alpha]$ contains $m+1$ copies of each clause gadget in the clause gadget order. With the definition of the blocks b^k, $1 \le k \le m+1$ we now "virtually" divide the whole instance $[P_\alpha, R_\alpha]$ into parts of length $5n$ such that every block b^k contains the copy G_i^k of any clause gadget. As we know from Lemma 2, we can only have one parity change *or* one non-induced segment on each of the m variable gadgets, but we constructed $m+1$ blocks b^k. Thus it is impossible to scatter the m possible parity changes/non-induced segments over all blocks and we obtain at least one block b^* containing the clause gadgets G_i^* without parity changes and non-induced segments.

□

Any truth assignment derived only from parities of the sets V_j in b^* must consequently be consistent by Lemma 3.

Now we determine the minimal length of the last set C_M, which consists of parts of the paths $\pi^*_{p_i^k q_i^k}$. As they are mutually geometrically independent, we consider each $\pi^*_{p_i^k q_i^k}$ separately. The length of each $\pi^*_{p_i^k q_i^k}$ comprises of the distances between p_i^k and q_i^k in x- y- and z-direction. By definition of a L_1-shortest path: $|\pi_{p_i^k q_i^k}| = 2 + 4 + (m+1)$ and by the definition of C_M we only count line segments $s \in \pi^*_{p_i^k q_i^k}$ for min_C if $s \notin U \cup V_M$. Thus, apart from its pure length, we also have to consider the structure, i.e. geometric location, of such a path as we did for $\pi^*_{uv}, \{u, v\} \in A_j \times A_j$.

Generally $\pi^*_{p_i^k q_i^k}$ can only contain segments located inside G_i^k, which is spatially bounded by $\partial B(P_i^k)$. As U already contains the $2m$ line segments $L_j, U_j, 1 \le i \le m$ which cross each clause gadget completely in x-direction, $\pi^*_{p_i^k q_i^k}$ can run along one of these segments to bridge the x-distance between p_i^k and q_i^k. This will be the case in an optimal solution, as on the one hand the total length of the solution gets smaller. On the other hand, the paths contributing to C_M cannot influence U in turn, because the line segments of U are fixed by unique shortest paths.

In y-direction $\pi^*_{p_i^k q_i^k}$ has two possible courses: There may be a vertical segment $s \in V_M$ located inside G_i^k or not (see Fig. 10). In the first case, $\pi^*_{p_i^k q_i^k}$ can run along s, such that only half of the y-distance between p_i^k and q_i^k has to be bridged by line segments in C_M. Thus the length of s does not contribute to min_C. In the second case, $\pi^*_{p_i^k q_i^k}$ can still run along $\pi^*_{r_i^k, s_i^k} \in U$, which is also not counted for min_C. But this only saves C_M a line segment of length 1. Therefore, in any optimal solution, $\pi^*_{p_i^k q_i^k}$ will always run along $s \in V_M$ located in G_i^k, if such a segment exists (Fig. 10a) and along $\pi^*_{r_i^k, s_i^k} \in U$ if not (Fig.

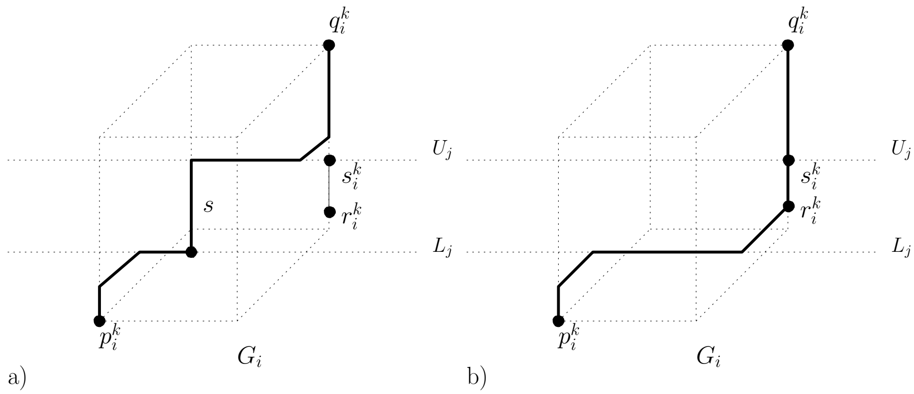

Fig. 10. A path $\pi_{p_i^k q_i^k}$.

10b). In turn C_M may influence the position of segments in V_M, but not their number, as an additional segment in V_M costs a length of 2, whereas it only saves a length of 1 in C_M, taking the use of $\pi^*_{r_i^k, s_i^k}$ into account. This leads to the following property which will be essential in the final step of the reduction.

Property 3 $|C_M| = min_C \Leftrightarrow \forall G_i^k\ \exists s \in V : s$ *is located in* G_i^k.

Since neither U nor V_M contains any line segment in z-direction, because the respective point pairs all share the same z-coordinate, C_M has to contain such segments of length $m+1$ for each $\pi^*_{p_i^k q_i^k}$ in any solution and the choice of those line segments is independent from U and V_M.

Finally each C_M must afford no line segments in x-direction, at least line segments of total length 2 in y-direction and $m+1$ in z-direction for each $\pi^*_{p_i^k q_i^k}$, which determines min_C to:

$$min_C = 2(m+1)n + (m+1)^2 n. \tag{6}$$

This completes the computation of the lower bound c_α to the length of an optimal solution TMMSN($[P_\alpha, R_\alpha]$) and by equations 4, 5 and 6 we obtain:

$$\begin{aligned} c_\alpha &= min_U + min_V + min_C & (7)\\ &= 10(m+1)mn + 4m + (m+1)n + 6(m+1)n & (8)\\ &\quad +2(m+1)n + (m+1)^2 n & (9)\\ &= (m+1)^2 n + 10(m+1)mn + 9(m+1)n + 4m. & (10) \end{aligned}$$

$|$TMMSN($[P_\alpha, R_\alpha]$)$|$ attending the lower bound. Let M be an optimal solution TMMSN($[P_\alpha, R_\alpha]$) and $V = V_M$, $C = C_M$. In the above paragraph we have determined the lower bound c_α on $|M|$, which is attended, if the length of each of the sets U, V and C equals its minimum length min_U, min_V, min_C over all M. For U and V, this is possible for every instance $[P_\alpha, R_\alpha]$, as min_U and min_V only depend on n and m. But with Property 6 $|C|$ also depends on the geometric structure of $V \subset$ TMMSN($[P_\alpha, R_\alpha]$). The latter again depends on the actual position of the segment inducing points $\bigcup_{1\leq j\leq m} A_j$, which is chosen due to the placement rule, i.e. due to the occurrence of literals in clauses of α. We show:

Theorem 4 *$|$TMMSN($[P_\alpha, R_\alpha]$)$| = c_\alpha \Leftrightarrow \alpha$ is satisfiable.*

PROOF.
$|$TMMSN($[P_\alpha, R_\alpha]$)$| = c_\alpha \Rightarrow \alpha$ is satisfiable. From $|$TMMSN($[P_\alpha, R_\alpha]$)$| = c_\alpha$ and Property 1 follows that $|U| = min_U$, $|V| = min_V$, $|C| = min_C$. With Property 3 there must be a segment $\overline{s} \in V$ located in every clause gadget G_i^k, especially in every clause gadget G_i^*. Considering b^* from Lemma 3, $\overline{s}$ must be an induced segment, as there are no non-induced segments in b^*. Let V_j^* be the subset of segments of V_j located inside b^*, then the following truth assignment ν is well defined and consistent, as no parity changes or induced segments occur inside b^*.

- $V_j^* = low_j \Rightarrow \nu(x_j) = 1$.
- $V_j^* = up_j \Rightarrow \nu(x_j) = 0$.

Let $\overline{s}$ located in G_i^* be induced by a point $s \in A_j$ then s is on L_j, if variable x_j occurs in clause c_i of α and on U_j, if $\neg x_j$ occurs in c_i due to the placement rule. Thus $\overline{s} \in low_j$ if x_j occurs in c_i and $\overline{s} \in up_j$ if $\neg x_j$ occurs in c_i. With the above truth assignment $\nu(x_j) = 1$ if x_j occurs in c_i and $\nu(x_j) = 0 \Leftrightarrow \nu(\neg x_j) = 1$ if $\neg x_j$ occurs in c_i. In both cases c_i is satisfied by ν. With Property 3 this holds for every clause of α and thus ν satisfies α.

$|$TMMSN($[P_\alpha, R_\alpha]$)$| = c_\alpha \Leftarrow \alpha$ is satisfiable. If α is satisfiable, there exists a truth assignment ν such that every clause of α is satisfied, i.e. for each clause c_i there exists one variable x_j occurring in c_i such that $\nu(x_j)$ satisfies c_i. We now construct TMMSN($[P_\alpha, R_\alpha]$) by choosing the line segments of the sets V and C partly depending on ν. (U is fixed by $[P_\alpha, R_\alpha]$.) Choose $V = \bigcup_{1\leq j\leq m} V_j$ as follows:

- $\nu(x_j) = 1 \Rightarrow V_j = low_j$.
- $\nu(x_j) = 0 \Rightarrow V_j = up_j$.

Thus $|V| = min_V$ is minimal due to Lemma 1. Moreover, C can also be chosen with minimum length: If x_j ($\neg x_j$) occurs in c_i, a point s is placed inside G_i^k on L_j (U_j). Let x_j be the variable satisfying c_i under ν, i.e. $\nu(x_j) = 1$ ($\nu(\neg x_j) = 1$), if x_j ($\neg x_j$) occurs in c_i. Then by the placement rule and the definition of V above the induced segment $\overline{s} \in V_j \subset V$ is located inside G_i^k. This holds for every G_i^k, $1 \leq i \leq n$, $1 \leq k \leq m+1$, and a line segment $\overline{s} \in V$

and by Property 3 $|C| = min_C$. Thus the sets U, V, C can be chosen such that they are of minimum length and $|\text{TMMSN}([P_\alpha, R_\alpha])|$ attains the lower bound $c_\alpha = min_U + min_V + min_C$.

□

Since the construction of the instance $[P_\alpha, R_\alpha]$ contains $4 + 2(m+1)occ(x_j)$ points for each variable x_j and $4(m+1)n$ points for each clause c_i of α and $m \leq 3n$, we need $4m+2(m+1)3n+4(m+1)n^2 \leq 12n+18(n+1)n+12(n+1)n^2 \in O(n^3)$ points. All points having integer coordinates within the range of their number, the construction is polynomial in the input size. Thus together with Theorem 4 we have proved the NP-hardness of the TMMSN decision problem. As the TMMSN decision problem also is in NP we state:

Proposition 5 *The TMMSN decision problem is NP-complete.*

4 Conclusion

We have introduced the TMMSN problem, a generalization of the MMN, but not the RSMA problem. We have proved the TMMSN problem to be NPO-complete in three dimensions, which is the complexity first result regarding > 2 dimensions.

5 Acknowledgements

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