

The ergodic decomposition of asymptotically mean stationary random sources

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Abstract. It is demonstrated how to represent asymptotically mean stationary (AMS) random sources with values in standard spaces as mixtures of ergodic AMS sources. This is an extension of the well known decomposition of stationary sources which has facilitated the generalization of prominent source coding theorems to arbitrary, not necessarily ergodic, stationary sources. Asymptotic mean stationarity generalizes the definition of stationarity and covers a much larger variety of real-world examples of random sources of practical interest. It is sketched how to obtain source coding and related theorems for arbitrary, not necessarily ergodic, AMS sources, based on the presented ergodic decomposition.

Keywords. Asymptotic mean stationarity, ergodicity, ergodic decomposition, ergodic theorem, source coding, stationarity.

1 Introduction

The main purpose of this paper is to demonstrate how to decompose asymptotically mean stationary (AMS) random sources into ergodic AMS sources. The issue was brought up in [10], as it is involved in a variety of aspects of substantial interest to information theory. To the best of our knowledge, it had remained unsolved since then.

The ergodic decomposition of AMS sources can be viewed as an extension of the ergodic decomposition of stationary sources which states that a stationary source can be decomposed into ergodic components or, in other words, that it is a mixture of stationary and ergodic sources. This was originally discussed in more abstract measure theoretic settings (see the subsequent remark 1).

The first result in information theory that builds on the idea of decomposing a source into ergodic components was obtained by Jacobs in 1963. He proved that the entropy rate of a stationary source is the average of the rates of its ergodic components [17]. In 1974, the ergodic decomposition of stationary sources was rigorously introduced to the community by Gray and Davisson [7] who also provided an intuitive proof for sources with values in a discrete alphabet. This turned out to be a striking success as prominent theorems from source coding theory and related fields could be extended to arbitrary,

not necessarily ergodic, stationary sources [8,19,23,27,22,5] (see the references therein as well as [11] for a complete list).

In general, these results underscore that ergodic and information theory have traditionally been sources of mutual inspiration.

Remark 1. The first variant of an ergodic decomposition of stationary sources (with values in certain topological spaces) was elaborated in a seminal paper by von Neumann [31]. Subsequently, Kryloff and Bogoliouboff [3] obtained the result for compact metric spaces, and it was further extended by Halmos [13,14] to normal spaces. In parallel, Rokhlin [29] proved the decomposition theorem for Lebesgue spaces, which still can be considered as one of the most general results. Oxtoby [24] further clarified the situation by demonstrating that Kryloff's and Bogoliouboff's results can be obtained as corollaries of Riesz' representation theorem. In ergodic theory, the corresponding idea is now standard [26,32].

Asymptotic mean stationarity was first introduced in 1952 by Dowker [4] and further studied by Rechar [28], but became an area of active research only in the early 1980s, thanks to a fundamental paper of Gray and Kieffer [9]. Asymptotic mean stationarity is a property that applies for a large variety of natural examples of sources of practical interest [9]. Reasons are:

1. Asymptotic mean stationarity is stable under conditioning (see [21], p. 33) whereas stationarity is not.
2. To possess ergodic properties w.r.t. bounded measurements is equivalent to asymptotic mean stationarity [4,9]. Note that Birkhoff's theorem (e.g. [21]) states that stationarity is sufficient to possess ergodic properties.
3. The Shannon-McMillan-Breiman (SMB) theorem was iteratively extended to finally hold for AMS discrete random sources in 1980 [9].

Note that an alternative, elegant proof of the SMB theorem can be achieved by employing the ergodic decomposition of stationary sources [1]. The second point gives evidence of the practical relevance of AMS sources, as to possess ergodic properties is a necessity in a wide range of real-world applications of stochastic processes. For example, asymptotic mean stationarity is implicitly assumed when relative frequencies along sequences emitted by a real-world process are to converge. See also [20,6] for expositions of large classes of AMS processes of practical interest. The validity of the SMB theorem is a further theoretical clue to the relevance of AMS sources in information theory.

The benefits of an ergodic decomposition of AMS sources are, on one hand, to arrange the theory of AMS sources and, on the other hand, to facilitate follow-up results in source coding theory and related fields (see the discussion section 7 for some immediate consequences). In [10], one can find a concise proof of the ergodic decomposition of stationary sources as well as the ergodic decomposition of two-sided AMS sources, both with values in standard spaces. The case of two-sided AMS sources, however, is a straightforward reduction to the stationary case which does not apply for arbitrary AMS

sources. As the result for arbitrary AMS sources would have been highly desirable, it was listed as an open question in the discussion section of [10].

The main purpose of this paper is to provide a proof of the ergodic decomposition of arbitrary (two-sided and one-sided) AMS sources with values in standard spaces which cover discrete-valued and all natural examples of topological spaces.

The paper is organized as follows. In section 2 we collect basic notations and state the two main results. The first one is the ergodic decomposition itself and the second one is an essential lemma that may be interesting in its own right. In section 3, we present basic definitions of probability and measure theory as well as a classical ergodic theorem (*Krengel's stochastic ergodic theorem*) required for our purposes. The statement of Krengel's theorem is intuitively easy to grasp and can be understood by means of basic definitions from probability theory only. In section 4 we give a proof of lemma 1. Both the statement and the proof of lemma 1 are crucial for the proof of the decomposition. In section 5, we list relevant basic properties of standard spaces (subsection 5.1) and regular conditional probabilities and conditional expectations (subsection 5.2). Finally, in section 6, we present the proof of the ergodic decomposition. For organizational convenience, we have subdivided it into three steps and collected the merely technical passages into lemmata which have been deferred to the appendices A and B. We conclude by outlining immediate consequences of our result and pointing out potential applications in source coding theory, in the discussion section 7.

2 Basic Notations and Statement of Results

Let (Ω, \mathcal{B}) be a measurable space and $T : \Omega \rightarrow \Omega$ a measurable function. In this setting (see [26,12]), a probability measure P is called *stationary* (relative to T), if

$$P(B) = P(T^{-1}B)$$

for all $B \in \mathcal{B}$. It is called *asymptotically mean stationary (AMS)* (relative to T), if there is a measure \bar{P} on (Ω, \mathcal{B}) such that

$$\forall B \in \mathcal{B} : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} P(T^{-i}B) = \bar{P}(B). \quad (1)$$

Clearly, the measure \bar{P} is stationary and it is therefore called the *stationary mean* of P . An event $I \in \mathcal{B}$ is called *invariant* (relative to T), if $T^{-1}I = I$. The set of invariant events is a sub- σ -algebra of \mathcal{B} which we will denote by \mathcal{I} . A probability measure P on (Ω, \mathcal{B}) is said to be *ergodic* (relative to T), if $P(I) \in \{0, 1\}$ for any such invariant $I \in \mathcal{I}$. Note that an AMS system is ergodic if and only if its stationary mean is.

In order to apply this theory to (*A-valued*) *random sources*, that is, discrete-time stochastic processes with values in a standard space A (for a definition of standard space see subsection 5.1), one sets

$$\Omega = A^I = \bigotimes_{i \in I} A$$

where $I \in \{\mathbb{N}, \mathbb{Z}\}$. That is, Ω is the space of one-sided ($I = \mathbb{N}$) or two-sided ($I = \mathbb{Z}$) A -valued sequences. \mathcal{B} then is set to be the σ -algebra generated by the cylinder sets of sequences. A random source is given by a probability measure P on (Ω, \mathcal{B}) . Further, $T : \Omega \rightarrow \Omega$ is defined to be the *left shift operator*, i.e.

$$(Tx)_n = x_{n+1}$$

for $x = (x_0, x_1, \dots, x_n, \dots) \in \Omega$ (one-sided case) or $x = (\dots, x_{-1}, x_0, x_1, \dots) \in \Omega$ (two-sided case).

The main contribution of this paper is to give a proof of the following theorem.

Theorem 1. *Let P be a probability measure on a standard space (Ω, \mathcal{B}) which is AMS relative to the measurable $T : \Omega \rightarrow \Omega$. Then there is a T -invariant set $E \in \mathcal{I}$ with $P(E) = 1$ such that for each $\omega \in E$ there is an **ergodic AMS** probability measure P_ω and the following properties apply:*

(a)

$$\forall B \in \mathcal{B} : P_\omega(B) = P_{T\omega}(B).$$

(b)

$$\forall B \in \mathcal{B} : P(B) = \int P_\omega(B) dP(\omega).$$

(c) *If $f \in L_1(P)$, then also $\omega \mapsto \int f dP_\omega \in L_1(P)$ and*

$$\int f dP(\omega) = \int \left(\int f dP_\omega \right) dP(\omega).$$

Replacing AMS by stationary yields the aforementioned and well-known theorem of the ergodic decomposition of stationary random sources (e. g. [10], th. 2.5).

The following lemma is a key observation for the proof of theorem 1 and may be interesting in its own right. It states that the convergence involved in the definition of AMS measures is uniform over the elements of \mathcal{B} . This may seem intuitively surprising, as the underlying measurable space does not even have to be standard.

Lemma 1. *Let P be an AMS measure on (Ω, \mathcal{B}) relative to T . Then*

$$\sup_{B \in \mathcal{B}} \left| \frac{1}{n} \sum_{i=0}^{n-1} P(T^{-i}B) - \bar{P}(B) \right| \xrightarrow[n \rightarrow \infty]{} 0.$$

In other words, the convergence of (1) is uniform over the events $B \in \mathcal{B}$.

3 Preliminaries

3.1 Convergence of Measures

Definition 1. *Let $(P_n)_{n \in \mathbb{N}}$ be a sequence of probability measures on a measurable space (Ω, \mathcal{B}) .*

- We say that the P_n converge **strongly** to a probability measure \bar{P} if the sequences $(P_n(B))_{n \in \mathbb{N}}$ converge to $\bar{P}(B)$ for all $B \in \mathcal{B}$.
- If this convergence happens to be uniform in $B \in \mathcal{B}$ we say that the P_n converge **Skorokhod weakly** to \bar{P} .

See [16] for history and detailed characterisations of these definitions. Obviously Skorokhod weak convergence implies strong convergence. Seen from this perspective, lemma 1 states that the measures $P_n = 1/n \sum_{t=0}^{n-1} P \circ T^{-t}$, where P is an AMS measure and $P \circ T^{-t}(B) := P(T^{-t}B)$, do not only converge strongly (which they do by definition), but also Skorokhod weakly to the stationary mean \bar{P} .

A helpful characterization of Skorokhod weak convergence is the following theorem. Therefore we recall that a probability measure Q is said to *dominate* another probability measure P (written $Q \gg P$) if $Q(B) = 0$ implies $P(B) = 0$ for all $B \in \mathcal{B}$. The theorem of Radon-Nikodym (e.g. [15]) states that in case of $Q \gg P$ there is a measurable function $f : \Omega \rightarrow \mathbb{R}$, called *Radon-Nikodym derivative* or simply *density*, written $f = \frac{dP}{dQ}$, such that

$$P(B) = \int_B f dQ$$

for all $B \in \mathcal{B}$. It holds that $P(f = g) = 1$ (hence $Q(f = g) = 1$) for two densities $f, g = \frac{dP}{dQ}$.

As usual,

$$L_1(Q) := L_1(\Omega, \mathcal{B}, Q)$$

denotes the (linear) space of Q -integrable functions on (Ω, \mathcal{B}) modulo the subspace of functions that are null almost everywhere. For technical convenience, we will sometimes identify elements of $f \in L_1(Q)$ with their representatives $f : \Omega \rightarrow \mathbb{R}$. As a consequence we have that $f = g$ in $L_1(Q)$ if and only if $Q(f = g) = 1$ for their representatives. That is, equality is in an almost-everywhere sense for the representatives. Therefore, in $L_1(Q)$, a density is unique. Furthermore, $L_1(Q)$ can be equipped with the norm

$$\|f\|_1 := \int_{\Omega} |f| dQ.$$

See standard textbooks (e.g. [15]) for details.

In this language, Skorokhod weak convergence has a useful characterisation.

Theorem 2 ([16]). *Let $(P_n)_{n \in \mathbb{N}}, \bar{P}$ be probability measures. Then the following statements are equivalent:*

- (i) *The P_n converge Skorokhod weakly to \bar{P} .*
- (ii) *There is a probability measure Q , which dominates \bar{P} and all of the P_n such that the densities $f_n := \frac{dP_n}{dQ}$ converge **stochastically** to the density $\bar{f} := \frac{d\bar{P}}{dQ}$, that is*

$$\forall \epsilon \in \mathbb{R}^+ : \quad Q(\{\omega : |f_n(\omega) - \bar{f}(\omega)| > \epsilon\}) \xrightarrow[n \rightarrow \infty]{} 0.$$

(iii) There is a probability measure Q , which dominates \bar{P} and all of the P_n such that the densities $f_n := \frac{dP_n}{dQ}$ converge **in mean** (in $L_1(Q)$) to the density $\bar{f} := \frac{d\bar{P}}{dQ}$, that is

$$\int |f_n - \bar{f}| dQ \xrightarrow{n \rightarrow \infty} 0.$$

Proof. See [16], pp. 6–7. ◇

3.2 Krengel's theorem

In few words, the *stochastic ergodic theorem* of Krengel states that the averages of densities which are obtained by iterative applications of a *positive contraction* in $L_1(Q)$ converge stochastically to a density that is invariant with respect to the positive contraction.

To be more precise, let (Ω, \mathcal{B}, P) be a measure space and U a positive contraction on $L_1(\Omega, \mathcal{B}, P)$, that is, $Uf \geq 0$ for $f \geq 0$ (positivity) and $\|Uf\|_1 \leq \|f\|_1$ (contraction). Then Ω can be decomposed into two disjoint subsets (uniquely determined up to P -nullsets)

$$\Omega = \tilde{C} \dot{\cup} \tilde{D},$$

where \tilde{C} is the maximal support of a $f_0 \in L_1(\Omega, \mathcal{B}, P)$ with $Uf_0 = f_0$. In other words, for all $f \in L_1$ with $Uf = f$, we have $f = 0$ on \tilde{D} and there is a $f_0 \in L_1$ such that both $Uf_0 = f_0$ and $f_0 > 0$ on \tilde{C} (see [21], p. 141 ff. for details). Krengel's theorem then reads as follows.

Theorem 3 (Stochastic ergodic theorem; Krengel). *If U is a positive contraction on L_1 of a σ -finite measure space (Ω, \mathcal{B}, Q) (e.g. a probability space, the definition of a σ -finite measure space [15] is not further needed here) then, for any $f \in L_1$, the averages*

$$A_n f := \frac{1}{n} \sum_{t=0}^{n-1} U^t f$$

converge stochastically to a U -invariant \bar{f} . Moreover, on \tilde{C} we have L_1 -convergence, whereas on \tilde{D} the $A_n f$ converge stochastically to 0. If $f \geq 0$ then

$$f = \liminf_{n \rightarrow \infty} A_n f \quad \text{in } L_1(Q). \quad (2)$$

Proof. [21], p.143. ◇

3.3 Finite Signed Measures

Let (Ω, \mathcal{B}) be a measurable space. A finite signed measure is a σ -additive, but not necessarily positive, finite set function on \mathcal{B} . The theorem of the Jordan decomposition ([15], p. 120 ff.) states that $P = P_+ - P_-$ for measures P_+, P_- . These measures are uniquely determined insofar as if $P = P_1 - P_2$ for measures P_1, P_2 then there is a measure δ such that

$$P_1 = P_+ + \delta \quad \text{and} \quad P_2 = P_- + \delta. \quad (3)$$

P_+, P_- and $|P| := P_+ + P_-$ are called *positive, negative* and *total variation* of P . We further define

$$\|P\|_{TV} := |P|(\Omega).$$

By “eventwise” addition and scalar multiplication the set of finite signed measures can be made a normed vector space equipped with the *norm of total variation* $\|\cdot\|_{TV}$, written $(\mathcal{P}, \|\cdot\|_{TV})$ or simply \mathcal{P} . The following observation about signed measures and measurable functions is crucial for this work.

Lemma 2. *Let P be a finite signed measure on (Ω, \mathcal{B}) and $T : \Omega \rightarrow \Omega$ a measurable function. Then $P \circ T^{-1}$ is a finite signed measure for which*

$$|P \circ T^{-1}|(B) \leq |P|(T^{-1}B)$$

for all $B \in \mathcal{B}$. In particular, $\|P \circ T^{-1}\|_{TV} \leq \|P\|_{TV}$.

Proof. Note that $P \circ T^{-1} = P_+ \circ T^{-1} - P_- \circ T^{-1}$ is a decomposition into a difference of measures. Because of the uniqueness property of the Jordan decomposition (3), there is a measure δ such that $P_+ \circ T^{-1} = (P \circ T^{-1})_+ + \delta$ and $P_- \circ T^{-1} = (P \circ T^{-1})_- + \delta$. Therefore $|P \circ T^{-1}|(B) = (P \circ T^{-1})_+(B) + (P \circ T^{-1})_-(B) \leq P_+(T^{-1}B) + P_-(T^{-1}B) = |P|(T^{-1}B)$. $B = \Omega$ yields the last assertion, as $T^{-1}\Omega = \Omega$. \diamond

We finally observe the following well known relationship between signed measures dominated by a measure Q and $L_1(Q)$. Therefore, as usual (e.g. [15]), we say that a finite, signed measure P is dominated by Q if its total variation is, that is, $|P| \ll Q$. Note that the set \mathcal{P}_Q of finite, signed measures that are dominated by Q is a linear subspace of \mathcal{P} .

Lemma 3. *Let Q be a measure on the measurable space (Ω, \mathcal{B}) and \mathcal{P}_Q be the linear space of the finite signed measures that are dominated by Q . If $P_f(B) := \int_B f dQ$ for $f \in L_1(Q)$, then*

$$\begin{array}{ccc} \Phi : (L_1(Q), \|\cdot\|_1) & \longrightarrow & (\mathcal{P}_Q, \|\cdot\|_{TV}) \\ f & \longmapsto & P_f \end{array}$$

establishes an isometry of normed vector spaces.

Proof. This is a consequence of the theorem of Radon-Nikodym, see [15], p. 128 ff. If P is a finite signed measure with $|P| \ll Q$ then also $P_+, P_- \ll Q$. Define $\Psi(P) := \frac{dP_+}{dQ} - \frac{dP_-}{dQ} \in L_1(Q)$ as the difference of the densities of P_+, P_- relative to Q . Then Ψ is just the inverse of Φ . It is straightforward to check that $\|f\|_1 = \|\Phi(f)\|_{TV}$. \diamond

4 Proof of Lemma 1

We start by illustrating one of the core techniques of this work. Let (Ω, \mathcal{B}) be a measurable space and $(Q_n)_{n \in \mathbb{N}}$ be a countable collection of probability measures on it. Then the set function defined by

$$Q(B) := \sum_{n \geq 0} 2^{-n-1} Q_n(B) \quad \forall B \in \mathcal{B} \quad (4)$$

is a probability measure which dominates all of the Q_n [16].

Let now $(\Omega, \mathcal{B}, P, T)$ be such that P is an AMS measure relative to the measurable $T : \Omega \rightarrow \Omega$. Define further P_n to be the measures given by

$$P_n(B) = \frac{1}{n} \sum_{t=0}^{n-1} P(T^{-t}B) \quad (5)$$

for $B \in \mathcal{B}$. As a consequence of (4), the set function Q defined by

$$Q(B) := \frac{1}{2}(\bar{P}(B) + \sum_{n \geq 0} 2^{-n-1} P(T^{-n}B)) \quad (6)$$

for $B \in \mathcal{B}$ is a probability measure which dominates all of the $P \circ T^{-n}$ as well as \bar{P} . Hence it also dominates all of the P_n . Accordingly, we write

$$f_n := \frac{dP_n}{dQ} \quad \text{and} \quad \bar{f} := \frac{\bar{P}}{dQ} \quad (7)$$

for the respective densities. Lemma 1 can be obtained as a corollary of the following result.

Lemma 4. *Let P be an AMS probability measure on (Ω, \mathcal{B}) relative to T with stationary mean \bar{P} . Let P_n , Q , f_n and \bar{f} as defined by equations (5), (6) and (7). Then the f_n converge stochastically to the density $\bar{f} := \frac{d\bar{P}}{dQ}$. Moreover,*

$$\bar{f} = \liminf_{n \rightarrow \infty} f_n \quad Q\text{-a.e.} \quad (8)$$

Proof. Let $f_1 = \frac{dP}{dQ}$. The road map of the proof is to construct a positive contraction U on $L_1(Q)$ such that

$$f_n = \frac{1}{n} \sum_{t=0}^{n-1} U^t f_1 =: A_n f_1.$$

As a consequence of Krengel's theorem we will obtain that the f_n converge stochastically to a U -invariant limit f^* . In a final step we will show that indeed $f^* = \bar{f}$ in $L_1(Q)$ (i.e. Q -a.e.), which completes the proof.

Our endomorphism U on $L_1(Q)$ is induced by the measurable function T . Let $f \in L_1(Q)$. We first recall that, by lemma 3, the set function $\Phi(f)$ given by

$$\Phi(f)(B) := \int_B f dQ$$

for $B \in \mathcal{B}$ and $f \in L_1(Q)$ is a finite, signed measure on (Ω, \mathcal{B}) whose total variation $|\Phi(f)|$ is dominated by Q .

We would like to define

$$Uf := \Phi^{-1}(\Phi(f) \circ T^{-1}),$$

which would be obviously linear. However, Φ^{-1} is only defined on \mathcal{P}_Q , that is, for finite signed measures that are dominated by Q . Therefore, we have to show that $\Phi(f) \circ T^{-1} \in \mathcal{P}_Q$ which translates to demonstrating that $|\Phi(f) \circ T^{-1}| \ll Q$. This does not hold in general (see [21]). However, in the special case of the dominating Q chosen here, it can be proven.

To see this let B such that $|\Phi(f) \circ T^{-1}|(B) > 0$ and we have to show that $Q(B) > 0$. Because of lemma 2

$$|\Phi(f)|(T^{-1}B) \geq |\Phi(f) \circ T^{-1}|(B) > 0.$$

As $|\Phi(f)| \ll Q$, we obtain $Q(T^{-1}B) > 0$. By definition of Q we thus either find an $N_0 \in \mathbb{N}$ such that $0 < P(T^{-N_0}(T^{-1}B)) = P(T^{-N_0-1}B)$ or we have that $0 < \bar{P}(T^{-1}B) = \bar{P}(B)$ because of the stationarity of \bar{P} . Both cases imply $Q(B) > 0$ which we had to show.

If $f \geq 0$ then $\Phi(f)$ is a measure. Hence also $\Phi(f) \circ T^{-1}$ is a measure which in turn implies $Uf = \frac{d(\Phi(f) \circ T^{-1})}{dQ} \geq 0$. Hence U is positive. It is also a contraction with respect to the L_1 -norm $\|\cdot\|_1$, as, because of the lemmata 2 and 3,

$$\|Uf\|_1 = \|\Phi(f) \circ T^{-1}\|_{TV} \leq \|\Phi(f)\|_{TV} = \|f\|_1.$$

For $f_1 = \frac{dP}{dQ}$ being the density of P relative to Q we obtain

$$U^n f_1 = \frac{d(P \circ T^{-n})}{dQ}$$

Hence the $f_n := A_n f_1 = 1/n \sum_{t=0}^{n-1} U^t f_1$ are the densities of the $P_n = \frac{1}{n} \sum_{t=0}^{n-1} P \circ T^{-t}$ relative to Q . An application of Krengel's theorem 3 then shows that the $A_n f_1$ converge stochastically to a U -invariant limit $f^* \in L_1(Q)$. Note that a positive U -invariant f just corresponds to a stationary measure.

It remains to show that $\bar{f} = f^*$ in $L_1(Q)$ or, equivalently, $\bar{f} = f^*$ Q -a.e. for their representatives (see the discussions in subsection 3.1). Let \tilde{D} , as described in subsection 3.2, be the complement of the maximal support of a U -invariant $g \in L_1(Q)$. We recall that stationary measures are identified with positive, U -invariant elements of $L_1(Q)$. Therefore, $\bar{f} = \frac{d\bar{P}}{dQ}$ is U -invariant which yields

$$Q(\{\bar{f} > 0\} \cap \tilde{D}) = 0$$

which implies $\bar{f} = 0$ Q -a.e. on \tilde{D} . Due to Krengel's theorem, it holds that also $f^* = 0$ Q -a.e. on \tilde{D} , and we obtain that

$$\bar{f} = 0 = f^* \quad Q - \text{a.e. on } \tilde{D}.$$

In order to conclude that

$$\bar{f} = f^* \quad Q - \text{a.e. on } \tilde{C}$$

it remains to show that $\int_B f^* dQ = \int_B \bar{f} dQ$ for events $B \subset \tilde{C} = \Omega \setminus \tilde{D}$ as two integrable functions coincide almost everywhere if their integrals over arbitrary events coincide ([15]) with which we will have completed the proof. From Krengel's theorem we know that, on \tilde{C} , we have L_1 -convergence of the f_n :

$$\lim_{n \rightarrow \infty} \int_{\tilde{C}} |f_n - f^*| dQ = 0. \quad (9)$$

Therefore, for $B \subset \tilde{C}$,

$$\int_B f^* dQ \stackrel{(9)}{=} \lim_{n \rightarrow \infty} \int_B f_n dQ = \lim_{n \rightarrow \infty} P_n(B) \stackrel{(**)}{=} \bar{P}(B) = \int_B \bar{f} dQ,$$

where $(**)$ follows from the asymptotic mean stationarity of P . We thus have completed the proof of the main statement of the lemma.

Finally, (8) is a direct consequence of (2) in Krengel's theorem. \diamond

In sum, we have shown that there is a measure Q that dominates all of the P_n as well as \bar{P} such that the densities of the P_n converge stochastically to the density of \bar{P} . According to theorem 2, this is equivalent to Skhorokhod weak convergence. Hence we obtain lemma 1 as a corollary.

5 Preliminaries II

In this section we will first review a couple of additional definitions that are necessary for a proof of theorem 1. In subsection 5.1 we give the definition of a standard space. The beneficial properties of standard spaces become apparent in subsection 5.2, where we shortly review conditional probabilities and expectation.

5.1 Standard spaces

See [25], ch. 3 or [12] for thorough treatments of standard spaces. In the following, a field \mathcal{F} is a collection of subsets of a set Ω that contains Ω and is closed with respect to complements and finite unions.

Definition 2. A field \mathcal{F} on a set Ω is said to have the **countable extension property** if the following two conditions are met.

1. \mathcal{F} has a countable number of elements.
2. Every nonnegative and finitely additive set function P on \mathcal{F} is continuous at \emptyset , that is, for a sequence of elements $F_n \in \mathcal{F}$ with $F_{n+1} \subset F_n$ such that $\bigcap_n F_n = \emptyset$ we have $\lim_{n \rightarrow \infty} P(F_n) = 0$.

Definition 3. A measurable space (Ω, \mathcal{B}) is called a **standard space**, if the σ -algebra \mathcal{B} is generated by a field \mathcal{F} which has the countable extension property.

Remark 2.

1. Most of the prevalent examples of measurable spaces in practice are standard. For example, any measurable space which is generated by a complete, separable, metric space (i.e. a *Polish space*) is standard. Moreover, standard spaces can be characterized as being isomorphic to subspaces $(B, \mathcal{B} \cap B)$ of Polish spaces (Ω, \mathcal{B}) where $B \in \mathcal{B}$ is a measurable set (see [25], ch. 3).
2. An alternative characterisation of standard spaces is that the σ -algebra \mathcal{B} possesses a *basis*. See [18], app. 6, for a discussion.

5.2 Conditional Probability and Expectation

See [25], ch. 6 or [12] for a discussion of conditional probability and expectation.

Definition 4. Let P be a probability measure on a measurable space (Ω, \mathcal{B}) and let $\mathcal{G} \subset \mathcal{B}$ be a sub- σ -algebra of \mathcal{B} . A function

$$\delta(\cdot, \cdot) : \mathcal{B} \times \Omega \rightarrow \mathbb{R},$$

is called a (version of the) **conditional probability** of P given \mathcal{G} , if

(CP1) $\delta(B, \cdot)$ is \mathcal{G} -measurable for all $B \in \mathcal{B}$ and

(CP2)

$$P(B \cap G) = \int_G \delta(B, \omega) dP(\omega)$$

for all $G \in \mathcal{G}, B \in \mathcal{B}$.

$\delta(\cdot, \cdot)$ is called a (version of the) **regular conditional probability** of P given \mathcal{G} , if, in addition to (CP1) and (CP2),

(RCP) $\delta(\cdot, \omega)$ is a probability measure on \mathcal{B} for all $\omega \in \Omega$.

We collect a couple of basic results about conditional probabilities. See [25] or [12] for details.

1. Let γ, δ be two versions of the conditional probability of P given \mathcal{G} . Then the \mathcal{G} -measurable functions $\gamma(B, \cdot), \delta(B, \cdot)$ agree almost everywhere for any given $B \in \mathcal{B}$, that is, we have

$$\forall B \in \mathcal{B} : P(\{\omega \mid \gamma(B, \omega) = \delta(B, \omega)\}) = 1. \quad (10)$$

2. Conditional probabilities always exist. Existence of regular conditional probabilities is not assured for arbitrary measurable spaces. However, for standard spaces (Ω, \mathcal{B}) existence can be proven.
3. Note that it cannot be shown for arbitrary measurable spaces that two versions δ, γ agree almost everywhere for all $B \in \mathcal{B}$, meaning that we do not have

$$P(\{\omega \mid \forall B \in \mathcal{B} : \gamma(B, \omega) = \delta(B, \omega)\}) = 1.$$

However, for standard spaces (Ω, \mathcal{B}) this beneficial property applies:

Lemma 5. *Let (Ω, \mathcal{B}) be a measurable space such that \mathcal{B} is generated by a countable field \mathcal{F} . Let P be a probability measure on it and assume that the regular conditional probability of P given a sub- σ -algebra \mathcal{G} exists. If δ, γ are two versions of it then the measures $\delta(\cdot, \omega)$ and $\gamma(\cdot, \omega)$ agree on a set of measure one, that is,*

$$P(\{\omega \mid \forall B \in \mathcal{B} : \gamma(B, \omega) = \delta(B, \omega)\}) = 1.$$

We display the proof, as its (routine) arguments are needed in subsequent sections.

Proof. Enumerate the elements of \mathcal{F} and write F_k for element No. k . According to (10) we find for each $k \in \mathbb{N}$ a set B_k of P -measure one on which $\delta(F_k, \cdot)$ and $\gamma(F_k, \cdot)$ agree. Hence, on $B := \bigcap_k B_k$, which is an event of P -measure one, all of the $\delta(F_k, \cdot)$ and the $\gamma(F_k, \cdot)$ coincide. Thus the measures $\delta(\cdot, \omega)$ and $\gamma(\cdot, \omega)$ agree on a generating field of \mathcal{B} for $\omega \in B$. As a measure is uniquely determined by its values on a generating field ([15]), we obtain that the measures $\delta(\cdot, \omega)$ and $\gamma(\cdot, \omega)$ agree on B , that is, P -almost everywhere. \diamond

We also give the definition of conditional expectations and point out their extra properties on standard spaces.

Definition 5. *Let (Ω, \mathcal{B}, P) be a probability space and $f \in L_1(P)$. Let $\mathcal{G} \subset \mathcal{B}$ be a sub- σ -algebra. If $h : \Omega \rightarrow \mathbb{R}$ is*

1. \mathcal{G} -measurable and
2. for all $G \in \mathcal{G}$ it holds that $\int_G f dP = \int_G h dP$

we say that h is a **version of the conditional expectation** of f given \mathcal{G} and write

$$h(\omega) = E(f|\mathcal{G})(\omega).$$

Conditional expectations always exist. In case of standard spaces they have an extra property which we rely on. See [25], ch. 6 for proofs of the following results.

Theorem 4. *Let (Ω, \mathcal{B}, P) be a probability space, \mathcal{G} a sub- σ -algebra of \mathcal{B} and $f \in L_1(P)$. Then there exists a version $E(f|\mathcal{G})$ of the conditional expectation. In case of a standard space (Ω, \mathcal{B}) it holds that*

$$E(f|\mathcal{G})(\omega) = \int f(x) d\delta_P(x, \omega) \tag{11}$$

where δ_P is a version of the regular conditional probability of P given \mathcal{G} .

Corollary 1. *Let (Ω, \mathcal{B}) be a standard space, P a probability measure on it and $f \in L_1(P)$. Let \mathcal{G} be a sub- σ -algebra and δ_P the regular conditional probability of P given \mathcal{G} . Then $\omega \mapsto \int f d\delta_P(\cdot, \omega)$ is \mathcal{G} -measurable (hence also \mathcal{B} -measurable) and*

$$\int_G f dP = \int_G \left(\int f d\delta_P(\cdot, \omega) \right) dP \tag{12}$$

for all $G \in \mathcal{G}$.

6 Proof of Theorem 1

We recall the notations of section 2 and that, according to the assumptions of theorem 1, P is a measure on a standard space (Ω, \mathcal{B}) that is AMS relative to the measurable $T : \Omega \rightarrow \Omega$.

6.1 Sketch of the Proof Strategy

The core idea for proving the theorem is to define the measures P_ω as being induced by the regular conditional probability measures of P given the invariant events \mathcal{I} . That is, we define

$$\forall B \in \mathcal{B} : P_\omega(B) := \delta_P(B, \omega) \quad (13)$$

where, here and in the following, δ refers to regular conditional probabilities given the invariant events \mathcal{I} . Note that, for arbitrary probability measures P on (Ω, \mathcal{B}) ,

$$\delta_P(B, \omega) = \delta_P(B, T\omega), \quad (14)$$

as, otherwise, $\delta_P(B, \cdot)^{-1}(y)$ would not be an invariant set for $y := \delta_P(B, T\omega)$ which would be a contradiction to the \mathcal{I} -measurability of $\delta_P(B, \cdot)$.

As a consequence of (14), we obtain property (a) of the theorem. Furthermore, (b) is the defining property (CP2) of a regular conditional probability (see Def. 4) and (c) is equation (12) from corollary 1 with $G = \Omega$. What remains to show is that, for ω in an invariant set E of P -measure one, the P_ω are ergodic and AMS.

We intend to do this by the following strategy. First, we recall that if, in theorem 1, AMS is replaced by *stationary*, we obtain the well known result of the ergodic decomposition of stationary measures (see the introduction for a discussion). If one follows the lines of argumentation of its proof (see [10], th. 2.5) one sees that, on an invariant set of P -measure one, the P_ω are just the regular conditional probabilities of the stationary P . Applying the ergodic decomposition of stationary measures to the stationary mean \bar{P} of P provides us with an invariant set \bar{E} of P -measure 1 such that

$$\omega \in \bar{E} \implies \bar{P}_\omega := \delta_{\bar{P}}(\cdot, \omega) \text{ is stationary and ergodic.} \quad (15)$$

We will show that, on an invariant set $E \subset \bar{E}$ of P -measure one, the P_ω converge Skorokhod weakly (hence strongly, see Def. 1) to the \bar{P}_ω , which translates to that the P_ω are AMS and have stationary means \bar{P}_ω . As an AMS measure is ergodic if its stationary mean is ergodic, we will have completed the proof.

Therefore, we will proceed according to the following steps:

Step 1 We construct measures Q_ω that dominate \bar{P}_ω and all of the

$$P_{n,\omega} := \frac{1}{n} \sum_{t=0}^{n-1} (P_\omega \circ T^{-t}), n \geq 0 \quad (16)$$

(note that $P_\omega = P_{1,\omega}$), which will provide us with densities

$$f_{n,\omega} := \frac{dP_{n,\omega}}{dQ_\omega} \quad \text{and} \quad \bar{f}_\omega := \frac{d\bar{P}_\omega}{dQ_\omega} \quad (17)$$

for all ω .

Step 2 We construct positive contractions U_ω on $L_1(Q_\omega)$ such that

$$U_\omega \frac{d(P_\omega \circ T^{-n})}{dQ_\omega} = \frac{d(P_\omega \circ T^{-n-1})}{dQ_\omega} \quad (18)$$

hence

$$A_n f_{1,\omega} := \frac{1}{n} \sum_{t=0}^{n-1} U_\omega^t f_{1,\omega} = f_{n,\omega} \quad (19)$$

We apply Krengel's theorem (th. 3) to obtain that the $f_{n,\omega}$ converge stochastically to a U_ω -invariant f_ω^* as well as $f_\omega^* = \liminf_{n \rightarrow \infty} f_{n,\omega}$ in $L_1(Q_\omega)$

Step 3 We show that, for ω in an invariant set E of P -measure one,

$$f_\omega^* = \bar{f}_\omega \quad \text{in } L_1(Q_\omega).$$

This completes the proof, as this states that the P_ω converge Skorokhod weakly to the \bar{P}_ω in E , hence that the P_ω are ergodic and AMS for ω in the invariant set E of P -measure one.

6.2 Step 1

We recall definitions (5) and (6) of P_n and Q . We define Q_ω as the probability measures induced by the regular conditional probability of Q given the invariant events \mathcal{I} , that is,

$$Q_\omega(B) := \delta_Q(B, \omega) \quad (20)$$

for $B \in \mathcal{B}$. It remains to show that, by choosing an appropriate version, Q_ω indeed dominates all of the $P_\omega \circ T^{-n}$ (hence all of the $P_{n,\omega}$) as well as \bar{P}_ω . This is established by the following lemma whose merely technical proof has been deferred to appendix A.

Lemma 6.

$$\alpha(B, \omega) := \frac{1}{2}(\bar{P}_\omega(B) + \sum_{n \geq 0} 2^{-n-1} P_\omega(T^{-n}B)) \quad (21)$$

is a version of the regular conditional probability of Q given \mathcal{I} .

Remark 3. In order to achieve that Q_ω dominates all of the $P_\omega \circ T^{-n}$ and \bar{P}_ω one could have defined Q_ω directly via (21). However, the observation that Q_ω is induced by the regular conditional probability of Q given \mathcal{I} is crucial for step 3.

6.3 Step 2

Construction of positive contractions U_ω on $L_1(Q_\omega)$ is achieved by, mutatis mutandis, reiterating the arguments accompanying the construction of U in the proof of lemma 4. In more detail, we replace $P, P_n, \bar{P}, Q, f_n, \bar{f}$ there by $P_\omega, \bar{P}_\omega, P_{n,\omega}, Q_\omega, f_{n,\omega}, \bar{f}_\omega$ (we recall (13),(15),(16),(20),(17) for the latter definitions) here. Note that choosing the version of Q_ω according to lemma 6 ensures that U_ω indeed maps $L_1(Q_\omega)$ onto $L_1(Q_\omega)$.

(18) and (19) then are a direct consequence of the definition of U_ω . Finally, application of Krengel's theorem 3 to the positive contraction U_ω on $L_1(Q_\omega)$ yields a U_ω -invariant f_ω^* to which the $f_{n,\omega}$ converge stochastically. Moreover, again by Krengel's theorem,

$$f_\omega^* = \liminf_{n \rightarrow \infty} f_{n,\omega} \quad \text{in } L_1(Q_\omega). \quad (22)$$

6.4 Step 3

We have to show that

$$f_\omega^* = \bar{f}_\omega \quad \text{in } L_1(Q_\omega)$$

for ω in an invariant set $E \subset \bar{E}$ with $Q(E) = 1$. In a first step, the following lemma will provide as with a useful invariant E^* where $E \subset E^* \subset \bar{E}$ and $Q(E^*) = 1$. We further recall the definitions of f_n and \bar{f} as the densities of P_n and \bar{P} w.r.t. Q (see (7)). Without loss of generality, we choose representatives that are everywhere nonnegative. Due to lemma 4,

$$\liminf_{n \rightarrow \infty} f_n = \bar{f} \quad \text{in } L_1(Q). \quad (23)$$

Lemma 7. *There is an invariant set E^* with $P(E^*) = Q(E^*) = 1$ such that, for $\omega \in E^*$,*

$$\liminf_{n \rightarrow \infty} f_n = \liminf_{n \rightarrow \infty} f_{n,\omega} \quad \text{in } L_1(Q_\omega) \quad (24)$$

and

$$\bar{f} = \bar{f}_\omega \quad \text{in } L_1(Q_\omega). \quad (25)$$

Proof. We have deferred the merely technical proof to appendix B. \diamond

We compute

$$\begin{aligned} \int_{E^*} \left(\int |f_\omega^* - \bar{f}_\omega| dQ_\omega \right) dQ &\stackrel{(22),(24),(25)}{=} \int_{E^*} \left(\int |\liminf_{n \rightarrow \infty} f_n - \bar{f}| dQ_\omega \right) dQ \\ &\stackrel{(*)}{=} \int_{E^*} |\liminf_{n \rightarrow \infty} f_n - \bar{f}| dQ \stackrel{(23)}{=} 0 \end{aligned}$$

where (*) follows from the defining properties of the conditional expectation $E(|\liminf_{n \rightarrow \infty} f_n - \bar{f}| \mid \mathcal{I})$ in combination with theorem 4. According to the last computation, we find a set $E \subset E^*$ with $Q(E) = 1$ such that

$$\omega \in E \implies \int |f_\omega^* - \bar{f}_\omega| dQ_\omega = 0.$$

The invariance of the regular conditional probabilities (see (14)) involved in the definitions of $f_\omega^*, \bar{f}_\omega$ implies

$$\int |f_\omega^* - \bar{f}_\omega| dQ_\omega = 0 \iff \int |f_{T\omega}^* - \bar{f}_{T\omega}| dQ_{T\omega} = 0.$$

This translates to that E is invariant such that E meets the requirements of theorem 1. \diamond

7 Discussion

We have demonstrated how to decompose AMS random sources, which encompass a large variety of sources of practical interest, into ergodic components. The result comes in the tradition of the ergodic decomposition of stationary sources. As outlined in the introduction, this substantially added to source coding theory by facilitating the generalization of a variety of prominent theorems to arbitrary, not necessarily ergodic, stationary sources.

Our result can be expected to yield similar contributions to the theory of AMS sources. An immediate clue is that the theorems developed in [10] for two-sided AMS sources are now valid for arbitrary AMS sources by replacing theorem 2.6 there by theorem 1 here.

Moreover, a couple of relevant quantities in information theory (e.g. entropy rate) are affine functionals that are upper semicontinuous w.r.t. the space of stationary random sources, equipped with the weak topology. Jacobs' theory of such functionals ([17], see also [5], th. 4) immediately builds on the ergodic decomposition of stationary sources. This theory should now be extendable to AMS sources.

We finally would like to mention that a certain class of source coding theorems for AMS sources were obtained by partially circumventing the lack of an ergodic decomposition. Schematically, this was done by a reduction from AMS sources to their stationary means and subsequent application of the ergodic decomposition for stationary sources in order to further reduce to ergodic sources. In these cases, our contribution would only be to simplify the theorems' statements and thus a merely esthetical one. However, in the remaining cases where the reduction from asymptotic mean stationarity to stationarity is not applicable, our result will be essential. The full exploration of related consequences seems to be a worthwhile undertaking.

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A Proof of lemma 6

In the following, according to the assumptions of theorem 1, P is a measure on a standard space (Ω, \mathcal{B}) that is AMS relative to the measurable $T : \Omega \rightarrow \Omega$. We further recall the notations of section 2 as well as equations (5) and (6) for the necessary definitions.

Lemma 8. *Let $g : \Omega \rightarrow \mathbb{R}$ be a T -invariant (that is, $g(\omega) = g(T\omega)$ for all $\omega \in \Omega$), measurable function. Then it holds that*

$$\int g dP = \int g d(P \circ T^{-n}) = \int g dP_n = \int g d\bar{P} = \int g dQ. \quad (26)$$

In particular, all of the integrals exist if one of the integrals exists.

Proof. Note that Q and all of the $P \circ T^{-n}$ and P_n , like P , are AMS with stationary mean \bar{P} , which is an obvious consequence of their definitions. Therefore, the claim of the lemma follows from the, intuitively obvious, observation that $\int g dP = \int g d\bar{P}$ for invariant g and general AMS P with stationary mean \bar{P} . See [12] for details. \diamond

Lemma 9. *The functions*

$$\zeta_n(B, \omega) := \delta_P(T^{-n}B, \omega) = P_\omega(T^{-n}B)$$

are versions of the regular conditional probabilities $\delta_{P \circ T^{-n}}$ of the $P \circ T^{-n}$ given \mathcal{I} .

Proof. The functions $\zeta_n(\cdot, \omega)$ are probability measures for fixed $\omega \in \Omega$ (this is (RCP) of definition 4) as the P_ω are, by the definition of δ_P . Again by the definition of δ_P , $\zeta_n(B, \cdot)$ is also \mathcal{I} -measurable in ω for fixed $B \in \mathcal{B}$, which is (CP1) of definition 4. For $I \in \mathcal{I}$ and $B \in \mathcal{B}$ we compute

$$\begin{aligned} \int_I \delta_P(T^{-n}B, \omega) d(P \circ T^{-n})(\omega) &\stackrel{(14),(26)}{=} \int_I \delta_P(T^{-n}B, \omega) dP(\omega) \\ &= P(I \cap T^{-n}B) \stackrel{T^{-n}I=I}{=} P(T^{-n}(I \cap B)) \\ &= \int_I \delta_{P \circ T^{-n}}(B, \omega) d(P \circ T^{-n})(\omega) \end{aligned}$$

where the first equation follows from the invariance of the integrands and lemma 8. We have thus shown (CP2) of definition 4. \diamond

We recall that, for lemma 6, we have to show that

$$\alpha(B, \omega) = \frac{1}{2}(\bar{P}_\omega(B) + \sum_{n \geq 0} 2^{-n-1} P_\omega(T^{-n}B))$$

is a version of the regular conditional probability δ_Q . Note first that \bar{P}_ω , according to our proof strategy outlined in subsection 6.1, was defined as $\delta_{\bar{P}}(\cdot, \omega)$ where $\delta_{\bar{P}}$ is the regular conditional probability of the stationary mean \bar{P} . Furthermore, as a consequence of lemma 9, we can identify the $P_\omega \circ T^{-n}$ with $\delta_{P \circ T^{-n}}(\cdot, \omega)$ and write

$$\alpha(B, \omega) = \frac{1}{2}(\delta_{\bar{P}}(B, \omega) + \sum_{n \geq 0} 2^{-n-1} \delta_{P \circ T^{-n}}(B, \omega)). \quad (27)$$

We will then exploit the defining properties of the δ s to finally show that α is a version of δ_Q .

Proof of lemma 6. We have to check properties (RCP), (CP1) and (CP2) of definition 4.

(RCP) : That $\alpha(\cdot, \omega)$ is a probability measure for fixed ω follows from an argumentation which is completely analogous to that at the beginning of section 4, surrounding equations (4) and (6).

(CP1) : As all of the δ 's involved in (27) are invariant in ω (see (14)), we know that $\alpha(B, \cdot)$ is measurable w. r. t. \mathcal{I} for any $B \in \mathcal{B}$ which is (CP1) of definition 4.

(CP2) : Fix $B \in \mathcal{B}$ and consider the functions

$$g_n(\omega) := \frac{1}{2}(\delta_{\bar{P}}(B, \omega) + \sum_{k=0}^n 2^{-k-1} \delta_{P \circ T^{-k}}(B, \omega)).$$

This is an increasing sequence of non-negative measurements which converges everywhere to the values $\alpha(B, \omega)$. Because of (14) the summands of g_n are invariant. As all of the summands are also integrable with respect to some $P \circ T^{-k}$ or \bar{P} they are also integrable with respect to Q , due to lemma 8. Therefore, also the g_n are integrable with respect to Q . The monotone convergence theorem of Beppo Levi (e.g. [15]) reveals that also $\alpha(B, \cdot)$ is and further, for $I \in \mathcal{I}$ and $B \in \mathcal{B}$:

$$\begin{aligned} \int_I \alpha(B, \omega) dQ(\omega) &= \int_I \lim_{n \rightarrow \infty} \frac{1}{2}(\delta_{\bar{P}}(B, \omega) + \sum_{k=0}^n 2^{-k-1} \delta_{P \circ T^{-k}}(B, \omega)) dQ(\omega) \\ &\stackrel{(a)}{=} \lim_{n \rightarrow \infty} \int_I \frac{1}{2}(\delta_{\bar{P}}(B, \omega) + \sum_{k=0}^n 2^{-k-1} \delta_{P \circ T^{-k}}(B, \omega)) dQ(\omega) \\ &\stackrel{(b)}{=} \lim_{n \rightarrow \infty} \frac{1}{2} \left(\int_I \delta_{\bar{P}}(B, \omega) d\bar{P}(\omega) \right. \\ &\quad \left. + \sum_{k=0}^n 2^{-k-1} \int_I \delta_{P \circ T^{-k}}(B, \omega) d(P \circ T^{-k})(\omega) \right) \\ &\stackrel{(c)}{=} \lim_{n \rightarrow \infty} \frac{1}{2} (\bar{P}(I \cap B) + \sum_{k=0}^n 2^{-k-1} P(T^{-k}(I \cap B))) \\ &= \frac{1}{2} (\bar{P}(I \cap B) + \sum_{n \geq 0} 2^{-n-1} P(T^{-n}(I \cap B))) \\ &= Q(I \cap B) \end{aligned}$$

where (a) follows from Beppo Levi's theorem, (b) follows from the invariance of the δ s and subsequent application of lemma 8 and (c) is just the defining property (CP2) of the conditional probabilities δ (definition 4). We thus have shown property (CP2) for α . \diamond

B Proof of Lemma 7

According to the assumptions of theorem 1, P is a measure on a standard space (Ω, \mathcal{B}) that is AMS relative to the measurable $T : \Omega \rightarrow \Omega$. We further recall the notations of section 2 as well as equations (5), (6), (7), (13), (15), (16), (17), (20) and the surrounding texts for the necessary definitions. We further remind that, without loss of generality, we had chosen representatives of the f_n and \bar{f} that are everywhere nonnegative. The following lemma will deliver the technical key to lemma 7.

Lemma 10. *For each $1 \leq n \in \mathbb{N}$ there is an invariant $E_n \in \mathcal{I} \subset \mathcal{B}$ with $P_n(E_n) = Q(E_n) = 1$ such that*

$$\omega \in E_n \implies f_{n,\omega} = f_n \quad \text{in } L_1(Q_\omega).$$

There is also an invariant E_∞ with $\bar{P}(E_\infty) = Q(E_\infty) = 1$ such that

$$\omega \in E_\infty \implies \bar{f}_\omega = \bar{f} \quad \text{in } L_1(Q_\omega).$$

Loosely speaking, the lemma reveals that the f_n and the $f_{n,\omega}$ as well as \bar{f} and \bar{f}_ω agree Q_ω -a.e. for Q -almost all $\omega \in \Omega$. This means that, for Q -almost all ω , they are equal on the parts of Ω considered relevant by the measures Q_ω .

Proof. Consider the functions

$$\beta_n(B, \omega) := \int_B f_{n,\omega} dQ_\omega \quad \text{and} \quad \gamma_n(B, \omega) := \int_B f_n dQ_\omega$$

By the definition of a density,

$$\delta_{P_n}(B, \omega) = \int_B f_{n,\omega} dQ_\omega.$$

Hence $\beta_n(B, \omega)$ is just the regular conditional probability of P_n given \mathcal{I} . We now show that γ_n is a version of the conditional probability of P_n given \mathcal{I} (but not necessarily a regular one). Note first that the $\gamma_n(B, \cdot)$ are \mathcal{I} -measurable as, according to (11), we have that $\gamma_n(B, \omega)$ agrees with the conditional expectation $E_Q(\mathbf{1}_B f_n | \mathcal{I})(\omega)$, which, by definition, is \mathcal{I} -measurable. Second, we observe that, for $I \in \mathcal{I}$ and $B \in \mathcal{B}$, as γ_n is invariant in ω (*),

$$\begin{aligned} \int_I \gamma_n(B, \omega) dP_n &\stackrel{(*), (26)}{=} \int_I \gamma_n(B, \omega) dQ \\ &= \int_I \left(\int_B f_n dQ_\omega \right) dQ \\ &= \int_I \left(\int \mathbf{1}_B f_n dQ_\omega \right) dQ \\ &\stackrel{(12)}{=} \int_I \mathbf{1}_B f_n dQ = \int_{I \cap B} f_n dQ \\ &= P_n(I \cap B), \end{aligned}$$

which shows the required property (CP2) of definition 4. Hence the γ_n 's are versions of the conditional probabilities of the P_n 's given \mathcal{I} .

Note that the $\gamma_n(\cdot, \omega)$ are measures because the f_n had been chosen nonnegative everywhere. If we follow the line of argumentation of lemma 5 we find a set E_n of P_n -measure one such that the measures $\beta_n(\cdot, \omega)$ and $\gamma_n(\cdot, \omega)$ agree for $\omega \in E_n$. Because of the invariance of β_n, γ_n the set E_n is invariant. Hence (lemma 8) also $Q(E_n) = 1$. Resuming we have

$$\omega \in E_n \implies \forall B \in \mathcal{B} : \int_B f_n dQ_\omega = \int_B f_{n,\omega} dQ_\omega.$$

As two functions agree almost everywhere if their integrals coincide over arbitrary events, we are done with the assertion of the lemma for the f_n .

We find an invariant set E_∞ with $\bar{P}(E_\infty) = Q(E_\infty) = 1$ such that

$$\bar{f}_\omega = \bar{f} \quad \text{in } L_1(Q_\omega)$$

for $\omega \in E_\infty$ by a completely analogous argumentation. ◇

Proof of lemma 7. Define

$$E^* := E_\infty \cap \left(\bigcap_{n \geq 1} E_n \right) \tag{28}$$

with E_∞ and the E_n from lemma 10. E^* is invariant and $Q(E^*) = 1$ as it applies to all sets on the right hand side of (28). We obtain

$$\forall n \in \mathbb{N} \quad f_n = f_{n,\omega} \quad \text{and} \quad \bar{f} = \bar{f}_\omega \quad \text{in } L_1(Q_\omega)$$

for $\omega \in E^*$. Therefore also

$$\liminf_{n \rightarrow \infty} f_n = \liminf_{n \rightarrow \infty} f_{n,\omega} \quad \text{in } L_1(Q_\omega)$$

for $\omega \in E^*$. ◇

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