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ON RECTANGULAR COVERING PROBLEMS*

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Many applications like picture processing, data compression or pattern recognition require a covering of a set of points most often located in the (discrete) plane by rectangles due to specific cost constraints. In this paper we provide exact dynamic programming algorithms for covering point sets by regular rectangles, that have to obey certain (parameterized) boundary conditions. The concrete representative out of a class of objective functions that is studied is to minimize sum of area, circumference and number of patches used. This objective function may be motivated by requirements of numerically solving PDE's by discretization over (adaptive multi-)grids.

More precisely, we propose exact deterministic algorithms for such problems based on a (set theoretic) dynamic programming approach yielding a time bound of $O(n^2 3^n)$. In a second step this bound is (asymptotically) decreased to $O(n^6 2^n)$ by exploiting the underlying rectangular and lattice structures. Finally, a generalization of the problem and its solution methods is discussed for the case of arbitrary (finite) space dimension.

Keywords: rectangular set cover; optimization problem; dynamic programming; closure operator; exact algorithm.

1. Introduction

We investigate a class of problems concerning covering grid points in the Euclidean plane by regular rectangles such that the overall area, the total circumference and the number of the rectangles used is minimized. Rectangular covering problems may arise for example in numerical analysis for solving partial differential equations by iterative multigrid methods.[2,14] For that purpose the equations are discretized and computed on grids. According to the values of error estimation functions it has to be decided iteratively whether the lattice has to be refined in certain regions meaning to modify the lattice spacing accordingly. Refinement steps may require a covering of indicated regions by e.g. regular, i.e., axis parallel rectangles optimized subjected

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to some reasonable constraints. Such computations can efficiently be performed only by parallel machines, where the communication amount of cooperating processors assigned to different lattice regions should be minimized. The specific choice of objective function as stated above aims at taking into account requirements of such parallel environments. Other applications for rectangular covering problems may be picture processing and data compression.[15,17]

Besides abstract set theoretic or graph theoretic covering problems,[1,10] there are numerous variants of related geometric covering or clustering problems most of them concerning points distributed in the Euclidean plane.[3,5] Many of them, as far as dealing with arbitrary many covering components are NP-hard optimization problems.[8,17] On the other hand, there are also certain partition or tiling problems, which could be related to partition variants of the problem at hand.[9,7,4,16]

In this paper we investigate the computational aspects of rectangular covering optimization problems from an abstract point of view. Namely, for the specific class of problems posing 1-sided boundary-constraints to rectangles, we provide exact deterministic algorithms for finding optimal rectangular coverings. In a first step, a (set theoretic) dynamic programming approach yielding a time bound of $O(n^2 3^n)$ is discussed. And in a second step this bound is (asymptotically) decreased to $O(n^6 2^n)$ by exploiting the underlying rectangular and lattice structures.

The rest of the paper is structured as follows. Section 2 is devoted to fix the notation used throughout and to explain basic notions. In Section 3 we consider the class of 1-sided problems and provide a procedure for computing an admissible rectangle that optimally covers a given point set. In Section 4 the 1-sided problem is solved by dynamic programming providing an exponential time bound. In Section 5 we investigate some structural features helping to improve the time bound which will be done in Section 6. Section 7 then presents a generalization of the concepts to the d-dimensional case, and, finally, in Section 8 some conclusions and open questions are stated.

2. Preliminary Notions and Notation

For $d \in \mathbb{N}$ fixed, let $\mathbb{E}^d$ be the d-dimensional Euclidean space which is the $\mathbb{R}^d$ equipped with the (orthogonal) standard basis $\boldsymbol{e}_i \in \mathbb{R}^d$ $(i = 1, \ldots, d)$ and the standard scalar product inducing the norm topology. Let us first consider the plane case $d = 2$ (the general case is treated in Section 7). Instead of $1, 2$ in this case coordinate(function)s are referred to as x, y. We sometimes also write $x(z)$, respectively, $y(z)$ for the coordinate values of a given $z \in \mathbb{E}^2$. Let $L = \mathbb{Z}\boldsymbol{e}_x\lambda + \mathbb{Z}\boldsymbol{e}_y\lambda$ be an *isothetical*, i.e., axis-parallel integer lattice (grid) embedded in $\mathbb{E}^2$ with lattice constant $\lambda \in \mathbb{R}_+$. (It may be convenient to set $\lambda = 1$ but in view of applications as mentioned above a lattice spacing parameter may be useful.) Recall that the (linear) lexicographic order on L is defined by $z_1 \leq_\ell z_2$ if either $x(z_1) < x(z_2)$ or $x(z_1) = x(z_2)$ and $y(z_1) \leq y(z_2)$. By translational invariance, w.l.o.g. it is sufficient to consider a bounded region in the first quadrant of the plane: $B := [0, N_x\lambda] \times [0, N_y\lambda] \subset \mathbb{E}^2$,

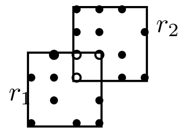

Fig. 1. Two rectangles r_1, r_2 whose intersection contains points (white dots) of the input set M (all dots); grid lines are omitted.

for $N_x, N_y \in \mathbb{N}$. For $n \in \mathbb{N}$, we write $[n] := \{1, \ldots, n\}$. Let $I := B \cap L$ denote the lattice part in the region of interest. For a set M we denote the collection of its r-subsets by $\binom{M}{r}$, its power set by 2^M, and let $\mathcal{B}_r(M) := \{S \subseteq M : |S| \leq r\}$.

Throughout we require that rectangles used for covering are placed isothetically in the plane which we call a *regular rectangle.* A regular rectangle $r = [x_d, x_u] \times [y_d, y_u] \subset \mathbb{E}^2$ is uniquely determined by its upper right $z_u(r) := (x_u(r), y_u(r)) \in \mathbb{E}^2$ and lower left $z_d(r) := (x_d(r), y_d(r)) \in \mathbb{E}^2$ diagonal points, which not necessarily coincide with grid points. r is considered as closed set in the norm topology; specifically meaning that points lying on the boundary of r are contained in r and thus are covered by r. Let $\mathcal{R}$ denote the set of all regular rectangles $r \subset B$ which can be placed in B; each represented by $(z_d(r), z_u(r)) \in B^2$. By $\ell_x(r)$, respectively, $\ell_y(r)$) the length of the x-parallel, respectively, y-parallel side of r is denoted. Moreover, let $a(r)$ be its area, $u(r)$ its circumference, and let ∂r be the boundary of r consisting of its four sides $\partial_i r$, $i \in \{1, \ldots, 4\}$.

Definition 1. An *objective function on rectangles* is a partial map $w : \mathcal{R} \to \mathbb{R}_+$ (the domain $D(w)$ will be made explicit by concrete problems), whose values $w(r)$ are assumed to be computable in constant time. Given w, a rectangle r is called *admissible* if $r \in D(w)$. To an objective function w assign the following $\mathbb{R}_+$-valued extension to sets defined by $w'(R) := \sum_{r \in R} w(r)$, for every $R \subset D(w)$. (Since the meaning should become clear from the context we also symbolize the set extension by w.) An objective function on rectangles is called *monotone* if it satisfies: $r \subseteq r' \Rightarrow w(r) \leq w(r')$, for all $r, r' \in D(w)$.

The monotonicity condition simply reflects the reasonable requirement that the costs contributed by a rectangle should not be decreased by a smaller rectangle.

Next we mention several basic rectangular covering problems differing with regard to their input parameters and their objective functions. Each of these problems searches for a certain subset $R \subset \mathcal{R}$ serving as a covering of a finite input set $M \subset I$ of lattice points meaning $M \subseteq \bigcup_{r \in R} r \cap I$. The rectangles of such a covering are permitted to overlap in any way, in contrast to the rules for tiling problems. It also may happen that $(r \cap r') \cap M \neq \emptyset$, i.e., there are points in M which are multiply covered, namely by r and r' (cf. Fig. 1). Such situations distinguish covering problems from rectangular partition problems allowing overlapping rectangles only in case of empty intersection with the input set M.

Definition 2. For a fixed real lattice constant $\lambda > 0$, let a point set $M = \{z_1, \ldots, z_n\} \subset I$ ($n \in \mathbb{N}$) and $t, t' \in \mathbb{R}_+, t' \geq t > 0$ be given.
(1) The *(fixed type) rectangular covering problem* $\mathrm{RC}^\lambda_{\mathrm{fix}}$ is the following search problem: Find a set R of isothetical rectangles each having two parallel sides of length t resp. t' such that $M \subset \bigcup_{r \in R} r \cap I$ and $|R|$ is minimized.
(2) The *(2-sided) rectangular covering problem* $\mathrm{RC}^\lambda(2)$ is the following search problem: Find a set R of isothetical rectangles whose side lengths lie in the closed interval $[k, k']$ such that $M \subset \bigcup_{r \in R} r \cap I$ and $|R|$ is minimized.
(3) Let w be an objective function. The *(2-sided) rectangular covering problem w.r.t. w* $\mathrm{RC}^\lambda_w(2)$ is the problem $\mathrm{RC}^\lambda(2)$ where (instead of $|R|$) $w(R)$ has to be minimized over all admissible coverings R of M.

Theorem 1. *All above defined rectangular covering problems are NP-hard.*[11] □

The following problem fixing only the left interval boundary for side lengths is a specialization of $\mathrm{RC}^\lambda_w(2)$:

Definition 3. Let w be an objective function for rectangles. The *(1-sided) rectangular covering problem w.r.t. w* $\mathrm{RC}^\lambda_w(1)$ is the following search problem: For a fixed real lattice constant $\lambda > 0$, let a point set $M = \{z_1, \ldots, z_n\} \subset I$ ($n \in \mathbb{N}$) and $k \in \mathbb{R}_+, k > 0$ be given. Find a set R of isothetical rectangles each having sides of length at least k such that $M \subset \bigcup_{r \in R} r \cap I$ and $w(R)$ is minimized.

Let $\mathrm{RC}^\lambda_{w_c}(1)$ be problem $\mathrm{RC}^\lambda_w(1)$ for the specific objective function w_c defined by $w_c(r) := a(r) + u(r) + c$, where $c > 0$ is a fixed constant and $r \in \mathcal{R}$ is an admissible rectangle.

We specifically have that w_c is monotone: Let $r' \subseteq r$ and assume that $(*)$: $w(r) < w(r')$ holds. Setting $\ell_j(r') = j_u(r') - j_d(r') \geq 0$ ($j \in \{x, y\}$) we obtain by some elementary arithmetics:

$$
\begin{aligned}
w(r) - w(r') &= a(r) - a(r') + u(r) - u(r') \\
&= (\ell_x(r) - \ell_x(r')) \left[\frac{1}{2}(\ell_y(r) + \ell_y(r')) + 2\right] \\
&\quad + (\ell_y(r) - \ell_y(r')) \left[\frac{1}{2}(\ell_x(r) + \ell_x(r')) + 2\right]
\end{aligned}
$$

Since all terms in the rectangular braces are strictly positive, relation $(*)$ implies that there is $j \in \{x, y\}$ such that $\ell_j(r) - \ell_j(r') < 0$ yielding a contradiction.

However, it is an open question whether $\mathrm{RC}^\lambda_{w_c}(1)$ is NP-hard. There is a closely related problem stemming from the application of data compression and mentioned to be NP-complete in:[6] For $n, m \in \mathbb{N}$, let $M \in \mathrm{GF}_2^{n \times m}$ be a matrix of binary entries. The associated search problem asks for a minimum cardinality set of rectangles *exactly* covering the 1-entries of M, which in a certain sense is related to the area and circumference constraints in the 1-sided rectangular covering problem with objective w_c. So we conjecture that $\mathrm{RC}^\lambda_{w_c}(1)$ also is NP-hard. That however cannot hold for $\mathrm{RC}_w^\lambda(1)$ for all w. E.g., if w is constant, obviously one admissible rectangle covering the input set is an optimal solution.

The rest of this paper is devoted to construct a deterministic algorithm for $\mathrm{RC}_{w_c}^\lambda(1)$ with exponential time bound, posing some additional boundary contraint as explained in the next section.

3. The 1-Sided Case and Optimal 1-Covers

Again let $M \subset I$ be an input set of lattice points. For given minimal side length parameter $k \in \mathbb{R}_+$ with $0 < k < N\lambda$ ($N := \min\{N_x, N_y\}$), call $r \in \mathcal{R}$ a *k-admissible* if $\ell_j(r) \geq k$, for $j \in \{x, y\}$. For some applications it may be required not to allow lattice points on the boundary of rectangles. This could be encountered by posing, in addition, a minimal distance condition for points covered by a rectangle: Given $\varepsilon \geq 0$ then a rectangle r is called *ε-admissible* if each point $z \in M \cap r$ has minimal euclidean distance ε to each part of the boundary ∂r of r. Finally, by a *(k,ε)-admissible rectangular covering of M* we mean a set $R \subset \mathcal{R}$ of regular rectangles which are (k,ε)-admissible such that $M \subset \bigcup_{r \in R} r \cap I$ and $\forall r \in R : r \cap M \neq \emptyset$. The totality of all (k,ε)-admissible rectangular coverings of M is denoted as $\mathcal{C}_{(k,\varepsilon)}(M) =: \mathcal{C}(M) \subseteq 2^{\mathcal{R}}$, where the indices (k,ε) are omitted since these values are fixed in a given problem class. For fixed $c \geq 1$ the objective function $w := w_c$ is given by $w : \mathcal{R} \ni r \mapsto w(r) =: a(r) + u(r) + c \in \mathbb{R}_+$. Extending to sets $R \subseteq \mathcal{R}$ we have $w(R) := a(R) + u(R) + c|R|$ where $a(R) := \sum_{r \in R} a(r)$, $u(R) := \sum_{r \in R} u(r)$. We will solve the following parameterized version of 1-sided problems:

Definition 4. For fixed $\lambda, k \in \mathbb{R}_+$ *k-RECTANGULAR COVER (k-RC)* is the following optimization problem: Given $N_x, N_y, c, \varepsilon \geq 0$ such that $\varepsilon < \lambda/2$, $0 < k < N := \min\{N_x, N_y\}\lambda$ and a finite set $M \subset I$. Find a (k,ε)-admissible covering R_0 of M with $\mathrm{opt}(k, M) := \min\{w(R) : R \in \mathcal{C}(M)\} = w(R_0)$. Such a covering R_0 is called an *optimal covering.*

For fixed lattice spacing λ input instances consist of the point set and the boundary parameters k, ε for rectangles. According to the relation of the value of k to λ several problem classes arise as discussed now. For arbitrarily fixed k ($0 < k < N\lambda$, $N := \min(N_x, N_y)$) there is a largest $\nu(k) \in \mathbb{N}_0 : k = \nu(k)\lambda + \alpha(k), 0 \leq \alpha(k) < \lambda$, hence, $\nu(k) = \lfloor \frac{k}{\lambda} \rfloor, \alpha(k) = k - \nu(k)\lambda$. Thus, we have the following classes partitioning the given interval: $\nu(k) = 0 \;\Leftrightarrow\; 0 < k < \lambda$ is the first class and $\nu(k) = i \;\Leftrightarrow\; i\lambda \leq k < (i+1)\lambda$ corresponds to class $i+1$ for $i \in \{1, \ldots, N-1\}$.

We now address the task to determine the smallest (k,ε)-admissible rectangle containing a given subset $S \subset M$ of the input set $M \subset I$. To that end, consider the map

$$b : 2^M \ni S \mapsto b(S) := \{z_d(S), z_u(S)\} \in \binom{I}{2}$$

where $z_d(S) := (x_d(S), y_d(S))$ and $z_u(S) := (x_u(S), y_u(S))$. Here $x_d(S) := \min_{z \in S} x(z)$, $y_d(S) := \min_{z \in S} y(z)$ and $x_u(S) := \max_{z \in S} x(z)$, $y_u(S) := \max_{z \in S}$

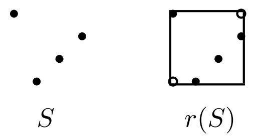

Fig. 2. Black dots represent points of S (left), white dots represent the base points $z_u(S), z_d(S) \in b(S)$ of the rectangle $r(S)$ enclosing S (right); grid lines are omitted.

$y(z)$ are the extremal coordinates of points in S. Hence, $b(S)$ in general contains no points of S or even M, but in any case lattice points.

Definition 5. The unique set $r(S) := [x_d(S), x_u(S)] \times [y_d(S), y_u(S)]$, with $r(\varnothing) := \varnothing$ is called the *rectangular base of* S. The extremal points (which coincide for a single element set) $z_d(S), z_u(S) \in b(S)$ are called the *(rectangular) base points* of S.

Clearly, we may identify the objects $r(S)$ tightly enclosing S and $b(S)$ (cf. Fig. 2). Obviously this construction violates the properness condition in the case that the points of S lie all on the same grid line, since then $r(S)$ corresponds to a line segment.

Lemma 1. *$r(S)$ is, w.r.t. to inclusion and also w.r.t. the objective function w, the smallest rectangular object containing $S \subseteq M$.*

Proof. The case $S = \varnothing$ is trivial, so assume $S \neq \varnothing$. We claim that for each rectangle $r \in \mathcal{R}$ with $S \subset r$ holds $r(S) \subseteq r$. Suppose the contrary, then there is $z' = (z'_x, z'_y) \in B \setminus S$ with $z' \in r(S)$ but $z' \notin r$. Thus there is $j \in \{x, y\}$ such that $z'_j < \min_{z \in r} z_j$ or $z'_j > \max_{z \in r} z_j$. But since $S \subset r$ we have $\min_{z \in r} z_j \leq \min_{z \in S} z_j$ and $\max_{z \in r} z_j \geq \max_{z \in S} z_j, \forall j \in \{x, y\}$. It follows that $z' \notin r(S)$ yielding a contradiction. So we are done by monotonicity of w. $\square$

Of course, for arbitrary k, ε in general $r(S)$ is not (k, ε)-admissible. Thus we need a procedure transforming $r(S)$ into a (w.r.t. w smallest) (k, ε)-admissible rectangle containing $r(S)$. For convenience, we define $\overline{x} := y$ and $\overline{y} := x$, and use a data structure **point** storing the components $x(z), y(z)$ of each $z \in M$. M is assumed to be represented as a one-dimensional **array** of length $n = |M|$ containing objects of type **point**. Furthermore, suppose that M_j ($j \in \{x, y\}$) is an array in which the elements of M are sorted by *lexicographic j-order* $\leq_{\ell_j}$, that is $\forall z_1, z_2 \in M$: $z_1 \leq_{\ell_j} z_2 \;\Leftrightarrow:\; j(z_1) \leq j(z_2)$ and if $j(z_1) = j(z_2)$ then $\overline{j}(z_1) \leq \overline{j}(z_2), j = x, y$, i.e., $\leq_\ell = \leq_{\ell_x}$, as earlier defined; $\leq_{\ell_y}$ is also called *anti-lexicographic order*. Hence, M_j is sorted by increasing j-coordinate values of its points. By construction of $r(S)$ holds $\ell_x(r(S)), \ell_y(r(S)) \in \mathbb{N}_0$ and $S \cap \partial_i r(S) \neq \emptyset, \forall i = 1, \ldots, 4$. We can consider the x, y-parallel sides of r independently, so for each of them two cases have to be distinguished:

a): $\ell_j(r(S)) \geq k, j \in \{x, y\}$, then we only have to enlarge each side of $r(S)$ at both ends by ε simultaneously, obviously resulting in a unique smallest (k, ε)-

admissible rectangle containing S.

b): There is $j \in \{x,y\} : \ell_j(r(S)) < k$. If $\gamma_j/\lambda := \nu(k) - \ell_j(r(S))/\lambda \in \mathbb{N}_0$, is odd then by extending $\ell_j(r(S))$ at both ends symmetrically by value $[\gamma_j + \alpha(k)]/2$ we simultaneously satisfy both the k- and ε-conditions, because by definition $\varepsilon < \lambda/2$. If γ_j/λ is even, we have to analyse the following subcases: (i): $\alpha(k)/2 \geq \varepsilon$, then symmetrically extending $\ell_j(r(S))$ at both ends about $[k - \ell_j(r(S))]/2$, we also satisfy the ε-condition; the rectangle achieved in this way may contain a larger set $M \supset S' \supset S$. (ii): $\alpha(k)/2 < \varepsilon$. In case we can find a value j in the interval $I_j^- := [j_d(S) - \gamma_j, j_d(S)]$ such that there is no intersection with M on the corresponding grid line parallel $\bar{j}$, then choosing this as the new boundary part we satisfy the ε-condition at that side. Similarly, we proceed for the interval $I_j^+ := [j_u(S), j_u(S) + \gamma_j]$. These tests can be executed in linear time as follows: Compute all j-values corresponding to points in M falling in the range I_j^- and store them in array S_j^-. Do the same for I_j^+ storing the values in S_j^+. For each $j \in S_j^-$, compute its *counterpart* $j + \gamma_j \in I_j^+$; storing them by increasing values in C_j^+. Then merge arrays S_j^+, C_j^+ resulting to array B_j^+. Finally check if there is a gap inside, i.e., whether there is a value in I_j^+ not contained in B_j^+. In the positive case obviously we can choose two new boundary parts parallel $\bar{j}$, s.t. at both ends the ε-conditions are satisfied. Notice that all these computations including the merge step can be done in $O(|S|)$ time because both arrays are assumed to be sorted by increasing values.

If there is no such choice we have to check if there can be chosen at least one boundary part not intersecting M. This can be done by first checking, whether there is $j \in S_j^-$ having no counterpart in S_j^+, and if the search yields no success then perform an analogous check for S_j^+. This simply can be done by two assistant arrays R_j^-, R_j^+ of length $|I_j^-|, |I_j^+|$ having coordinate values as indices. If that, finally, also is impossible, then we have to enlarge both sides of $r(S)$ about $\gamma_j/2 + \max\{\varepsilon, \alpha(k)/2\}$ to obtain (k,ε)-admissibility. The above discussion leads to the following algorithm:

`Algorithm (OPT1)`
`Input: rectangular base` $r(S) := [x_d(S), x_u(S)] \times [y_d(S), y_u(S)], S \subset M$
`Output: w.r.t.` w `smallest` (k,ε)`-admissible rectangle` $r_\delta(S)$ `containing` $r(S)$
`begin`
`for` $j = x, y$ `do`
  $\delta_j^d \leftarrow \delta_j^u \leftarrow 0, \gamma_j \leftarrow \nu(k)\lambda - \ell_j(S)$
  `if` $\ell_j(r(S)) \geq k$ `then` $\delta_j^d \leftarrow \delta_j^u \leftarrow \varepsilon$
  `else if` $\ell_j(r(S)) < k \wedge \alpha(k)/2 \geq \varepsilon$ `then` $\delta_j^d \leftarrow \delta_j^u \leftarrow [k - \ell_j(r(S))]/2$
  `else` $(*\ell_j(r(S)) < k \wedge \alpha(k)/2 < \varepsilon*)$
    `if` $\gamma_j/\lambda \mod 2 = 1$ `then` $\delta_j^d \leftarrow \delta_j^u \leftarrow [\gamma_j + \alpha_j(k)]/2$
    `else` $(*\gamma_j/2$ `is even: compute arrays` $S_j^-, R_j^-, C_j^+ *$`:)`
      `for all` $z_i \in M_j : j_d(S) - \gamma_j \leq j(z_i) \leq j_d(S)$ `do`
        `if` $\bar{j}_d(S) \leq \bar{j}(z_i) \leq \bar{j}_u(S)$ `then`
          $S_j^-[i] \leftarrow j(z_i), R_j^-[j(z_i)] \leftarrow j(z_i), C_j^+[i] \leftarrow j(z_i) + \gamma_j$
      `od` $(*$ `compute arrays` $S_j^+, R_j^+ *$`:)`

```
for all z_i ∈ M_j : j_u(S) ≤ j(z_i) ≤ j_u(S) + γ_j do
  if j̄_d(S) ≤ j̄(z_i) ≤ j̄_u(S) then S_j^+[i] ← j(z_i), R_j^+[j(z_i)] ← j(z_i)
end do
(* merge S_j^+, C_j^+ into B_j^+ *)
for all j_i ∈ B_j^+ do (* check if there are two sides not intersecting M*)
  if j_{i-1} ∉ B_j^+ then
    δ_j^d ← j_d(S) − j_{i-1} − γ_j + α(k)/2, δ_j^u ← j_{i-1} + α(k)/2 − j_u(S)
    break
  if j_{i+1} ∉ B_j^+ then
    δ_j^d ← j_d(S) − j_{i+1} − γ_j + α(k)/2, δ_j^u ← j_{i+1} + α(k)/2 − j_u(S)
    break
end do
if δ_j^d = δ_j^u = 0 then (* check if there is one side not intersecting M*)
  for all j ∈ S_j^- do (* check for S_j^- *:)
    if R_j^+[j + γ_j] = nil then
      δ_j^d ← x_d(S) − j + max{ε, α(k)}, δ_j^u ← j + γ_j − x_u(S)
      break
  end do
  if δ_j^d = δ_j^u = 0 then
    for all j ∈ S_j^+ do (* check for S_j^+ *:)
      if R_j^-[j − γ_j] = nil then
        δ_j^d ← x_d(S) − j + γ_j, δ_j^u ← j + max{ε, α(k)} − x_u(S)
        break
    end do
if δ_j^d = δ_j^u = 0 then (* both sides intersect M*)
  δ_j^d ← δ_j^u ← γ_j/2 + max{ε, α(k)/2}
end do
r_δ(S) ← [x_d(S) − δ_x^d, x_u(S) + δ_x^u] × [y_d(S) − δ_y^d, y_u(S) + δ_y^u]
end
```

Summarizing the argumentation above, we have:

Lemma 2. *Let (M, k) be an instance of k-RC, that is sorted lexicographically. For each $S \in 2^M \setminus \{\emptyset\}$, let $r(S)$ be the rectangular base corresponding to $\rho(S)$ as defined in Lemma 1. Then Algorithm OPT1 correctly computes $r_\delta(S) \in \mathcal{R}(k, M)$ which, w.r.t. w, is a smallest (k, ε)-admissible rectangle containing S, i.e., $\mathrm{opt}_1(k, S) := \min\{w(r); r \supseteq S, r \in \mathcal{R}(k, M)\} = w(r_\delta(S))$. In general, $r_\delta(S)$ is not unique. Algorithm OPT1 runs in $O(|S|)$ time.* □

4. A Dynamic Programming Approach

A first reasonable time bound for k-RC can be achieved using dynamic programming on a pure set theoretical base. The idea is to construct under the constraints minimal rectangular coverings systematically for all subsets of a subset of M and for all

reasonable cardinalities $|R| \in \{1, 2, \dots, n\}$ of coverings. For a fixed subset $\emptyset \neq S \subseteq M, |M| := n \in \mathbb{N}$, let $\mathrm{opt}_j(k, S) := \min\{w(R); R \in \mathcal{C}(S), |R| = j\}$. Suppose we can effectively compute $\mathrm{opt}_j(k, M), 2 \leq j \leq n$. Then $\mathrm{opt}(k, M) = \min_{j\in[n]} \mathrm{opt}_j(k, M)$ is the optimal value for k-RC with regard to the objective function.
Now suppose, for each subset $S \in 2^M \setminus \{\emptyset\}$, we have computed $\mathrm{opt}_1(k, S) = w(r_\delta(S))$ as well as $\mathrm{opt}_j(k, S), 2 \leq j \leq i - 1$. As induction step on that basis, for each fixed $\emptyset \neq S \in 2^M$, holds $\mathrm{opt}_j(k, S) = \min\{w(r_\delta(T)) + \mathrm{opt}_{j-1}(T'); \emptyset \neq T \in 2^S\}$, $\forall \emptyset \neq S \in 2^M, T' := S \setminus T$ forming the Bellman optimality equations in our context. Of course, we touched too many subsets, needed are only those, having sufficient cardinality: $\mathcal{S}_j := \{S \in 2^M; |S| \geq j\}$, and given $S \in \mathcal{S}_j$, for computing $\mathrm{opt}_j(k, S)$ it is sufficient to consider each element of $\mathcal{T}_j(S) := \{T \in 2^S \setminus \{\emptyset\}; |T| \leq |S| - (j - 1)\}$.
Before precisely stating the procedure, we explain the data structures used: Rectangles will be represented by their diagonal points in a data type **rectangle** storing objects of type **point**. Thinking of M as a sorted alphabet, each subset $S \subset M$ corresponds to a unique word over M, denoted $word(S)$ or S for short, thus 2^M may be sorted by the corresponding lexicographic order. For each S, there can be determined an unique index $ind(S)$ according to this order. A datatype **subset** is used for storing a rectangle and an integer. Then in a preprocessing step for each $S \subseteq M$ there can be defined **subset** A_S holding $ind(S)$ and also $r_\delta(S)$ such that it is possible to read each of them in constant time. We make use of two further container arrays $Opt_i, Rect_i$ for $i = 0, 1$, each sorted by increasing $ind(S)$. Two of each kind are needed, because during the algorithm they may be read and filled up alternately. The arrays $Opt_i, i = 0, 1$, shall store the intermediately computed $\mathrm{opt}_j(k, S)$-values. The other two arrays $Rect_i$ of dynamic length have the task to hold at each index $ind(S)$ a set $R_0^{(j)}(S)$ for storing the intermediately computed rectangles covering S. These arrays are also (re-)used alternately. By the common order of these arrays the task of determining for a given set $T \subset M$ its array position is solved in $O(1)$ by referring to $A\_S.ind = ind(S)$. Finally, we make use of two arrays $Subs_i, i = 0, 1$, of dynamic length. The first one shall store $word(T)$ and the second $word(T')$ for each subset T of the current $S \subset M$. These arrays may be sorted by lexicographic order.

`Algorithm (`k`-RC)`
`Input: set of points in the plane` M `as array of points array of values` $r_\delta(S), w(r_\delta(S))$`, for all` $S \in 2^M$ `computed by algorithm OPT1`
`Output: optimal covering value` $\mathrm{opt}(k, M)$`, optimal covering` $R_0(M)$
`begin`
`if` $n = |r_\delta(M) \cap I|$ `then` $\mathrm{opt}(k, M) \leftarrow w(r_\delta(M)),\ R_0(M) \leftarrow \{r_\delta(M)\}$
`else`
  $\mathrm{opt}(k, M) \leftarrow \infty,\ R_0(M) \leftarrow \emptyset$
  `sort` $2^M \setminus \{\emptyset\}$ `by lexicographic order, thereby:`
  $\forall S \in 2^M \setminus \{\emptyset\}$ `: compute` $r_\delta(S), ind(S)$ `and fill` A_S
  $\forall S \in 2^M \setminus \{\emptyset\} : Opt_0[ind(S)] \leftarrow w(r_\delta(S)),\ Rect_0[ind(S)] \leftarrow \{r_\delta(S)\}$

```
  opt(k,M) ← Opt_0[ind(M)], Rect_0[ind(M)] ← {r_δ(M)}
  if n ≥ 3 then
    for j = 2 to n − 1 do
      for all S ∈ S_j := {S ∈ 2^M \ {∅}; j ≤ |S|} do
        sort 2^S \ {∅} by lexicographic order, thereby:
        ∀T ∈ 2^S \ {∅} : Subs_0[ind(T)] ← word(T), Subs_1[ind(T)] ← word(T')
        Opt_{(j−1) mod 2}[ind(S)] ← ∞ (*opt_j(k,S) ← ∞*)
        for all T ∈ T_j(S) := {T ∈ 2^S \ {∅}; 1 − j + |S| ≥ |T|}*) do
          temp = w(r_δ(T)) + Opt_{j mod 2}[ind(T')] (*opt_{j−1}(k,T')*)
          if temp < Opt_{(j−1) mod 2}[ind(S)] then
            Opt_{(j−1) mod 2}[ind(S)] ← temp
            Rect_{(j−1) mod 2}[ind(S)] ← {r_δ(T)} ∪ Rect_{j mod 2}[ind(T')]
        end do (* now: Opt_{(j−1) mod 2}[ind(S)] = opt_j(k,S)*)
      end do
      if Opt_{(j−1) mod 2}[ind(M)] < opt(k,M) then
        opt(k,M) ← Opt_{(j−1) mod 2}[ind(M)], R_0(M) ← Rect_{(j−1) mod 2}[ind(M)]
    end do
  Opt_{(n−1) mod 2}[ind(M)] ← ∞, Rect_{(n−1) mod 2}[ind(M)] ← ∅
  for all T ⊂ M : |T| = 1 do
    temp = w(r_δ(T)) + Opt_{n mod 2}(ind(T')) (*opt_{n−1}(k,T')*)
    if temp < Opt_{(n−1) mod 2}[ind(M)] then
      Opt_{(n−1) mod 2}[ind(M)] ← temp
      Rect_{(n−1) mod 2}[ind(M)] ← {r_δ(T)} ∪ Rect_{n mod 2}[ind(T')]
  end do (* now: opt_n(k,M) = min{w(r_δ(T)) + opt_{n−1}(k,T'); T ∈ T_n(M)}*)
  if Opt_{(n−1) mod 2}[ind(M)] < opt(k,M) then
    opt(k,M) ← Opt_{(n−1) mod 2}[ind(M)], R_0(M) ← Rect_{(n−1) mod 2}[ind(M)]
(* now: opt(k,M) = min{opt_n(k,M); n ∈ M}*)
end
```

Theorem 2. *For input (M,k) with $n := |M|$, Algorithm k-RC correctly computes $\mathrm{opt}(k,M) = \min_{i\in[n]} \mathrm{opt}_i(k,M)$ and $R_0 \in \mathcal{C}(M)$ such that $\mathrm{opt}(k,M) = w(R_0)$ in $O(n^2 3^n)$ time.*

Proof. Let $n := |M|$. For proving correctness we first show that $\mathrm{opt}(k,M) = \min_{i\in[n]} \mathrm{opt}_i(k,M)$ holds true, where $(*)$: $\mathrm{opt}_i(k,M) := \min\{w(R); R \in \mathcal{C}_i(M)\}$, $i \in [n]$ and $\mathcal{C}_i(M) := \{R \in \mathcal{C}(M); |R| = i\}$. In the second step it is verified that the dynamic program correctly computes $(*)$ by induction on $n \in \mathbb{N}$. Clearly, as disjoint union $\mathcal{C}(M) = \bigcup_{i\in\mathbb{N}} \mathcal{C}_i(M)$ ($|R| = 0$ is impossible). Obviously, we never need more covering components than there are elements in M, thus

$$\mathrm{opt}(k,M) = \min \bigcup_{i=1}^{n} \{w(R); R \in \mathcal{C}_i(M)\} = \min_{i\in[n]} \mathrm{opt}_i(k,M)$$

Next, we have to show that the dynamic program will reproduce $\mathrm{opt}_i(k,M)$, $1 \leq i \leq n \in \mathbb{N}$, as in $(*)$. First of all, by directly applying the above argumentation it

is clear that we never have to take into consideration more covering components than there are elements in a set $S \subseteq M$ for finding $\mathrm{opt}(k,S)$. Which means, for $|S| = j$ it suffices to compute $\mathrm{opt}_l(k,S), l \leq j$. In other words $\mathrm{opt}_j(k,S)$ has to be computed only for all elements of $\mathcal{S}_j := \{S \in 2^M \setminus \{\emptyset\}; j \leq |S|\}$. In the same way it is clear that in the most inner loop of the algorithm it suffices to consider only $\mathcal{T}_j(S) := \{T \in 2^S \setminus \{\emptyset\}; 1 - j + |S| \geq |T|\}$, as then $|T'| = |S \setminus T| \geq j-1, |S| \geq j$.
Let us proceed by induction on n. By Lemma 2, $\mathrm{opt}_1(k,S) = w(r_\delta(S))$ for each $\emptyset \neq S \subseteq M$, especially $\mathrm{opt}_1(k,M) = w(r_\delta(M)), \forall |M| \in \mathbb{N}$, which is also the basis of the dynamic program. Now suppose that for each $|M| \leq n$ holds

$$\mathrm{opt}_i(k,S) = \min\{w(r_\delta(T)) + \mathrm{opt}_{i-1}(k,T'); T \in \mathcal{T}_i(S)\}$$

$\forall S \subseteq M, |S| \geq i, \forall i \in \{2,\dots,n\}$. Consider the case $|M| = n+1$. Then for $S \subsetneqq M$ we are ready by induction, because $|S| \leq n$. Moreover, by definition $\mathrm{opt}_i(k,M) = \{w(R); R \in \mathcal{C}_i(M)\}$, for each $i \in \{2,\dots,|M|\}$. Thus, for each $S \subsetneqq M, |S| \geq j$ and $|M| - |S| \geq i - j$, we have $R_1 \cup R_2 \in \mathcal{C}_i(M)$ whenever $R_1 \in \mathcal{C}_j(S), R_2 \in \mathcal{C}_{i-j}(S')$, hence

$$\begin{aligned}
\mathrm{opt}_i(k,M) &= \min_{S \subset M : i + |S| - |M| \leq j \leq |S|} \{w(R_1) + w(R_2); R_1 \in \mathcal{C}_j(S), R_2 \in \mathcal{C}_{i-j}(S')\} \\
&= \min_{S \subset M : i + |S| - |M| \leq j \leq |S|} \left(\min_{R \in \mathcal{C}_j(S)} w(R) + \min_{R \in \mathcal{C}_{i-j}(S')} w(R) \right) \\
&= \min_{S \subset M : i + |S| - |M| \leq j \leq |S|} (\mathrm{opt}_j(k,S) + \mathrm{opt}_{i-j}(k,S'))
\end{aligned}$$

Now we state the following simple but helpful claim: For $S \subsetneqq M$ and l with $i + |S| - |M| \leq l \leq |S|$ there is a $T \subset S$ such that

$$\mathrm{opt}_i(k,M) \leq w(r_\delta(T)) + \mathrm{opt}_{i-1}(k,T') \leq \mathrm{opt}_l(k,S) + \mathrm{opt}_{i-l}(S')$$

From this directly follows $\mathrm{opt}_i(k,M) = \{w(r_\delta(T)) + \mathrm{opt}_{i-1}(k,T'); \emptyset \neq T \subseteq M, |T| \leq n - i + 1\}$, which is what has been stated.

Finally, for proving the claim take an arbitrary fixed $S \subsetneqq M$, then the case $l = 1$ is clear by setting $T = S$. For $|S| \geq l > 1$ by induction there is a $T_0 \subset S : |T_0| \leq |S| - l + 1$ with $\mathrm{opt}_l(k,S) = w(r_\delta(T_0)) + \mathrm{opt}_{l-1}(k, S \setminus T_0)$ hence

$$\mathrm{opt}_l(k,S) + \mathrm{opt}_{i-l}(k,S') = w(r_\delta(T_0)) + \underbrace{\mathrm{opt}_{l-1}(k, S \setminus T_0) + \mathrm{opt}_{i-l}(k,S')}_{\text{covering } S' \cup (S \setminus T_0) = M \setminus T_0}$$

thus $\mathrm{opt}_{l-1}(k, S \setminus T_0) + \mathrm{opt}_{i-l}(k,S') \geq \mathrm{opt}_{i-1}(k, M \setminus T_0)$. Therefore the choice T_0 establishes the claim.
Addressing the running time only the else-part is of interest. First of all there is the preprocessing step consisting of sorting $2^M \setminus \{\emptyset\}$ thereby computing $r_\delta(S), ind(S)$. For fixed S computing $r(S)$ needs $O(|S|)$ time and from this according to Lemma 2 we can compute $r_\delta(S)$ also in $O(|S|)$ time which also holds for computing $ind(S)$, hence this step delivers an additive term of $O(n2^n)$. Next, there is the dominating part consisting of two nested loops. The inner loop considers all subsets $S \subset M$ such

that $|S| = p \geq j$; for each p we have $\binom{n}{p}$ such sets. For each of which there is computed the body containing two further loops. In the first for each $S, |S| = p \geq j$, the set 2^S is constructed, and for each fixed $T \subset S$ corresponding $word(T)$ and $word(T')$ are computed needing $O(|S|)$ time thus yielding $O(p2^p)$. In the second loop each relevant $T \subseteq S$ is considered where all operations are of $O(1)$. Therefore and because of $\sum_{k=1}^{p-j+1} \binom{p}{k} \leq 2^p \leq p2^p$ it also contributes $O(p2^p)$. Hence, by the binomial theorem, we get for the inner loop $O(\sum_{p=j}^{n} \binom{n}{p} p2^p \leq n3^n)$. Finally, the outer loop is iterated less than n times leading altogether to $O(n^2 3^n)$ also dominating the bound of the preprocessing step. The last step contains a loop of $O(n)$ iterations during each of which $O(n)$ time is needed for computing $word(T')$ thus it contributes additively $O(n^2)$ also dominated by $O(n^2 3^n)$ completing the proof. $\square$

Proposition 1. *Algorithm k-RC has a space requirement of* $O(2^{|M|}|M| \log(\max\{N_x, N_y\}))$.

Proof. First there are $2^{|M|}$ **subset** types, for all $S \subset M$, each holding a rectangle consisting of four real numbers not greater than $O(\max\{N_x, N_y\})$ thus needing space of at most $O(\log(\max\{N_x, N_y\}))$. Further an integer is stored representing $ind(S) \leq 2^{|M|}$ therefore requiring $O(|M|)$ thus the term $O([|M| + \log(\max\{N_x, N_y\})]2^{|M|})$ is contributed additively. Next, consider the two arrays Opt_0, Opt_1 of length $O(2^{|M|})$ for storing the $\mathrm{opt}_i(k, S)$-values. In each component of $Opt_i, i = 0, 1$, we have to store a real number of worst case bound $O(w(r_\delta(M))|M|)$ with $w(r_\delta(M)) \in O(\max\{N_x, N_y\}^2 + 4\max\{N_x, N_y\} + |M|)$ and by noting $|M| \leq \max\{N_x, N_y\}^2$ we have to store numbers of length $O(\log(\max\{N_x, N_y\}))$ leading to a worst case space complexity of $O(\log(\max\{N_x, N_y\})2^{|M|})$. In a similar manner for $Rect_i, i = 0, 1$, holding in each component a set of rectangles consisting of maximally $4|M|$ numbers whose lengths are bounded by $O(\log(\max\{N_x, N_y\}))$ one is lead to $O(|M| \log(\max\{N_x, N_y\})2^{|M|})$ as space complexity contribution. The arrays $Subs$ of maximal length $2^{|M|}$ holding in each component a word of maximal length $|M|$ delivering space length $O(\log(\max\{N_x, N_y\}))$ thus contributes additively the term $O(2^{|M|} \log(\max\{N_x, N_y\}))$. By collecting all terms the claimed space bound is obtained. $\square$

5. Underlying Structural Features

This section is devoted to enumerate possible covering patches based on grid orders and to investigate the rectangular structure. Both features may be used to improve the time bound obtained in the previous section.

5.1. *Grid orderings*

There is a natural (partial) order $\leq_L$ on the lattice given by $z_1 \leq_L z_2 \Leftrightarrow x(z_1) \leq x(z_2) \wedge y(z_1) \leq y(z_2)$ $(\forall z_1, z_2 \in L)$. Recall that $\leq_L$ is not a linear order on the grid points as, for example, the points $z_1 = (2, 5)$ and $z_2 = (3, 4)$ are not comparable

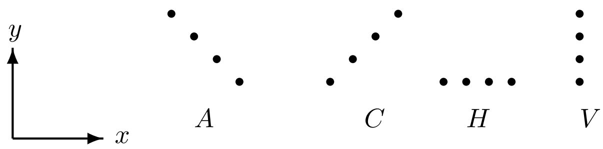

Fig. 3. An antichain A, a strict chain C, and two chains H, V (grid lines are omitted).

with respect to $\leq_L$, whereas $z_1 \leq_\ell z_2$ lexicographically holds true. We call a $\leq_\ell$-sequence $\{z_1, \ldots, z_n\} \subset L$ a *chain* if $z_i \leq_L z_{i+1}, 1 \leq i \leq n-1$, and call it *strict* if $\leq_L$ is replaced by $<_L$ at any position. A $\leq_\ell$-sequence $\{z_1, \ldots, z_n\} \subset L$ is called an *antichain* if $x(z_i) < x(z_{i+1})$ and $y(z_{i+1}) < y(z_i), 1 \leq i \leq n-1$ (cf. Fig. 3).

By some simple combinatorics, first of all, we can enumerate the set of all possible rectangular bases in the bounded grid region under consideration.

Lemma 3. *For $I = I_x \times I_y$, let $\leq_j$ denote the restriction of the usual order $\leq$ to I_j^2, $j \in \{x, y\}$. Then there is a bijection between $\leq_L$ and $\leq_x \times \leq_y$.*

Proof. Let $z_i = (x_i, y_i)$, $i = 1, 2$, then $(z_1, z_2) \in \leq_L \Leftrightarrow x_1 \leq x_2 \wedge y_1 \leq y_2$. Thus $(z_1, z_2) \leftrightarrow ((x_1, x_2), (y_1, y_2))$ yields the desired bijection. □

Lemma 4. *Let $P(I) := \{(z_1, z_2) \in I^2; z_1 \leq_L z_2\}$, then $(*) : |P(I)| = \frac{1}{4}|I_x||I_y|(|I_x|+1)(|I_y|+1)$ and the following defines a partial order $\leq_P$ on $P(I)$:*

$$\forall p = (z_1, z_2), p' = (z_1', z_2') \in P(I) : p \leq_P p' \Leftrightarrow_{\mathrm{def}} z_1' \leq_L z_1 \wedge z_2 \leq_L z_2'$$

Moreover the map

$$\rho : 2^I - \{\varnothing\} \ni S \mapsto \rho(S) := (z_d(S), z_u(S)) \in P(I)$$

(where the points $z_d(S) := (x_d(S), y_d(S)), z_u(S) := (x_u(S), y_u(S))$ are determined as shown above Definition 5) is well defined, surjective and order preserving: $\forall S, S' \in 2^I : S \subseteq S' \Rightarrow \rho(S) \leq_P \rho(S')$, hence transporting the lattice structure of $(2^I, \subseteq)$ to $(P(I), \leq_P)$.

Proof. The first claim immediately follows from the existence of a bijection between $\leq_x \times \leq_y$ and $\leq_L$ (cf. Lemma 3): As $\leq_j$ is a total order on I_j, $|\leq_j| = \frac{1}{2}|I_j|(|I_j|+1)$ holds for $j = x, y$ establishing $(*)$.

Let $p = (z_1, z_2), p' = (z_1', z_2'), p'' = (z_1'', z_2'') \in P(I)$ be arbitrarily chosen, then because $\leq_L$ is a partial order we have $p \leq_P p$ meaning reflexivity. Let $p \leq_P p'$ and $p' \leq_P p$, then by definition $z_1' \leq_L z_1 \wedge z_2 \leq_L z_2'$ and $z_1 \leq_L z_1' \wedge z_2' \leq_L z_2$. By the antisymmetry property of $\leq_L$ immediately follows $p = p'$ which is antisymmetry of $\leq_P$. Finally, $p \leq_P p' \wedge p' \leq_P p''$ by definition means $z_1' \leq_L z_1 \wedge z_2 \leq_L z_2'$ and $z_1'' \leq_L z_1' \wedge z_2' \leq_L z_2''$. Now making use of the $\leq_L$-transitivity one obtains $p \leq_P p''$, thus $\leq_P$ is a partial order on $P(I)$.

Let $S \subseteq I$ then the points $x_d(S) := \min_{i \in S} x_i, x_u(S) := \max_{i \in S} x_i$ and $y_d(S) := \min_{i \in S} y_i, y_u(S) := \max_{i \in S} y_i$ are uniquely determined from S, they may coincide but they obviously fulfill $x_d(S) \leq x_u(S)$ and $y_d(S) \leq y_u(S)$. Thus by Lemma 3 we have $z_d \leq_L z_u$ hence $(z_d, z_u) \in P(I)$ and ρ is well defined. For arbitrary $(z_1, z_2) \in P(I)$, clearly $\{z_1, z_2\} \in \rho^{-1}(z_1, z_2)$ showing surjectivity of π. Now let $S, S' \in 2^I$: $S \subseteq S'$ then holds $\min_{i\in S} x_i \geq \min_{i\in S'} x_i \Rightarrow x_d(S) \geq x_d(S')$ and $\min_{i\in S} y_i \geq \min_{i\in S'} y_i \Rightarrow y_d(S) \geq y_d(S')$ establishing (1): $z_d(S') \leq_L z_d(S)$. Moreover, we have $\max_{i\in S} x_i \leq \max_{i\in S'} x_i \Rightarrow x_u(S) \leq x_u(S')$ and $\max_{i\in S} y_i \leq \max_{i\in S'} y_i \Rightarrow y_u(S) \leq y_u(S')$ establishing (2): $z_u(S) \leq_L z_u(S')$. Now (1), (2) by definition imply $\rho(S) \leq_P \rho(S')$ proving the partial order preserving property of ρ. □

Of course, the set $P(I)$ is bijective to the set $B(I)$ of all possible rectangular bases in the region I by $P(I) \ni (z_1, z_2) \leftrightarrow [x_1, x_2]\times[y_1, y_2] \in B(I)$, with $z_i = (x_i, y_i)$.

According to the problem parameter k, we get a hierarchy of irreflexive transitive binary relations on I, the class $k < \lambda \Leftrightarrow \nu(k) = 0$ corresponds to the reflexive order $\leq_L$:

Lemma 5. *Let $k \in [\lambda, N\lambda) \subset \mathbb{R}_+$, ($N := \min\{N_x, N_y\}$) be fixed, then $z_1 <_{\nu(k)} z_2 \Leftrightarrow_{\text{def}} x_2 - x_1 \geq \nu(k)\lambda \wedge y_2 - y_1 \geq \nu(k)\lambda$ is a transistive binary relation. Setting $P_0(I) := P(I), P_{\nu(k)}(I) := \{(z_1, z_2) \in I^2; z_1 <_{\nu(k)} z_2\}$, then $P_0(I) \supset P_1(I) \supset \cdots \supset P_{N-1}(I)$. Similarly we have a hierarchy of rectangular base sets: $B(I) =: B_0(I) \supset B_1(I) \supset \cdots \supset B_{N-1}(I)$, $B_i(I)$ corresponding to $P_i(I)$ in obvious manner. Finally, for each $i \in \{1, \ldots, N-1\}$:*

$$|B_i(I)| = |P_i(I)| = \frac{1}{4}(|I_x| - i)(|I_y| - i)(|I_x| - i + 1)(|I_y| - i + 1)$$

Proof. The transitivity of $<_i$ is obvious, as well as the indicated set hierarchies. For the last claim observe that $<_i$ is bijective to $P_i(I_x) \times P_i(I_y)$ with $P_i(I_j) := \{(j_1, j_2) \in I_j^2; j_2 - j_1 \geq i\lambda\}$ and obviously $|P_i(I_j)| = \frac{1}{2}(|I_j| - i)(|I_j| - i + 1), j = x, y$. □

Remark 1. For the sets $P_i(M)$ induced by points of M we have $P_i(M) := P_i(I) \cap M^2$ and $B_i(M) := B_i(I) \cap M^2$, as the corresponding rectangular bases. Consequently, specifically for $i = 0$ holds $|B(M)| \leq \frac{1}{4}(N_x + 1)(N_y + 1)(N_x + 2)(N_y + 2)$, because $|I_j| = N_j + 1, j = x, y$.

5.2. *Rectangular subset classes*

In the discussion of Section 4 almost all subsets $S \in 2^M$ have been considered, but many of these subsets can be identified in the sense that they lead to the same rectangular base: $r(S) = r(S'), S, S' \in 2^M$. Now, independently of the grid structure we analyse some kind of rectangular structure inherent in the discrete point set M itself.

Definition 6. A set $S \in 2^M \setminus \{\emptyset\}$ is called *admissible rectangular subset of M* if $r(S) \cap M = S$. Let $\mathcal{A}(M) \subset 2^M \setminus \{\emptyset\}$ denote the set of all admissible rectangular subsets of M. Given k, we define $\mathcal{A}_{\nu(k)}(M) := \{A \in \mathcal{A}(M) | \ell_x(r(A)) \geq \nu(k)\lambda, \ell_y(r(A)) \geq \nu(k)\lambda\} \subset \mathcal{A}(M) =: \mathcal{A}_0(M)$, for each $\nu(k) \in \{0, \ldots, N-1\}$.

Proposition 2. *Let (M, k) be an input instance of k-RC and $\mathcal{A}(M)$ as above, then*

(i) $S_1 \sim_r S_2 \Leftrightarrow_{\text{def}} r(S_1) = r(S_2), \forall \emptyset \neq S_1, S_2 \in 2^M$ *defines an equivalence relation on* $2^M \setminus \{\emptyset\}$*. We write* $\mathcal{M} := [2^M \setminus \{\emptyset\}]/\sim_r$.

(ii) The map $\sigma : 2^M \ni S \mapsto \sigma(S) := r(S) \cap M \in 2^M$ *(*$r(\emptyset) := \emptyset$*) is a closure operator (*rectangular closure*) having image* $\sigma(2^M) = \mathcal{A}(M) \cup \{\emptyset\}$.

(iii) The sets $\mathcal{A}(M)$ *and* $\mathcal{M}$ *are isomorphic, let the corresponding bijection be denoted by* $\mu : \mathcal{A}(M) \to \mathcal{M}$*. Thus each* $A \in \mathcal{A}(M)$ *defines a class of subsets* $\mu(A)$ *called* rectangular subset class.

(iv) $\forall S \subseteq M$ *we have* $\mu^{-1}([T]) \subseteq \mu^{-1}([S]), \forall T \subseteq S$*. Especially* $\forall A \in \mathcal{A}(M)$ *holds* $\mu^{-1}(T) \subseteq A, \forall T \subseteq A$*. Moreover for each* $[S] \in \mathcal{M}$ *we have* $\mu^{-1}([S]) \in [S]$ *is according to* $\subseteq$ *the greatest element of* $[S]$ *(which means* $\forall T \in [S] : T \subseteq A := \mu^{-1}([S]) \in \mathcal{A}(M)$*).*

(v) For $A \in \mathcal{A}(M)$ *we have* $\mu(A) = \{T \subseteq A : T \cap \partial_i r(A) \neq \emptyset, i = 1, \ldots, 4\}$.

(vi) Let $\mathcal{A}_i(M) := \{A \in \mathcal{A}(M) : r(A) \in B_i(I)\}, i \in \{1, \ldots, N-1\}$*, then* $\mathcal{A}(M) =: \mathcal{A}_0(M) \supset \mathcal{A}_1(M) \supset \cdots \supset \mathcal{A}_{N-1}$ *and* $|\mathcal{A}_{\nu(k)}(M)| \leq |B_{\nu(k)}(I)| = |P_{\nu(k)}(I)|$.

Proof. Part (i) is obvious. For proving (ii) recall that a closure operator $\omega : 2^I \to 2^I$ has the following defining properties: (a) $\forall S \subseteq I$ holds $S \subseteq \omega(S)$, (b) $\forall S_1, S_2 \subseteq I$ with $S_1 \subseteq S_2$ holds $\omega(S_1) \subseteq \omega(S_2)$, and (c) $\forall S \subseteq I$ we have $\omega(\omega(S)) = \omega(S)$. To show now that σ has the above properties, we first observe that $S \subseteq r(S)$ implies $S \subseteq \sigma(S), \forall S \in 2^I$, i.e., the first condition is fulfilled. Also it is quite obvious that $r(S_1) \subseteq r(S_2)$ whenever $S_1 \subseteq S_2 \subseteq I$ thus we have also $\sigma(S_1) \subseteq \sigma(S_2)$ which is (ii). Addressing (iii) we have to show that if $T = \sigma(S)$ then $\sigma(T) = T$ for an arbitrary $S \subseteq M$. By definition we have $\sigma(S) = T = r(S) \cap M$ so that obviously $r(T) = r[r(S) \cap M] = r(S)$ implying $\sigma(T) = r(T) \cap M = r(S) \cap M = T$. The claim concerning the image then directly follows from the closure operator property $\sigma(\sigma(S)) = \sigma(S)$.

To justify (iii), we first have to show that $\forall A_1, A_2 \in \mathcal{A}(M) : A_1 \neq A_2$ implies $[A_1] \neq [A_2]$. But this is true, because $A_i = r(A_i) \cap M, i = 1, 2$, and $r(A_1) \cap M \neq r(A_2) \cap M \Leftrightarrow r(A_1) \neq r(A_2)$, thus $[A_1] \neq [A_2]$. Second, for $\varnothing \neq A \subset M$, let $[A] \in \mathcal{M}$, then $\sigma(A) \in \mathcal{A}(M)$ and $[\sigma(A)] = [A]$, because $r(A) = r(r(A) \cap M)$. For showing part (iv) we have $\mu^{-1}([T]) = \sigma(T'), \forall T' \in [T]$, because $r(T') = r(T) \Rightarrow \sigma(T') = \sigma(T) \in \mathcal{A}(M)$, as $T', T \subseteq M$, especially $\mu^{-1}([T]) = \sigma(T)$. By this all claims of (iv) readily follow from the closure operator properties of σ. Part (v) is obvious. The last part (vi) directly follows because $A \neq A' \Leftrightarrow r(A) \neq r(A') \in B(I), \forall A, A' \in \mathcal{A}(M)$. That means $\mathcal{A}_0(M) = \mathcal{A}(M)$ and $|\mathcal{A}_{\nu(k)}(M)| \leq |B_i(I)|$, for $\nu(k) \in \{0, \ldots, N-1\}$. □

We have another useful lemma:

Lemma 6. *For* $(M, k), \mathcal{A}(M)$ *as above,* $|\mathcal{A}(M)| \in O(|M|^4)$ *(hence* $|\mathcal{A}_{\nu(k)}(M)| \in O(|M|^4), \forall \nu(k) \in \{0, \ldots, N-1\}$*).*

Proof. For $\mathcal{B}_4(M) = \{\emptyset \neq T \subseteq M; |T| \leq 4\}$ we first prove that the map

$$\tau : \mathcal{B}_4(M) \ni T \mapsto \tau(T) := \mu^{-1}([T]) \in \mathcal{A}(M)$$

is well defined and is surjective. Assume $M \neq \emptyset$ otherwise the situation is trivial. Because $\mathcal{B}_4(M) \subseteq 2^M$, for each T, there is $[T] \in \mathcal{M}$ due to Proposition 2, and therefore $\mu^{-1}([T]) = \tau(T) \in \mathcal{A}(M)$. To complete the proof that τ is well defined, observe that each T lies in exactly one equivalence class $[T]$. Now let $A \in \mathcal{A}(M)$ be arbitrarily chosen. Then either (i): $|A| \leq 4$ or (ii): $|A| \geq 5$. In case (i) we have $A \in \mathcal{B}_4(M) \cap \mathcal{A}(M)$ and therefore $A = \mu^{-1}([A]) = \tau(A)$ thus $A \in \tau^{-1}(A)$. We observe that $\tau|\mathcal{B}_4(M) \cap \mathcal{A}(M) = \mathrm{id}_{\mathcal{B}_4(M)\cap\mathcal{A}(M)}$, where $\tau|\mathcal{B}_4(M) \cap \mathcal{A}(M)$ denotes the corresponding restriction of τ. Next, let A satisfy case (ii). We claim $(*)$: there exists $T \in \mathcal{B}_4(M)$ such that $r(T) = r(A)$. From $(*)$ the assertion follows since then $T \in [A]$ which is the same as $A \in [T]$ thus $A = \mu^{-1}([T]) = \tau(T)$. Hence, $T \in \tau^{-1}(A)$ establishing that τ is surjective.

To justify $(*)$, let $z_d(A) = (x_d(A), y_d(A))$ and $z_u(A) = (x_u(A), y_u(A))$ be the base points of A, where $x_d(A) = \min_{z\in A} x(z), y_d(A) = \min_{z\in A} y(z)$ and $x_u(A) = \max_{z\in A} x(z), y_u(A) = \max_{z\in A} y(z)$. Hence, there must exist members $z_1, z_2, z_3, z_4 \in A$, at least one, for determining each of these extremal values: $x(z_1) = x_d(A)$, $y(z_2) = y_d(A)$, $x(z_3) = x_u(A)$, $y(z_4) = y_u(A)$. In conclusion, for the set $T := \bigcup_{i=1}^4 \{z_i\}$, holds $|T| \leq 4 \Rightarrow T \in \mathcal{B}_4(M)$ and by construction $r(T) = r(A)$ which has been claimed. Finally, by surjectivity we obtain

$$\tau(\mathcal{B}_4(M)) = \mathcal{A}(M) \Rightarrow |\mathcal{A}(M)| \leq |\mathcal{B}_4(M)| = \bigcup_{i=1}^{4} \binom{|M|}{i} \in O\left(\sum_{i=1}^{4} |M|^i\right)$$

therefore $|\mathcal{A}(M)| \in O(|M|^4)$. □

5.3. *The rectangular subset closure*

From a slightly different point of view we have that the pair $G_I := (I, P(I))$ is a (plane-embedded) acyclic directed graph (DAG) and $G_M = (M, P(M))$, where $P(M) = P(I) \cap M^2$, is a (plane-embedded) induced DAG. Unfortunately the digraph $(M, P(M))$ cannot directly be used to enumerate the set of covering components, because it will not determine all necessary rectangular bases. Intuitively, we are looking for some kind of a "covering hull" which means adding the minimal number of necessary vertices (i.e. points) from $I \setminus M$ to M and also the corresponding edges to $P(M)$ such that all rectangular bases needed for covering are determined.

The concept we need for that is the *rectangular subset closure*.[12] To make the presentation more self-contained we shall briefly explain the main features. The rectangular subset closure of a point set M naturally appears as the smallest superset of M containing the base points of all subsets of M.

Definition 7. For $M \subset L$ finite, the *rectangular subset closure* $\mathrm{RS}(M) \subset L$ is defined by $\mathrm{RS}(M) := \bigcap\{L \supseteq M' \supseteq M : b(S) \subset M', \forall S \in 2^M\}$.

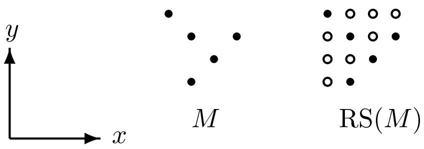

Fig. 4. A set M and its rectangular subset closure $\mathrm{RS}(M)$ (grid lines are omitted).

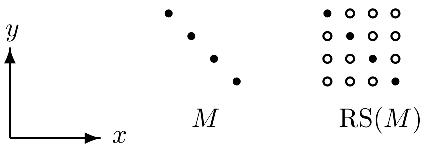

Fig. 5. An antichain M of $n = 4$ points yielding a rectangular subset closure $\mathrm{RS}(M)$ of $n^2 = 16$ points (grid lines are omitted).

Definition 7 obviously is equivalent to:

Lemma 7. *For $M \subset L$ finite, we have $\mathrm{RS}(M) = M \cup \bigcup_{S \subseteq M} b(S)$.*[12] □

As an example consider Figure 4, where all *additional* base points, for a given set M, contained in the corresponding $\mathrm{RS}(M)$ are represented as white dots.

The rectangular subset closure as introduced above gives rise to a closure operator defined for a fixed finite rectangular grid region $I \subset L$.

Proposition 3. $\mathrm{RS} : 2^I \to 2^I$ *is a closure operator.*[12]

The rectangular subset closure of a $M \subset I$ gives rise to a (plane-embedded) directed acyclic graph. Its vertex set is $\mathrm{RS}(M)$ and each chain $(z_d, z_u) \in \mathrm{RS}(M)^2$ forms an edge if and only if they appear as base points of some $S \subseteq M$. Such a graph has loops (z, z) corresponding to the single element subsets, i.e., to the points $z \in M$.

Moreover we have:[12]

Theorem 3. *For finite $M \subset L$ holds:*
(1) $|\mathrm{RS}(M)| \leq |M|^2$, *and* $|\mathrm{RS}(M)| = |M|^2$ *if and only if M is an antichain (cf. Fig. 5).*
(2) $\mathrm{RS}(M)$ *can be computed in time* $O(|M|^2)$.
(3) $|\mathcal{A}(M)| \leq |\mathrm{RS}(M)|^2$.
(4) $\mathcal{A}(M)$ *can be computed in time* $O(|M|^4)$.

Observe that the bound for $|\mathcal{R}(M)|$, in many cases, namely when $|\mathrm{RS}(M)| \in O(|M|)$ (which, e.g., is the case for chains) holds, is much better than that stated above. Moreover, observe that in case M is an antichain, i.e., the most extremely class regarding the size of $\mathrm{RS}(M)$, then for computing $\mathcal{A}(M)$ only all subsets of size two of M are needed. Hence, for an antichain M, holds $\mathcal{A}(M) \in O(|M|^2)$.

6. Improving Time Bounds

The structural features provided previously next will be used to improve the bound for solving k-RCn stated in Theorem 2.

Lemma 8. *Let* (M, k) *be an input instance of* k-RC. *If, for each* $S \in \mathcal{S}_j$ *one replaces* $\mathcal{T}_j(S)$ *by the set* $\hat{\mathcal{T}}_j(S) := \mathcal{T}_j(S) \cap \mathcal{A}_{\nu(k)}(M), j \in \{2, \dots, |M|\}$, *in Algorithm* k-RC, *then it still works correctly.*

Proof. We have to show that

$$(*):\ \mathrm{opt}_j(k, S) = \min\{w(r_\delta(T)) + \mathrm{opt}_{j-1}(k, S') : T \in \hat{\mathcal{T}}_j(S)\}$$

holds, for $2 \leq j \leq |M|$; in any case it is necessary to compute $\mathrm{opt}_1(k, S), \forall S \in 2^M \setminus \{\emptyset\}$. By Theorem 2 we have $\mathrm{opt}_j(k, S) = \min\{f_S^j(T) : T \in \mathcal{T}_j(S)\}$ defining $f_S^j(T) := w(r_\delta(T)) + \mathrm{opt}_{j-1}(k, S')$. Now we claim that for each $T \in \mathcal{T}_j(S)$ there is $A \in \hat{\mathcal{T}}_j(S) : f_S^j(A) \leq f_S^j(T)$, from which the lemma immediately follows, because in that case we do not miss a relevant candidate when $\mathrm{opt}_j(k, S)$ is computed according to $(*)$. To show the claim consider any $T \in \mathcal{T}_j(S)$; if $T \in \hat{\mathcal{T}}_j(S)$ we are ready by setting $A := T$ implying $f_S^j(A) = f_S^j(T)$. Moreover, $T \in \mathcal{T}_j(S) \setminus \hat{\mathcal{T}}_j(S)$ implies $T \notin \mathcal{A}_{\nu(k)}(M)$, and we set $A := A(T) := r(T) \cap M \in \mathcal{A}(M)$, and obviously $w(r_\delta(T)) = w(r_\delta(A))$. From Proposition 2, (iv), we obtain $S \setminus A \subseteq S \setminus T$, because $T \subseteq A(T)$, which in case $|S \setminus A| \geq j - 1$ directly implies $f_S^j(A) \leq f_S^j(T)$. In the remaining case $|S \setminus A| < j - 1$, we have

$$\begin{aligned}
\mathrm{opt}_{j-1}(k, S \setminus A) &= \mathrm{opt}_l(k, S \setminus T) + \mathrm{opt}_{j-1-l}(k, \emptyset) \\
&\leq \mathrm{opt}_l(k, S \setminus T) + \mathrm{opt}_{j-1-l}(k, \emptyset) \\
&\leq \mathrm{opt}_{j-1}(k, S \setminus T)
\end{aligned}$$

where the last inequality holds because $|S \setminus T| \geq j - 1$ and $\mathrm{opt}_{j-1-l}(k, \emptyset)$ means the value of w for $j - 1 - l$ rectangles being smallest according to k, from which the claim and also the lemma follow. □

Defining $f(k, |M|, N_x, N_y) := \min\{|\mathcal{B}_4(M)|, |P_{\nu(k)}(I)|\}$, with $|\mathcal{B}_4(M)| \in O(|M|^4), |P_{\nu(k)}(I)| \in O([N_x - \nu(k)]^2[N_y - \nu(k)]^2)$ one obtains:

Theorem 4. *For input* (M, k) *problem k-RC can be solved in time* $O(|M|^2 f(k, |M|, N_x, N_y) 2^{|M|}) \in O(|M|^6 2^{|M|})$.

Proof. The correctness directly follows from Lemma 8. To verify the time bound first observe that from the proof of Theorem 2 follows that for the most inner loops instead of considering each element of $\mathcal{T}_j(S)$ we have to consider only those also being elements of $\mathcal{A}_{\nu(k)}(M)$. Thus, instead of $p2^p$, for fixed $S \subset M$ with $|S| = p$, one obtains $\sum_{p=j}^{n} \binom{n}{p} p |\mathcal{A}_{\nu(k)}(M)| \leq n 2^n |\mathcal{A}_{\nu(k)}(M)|$ and the outer loop never is iterated more than n times yielding another factor $n := |M|$. Finally, realizing $|\mathcal{A}(M)| \leq \min\{|\mathcal{B}_4(M)|, |P_{\nu(k)}(I)|\}$ which directly follows from Lemma 6 and Proposition 2, (vi), finishes the proof. □

Observe that there are many situations where $\mathrm{RS}(M)$ is much less than $|M|^2$. E.g. for an chain M, we even have $\mathrm{RS}(M) = M$. In such cases we achieve a smaller exponent of $|M|$ than that in the bound stated above, which could shrink from 6 to 4, if $|\mathrm{RS}(M)|$ is considered instead of $|\mathcal{B}_4(M)|$.

Consider the following parameterized variant of the problem at hand: For fixed $p \in \mathbb{N}$, let k-RC_p be the problem of solving k-RC with at most p covering components. For this situation we have:

Theorem 5. *For fixed $p \in \mathbb{N}$, and input (M, k), k-RC_p can be solved in time $O(p|M|f(k, |M|, N_x, N_y)^p)$.*

Proof. Even as brute force search: we only have to check each covering candidate R in the set

$$\bigcup_{i=1}^{p} \binom{\mathcal{A}_{\nu(k)}(M)}{p}$$

whose cardinality is in $O(|\mathcal{A}_{\nu(k)}(M)|^p)$, hence the bound follows, because $|\mathcal{A}_{\nu(k)}(M)| \leq \min\{|\mathcal{B}_4(M)|, |P_{\nu(k)}(I)|\}$ and $w(\{r_\delta(A), A \in R\})$ can be computed in worst case time $O(p|M|)$. $\square$

7. Generalization to the d-Dimensional Case

The setup described in the preceeding section will be generalized in the sequel to the d-dimensional case for $2 \leq d \in \mathbb{N}$. This generalization is not only interesting from an abstract point of view but it may be profitable also for modeling higher dimensional applications.

For fixed $1 < d \in \mathbb{N}$, let E^d be the Euclidean space in d dimensions with fixed (orthogonal) standard basis $B^d = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$. For the (orthogonal) integer lattice $L^d = \mathbb{Z}\boldsymbol{e}_1\lambda + \cdots + \mathbb{Z}\boldsymbol{e}_d\lambda$ with lattice constant $0 < \lambda \in \mathbb{R}$ by the vector $\boldsymbol{N} := (N^1, \ldots, N^d) \in \mathbb{N}^d$ we fix the bounded region

$$I^d = ([0, N^1\lambda] \times \cdots \times [0, N^d\lambda]) \cap L^d$$

Let $M = \{\boldsymbol{m}_1, \ldots, \boldsymbol{m}_n\} \subset I^d$, where each $\boldsymbol{m}_i = (m_i^1, \ldots, m_i^d)$ is represented by its coordinate values with respect to B^d. We are searching for a covering of M, by regular, i.e., B^d-parallel d-boxes of minimal fixed side lengths k with $0 < k < \min_{1 \leq i \leq d} N_i\lambda$, s.t. the overall volume, boundary volume and number of boxes used are minimized. Let r be a d-box with side-length vector $(\ell^1, \ldots, \ell^d), \ell_i \geq k$, then its volume is given by $\mathrm{vol}(r) = \prod_{i=1}^d \ell^i$, and the volume of its boundary is $\mathrm{vol}(\partial r) = 2\sum_{i=1}^d \prod_{i \neq j}^d \ell^j$. Here ∂r denotes the boundary of r topologically viewed as closed set. Generalizing Definition 4, more precisely we define:

Definition 8. Let $\boldsymbol{N} \in \mathbb{N}^d$, and $k, \lambda, \varepsilon \in \mathbb{R}_+ - \{0\}$ such that $k < N := \min\{N^i\lambda : i \in [d]\}$, and $\lambda/2 > \varepsilon$ be fixed. A *(k,ε)-admissible d-box* is a regular d-box $r \subset [-\varepsilon, N^1\lambda + \varepsilon] \times \cdots \times [-\varepsilon, N^d\lambda + \varepsilon]$ and $k \leq \ell^i(r)$ $(1 \leq i \leq d)$ such that each $\boldsymbol{m} \in r \cap M$ has minimal (Euclidean) distance ε to $\partial_i r, \forall i = 1, \ldots, 2d$. Let $\mathcal{R}_d$

be the set of all (k,ε)-admissible d-boxes. A *(k,ε)-admissible d-box covering* of M is a set $R \subset \mathcal{R}_d$ such that $M \subseteq \bigcup_{r\in R} r \cap I^d$ and for each $r \in R : r \cap M \neq \emptyset$. The set of all such coverings of M is denoted as $\mathcal{C}(M) \subseteq 2^{\mathcal{R}(k,M)}$. As objective function define $w : \mathcal{R}_d \ni r \mapsto w(r) := \mathrm{vol}(r) + \mathrm{vol}(\partial r) + c \in \mathbb{R}_+$, for fixed $c \geq 1$, and by extension $R \in \mathcal{C}(M) : w(R) := \sum_{r\in R} w(r)$.
k,d-RECTANGULAR COVER (k-RC^d) is the following optimization problem: Given $M \subset I^d$, find $\mathrm{opt}(k,M) := \min\{w(R); R \in \mathcal{C}(M)\}$ and $R_0 \in \mathcal{C}(M)$, with $w(R_0) = \mathrm{opt}(k,M)$, which is called an *optimal d-covering*.

Given $\boldsymbol{m} \in I^d$ and $\boldsymbol{e}_i \in B^d$, there is a unique hyperplane $H_m(\boldsymbol{e}_i) \subset E^d$ containing $\boldsymbol{m}$ and being orthogonal to $\boldsymbol{e}_i$, which is given by $H_m(\boldsymbol{e}_i) := \{m^i\boldsymbol{e}_i + \sum_{j\neq i} \alpha_j \boldsymbol{e}_j : \alpha_j \in \mathbb{R}\}$. Hence given $S \in 2^M$, by $b_d(S) := \{\boldsymbol{m}_a(S), \boldsymbol{m}_b(S)\} \in \binom{I^d}{2}$, a unique *$d$-box base* $r_d(S)$ is determined in time $O(d|M|)$ via the intersections of the corresponding hyperplanes, where $m_a^i(S) := \min\{m^i : \boldsymbol{m} \in S\}$ and $m_b^i(S) := \max\{m^i : \boldsymbol{m} \in S\}$, $1 \leq i \leq d$. Thus, by slightly modifying Algorithm OPT1 resulting to Algorithm OPT1^d considering all d directions, we obtain the time bound $O(d|M|)$ for computing a (k,ε)-admissible d-box covering S from $b_d(S)$, if S is assumed to be ordered lexicographically. Similarly, Algorithm k-RC may be modified to Algorithm k-RC^d only by incorporating OPT1^d yielding the worst case time bound of $O(d|M|^2 3^{|M|})$ for solving k-RC^d.

This bound can be improved by generalizing the structural features discussed in Section 5 to the d-dimensional case. The equivalence relation $\sim$ on the power set 2^M can also be generalized to the d-dimensional case where $M \subset L^d$:

$$S_1 \sim_d S_2 \Leftrightarrow_{\mathrm{def}} b_d(S_1) = b_d(S_2), \forall S_1, S_2 \in 2^M$$

with classes $[S]_d$. Defining $\mathcal{M}_d := 2^M/\sim_d$ as well as

$$\sigma_d : 2^M \ni S \mapsto \sigma_d(S) := r_d(S) \cap M \in 2^M$$

$(r_d(\varnothing) := \varnothing)$ and $\mathcal{A}_d(M) := \{S \subseteq M : \sigma_d(S) = S\}$ we arrive at:

Proposition 4. *$\sigma_d : 2^M \to 2^M$ is a closure operator and there is a bijection $\mu_d : \mathcal{A}_d(M) \to \mathcal{M}_d$ defined by $S \mapsto \mu_d(S) := [S]_d, S \in \mathcal{A}_d(M)$.* □

Instead of $\mathcal{B}_4$ we now have to consider the set $\mathcal{B}_{2d} = \{S \subset M : |S| \leq 2d\}, |\mathcal{B}_{2d}| \in O(|M|^{2d})$ which for fixed d defines a polynomial bound. As in the plane case we have for the sets $\mathcal{A}_{d_{\nu(k)}}(M) = \{A \in \mathcal{A}_d(M) : \ell^i(r_d(A)) \geq \nu(k)\lambda, i \in [d]\}$ that $|\mathcal{A}_{d_{\nu(k)}}(M)| \leq |\mathcal{B}_{2d}|$, for all $\nu(k) \in \{0,\ldots,N-1\}$.

Furthermore there is a straightforward generalization of the relations $\leq_{\nu(k)} \subseteq I^2$ (cf. Lemmata 4 and 5), to $I^d \times I^d$ via $\boldsymbol{m}_1 \leq_{\nu(k)} \boldsymbol{m}_2 \Leftrightarrow_{\mathrm{def}} m_2^i - m_1^i \geq \nu(k)\lambda$, $1 \leq i \leq d$. The corresponding sets $P_i(I^d), B_i(I^d)$ are defined analogously to $P_i(I), B_i(I)$, and for each $i \in \{1,\ldots,\min_{1\leq j\leq d} N_j - 1\}$ we have:

$$|P_i(I^d)| = |B_i(I^d)| = \prod_{l=1}^{d} \frac{1}{2}(N_l - i)(N_l - i + 1)$$

Defining $f_d(k, |M|, \boldsymbol{N}) := \min\{|\mathcal{B}_{2d}(M)|, |P_{\nu(k)}(I^d)|\}$, where $|\mathcal{B}_{2d}(M)| \in O(|M|^{2d}), |P_{\nu(k)}(I^d)| \in O(\prod_{i=1}^{d}[N^i - \nu(k)]^2)$ and Collecting all parts of the preceeding discussion we obtain the result:

Theorem 6. *A worst case time bound for exactly solving k-RC^d for input (M, k) is $O(d|M|^2 f_d(k, |M|, \boldsymbol{N})2^{|M|})$.*

We also have a generalization of the notion of the rectangular subset closure of a point set:[12] The *d-dimensional rectangular subset closure* $\mathrm{RS}_d(M)$ of a finite point set $M \subset L^d$ is given by $\mathrm{RS}_d(M) = M \cup \bigcup_{S \subseteq M} b_d(S)$. Moreover one has:[12]:

Theorem 7. *For $M \subset L(d)$ finite, we have*

Similarly, we can derive generalized results concerning the set $\mathcal{R}_d(M)$ of representatives according to relation $\sim_d$, for a finite set $M \subset L(d)$.

Theorem 8. *For $M \subset L(d)$ finite, we have*
(1) $|\mathrm{RS}_d(M)| \in O(|M|^2)$, and $\mathrm{RS}_d(M)$ can be computed in time $O(d|M|^2)$.
(2) $|\mathcal{A}_d(M)| \in O(|\mathrm{RS}_d(M)|^2)$.

As in the plane case the exponent of $|M|$ in the bound stated in Theorem 6 may be decreased when $|\mathrm{RS}_d(M)|$ is considered instead of $|\mathcal{B}_{2d}(M)|$.

For fixed $p \in \mathbb{N}$, let k-RC_p^d be the problem of solving k-RC^d with at most p covering components, we have generalizing the proof of Theorem 5:

Theorem 9. *For input (M, k) problem k-RC$_p^d$ can be solved in time $O(dp|M|f(k, |M|, \boldsymbol{N})^p)$.*

8. Concluding Remarks and Open Problems

We provided algorithms for covering a given set of n points in the discrete d-dimensional Euclidean space, $d \geq 2$, by regular rectangles due to an objective function minimizing sum of areas, circumferences and number of rectangles used. A first dynamic programming approach yielding an appropriate covering algorithm of running time $O(dn^2 3^n)$ could be improved to $O(dn^6 2^n)$ by slight modifications exploiting the underlying rectangular structure.

The methods provided can be easily adapted to 1-sided (parameterized) rectangular covering problems defined for any monotone objective function w. An important open question is whether the rectangular problems with 1-sided boundary constraints as discussed are NP-hard.

Closely related is the question whether the methods presented here also apply to the variants of 2-sided boundary constraints which are known to be NP-hard.

Finally, it might be worthwhile also to formulate the problem in terms of linear/integer programming with appropriate relaxations. However to fixing concrete time bounds and to encounter the presented structural features seems not to be as accessible as in the dynamic programming environment discussed here.

References

1. E. M. Arkin, R. Hassin, Minimum-Diameter Covering Problems, Networks 36 (2000) 147-155.
2. P. Bastian, Load Balancing for Adaptive Multigrid Methods, SIAM Journal on Scientific Computing, 19 (1998) 1303-1321.
3. E. Boros, P. L. Hammer, On Clustering Problems with Connected Optima in Euclidean Spaces, Discrete Mathematics 75 (1989) 81-88.
4. F. C. Calheiros, A. Lucena, C. C. de Souza, Optimal Rectangular Partitions, Networks 41 (2003) 51-67.
5. J. C. Culberson, R. A. Reckhow, Covering Polygons is Hard, Proceedings of the twenty-ninth IEEE Symosium on Foundations of Computer Science, 1988, pp. 601-611.
6. M. R. Garey, D. S. Johnson, Computers and Intractability, Freeman, New York, 1979.
7. J. Hershberger, S. Suri, Finding Tailored Partitions, Journal of Algorithms 12 (1991) 431-463.
8. D. S. Hochbaum, Approximation Algorithms for NP-hard problems, PWS Publishing, Boston, Massachusetts, 1996.
9. M. N. Kolountzakis, On the Structure of Multiple Translational Tilings by Polygonal Regions, Discrete Comput. Geom. 23 (2000) 537-553.
10. B. Monien, E. Speckenmeyer, O. Vornberger, Upper Bounds for Covering Problems, Methods of Operations Research 43 (1981) 419-431.
11. S. Porschen, On the Time Complexity of Rectangular Covering Problems in the Discrete Plane, Proceedings of 4th ICCSA 2004/CGA 2004, Lect. Notes in Comp. Sci., Vol. 3045, pp. 137-1465, 2004.
12. S. Porschen, On the Rectangular Subset Closure of Point Sets, Proceedings of 5th ICCSA 2005/CGA 2005, Lect. Notes in Comp. Sci., Vol. 3480, pp. 796-805, 2005.
13. S. Porschen, Algorithms for Rectangular Covering Problems, Proceedings of 6th ICCSA 2006/CGA 2006, Lect. Notes in Comp. Sci., Vol. 3980, pp. 40-49, 2006.
14. S. J. Plimpton, B. Hendrickson, J. R. Stewart, A parallel rendezvous algorithm for interpolation between multiple grids, J. Parallel Distrib. Comput. 64 (2004) 266-276.
15. S. S. Skiena, Probing Convex Polygons with Half-Planes, Journal of Algorithms 12 (1991) 359-374.
16. A. Smith, S. Suri, Rectangular Tiling in Multidimensional Arrays, Journal of Algorithms 37 (2000) 451-467.
17. S. L. Tanimoto, R. J. Fowler, Covering Image Subsets with Patches, Proceedings of the fifty-first International Conference on Pattern Recognition, 1980, pp. 835-839.