Note on Representations of Ordered Semirings

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Abstract

The article studies ordered semigroups and semirings with respect to their representations in lattices. Such structures are essentially the pseudolattices of Dietrich and Hoffman. It is shown that a subadditive representation implies the semigroup to be a lattice in its own right. In particular, distributive lattices can be characterized as semirings admitting subadditive supermodular representations. The cover problem asks for a minimal cover of a ground set by representing sets with respect to a semiring. A greedy algorithm is exhibited to solve the cover problem for the class of lattices with weakly subadditive and supermodular representation.

1 Introduction

Pseudolattices were recently introduced by Dietrich and Hoffman [2] as very general (finite) algebraic lattice-type structures on (partially) ordered ground sets. Indeed, every ordered set with a unique lower bound and a unique upper bound can be endowed with such a structure. Combinatorial interest in pseudolattices,

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however, arises not so much from their abstract structure but from their representations in lattices in general and Boolean algebras in particular. In this note, we study such structures under the assumption that they admit representations of certain types.

In our discussion, we prefer the terminology of ordered semigroups and semirings, which we define in Section 2. We introduce the characteristic of a representation and show that ordered semigroups with a subadditive characteristic are lattices in their own right. In particular, distributive lattices may be viewed as semirings admitting a subadditive and supermodular representation in some lattice. In Section 3, we turn to the cover problem, which assumes a given representation of a semiring in a set system (Boolean algebra). The problem consists in identifying a minimal cover of the ground set by representing sets. This problem is dual to the packing problem by representing sets, which is known to be solvable by Frank's [7] greedy algorithm if the representation is submodular. Generalizing the approach of [4], [5] we establish an analogous greedy-type (Monge) algorithm for the cover problem relative for a class of lattices with weakly subadditive and supermodular representations.

2 Representations of Ordered Semigroups

Let L be a finite lattice. So L is equipped with a partial order $\preceq$ so that for any two $x, y \in L$ their supremum $x \vee y$ exists in L. It is well-known that the existens of suprema in L implies the existence of infima in L. Indeed, the infimum of x and y is

$$x \wedge y = \bigvee \{z \in L \mid z \preceq x, y\}.$$

Let $(P, \leq)$ be an arbitrary finite (partially) ordered set. By a *representation* of P in L we understand a map $\chi : P \to L$ that is order-compatible in the sense that the following two conditions are satisfied for all $a, b, c \in P$:

(C0) $\chi(a) \preceq \chi(b) \quad \Longrightarrow \quad a \leq b.$

(C1) $a \leq b \leq c \quad \Longrightarrow \quad \chi(a) \wedge \chi(c) \ \preceq \ \chi(b).$

Note that (C0) implies that the representation $\chi : P \to L$ is injective and the inverse map $\chi^{-1} : \chi(P) \to P$ is an order-homomorphism (but not necessarily an order-isomorphism!). (C1) is the so-called *consecutive ones property* (*cf.* [2]). Clearly (C0) holds, for example, whenever χ is an order-homomorphism.

Observe that any ordered set P always admits a representation χ in the lattice $\mathcal{B}(P) = (2^P, \subseteq)$ of subsets of P with

$$\chi(a) = \{a' \in P \mid a' \leq a\}.$$

Recall that an element u is called *(join-)irreducible* in the finite lattice L if u has precisely one lower neighbor in $(L, \preceq)$. Denote by $J = J(L)$ the set of all irreducible elements. Then each $x \in L$ is characterized by the associated subset

$$J(x) = \{u \in J \mid u \preceq x\},$$

which is an order ideal in $(J, \preceq)$. It follows that the structure of L is determined by the *characteristic functions* $\mu_u : L \to \{0, 1\}$, where $u \in J(L)$ and

$$\mu_u(x) = \begin{cases} 1 & \text{if } u \preceq x \\ 0 & \text{if } u \not\preceq x. \end{cases}$$

If $\chi : P \to L$ is a representation of the ordered set P in the lattice L, we thus obtain the *characteristic functions of the representation* $\chi_u : P \to \{0, 1\}$, where $u \in J(L)$ and

$$\chi_u(a) = \mu_u(\chi(a)) \quad \text{for all } a \in P.$$

2.1 Ordered Semigroups

We call $(P, \leq)$ an *ordered semigroup* if there is binary operation $(a, b) \mapsto a \oplus b$ on P such that

$$a, b \leq a \oplus b \quad \text{for all } a, b \in P.$$

Consider the representation $\chi : P \to L$ of the (ordered) semigroup P in the lattice L. We say that (the characteristic of) χ is *subadditive* if the inequality

$$\chi_u(a \oplus b) \leq \chi_u(a) + \chi_u(b)$$

holds for all irreducibles $u \in J(L)$ and elements $a, b \in P$. Subadditivity is equivalent with the property

$$J(\chi(a \oplus b)) \ \subseteq \ J(\chi(a)) \cup J(\chi(b))$$

and imposes a strong condition on the structure of P.

Theorem 2.1 *Assume that the ordered semigroup P admits a subadditive representation $\chi : P \to L$ in the lattice L. Then for all $a, b \in P$, $a \oplus b$ is the supremum of a, b in $(P, \leq)$. In particular, $(P, \leq)$ is a lattice.*

Proof. Suppose the Theorem is false and $c = a \oplus b$ is not the supremum of a and b in P, *i.e.*, there exists some $d \in P$ with $d \geq a, b$ and $d \not\geq c$. Consider $c \oplus d$ and observe that $\chi(c \oplus d) \not\preceq \chi(d)$ must hold since otherwise (C0) would imply the contradiction

$$c \leq c \oplus d \leq d.$$

So there exists an irreducible $u \in J(\chi(c \oplus d)) \setminus J(\chi(d))$. Since χ is subadditive, we know $u \in J(\chi(c))$ and hence $u \in J(\chi(a))$ or $u \in J(\chi(b))$. Assume $u \in J(\chi(a))$, for example, and recall the relation

$$a \leq d \leq c \oplus d.$$

Now $u \preceq \chi(a) \wedge \chi(c \oplus d)$ holds while $u \preceq \chi(d)$ is not satisfied, which contradicts property (C1) of χ. Consequently, no counterexample to the claim of the Theorem can exist.

$\diamond$

2.2 Ordered Semirings and Pseudolattices

Let P be an ordered set as before and assume that there are two binary operations $a \oplus b$ and $a \odot b$ on P such that

$$a \odot b \;\leq\; a, b \;\leq\; a \oplus b \quad \text{for all } a, b \in P.$$

We then call $(P, \oplus, \odot)$ an *ordered semiring*. Note that our "ordered semirings" are essentially the *pseudolattices* of Dietrich and Hoffman [2], whose definition stipulates the additional property

$$a \oplus b = b \quad \text{and} \quad a \odot b = a \quad \text{whenever } \; a \leq b.$$

Given the ordered semiring P, we extend the terminology for the characteristics of representations $\chi : P \to L$ and call χ *supermodular* if

$$\chi_u(a \oplus b) + \chi_u(a \odot b) \geq \chi_u(a) + \chi_u(b)$$

holds for all $u \in J(L)$ and $a, b \in P$. χ is *submodular* if the reverse inequality

$$\chi_u(a \oplus b) + \chi_u(a \odot b) \leq \chi_u(a) + \chi_u(b)$$

is always true. Note that the submodularity of χ implies in particular that χ is subadditive.

We say that χ is *modular* if χ is both super- and submodular. The ordered semirings (or pseudolattices) that admit a modular representation in a Boolean lattice are central in the investigation [2].

It is well-known that the characteristic functions μ_u of a distributive lattice L are modular and form the basis for the vector space of *valuations* of L. Valuations play an important role in combinatorial analysis (*cf.* [11]). One might expect that representations on a distributive lattice therefore always have a modular characteristic. However, this is not the case as the following example shows.

Example 2.1 *Let $(N, \leq)$ be an ordered set and $\mathcal{A}$ its collection of antichains. With each antichain $A \in \mathcal{A}$ we associate the order ideal*

$$\chi(A) = \{s \in N \mid s \leq a \text{ for some } a \in A\}.$$

$P = (\mathcal{A}, \leq)$ is a distributive lattice under the ordering

$$A \leq B \quad \Longleftrightarrow \quad \chi(A) \subseteq \chi(B).$$

Moreover, χ yields a representation of P in the (distributive) lattice of subsets $\mathcal{B}(N) = (2^N, \subseteq)$. P is an ordered semiring under the operations

$$A \oplus B = MAX(A \cup B) \quad \text{and} \quad A \odot B = A \cap B,$$

where $MAX(S)$ denotes the set of maximal elements of a set $S \subseteq N$. It is straightforward to check that the representation $\chi : P \to \mathcal{B}$ is generally submodular but not necessarily modular. Following [9], let us modify the multiplication in P to

$$A \sqcap B = MAX(\chi(A) \cap \chi(B)).$$

Then $(P, \oplus, \sqcap)$ is an ordered semiring with respect to which the representation χ is modular.

2.3 Representations of Closure Systems

Let $\mathcal{F} \subseteq 2^N$ be a closure system on N, *i.e.,* an intersection-closed family of subsets with $N \in \mathcal{F}$. As usual, we denote the *closure* of a set $S \subseteq N$ by

$$\overline{S} = \bigcap \{F \in \mathcal{F} \mid S \subseteq F\}.$$

Then $(\mathcal{F}, \subseteq)$ is a lattice and an ordered semiring with respect to the operations $S \oplus T = \overline{S \cup T}$ and $S \odot T = S \cap T$. Because $S \cup T \subseteq \overline{S \cup T}$, it is clear that the identity map $\iota(S) = S$ yields a supermodular representation of $(\mathcal{F}, \oplus, \odot)$ in the lattice $\mathcal{B}(N) = (2^N, \subseteq)$ of all subsets of N:

$$\mu_u(S \oplus T) + \mu_u(S \cap T) \ \geq \ \mu_u(S \cup T) + \mu_u(S \cap T) \ = \ \mu_u(S) + \mu_u(T).$$

As indicated in Example 2.1, it is quite possible that a closure system $\mathcal{F}$ admits a submodular representation as well. Assume, for example, that $\mathcal{G}$ is a family of subsets of the groundset N with the property that for each closed sets $S \in \mathcal{F}$ a unique set $G_S \in \mathcal{G}$ exists such that for all $S, T, V \in \mathcal{F}$,

(G0) $\overline{G_S} = S$.

(G1) $S \subseteq T \subseteq V \quad \Longrightarrow \quad G_S \cap G_V \subseteq G_T$.

(G2) $G_{S \oplus T} \subseteq G_S \cup G_T$ and $G_{S \cap T} \subseteq G_S \cap G_T$.

For instance, if $\mathcal{F}$ is the system of order ideals of the partially ordered set $(N, \leq)$, the collection $\mathcal{A}$ of antichains satisfies (G0)-(G1).

Consider generally the map $\chi : \mathcal{F} \to \mathcal{B}(N)$, given by $\chi(S) = G_S$. Then (G0) implies property (C0) and (G1) implies (C1). Moreover, (G2) says that χ is submodular:

$$\chi_u(S \oplus T) + \chi_u(S \odot T) \leq \mu_u(G_S \cup G_T) + \mu_u(G_S \cap G_T) = \chi_u(S) + \chi_u(T).$$

Hence we have

Proposition 2.1 *Assume that the closure system $\mathcal{F}$ on the set N admits a family $\mathcal{G}$ with property (G0)-(G2). Then $S \mapsto G_S$ yields a submodular representation of $\mathcal{F}$ in the lattice $\mathcal{B}(N)$ of all subsets of N.*

$\diamond$

Further examples of closure systems $\mathcal{F}$ satisfying (G0)-(G1) arise from so-called *convex geometries* (*cf.* [3]), where one may take G_S as the set of vertices of the closed set $S \in \mathcal{F}$. (Recall that a vertex of S is a point $v \in S$ with the property $v \notin \overline{S \setminus \{v\}}$ and that each closed set of a convex geometry is the closure of its set of vertices.)

Our next example of the closure system $\mathcal{N}_5$ shows that the class of closure systems satisfying (G0)-(G2) is strictly larger than the class of convex geometries.

Example 2.2 *Let* $N = \{a, b, c, d\}$ *and* $\mathcal{N}_5 = \{\emptyset, \{a\}, \{b, d\}, \{b, c, d\}, N\}$*.* $\mathcal{N}_5$ *is a closure system and* $\mathcal{G} = \{\emptyset, \{a\}, \{b, d\}, \{b, c\}, \{a, b\}\}$ *satisfies (G0)-(G2). We remark that* $\mathcal{N}_5$ *fails to satisfy all the requirements of a convex geometry in the sense of [3].*

2.3.1 Co-closure Systems

A family $\mathcal{F}$ of subsets of the set N is a *co-closure system* if

$$\emptyset \in \mathcal{F} \quad \text{and} \quad S \cup T \in \mathcal{F} \quad \text{for all } S, T \in \mathcal{F}.$$

$\mathcal{F}$ is an ordered semiring with respect to the operations

$$S \oplus T = S \cup T \quad \text{and} \quad S \odot T = \cup\{A \in \mathcal{F} \mid A \subseteq S \cap T\}.$$

Clearly, the identity map $\chi(S) = S$ provides a submodular representation of $(\mathcal{F}, \subseteq)$ in the lattice $\mathcal{B}(N)$.

2.4 Distributive Lattices

We want to characterize distributive lattices in terms of their representability as ordered semirings. Our proof is based on the well-known characterization of a distributive lattice as a lattice that admits neither a substructure of type N_5 nor a substructure of type M_3 (*cf.* [1]).

Theorem 2.2 *Let* $(P, \oplus, \odot)$ *be an ordered semiring. Then* P *is a distributive lattice if and only if* P *admits a subadditive and supermodular representation* $\chi : P \to L$ *in some lattice* L*.*

Proof. The necessity of the condition is obvious since the identity $\chi(a) = a$ provides a representation of P in $L = P$ of the desired kind if P is a distributive lattice. We prove that the condition is sufficient for P to be a distributive lattice.

From the subadditivity of χ, we know that P is a lattice with $\sup(a, b) = a \oplus b$. Suppose P is not distributive. Then P contains either a sublattice of type N_5 or of type M_3. Assume first that there exists a subset $N_5 = \{a, b, c, d, e\} \subseteq P$ so that

$$c < d, \ e = b \oplus c = b \oplus d \text{ and } a = \inf(b, c) = \inf(b, d).$$

(C0) guarantees the existence of an irreducible element $u \in J(\chi(d)) \setminus J(\chi(c))$. Because $b \odot d \leq a \leq c \leq d$, (C1) then implies $\chi_u(b \odot d) = 0$ and the supermodularity of χ yields

$$1 \geq \chi_u(b \oplus d) + \chi_u(b \odot d) \geq \chi_u(b) + \chi_u(d) \geq 1.$$

Hence we conclude $\chi_u(e) = 1$ and $\chi_u(b) = 0$, which however contradicts the subadditivity of χ_u:

$$\chi_u(e) = \chi_u(b \oplus c) \leq \chi_u(b) + \chi_u(c) = 0.$$

Therefore, we conclude that N_5 cannot occur in $(P, \leq)$.

Assume finally that P contains a subset $M_3 = \{a, b, c, d, e\}$ with the property

$$e = b \oplus c = b \oplus d = c \oplus d \quad \text{and} \quad a = \inf(b, c) = \inf(b, d) = \inf(c, d).$$

Choose some $u \in J(\chi(e)) \setminus J(\chi(b))$. Then the subadditivity of χ implies

$$\chi_u(c) = \chi_u(d) = 1.$$

The supermodularity then guarantees $\chi_u(c \odot d) = 1$. Hence $c \odot d \leq a < b < e$ yields a contradiction to the consecutive property (C1) with respect to u. It follows that also M_3 cannot occur in P.

$\diamond$

Corollary 2.1 ([?]) *Every pseudolattice with modular representation is a distributive lattice.*

$\diamond$

3 The Covering Problem

Let $\chi : P \to L$ be a representation of the ordered set in the lattice $\mathcal{B}(U) = (2^U, \subseteq)$ of subsets of the set U. The *covering problem* we consider here consists in the identification of a subset $C \subseteq P$ of minimal cardinality such that $\chi(C)$ covers all of U, *i.e.*,

(C) $\min |C|$ such that $U = \bigcup_{a \in C} \chi(a)$.

To avoid trivial cases, we assume throughout that the covering problem has a solution, *i.e.*

$$U = \bigcup_{a \in P} \chi(u).$$

The covering problem is dual to the *packing problem*

(C*) $\max |C^*|$ such that $\chi(a) \cap \chi(b) = \emptyset$ for all $a \neq b \in C^*$.

The *partition problem* consists in finding a minimal cover that is also a packing. A χ-partition of U does not necessarily exist. In the case of a pseudolattice P with modular representation, the greedy type algorithm of [2] provides an optimal partition solution if one exists. It is pointed out in [6] that Frank's [7] greedy-type algorithm may be used to solve the packing problem if the characteristic of the representation χ is submodular.

We treat the covering problem in the framework of linear programming. We formulate the covering problem as

$$\min \sum_{a \in P} x_a \quad \text{s.t.} \quad x_a \in \{0,1\} \quad \text{and} \quad \sum_{\chi(a) \ni u} x_a \geq 1 \text{ for all } u \in U, a \in P \tag{1}$$

and associate with it the dual problem

$$\max \sum_{u \in U} y_u \quad \text{s.t.} \quad y_u \in \{0,1\} \quad \text{and} \quad \sum_{u \in \chi(a)} y_u \leq 1 \text{ for all } u \in U, a \in P. \tag{2}$$

If we can find feasible solutions x for (1) and y for (2) such that

$$\sum_{a \in P} x_a \;\leq\; \sum_{u \in U} y_u, \tag{3}$$

it follows from the well-known duality theory of linear programming that x and y are optimal for the respective problems.

Note that (2) is a combinatorial *matching problem*: one seeks a maximal subset $X \subseteq U$ that contains from each $\chi(a)$ at most one representative. Hence it can in principle be solved with matching algorithms from combinatorial optimization. We want to show here that under additional assumptions a simple greedy-type algorithm exists for the covering problem. We will make the following assumptions on the ordered semiring $(P, \oplus, \odot)$:

(A1) $(P, \leq)$ is a lattice with $a \oplus b = \sup(a, b)$.

(A2) If a and b are lower neighbors of $a \oplus b$, then $\chi(a \oplus b) \subseteq \chi(a) \cup \chi(b)$.

(A3) The characteristic of $\chi : P \to \mathcal{B}(U)$ is supermodular.

Our requirements allow P to be still a more general structure than a distributive (pseudo)lattice. For example, the identity representation of the (generally non-distributive) system of closed sets of a convex geometry in the sense of [3] can be shown to satisfy (A1)-(A3).

3.1 The Monge Algorithm

We now present the Monge algorithm with the goal to compute a feasible solution for (1) in a straightforward manner. At every stage of the algorithm, the elements $u \in U$ will carry labels $c_u \in \mathbb{R}$. u is *covered* once its label is nonpositive (*i.e.*, $c_u \leq 0$). The algorithm will reduce the lattice P iteratively until all elements $u \in U$ are covered.

Let m be the maximal element of the lattice $(P, \leq)$ currently under consideration. Denote by $\ell(m)$ its collection of lower neighbors and compute the parameter

$$c^* = \min_{m' \in \ell(m)} \max\{c_u \mid u \in \chi(m) \setminus \chi(m')\} \quad \text{for all } m' \in \ell(m).$$

A pair (u, m^*) with $m^* \in \ell(m)$ and $u \in \chi(m) \setminus \chi(m^*)$ is a called a *Monge pair* if $c_u = c^*$. If $c^* \geq 0$, then m is said to be *active* and $u \in \chi(m) \setminus \chi(m^*)$ is the corresponding *representative*. With this terminology, the *Monge algorithm* is now easy to describe as the following iterative procedure:

(M_0) INITIALIZE: Set $x_a = 0$ for all $a \in P$ and label the $u \in U$ with $c_u = 1$. Then modify x iteratively as follows.

(M_1) Consider the maximal member $m \in P$ and select a Monge pair (u, m^*).

(M_2) Set $x_m = \max\{c^*, 0\}$ and subtract x_m from all c_v with $v \in \chi(m)$.

(M_3) Replace P by $P^* = \{a \in P \mid m \leq m^*\}$.

(M_4) IF $|P| \geq 2$ THEN GO TO (M_1). ELSE return x and END.

Proposition 3.1 *Assume (A1)-(A3). Then the Monge algorithm returns a feasible solution x for (1).*

Proof. It follows directly from the Monge algorithms that x has only $(0, 1)$-components. So x is feasible if and only if all elements $u \in U$ are covered when the Monge algorithm ends. Consider any currently not covered $u \in U$ (*i.e.* $c_u > 0$). If $u \in \chi(a)$ holds for some $a \in P^*$, then u will be covered at a later stage of the algorithm. If $u \in \chi(a)$ is only true for $a \in P \setminus P^*$, then supermodularity yields

$$\chi_u(m) = \chi_u(a \oplus m^*) \geq \chi_u(a) + \chi_u(m^*) - \chi_u(a \odot m^*) = \chi_u(a) = 1.$$

So u occurs in $\chi(m)$, *i.e.*, $u \in \chi(m) \setminus \chi(m^*)$ holds and the Monge algorithm implies $c^* \geq c_u > 0$. Hence u will be covered in the present iteration.

$\diamond$

Collect all the active elements m_j encountered in the course of the algorithm and the corresponding representatives $u_j \in \chi(m_j) \setminus \chi(m_j^*)$ into the *Monge chain* $\mathcal{M}$ and *Monge sequence* π, where

$$\mathcal{M} = \{m_1 < \ldots < m_k\} \quad \text{and} \quad \pi = u_1 \ldots u_k \quad \text{with } u_j \in \chi(m_j) \setminus \chi(m_j^*)$$

for $j = 1 \ldots k$ and let m_0 be the minimal element of P. Since the algorithm starts with $c \equiv 1$, m_k will be the maximal element of the lattice P. Considering the intervals

$$[m_{j-1}, m_j] = \{a \in P \mid m_{j-1} \leq a \leq a_j\} \quad (j = 1, \ldots, k),$$

the crucial technical observation is the following.

Lemma 3.1 *Consider any* $a \in [m_{j-1}, m_j]$. *Then either* $a = m_j$ *or* $u_j \notin \chi(a)$.

Proof. Suppose the Lemma is false and $m_{j-1} \leq a < m_j$ exhibits $u_j \in \chi(a)$. So $a \not\leq m_j^*$ holds and, by (C1), we may assume that a is a lower neighbor of m_j. Because the Monge algorithm selected m_j^* (and not a), there must be some element $v \in \chi(m_j) \setminus \chi(a)$ with $c_v \geq c^* \geq 0$.

Because of $m_{j-1} < a < m_j$, (C1) implies $v \notin \chi(m_{j-1})$. Hence v must have left the algorithm at some m_i with $v \in \chi(m_i) \setminus \chi(m_i^*)$ and $m_{j-1} < m_i \leq m_j$. $c_v \geq 0$ says that m_i is an active element. So we must have $m_i = m_j$ and $v \notin \chi(m_j^*)$. But now we have a contradiction to (A2): a and m_j^* are lower neighbors of m_j but $v \in \chi(m_j) \setminus \chi(a \oplus m_j^*)$.

$\diamond$

3.2 The Greedy Algorithm

Based on the Monge chain $\mathcal{M}$ and the Monge sequence π, the greedy algorithm constructs a *greedy vector* $y^\pi : U \to \mathbb{R}$ by modifying the components of the zero vector $y = \mathbf{0}$ iteratively as follows:

- $y^\pi_{u_1} = 1$.
- $y^\pi_{u_j} = 1 - \sum \{y^\pi_{u_i} \mid i < j, u_i \in \chi(m_j)\} \quad (j = 2, \ldots, k)$.

Obviously, all components of y^π are integer-valued. Moreover, since $u_i \notin \chi(m_j)$ if $i > j$, we observe

$$y^\pi(m_j) = \sum \{y^\pi_{u_i} \mid u_i \in \chi(m_j)\} = 1 \quad \text{for all } j = 1, \ldots, k.$$

To prove that y^π is a feasible solution for (2), we first show that y^π is a binary vector.

Lemma 3.2 $y^\pi \in \{0,1\}^{|U|}$.

Proof. It remains to show that no component of y^π is negative. The algorithm yields $y^\pi_{u_1} = 1$ by construction. Since each $u_i \in \chi(m_j)$ with $i < j$ must also lie in $\chi(m_{j-1})$, we furthermore find iteratively

$$\begin{aligned} y^\pi_{u_j} &= 1 - \sum \{y^\pi_{u_i} \mid i < j, u_i \in \chi(m_j)\} \\ &\geq 1 - \sum \{y^\pi_{u_i} \mid u_i \in \chi(m_{j-1})\} \\ &= 1 - y^\pi(m_{j-1}) \; = \; 0\,. \end{aligned}$$

$\diamond$

As the characteristic of $\chi : P \to \mathcal{B}(U)$ is supermodular, it follows that the non-negative vector y^π gives rise to a supermodular function on P *via*

$$y^\pi(a) = \sum \{y^\pi_u \mid u \in \chi(a)\}.$$

Recalling that y^π is constructed from the Monge chain $\mathcal{M} = \{m_1 < \ldots < m_k\}$, we next observe

Lemma 3.3 *For all* $m_j \in \mathcal{M}$ *and* $a \in P$*,*

$$\begin{array}{lcl} m_{j-1} \leq a \leq m_j & \Longrightarrow & \mathbf{y}^\pi(a) \leq 1 \quad (j = 1, \ldots, k) \\ a \geq m_k & \Longrightarrow & \mathbf{y}^\pi(a) \leq 1. \end{array}$$

Proof. Assume $m_{j-1} \leq a \leq m_j$. In the case $a = m_j$, we already know $y^\pi(a) = 1$. If $a < m_j$, then $u_j \notin \chi(a)$ by Lemma 3.1. By the consecutive property (C1), $u_i \in \chi(a)$ implies $u_i \in \chi(m_{j-1})$ for all i. So $y^\pi \geq 0$ yields

$$y^\pi(a) \leq y^\pi(m_{j-1}) = 1.$$

The case $a \geq m_k$ is analyzed the same way.

$\diamond$

Observe that the greedy vector y^π is feasible if and only if the submodular function $h(a) = 1 - y^\pi(a)$ is nonnegative for each $a \in P$.

Proposition 3.2 *Assume (A1)-(A3) and let $\mathcal{M} = \{m_1 < \ldots < m_k\}$ be a Monge chain with Monge sequence $\pi = u_1 \ldots u_k$. Then the greedy vector y^π is a feasible solution for (2).*

Proof. Suppose the Proposition is false and a is a minimal counterexample. So $a \not\leq m_1$ and $a \not\geq m_k$. Observe that $h(m_i) = 0$ holds for each $m_i \in \mathcal{M}$. If $a \not\leq m_k$, then

$$\begin{aligned} h(a) &\geq h(a \odot m_k) + h(a \oplus m_k) - h(m_k) \\ &= h(a \odot m_k) + h(a \oplus m_k) \ \geq \ 0 \ , \end{aligned}$$

as $a \oplus m_k \geq m_k$ and $a \odot m_k < a$ imply that both additive terms are nonnegative. Hence there must exist some $j > 1$ such that

$$a \not\leq m_{j-1} \quad \text{and} \quad a \leq m_j .$$

Noting $m_{j-1} \leq a \oplus m_{j-1} \leq m_j$, we then arrive at a contradiction in a similar way through the submodular expansion

$$\begin{aligned} h(a) &\geq h(a \odot m_{j-1}) + h(a \oplus m_{j-1}) - h(m_{j-1}) \\ &= h(a \odot m_{j-1}) + h(a \oplus m_{j-1}) \ \geq \ 0 \ , \end{aligned}$$

◇

Theorem 3.1 *Assume (A1)-(A3). Let $x : P \to \{0,1\}$ be the solution returned by the Monge algorithm, $\mathcal{M} = \{m_1 < \ldots < m_k\}$ be the Monge chain with Monge sequence $\pi = u_1 \ldots u_k$, and $y^\pi : U \to \{0,1\}$ be the corresponding greedy solution. Then x and y^π are optimal solutions to (1) and (2), respectively.*

Proof. As $\mathbf{x}$ and $\mathbf{y}^\pi$ are feasible solutions, by duality theory, it remains to show that

$$\sum_{a \in P} x_a \leq \sum_{u \in U} y_u^\pi .$$

Consider the cover $C = \{a \in P \mid x_a = 1\}$. Since $C \subseteq \mathcal{M}$, the construction of y^π guarantees for each $m_i \in C$ at least one $u \in U$ with $u \in \chi(m_i)$ and $y_u^\pi = 1$. In particular, $u \in \pi$.

On the other hand, if there exists some $u \in U$ with $u \in \chi(a) \cap \chi(b)$ for two cover elements $a \neq b$, the weight c_u becomes negative at some iteration of the Monge algorithm. Hence $u \notin \pi$ and $y_u^\pi = 0$.

It follows that for each $a \in C$ there exists exactly one $u \in U$ with $y_u^\pi = 1$. Therefore,

$$|C| = \sum_{a \in P} x_a = \sum_{u \in U} y_u^\pi .$$

◇

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