Cooperative Games with Lattice Structure

Ulrich Faigle [*] Britta Peis [†]

Zentrum für Angewandte Informatik (ZAIK)
Universität zu Köln
Weyertal 80
D-50931 Köln, Germany.

February 7, 2006

Abstract

A general model for cooperative games with possibly restricted and hierarchically ordered coalitions is introduced and shown to have lattice structure under quite general assumptions. Moreover, the core of games with lattice structure is investigated. Within a general framework that includes the model of classical cooperative games as a special case, it is proved algorithmically that monotone convex games have a non-empty core. Finally, the solution concept of the Shapley value is extended to the general class of cooperative games with restricted cooperation. It is shown that several generalizations of the Shapley value that have been proposed in the literature are subsumed in this model.

1 Introduction

The standard model of a cooperative game (N, v) involves a ground set N of players that are assumed to be free to form arbitrary coalitions $S \subseteq N$. The function v represents the value $v(S)$ generated when players cooperate in

[*]Email: faigle@zpr.uni-koeln.de
[†]Email: peis@zpr.uni-koeln.de

the coalition S . Often, however, cooperative games carry natural hierarchical structures which exclude certain coalitions from being feasible. It turns out that solution concepts like the core are also meaningful in the context of games with restricted cooperation (*cf.* [7]). In particular, games where the restriction on the formation of coalitions results from some order or permission structure on the set N have recently received attention (see, *e.g.*, [8, 14, 9, 10, 11]) and have been extended to so-called convex geometries and antimatroids (see [3, 1, 2]).

We introduce here a general model for cooperative games with dominance relations on the collection of feasible coalitions, where we do *not* need to start from a given relation on the set of players. We show that the collection $\mathcal{F}$ of feasible coalitions has a lattice structure (in the strict sense) under fairly mild assumptions and we proceed to define and analyze the core of more general lattice-structured cooperative games. There has been some recent progress in the algorithmic solution of linear programs over so-called lattice polyhedra ([13, 5, 12]), which were introduced by Hoffman and Schwartz [15] as generalizations of (poly-)matroids (*cf.* [21]) or, in game-theoretic language, convex cooperative games in the sense of Shapley [20]. For example, it has been realized that the ”greedy algorithm” is actually an algorithm that depends on a ”Monge algorithm” for a preprocessing phase (*cf.* [9, 11]). Frank [13] has generalized the latter idea to solve the corresponding minimization problem with monotone decreasing convex constraints. We show how this general two-phase algorithm allows us to compute core vectors in the present context (Section 4) in a far-reaching generalization of Shapley's [20] classical model.

We finally argue that also the solution concept of a ”Shapley value” admits a natural extension to the general framework here. We provide an algorithmic definition for a Shapley value for general cooperative games and point out how other generalized Shapley values for extensions of classical cooperative games to partially ordered sets and convex geometries are subsumed under our model.

2 Lattices of Feasible Coalitions

We assume throughout that the set N of players under consideration is finite and that a collection $\mathcal{F}$ of *feasible* coalitions $S \subseteq N$ is specified. The crucial aspect in our model is that $\mathcal{F}$ is allowed to carry a (partial) order $(\mathcal{F}, \leq)$ which might reflect a dominance relation among feasible coalitions. We assume the dominance relation to be *compatible* with the natural set-theoretic order on $\mathcal{F}$ in the following sense. For all feasible coalitions $S, T, U \in \mathcal{F}$, we assume

(C1) $S \leq U \leq T \implies S \cap T \subseteq U$.

Axiom (C1) stipulates that any player $p \in S$ that is part of a dominating feasible coalition T is also a member of every feasible coalition U that dominates S and is dominated by T.

We say that $(\mathcal{F}, \leq)$ is a *weakly submodular lattice* if (C1) holds and from among the members of any two feasible coalitions $S, T \in \mathcal{F}$ feasible coalitions U and V can be formed that dominate S and T, resp. are dominated by S and T:

(C2) $U \leq S, T \leq V$ for some $U, V \in \mathcal{F}(S \cup T)$,

where we have used the notation $\mathcal{F}(X) = \{S \in \mathcal{F} \mid S \subseteq X\}$ for any subset $X \subseteq N$. Directly from the definitions, we find

Lemma 2.1 *Assume that $(\mathcal{F}, \leq)$ is a weakly submodular lattice and let $X \subseteq N$ be arbitrary. Then either $\mathcal{F}(X) = \emptyset$ or $(\mathcal{F}(X), \leq)$ has a unique maximal and a unique minimal member.*

Proof. Suppose the Lemma is false and $\mathcal{F}(X)$ contains the maximal members $M_1 \neq M_2$, for example. Then (C2) guarantees some $M \in \mathcal{F}$ such that

$$M_1, M_2 \leq M \subseteq M_1 \cup M_2 \subseteq X \quad \text{and hence} \quad M \in \mathcal{F}(X),$$

which contradicts the maximality of M_1 and M_2.

$\diamond$

The particular choice $X = N$ in Lemma 2.1 shows that $(\mathcal{F}, \leq)$ itself has a unique maximal and a unique minimal element. We stress, however, that the "grand coalition" N need not be feasible or maximal in our model. For example, the maximal element M in $(\mathcal{F}, \leq)$ may very well be a coalition that consists of just one "dictator" that dominates all other coalitions.

Proposition 2.1 *Assume $N \in \mathcal{F}$. Then N is the maximal element of $(\mathcal{F}, \leq)$ if and only if the order $(\mathcal{F}, \leq)$ coincides with the set-theoretic order $(\mathcal{F}, \subseteq)$:*

$$S \leq T \iff S \subseteq T \quad \textit{for all } S, T \in \mathcal{F}.$$

Proof. If $(\mathcal{F}, \leq) = (\mathcal{F}, \subseteq)$, then N is obviously the maximal member of $\mathcal{F}$. Conversely, if N is maximal then any relation $S \leq T$ in $(\mathcal{F}, \leq)$ yields, in view of (C1):

$$S \leq T \leq N \quad \text{and hence} \quad S = S \cap N \subseteq T.$$

$\diamond$

2.1 Characteristic Functions

Relative to the collection $\mathcal{F}$ of feasible coalitions, a player $p \in N$ gives rise to the *characteristic function* $\chi_p : \mathcal{F} \to \{0, 1\}$, where

$$\chi_p(S) = 1 \quad \Longleftrightarrow \quad p \in S.$$

Let now $S, T, U, V \in \mathcal{F}$ be such that $U \leq S, T \leq V$. Property (C1) guarantees

$$p \in U \cap V \quad \Longrightarrow \quad p \in S \cap T.$$

In view of $U, V \subseteq S \cup T$, we obtain the so-called *submodular inequality*

$$\chi_p(U) + \chi_p(V) \leq \chi_p(S) + \chi_p(T). \tag{1}$$

This is the reason why we refer to a coalition structure $(\mathcal{F}, \leq)$ with the properties (C1) and (C2) as ”weakly submodular”.

Every vector $x \in \mathbb{R}^N$ defines a function $x : \mathcal{F} \to \mathbb{R}$ *via*

$$x(S) = \sum_{p \in S} x(p) = \sum_{p \in N} x(p)\chi_p(S) \quad \text{for all } S \in \mathcal{F}.$$

If $x \geq 0$, all coefficients $x(p)$ are nonnegative. Consequently, the function $x : \mathcal{F} \to \mathbb{R}_+$ is a nonnegative linear combination of the characteristic functions χ_p, which means that the submodular inequality is preserved:

Lemma 2.2 *If $\mathcal{F}$ is a weakly submodular lattice and $x \geq 0$, then we have for all coalitions $S, T \in \mathcal{F}$ and $U, V \in \mathcal{F}(S \cup T)$,*

$$U \leq S, T \leq V \quad \Longrightarrow \quad x(U) + x(V) \leq x(S) + x(T).$$

$\diamond$

REMARK. If $\mathcal{F}$ contains the empty set $\emptyset$ as its minimal element (which is the case in practically all of the examples below), one may always choose $U = \emptyset$ in (C2). Submodularity of the characteristic functions thus becomes equivalent with the seemingly weaker property

For all $S, T \in \mathcal{F}$, there is some $V \in \mathcal{F}(S \cup T)$ such that $S, T \leq V$.

2.2 Proper Lattices and Duality

A partially ordered set $(\mathcal{F}, \leq)$ is usually said to be a ”lattice” if $\mathcal{F}$ has a unique minimal element and for each $S, T \in \mathcal{F}$ there exists a *supremum*, *i.e.* a member $S \vee T \in \mathcal{F}$ such that

$$V \geq S, T \quad \Longrightarrow \quad V \geq S \vee T \quad \text{for all } V \in \mathcal{F}.$$

Let us call such structures *proper lattices* in our context. Note that the supremum $S \vee T$, if it exists, is unique. It is easy to see that a weakly submodular lattice, as introduced earlier, is not necessarily a ”lattice” in this strict sense. Lemma 2.1 yields a weak analog of the supremum property.

Example 2.1 *Let $N = \{a, b, c, d\}$ and consider the collection*

$$\mathcal{F} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}\}$$

with dominance relation

$$\emptyset < \{b\}, \{a, d\} < \{a, c\}, \{a, b\} < \{a\}.$$

Then $(\mathcal{F}, \leq)$ is a weakly submodular lattice. Yet, $(\mathcal{F}, \leq)$ is not a proper lattice since $S = \{a, d\}$ and $T = \{b\}$, for example, do not have a (unique) supremum $S \vee T$ relative to $(\mathcal{F}, \leq)$ (see Fig. 1).

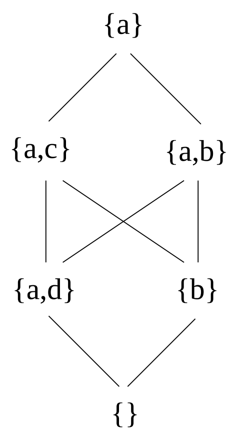

Figure 1: A weakly submodular but not proper lattice.

An additional condition, which is often satisfied in practice, namely

(C0) $T > S \quad \Longrightarrow \quad p \in T \setminus S$ for some $p \in N$,

guarantees a weakly submodular lattice to be also a proper lattice. (C0) stipulates that a coalition $T \in \mathcal{F}$ can only dominate the coalition $S \in \mathcal{F}$ if T comprises at least one player p that is not already part of S.

Theorem 2.1 *Assume that $(\mathcal{F}, \leq)$ satisfies (C0)-(C2) and let $S, T, V \in \mathcal{F}$ be coalitions such that $S, T \leq V \subseteq S \cup T$ holds. Then V is the unique minimal coalition in $\mathcal{F}$ that dominates both S and T (and hence the supremum $S \vee T$).*

Proof. Consider any feasible coalition $U \geq S, T$ in $\mathcal{F}$. We must show that $U \geq V$ holds. Suppose this is not the case. So (C2) guarantees a $W \in \mathcal{F}$ such that

$$S, T \ \leq \ U \ < \ W \subseteq U \cup V.$$

By (C0), we should be able to find some $w \in W \setminus U$. From (C1), we know that w lies neither in S nor in T, *i.e.* $w \notin S \cup T$. Because of $w \in U \cup V$, however, $w \notin U$ implies $w \in V \subseteq S \cup T$, which is a contradiction.

$\diamond$

REMARK (SINGULARITY OF DICTATORS). Suppose that the coalition structure $(\mathcal{F}, \leq)$ satisfies (C0) and that the maximal coalition M represents a dictator, *i.e.*, $M = \{d\}$. Then $d \in M \setminus S$ holds for all coalitions $S \neq M$, which means that d cannot be part of any other coalition.

Relative to $(\mathcal{F}, \leq)$ we define the *dual order* as the coalition structure $(\mathcal{F}, \leq^d)$, where for all $S, T \in \mathcal{F}$,

$$S \leq^d T \quad \Longleftrightarrow \quad T \leq S.$$

It is clear that property (C1) is preserved under (order) duality and so is property (C2). In other words: *$(\mathcal{F}, \leq)$ is a weakly submodular lattice if and only if $(\mathcal{F}, \leq^d)$ is a weakly submodular lattice.*

NOTA BENE. Property (C0) is generally *not* preserved under duality. Nevertheless, it is a well-known fact that $(\mathcal{F}, \leq)$ is a proper lattice if and only if $(\mathcal{F}, \leq^d)$ is a proper lattice.

2.3 Examples

The system $\mathcal{F}$ of sets is called a *ring family* if it is intersection and union closed:

$$S, T \in \mathcal{F} \quad \Longrightarrow \quad S \cap T, S \cup T \in \mathcal{F}.$$

If $\mathcal{F}$ is a ring family, then $(\mathcal{F}, \subseteq)$ satisfies (C0)-(C2) and hence is a proper lattice with supremum operation $S \vee T = S \cup T$. Moreover, the characteristic functions are *modular* with respect to intersection and union, *i.e.*, satisfy the submodular inequality with equality:

$$\chi_p(S \cap T) + \chi_p(S \cup T) = \chi_p(S) + \chi_p(T). \tag{2}$$

The standard model of cooperative games takes $\mathcal{F}$ as the collection of all subsets of N, which obviously constitutes a ring family.

2.3.1 Precedence Constraints

The standard cooperative game model can be considerably extended if the set N of players is allowed to carry a (partial) precedence ordering $(N, \leq)$ (*cf.* [8, 9, 10]). Depending on the interpretation, a coalition $S \subseteq N$ is now "feasible" if S is an *ideal* in $(N, \leq)$, *i.e.*, if

$$q \leq p \quad \Longrightarrow \quad q \in S \quad \text{for all } p \in S \text{ and } q \in N.$$

The collection $\mathcal{I}$ of ideals of $(N, \leq)$ is easily recognized as a ring family. $(\mathcal{I}, \subseteq)$ thus is a proper lattice with modular characteristic functions.

In a related (but different) model only those subsets $A \subseteq N$ are considered to form feasible coalitions in which no player $p \in A$ is dominated by another player $q \in A$. Those subsets are the so-called *antichains* of $(N, \leq)$. Let $\mathcal{A}$ be the collection of antichains of $(N, \leq)$. A natural partial order $(\mathcal{A}, \leq)$ is induced by $(\mathcal{I}, \subseteq)$ as follows. Associate with any $S \subseteq N$ the ideal

$$\overline{S} = \{q \in N \mid q \leq p \text{ for some } p \in S\}$$

and set

$$A \leq B \iff \overline{A} \subseteq \overline{B} \quad \text{for all } A, B \in \mathcal{A}.$$

It is straightforward to check that $(\mathcal{A}, \leq)$ enjoys the properties (C0) and (C1). Moreover, denoting by $\max(S)$ the set of maximal elements of the set S, the antichains $A \sqcup B = \max(A \cup B)$ and $A \cap B$ are contained in $A \cup B$ and satisfy $A \cap B \leq A, B \leq A \sqcup B$. So property (C2) holds as well, which tells us that the characteristics of $(\mathcal{A}, \leq)$ are submodular (*cf.* (1) above). In fact, the following modular equality

$$\chi_p(A \sqcap B) + \chi_p(A \sqcup B) = \chi_p(A) + \chi_p(B)$$

holds relative to the operation $A \sqcap B = \max(\overline{A} \cap \overline{B})$ in $(\mathcal{A}, \leq)$ (see [17]).

NOTA BENE. $A \cap B \subseteq A \sqcap B \subseteq A \cup B$ is true for all antichains $A, B \in \mathcal{A}$. However, $A \cap B \neq A \sqcap B$ may be quite possible.

2.3.2 Convex Geometries

As a further generalization of the coalition model for cooperative games so-called convex geometries are suggested in [1, 2, 3]. Here a *convex geometry* is an intersection closed collection $\mathcal{F}$ of subsets of N such that

(CG$_0$) $\emptyset, N \in \mathcal{F}$.

(CG$_1$) For all $S, T \in \mathcal{F}$ either $S \subseteq T$ or there exists some $p \in S \setminus T$ such that $S \setminus \{p\} \in \mathcal{F}$.

Such structures were introduced in [6] as a discrete analog of convex sets in Euclidean spaces. For example, the collection $\mathcal{I}$ of ideals relative to a precedence order $(N, \leq)$ satisfies (CG$_0$) and (CG$_1$) and thus is a convex geometry. The model of a convex geometry is more general, however. While $(\mathcal{F}, \subseteq)$ is a proper lattice with supremum operation

$$S \vee T = \bigcap\{U \in \mathcal{F} \mid (S \cup T) \subseteq U\},$$

$S \vee T \neq S \cup T$ may happen, which means that (C2) cannot hold. In fact, it is not difficult to check the validity of the following *supermodular* inequality:

$$\chi_p(S \cap T) + \chi_p(S \vee T) \;\geq\; \chi_p(S) + \chi_p(T),$$

which can be strict.

Assuming $(\mathcal{F}, \subseteq)$ to be a convex geometry, consider any feasible coalition $S \in \mathcal{F}$. Geometrically speaking, a player $p \in S$ is an *extreme point* of S if and only if $S \setminus \{p\} \in \mathcal{F}$. It is readily proved (by induction, for example) that a feasible set $S \in \mathcal{F}$ is uniquely determined by its set $\mathrm{ext}(S)$ of extreme points and one finds

$$S = \bigcap\{T \in \mathcal{F} \mid \mathrm{ext}(S) \subseteq T\}. \tag{3}$$

Generalizing the transition from ideals to antichains relative to precedence orders, one could now argue that $\mathrm{ext}(S)$ is the set of the truly relevant players in S. This would suggest to consider the collection

$$\mathcal{E} = \{\mathrm{ext}(S) \mid S \in \mathcal{F}\}$$

of relevant coalitions with the induced order

$$\mathrm{ext}(S) \leq \mathrm{ext}(T) \;\text{ in }\; (\mathcal{E}, \leq) \quad \Longleftrightarrow \quad S \subseteq T \;\text{ in }\; (\mathcal{F}, \subseteq).$$

Having noted that a convex geometry is *not* necessarily weakly submodular, the next result is perhaps a bit surprising.

Proposition 2.2 *If $(\mathcal{F}, \subseteq)$ is a convex geometry, then the coalition structure $(\mathcal{E}, \leq)$ has the properties (C0)-(C2) and hence is a lattice with submodular characteristics.*

Proof. (C0) is a direct consequence of (3). To see (C1), observe that the extreme point property is preserved when one passes to a smaller coalition, *i.e.* if $S, T \in \mathcal{F}$ are such that $S \subseteq T$ and $p \in S$ is an extreme point of T, then p is also an extreme point of S because

$$S \setminus \{p\} = S \cap (T \setminus \{p\}) \in \mathcal{F}.$$

Hence $S \subseteq U \subseteq T$ implies $\mathrm{ext}(S) \cap \mathrm{ext}(T) \subseteq \mathrm{ext}(U)$. To verify (C2) let $S, T \in \mathcal{F}$ be arbitrary and consider the set

$$V = \bigcap \{W \in \mathcal{F} \mid (\mathrm{ext}(S) \cup \mathrm{ext}(T)) \subseteq W\} \in \mathcal{F}.$$

Clearly, every $p \in \mathrm{ext}(V)$ must lie in $\mathrm{ext}(S) \cup \mathrm{ext}(T)$ as otherwise

$$\mathrm{ext}(S) \cup \mathrm{ext}(T)) \subseteq V \setminus \{p\} \in \mathcal{F}$$

would contradict the definition of V. So $\emptyset \in \mathcal{F}$ yields the desired property

$$\emptyset \;\leq\; \mathrm{ext}(S), \mathrm{ext}(T) \;\leq\; \mathrm{ext}(V) \;\subseteq\; \mathrm{ext}(S) \cup \mathrm{ext}(T).$$

◇

3 The Core of a Cooperative Game

Let $\mathcal{F}$ be the collection of feasible coalitions relative to the set N of players and assume throughout that the properties (C1) and (C2) hold relative to a given partial order $(\mathcal{F}, \leq)$. A *cooperative game* in our model is a triple $(N, \mathcal{F}, v)$, where $v : \mathcal{F} \to \mathbb{R}_+$ is some function that takes nonnegative real values. We interpret the parameter $v(S)$ as the *value* the feasible coalition S can generate. As usual, we assume that v is *normalized* in the sense

$$v(\emptyset) = 0 \quad \text{if } \emptyset \in \mathcal{F}.$$

3.1 The Extended Game and the Core

While an arbitrary subset $S \subseteq N$ of players need not form a feasible coalition in its own right (*i.e.*, $S \notin \mathcal{F}$), it is natural to assign to S the maximal possible value, non-overlapping feasible coalitions with players from S could jointly achieve. So we define

$$v^*(S) = \max\{\sum_{i=1}^{k} v(S_i) \mid S_i \in \mathcal{F}(S) \text{ and } S_i \cap S_j = \emptyset \text{ if } i \neq j\}.$$

Here we assume w.l.o.g. that the empty set is considered feasible, *i.e.*, $\emptyset \in \mathcal{F}$ holds. Otherwise, we add $\emptyset$ as the new minimal element to $\mathcal{F}$. So (N, v^*) is a classical cooperative game, where $v^*(S)$ is defined for all subsets $S \subseteq N$. We refer to (N, v^*) as the *extension* of the cooperative game $(N, \mathcal{F}, v)$. Recall that the core of (N, v^*) is the following set of nonnegative allocation vectors $x \in \mathbb{R}^N$:

$$\text{core}(v^*) = \{x \geq \mathbf{0} \mid x(N) = v^*(N), x(S) \geq v^*(S) \text{ for all } S \subseteq N\}$$

with the notation $x(S) = \sum_{p \in S} x(p)$.

Lemma 3.1 *Assume* $x \geq \mathbf{0}$. *Then* $x \in \text{core}(v^*)$ *holds if and only if*

$$x(N) = v^*(N) \quad \textit{and} \quad x(F) \geq v(F) \quad \textit{for all } F \in \mathcal{F}.$$

Proof. The condition is obviously necessary for $x \geq \mathbf{0}$ to lie in $\text{core}(v^*)$. It is also sufficient because the nonnegativity of x implies for any subset $S \subseteq N$ and collection of pairwise disjoint feasible coalitions $S_i \in \mathcal{F}(S)$:

$$x(S) \geq \sum_i x(S_i) \geq \sum_i v(S_i) \; .$$

$\diamond$

We refer to the parameter $v^* = v^*(N)$ as the *(total) value* of the cooperative game $(N, \mathcal{F}, v)$ and define its *core* as

$$\text{core}(v) = \{x \in \mathbb{R}^N_+ \mid \sum_{p \in N} x(p) = v^*, \; \sum_{p \in S} x(p) \geq v(S) \;\text{ for all } S \in \mathcal{F}\}.$$

Lemma 3.1 says that $\text{core}(v)$ is identical with $\text{core}(v^*)$, *i.e.*, the core of $(N, \mathcal{F}, v)$ coincides with the core of the extension (N, v^*).

We now establish the appropriate generalization of Shapley's [20] construction of core vectors for convex games to the present model.

3.2 Monotone Convex Games

The cooperative game $(N, \mathcal{F}, v)$ is said to be *monotone (increasing)* if

$$S \leq T \quad \Longrightarrow \quad v(S) \leq v(T) \quad \text{for all } S, T \in \mathcal{F}.$$

REMARK. v is monotone *decreasing* relative to $(\mathcal{F}, \leq)$ if v is monotone increasing relative to the order dual $(\mathcal{F}, \leq^d)$. Without loss of generality, we take the term "monotone" to mean monotone increasing in the following discussion. The analogous statements for monotone decreasing functions are obtained by simply dualizing the model.

The cooperative game $(N, \mathcal{F}, v)$ is said to be *convex* if

(Cv) For all $S, T \in \mathcal{F}$ with $v(S) > 0$ and $v(T) > 0$, there exist coalitions $U, V \in \mathcal{F}(S \cup T)$ such that

$$U \leq S, T \leq V \quad \text{and} \quad v(U) + v(V) \geq v(S) + v(T).$$

The general form of Shapley's theorem now becomes:

Theorem 3.1 *Let* $(\mathcal{F}, \leq)$ *be a weakly submodular lattice of coalitions and* $(N, \mathcal{F}, v)$ *a monotone convex cooperative game. Then* $\text{core}(v) \neq \emptyset$*. Moreover, a core vector can be constructed with a greedy algorithm.*

It turns out that Frank's [13] greedy algorithm can be used to construct core vectors for monotone convex games. We will give a simplified proof of its correctness and of Theorem 3.1 in the next section. Shapley [20] considers the situation where $\mathcal{F}$ consists of all subsets of N and is ordered by set-theoretic containment. Convexity there is meant to satisfy the inequality

$$v(S \cap T) + v(S \cup T) \geq v(S) + v(T) \quad \text{for all } S, T \subseteq N.$$

Shapley's convex function v is assumed to be monotone increasing. In this situation, one obviously has

$$v^* = v(N)$$

and retrieves the classical results as special cases within our model.

3.2.1 Extensions of Monotone Convex Games

A direct consequence of the definition is the observation that the extension (N, v^*) of an arbitrary cooperative game $(N, \mathcal{F}, v)$ is monotone and subadditive, *i.e.*,

$$\begin{array}{ccccc} S \subseteq T & \Longrightarrow & v^*(S) & \leq & v^*(T) \\ S \cap T = \emptyset & \Longrightarrow & v^*(S) + v^*(T) & \leq & v^*(S \cup T). \end{array}$$

One might suspect that the extension (N, v^*) of a convex game $(N, \mathcal{F}, v)$ is a convex game in the classical sense (and hence a core vector could be constructed with Shapley's [20] procedure). However, this is not necessarily the case as the next example shows.

Example 3.1 *Let* $N = \{a, b, c\}$ *and* $\mathcal{F} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. *Assume the order* $(\mathcal{F}, \leq)$ *to be given by (see Fig. 2)*

$$\{a, b\} < \{a, c\} \quad \textit{and} \quad S \leq T \;\; \textit{if} \;\; S \subseteq T.$$

The function $v : \mathcal{F} \to \mathbb{R}_+$ *with* $v(\emptyset) = v(a) = v(b) = v(c) = 0$ *and* $v(a, b) = v(a, c) = 1$ *is monotone and convex. The extension* v^* *yields*

$$v^*(N) + v^*(a) = 1 \;<\; 2 = v^*(a, b) + v^*(a, c).$$

So the condition $v^*(S \cup T) + v^*(S \cap T) \geq v^*(S) + v^*(T)$ *is not satisfied.*

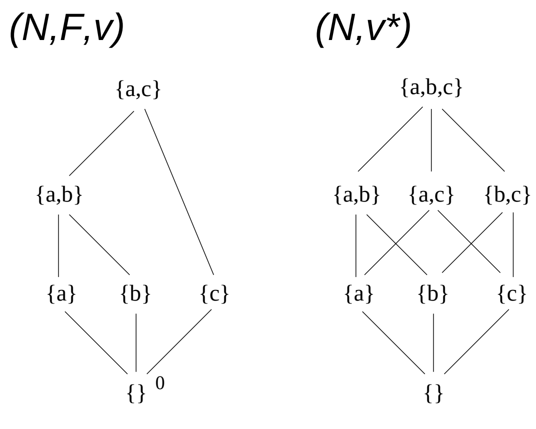

Figure 2: The extension of a coalition structure

A further problem we have not yet commented on arises from the computation of the values $v^*(S)$ of the extended game. Here, we assume that a procedure (or "oracle" v) is available that allows us to evaluate feasible coalitions:

$$S \in \mathcal{F} \;\rightarrow\; \boxed{\;v\;} \;\rightarrow\; v(S) \; .$$

There is no generally efficient method known that would allow us to evaluate $v^*(S)$ for arbitrary subsets $S \subseteq N$ on the basis of an oracle for v. The algorithms in the next section show, however, that the situation is much better for monotone convex games. Given an oracle for v and an oracle for finding a maximal feasible coalition M in a subcollection $\mathcal{F}(X) \subseteq \mathcal{F}$, we can efficiently evaluate the parameters $v^*(S)$ and construct allocation vectors in $\text{core}(v) = \text{core}(v^*)$.

4 The Greedy Algorithm for Convex Games

We assume throughout that $(\mathcal{F}, \leq)$ is weakly submodular. We now compute a heuristic value for v^* in a straightforward fashion and then show that the computation is exact under the hypothesis of Theorem 3.1.

4.1 The Monge Algorithm

The algorithm works with labels $w(p) \in \{0, 1\}$ for the players $p \in N$. In each iteration, labels are possibly reduced.

Assume that v is monotone and assign initially the label $w(p) = 1$ to each of the players $p \in N$. The algorithm works as follows:

(M_0) Let M be the maximal coalition of $\mathcal{F}$ and assume $v(M) > 0$. Choose some player $p^* \in M$ as a representative. Reduce the labels of all $p \in M$ by $w(p^*)$. Set $\Pi = \{p^*\}$ and replace $\mathcal{F}$ by

$$\mathcal{F}^* = \{S \in \mathcal{F} \mid v(S) > 0, p^* \notin S\}.$$

(M_1) Let M be the maximal coalition of $\mathcal{F}$ and choose some player $p^* \in M$ with smallest label $w(p^*)$ as a representative. Reduce the labels of all $p \in M$ by $w(p^*)$ and update:
$\Pi \;\leftarrow\; \Pi \cup \{p^*\}, \; \mathcal{F} \;\leftarrow\; \mathcal{F}^* = \{S \in \mathcal{F} \mid v(S) > 0, p^* \notin S\}.$

(M_2) Iterate (M_1) until $\mathcal{F} = \emptyset$.

Since $(\mathcal{F}, \leq)$ is weakly submodular and v monotone increasing, the reduced set $\mathcal{F}^* = \{S \in \mathcal{F} \mid v(S) > 0, p^* \notin S\}$ admits indeed a unique maximal coalition (*cf.* Lemma 2.1). The maximal coalitions considered in the course of the algorithm form a chain

$$\mathcal{M} = \{M_1 < \ldots < M_k\} \subseteq \mathcal{F},$$

where M_k is the maximal coalition in the original coalition structure $\mathcal{F}$. We refer to $\mathcal{M}$ as a *Monge chain*. The corresponding representatives $p_j \in M_j$ constitute the set $\Pi = \{p_1, \ldots, p_k\}$. The collection

$$\mathcal{P} = \{M_j \in \mid w(p_j) = 1\}$$

of those coalitions where representatives p^* with the label $w(p^*) = 1$ were chosen must be pairwise disjoint since the algorithm always chooses a representative with a currently *minimal* possible label. Moreover, each representative $p_i \in \Pi$ belongs to some $M_j \in \mathcal{P}$. So we find

$$\Pi \subseteq \bigcup_{M_j \in \mathcal{P}} M_j \quad \text{and} \quad v(\mathcal{P}) = \sum_{M_j \in \mathcal{P}} v(M_j) \leq v^*. \tag{4}$$

An important (technical) observation is the following:

Lemma 4.1 *If $S \in \mathcal{F}$ satisfies $S < M_j$ and $S \not\leq M_{j-1}$, then $p_j \in S$.*

Proof. Assume that the Monge algorithm has already constructed the set $\Pi = \{p_k, \ldots, p_{j+1}\}$ of representatives and M_j is the largest coalition with $M_j \cap \Pi = \emptyset$. Because of the consecutive property (C1), we know from $S < M_j < \ldots < M_k$ that $S \cap \Pi = \emptyset$ must hold.

By (C2), there exists some $V \in \mathcal{F}(M_{j-1} \cup S)$ which strictly dominates M_{j-1}. Now $p_j \notin S$ would imply $p_j \notin V$, so that the Monge algorithm should have chosen V instead of M_{j-1} in the next iteration.

$\diamond$

4.2 The Greedy Algorithm

Assume that v is monotone increasing and $\mathcal{M} = \{M_1 < \ldots < M_k\}$ the Monge chain with set $\Pi = \{p_1, \ldots, p_k\}$ of representatives constructed by the algorithm in the previous section. Given these data, the *greedy algorithm* constructs an allocation vector $x \in \mathbb{R}^N$ by an iterative procedure:

(G_0) Let $x(p) = 0$ for all $p \in N$ and modify x iteratively as follows.

(G$_1$) $x(p_1) = v(M_1)$ and
$x(p_j) = v(M_j) - \sum\{x(p_i) \mid i < j, p_i \in M_j\} \quad (j = 2, \ldots, k)$.

Since no representative $p_i \in M_i$ with $i > j$ belongs to M_j (by the algorithmic construction of M_j), we immediately find for the resulting greedy vector x:

$$x(M_j) = \sum_{p \in M_j} x_p = v(M_j) \quad \text{for all } j = 1, \ldots, k.$$

Lemma 4.2 $x(p) \geq 0$ *for all* $p \in N$ *and* $\sum_{p \in N} x(p) = v(\mathcal{P})$.

Proof. If the greedy algorithm does not modify the component $x(p)$, we have $x(p) = 0$. Now $x(p_1) = v(M_1) \geq 0$ follows directly from (G$_1$) and the nonnegativity of v. The monotone property of v furthermore yields inductively:

$$\begin{aligned} x(p_j) &= v(M_j) - \sum\{x(p_i) \mid i < j, p_i \in M_j\} \\ &\geq v(M_j) - \sum\{x(p_i) \mid p_i \in M_{j-1}\} \\ &= v(M_j) - v(M_{j-1}) \geq 0. \end{aligned}$$

From (4) we know $\Pi \subseteq \bigcup \mathcal{P}$. Since the coalitions in $\mathcal{P}$ are pairwise disjoint, we thus find

$$\sum_{p \in N} x(p) = \sum_{M_j \in \mathcal{P}} \sum_{p \in M_j} x(p) = \sum_{M_j \in \mathcal{P}} v(M_j) = v(\mathcal{P}).$$

$\diamond$

Lemma 4.3 *If* v *is monotone increasing and convex, then*

$$x(S) = \sum_{p \in S} x(p) \geq v(S) \quad \textit{for all } S \in \mathcal{F}.$$

Proof. We already know $x(S) = v(S)$ if $S \in \mathcal{M}$. Consider now an arbitrary $S \in \mathcal{F} \setminus \mathcal{M}$ and suppose that S were a minimal counterexample to the claim of the Lemma. So $v(S) > 0$ must hold (otherwise $x(S) \geq 0 = v(S)$ follows trivially). If $S < M_1$, the Monge algorithm implies $v(S) \leq 0$, which would contradict the choice of S as a counterexample.

Let $j \leq k$ be the smallest index such that $S < M_j$. If $M_{j-1} < S < M_j$ for some index j. Then Lemma 4.1 guarantees $p_j \in S$. Moreover, (C1) implies

$$p_i \in S \quad \text{if } i < j \text{ and } p_i \in M_j.$$

Hence we deduce from $x \geq 0$ a contradiction to our choice of S:

$$x(S) \geq x(M_j) = v(M_j) \geq v(S).$$

In the remaining case $S \not\geq M_{j-1}$, the convexity of v guarantees coalitions $U, V \in \mathcal{F}(S \cup M_{j-1})$ with

$$U < S, M_{j-1} < V \quad \text{and} \quad v(U) + v(V) \geq v(S) + v(M_{j-1}).$$

(C1) implies $V \cap \{p_{j+1}, \ldots, p_k\} = \emptyset$. So $M_{j-1} < V \leq M_j$ holds, which guarantees $x(V) \geq v(V)$ by the preceding argument. Moreover, $x(U) \geq v(U)$ is implied by the choice of S as a minimal counterexample. So the submodularity of the nonnegative vector $x \geq 0$ yields the contradiction

$$\begin{aligned} x(S) &\geq x(U) + x(V) - x(M_{j-1}) \\ &\geq v(U) + v(V) - v(M_{j-1}) \;\geq\; v(S). \end{aligned}$$

◇

Recalling the value $v(\mathcal{P}) \leq v^*$, computed in the previous section from the Monge algorithm, we can now finish the proof of Theorem 3.1.

Lemma 4.4 *Let v be convex and monotone increasing and x the greedy vector computed from the Monge chain $\mathcal{M}$ with representative set Π. Then*

$$v^* = v(\mathcal{P}) \quad \textit{and} \quad x \in \text{core}(v).$$

Proof. In view of the preceding Lemmas, it suffices to establish the inequality $v^* \leq v(\mathcal{P})$. So let $\mathcal{P}' = \{S'_1, \ldots, S'_\ell\}$ be a v-optimal collection of pairwise disjoint coalitions S'_i. Then we find

$$v^* = v(\mathcal{P}') = \sum_{i=1}^{\ell} v(S'_i) \leq \sum_{i=1}^{\ell} x(S'_i) \leq \sum_{p \in N} x(p) = v(\mathcal{P}) \leq v^*.$$

◇

REMARK. Frank [13] proves that his algorithm actually solves the weighted optimization problem

$$\min_{x \geq 0} \sum_{p \in N} w(p)x(p) \quad \text{s.t. } \textstyle\sum_{p \in S} x(p) \geq v(S) \;\text{ for all } S \in \mathcal{F}$$

for arbitrary weights $w(p) \geq 0$ and convex decreasing v. The same can be shown for our version of the algorithm.

5 Cost Games

Assume that $(N, \mathcal{F}, c)$ is a *cooperative cost game*, where $c : \mathcal{F} \to \mathbb{R}_+$ describes the costs $c(S)$ feasible coalitions S generate. In the classical situation, where $\mathcal{F}$ comprises all subsets $S \subseteq N$, it is well-known that each set of vectors that occurs as the core of a cooperative cost game also occurs as the core of a cooperative value game. So structural investigations into the classical core do not need to distinguish between "value games" and "cost games".

In the present framework, this is no longer the case. In order to define the core of the cost game $(N, \mathcal{F}, c)$, we introduce the *total cost* as the parameter

$$c^* = \min\{\sum_{i=1}^{\ell} c(S_i) \mid N \subseteq \bigcup_{i=1}^{\ell} S_i, S_i \in \mathcal{F}\}$$

and consider the corresponding nonnegative core-type allocations:

$$\text{core}(c) = \{x \in \mathbb{R}^N_+ \mid \sum_{p \in N} x(p) = c^*, \sum_{p \in S} x(s) \leq c(S) \text{ for all } S \in \mathcal{F}\}.$$

If the characteristic functions satisfy a modular equality and the cost function c is submodular relative to that modular equality, *i.e.*,

$$c(U) + c(V) \leq c(S) + c(T)$$

holds for suitable $U, V \in \mathcal{F}(S \cup T)$, an analogue of Theorem 3.1 can be established on the basis of the greedy algorithm of [12]. We will not go into details here.

6 The Shapley Value

In the classical model, the Shapley value tries to assess the average marginal value of a player (*cf.* [19], [18]). In our present model $(N, \mathcal{F}, v)$ of a cooperative game, however, the notion of the "marginal value" of a player needs some further clarification.

To this end, we consider an *elimination sequence* for N relative to the coalition structure $(\mathcal{F}, \leq)$, which is a sequence $\pi = p_k p_{k-1} \ldots p_1$ of players such that

$$p_j \in S_j \quad (j = k, k-1, \ldots, 1), \tag{5}$$

where S_k is the maximal coalition of $\mathcal{F} = \mathcal{F}(N)$ and S_j is the maximal coalition of

$$\mathcal{F}(N \setminus \{p_k, \ldots, p_{j+1}\}) = \{S \in \mathcal{F} \mid S \cap \{p_k, \ldots, p_{j+1}\} = \emptyset\}.$$

It is natural to assess the marginal values $\partial^\pi(p)$ of the players $p \in N$ relative to the elimination sequence $\pi = p_k \ldots p_1$ in the following fashion:

(ES) $\partial^\pi(p_1) = v(S_1)$
$\partial^\pi(p_j) = v(S_j) - \sum\{\partial^\pi(p_i) \mid i < j, p_i \in S\} \quad (j = 2, \ldots, k)$
$\partial^\pi(p) \ = \ 0$ otherwise.

(ES) guarantees that the value of each coalition S_j associated with π is the sum of the corresponding marginal values:

$$v(S_j) = \sum_{p \in S_j} \partial^\pi(p) \quad (j = 1, \ldots, k).$$

As the *Shapley value* $\Phi(v)$ of the cooperative game $(N, \mathcal{F}, v)$ we propose the allocation vector that assigns to each player its average marginal elimination value. Letting Π be the collection of all elimination sequences, we thus have

$$\Phi_p(v) = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} \partial^\pi(p) \quad \text{for all } p \in N. \tag{6}$$

Theorem 6.1 *Assume that* $(\mathcal{F}, \leq)$ *is a weakly submodular lattice and that the cooperative game* $(N, \mathcal{F}, v)$ *is convex and monotone. Then* $\Phi(v) \in$ core(v).

Proof. Under the hypotheses of the Theorem, the elimination sequences are exactly the Monge sequences and the marginal elimination vectors are the associated greedy vectors (see Section 4). Hence each of these lies in core(v). So $\Phi(v)$ is a convex combination of core vectors and thus a member of the convex polyhedron core(v) as well.

$\diamond$

If $\mathcal{F}$ consists of all subsets $S \subseteq N$, an elimination sequence relative to $(\mathcal{F}, \subseteq)$ is just a permutation $\pi = p_n p_{n-1} \ldots p_1$ of the ground set N. In this case, the Shapley value given by (6) coincides with classical Shapley value. It also extends other models for the Shapley value that have been proposed in the literature.

6.0.1 Precedence Constraints

As in Section 2.3.1, assume that a precedence structure $(N, \leq)$ on the set of players is given. In the hierachical model of [8] the collections of maximal elements of the ideal of $(N \ \leq)$, *i.e.* the antichains of $(N, \leq)$, form the relevant coalitions. The maximal coalition in the corresponding coalition structure $(\mathcal{A}, \leq)$ consists of the maximal elements of N. Let $p_n \in N$ be one of them. Then the maximal coalition in $\mathcal{A}$ that does not contain p_n is the set $\max(N \setminus \{p_n\})$ *etc.* Hence the elimination sequences are exactly the *linear extensions* of $(N, \leq)$, *i.e.*, those permutations $\pi = p_n p_{n-1} \ldots p_1$ of N with the property

$$p_j \text{ is a maximal element of } N \setminus \{p_n, \ldots, p_{j+1}\} \quad (j = n, \ldots, 1).$$

A Shapley value is introduced axiomatically in [8]) and it is shown that its computation amounts to (6) relative to the set of linear extensions of $(N, \leq)$. Hence (6) also generalizes the Shapley value of cooperative games with precedence constraints on the players.

6.0.2 Convex Geometries

Marginal operators and Shapley values are studied in [2] for cooperative games $(N, \mathcal{F}, v)$, where $\mathcal{F}$ is a convex geometry (*cf.* Section 2.3.2). So each non-empty coalition $S \in \mathcal{F}$ contains a feasible coalition $T \subset S$ with $|S \setminus T| = 1$. The Shapley value suggested in [1] is computed according to the scheme (6) relative to the permutations $\pi = p_n p_{n-1} \ldots, p_1$ of N with the property

$$N \setminus \{p_n \ldots p_{j+1}\} = \{p_1, \ldots, p_j\} \in \mathcal{F} \quad (j = n, \ldots, 1). \tag{7}$$

Every such permutation π is also an elimination sequence in the sense of (5). However, *not every* elimination sequence is of type (7). Hence our approach will not necessarily result in the Shapley value of [2]. It turns out that the dilemma is easily remedied by clarifying the model of games on convex geometries.

Basing the definition of the marginal values of the players exclusively on sequences of type (7) says implicitly that in a coalition $S \in \mathcal{F}$ *only* those players are considered relevant and valuable that correspond to extreme points in the geometric interpretation. Hence it appears appropriate to model the cooperative game in question not on $\mathcal{F}$ but on the collection

$$\text{ext}(\mathcal{F}) = \{\text{ext}(S) \mid S \in \mathcal{F}\}$$

with the induced order relation $(\mathrm{ext}(\mathcal{F}), \leq)$ as in Section 2.3.2. It is straightforward to check that the sequences of type (7) are *precisely* the elimination sequences relative to $(\mathrm{ext}(\mathcal{F}), \leq)$ in the sense (5).

Final Remark. Note that our approach to the Shapley value does not require $(\mathcal{F}, \leq)$ to be a lattice. It thus provides a solution concept for very general cooperative games with (possibly) restricted collections of feasible coalitions. Classes of combinatorial structures with particularly "well-behaved" elimination sequences are, *e.g.*, greedoids (*cf.* [16]), of which matroids are special cases.

References

[1] E. Algaba, J.M Bilbao, R. van den Brink and A. Jiménez-Losada: *Cooperative games on antimatroids*, Discr. Mathematics 282 (2004), 1-15.

[2] J.M. Bilbao, N. Jiménez, E. Lebrón and J.J López: *The marginal operators for games on convex geometries*, to appear: Intern. Game Theory Review 8 (2006).

[3] J.M. Bilbao, E. Lebrón and N. Jiménez: *The core of games on convex geometries*, Europ. J. Operational Research 119 (1999), 365-372.

[4] J. Derks and R.P. Gilles: *Hierarchical organization structures and constraints in coalition formation*, Intern. J. Game Theory 24 (1995), 147-163.

[5] B.L. Dietrich and A.J. Hoffman: *On greedy algorithms, partially ordered sets, and submodular functions.* IBM J. Res. & Dev. 47 (2003), 25-30.

[6] P.H. Edelman and R.E. Jamison: *The theory of convex geometries*, Geometriae Dedicata 19 (1985), 247-270.

[7] U. Faigle: *Cores of games with restricted cooperation.* Methods and Models of Operations Research 33 (1989), 405-422.

[8] U. Faigle and W. Kern: *The Shapley value for cooperative games under precedence constraints.* Intern. J. Game Theory 21 (1992), 249-266.

[9] U. Faigle and W. Kern: *Submodular linear programs on forests.* Math. Programming 72 (1996), 195-206.

[10] U. Faigle and W. Kern: *On the core of ordered submodular cost games.* Math. Programming 87 (2000), 483-489.

[11] U. Faigle and W. Kern: *An order-theoretic framework for the greedy algorithm with applications to the core and Weber set of cooperative games.* Order 17 (2000), 353-375.

[12] U. Faigle and B. Peis: *Note on pseudolattices, lattices and submodular linear programs.* Preprint, ZAIK, University of Cologne, 2006.

[13] A. Frank: *Increasing the rooted-connectivity of a digraph by one.* Math. Programming 84 (1999), 565-576.

[14] R.P. Gilles, G. Owen and R. van den Brink: *Games with permission structures: the conjunctive approach.* Intern. J. Game Theory 20 (1992), 277-293.

[15] A.J. Hoffman and D.E. Schwartz: *On lattice polyhedra*, in: *Proc. 5th Hungarian Conference in Combinatorics*, A. Hajnal and V.T. Sós eds., North-Holland, Amsterdam, 1978, 593-598.

[16] B. Korte, L. Lóvász and R. Schrader, *Greedoids*, Springer-Verlag, Heidelberg, 1991.

[17] U. Krüger: *Structural aspects of ordered polymatroids.* Discr. Appl. Math. 99 (2000), 125-148.

[18] A.E. Roth: *The Shapley value.* In: *Essays in Honor of L.S. Shapley*, A.E. Roth ed., Cambridge University Press, 1988.

[19] L.S. Shapley: *A value for n-person games.* In: *Contributions to the Theory of Games*, H.W. Kuhn and A.W. Tucker eds., Ann. Math. Studies 28, Princeton University Press, 1953, 307-317.

[20] L.S. Shapley: *Cores of convex games.* Intern. J. Game Theory 1 (1971), 12-26.

[21] D.J.A. Welsh, *Matroid Theory*, Academic Press, London, 1976.