On a relation between the domination number and a strongly connected bidirection of an undirected graph

Martin Lätsch, Britta Peis

Center for Applied Computer Science Cologne
Universität zu Köln, Weyertal 80, D-50931 Köln

Abstract

As a generalization of directed and undirected graphs, Edmonds and Johnson [6] introduced bidirected graphs. A bidirected graph is a graph each arc of which has either two positive end-vertices (tails), two negative end-vertices (heads), or one positive end-vertex (tail) and one negative end-vertex (head). We extend the notion of directed paths, distance, diameter and strong connectivity from directed to bidirected graphs and characterize those undirected graphs that allow a strongly connected bidirection. Considering the problem of finding the minimum diameter of all strongly connected bidirections of a given undirected graph, we generalize a result of Fomin et al. [1] about directed graphs and obtain an upper bound for the minimum diameter which depends on the minimum size of a dominating set and the number of bridges in the undirected graph.

Key words: bidirected graphs, domination number, diameter, strong connectivity

1 Introduction

A *direction* or *orientation* $\vec{G}$ of an undirected graph $G = (V, E)$ is an assignment of the edges such that each edge has exactly one positive end-vertex (the tail) and one negative end-vertex (the head). The *distance* $d_{\vec{G}}(u, v)$ in a directed graph $\vec{G}$ denotes the length of a shortest directed path from vertex u to vertex v. The undirected distance $d_G(u, v)$ is the length of a shortest undirected path between u and v in G. The *diameter* $diam(\vec{G})$ of a directed graph

Email address: laetsch@zpr.uni-koeln.de, peis@zpr.uni-koeln.de (Martin Lätsch, Britta Peis).

$\vec{G}$ is the maximal distance between two vertices. We call $\vec{G}$ *strongly connected* if its diameter is finite. Note that it is an $\mathcal{NP}$-hard problem to determine a direction of minimal diameter [5]. At least, it is an easy task to decide whether a graph G admits a strongly connected bidirection at all:

Given a connected undirected graph G, edge $e \in E$ is called a *bridge* if $G \setminus e$ is not connected. In 1939, Robbins proved:

Theorem 1 (Compare Robbins [3]) *An undirected graph G allows a strongly connected direction if and only if G is connected and bridgeless.*

Chung et al. provided a linear-time algorithm for testing whether a graph has a strong direction and finding one if it does [4].

Fomin et al. [1] discovered an upper bound for the minimal diameter of all directions of a connected bridgeless graph G which depends on the *domination number* $\gamma(G)$, defined as follows:

A vertex set $D \subseteq V(G)$ of a graph G is said to be a *dominating set* if for any vertex $v \in V(G) \setminus D$ there exists at least one (undirected) edge (w, v) with $w \in D$. The minimal cardinality of a dominating set $\gamma(G)$ is called the *domination number*. In case the vertex-set of a subgraph $G_D \subseteq G$ is a dominating set of G, we say that G_D *dominates* G. Fomin et al. [1] proved:

Theorem 2 (Compare Theorem 3 in [1]) *Every connected bridgeless graph G has a direction $\vec{G}$ such that $diam(\vec{G}) \leq 5\gamma(G) - 1$.*

In this article we will generalize the two results above to bidirected graphs. Bidirected graphs were investigated by Edmonds and Johnson [6] as a generalization of directed and undirected graphs to illustrate a generalized matching problem. A bidirected graph $\bar{G}$ is a graph together with an assignment of the edges such that for each edge the two end-vertices are either both positive (tails), both negative (heads) or one end-vertex is positive (a head) and one end-vertex is negative (a tail). Bidirected graphs are closely related to signed graphs which were extensively studied by Zaslavsky [7], [8].

We will extend the concept of directed paths, distance, diameter and strong connectivity from directed to bidirected graphs and, as a generalization of Robbin's result, show that G allows a strongly connected bidirection if and only if either G consists of only one vertex, or G is connected and every vertex has degree at least two.

Let $\mathcal{B}(G)$ denote the set of bridges in G and define

$$b(G) = \begin{cases} 0 & : \quad G \text{ bridgeless} \\ 1 & : \quad \text{otherwise} \end{cases}$$

to indicate whether G is bridgeless or not.

In Section 3 we generalize the result of Fomin et al. and show that any connected graph with minimal degree at least two admits a bidirection $\bar{G}$ such that

$$diam(\bar{G}) \leq \min\{2|\mathcal{B}(G)| + 2b(G) + 5\gamma(G) - 1,\ 6\gamma(G) + 3\}.$$

We provide constructive proofs of each of these two upper bounds. The bidirection $\bar{G}$ constructed in the proof of the first upper bound assures to be a common direction in case G is bridgeless. Whereas the bidirection constructed in the second proof might consist of edges with two tails or two heads even if G is bridgeless.

2 Characterization of graphs that allow a strongly connected bidirection

To distinguish between undirected and bidirected edges, we call bidirected edges “arcs”. Given a bidirected graph $\bar{G} = (V, A)$ let G denote the underlying undirected graph. In this article we only consider the case where G has neither loops nor multiple edges.

If arc a has a positive (negative) end-vertex u, we say that a is *positively (negatively) incident* to u. If two arcs a and a' are, respectively, positively and negatively incident to a common node u, we say that a and a' are *oppositely incident* to u.

Definition 3 *A* bidirected path *in $\bar{G} = (V, A)$ is an alternate sequence $P = (v_0, a_1, v_1, a_2, ..., a_k, v_k)$ of vertices $v_i (i = 0, ..., k)$ and arcs $a_i (i = 1, .., k)$ for any integer $k \geq 1$ such that a_1 is positively incident to v_0, a_k is negatively incident to v_k, and for each $i \in \{1, .., k-1\}$ the arcs a_i and a_{i+1} are oppositely incident to v_i.*

Note that we allow repetitions of arcs in a bidirected path. The *length* of a path is the number of arcs, i.e. $|P| = k$.

We define the *distance* $d_{\bar{G}}(u, v)$ between two vertices to be the length of a shortest bidirected path in $\bar{G}$ starting in u and ending in v. The *diameter*

$diam(\bar{G})$ denotes the maximal distance between two vertices in $\bar{G}$.

Definition 4 *A bidirected graph $\bar{G}$ is strongly connected if its diameter is finite.*

While any strongly connected direction is a strongly connected bidirection, there exist graphs that allow a strongly connected bidirection but no strongly directed direction. (See for example Figure 1.)

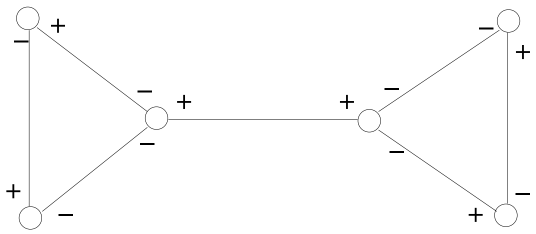

Fig. 1. Strongly connected bidirection of a graph having a bridge.

Graphs that allow a strongly connected bidirection can be characterized as follows:

Theorem 5 *An undirected graph G admits a strongly connected bidirection $\bar{G}$ if and only if either G consists of only one vertex, or G is connected and every vertex has degree at least two.*

PROOF. $'' \Rightarrow$:" Let $\bar{G} = (V, A)$ be a strongly connected bidirection of G with $|V| \geq 2$. According to the definition, between any two vertices u, v there exist finite bidirected paths from u to v and from v to u. Therefore G has to be connected and for any vertex v there exist two arcs that are oppositely incident to v. Hence the degree of each vertex is at least two.

$'' \Leftarrow$:" Let G be connected with $|V| \geq 2$ such that each vertex has degree at least two. By shrinking the maximal bridgeless components of G we obtain a tree T whose edges correspond to the bridges of G and whose vertices correspond to bridgeless components that consist of either one vertex (which we call "trivial") or at least three vertices (which we call "proper"). Moreover, the leaves of the tree are proper. Choose a proper component Q_r as the "root" of T.

We bidirect G in two steps: In a first step, we determine for each proper component a strongly connected direction. Note that this can be done in linear time, as each component is bridgeless. In a second step, we modify the direction inside the components such that any end-vertex of a bridge ("bridge-vertex") has only negatively incident arcs inside the component. We then bidirect the bridges such that for any proper component Q there exist a path from Q to Q_r

such that the starting- and ending arcs are positively incident to the terminal bridge-vertices. See Figure 2 for an example.

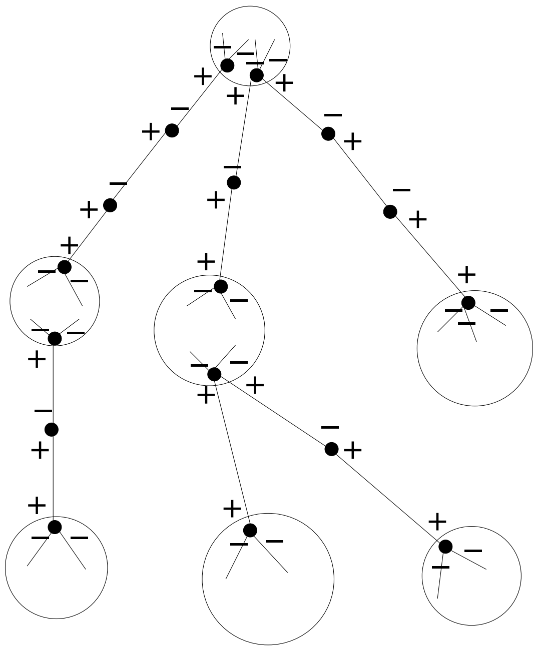

Fig. 2. Strongly connected bidirection.

It is easy to see that between any two vertices u, v there exists a finite bidirected path P from u to v:

- If u and v belong to the same component Q in $G \setminus \mathcal{B}(G)$, there exists a directed path $\vec{P}$ from u to v in the direction found in the first step of the proof. If no vertex of $\vec{P}$ is a bridge-vertex, take $P = \vec{P}$. Otherwise follow $\vec{P}$ and, whenever a bridge-vertex b is reached, walk along a closed path outside Q whose starting- and ending arcs are positively incident to b, and keep on following $\vec{P}$ to obtain P. Note that such a closed path outside Q always exists: You may simply follow a path from b to a bridge-vertex b_r of the root component Q_r, choose a closed path inside Q_r whose starting- and ending arcs are negatively incident to b_r, and walk on the same way back to b.
- If u and v are in different components, let b_u resp. b_v be the bridge-vertex in the component containing u resp. v. There exist directed paths $\vec{P_u}$ from u to b_u and $\vec{P_v}$ from b_v to v in the direction found in the first step of the proof. Follow $\vec{P_u}$ and, whenever a bridge-vertex $b \neq b_u$ is reached, walk an additional closed path, whose starting and ending arc are positively incident to b. As soon as b_u is reached, walk the way from b_u to b_v and follow $\vec{P_v}$ (with possible additional closed paths at bridge-vertices) to obtain the bidirected path P from u to v.
- If u is a bridge-vertex and the unique way from u to v in the tree starts with an arc which is negatively incident to u, we may first walk a closed

walk, whose starting- and ending arcs are positively incident to u, and then walk the path to v whose starting arc is negatively incident to u.

- If v is a bridge-vertex and the unique way from u to v in the tree ends with an arc which is positively incident to v, we add a closed path whose starting- and ending arcs are negatively incident to v.

We extend the upper bound

$$diam(\vec{G}) \leq |V| - 1$$

known for directions $\vec{G}$ of connected bridgeless graphs G and observe:

Lemma 6 *Let $\bar{G}$ be the bidirection constructed in the proof of Theorem 5. Then*

$$diam(\bar{G}) \leq |V| + 2|\mathcal{B}(G)| - 1.$$

PROOF. Let us call the bridgeless components containing exactly one bridge-vertex "leaf components", and the remaining bridgeless proper components "inner components".

It is easy to see that the greatest distance between two vertices in $\bar{G}$ is adopted by two adjacent vertices of the same leaf component. Let us assume $diam(\bar{G}) = d_{\bar{G}}(u, v)$ for two adjacent vertices u and v of leaf component G_1.

Obviously, a shortest path from u to v first goes to the unique bridge-vertex b_1 of G_1, traverses the inner components $\{G_2, ..., G_{k-1}\}$ in this order to reach the leaf component G_k at its unique bridge-vertex b_k. After reaching b_k the first time, the shortest path follows a circuit back to b_k and returns, traversing the inner components in reverse order, to b_1, before it finally follows a path from b_1 to v.

We show that there exists a path P from u to v in $\bar{G}$ of length at most $|V| + 2|\mathcal{B}(G)| - 1$:

For each bridgeless component $G_i, i = 1, .., k$, consider a direction $\vec{G}_i$. Let b_i and b'_i denote the bridge-vertices of the inner component $G_i, i = 2, .., k-1$, traversed by a path from u to v in G. Clearly, for each inner component G_i holds

$$\min\{d_{\vec{G}_i}(b_i, b'_i), d_{\vec{G}_i}(b'_i, b_i)\} \leq \frac{|V(G_i)|}{2}.$$

Let P_i be an (undirected) path from b_i to b'_i with reverse path P_i^- such that either $\vec{P}_i$ or $\vec{P}_i^-$ attains the minimum above. Moreover, there exist a path $P(u, b_1)$ from u to b_1 and a path $P(b_1, v)$ from b_1 to v in $\vec{G}_1$ such that the length of $P(u,b_1)+P(b_1,v)$ is at most $|V(G_1)|-1$. And finally there exists a circuit C_k starting and ending in b_k in $\vec{G}_k$ whose length is at most $|V(G_k)|$.

Let $\mathcal{B}_i,\ i=1,\dots,k-1$ denote the set of bridges linking component G_i and G_{i+1}. By construction of $\bar{G}$, we know that

$$P = P(u,b_1)+\mathcal{B}_1+P_2+...+\mathcal{B}_{k-1}+C_k+\mathcal{B}_{k-1}^{-}+P_{k-1}^{-}+...+\mathcal{B}_1^{-}+P(b_1,v)$$

corresponds to a bidirected path from u to v of length at most

$$\begin{aligned}
&|V(G_1)|-1+\sum_{i=1}^{k-1}2|\mathcal{B}_i|+\sum_{i=2}^{k-1}|V(G_i)|+|V(G_k)|\\
=&|V(G_1)|-1+\sum_{i=1}^{k-1}(|\mathcal{B}_i|+|V(\mathcal{B}_i)|-1)+\sum_{i=2}^{k-1}|V(G_i)|+|V(G_k)|\\
\le&|V(G)|-1+|\mathcal{B}(G)|+k-1\\
\le&|V(G)|+2|\mathcal{B}(G)|-1.
\end{aligned}$$

(Compare Figure 3.)

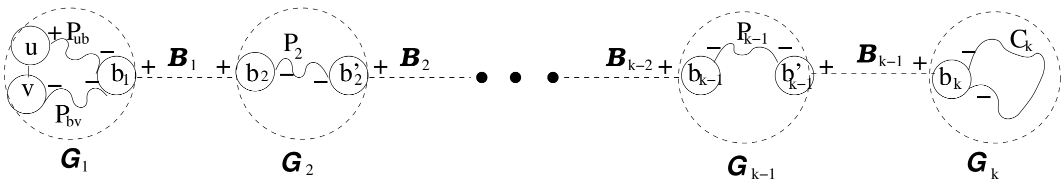

Fig. 3. Path P from u to v.

The question is, whether there exists a connected bridgeless graph G such that G admits a bidirection $\bar{G}$ whose diameter is smaller than the minimal diameter of all possible directions $\vec{G}$ of G. Our conjecture is that this is not possible.

3 An upper bound for the minimal diameter of possible strongly connected bidirections

To shorten notations let us call an undirected graph G *feasible* if either G consists of only one vertex, or G is connected and every vertex has degree at least two.

In this Section we extend results of Fomin et al. [1] about the relation between the minimal diameter of directed graphs and the minimal size $\gamma(G)$ of a dominating set of the underlying undirected graph to the relation between the minimal diameter of bidirected graphs and $\gamma(G)$. The main idea in order to find a bidirection of "small" diameter is to determine a dominating subgraph with certain properties, assign a bidirection of this subgraph and extend this bidirection to the whole graph.

Let us construct an extension of the bidirection of a feasible dominating subgraph to the whole graph such that the diameter increases at most by 4:

Lemma 7 *Let G and G_D be feasible graphs such that G_D is a dominating subgraph of G. Then for any strongly connected bidirection $\bar{G}_D$ of G_D there is an bidirection $\bar{G}$ of G such that*

$$diam(\bar{G}) \leq diam(\bar{G}_D) + 4.$$

PROOF. For each connected component Q in $G \setminus V(G_D)$ direct the edges having ends in Q as follows:

- Suppose Q consists of only one vertex q. Since the degree of q is at least two, we know that q is adjacent to at least two vertices $u, v \in V(G_D)$. Direct the edges (q, u) and (q, v) such that (q, u) is positively incident to q and negatively incident to u, while (q, v) is negatively incident to q and positively incident to v. All other edges incident to q may be directed arbitrarily. This way we assured the existence of vertices $u, v \in V(G_D)$ such that $d_{\bar{G}}(q, u) = 1$ and $d_{\bar{G}}(v, q) = 1$.
- In case Q consists of at least two vertices, choose a spanning tree T in this component rooted in a vertex r. For any vertex $x \in Q$ let $(x, \tilde{x})$ be the edge in T, which is incident to x and on the (unique) path to r. Since G_D dominates G, any $x \in Q$ is adjacent to at least one vertex $x' \in V(G_D)$. Direct the edges with end-vertices in Q as follows:
  · If the length of the path from x to r is odd, let $(x, \tilde{x})$ be negatively incident in x and positively incident in $\tilde{x}$ and (x, x') be positively incident in x and negatively incident in x'.
  · Otherwise let $(x, \tilde{x})$ be positively incident in x and negatively incident in $\tilde{x}$ and (x, x') be negatively incident in x and positively incident in x'. All other edges with end-vertex in Q may be directed arbitrarily.

  See Figure 4 for illustration.

In such a bidirection $\bar{G}$, for every vertex $x \in Q$ there are vertices $u, v \in V(G_D)$ such that $d_{\bar{G}}(x, v) \leq 2$ and $d_{\bar{G}}(u, x) \leq 2$. Hence, for every $x, y \in V(G)$ the distance between x and y in $\bar{G}$ is at most $diam(\bar{G}_D) + 4$. (On a path P from $x \in Q$ to $y \in Q$, the arcs between Q and the first vertex $u \in V(G_D) \cap P$ and the last vertex $v \in V(G_D) \cap P$ are oppositely incident.)

Note that if G_D is bridgeless and $\bar{G}_D$ is a strongly connected direction, the

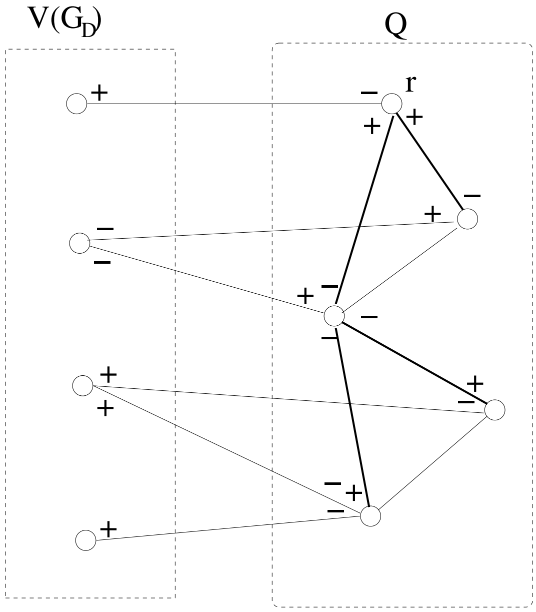

Fig. 4. Bidirection of edges with end-vertex in Q.

bidirected graph $\bar{G}$ constructed in the proof is also a strongly connected direction.

For the proof of the second upper bound on the minimal diameter, we need to show that each feasible graph contains a dominating tree with not to many vertices:

Lemma 8 *Let $G = (V, E)$ be a feasible graph and $D \subseteq V$ be a dominating set with $|D| = \gamma(G)$. Then there exists a tree $T \subset G$ with $D \subseteq V(T)$ such that:*

$$|V(T)| \leq \min\{3\gamma(G) - 2, \gamma(G) + \sum_{v \in D} deg_T(v)\}$$

PROOF. In case $\gamma(G) = 1$ set $T = D$. Thus

$$|V(T)| = 1 \leq \min\{3 - 2, 1\} = 1.$$

If $\gamma(G) \geq 2$, take any dominating set D of size $\gamma(G)$. Iteratively, for $k = 1, .., |D|$, construct trees T_k as follows: Choose $x_1 \in D$ arbitrary and set $T_1 = \{x_1\}$. After tree T_k has been constructed, choose $x_{k+1} \in D \setminus \{x_1, .., x_k\}$ with minimal (undirected) distance to T_k. Let P_k denote such a shortest path between x_{k+1} and T_k and set $T_{k+1} = T_k \cup P_k$. Since D is a dominating set, the length of P_k is at most three. At the last step, we obtain a tree T with $D \subseteq T$ and with $|V(T)| \leq 2(|D| - 1) + |D| = 3|D| - 2$. If the length of P_k equals three, then P_k is incident to one of the vertices of the set $\{x_1, \ldots, x_k\}$. Therefore the sum of the degrees of the dominating vertices will increase by two. Hence: $|V(T)| \leq \gamma(G) + \sum_{v \in D} deg_T(v)$

We now determine a feasible dominating subgraph G_D of a feasible graph G whose size is bounded by a function of the domination number. The proof is similar to the proof of Lemma 2 in [1].

Lemma 9 *Every feasible graph G has a feasible dominating subgraph G_D such that*

$$|V(G_D)| \leq 5\gamma(G) - 4 + 2b(G).$$

Moreover, in case G is bridgeless, G_D is bridgeless, too.

PROOF. In case $\gamma(G) = 1$ the unique dominating subgraph $G_D = G$ satisfies

$$1 = |V(G_D)| \leq 5\gamma(G) - 4 = 1.$$

If $\gamma(G) \geq 2$, take any dominating set D of size $\gamma(G)$. Construct the dominating tree T with paths $P_k,\ k = \{1, \ldots, \gamma(G)\}$, according to Lemma 8.
We now transform T into a feasible subgraph G_D which is bridgeless, in case G is bridgeless. For this purpose, we iteratively for $k = 1, .., |D|$ construct subgraphs G_k each containing T.

We call a vertex $x_j \in D$ *fixed* in G_k, if for each vertex $v \in P_{j-1} (j > 1)$ holds: either v lies on an (undirected) circuit or handcuff in G_k. (Recall that a handcuff consists of two circuits joined by a path.)

Let $F(G_k)$ denote the set of fixed vertices in G_k and $N(G_k) = |V(G_k) \setminus V(T)|$ denote the number of vertices that were added to T to obtain G_k.

We prove inductively for $k = 1, .., |D|$ that there exists a subgraph G_k with $T \subseteq G_k$, $\{x_1, ..., x_k\} \subseteq F(G_k)$ and $N(G_k) \leq 2(|F(G_k)| - 1 + b(G))$.

We set $G_1 = T$. Then $x_1 \in F(G_1)$ (since P_0 is not defined) and $N(G_1) = 0$. Assume we have constructed the subgraph G_k such that the induction hypothesis is satisfied. If x_{k+1} is already fixed in G_k, we set $G_{k+1} = G_k$ and the induction hypothesis is satisfied. Otherwise we add a subgraph M to G_k to obtain G_{k+1}. We require that x_{k+1} is fixed in $G_{k+1} = G_k \cup M$ and the number of fixed vertices increases with the number of new vertices in M as follows:

$$|F(G_{k+1})| - |F(G_k)| \geq \begin{cases} \left\lceil \frac{|V(M) \setminus V(G_k)| - 2}{2} \right\rceil & : \ P_1 \text{ contains a bridge of } G \\ \left\lceil \frac{|V(M) \setminus V(G_k)|}{2} \right\rceil & : \ \text{otherwise.} \end{cases}$$

Note that this is sufficient to prove $N(G_{k+1}) \leq 2(|F(G_{k+1})| - 1 + b(G))$ since $N(G_{k+1}) = N(G_k) + |V(M) \setminus V(G_k)|$ and $N(G_k) \leq 2(|F(G_k)| - 1 + b(G))$ by induction.

We only consider the case where P_k is of length three. The other cases can be done similarly. Let us assume that P_k is given by $P_k = \{x_{k+1}, u, v, x_j\}$ with $u, v \notin D$ and $j \leq k$. If we remove the edges $e = (x_{k+1}, u), e' = (u, v)$ and $e'' = (v, x_j)$ from T, we obtain four subtrees T^1, T^2, T^3 and T^4 containing x_{k+1}, u, v and x_j respectively.

- Suppose P_k contains a bridge. By induction, vertex x_k is fixed in G_k for any $k > 1$. Therefore G_k contains a circuit for $k > 1$.
  - If e is a bridge, choose a shortest path P such that $P \cup T_1$ contains a circuit. In case $k = 1$, choose a shortest path Q such that $Q \cup T_2 \cup T_3 \cup T_4 \cup \{e', e''\}$ contains a circuit.
  - Else, if e' is a bridge, choose a shortest path P such that $P \cup T_1 \cup T_2 \cup \{e\}$ contains a circuit. In case $k = 1$, choose a shortest path Q such that $Q \cup T_3 \cup T_4 \cup \{e''\}$ contains a circuit.
  - Else, if e'' is a bridge, choose a shortest path P such that $P \cup T_1 \cup T_2 \cup T_3 \cup \{e, e'\}$ contains a circuit. In case $k = 1$, choose a shortest path Q such that $Q \cup T_4$ contains a circuit.

  In case $k = 1$, set $M = P \cup Q$. Otherwise, set $M = P$. Since D is a dominating set, P and Q are each of length at most three. By construction, x_{k+1} is fixed in G_{k+1}. Hence,

  $$|F(G_{k+1})| - |F(G_k)| \geq 1 \geq \begin{cases} \left\lceil \frac{|V(M) \setminus V(G_k)| - 2}{2} \right\rceil & : \; P_1 \text{ contains a bridge of } G \\ \left\lceil \frac{|V(M) \setminus V(G_k)|}{2} \right\rceil & : \; \text{otherwise.} \end{cases}$$

  is satisfied.
- Now suppose P_k contains no bridge.

  Notice that all vertices $x_j \in D \cap (T^2 \cup T^3)$ have $j > k + 1$ since the subtrees T^2 and T^3 were built after step $k + 1$ in the construction of T.

  Among all shortest paths in $G \setminus e$ connecting T^1 with $T^2 \cup T^3 \cup T^4$, we select P as one whose last vertex belongs to T^i with i maximum. If no such path exists, e is a bridge in G.

  Among all shortest paths in $G \setminus e''$ connecting T^4 with $T^1 \cup T^2 \cup T^3$, we select Q as one whose first vertex belongs to T^i with i minimum.

  Let R be any shortest path in $G \setminus e'$ connecting $T^1 \cup T^2$ with $T^3 \cup T^4$.

  Since T dominates G, the paths P, Q or R are of length at most three each. Moreover, if the length of one of these paths is three, its two end-vertices belong to D.

  We define M as follows: If the last vertex of P belongs to T^4, we set $M = P$. If the last vertex of P belongs to T^3, or it belongs to T^2 and the first vertex of Q belongs to T^2, we set $M = P \cup Q$. If none of the previous cases holds, the first vertex of R belongs to T^2 and the last one belongs to T^3 and we set $M = P \cup Q \cup R$.

  By construction, x_{k+1} is fixed in $G_{k+1} = G_k \cup M$. Therefore, if $|V(M) \setminus V(G_k)| \leq 2$, we are done. If $6 \geq |V(M) \setminus V(G_k)| \geq 5$, at least two of the three paths have length three and $R \neq \emptyset$. Therefore, there exist $x_i \in D \cap T^2$ and $x_l \in D \cap T^3$ with $i, l > k+1$ which are end-vertices of these paths. Moreover, the vertices $x_{i'} \in D \cap T^2$ closest to u and $x_{l'} \in D \cap T^3$ closest to v are not fixed in G_k but are fixed in G_{k+1}. Thus, in this case three more vertices are fixed in G_{k+1}. If $4 \geq |V(M) \setminus V(G_k)| \geq 3$ and one of the three paths P, Q or R has length three, then there exists $x_i \in D \cap (T^2 \cup T^3)$ with $i > k+1$. As before, the vertex $x_{i'} \in D \cap (T^2 \cup T^3)$ closest to u or v is not fixed in G_k but is fixed in G_{k+1}. Therefore, two more vertices are fixed in G_{k+1}. If $|V(M) \setminus V(G_k)| = 3$ and all paths P, Q and R have length two, we know that one of the end-vertices of R is a vertex $x_i \in D \cap (T^2 \cup T^3)$ with $i > k+1$. It is clear that as above a vertex $x_{i'} \in D$ which is not fixed in G_k becomes fixed in G_{k+1}.

Summarizing, in G_{k+1} we have $\{x_1, .., x_k\} \subseteq F(G_{k+1})$ and

$$\begin{aligned} N(G_{k+1}) &= N(G_k) + |V(M) \setminus V(G_k)| \\ &\leq 2(|F(G_k)| - 1 + b(G)) + 2(|F(G_{k+1})| - |F(G_k)|) \\ &= 2(|F(G_{k+1})| - 1 + b(G)). \end{aligned}$$

In the last step we obtain a feasible subgraph G_D which is bridgeless in case G is bridgeless. Furthermore, G_D satisfies: $|F(G_D)| = D$ and $N(G_D) \leq 2(|D| - 1 + b(G))$. Since $|V(T)| \leq 3|D| - 2$ (see Lemma 8) we conclude that

$$|V(G_D)| \leq 5\gamma(G) - 4 + 2b(G).$$

For the bridgeless case, the bound is sharp: Consider the graphs $C_6[n]$ obtained from an $n-$vertex path P_n by replacing each edge by two internally disjoint length-3 paths (see Figure 5). The unique bridgeless connected dominating subgraph is the graph itself. Hence, $5(n-1)+1 = |V(C_6[n])| = 5n - 4 = 5\gamma(C_6[n]) - 4$. The non-bridgeless graph in Figure 6 shows the sharpness of the

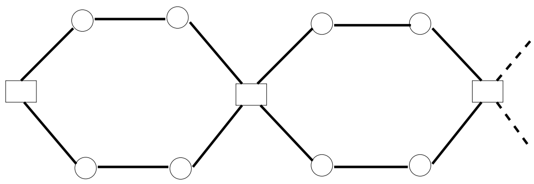

Fig. 5. Bridgeless graph where the bound of Lemma 9 is sharp.

bound in the non-bridgeless case: The unique minimum dominating set consists of the vertices indicated through boxes. The unique feasible dominating subgraph is the graph itself. Hence $8 = |V(G)| = 5 * 2 - 2$.

We use Lemma 7 and 9 to obtain the first upper bound on the minimum diameter.

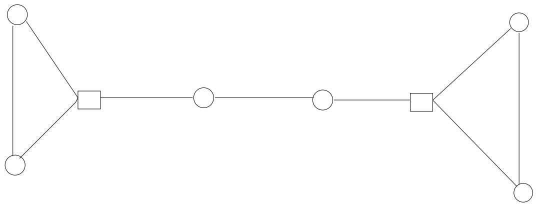

Fig. 6. Non-bridgeless graph where the bound of Lemma 9 is sharp.

Theorem 10 *Every feasible graph G admits a bidirection $\bar{G}$ such that*

$$diam(\bar{G}) \leq 2|\mathcal{B}(G)| + 2b(G) + 5\gamma(G) - 1.$$

In case G is bridgeless, $\bar{G}$ is a direction.

PROOF.

Let G_D be the graph constructed in Lemma 9. In case G_D is bridgeless, find a strongly directed direction $\vec{G}_D$ and extend it to a strongly directed direction $\vec{G}$ of G according to Lemma 7. Thus $\vec{G} = \bar{G}$ resp. $\vec{G}_D = \bar{G}_D$ is a stronlgy directed bidirection of G resp. G_D.

Otherwise G_D is feasible and we can determine a strongly connected bidirection $\bar{G}_D$ and extend it to a bidirection $\bar{G}$ according to Lemma 7. Then

$$\begin{aligned} diam(\bar{G}) &\overset{Lemma\ 7}{\leq} diam(\bar{G}_D) + 4 \\ &\overset{Lemma\ 6}{\leq} 2|\mathcal{B}(G_D)| + |V(G_D)| - 1 + 4 \\ &\overset{Lemma\ 9}{\leq} 2|\mathcal{B}(G)| + 2b(G) + 5\gamma(G) - 1. \end{aligned}$$

So far, we required that the bidirection of a connected bridgeless graph should be a common direction. The following construction of a dominating feasible subgraph G_D abstains from this requirement and provides a different bound on the minimal diameter:

Theorem 11 *Every feasible graph G admits a bidirection $\bar{G}$ such that*

$$diam(\bar{G}) \leq 6\gamma(G) + 3.$$

PROOF. Since a direction is a bidirection, see [1] or Theorem 10 for the bridgeless case.
Let G be a non-bridgeless feasible graph. Clearly $\gamma(G) \geq 2$.
Take any dominating set D of size $\gamma(G)$ and construct a tree T with $D \subseteq V(T)$

according to Lemma 8. Let $L(T)$ denotes the set of leaves in T.
Obviously each leaf is a dominating vertex. Since G is feasible, each vertex of T has degree at least two in G. As D dominates G for each leaf of the tree there exists a path $Q \in E(G)\backslash E(T)$ of length at most three connecting the leaf with a vertex of T. Adding these paths we therefore obtain a feasible dominating graph G_D such that

$$|V(G_D)| \leq |V(T)| + 2|L(T)|.$$

Graph G_D has the property that each strongly connected component of G_D contains at least one leaf vertex of T. Let u, v be two vertices of G_D, and $\bar{G}_D$ a bidirection of G_D constructed as in Theorem 5. We claim:

$$d_{\bar{G}_D}(u,v) \leq 3 + 3(\gamma(G) - |L(T)|) + |V(G_D)| \quad \forall\ u, v \in V.$$

This holds, since if all dominating vertices are leaves of the tree T, then the shortest connecting bidirected path between u and v has length at most $3 + |V(G_D)|$. For each additional dominating vertex, which is not a leaf vertex, the path lengthened at most by three edges.
We have:

$$\begin{array}{rcl} diam(\bar{G}) & \overset{Lemma\ 7}{\leq} & diam(\bar{G}_D) + 4 \\ & \leq & 3 + 3(\gamma(G) - |L(T)|) + |V(G_D)| + 4 \\ & \leq & 3\gamma(G) - 3|L(T)| + |V(T)| + 2|L(T)| + 7 \\ & \overset{Lemma\ 8}{\leq} & 3\gamma(G) - |L(T)| + 7 + \min\{3\gamma(G) - 2, \gamma(G) + \sum\limits_{v \in D} deg_T(v)\} \\ & \leq & \begin{cases} 3\gamma(G) - \gamma(G) + 7 + 2\gamma(G) = 4\gamma(G) + 7 & : \quad |L(T)| = \gamma(G) \\ 3\gamma(G) - 2 + 7 + 3\gamma(G) - 2 = 6\gamma(G) + 3 & : \quad |L(T)| \leq |\gamma(G) - 1|. \end{cases} \end{array}$$

Since $6\gamma(G) + 3 \geq 4\gamma(G) + 7 \quad \forall\ \gamma(G) \geq 2$ the Theorem is shown.

The following Corollary is a direct consequence of Theorems 10 and 11.

Corollary 12 *Every feasible graph G admits a bidirection $\bar{G}$ such that*

$$diam(\bar{G}) \leq \min\{2|\mathcal{B}(G)| + 2b(G) + 5\gamma(G) - 1,\ 6\gamma(G) + 3\}$$

It is easy to see that the bound is sharp for the graph in Figure 7.

For non-bridgeless graphs, the bound of Corollary 12 is at least asymptotically tight. See the graph in Figure 8, where the bridge component has length

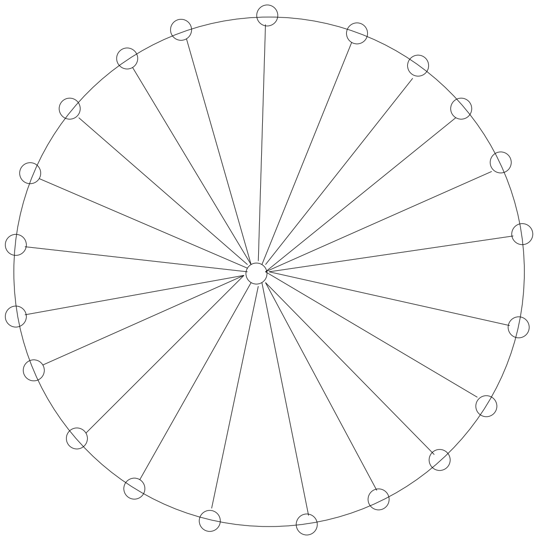

Fig. 7. Bridgeless graph where the bound of Corollary 12 is sharp.

$3(\gamma(G)-1)$. The distance between the vertices a and b is:

$$d_{\bar{G}}(a,b) = 2 + 3(\gamma(G)-1) + 3 + 3(\gamma(G)-1) + 2 = 6\gamma(G) + 1$$

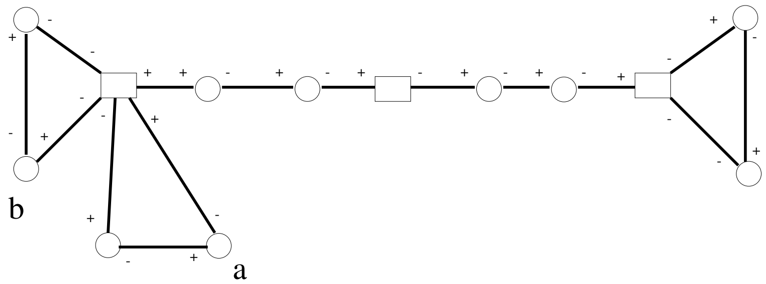

Fig. 8. Non-bridgeless graph where the bound of Corollary 12 is asymptotically sharp.

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