Note on Pseudolattices, Lattices and Submodular Linear Programs

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Abstract

A pseudolattice L is a poset with lattice-type binary operations. Assuming that the pseudolattice permits a modular representation as a family of subsets of a set U with certain compatibility properties, we show that L actually is a distributive lattice with the same supremum operation. Given a submodular function $r : L \to \mathbb{R}$, we prove that the corresponding unrestricted linear program relative to the representing set family can be solved by a greedy algorithm. This complements the Monge algorithm of Dietrich and Hoffman for the associated dual linear program. We furthermore show that our Monge and greedy algorithm is generally optimal for nonnegative submodular linear programs and their duals (relative to L).

1 Introduction

The greedy algorithm is a heuristic procedure for discrete optimization problems that has long been recognized not only to solve certain problems optimally but also to be a basic subroutine in other efficient algorithms, when

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cast into the framework of linear programming (see, *e.g.*, Hoffman [13]). It can be used to provide an algorithmic characterization of matroids. Moreover, many generalizations of matroids have turned out to be accompanied by corresponding greedy algorithms (see, *e.g.*, [4, 18, 10]).

From the linear programming point of view, these greedy algorithms can be viewed as primal solutions associated with feasible solutions for the dual linear program. These dual solutions can often be constructed by an algorithmic procedure that goes back to Monge [16] and is of interest for the analysis of many optimization structures (see, *e.g.*, [2]). For example, the greedy algorithms of [9, 6, 7] follow this principle.

A powerful concept for the analysis of integral linear programs are Hoffman's *lattice polyhedra* (see, *e.g.*, [14]) that generalize matroid polyhedra by allowing an order structure on the feasible sets that need not coincide with the ”natural” set-theoretic ordering by containment. It appears difficult however, to identify appropriate greedy algorithms for general lattice polyhedra. Frank [9] could provide such an algorithm relative to a class of lattice polyhedra that arise from nonnegative and monotone decreasing supermodular functions with a submodular set-theoretic presentation. Recently, Dietrich and Hoffman [3] have established an optimal Monge algorithm for a class of lattice polyhedra relative to general sub- and supermodular functions with a modular presentation.

In the present article, we analyze the Dietrich and Hoffman model and show that the underlying orders of *pseudolattices* are actually distributive lattices in the usual sense, which relates the model to classical matroid structures. We exhibit a (primal) greedy algorithm that complements the Monge algorithm optimally. Furthermore, we show that these Monge and greedy algorithms can be specified in such a way that also the corresponding linear programs under nonnegativity restrictions are optimally solved.

An elegant model for the analysis of Monge algorithms has recently been proposed by Fujishige [11] (see also [12]). The approach differs from ours, however, in that [11] assumes an (in terms of certain ”choice functions”) well-defined Monge algorithm to be given. The question then is under which conditions it is optimal. We, on the other hand, start from a combinatorial optimization problem and try to identify appropriate Monge and greedy algorithms for it. Moreover, our algorithms do seem to be subsumed by Fujishige's model in an obvious way.

2 Lattices and Pseudolattices

Let $(L, \leq)$ be a finite (partially) ordered set. L is a *lattice* if for all $a, b \in L$ there are unique elements $\sup(a, b), \inf(a, b) \in L$ such that for all $c \in L$

$$\begin{aligned} c \geq a, b, &\iff c \geq \sup(a, b) \\ c \leq a, b, &\iff c \leq \inf(a, b). \end{aligned}$$

It is well-known that suprema always exist in L if and only if infima always exist. In fact, one has for any $A \subseteq L$,

$$\inf A = \sup\{c \in L \mid c \leq a \text{ for all } a \in A\}.$$

The ordered set $(L, \leq)$ is a *pseudolattice* if for all $a, b \in L$, there exist elements $a \wedge b, a \vee b \in L$ such that

$$a \wedge b \leq a, b \leq a \vee b\ .$$

Note that a pseudolattice necessarily has a unique maximal and a unique minimal element. However, a pseudolattice need not be a lattice. When the pseudolattice L is a lattice, one has

$$a \wedge b \leq \inf(a, b) \leq a, b \leq \sup(a, b) \leq a \vee b\ .$$

It is quite possible, however, that all of the inequalities are strict.

Let U be a (finite) set. A *set representation* of $(L, \leq)$ is a map

$$\chi : L \to 2^U$$

into the collection of subsets of U such that for all $a, b, c \in L$

(C0) $a \neq b \implies \chi(a) \neq \chi(b)$ (*i.e.* χ is injective).

(C1) $a \leq b \leq c \implies \chi(a) \cap \chi(c) \subseteq \chi(b)$ (*i.e.*, χ has the *consecutive ones property*).

(C2) $\chi(a) \subseteq \chi(b) \implies a \leq b$ in L.

For any $u \in U$, we define the *characteristic function*

$$\chi(a, u) = \begin{cases} 1 & \text{if } u \in \chi(a) \\ 0 & \text{if } u \notin \chi(a). \end{cases}$$

Provided L is a pseudolattice, we call the representation χ *modular* if for all $u \in U$ and $a, b \in L$,

$$\chi(a \wedge b, u) + \chi(a \vee b, u) = \chi(a, u) + \chi(b, u)\ .$$

Theorem 2.1 *Assume that L is a pseudolattice with a modular representation χ. Then L is a lattice with $a \vee b = \sup(a, b)$ for all $a, b \in L$.*

Proof. We claim that $\sup(a, b)$ exists and equals $a \vee b$ for all $a, b \in L$. So consider any $c \geq a, b$. We must show that $c \geq a \vee b$ is true. Suppose this is not the case and let $d = a \vee b$. Then we have

$$a, b \leq c < c \vee d \; .$$

By (C2), there exists some $u \in \chi(c \vee d) \setminus \chi(c)$. Because $u \in \chi(c \vee d)$, the modularity of χ implies $u \in \chi(d) = \chi(a \vee b)$ and hence $u \in \chi(a) \cup \chi(b)$. In view of $u \notin \chi(c)$, on the other hand, the consecutive property (C1) yields $u \notin \chi(a) \cup \chi(b)$, which is a contradiction.

$\diamond$

If L is a pseudolattice with modular representation χ and $u \in U$ an arbitrary element, we define the *u-reduction* of L to be the ordered set

$$L \setminus u = \{a \in L \mid u \notin \chi(a)\}.$$

It is straightforward to check that $L \setminus u$ is a pseudolattice and that χ yields a modular representation with respect to the reduced set $U \setminus \{u\}$.

2.1 Distributivity

We now show, more specifically, that a pseudolattice with modular representation such that (C0)-(C2) hold is, in fact, a distributive lattice. To see this, we use the well-known fact that a distributive lattice is characterized by not admitting N_5 or M_3 (see Fig. 1) as a sublattice (*cf.* [1]).

Theorem 2.2 *Assume that L is a pseudolattice with a modular representation χ. Then L is a distributive lattice.*

Proof. Suppose that the Theorem is false and there exists a sublattice $N_5 = \{a, b, c, d, e\}$ such that $b < c$, $e = b \vee d = c \vee d$ and $a = \inf(b, d) = \inf(c, d)$.

By (C2), we may choose an element $u \in \chi(c) \setminus \chi(b)$. Property (C1) implies $\chi(c \wedge d, u) = 0$. Hence, the modularity of χ implies $\chi(c \vee d, u) = 1$ and $\chi(d, u) = 0$. So

$$\chi(d, u) + \chi(b, u) = 0 < \chi(b \vee d, u) = \chi(c \vee d, u) = 1$$

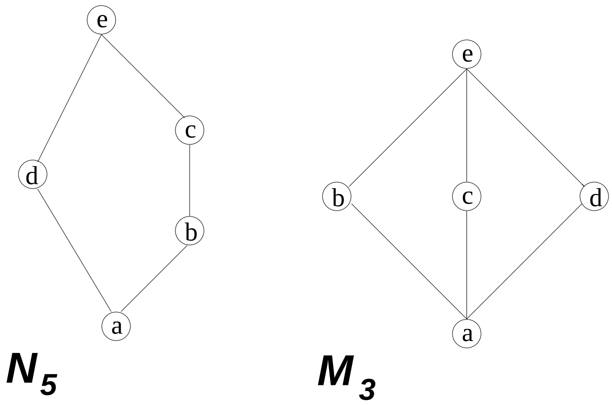

Figure 1: Minimal non-distributive lattices.

yields a contradiction to the modularity of χ.

Assume now that L contains a sublattice $M_3 = \{a, b, c, d, e\}$ such that $e = b \vee c = b \vee d = c \vee d$ and $a = \inf(b, c) = \inf(b, d) = \inf(c, d)$. Choose an element $u \in \chi(e) \setminus \chi(b)$. The modularity of χ implies

$$\chi(c, u) = \chi(d, u) = \chi(\inf(c, d), u) = 1.$$

Hence $\inf(c, d) \leq a < b < e$ yields a contradiction to property (C1).

$\diamond$

By Birkhoff's Theorem [1], a (finite) distributive lattice L admits a particular representation as a union- (and intersection-) closed system of sets in the following way. Call $p \in L$ *irreducible* if p has precisely one lower neighbor in L, where we say that a lattice element $q < p$ is a *lower neighbor* of p if there is no $a \in L$ with $q < a < p$. Let $P = P(L)$ denote the set of all irreducibles of L and represent each $a \in L$ by the set

$$\overline{a} = \{p \in P \mid p \leq a\} \subseteq P.$$

The equality $a = \sup(\overline{a})$ always holds. The distributivity of L, however, is equivalent with

$$\overline{a \vee b} = \overline{a} \cup \overline{b} \quad \text{for all } a, b \in L.$$

and satisfies $\overline{a} \cap \overline{b} = \overline{\inf(a, b)}$. So the "canonical" *Birkhoff representation* $\overline{\chi}(a) = \overline{a}$ is modular with respect to the lattice-theoretic operations, *i.e.*,

$$\overline{\chi}(\inf(a, b)) + \overline{\chi}(a \vee b) = \overline{\chi}(a) + \overline{\chi}(b),$$

and trivially has the properties (C0)-(C2).

3 The Monge Algorithm

We assume in the following always that the order $(L \leq)$ is a (pseudo)lattice with modular representation χ relative to the ground set U which satisfies the set theoretic compatibility properties (C0)-(C2). Without loss of generality, let us also assume that each $u \in U$ occurs in at least one representing set, *i.e.*,

$$U = \bigcup_{a \in L} \chi(a).$$

Given weights $c_u \in \mathbb{R}$ on the elements of U, we want to find parameters $y_a \in \mathbb{R}$ such that the following linear inequalities are satisfied

$$\text{(M)} \quad y_a \geq 0 \quad \text{for all } a \in L \quad \text{and} \quad \sum_{\chi(a) \ni u} y_a \geq c_u \quad \text{for all } u \in U.$$

In view of our assumption on U, it is clear that (M) has a feasible solution. The *Monge algorithm* computes a particular solution in a straightforward iterative procedure. To formulate it, we denote by $\ell(m)$ the set of all lower neighbors of m. The algorithm works as follows:

(M_1) Let $m \in L$ be maximal and choose some lower neighbor $m^* \in \ell(m)$ and $u^* \in \chi(m) \setminus \chi(m^*)$ such that

$$c^* = \min_{m' \in \ell(m)} \max \{c_u \mid u \in \chi(m) \setminus \chi(m')\} = c_{u^*}.$$

(M_2) Set $y_m = \max\{0, c^*\}$ and subtract y_m from all c_u with $u \in \chi(m)$.

(M_3) Replace L by $L^* = \{a \in L \mid a \leq m^*\}$.

(M_4) Iterate until $L = \emptyset$.

Note that (in view of our assumptions on χ) L^* is precisely the u^*-reduction of L:

$$L^* = L \setminus u^* = \{a \in L \mid u^* \notin \chi(a)\}.$$

We refer to $u^* \in \chi(m)$ as the *representative* of m with respect to the Monge algorithm. The crucial point is that for the reduced weight function c in the current step of the algorithm holds:

$$c^* \geq 0 \quad \Longrightarrow \quad y_m = c_{u^*} \text{ and } y_m \geq c_u \text{ for all } u \in \chi(m) \setminus \chi(m^*).$$

We call m *active* (in the Monge algorithm) if $c_{u^*} \geq 0$ holds and collect all active elements m_j into the *Monge chain*

$$M = \{m_1 < \ldots < m_k\} \subseteq L.$$

Let $u_j \in \chi(m_j)$ denote the representative of the active element $m_j \in M$. Recall that (C1) implies $u_i \notin \chi(m_j)$ for all $i < j$. So we find for all $u \in U$ and all representatives u_j:

$$\sum_{i=1}^{m} y_{m_i}\chi(m_i, u) \geq c_u \quad \text{and} \quad \sum_{i=1}^{m} y_{m_i}\chi(m_i, u_j) = c_{u_j}.$$

In particular, the resulting vector $\mathbf{y}^M$ solves (M) with components y_a^M given as

$$y_a^M = \begin{cases} y_a & \text{if } a \in M \\ 0 & \text{if } a \notin M. \end{cases}$$

Proposition 3.1 *Assume that all weights c_u are integers. Then all components y_a of the Monge solution $\mathbf{y}$ are integral.*

$\diamond$

Note that an iteration relative to an inactive element $m \in L$ does not affect the weights of the remaining elements $u \in U$. It follows that an element $u \in U$ with nonnegative weight c_u must be removed at an iteration involving an active element. Hence we find

Lemma 3.1 *If $m_i \in M$ is an active element such that the Monge algorithm chooses an inactive lower neighbor $m^* \in \ell(m)$, then m^* is the unique lower neighbor of m.*

Proof. Suppose that the Lemma is false and m_i has at least two lower neighbors $m^*, m' \in \ell(m_i)$. W.l.o.g., let m^* be such that $u_i \in \chi(m) \setminus \chi(m^*)$.

As the Monge algorithm chose m^* (instead of m'), we know that there must be some $u \in \chi(m) \setminus \chi(m')$ with (reduced) weight

$$c'_u \geq c'_{u_i} \geq 0.$$

Since $m' \not\leq m^*$, u is still in the representative set for the reduced lattice $\{a \in L \mid a \leq m^*\}$ and has nonnegative reduced weight. Moreover, if there exist an active element $m_{i-1} \in M$, the modularity of χ implies $u \notin \chi(m_{i-1})$.

So the Monge algorithm must produce at least one more active element m_k such that $m_k < m_i$ and, if $m_{i-1} \in M$ exists, $m_{i-1} < m_k < m_i$. A contradiction.

$\diamond$

Note that, in particular, the minimal active element m_1 in the Monge algorithm has at most one lower neighbor.

3.1 Equality Constraints

Consider the linear system

$$(\mathrm{M}^{=}) \quad y_a \geq 0 \quad \text{for all } a \in L \quad \text{and} \quad \sum_{\chi(a) \ni u} y_a = c_u \quad \text{for all } u \in U.$$

While (M) is always solvable, (M$^{=}$) might be infeasible. Dietrich and Hoffman [3], however, have observed that a greedy-type algorithm finds a feasible solution for (M$^{=}$), provided one exists at all:

(M1) Let $m \in L$ be maximal and choose $\bar{u} \in \chi(m)$ such that

$$\bar{c} = c_{\bar{u}} = \min \{c_u \mid u \in \chi(m)\}.$$

(M2) Set $y_m = \bar{c}$ and subtract y_m from all c_u with $u \in \chi(m)$.

(M3) Replace L by the $\bar{u}$-reduction $\bar{L} = L \setminus \bar{u} = \{a \in L \mid \bar{u} \notin \chi(a)\}$.

(M4) Iterate until $L = \emptyset$.

We claim that, in the equality case, our Monge algorithm may be interpreted as a special version of the Dietrich-Hoffman algorithm.

Theorem 3.1 *If (M$^{=}$) has a feasible solution at all, the Monge algorithm relative to (M) computes a feasible solution* $\mathbf{y}$ *for (M$^{=}$).*

Proof. Assuming (M$^{=}$) to be feasible, we first show $c^* = \bar{c}$. The inequality $\bar{c} \leq c^*$ follows from the definition. Let $\bar{m}$ be the maximal element of $\bar{L}$ and choose some $m' \in \ell(m)$ such that $\bar{m} \leq m' \leq m$.

Let $u' \in \chi(m) \setminus \chi(m')$ be an arbitrary element with weight c'. (C1) implies $u' \notin \chi(a)$ for all $a \in \bar{L}$. So the feasibility of the Dietrich-Hoffman algorithm yields

$$c^* \; \geq \; \bar{c} \; = \; y_m \; = \; c' \; \geq \; c^*.$$

So we can choose $\bar{u} = u^*$ in the Dietrich-Hoffman algorithm. Consider any $a \in L$. If $a \in L^*$, then (C1) implies $u^* \notin \chi(a)$. On the other hand, if $a \notin L^*$, we have $a \vee m^* = m$. Hence the modularity of χ yields $u^* \in \chi(a)$. Consequently, we find $\bar{L} = L^*$.

$\diamond$

Corollary 3.1 *If (M$^=$) is feasible and $M \subseteq L$ the chain of active elements in the Monge algorithm, then M is a maximal (*i.e.*, at most trivially extendible) chain in L.*

Proof. If (M$^=$) is feasible, all elements m considered in the Monge algorithm are active. Since successive elements are neighbors, the resulting chain is non-extendible if no element of L is represented by the empty set $\emptyset$. Otherwise, M can be trivially extended by the minimal element of L.

$\diamond$

3.1.1 The Monge Algorithm and the Birkhoff Representation

In the case of the Birkhoff representation $\overline{\chi}(a)$ of the distributive lattice L by subsets of the ordered set $(P, \leq)$ of irreducibles of L, one wants to solve

$$(\overline{\mathrm{M}}) \quad y_a \geq 0 \quad \text{for all } a \in L \quad \text{and} \quad \sum_{\overline{\chi}(a) \ni p} y_a = \overline{c}_p \quad \text{for all } p \in P.$$

for a given $\overline{\mathbf{c}} : P \to \mathbb{R}$. The Monge algorithm successively removes maximal elements $p_n, p_{n-1}, \ldots$ of minimal weight and thus generates a *linear extension* $\overline{\pi}$ of P, where

$$\overline{\pi} = p_1 p_2 \ldots p_{n-1} p_n \quad \text{such that} \quad p_i \leq p_j \;\Rightarrow\; i \leq j\ .$$

This Monge algorithm is the basis of the generalized (poly-)matroid greedy algorithms (see,*e.g.*, [6, 7, 17, 15]). It produces the solution

$$y_{m_k} = \overline{c}_{p_k} - \overline{c}_{p_{k+1}}.$$

The solution is feasible if and only if $\overline{\mathbf{c}} : P \to \mathbb{R}$ is nonnegative with the *antitone* property (*cf.* [5]):

$$p \leq q \quad \Longrightarrow \quad \overline{c}_p \geq \overline{c}_q \geq 0.$$

The classical (poly-)matroid case of Edmonds [4] (see also [10]) corresponds to P being trivially ordered.

Assume that $\mathbf{y} \geq \mathbf{0}$ is a feasible solution for ($\mathrm{M}^=$), relative to the weight function $\mathbf{c} : U \to \mathbb{R}$. The Monge algorithm produces the maximal chain $M = \{m_1 < \ldots < m_n\}$. Since L is distributive, P can be arrranged in a (unique) linear extension $\overline{\pi} = p_1 \ldots p_n$ such that

$$m_1 = p_1 \quad \text{and} \quad m_j = m_{j-1} \vee p_j \quad (j = 2, \ldots, n).$$

So the Monge solution induces a weighting $\overline{\mathbf{c}} : P \to \mathbb{R}$, *via*

$$\overline{c}_{p_j} = \sum_{k=j}^{n} y_{m_k} \quad (j = 1, \ldots, n),$$

and $\mathbf{y}$ turns into a feasible solution of the corresponding system ($\overline{\mathrm{M}}$) as well. Hence (with hindsight) the Monge algorithm relative to ($\mathrm{M}^=$) permits an interpretation within the framework ($\overline{\mathrm{M}}$) of the Birkhoff representation.

4 The Greedy Algorithm

Under the same assumptions on the (pseudo)lattice L, we now consider a function $r : L \to \mathbb{R}$ and the linear system

$$(\mathrm{P}) \qquad \mathbf{x}(a) \leq r(a) \quad \text{for all } a \in L,$$

with the understanding that $\mathbf{x}$ is a vector with components x_u and

$$\mathbf{x}(a) = \sum_{u \in U} x_u \chi(a, u) = \sum_{\chi(a) \ni u} x_a \ .$$

Again, it is clear that (P) always is feasible while its nonnegative version may be infeasible:

$$(\mathrm{P}^+) \quad x_u \geq 0 \text{ for all } u \in U \quad \text{and} \quad \mathbf{x}(a) \leq r(a) \text{ for all } a \in L,$$

Motivated by the Monge algorithm, we consider an arbitrary chain $M = \{m_1 < \ldots < m_k\} \subseteq L$ such that m_1 has at most one lower neighbor m_0 in L. Moreover, we select a sequence $\pi = u_1 \ldots u_k$ of representatives

$$u_j \in \chi(m_j) \setminus \chi(m_{j-1}) \quad (j = 1, \ldots, k).$$

Let us generally call such a pair (M, π) a *Monge pair*. The *greedy algorithm* computes a candidate solution $\mathbf{x}^\pi$ for (P) from the Monge pair (M, π) by modifying the components of the zero vector $\mathbf{x} = \mathbf{0}$ iteratively as follows:

(G$_1$) $x_{u_1} = r(m_1)$.

(G$_2$) $x_{u_j} = r(m_j) - \sum \{x_{u_i} \mid i < j, u_i \in \chi(m_j)\} \quad (j = 2, \ldots, k)$.

The algorithm yields immediately

Proposition 4.1 *Assume that r is integer-valued. Then every component of the greedy vector $\mathbf{x}^\pi$ is an integer.*

$\diamond$

Since $u_i \notin \chi(m_j)$ if $i > j$, we observe for the greedy vector $\mathbf{x}^\pi$ thus constructed:

$$\mathbf{x}^\pi(m_j) = r(m_j) \quad \text{for all } j = 1, \ldots, k.$$

With respect to $\mathbf{x}^\pi$ being a candidate solution (P$^+$), we note

Lemma 4.1 *Assume that $r : L \to \mathbb{R}$ is nonnegative and monotone increasing. Then the greedy vector $\mathbf{x}^\pi$ is nonnegative.*

Proof. The algorithm yields $x_{u_1} = r(m_1) \geq 0$ directly. Since each $u_i \in \chi(m_j)$ with $i < j$ must also lie in $\chi(m_{j-1})$, we furthermore find iteratively

$$\begin{aligned} x_{u_j} &= r(m_j) - \sum \{x_{u_i} \mid i < j, u_i \in \chi(m_j)\} \\ &\geq r(m_j) - \sum \{x_{u_i} \mid u_i \in \chi(m_{j-1})\} \\ &\geq r(m_j) - r(m_{j-1}) \geq 0 . \end{aligned}$$

$\diamond$

Before discussing sufficient conditions for the feasibility of $\mathbf{x}^\pi$, we derive further feasibility properties.

Lemma 4.2 *Let $r : L \to \mathbb{R}$ be nonnegative and monotone increasing. Let furthermore $\mathbf{x}^\pi$ be the greedy vector relative to the Monge pair (M, π) and consider the lattice element $a \in L$. Then*

$$\begin{array}{rcll} a < m_1 & \Longrightarrow & \mathbf{x}^\pi(a) = 0 \leq r(a) & \\ m_{j-1} < a < m_j & \Longrightarrow & \mathbf{x}^\pi(a) \leq r(a) & (j = 2, \ldots, k) \\ a > m_k & \Longrightarrow & \mathbf{x}^\pi(a) \leq r(a). & \end{array}$$

Proof. Assume $m_{j-1} < a < m_j$. From Lemma 3.1 we know $m_{j-1} < a < m^*$ for the unique lower neighbor $m^* \in \ell(m_j)$ chosen by the Monge algorithm.

By the consecutive property (C1), $u_i \in \chi(a)$ implies $u_i \in \chi(m_{j-1})$ for all i. So $\mathbf{x}^\pi \geq \mathbf{0}$ yields

$$\mathbf{x}^\pi(a) \leq x^\pi(m_{j-1}) = r(m_{j-1}) \leq r(a).$$

The case $a \geq m_k$ is analyzed the same way. Let finally m_0 be the lower neighbor of m_1 and assume $a < m_1$. Then $a \leq m_0$ and $u_1 \notin \chi(m_0)$ yields $u_1 \notin \chi(a)$. Similarly, no other u_j can lie in $\chi(a)$, *i.e.*, $\chi(a) = 0$.

$\diamond$

5 Submodular Functions

Let L be a pseudolattice with modular representation χ, satisfying (C0)-(C2), as before and assume that $r : L \to \mathbb{R}$ is *submodular, i.e.*,

$$r(a \wedge b) + r(a \vee b) \leq r(a) + r(b) \quad \text{for all } a, b \in L.$$

It follows that any $\mathbf{x} \in \mathbb{R}^U$ induces a submodular function $h = r - \mathbf{x}$, where

$$h(a) = r(a) - \mathbf{x}(a) \quad \text{for all } a \in L.$$

To say that $\mathbf{x}$ is a feasible solution for the linear system (P) of the previous section is equivalent to saying that h is nonnegative.

Theorem 5.1 *Let $M = \{m_1 < \ldots < m_k\}$ be an arbitrary chain in L. Assume that $h : L \to \mathbb{R}$ is a submodular function with the properties*

(1) $h(m_j) = 0$ for all $m_j \in M$.

(2) $h(a) \geq 0$ if $m_{j-1} \leq a \leq m_j$ for some j.

(3) $h(a) \geq 0$ if $a \leq m_1$ or $a \geq m_k$.

Then $h(a) \geq 0$ holds for all $a \in L$.

Proof. Suppose the Theorem is false and a a minimal counterexample. So $a \not\leq m_1$ and $a \not\geq m_k$. If $a \not\leq m_k$, then

$$\begin{aligned} h(a) &\geq h(a \wedge m_k) + h(a \vee m_k) - h(m_k) \\ &= h(a \wedge m_k) + h(a \vee m_k) \ \geq\ 0\ , \end{aligned}$$

as $a \vee m_k \geq m_k$ and $a \wedge m_k < a$ imply that both additive terms are non-negative. Hence there must exist some $j > 1$ such that

$$a \not\leq m_{j-1} \quad \text{and} \quad a \leq m_j .$$

Noting $m_{j-1} \leq a \vee m_{j-1} \leq m_j$, we then arrive at a contradiction in a similar way through the submodular expansion

$$\begin{aligned} h(a) &\geq h(a \wedge m_{j-1}) + h(a \vee m_{j-1}) - h(m_{j-1}) \\ &= h(a \wedge m_{j-1}) + h(a \vee m_{j-1}) \ \geq \ 0 \ , \end{aligned}$$

$\diamond$

Corollary 5.1 *Let (M, π) be the Monge pair obtained from the Monge algorithm relative to some weight function $\mathbf{c} : U \to \mathbb{R}$ and $\mathbf{x}^\pi$ the associated greedy vector relative to the submodular function $r : L \to \mathbb{R}$. Then we have*

$$\mathbf{x}^\pi(a) \leq r(a) \quad \textit{for all } a \in L$$

provided r is nonnegative and monotone increasing or the Monge algorithm solves the equality constrained system ($M^=$) .

Proof. Consider the submodular function $h(a) = r(a) - \mathbf{x}^\pi(a)$. In either case of the Corollary, h satisfies the hypothesis of Theorem 5.1. Hence we conclude

$$h(a) \geq 0 \quad \text{or} \quad \mathbf{x}^\pi(a) \leq r(a).$$

$\diamond$

5.1 Submodular Linear Programs

Under the same assumptions on L and U as before, consider a function $r : L \to \mathbb{R}$, a weighting $\mathbf{c} : U \to \mathbb{R}$ and the linear program

$$\text{(P)} \qquad \max \sum_{u \in U} c_u x_u \quad \text{such that} \quad \mathbf{x}(a) \leq r(a) \quad \text{for all } a \in L$$

with dual

$$\text{(D)} \qquad \min_{\mathbf{y} \geq \mathbf{0}} \sum_{a \in K} r(a) y_a \quad \text{such that} \quad \sum_{\chi(a) \ni u} y_a = c_u \quad \text{for all } u \in U.$$

(P) is always feasible. From linear programming duality, we therefore know that an optimal solution exists if and only if (D) is feasible. So the Monge algorithm can be used to decide whether an optimal solution exists at all. Let $\mathbf{y} \geq \mathbf{0}$ be the feasible solution computed by the Monge algorithm for (D) and construct the greedy vector $\mathbf{x}^\pi$ from the associated Monge pair (M, π).

Consider the matrix $\mathcal{M}$ with elements $\chi(m, u)$ for $m \in M$ and $u \in \pi$. Denoting by $\bar{\mathbf{y}}$ and $\bar{r}$ the restrictions of $\mathbf{y}$ and r to M and by $\bar{\mathbf{c}}$ and $\bar{\mathbf{x}}$ the restrictions of $\mathbf{c}$ and $\mathbf{x}^\pi$ to π, we find

$$\bar{\mathbf{y}}^T \mathcal{M} = \bar{\mathbf{c}}^T \quad \text{and} \quad \mathcal{M}\bar{\mathbf{x}} = \bar{r}$$

and thus conclude

$$\sum_{a \in L} r(a) y_a = \sum_{m \in M} r(m) y_m = \bar{\mathbf{y}}^T \mathcal{M} \bar{\mathbf{x}} = \sum_{u \in \pi} c_u \bar{x}_u = \sum_{u \in U} c_u x_u^\pi. \tag{1}$$

So $\mathbf{x}^\pi$ and $\mathbf{y} \geq \mathbf{0}$ are optimal solutions for (P) and (D) precisely when $\mathbf{x}^\pi$ is feasible for (P).

The same argument applies to the nonnegative version

$$(\mathrm{P}^+) \qquad \max_{\mathbf{x} \geq \mathbf{0}} \sum_{u \in U} c_u x_u \quad \text{such that} \quad \mathbf{x}(a) \leq r(a) \quad \text{for all } a \in L$$

of (P) with dual

$$(\mathrm{D}^+) \qquad \min_{\mathbf{y} \geq \mathbf{0}} \sum_{a \in K} r(a) y_a \quad \text{such that} \quad \sum_{\chi(a) \ni u} y_a \geq c_u \quad \text{for all } u \in U.$$

Hence we obtain our main result:

Theorem 5.2 *If $r : L \to \mathbb{R}$ is submodular, the Monge and the greedy algorithm construct optimal solutions for (D) and (P) or demonstrate that no optimal solution exists.*

If $r : L \to \mathbb{R}$ is submodular, nonnegative and monotone increasing, the Monge and the greedy algorithm construct optimal solutions for (D$^+$) and (P$^+$).

◇

6 Supermodular Functions

The function $p : L \to \mathbb{R}$ is said to be *supermodular* on the pseudolattice L if its negative $r = -p$ is submodular. Now the linear program (P) of the previous section is equivalent with the linear program

$$(\mathrm{Q}) \qquad \min \sum_{u \in U} c_u x_u \quad \text{such that} \quad \mathbf{x}(a) \geq p(a) \quad \text{for all } a \in L.$$

Hence the Monge and greedy algorithm also solves a linear program of type (Q) optimally if p is supermodular.

A curious situation arises from the nonnegative version

$$(\mathrm{Q}^+) \qquad \min_{\mathbf{x} \geq \mathbf{0}} \sum_{u \in U} c_u x_u \quad \text{such that} \quad \mathbf{x}(a) \geq p(a) \quad \text{for all } a \in L.$$

Frank [9] establishes a greedy algorithm to solve (Q^+) in the case where the supermodular function p is nonnegative and monotone decreasing. His algorithm is quite similar in spirit to our algorithm for the solution of (P^+) with a submodular and nonnegative monotone increasing r. Yet, we do not see a direct way to derive Frank's algorithm from our approach. Nor does Frank's algorithm appear to be applicable to (P^+).

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